Notes on simplicial homotopy theory

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The notes contained in this booklet were printed directly from files supplied by the authors before the course.

Contents

Introduction

These notes were used by the second author in a course on simplicial homotopy theory given at the CRM in February 2008 in preparation for the advanced courses on simplicial methods in higher categories that followed. They form the first four chapters of a book on simplicial homotopy theory, which we are currently preparing.

What appears here as Appendix A on Quillen model structures will, in fact, form a new chapter 2. The material in the present chapter 2 will be moved elsewhere. The second author apologizes for the resulting organizational and notational confusion, citing lack of time as his only excuse.

Both authors warmly thank the CRM for their hospitality during this period.

Chapter 1

Simplicial sets

This chapter introduces simplicial sets. A simplicial set is a combinatorial model of a topological space formed by gluing simplices together along their faces. This topological space, called the *geometric realization* of the simplicial set, is defined in section 3. Also in section 3, we introduce the fundamental category of a simplicial set, and the nerve of a small category. Section 2 is concerned with the skeleton decomposition of a simplicial set. Finally, in section 4 we give presentations of the nerves of some useful partially ordered sets.

1.1 Definitions and examples

The *simplicial category* $\mathbf{\Delta}$ has objects $[n] = \{0, \dots, n\}$ for $n \geq 0$ a nonnegative integer. A map $\alpha\colon [n] \to [m]$ is an order preserving function.

Geometrically, an *n-simplex* is the convex closure of $n+1$ points in general position in a euclidean space of dimension at least n. The *standard, geometric n-simplex* Δ_n is the convex closure of the standard basis $e_0, \dots, e_n$ of $\mathbf{R}^{n+1}$. Thus, the points of Δ_n consists of all combinations

$$p = \sum_{i=0}^{n} t_i e_i$$

with $t_i \geq 0$, and $\sum_{i=0}^{n} t_i = 1$. We can identify the elements of $[n]$ with the vertices $e_0, \dots, e_n$ of Δ_n. In this way a map $\alpha\colon [n] \to [m]$ can be linearly extended to a map $\Delta_\alpha\colon \Delta_n \to \Delta_m$. That is,

$$\Delta_\alpha(p) = \sum_{i=0}^{n} t_i e_{\alpha(i)}.$$

Clearly, this defines a functor $r\colon \mathbf{\Delta} \to Top$.

A *simplicial set* is a functor $X\colon \mathbf{\Delta}^{op} \to Set$. To conform with traditional notation, when $\alpha\colon [n] \to [m]$ we write $\alpha^*\colon X_m \to X_n$ instead of $X_\alpha\colon X[m] \to X[n]$. The elements of X_n are called the n-simplices of X.

Many examples arise from classical simplicial complexes. Recall that a *simplicial complex* K is a collection of non-empty, finite subsets (called *simplices*) of a given set V (of *vertices*) such that any non-empty subset of a simplex is a simplex. An *ordering* on K consists of a linear ordering $O(\sigma)$ on each simplex σ of K such that if $\sigma' \subseteq \sigma$ then $O(\sigma')$ is the ordering on σ' induced by $O(\sigma)$. The choice of an ordering for K determines a simplicial set by setting

$$K_n = \{(a_0, \ldots, a_n) | \sigma = \{a_0, \ldots, a_n\} \text{ is a simplex of K} \\ \text{and } a_0 \le a_1 \le \ldots \le a_n \text{ in the ordering } O(\sigma).\}$$

For $\alpha\colon [n] \to [m]$, $\alpha^*\colon K_m \to K_n$ is $\alpha^*(a_0, \ldots, a_m) = (a_{\alpha(0)}, \ldots, a_{\alpha(n)})$.

Remark. An $\alpha\colon [n] \to [m]$ in $\mathbf{\Delta}$ can be decomposed uniquely as $\alpha = \varepsilon\eta$, where $\varepsilon\colon [p] \to [m]$ is injective, and $\eta\colon [n] \to [p]$ is surjective. Moreover, if $\varepsilon^i\colon [n-1] \to [n]$ is the injection which skips the value $i \in [n]$, and $\eta^j\colon [n+1] \to [n]$ is the surjection covering $j \in [n]$ twice, then $\varepsilon = \varepsilon^{i_s} \ldots \varepsilon^{i_1}$ and $\eta = \eta^{j_t} \ldots \eta^{j_1}$ where $m \ge i_s > \ldots > i_1 \ge 0$, and $0 \le j_t < \ldots < j_1 < n$ and $m = n-t+s$. The decomposition is unique, the i's in $[m]$ being the values not taken by α, and the j's being the elements of $[m]$ such that $\alpha(j) = \alpha(j+1)$. The ε^i and η^j satisfy the following relations:

$$\varepsilon^j \varepsilon^i = \varepsilon^i \varepsilon^{j-1} \qquad i < j$$

$$\eta^j \eta^i = \eta^i \eta^{j+1} \qquad i \le j$$

$$\eta^j \varepsilon^i = \begin{cases} \varepsilon^i \eta^{j-1} & i < j \\ id & i = j \text{ or } i = j+1 \\ \varepsilon^{i-1} \eta^j & i > j+1 \end{cases}$$

Thus, a simplicial set X can be considered to be a graded set $(X_n)_{n \ge 0}$ together with functions $d^i = \varepsilon^{i*}$ and $s^j = \eta^{j*}$ satisfying relations dual to those satisfied by the $\varepsilon^{i's}$ and $\eta^{j's}$. Namely,

$$d^j d^i = d^i d^{j-1} \qquad i < j$$

$$s^j s^i = s^i s^{j+1} \qquad i \le j$$

$$s^j d^i = \begin{cases} d^i s^{j-1} & i < j \\ id & i = j \text{ or } i = j+1 \\ d^{i-1} s^j & i > j+1 \end{cases}$$

This point of view is frequently adopted in the literature.

The category of simplicial sets is $[\mathbf{\Delta}^{op}, Set]$, which we often denote simply by $\mathbf{S}$. Again for traditional reasons, the representable functor $\mathbf{\Delta}(\ \ ,[n])$ is written $\Delta[n]$ and is called the *standard (combinatorial) n-simplex*. Conforming to this usage, we use $\Delta\colon \mathbf{\Delta} \to \mathbf{S}$ for the Yoneda functor, though if $\alpha\colon [n] \to [m]$, we write simply $\alpha\colon \Delta[n] \to \Delta[m]$ instead of $\Delta\alpha$.

Remark. We have

$$\Delta[n]_m = \Delta([m],[n]) = \{(a_0,\ldots,a_m) | 0 \le a_i \le a_j \le n \text{ for } i \le j\}$$

Thus, $\Delta[n]$ is the simplicial set associated to the simplicial complex whose simplices are all non-empty subsets of $\{0,\ldots,n\}$ with their natural orders. The *boundary* of this simplicial complex has all *proper* subsets of $\{0,\ldots,n\}$ as simplices. Its associated simplicial set is a simplicial $(n-1)$-sphere $\partial\Delta[n]$ called the *boundary* of $\Delta[n]$. Clearly, we have

$$\partial\Delta[n]_m = \{\alpha\colon [m] \to [n] | \alpha \text{ is not surjective}\}$$

$\partial\Delta[n]$ can also be described as the union of the $(n-1)$-faces of $\Delta[n]$. That is,

$$\partial\Delta[n] = \bigcup_{i=0}^{n} \partial^i\Delta[n]$$

where $\partial^i\Delta[n] = im(\varepsilon^i\colon \Delta[n-1] \to \Delta[n])$. Recall that the union is calculated pointwise, as is any colimit (or limit) in $[\mathbf{\Delta}^{op}, Set]$ [Appendix A 4.4].

1.2 The skeleton of a simplicial set

The relations between the ϵ^i and η^j of section 1 can be expressed by a diagram

$$\begin{tikzcd}
{[n]} \arrow[r, shift left, "\eta^j"] \arrow[d, shift right, "\eta^i"'] & {[n-1]} \arrow[l, shift left, "\varepsilon^j"] \arrow[d, shift left, "\eta^i"] \\
{[n-1]} \arrow[u, shift right, "\varepsilon^i"'] \arrow[r, shift right, "\eta^{j-1}"'] & {[n-2]} \arrow[l, shift right, "\varepsilon^{j-1}"'] \arrow[u, shift left, "\varepsilon^i"]
\end{tikzcd}$$

in $\mathbf{\Delta}$, for $n \ge 2$, in which

$$\begin{array}{ll}
\varepsilon^j\varepsilon^i = \varepsilon^i\varepsilon^{j-1} & i < j \\
\eta^{j-1}\eta^i = \eta^i\eta^j & i < j \\
\eta^j\varepsilon^i = \varepsilon^i\eta^{j-1} & i < j \\
\eta^i\varepsilon^i = id & \\
\eta^i\varepsilon^j = \varepsilon^{j-1}\eta^i & i < j-1 \\
\eta^i\varepsilon^{i+1} = id & i = j-1.
\end{array}$$

Note that we have interchanged i and j in the next to last equation. Now in such a diagram, the square of η's is an absolute, equational pushout, and the square of ε's is an absolute, equational pullback. In fact we have

Proposition 1.2.1. *Let*

$$\begin{tikzcd} C \arrow[r, shift left, "p_4"] \arrow[d, shift right, "p_3"'] & A \arrow[l, shift left, "s_4"] \arrow[d, shift left, "p_1"] \\ B \arrow[u, shift right, "s_3"'] \arrow[r, shift right, "p_2"'] & D \arrow[l, shift right, "s_2"'] \arrow[u, shift left, "s_1"] \end{tikzcd}$$

be a diagram in a category $\boldsymbol{A}$ *in which*

$$\begin{aligned} & s_4 s_1 = s_3 s_2 \\ & p_1 p_4 = p_2 p_3 \\ & p_4 s_3 = s_1 p_2 \\ & p_i s_i = id \qquad\qquad i = 1, 2, 3, 4 \\ & p_3 s_4 = s_2 p_1 \end{aligned}$$

or, instead of the last equation, $p_1 = p_2$, $s_1 = s_2$ *and* $p_3 s_4 = id$. *Then the square of s's is a pullback, and the square of p's is a pushout.*

Proof. We prove the pushout statement, as it is this we need, and leave the pullback part as an exercise. Thus, let $g_1 \colon A \to X$ and $g_2 \colon B \to X$ satisfy $g_1 p_4 = g_2 p_3$. Then, $g_1 = g_1 p_4 s_4 = g_2 p_3 s_4$, so $g_1 s_1 = g_2 p_3 s_4 s_1 = g_2 p_3 s_3 s_2 = g_2 s_2$ as a map from D to X. Furthermore, $(g_1 s_1) p_2 = g_1 p_4 s_3 = g_2 p_3 s_3 = g_2$ and $(g_2 s_2) p_1 = g_2 p_3 s_4 = g_1 p_4 s_4 = g_1$. The map $g_1 s_1 = g_2 s_2$ is unique, for if $a \colon D \to X$ satisfies $g_1 = a p_1$ and $g_2 = a p_2$, then $g_1 s_1 = a p_1 s_1 = a$ and $g_2 s_2 = a p_2 s_2 = a$. If we are in the alternate situation $p_1 = p_2$, $s_1 = s_2$ and $p_3 s_4 = id$, then $g_1 = g_2 p_3 s_4 = g_2$ as above, and $g_2 = g_1 s_1 p_2$, $g_1 = g_2 s_2 p_1$ and $g_1 s_1 = g_2 s_2$ is unique again as above. $\square$

If $i = j$, then

$$\begin{tikzcd} {[n+1]} \arrow[r, "\eta^i"] \arrow[d, "\eta^i"'] & {[n]} \arrow[d, "id_{[n]}"] \\ {[n]} \arrow[r, "id_{[n]}"'] & {[n]} \end{tikzcd}$$

is a pushout —since η^i is surjective— and equational since η^i has a section. There is a similar pullback for ε^i. Notice, however, that the pullback

$$\begin{tikzcd} & {[0]} \arrow[d, "\varepsilon^1"] \\ {[0]} \arrow[r, "\varepsilon^0"'] & {[1]} \end{tikzcd}$$

does not exist in $\mathbf{\Delta}$, though, of course morally, it is empty.

Any surjection can be factored uniquely into a composite of η^i's, and any injection into a composite of ε^i's so we have the following

Theorem 1.2.1. *$\mathbf{\Delta}$ has absolute pushouts of surjections, and absolute non-empty intersections of injections.*

An n-simplex x of a simplicial set X is said to be *degenerate* if there is a surjection $\eta\colon [n] \to [m]$ with $m < n$ and an m-simplex y such that $x = \eta^* y$. An important consequence of Theorem 1.2.1 is the following

Proposition 1.2.2 (The Eilenberg-Zilber Lemma)**.** *For each $x \in X_n$ there exists a unique surjection $\eta\colon [n] \to [m]$ and a unique non-degenerate $y \in X_m$ such that $x = \eta^* y$.*

Proof. The existence of such a pair (η, y) is clear, so suppose (η, y) and $(\eta^{'}, y')$ are two such. The pushout

$$\begin{tikzcd}
{[n]} \arrow[r, "\eta"] \arrow[d, "\eta^{'}"'] & {[m]} \arrow[d, "\eta_2"] \\
{[m']} \arrow[r, "\eta_1"'] & {[p]}
\end{tikzcd}$$

in $\mathbf{\Delta}$ is preserved by Δ, so we have a pushout

$$\begin{tikzcd}
\Delta[n] \arrow[r, "\eta"] \arrow[d, "\eta^{'}"'] & \Delta[m] \arrow[d, "\eta_2"] \\
\Delta[m'] \arrow[r, "\eta_1"'] & \Delta[p]
\end{tikzcd}$$

in $\mathbf{S}$. $\eta^* y = \eta^{'*} y' = x$, so there is $z\colon \Delta[p] \to X$ such that $y' = \eta_1^* z$, and $y = \eta_2^* z$. Since y and y' are non-degenerate, $\eta_1 = \eta_2 = id, y = y'$ and $\eta = \eta^{'}$. $\square$

Let $\mathbf{\Delta}_n$ be the full subcategory of $\mathbf{\Delta}$ whose objects are the $[m]$ for $m \leq n$. A functor $X\colon \mathbf{\Delta}_n^{op} \to Set$ is called an *n-truncated* simplicial set. The inclusion $\mathbf{\Delta}_n \to \mathbf{\Delta}$ induces a restriction functor

$$tr^n\colon [\mathbf{\Delta}^{op}, Set] \to [\mathbf{\Delta}_n^{op}, Set]$$

which truncates a simplicial set X at n. tr^n has a left adjoint sk^n, which is given as follows. If X is an n-truncated simplicial set,

$$sk^n X = \varinjlim_{\Delta_n[p] \to X} \Delta[p].$$

Since the inclusion $\mathbf{\Delta}_n \to \mathbf{\Delta}$ is full and faithful, it follows that $(sk^n X)_m \simeq X_m$ for $m \leq n$ so that the adjunction $X \to tr^n sk^n X$ is an isomorphism, and sk^n

is full and faithful. Since $sk^n X$ is a quotient of a sum of $\Delta[p]$ for $p \leq n$, and the m-simplices of $\Delta[p]$ are degenerate for $m > n$, it follows that $(sk^n X)_m$ consists entirely of degenerate simplices for $m > n$. From this we can conclude

Proposition 1.2.3. *The adjunction $sk^n tr^n X \to X$ is a monomorphism.*

Proof. By the above, the map $(sk^n tr^n X)_m \to X_m$ is a bijection for $m \leq n$. It will thus suffice to prove that if $f\colon Y \to X$ is a map of simplicial sets such that $f_m\colon Y_m \to X_m$ is injective for $m \leq n$ and the m-simplices of Y are degenerate for $m > n$ then f_m is injective for all m. In fact, let $y, y' \in Y_m$ for $m > n$. By the Eilenberg-Zilber Lemma, there are surjections $\eta\colon [m] \to [p]$ and $\eta'\colon [m] \to [p']$ and non-degenerate simplices z and z' such that $\eta^* z = y$, $\eta^{'*} z' = y'$. Since $p, p' \leq n$, and $f_p, f_{p'}$ are injective, it follows that $f_p(z)$ and $f_{p'}(z')$ are non-degenerate. Indeed, suppose, say, $f_p(z) = \alpha^* x$ where $\alpha\colon [p] \to [q]$ is surjective. α has a section $\epsilon\colon [q] \to [p]$ such that $\alpha\epsilon = id$. It follows that $\epsilon^* f_p(z) = \epsilon^* \alpha^* x = x$, so $\alpha^* \epsilon^* f_p(z) = f_p(\alpha^* \epsilon^* z) = \alpha^* x = f_p(z)$ But then $z = \alpha^* \epsilon^* z$ making z degenerate. Since $f_m(y) = \eta^* f_p(z)$ and $f_m(y') = \eta^{'*} f_{p'}(z')$, by the Eilenberg-Zilber Lemma, if $f_m(y) = f_m(y')$ we have $\eta = \eta'$ and $f_p(z) = f_{p'}(z')$. Thus, $p = p', z = z'$, and $y = y'$. □

We will write $Sk^n X$ for the image of $sk^n tr^n X$ in X, and call it the *n-skeleton* of X. Clearly, $(Sk^n X)_m = X_m$ for $m \leq n$, and for $m > n$ $(Sk^n X)_m$ consists of those m-simplices x of X_m for which there is a surjection $\eta\colon [m] \to [p]$ with $p \leq n$ and a $y \in X_p$ such that $x = \eta^* y$.

We say X is of *dimension* $\leq n$ if $Sk^n X = X$. Notice that the adjoint pair $sk^n \dashv tr^n$ provides an equivalence between the full subcategory of simplicial sets of dimension $\leq n$ and the category of n-truncated simplicial sets.

We remark that tr^n also has a right adjoint csk^n, which is given as follows:

$$(csk^n X)_m \simeq hom(\Delta[m], csk^n X) \simeq hom(tr^n \Delta[m], X).$$

Since $tr^n \Delta[m] = \Delta_n[m]$ for $m \leq n$, it follows that the adjunction $tr^n csk^n X \to X$ is an isomorphism and csk^n is full and faithful. We write $Csk^n X$ for $csk^n tr^n X$ and call it the *n-coskeleton* of X.

$$(Csk^n X)_m \simeq hom(Sk^n \Delta[m], X)$$

so that $(Csk^n X)_m = X_m$ for $m \leq n$. The adjuntion $X \to Csk^n X$ takes an m-simplex $\Delta[m] \to X$ of X to its restriction $Sk^n \Delta[m] \to \Delta[m] \to X$.

A simplicial set X is the union of its skeletons

$$Sk^0 X \subseteq Sk^1 X \ldots Sk^{n-1} X \subseteq Sk^n X \ldots$$

where $Sk^0 X$ is a discrete simplicial set...a sum of $\Delta[0]$'s corresponding to the 0-simplices of X. Furthermore, we claim $Sk^n X$ can be obtained from $Sk^{n-1} X$ as follows. Let $e(X)_n$, $n \geq 0$, denote the set of non-degenerate n-simplices of X. For each $x \in e(X)_n$, $n \geq 1$, consider the diagram

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d] & \Delta[n] \arrow[d, "x"] \\ Sk^{n-1}X \arrow[r] & Sk^n X \end{tikzcd}$$

Summing over x yields a diagram

$$\begin{tikzcd} \sum_{e(X)_n} \partial\Delta[n] \arrow[r] \arrow[d] & \sum_{e(X)_n} \Delta[n] \arrow[d] \\ Sk^{n-1}X \arrow[r] & Sk^n X \end{tikzcd}$$

Proposition 1.2.4. *The above diagram is a pushout.*

Proof. Since all of the simplicial sets involved are of dimension $\leq n$, it suffices to verify that the diagram is a pushout after application of tr^n, i.e. that

$$\begin{tikzcd} \sum_{e(X)_n} \partial\Delta[n]_m \arrow[r] \arrow[d] & \sum_{e(X)_n} \Delta[n]_m \arrow[d] \\ (Sk^{n-1}X)_m \arrow[r] & (Sk^n X)_m \end{tikzcd}$$

is a pushout in *Set* for $m \leq n$. For $m \leq n-1$ this is clear, since then the two horizontal maps are isomorphisms. For $m = n$, the complement of $\partial\Delta[n]_n$ in $\Delta[n]_n$ consists of one element, $id_{[n]}$. Thus, the complement of $\sum_{e(X)_n} \partial\Delta[n]_n$ in $\sum_{e(X)_n} \Delta[n]_n$ is isomorphic to $e(X)_n$. But $(Sk^n X)_n = (Sk^{n-1}X)_n \bigcup e(X)_n$ so the diagram is also a pushout for $m = n$. $\square$

1.3 Geometric realization and the fundamental category

Using the universal property of $\mathbf{\Delta} - Set$, the functor $r\colon \mathbf{\Delta} \to Top$ can be extended to a functor $|\ \ |\colon \mathbf{S} \to Top$, called the *geometric realization*. Thus, we have a commutative triangle

$$\begin{tikzcd} \mathbf{\Delta} \arrow[rr, "\Delta"] \arrow[dr, "r"'] & & \mathbf{S} \arrow[dl, "|\ \ |"] \\ & Top & \end{tikzcd}$$

where

$$|X| = \varinjlim_{\Delta[n]\to X} \Delta_n \,.$$

Another useful way of writing the geometric realization is as a coend

$$|X| = \int^n X_n \times \Delta_n.$$

$|\quad|$ has a right adjoint s. For any topological space T, sT is the *singular complex* of T.

$$(sT)_n = Top(\Delta_n, T).$$

Since a left adjoint preserves colimits , the geometric realization $|\quad| : \mathbf{S} \to Top$ is colimit preserving. A consequence of this is that $|X|$ is a $CW-$complex. To see this, we remark first that for $n \geq 2$ the union $\partial\Delta[n]$ is computed as the coequalizer of ϕ and ψ in the diagram

$$\begin{tikzcd}
\sum_{i<j} \Delta[n-2]_{ij} \arrow[r, shift left, "\phi"] \arrow[r, shift right, "\psi"'] & \sum_{0\le i\le n} \Delta[n-1]_i \arrow[rr] \arrow[dr, "\zeta"'] & & \partial\Delta[n] \arrow[dl] \\
& & \Delta[n] &
\end{tikzcd}$$

Here $\Delta[n-1]_i$ is a copy of $\Delta[n-1]$ and $\Delta[n-2]_{ij}$ is a copy of $\Delta[n-2]$. If we denote the injections of the sum by μ then ζ, ϕ and ψ are the unique maps dedtermined by $\zeta\mu_i = \epsilon^i$, $\phi\mu_{ij} = \mu_i\epsilon^{j-1}$ and $\psi\mu_{ij} = \mu_j\epsilon^i$. This coequalizer *is* $\partial\Delta[n]$ because for $i < j$ the diagram

$$\begin{tikzcd}
{[n-2]} \arrow[r, "\epsilon^{j-1}"] \arrow[d, "\epsilon^i"'] & {[n-1]} \arrow[d, "\epsilon^i"] \\
{[n-1]} \arrow[r, "\epsilon^j"'] & {[n]}
\end{tikzcd}$$

is an absolute intersection in $\mathbf{\Delta}$ (Theorem 1.2.1). It follows that $|\partial\Delta[n]| = \partial\Delta_n$. Then, by applying $|\quad|$ to the pushout of Proposition 1.2.4, we see that $|X|$ has a complete filtration given by $|X|^n = |Sk^n X|$, and $|X|^n$ arises by attaching n-cells to $|X|^{n-1}$. That is, $|X|$ is a CW-complex.

Furthermore, if Top is replaced by Top_c —the category of compactly generated spaces— then $|\quad|$ is also left-exact, i.e. preserves all finite limits.

Denote by Cat the category of small categories and functors. There is a functor $\mathbf{\Delta} \to Cat$ which sends $[n]$ into $[n]$ regarded as a category via its natural ordering. Again, by the universal property of $\mathbf{\Delta} - Set$ this functor can be extended to a functor $\tau_1 \colon \mathbf{S} \to Cat$ so as to give a commutative triangle

$$\begin{tikzcd} \boldsymbol{\Delta} \arrow[rr, "\Delta"] \arrow[dr] & & \mathbf{S} \arrow[dl, "\tau_1"] \\ & Cat & \end{tikzcd}$$

where

$$\tau_1 X = \varinjlim_{\Delta[n] \to X} [n]$$

$\tau_1 X$ is called the *fundamental category* of X. As before, τ_1 has a right adjoint N. If $\boldsymbol{A}$ is a small category, $N\boldsymbol{A}$ is the *nerve* of $\boldsymbol{A}$ and

$$(N\boldsymbol{A})_n = Cat([n], \boldsymbol{A}).$$

For future applications, we will need a more explicit description of $\tau_1 X$. Start by letting $\boldsymbol{\Delta}_2 \to Cat$ be the composite $\boldsymbol{\Delta}_2 \to \boldsymbol{\Delta} \to Cat$. As above, this can be extended to $[\boldsymbol{\Delta}_2^{op}, Set]$ giving a commutative diagram

$$\begin{tikzcd} \boldsymbol{\Delta_2} \arrow[rr, "\Delta_2"] \arrow[dr] & & {[\boldsymbol{\Delta}_2^{op}, Set]} \arrow[dl, "\tau_1^2"] \\ & Cat & \end{tikzcd}$$

$$\tau_1^2 X = \varinjlim_{\Delta_2[p] \to X} [p] \ .$$

Let $N^2 \colon Cat \to [\boldsymbol{\Delta}_2^{op}, Set]$ denote the right adjoint to τ_1^2. If $\boldsymbol{A}$ is a small category,

$$(N^2\boldsymbol{A})_p = Cat([p], \boldsymbol{A}) \ \ p \leq 2$$

Evidently, $N^2\boldsymbol{A} = tr^2 N\boldsymbol{A}$, from which it follows that $\tau_1^2 X \simeq \tau_1 sk^2 X$.

Proposition 1.3.1. *If $\boldsymbol{A}$ is a small category, the canonical map $N\boldsymbol{A} \to Csk^2 N\boldsymbol{A}$ is an isomorphism.*

Proof. Let us write ρ for the map $N\boldsymbol{A} \to Csk^2 N\boldsymbol{A}$. It is bijective iff every map $a \colon Sk^2\Delta[n] \to N\boldsymbol{A}$ can be extended uniquely to a map $\Delta[n] \to N\boldsymbol{A}$. Notice first that a map $a \colon Sk^1\Delta[n] \to \boldsymbol{A}$ is just a mapping of graphs with unit: it assigns to each $i \in [n]$ an object $a_i \in A_0$ and to each pair $i < j$ an arrow $a_{ij} \colon a_i \to a_j$. a can be extended to $Sk^2\Delta[n]$ iff for each triple $i < j < k$ we have $a_{jk}a_{ij} = a_{ik}$, from which it is clear that a map $a \colon Sk^2\Delta[n] \to N\boldsymbol{A}$ extends uniquely to a map $\Delta[n] \to N\boldsymbol{A}$. $\square$

Since $Sk^2 \dashv Csk^2$, and $\tau_1 \dashv N$, by uniqueness of adjoints we obtain

Corollary 1.3.1. *If X is a simplicial set, the canonical functor $\tau_1 Sk^2 X \to \tau_1 X$ is an isomorphism.*

Since $\tau_1 sk^2 \simeq \tau_1^2$ we have

Proposition 1.3.2. *The functor $\tau_1^2 tr^2 X \to \tau_1 X$ is an isomorphism.*

Let X be a 2-truncated simplicial set. Define a category $\tau_1' X$ as follows. Let FX be the free category on the graph

$$\begin{tikzcd} X_1 \arrow[r, shift left, "d^1"] \arrow[r, shift right, "d^0"'] & X_0 \end{tikzcd}$$

with units $s^0 \colon X_0 \to X_1$. Then, let $\tau_1' X$ be the quotient of FX by the relations $d^1 x \equiv d^0 x \circ d^2 x$ for $x \in X_2$.

Theorem 1.3.1. *$\tau_1^2 X$ is isomorphic to $\tau_1' X$.*

Proof. Let $\boldsymbol{A}$ be a small category. $(N^2\boldsymbol{A})_0 = \boldsymbol{A}_0$ —the objects of $\boldsymbol{A}$, and $(N^2\boldsymbol{A})_1 = \boldsymbol{A}_1$— the arrows of $\boldsymbol{A}$. Thus, a map $f \colon X \to N^2\boldsymbol{A}$ consists of three mappings $f_0 \colon X_0 \to \boldsymbol{A}_0, f_1 \colon X_1 \to \boldsymbol{A}_1$ and $f_2 \colon X_0 \to (N^2\boldsymbol{A})_2$ which commute with all possible faces and degeneracies. For levels 0 and 1, the diagram

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, shift right=2, "d^0"'] \arrow[d, shift left=2, "d^1"] & \boldsymbol{A}_1 \arrow[d, shift right=2, "d^0"'] \arrow[d, shift left=2, "d^1"] \\ X_0 \arrow[r, "f_0"'] \arrow[u, "s^0"] & \boldsymbol{A}_0 \arrow[u, "s^0"] \end{tikzcd}$$

is equivalent to a functor $FX \to \boldsymbol{A}$.

$(N^2\boldsymbol{A})_2$ is the pullback

$$\begin{tikzcd} \boldsymbol{A}_1 \times_{\boldsymbol{A}_0} \boldsymbol{A}_1 \arrow[r, "d^0"] \arrow[d, "d^2"'] & \boldsymbol{A}_1 \arrow[d, "d^1"] \\ \boldsymbol{A}_1 \arrow[r, "d^0"'] & \boldsymbol{A}_0 \end{tikzcd}$$

consisting of the composable pairs of $\boldsymbol{A}$. Thus, for levels 1 and 2 of the map $f \colon X \to N^2\boldsymbol{A}$ we have the diagram

$$\begin{tikzcd} X_2 \arrow[r, "f_2"] \arrow[d, shift right=3, "d^0"'] \arrow[d, "d^1"] \arrow[d, shift left=3, "d^2"] & \boldsymbol{A}_1 \times_{\boldsymbol{A}_0} \boldsymbol{A}_1 \arrow[d, shift right=3, "d^0"'] \arrow[d, "d^1"] \arrow[d, shift left=3, "d^2"] \\ X_1 \arrow[r, "f_1"'] & \boldsymbol{A}_1 \end{tikzcd}$$

It follows that $f_2 = (f_1 d^2, f_1 d^0)$ since $f_1 d^0 = d^0 f_2, f_1 d^2 = d^2 f_2$ and $(N_2\boldsymbol{A})_2$ is a pullback. But then, the equations $f_2 s^0 = (f_1 d^2 s^0, f_1 d^0 s^0) = (f_1 s^0 d^1, f_1) = s^0 f_1$ and $f_2 s^1 = (f_1 d^2 s^1, f_1 d^0 s^1) = (f_1, f_1 s^0 d^0) = s^1 f_1$ follow from the simplicial identities. Finally, we have $f_1 d^1 = d^1 f_2$ iff for each $x \in X_2$, $f_1 d^1 x = d^1(f_1 d^2 x, f_1 d^0 x) = (f_1 d^0 x) \circ (f_1 d^2 x)$. Thus, the map $f\colon X \to N^2\boldsymbol{A}$ is equivalent to a functor $\tau_1' X \to \boldsymbol{A}$. The theorem then follows by uniqueness of adjoints. $\square$

From Proposition 1.3.2 and Theorem 1.3.1 we see that $\tau_1(X)$ can be described as follows: its objects are the 0-simplices X_0 of X, and its arrows are generated by the 1-simplices X_1 with $sx = d^1 x$ and $tx = d^0 x$ for $x \in X_1$, subject to the relations $s^0 x \equiv id_x$ for $x \in X_0$ and $d^1 x \equiv d^0 x \circ d^2 x$ for $x \in X_2$.

Corollary 1.3.2. *If $\boldsymbol{A}$ is a small category, the adjunction $\tau_1 N\boldsymbol{A} \to \boldsymbol{A}$ is an isomorphism.*

Corollary 1.3.3. *If X and Y are simplicial sets, the canonical functor $\tau_1(X \times Y) \to \tau_1 X \times \tau_1 Y$ is an isomorphism.*

Proof. In $\mathbf{S}$ as well as Cat products commute with colimits since each is cartesian closed. X and Y are colimits of simplices, so it is enough to show that $\tau_1(\Delta[n] \times \Delta[m]) \to \tau_1\Delta[n] \times \tau_1\Delta[m]$ is an isomorphism for $n, m \geq 0$. But $\tau_1(\Delta[n] \times \Delta[m]) = \tau_1(N[n] \times N[m]) \simeq \tau_1(N([n] \times [m])) \simeq [n] \times [m] \simeq \tau_1\Delta[n] \times \tau_1\Delta[m]$. $\square$

Remark. τ_1 does not preserve all finite limits. In fact, τ_1 does not preserve pullbacks. For an example, consider the pushout

$$\begin{tikzcd}
\partial\Delta[2] \arrow[r, "i"] \arrow[d, "i"'] & \Delta[2] \arrow[d] \\
\Delta[2] \arrow[r] & S
\end{tikzcd}$$

where i is the inclusion. Since i is a monomorphism, the diagram is also a pullback (this is true in any topos). Applying τ_1 we obtain a pushout

$$\begin{tikzcd}
\tau_1\partial\Delta[2] \arrow[r] \arrow[d] & {[2]} \arrow[d] \\
{[2]} \arrow[r] & \tau_1 S
\end{tikzcd}$$

$\tau_1\partial\Delta[2]$ has three objects 0, 1, 2, and four morphism (besides identities), $a\colon 0 \to 1$, $b\colon 1 \to 2, c$ and $b \circ a\colon 0 \to 2$. The functor $\tau_1\partial\Delta[2] \to [2]$ identifies c and $b \circ a$, and is thus surjective on objects and morphisms. It follows that it is an epimorphism in Cat, so the functors $[2] \to \tau_1 S$ are isomorphisms. Hence, the diagram is not a pullback.

1.4 The nerve of a partially ordered set

We consider here the nerve NP of a partially ordered set P. An n-simplex of NP is an order-preserving mapping $x\colon [n] \to P$ which is non-degenerate iff it is injective. Since N preserves monomorphisms, the singular n-simplex $\Delta[n] \to NP$ associated to x is also injective. Thus, the image of a non-degenerate n-simplex of NP is a standard n-simplex.

Suppose P is finite. Call a totally ordered subset c of P a *chain* of P. Then there are a finite number $c_1 \dots c_r$ of maximal chains of P, and every chain c is contained in some c_i. If c_i contains $n_i + 1$ elements we can associate to it a unique non-degenerate simplex $x_i\colon [n_i] \to P$ whose image in P is c_i. Each non-degenerate simplex of NP is a face of some x_i. The x_i together yield a commutative diagram

$$\sum_{1 \le i < j \le r} [n_{ij}] \underset{\nu}{\overset{\mu}{\rightrightarrows}} \sum_{1 \le i \le r} [n_i] \longrightarrow P$$

where $n_{ij} + 1$ is the number of elements in $c_i \cap c_j$, and μ, respectively ν, is defined by the inclusion of $c_i \cap c_j$ in c_i, respectively c_j. Moreover, applying N, we obtain an *exact* diagram

$$\sum_{1 \le i < j \le r} \Delta[n_{ij}] \rightrightarrows \sum_{1 \le i \le r} \Delta[n_i] \longrightarrow NP$$

in $\mathbf{S}$, i.e. a diagram which is both a coequalizer and a kernel pair. To see this, simply evaluate the diagram at any $m \geq 0$, and check that the result is exact as a diagram in Set. We call this the *finite presentation of NP corresponding to the maximal chains of P.*

We examine in detail the example $P = [p] \times [q]$ as this will be of use to us later. We claim first that a maximal chains of $[p] \times [q]$ can be pictured as a path from $(0, 0)$ to (p, q) in the lattice of points (m, n) in the plane with integral coordinates where at each point (i, j) on the path, the next point is either immediately to the right, or up. An example follows for $p = 3$, $q = 2$.

$$\begin{tikzcd}
(0,2) & (1,2) & (2,2) & (3,2) \\
(0,1) & (1,1) \arrow[r] & (2,1) \arrow[r] & (3,1) \arrow[u] \\
(0,0) \arrow[r] & (1,0) \arrow[u] & (2,0) & (3,0)
\end{tikzcd}$$

The number of elements in each of these chains is $p + q + 1$ and they are clearly the maximal ones, since any maximal chain must contain $(0, 0)$ and (p, q), and whenever it contains (i, j) it must contain either $(i + 1, j)$ or $(i, j + 1)$.

We can identify these chains with (p,q)-*shuffles* $(\sigma;\tau)$, which are partitions of the set $(1,2,\ldots,p+q)$ into two disjoint subsets $\sigma = (\sigma_1 < \ldots < \sigma_p)$ and $\tau = (\tau_1 < \ldots < \tau_q)$. Such a partition describes a shuffling of a pack of p cards through a pack of q cards, putting the cards of the first pack in the positions $\sigma_1 < \ldots < \sigma_p$, and those of the second in the positions $\tau_1 < \ldots < \tau_q$. The identification proceeds as follows. Given a maximal chain c, let the σ's be the sums of the coordinates of the right-hand endpoints of the horizontal segments. The τ's are then the sums of the coordinates of the upper endpoints of the vertical segments. Thus, the shuffle associated to the maximal chain above is $(1,3,4;2,5)$. On the other hand, given a (p,q) shuffle $(\sigma;\tau)$, form a chain by selecting $(0,0)$, then $(0,1)$ if $1 \in \sigma$ and $(0,1)$ if $1 \in \tau$. In general, if (i,j) has been selected, select $(i+1,j)$ if $i+j+1 \in \sigma$ and $(i,j+1)$ if $i+j+1 \in \tau$. Since the correspondence is clearly $1-1$, and there are $\binom{p+q}{p}$ such shuffles, this is also the number of maximal chains. We thus obtain the presentation

$$\sum_{1\le i<j\le r} \Delta[n_{ij}] \rightrightarrows \sum_{1\le i\le r} \Delta[p+q] \longrightarrow \Delta[p]\times\Delta[q]$$

of $N([p]\times[q]) \simeq \Delta[p]\times\Delta[q]$, where $r = \binom{p+q}{p}$. Taking the geometric realization of this presentation yields a triangulation of $\Delta_p \times \Delta_q$ into $\binom{p+q}{p}$ $(p+q)$-simplices.

Chapter 2

Quillen homotopy structures

This chapter introduces Quillen homotopy structures, which, in one form or another, will occupy us for the rest of the book. We give the definition and some examples, whose proof will comprise chapter 3, in section 1. In section 2 we establish the Quillen structure on the category of small groupoids. Section 3 is concerned with cofibration and fibration structures. These are weaker than Quillen structures, but still have important consequences such as the "Glueing lemma". In section 4 we apply the results of section 3 to the fundamental groupoid of a simplicial set, obtaining a conceptual proof of the Van Kampen Theorem and a simple treatment of coverings. See Topic E for the development of basic abstract homotopy theory in the context of a category provided with a Quillen homotopy structure.

2.1 Homotopy structures

Definition 2.1.1. *Let $\mathcal{K}$ be a category with finite limits and colimits. A* Quillen homotopy structure *on $\mathcal{K}$ (also called a* Quillen model structure *on $\mathcal{K}$) consists of three classes $(\mathcal{F}, \mathcal{C}, \mathcal{W})$ of mappings of $\mathcal{K}$ called* fibrations*,* cofibrations*, and* weak equivalences *respectively. These are subject to the following axioms:*

Q1.(Saturation) If $f\colon X \to Y$ and $g\colon Y \to Z$ are mappings of $\mathcal{K}$, and any two of f, g or gf belong to $\mathcal{W}$, then so does the third. This is sometimes called the "three for two" property of weak equivalences.

Q2.(Retracts) Let

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & X' \arrow[r, "u"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y \arrow[r, "j"'] & Y' \arrow[r, "v"'] & Y \end{tikzcd}$$

be a commutative diagram with $ui = id_X$ and $vj = id_Y$. Then if f' is a fibration, cofibration or weak equivalence, so is f.

Q3.(Lifting) If

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

is a commutative diagram in which i is a cofibration and f is a fibration, then if i or f belongs to $\mathcal{W}$, there is a diagonal filler making both triangles commute.

Q4.(Factorization) Any map $f\colon X \to Y$ can be factored as

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "f"'] & & E \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

where i is a cofibration and p is a fibration in two ways: one in which i is in $\mathcal{W}$, and one in which p is in $\mathcal{W}$.

The homotopy structure is said to be *proper*, if, in addition, the following axiom is satisfied.

Q5. If

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "w'"'] & X \arrow[d, "w"] \\ Y' \arrow[r, "f"'] & Y \end{tikzcd}$$

is a pullback diagram, in which f is a fibration and w is in $\mathcal{W}$, then w' is in $\mathcal{W}$. Dually, the pushout of a weak equivalence by a cofibration is a weak equivalence.

We obtain three trivial examples of a Quillen homotopy structure on any category $\mathcal{K}$ by taking $\mathcal{F}$, $\mathcal{C}$ and $\mathcal{W}$, respectively, to be the isomorphisms, and the other two classes to be all maps. Two non-trivial examples are the following.

First, let $\mathcal{K}$ be Top_c. The class $\mathcal{W}$ of weak equivalences consists of all maps $f\colon X \to Y$ such that $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ is a bijection, and for $n \geq 1$ and $x \in X$, $\pi_n(f)\colon \pi_n(X, x) \to \pi_n(Y, fx)$ is an isomorphism. The class $\mathcal{F}$ is the class of Serre fibrations, i.e. maps $p\colon E \to X$ with the *covering homotopy property* (CHP) for each n-simplex Δ_n, $n \geq 0$. This means that if $h\colon \Delta_n \times I \to X$ is a homotopy ($I = [0, 1]$), and $f\colon \Delta_n \to E$ is such that $pf = h_0$, then there is a "covering homotopy" $\bar{h}\colon \Delta_n \times I \to E$ such that $\bar{h}_0 = f$, and $p\bar{h} = h$. The cofibrations $\mathcal{C}$ are mappings $i\colon A \to B$ having the *left lifting property* (LLP) with respect to those fibrations $p\colon E \to X$ which are also weak equivalences. That is, if

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & E \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

is a commutative diagram where p is a fibration and a weak equivalence, then there is a diagonal filler making both triangles commute.

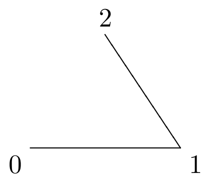
Theorem 2.1.1. *The fibrations, cofibrations and weak equivalences defined above form a Quillen homotopy structure on* Top_c

Our principal, and most difficult, example is on **S**, the category of simplicial sets. Here, the weak equivalences are *geometric homotopy equivalences*, by which we mean a map $f\colon X \to Y$ such that $|f|\colon |X| \to |Y|$ is a homotopy equivalence, i.e. there is a map $f'\colon |Y| \to |X|$ such that $f'|f|$ is homotopic to $id_{|X|}$, and $|f|f'$ is homotopic to $id_{|Y|}$. The cofibrations are monomorphisms.

To define the fibrations, recall that the i^{th} face of $\Delta[n]$ $(n \geq 1, 0 \leq i \leq n)$ is $\Delta^i[n] = im(\varepsilon^i\colon \Delta[n1] \to \Delta[n])$. The k^{th} *horn* of $\Delta[n]$ is

$$\Lambda^k[n] = \bigcup_{i \neq k} \Delta^i[n].$$

The geometric realization of $\Lambda^k[n]$ is the union of all those $(n-1)$-dimensional faces of Δ_n that contain the k^{th} vertex of Δ_n. For example,



is the geometric realization of $\Lambda^1[2]$.

Definition 2.1.2. *A Kan fibration is a map $p\colon E \to X$ of simplicial sets having the right lifting property (RLP) with respect to the inclusions of the horns $\Lambda^k[n] \to \Delta[n]$ for $n \geq 1$, and $0 \leq k \leq n$.*

That is, if

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

is a commutative diagram with $n \geq 1$, and $0 \leq k \leq n$, then there is a diagonal filler making both triangles commute. We express this by saying "any horn in E which can be filled in X, can be filled in E". For example, $p\colon E \to X$ is a Serre fibration in Top_c, iff $sp\colon sE \to sX$ is a Kan fibration in $\mathbf{S}$. This is so, because a diagram

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d] & sE \arrow[d, "sp"] \\ \Delta[n] \arrow[r] & sX \end{tikzcd}$$

is equivalent to a diagram

$$\begin{tikzcd} |\Lambda^k[n]| \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ |\Delta[n]| \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

and $|\Lambda^k[n]| \to |\Delta[n]|$ is homeomorphic to $\Delta_{n-1} \to \Delta_{n-1} \times I$.

Most of chapter 3 will be devoted to the proof of the following theorem.

Theorem 2.1.2. *The fibrations, cofibrations and weak equivalences defined above form a proper Quillen homotopy structure on* $\mathbf{S}$.

2.2 The Quillen structure on groupoids

Let *Gpd* denote the category of small groupoids. A functor $w\colon \boldsymbol{G} \to \boldsymbol{H}$ will be said to be a *weak equivalence* if it is a categorical equivalence, i.e. full, faithful and representative, meaning that for each object h of $\boldsymbol{H}$ there is an object g of $\boldsymbol{G}$ and an arrow $h \to w(g)$. Equivalently, w is a weak equivalence if it has a *quasi-inverse*, i.e. a functor $w'\colon \boldsymbol{H} \to \boldsymbol{G}$ together with isomorphisms $ww' \to id_{\boldsymbol{H}}$ and $id_{\boldsymbol{G}} \to w'w$. We call a functor $i\colon \boldsymbol{A} \to \boldsymbol{B}$ a *cofibration* if it is injective on objects. A *fibration* will be a Grothendieck fibration, which for groupoids just means a functor $p\colon \boldsymbol{E} \to \boldsymbol{C}$ such that if e is an object of $\boldsymbol{E}$ and $\gamma\colon c \to p(e)$ is an arrow of $\boldsymbol{C}$ then there is an arrow $\epsilon\colon e' \to e$ of $\boldsymbol{E}$ such that $p(\epsilon) = \gamma$.

Theorem 2.2.1. *The fibrations, cofibrations and weak equivalences defined above form a Quillen homotopy structure on Gpd.*

We digress for a moment to introduce an auxilary concept, which will be useful in the proof of the theorem. Namely, we call a functor $p\colon \boldsymbol{E} \to \boldsymbol{C}$ a *trivial fibration* if it has the right lifting property with respect to the cofibrations.

Lemma 2.2.1. $p\colon \boldsymbol{E} \to \boldsymbol{C}$ *is a trivial fibration iff it is a weak equivalence and* $p_0\colon E_0 \to C_0$ *is surjective.*

Proof. Let $p\colon \boldsymbol{E} \to \boldsymbol{C}$ be a trivial fibration. Denoting the empty groupoid by $\mathbf{0}$, the unique functor $\mathbf{0} \to \boldsymbol{C}$ is a cofibration. It follows that there is a diagonal filler s in the diagram

$$\begin{tikzcd}
\mathbf{0} \arrow[r] \arrow[d] & \boldsymbol{E} \arrow[d, "p"] \\
\boldsymbol{C} \arrow[r, "id"'] \arrow[ur, dashed, "s"] & \boldsymbol{C}
\end{tikzcd}$$

so that $p_0\colon E_0 \to C_0$ is surjective. Let $\boldsymbol{I}$ be the groupoid with two objects 0 and 1 and one isomorphism between them in addition to identities. In the commutative square

$$\begin{tikzcd}
(\boldsymbol{E} \times 0) + (\boldsymbol{E} \times 1) \arrow[rr, "(sp{,}id)"] \arrow[d] & & \boldsymbol{E} \arrow[d, "p"] \\
\boldsymbol{E} \times \boldsymbol{I} \arrow[r, "\pi"'] \arrow[urr, dashed, "u"] & \boldsymbol{E} \arrow[r, "p"'] & \boldsymbol{C}
\end{tikzcd}$$

where π is the first projection, the left-hand vertical functor is a cofibration, so the square has a diagonal filler u, which provides an isomorphism $sp \simeq id$. Thus, p is a weak equivalence.

On the other hand, suppose $p\colon \boldsymbol{E} \to \boldsymbol{C}$ is a weak equivalence and $p_0\colon E_0 \to C_0$ is surjective. Let $\boldsymbol{A} \to \boldsymbol{B}$, be a cofibration, and suppose

$$\begin{tikzcd}
\boldsymbol{A} \arrow[r, "g"] \arrow[d] & \boldsymbol{E} \arrow[d, "p"] \\
\boldsymbol{B} \arrow[r, "f"'] & \boldsymbol{C}
\end{tikzcd}$$

commutes. In

$$\begin{tikzcd}
A_0 \arrow[r, "g_0"] \arrow[d] & E_0 \arrow[d, "p_0"] \\
B_0 \arrow[r, "f_0"'] \arrow[ur, dashed, "\overline{f_0}"] & C_o
\end{tikzcd}$$

define $\overline{f_0}$ as g_0 on A_0, and sf_0 on $B_0 - A_0$, where $s\colon C_0 \to E_0$ is a section to p_0, i.e. $p_0 s = id$. Suppose $b \to b'$ is an arrow of $\boldsymbol{B}$. $f(b \to b') = fb \to fb' = p\overline{f}b \to p\overline{f}b'$. Since p is full and faithful, there is a unique arrow $\overline{f}b \to \overline{f}b'$ in $\boldsymbol{E}$ such that $p(\overline{f}b \to \overline{f}b') = fb \to fb'$. Set $\overline{f}(b \to b') = \overline{f}b \to \overline{f}b'$. Thus, the original diagram has a filler $\overline{f}$, so p is a trivial fibration. $\square$

Proof of Theorem 2.2.1. Q1 and Q2 are easy and straightforward, and are left to the reader as exercises.

To establish the first half of Q3, let

$$\begin{tikzcd} \boldsymbol{A} \arrow[r, "u"] \arrow[d, "i"'] & \boldsymbol{E} \arrow[d, "p"] \\ \boldsymbol{B} \arrow[r, "v"'] \arrow[ur, dashed, "\overline{v}"] & \boldsymbol{C} \end{tikzcd}$$

be a commutative diagram in which i is a cofibration weak equivalence, and p is a fibration. We want a diagonal filler $\overline{v}\colon \boldsymbol{B} \to \boldsymbol{E}$ making both triangles commute. We begin by choosing, for each object b of $\boldsymbol{B}$, an arrow $\beta\colon b \to ia$ where a is in $\boldsymbol{A}$, choosing β to be the identity when $b = ia$. Given an arrow $b \to b'$ of $\boldsymbol{B}$ there is a unique $\alpha\colon a \to a'$ in $\boldsymbol{A}$ making

$$\begin{tikzcd} b \arrow[r] \arrow[d, "\beta"'] & b' \arrow[d, "\beta'"] \\ ia \arrow[r, "i\alpha"'] & ia' \end{tikzcd}$$

commute since i is full and faithful. Applying v we obtain the square

$$\begin{tikzcd} vb \arrow[r] \arrow[d, "v\beta"'] & vb' \arrow[d, "v\beta'"] \\ pua \arrow[r, "u\alpha"'] & pua' \end{tikzcd}$$

in $\boldsymbol{C}$. Since p is a fibration, we can lift $v\beta$ and $v\beta'$ to a diagram

$$\begin{tikzcd} e \arrow[d, "\epsilon"'] & e' \arrow[d, "\epsilon'"] \\ ua \arrow[r, "u\alpha"'] & ua' \end{tikzcd}$$

in $\boldsymbol{E}$. We thus obtain an arrow $e \to e'$ in $\boldsymbol{E}$, and we put $\overline{v}(b \to b') = e \to e'$.

To establish the first half of Q4, let $f\colon \boldsymbol{G} \to \boldsymbol{H}$ be an arbitrary functor, and consider the triangle

$$\begin{tikzcd} \boldsymbol{G} \arrow[rr, "i"] \arrow[dr, "f"'] & & (\boldsymbol{H}, f) \arrow[dl, "p"] \\ & \boldsymbol{H} & \end{tikzcd}$$

where the objects of $(\boldsymbol{H}, f)$ are triples (y, α, x) with y in $\boldsymbol{H}$, x in $\boldsymbol{G}$ and $\alpha\colon y \to fx$.

An arrow $(y, \alpha, x) \to (y', \alpha', x')$ is a pair (β, γ) where $\beta\colon y \to y'$, $\gamma\colon x \to x'$ and

$$\begin{tikzcd} y \arrow[r, "\alpha"] \arrow[d, "\beta"'] & fx \arrow[d, "f\gamma"] \\ y' \arrow[r, "\alpha'"'] & fx' \end{tikzcd}$$

commutes. Let $p(y, \alpha, x) = y$, and $ix = (fx, id_{fx}, x)$, so $pi = f$. Define $r\colon (\boldsymbol{H}, f) \to \boldsymbol{G}$ by $r(y, \alpha, x) = x$. It is easy to check that $r \vdash i$, and $ri = id_{\boldsymbol{G}}$, so i is a cofibration weak equivalence. Moreover, if $p(y', \alpha, x) = y'$ and $\beta\colon y \to y'$ then $(\beta, id)\colon (y, \alpha\beta, x) \to (y', \alpha, x)$ is a lifting of β, so p is a fibration.

To establish the second part of Q3 and Q4, we show that we can factor any functor $f\colon \boldsymbol{G} \to \boldsymbol{H}$ as a cofibration followed by a trivial fibration. In fact, factor $f_0\colon G_0 \to H_0$ as

$$\begin{tikzcd} G_0 \arrow[rr, "i_0"] \arrow[dr, "f_0"'] & & E_0 \arrow[dl, "p_0"] \\ & H_0 & \end{tikzcd}$$

where p_0 is surjective, and i_0 is injective. For example, take $E_0 = G_0 + H_0$ and $p_0 = (f_0, id)$. Now take a pullback

$$\begin{tikzcd} E_1 \arrow[r] \arrow[d] & H_1 \arrow[d] \\ E_0 \times E_0 \arrow[r, "p_0 \times p_0"'] & H_0 \times H_0 \end{tikzcd}$$

giving a diagram

$$\begin{tikzcd} \boldsymbol{G} \arrow[rr, "i"] \arrow[dr, "f"'] & & \boldsymbol{E} \arrow[dl, "p"] \\ & \boldsymbol{H} & \end{tikzcd}$$

with i a cofibration, and and p full and faithful with p_0 surjective, hence a trivial fibration by Lemma 2.2.1.

Now, if $p\colon \boldsymbol{E} \to \boldsymbol{H}$ is a trivial fibration, then p is a weak equivalence and p is also a fibration, since if $y \to y'$ is an arrow of $\boldsymbol{H}$ and e' is an object of $\boldsymbol{E}$ such that $pe' = y'$, let e be such that $pe = y$ (p_0 is surjective). p is full and faithful, so there is a unique arrow $e \to e'$ such that $p(e \to e') = y \to y'$. This finishes the second half of Q4.

Let $p\colon \boldsymbol{E} \to \boldsymbol{B}$ be a fibration weak equivalence. Factor p as

$$\begin{tikzcd} \boldsymbol{E} \arrow[rr, "i"] \arrow[dr, "p"'] & & \boldsymbol{E}' \arrow[dl, "p'"] \\ & \boldsymbol{B} & \end{tikzcd}$$

where i a cofibration and p' is a trivial fibration. By Q1, i is a cofibration weak equivalence, so there is a diagonal filler r in the diagram

$$\begin{tikzcd} \boldsymbol{E} \arrow[r, "id"] \arrow[d, "i"'] & \boldsymbol{E} \arrow[d, "p"] \\ \boldsymbol{E}' \arrow[r, "p'"'] \arrow[ur, dashed, "r"] & \boldsymbol{B} \end{tikzcd}$$

It follows that p is a retract of p', and hence a trivial fibration, since these are stable under retracts. Thus, $p\colon \boldsymbol{E} \to \boldsymbol{B}$ is a fibration weak equivalence iff it is a trivial fibration, which finishes the second half of Q3. $\square$

2.3 Cofibration and fibration structures

We say that a class of maps $\mathcal{C}$ in a category $\mathcal{K}$ *admits cobase change* if the pushout $C \to B +_A C$ of a map $A \to B$ in $\mathcal{C}$ along any map $A \to C$ exists, and belongs to $\mathcal{C}$. Dually, a class $\mathcal{F}$ *admits base change* if the pullback $A \times_B E \to A$ of a map $E \to B$ in $\mathcal{F}$ along any map $A \to B$ exists, and belongs to $\mathcal{C}$.

Definition 2.3.1. *A cofibration structure on a category $\mathcal{K}$ is a pair $(\mathcal{C}, \mathcal{W})$ of classes of maps containing the isomorphisms and closed under composition such that:*

C1.(Saturation) If $f\colon X \to Y$ and $g\colon Y \to Z$ and any two of f, g or gf belong to $\mathcal{W}$ then so does the third ("three for two").

C2.(Cobase change) The classes $\mathcal{C}$ and $\mathcal{C} \cap \mathcal{W}$ admit cobase change.

C3.(Factorization) Every map f in $\mathcal{K}$ can be factored as $f = pi$, where $i \in \mathcal{C}$ and $p \in \mathcal{W}$.

The maps in $\mathcal{C}$ are called cofibrations, and the maps in $\mathcal{W}$ are called weak equivalences. An object A of $\mathcal{K}$ is called cofibrant if $0 \to A$ is a cofibration.

Definition 2.3.2. *A fibration structure on a category $\mathcal{K}$ is a pair $(\mathcal{F}, \mathcal{W})$ of classes of maps containing the isomorphisms and closed under composition such that:*

F1.(Saturation) $\mathcal{W}$ satisfies "three for two".

F2.(Base change) The classes $\mathcal{F}$ and $\mathcal{F} \cap \mathcal{W}$ admit base change.

C3.(Factorization) Every map f in $\mathcal{K}$ can be factored as $f = pi$, where $i \in \mathcal{W}$ and $p \in \mathcal{F}$.

The maps in $\mathcal{F}$ are called fibrations, and the maps in $\mathcal{W}$ are called weak equivalences. An object X of $\mathcal{K}$ is called fibrant if $X \to 1$ is a fibration.

If $i\colon A \to B$ and $f\colon X \to Y$ are maps in a category $\mathcal{K}$, we say i is *left orthogonal* to f, or f is *right orthogonal* to i, if any commutive diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

has a diagonal filler making both triangles commute. In the literature this is often expressed by saying i has the *left lifting property* (LLP) with respect to f and f has the *right lifting property* (RLP) with respect to i.

A class of maps $\mathcal{M}$ in $\mathcal{K}$ is said to be *stable under retracts* if every retract of a map in $\mathcal{M}$ (in the sense of **Q3**) belongs to $\mathcal{M}$. $\mathcal{M}$ is said to be *stable under pushouts* if whenever the the pushout $C \to B +_A C$ of a map $A \to B$ in $\mathcal{M}$ along a map $A \to C$ in $\mathcal{K}$ exists, it belongs to $\mathcal{M}$. We have also the dual concept: $\mathcal{M}$ is *stable under pullbacks.*

Let $\mathcal{M}$ be a class of maps in $\mathcal{K}$. We denote by ${}^{\perp}\mathcal{M}$ (respectively $\mathcal{M}^{\perp}$) the class of maps left (respectively right) orthogonal to $\mathcal{M}$. An easy argument gives

Proposition 2.3.1. *$\mathcal{M}^{\perp}$ contains the isomorphisms, is closed under composition, and is stable under retracts and pullbacks. Dually, ${}^{\perp}\mathcal{M}$ contains the isomorphisms, is closed under composition, and is stable under retracts and pushouts.*

Let $(\mathcal{F}, \mathcal{C}, \mathcal{W})$ be a Quillen homotopy structure on $\mathcal{K}$.

Proposition 2.3.2. *$\mathcal{C} =^{\perp} (\mathcal{F} \cap \mathcal{W})$, $\mathcal{F} = (\mathcal{C} \cap \mathcal{W})^{\perp}$, $\mathcal{C} \cap \mathcal{W} =^{\perp} \mathcal{F}$ and $\mathcal{F} \cap \mathcal{W} = \mathcal{C}^{\perp}$.*

Proof. This is called the *retract argument.* By **Q3** we know $\mathcal{C} \subseteq^{\perp} (\mathcal{F} \cap \mathcal{W})$, so let $i\colon A \to B$ be in ${}^{\perp}(\mathcal{F} \cap \mathcal{W})$. By **Q4**, factor i as $i = pj$ where $j \in \mathcal{C}$ and $p \in \mathcal{F} \cap \mathcal{W}$. The square

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ B \arrow[r, "id_B"'] \arrow[ur, dashed, "s"] & B \end{tikzcd}$$

has a diagonal filler s, so i is a retract of j. But $\mathcal{C}$ is stable under retracts, so $i \in \mathcal{C}$. The others are similar. $\square$

Corollary 2.3.1. *$(\mathcal{C}, \mathcal{W})$ is a cofibration structure on $\mathcal{K}$, and $(\mathcal{F}, \mathcal{W})$ is a fibration structure.*

Let $(\mathcal{C}, \mathcal{W})$ be a cofibration structure on a category $\mathcal{K}$ with initial object 0. Then if A and B are cofibrant, $A+B$ exists, and the canonical injections $A \to A+B$ and $B \to A+B$ are cofibrations. A *mapping cylinder* for a map $f\colon A \to B$ is obtained by factoring the map $(f, id_B)\colon A+B \to B$ as a cofibration $(i_A, i_B)\colon A+B \to Z_f$ followed by a weak equivalence $p\colon Z_f \to B$. We have $f = pi_A$ and $pi_B = id_B$, where i_A is a cofibration, and i_B is a cofibration weak equivalence by $C1$.

Lemma 2.3.1 (Ken Brown's Lemma)**.** *Let $(\mathcal{C}, \mathcal{W})$ be a cofibration structure on a category $\mathcal{K}$ with initial object 0. Let $F\colon \mathcal{K} \to \mathcal{L}$ be a functor from $\mathcal{K}$ to a category $\mathcal{L}$ equipped with a concept of weak equivalence satisfying the "three for two" property. Then if F takes cofibration weak equivalences between cofibrant objects to weak equivalences, it takes weak equivalences between cofibrant objects to weak equivalences.*

Proof. Let $f\colon A \to B$ be a weak equivalence with A and B cofibrant. Let $f = pi_A$ be a mapping cylinder factorization of f with $pi_B = id_B$. Then, by $C1$, both i_A and i_B are cofibration weak equivalences. Thus, $F(i_A)$ and $F(i_B)$ are weak equivalences, and $F(p)$ is a weak equivalence by "three for two" in $\mathcal{L}$. It follows that $F(f) = F(p)F(i_A)$ is a weak equivalence. □

Let $(\mathcal{C}, \mathcal{W})$ be a cofibration structure on $\mathcal{K}$ and A an object of $\mathcal{K}$. Then $A/\mathcal{K}$ has an induced cofibration structure, in which a map is a cofibration or weak equivalence iff its projection in $\mathcal{K}$ is such.

Corollary 2.3.2. *Let $(\mathcal{C}, \mathcal{W})$ be a cofibration structure on $\mathcal{K}$, and $w\colon X \to Y$ a weak equivalence between cofibrant objects of $A/\mathcal{K}$. Then for any map $A \to B$ the pushout $B +_A w\colon B +_A X \to B +_A Y$ is a weak equivalence in $B/\mathcal{K}$.*

Proof. The functor $B +_A ()\colon A/\mathcal{K} \to B/\mathcal{K}$ satisfies the hypothesis of Ken Brown's Lemma. □

Theorem 2.3.1. *Let $(\mathcal{C}, \mathcal{W})$ be a cofibration structure on a category $\mathcal{K}$ with initial object 0. If $f\colon A \to B$ is a weak equivalence between cofibrant objects of $\mathcal{K}$ and $i\colon A \to C$ is a cofibration then in the pushout f' is a weak equivalence.*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d] \\ C \arrow[r, "f'"'] & C +_A B \end{tikzcd}$$

Proof. Let $f = pi_A$ be a mapping cylinder factorization of f with $pi_B = id_B$. Since cofibration weak equivalences admit cobase change, we see it is enough to prove the theorem for a weak equivalence between cofibrant objects having a section which is a cofibration weak equivalence. So, let $f\colon A \to B$ be such, with section $j\colon B \to A$, and suppose $i\colon A \to C$ is a cofibration. Consider the diagram

$$\begin{tikzcd}
B \arrow[r, "j"] \arrow[d, "ij"'] & A \arrow[rr, "f"] \arrow[d, "i'"'] \arrow[ddr, "i"] & & B \arrow[d] \arrow[ddr] & \\
C \arrow[r, "j'"] \arrow[drr, "id_C"'] & C' \arrow[rr, "\overline{f}"] \arrow[dr, "w"] & & C \arrow[dr, "w'"] & \\
& & C \arrow[rr, "f'"'] & & C +_A B
\end{tikzcd}$$

in which all the squares are pushouts. $w\colon C' \to C$ is defined by $wj' = id_C$, and $wi' = i$. i' is a cofibration, and j' is a cofibration weak equivalence. $\overline{f}j' = id_C$, so $\overline{f}$ and w are weak equivalences by $C1$. w is now a weak equivalence between cofibrant objects of $A/\mathcal{K}$, so it's pushout w' is a weak equivalence by Corollary 2.3.2. It follows that f' is a weak equivalence by $C1$. $\square$

Dually, we have

Theorem 2.3.2. *Let $(\mathcal{F}, \mathcal{W})$ be a fibration structure on a category $\mathcal{K}$ with terminal object* 1. *Then if $f\colon X \to Y$ is a weak equivalence between fibrant objects of $\mathcal{K}$ and $p\colon E \to Y$ is a fibration, then in the pullback f' is a weak equivalence.*

$$\begin{tikzcd}
E \times_Y X \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
E \arrow[r, "p"'] & Y
\end{tikzcd}$$

Corollary 2.3.3. *The Quillen structure of Section 2.2 on Gpd is proper, i.e.* $Q5$ *is satisfied.*

Proof. Every object of *Gpd* is both fibrant and cofibrant. $\square$

The following result is often called the "Gluing Lemma".

Theorem 2.3.3. *Let $(\mathcal{C}, \mathcal{W})$ be a cofibration structure on a category $\mathcal{K}$ with initial object. Suppose is a commutative cube in $\mathcal{K}$ such that the front and back faces are pushouts, i and j are cofibrations, and X_0, Y_0, X_2 and Y_2 are cofibrant. Then if*

w_0, w_1 and w_2 are weak equivalences, so is w.

$$\begin{tikzcd}
& X_0 \arrow[rr, "i"] \arrow[dl, "w_0"'] \arrow[dd] & & X_1 \arrow[dl, "w_1"'] \arrow[dd] \\
Y_0 \arrow[rr, "j"] \arrow[dd] & & Y_1 \arrow[dd] & \\
& X_2 \arrow[rr] \arrow[dl, "w_2"'] & & X \arrow[dl, "w"'] \\
Y_2 \arrow[rr] & & Y &
\end{tikzcd}$$

Proof. Pushout the top and bottom faces of the cube, obtaining a cube in which

$$\begin{tikzcd}
& X_0 \arrow[rr, "i"] \arrow[dl, "w_0"'] \arrow[dd] & & X_1 \arrow[dl, "q"'] \arrow[dd] \\
Y_0 \arrow[rr, "k"] \arrow[dd] & & Q \arrow[dd] & \\
& X_2 \arrow[rr] \arrow[dl, "w_2"'] & & X \arrow[dl, "r"'] \\
Y_2 \arrow[rr] & & R &
\end{tikzcd}$$

now also the top and bottom faces are pushouts, and q and r are weak equivalences by Theorem 2.3.1. Together, The front and right-hand faces of the original cube yield two diagrams

$$\begin{tikzcd}
& Q \arrow[dr, "u"] \arrow[dd] & \\
Y_0 \arrow[ur, "k"] \arrow[rr, "j"] \arrow[dd] & & Y_1 \arrow[dd] \\
& R \arrow[dr, "v"] & \\
Y_2 \arrow[ur] \arrow[rr] & & Y
\end{tikzcd}$$

and

$$\begin{tikzcd}
& Q \arrow[dr, "u"] \arrow[dd] & \\
X_1 \arrow[ur, "q"] \arrow[rr, "w_1"] \arrow[dd] & & Y_1 \arrow[dd] \\
& R \arrow[dr, "v"] & \\
X \arrow[ur, "r"] \arrow[rr, "w"] & & Y
\end{tikzcd}$$

Since w_1 is a weak equivalence, u is by $C1$. It follows that w is a weak equivalence iff v is. Thus, we have reduced to the case $w_0 = w_2 = id$, i.e. to the case of the first of the two diagrams above. To show that v is a weak equiavalence, factor $Y_0 \to Y_2$ as $Y_0 \to Y_0' \to Y_2$ where $Y_0 \to Y_0'$ is a cofibration and $Y_0' \to Y_2$ is a weak equivalence. Then pushout in stages, yielding

$$\begin{tikzcd}
& Q \arrow[dr, "u"] \arrow[dd] & \\
Y_0 \arrow[ur, "k"] \arrow[rr, "j"] \arrow[dd] & & Y_1 \arrow[dd] \\
& Q' \arrow[dr] \arrow[dd] & \\
Y_0' \arrow[ur] \arrow[rr] \arrow[dd] & & Y_1' \arrow[dd] \\
& R \arrow[dr, "v"] & \\
Y_2 \arrow[ur] \arrow[rr] & & Y
\end{tikzcd}$$

Now $Y_0' \to Q'$, $Y_0' \to Y_1'$ and $Q \to Q'$ are cofibrations, so $Q' \to R$, $Y_1' \to Y$ and $Q' \to Y_1'$ are weak equivalences by Theorem 2.3.1. It follows that v is a weak equivalence by $C1$. $\square$

Dually we have

Theorem 2.3.4. *Let $(\mathcal{F}, \mathcal{W})$ be a fibration structure on a category $\mathcal{K}$ with terminal* *object. Suppose*

$$\begin{tikzcd}
& X \arrow[rr] \arrow[dl, "w"'] \arrow[dd] & & X_2 \arrow[dl, "w_2"'] \arrow[dd] \\
Y \arrow[rr, crossing over] \arrow[dd] & & Y_2 & \\
& X_1 \arrow[rr, "f"] \arrow[dl, "w_1"'] & & X_0 \arrow[dl, "w_0"] \\
Y_1 \arrow[rr, "g"] & & Y_0 \arrow[from=uu, crossing over] &
\end{tikzcd}$$

is a commutative cube in $\mathcal{K}$ such that f and g are fibrations, X_0, Y_0, X_2 and Y_2 are fibrant, and the front and back faces are pullbacks. Then if w_0, w_1, and w_2 are weak equivalences, so is w.

2.4 The fundamental groupoid and the Van Kampen theorem

Definition 2.4.1. *Let X be a simplicial set. In the coequalizer*

$$X_1 \underset{d^1}{\overset{d^0}{\rightrightarrows}} X_0 \longrightarrow \pi_0(X)$$

$\pi_0(X)$ is called the set of connected components of X, and X is said to be connected when $\pi_0(X) = 1$.

We remark that this $\pi_0(X)$ is the same as the set of connected components of X considered as a set-valued functor and defined in Topic A.5, i.e. $\pi_0(X) = \varinjlim X$. This follows by uniqueness of adjoints, since it is easy to see that $\pi_0 \dashv dis$, where $dis\colon Set \to \mathbf{S}$ is the functor taking a set S to the constant, discrete simplicial set at S all of whose faces and degeneracies are id_S.

Let us write the relation on X_0 determined by d^0 and d^1 as $x \sim y$, saying "x is connected to y by a path". That is, writing I for $\Delta[1]$ and $(0) \subseteq I$, respectively $(1) \subseteq I$, for the images of $\varepsilon^1\colon \Delta[0] \to \Delta[1]$ and $\varepsilon^0\colon \Delta[0] \to \Delta[1]$, then $x \sim y$ iff there is a map $\alpha\colon I \to X$ such that $\alpha(0) = x$, and $\alpha(1) = y$. In general, $x \sim y$ is not an equivalence relation, so that if we denote the map $X_0 \to \pi_0(X)$ by $x \mapsto \bar{x}$ then $\bar{x} = \bar{y}$ iff there is a "path of length n' connecting x and y", i.e, we have a diagram of the form

$$x \to x_1 \leftarrow x_2 \to x_3 \cdots x_{n-1}. \leftarrow y$$

Definition 2.4.2. *Let $\boldsymbol{G}$ be a groupoid. In the coequalizer*

$$\begin{tikzcd} G_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & G_0 \arrow[r] & c(\boldsymbol{G}) \end{tikzcd}$$

$c(\boldsymbol{G})$ is called the set of (algebraic) connected components of $\boldsymbol{G}$, and $\boldsymbol{G}$ is said to be connected when $c(\boldsymbol{G}) = 1$.

Clearly, $c(\boldsymbol{G}) = \pi_0(N\boldsymbol{G})$, and $\boldsymbol{G}$ is connected iff $(s,t)\colon G_1 \to G_0 \times G_0$ is surjective or iff for each object $x \in G_0$, $\boldsymbol{G}$ is equivalent to the group $\boldsymbol{G}(x,x)$ of maps from x to itself.

Let $\boldsymbol{A}$ be a small category, and define $G\colon Cat \to Gpd$ by $G\boldsymbol{A} = \boldsymbol{A}[\Sigma^{-1}]$ where $\Sigma = A_1$ —the set of all arrows of $\boldsymbol{A}$. If $I\colon Gpd \to Cat$ denotes the inclusion, then $G \dashv I$.

Definition 2.4.3. *Let X be a simplicial set. The fundamental groupoid of X is*

$$\pi_1(X) = G\gamma(X).$$

By the remark following the proof of Theorem 1.3.1 we can describe $\pi_1(X)$ as follows: its objects are the vertices X_0 of X, and its arrows are generated by the 1-simplices X_1 and their formal inverses. If $x \in X_1$, $sx = d^1x$ and $tx = d^0x$. The relations between the generating arrows are generated by the relations $s^0x = id_x$ for $x \in X_0$ and $d^1x = d^0x \circ d^2x$ for $x \in X_2$. Clearly, X is connected as a simplicial set iff $\pi_1(X)$ is connected as a groupoid.

Definition 2.4.4. *Let X be a simplicial set, and $x \in X_0$. We denote the group $\pi_1(X)(x,x)$ by $\pi_1(X,x)$ and call it the fundamental group of X at x.*

The following theorem is very useful for computing $\pi_1(X,x)$.

Theorem 2.4.1. *(Van Kampen) Let*

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

be a pushout of simplicial sets such that $W_0 \to V_0$ is injective, and U, V and W are connected. Let $w \in W_0$ and let u, v and x be the images of w in U, V and X respectively. Then

$$\begin{tikzcd} \pi_1(W,w) \arrow[r] \arrow[d] & \pi_1(V,v) \arrow[d] \\ \pi_1(U,u) \arrow[r] & \pi_1(X,x) \end{tikzcd}$$

is a pushout of groups.

Proof. Consider the commutative cube of groupoids

$$\begin{tikzcd}
& \pi_1(W,w) \arrow[rr] \arrow[dl] \arrow[dd] & & \pi_1(V,v) \arrow[dl] \arrow[dd] \\
\pi_1(W) \arrow[rr] \arrow[dd] & & \pi_1(V) \arrow[dd] & \\
& \pi_1(U,u) \arrow[rr] \arrow[dl] & & \pi_1(U,u) +_{\pi_1(W,w)} \pi_1(V,v) \arrow[dl] \\
\pi_1(U) \arrow[rr] & & \pi_1(X) &
\end{tikzcd}$$

Since π_1 is a left adjoint, the front face is a pushout, as is the back by definition. $\pi_1(W) \to \pi_1(V)$ is a cofibration by assumption, as is $\pi_1(W,w) \to \pi_1(V,v)$. Since U, V and W are connected, each of $\pi_1(U,u) \to \pi_1(U)$, $\pi_1(V,v) \to \pi_1(V)$ and $\pi_1(W,w) \to \pi_1(W)$ is an equivalence of groupoids. By the Glueing Lemma- Theorem 2.3.3 - $\pi_1(U,u) +_{\pi_1(W,w)} \pi_1(V,v) \to \pi_1(X)$ is an equivalence. There is a commutative triangle

$$\begin{tikzcd}
\pi_1(U,u) +_{\pi_1(W,w)} \pi_1(V,v) \arrow[rr] \arrow[dr] & & \pi_1(X,x) \arrow[dl] \\
& \pi_1(X) &
\end{tikzcd}$$

in which $\pi_1(X,x) \to \pi_1(X)$ is an equivalence since X is connected (π_0 is a left adjoint). But then, by "three for two", $\pi_1(U,u) +_{\pi_1(W,w)} \pi_1(V,v) \to \pi_1(X,x)$ is an equivalence which is the identity on objects, i.e. an isomorphism. $\square$

In most applications of the Van Kampen Theorem U and V are subcomplexes of X with $W = U \cap V$, in which case $X = U \cup V$.

2.5 Coverings

We begin this section by deriving an equivalent form of the fundamental groupoid, which will be useful in the discussion of coverings.

Let X be a simplicial set, and consider the category $\mathbf{\Delta}/X$ of elements of X. It's objects are pairs $([n], x)$ where $[n] \in \mathbf{\Delta}$ and $x \colon \Delta[n] \to X$. A mapping $\alpha \colon ([n], x) \to ([m], y)$ is a map $\alpha \colon [n] \to [m]$ in $\mathbf{\Delta}$ such that $x = y\alpha$. There is a functor

$$\phi_X \colon \mathbf{\Delta}/X \to \pi_1(X)$$

described as follows: $\phi_X([n], x) = x(n) \in X_0$, where $n \colon [0] \to [n]$ is the inclusion. If $\alpha \colon ([n], x) \to ([m], y)$, then $\alpha(n) \leq m$, and these two elements define a mapping

$(\alpha(n), m)\colon \Delta[1] \to \Delta[n]$, and $\phi_X \alpha = y(\alpha(n), m) \in X_1$. Since $\pi_1(X)$ is a groupoid, ϕ_X passes to a functor, with the same name,

$$\phi_X\colon G(\mathbf{\Delta}/X) \to \pi_1(X).$$

Theorem 2.5.1. $\phi_X\colon G(\mathbf{\Delta}/X) \to \pi_1(X)$ *is an equivalence of groupoids.*

Proof. First, consider, for $n \geq 0$,

$$\phi_n\colon G(\mathbf{\Delta}/\Delta[n]) \to \pi_1(\Delta[n]).$$

We may identify $\mathbf{\Delta}/\Delta[n]$ with $\mathbf{\Delta}/[n]$, and $\pi_1(\Delta[n])$ with $G[n]$. The functor $\mathbf{\Delta}/[n] \to [n]$ sending $\beta\colon [m] \to [n]$ to $\beta(m)$, is natural in $[n]$, and yields ϕ_n upon application of G. It has a right adjoint $[n] \to \mathbf{\Delta}/[n]$ given by sending $k \in [n]$ to the inclusion $[k] \to [n]$, which is, however, not natural in n. In any case, since adjoint pairs are sent to equivalences by G, we see that ϕ_n is an equivalence.

Call a groupoid $\boldsymbol{G}$ *contractible* if $(s,t)\colon G_1 \to G_0 \times G_0$ is $(\pi_1, \pi_2)\colon G_0 \times G_0 \to G_0 \times G_0$ Notice that it is equivalent to say that the canonical functor $\boldsymbol{G} \to \mathbf{1}$ is a weak equivalence in the Quillen structure of section 2.2 on *Gpd*, where $\mathbf{1}$ is the terminal groupoid with one object and one identity isomorphism, hence the name. For example, the groupoid $G[n]$ is contractible. Since $G(\mathbf{\Delta}/[n])$ is equivalent to $G[n]$, it too is contractible. But then we can define a new pseudo-inverse $\sigma_n\colon G[n] \to G(\mathbf{\Delta}/[n])$ by letting $\sigma_n(k) = k\colon [0] \to [n]$ for $k \in [n]$, and extending to morphisms in the only possible way. Moreover, σ_n is natural in $[n]$, for if $\alpha\colon [n] \to [m]$, the composite of $k\colon [0] \to [n]$ and $\alpha\colon [n] \to [m]$ is $\alpha(k)\colon [0] \to [m]$.

Now recall that by the universal property of $[\mathbf{\Delta}^{op}, Set]$, a functor $F\colon \mathbf{\Delta} \to Gpd$ can be uniquely extended to a colimit-preserving functor $F^+\colon \mathbf{S} \to Gpd$. In fact, if $X \in \mathbf{S}$

$$F^+X = \varinjlim_{\Delta[n]\to X} F[n]$$

$G(\mathbf{\Delta}/X)$ is colimit preserving in X, since the functor $\mathbf{\Delta}/(\ )\colon \boldsymbol{S} \to Cat$ has a right adjoint $R\colon Cat \to \boldsymbol{S}$ given by $(R\boldsymbol{A})_n = Cat(\mathbf{\Delta}/[n], \boldsymbol{A})$ for $\boldsymbol{A} \in Cat$. Thus, $G(\mathbf{\Delta}/X)^+ \simeq G(\mathbf{\Delta}/X)$ and similarly for π_1. It follows that ϕ_n extends to $\phi_X\colon G(\mathbf{\Delta}/X) \to \pi_1(X)$ and σ_n extends to a transformation $\sigma_X\colon \pi_1(X) \to G(\mathbf{\Delta}/X)$. $\phi_n\sigma_n = id$ for $n \geq 0$ so $\phi\sigma = id$. We have a natural isomorphism $\sigma_n\phi_n \to id$, which can be considered to be a functor $\boldsymbol{I} \times G(\mathbf{\Delta}/[n]) \to G(\mathbf{\Delta}/[n])$ where $\boldsymbol{I}$ is the groupoid with two objects and one non-identity isomorphism between them. Since $\boldsymbol{I} \times G(\mathbf{\Delta}/X)$ is also colimit preserving, this extends to a functor $\boldsymbol{I} \times G(\mathbf{\Delta}/X) \to G(\mathbf{\Delta}/X)$, providing an isomorphism $\sigma\phi \to id$. $\square$

Definition 2.5.1. *A map $p\colon E \to X$ is called a covering if any commutative diagram*

$$\begin{tikzcd} \Delta[0] \arrow[r, "e"] \arrow[d, "k"'] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r, "x"'] \arrow[ur, dashed] & X \end{tikzcd}$$

has a unique diagonal filler.

We call a map $p\colon E \to X$ *cartesian* if for every $\alpha\colon [n] \to [m]$ in $\mathbf{\Delta}$ the diagram

$$\begin{tikzcd} E_m \arrow[r, "\alpha^*"] \arrow[d, "p_m"'] & E_n \arrow[d, "p_n"] \\ X_m \arrow[r, "\alpha^*"'] & X_n \end{tikzcd}$$

is a pullback. Notice that cartesian maps make sense in any category $[\boldsymbol{A}^{op}, Set]$.

Proposition 2.5.1. *In* $\mathbf{S}$ *the coverings are exactly the cartesian maps.*

Proof. A map $p\colon E \to X$ is a covering iff for any $k\colon \Delta[0] \to \Delta[n]$

$$\begin{tikzcd} E_n \arrow[r, "k^*"] \arrow[d, "p_n"'] & E_0 \arrow[d, "p_0"] \\ X_n \arrow[r, "k^*"'] & X_0 \end{tikzcd}$$

is a pullback. If $\alpha\colon [n] \to [m]$ and $p\colon E \to X$ is a covering, consider the diagram

$$\begin{tikzcd} E_m \arrow[r, "\alpha^*"] \arrow[d, "p_m"'] & E_n \arrow[r, "k^*"] \arrow[d, "p_n"] & E_0 \arrow[d, "p_0"] \\ X_m \arrow[r, "\alpha^*"'] & X_n \arrow[r, "k^*"'] & X_0 \end{tikzcd}$$

The whole rectangle and the right-hand square are pullbacks, so the left-hand square is also a pullback. Thus, a covering is cartesian. $\square$

If X is a simplicial set, there is an equivalence

$$[\mathbf{\Delta}^{op}, Set]/X \simeq [(\mathbf{\Delta}/X)^{op}, Set]$$

given by sending a map $p\colon E \to X$ to the functor $E\colon (\mathbf{\Delta}/X)^{op} \to Set$ defined by $E(x) = \{e \in E_n | p_n(e) = x\}$ for $x\colon \Delta[n] \to X$ an object of $\mathbf{\Delta}/X$.

Proposition 2.5.2. *Under the above equivalence, the full subcategory of cartesian maps over* X *corresponds to the full subcategory of functors* $G(\mathbf{\Delta}/X)^{op} \to Set$.

Proof. Let $\alpha\colon \Delta[m] \to \Delta[n]$ and suppose

$$\begin{tikzcd} & \Delta[m] \arrow[dl, "\alpha"'] \arrow[dr, "e'"] \arrow[dd, "y", near end] & \\ \Delta[n] \arrow[rr, "e" near start] \arrow[dr, "x"'] & & E \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

commutes. Then under the above equivalence, $E_\alpha \colon E(x) \to E(y)$ is given by $e \longmapsto e' = e\alpha$. Thus, E_α is an isomorphism iff for each $e' \in E(y)$ there is a unique $e \in E(x)$ such that $e' = e\alpha$. That is, iff in the commutative square

$$\begin{tikzcd} \Delta[m] \arrow[r, "e'"] \arrow[d, "\alpha"'] \arrow[dr, "y"] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r, "x"'] \arrow[ur, dashed, "e"] & X \end{tikzcd}$$

there is a unique diagonal filler $e \colon \Delta[n] \to E$, i.e. iff $p \colon E \to X$ is cartesian. □

Notice that the above proposition is valid for any small category $\boldsymbol{A}$. No use is made, specifically, of $\boldsymbol{\Delta}$.

We give one more interpretation of coverings, which may aide the reader's understanding.

Definition 2.5.2. *A mapping $p \colon E \to X$ is called a bundle if for every n-simplex $x \colon \Delta[n] \to X$ there is a vertex $\Delta[0] \to X$ such that the pullback of p over x is isomorphic to its pullback over $\Delta[n] \to \Delta[0] \to X$.*

Note that the fibers of such a bundle are constant on any connected component of X, though they may vary over different components.

Proposition 2.5.3. *$p \colon E \to X$ is a covering iff it is a bundle with discrete fibers.*

Proof. A bundle with discrete fibers is evidently a covering, so we prove the converse. Clearly it suffices to show that any covering $p \colon E \to \Delta[n]$ is trivial with discrete fiber. For this, suppose $k \in \Delta[n]_0$ and take the pullback

$$\begin{tikzcd} E_k \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ \Delta[0] \arrow[r, "k"'] & \Delta[n] \end{tikzcd}$$

Now, for any $\Delta[0] \to \Delta[m]$

$$\begin{tikzcd} (E_k)_m \arrow[r] \arrow[d] & (E_k)_0 \arrow[d] \\ \Delta[0]_m \arrow[r] & \Delta[0]_0 \end{tikzcd}$$

is a pullback, so $E_k \simeq Sk^0 E_k$ is discrete. Also, If $i, j \colon \Delta[0] \to \Delta[n]$, any diagram

$$\begin{tikzcd} & \Delta[0] \arrow[r] \arrow[d, "i"] & E \arrow[d, "p"] \\ \Delta[0] \arrow[r, "j"'] & \Delta[n] \arrow[r, "id"'] \arrow[ur, dashed] & \Delta[n] \end{tikzcd}$$

has a unique diagonal filler, which, when composed with j defines an isomorphism $(E_i)_0 \to (E_j)_0$. Let F denote any one of these sets. Then for any $\Delta[0] \to \Delta[m]$ we have a pullback

$$\begin{tikzcd} E_m \arrow[r] \arrow[d] & E_0 \arrow[d] \\ \Delta[n]_m \arrow[r] & \Delta[n]_0 \end{tikzcd}$$

But $E_0 \simeq \sum_{i \in \Delta[n]_0} (E_i)_0 \simeq \Delta[n]_0 \times F$. Thus, $E_m \simeq \Delta[n]_m \times F$, and $E \simeq \Delta[n] \times disF$. $\square$

In Topic D we will discuss further the relationship between simplicial coverings and topological coverings under geometric realization.

Chapter 3

The homotopy theory of simplicial sets

Our principal goal in this chapter is to establish the existence of the classical Quillen homotopy structure on **S**, i.e. to prove Theorem 2.1.2. To that end we will need to develop most of the the basic ingredients of homotopy theory in the context of simplicial sets. Thus, In section 1 we study fibrations, introduced in section 2.1, and the extremely useful concept of anodyne extension due to Gabriel and Zisman [2]. Section 2 is concerned with the homotopy relation between maps. Next, section 3 contains an exposition of the theory of minimal complexes and fibrations. These are then used in section 4 to establish the main theorem. The proof we give is different from those in the literature, [11] or [3]; from [3], for example, in that it is purely combinatorial, making no use of geometric realization.

3.1 Anodyne extensions and fibrations

A class $\mathcal{A}$ of monomorphisms in **S** is said to be *saturated* if it satisfies the following conditions:

(i.) $\mathcal{A}$ contains all isomorphisms.

(ii.) $\mathcal{A}$ is closed under pushouts. That is, if

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & A' \arrow[d, "i'"] \\ B \arrow[r, "f"'] & B' \end{tikzcd}$$

is a pushout diagram, and $i \in \mathcal{A}$, then $i' \in \mathcal{A}$.

(iii.) $\mathcal{A}$ is closed under retracts. That is, if

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"] & A' \arrow[r, "u"] \arrow[d, "i'"] & A \arrow[d, "i"] \\ B \arrow[r, "k"'] & B' \arrow[r, "v"'] & B \end{tikzcd}$$

is a commutative diagram with $uj = id_A$, $vk = id_B$ and $i' \in \mathcal{A}$, then $i \in \mathcal{A}$.

(iv.) $\mathcal{A}$ is closed under coproducts. That is, if $(A_l \xrightarrow{i_l} B_l | l \in L)$ is a family of monomorphisms with $i_l \in \mathcal{A}$ for each $l \in L$, then

$$\sum_{l \in L} i_l \colon \sum_{l \in L} A_l \longrightarrow \sum_{l \in L} B_l$$

is in $\mathcal{A}$.

(v.) $\mathcal{A}$ is closed under ω-composites. That is, if

$$(A_n \to A_{n+1} | n = 1, 2, \ldots)$$

is a countable family of morphisms of $\mathcal{A}$, then

$$\mu_1 \colon A_1 \longrightarrow \varinjlim_{n \geq 1} A_n$$

is a morphism of $\mathcal{A}$.

The intersection of all saturated classes containing a given set of monomorphisms Γ is called the *saturated class generated by* Γ.

For example, if $m \colon A \to X$ is an arbitrary monomorphism of **S**, then (Topic D.4.9)

$$\begin{tikzcd} \sum\limits_{e(X-A)_n} \partial\Delta[n] \arrow[r] \arrow[d] & \sum\limits_{e(X-A)_n} \Delta[n] \arrow[d] \\ Sk^{n-1}(X) \cup A \arrow[r] & Sk^n(X) \cup A \end{tikzcd}$$

is a pushout for $n \geq 0$, where $e(X-A)_n$ is the set of non-degenerate n-simplices of X which are not in A. Furthermore,

$$X = \varinjlim_{n \geq -1} (Sk^n(X) \cup A) \quad \text{and} \quad \mu_{-1} \colon Sk^{-1}(X) \cup A \longrightarrow \varinjlim_{n \geq -1} (Sk^n(X) \cup A)$$

is $m\colon A \to X$. Thus, the saturated class generated by the family

$$\{\partial\Delta[n] \to \Delta[n] | n \geq 0\}$$

is the class of all monomorphisms.

The saturated class $\mathcal{A}$ generated by the family

$$\{\Lambda^k[n] \to \Delta[n] | 0 \leq k \leq n, n \geq 1\}$$

is called the class of *anodyne extensions.*

If $\mathcal{M}$ is any class of maps in $\mathbf{S}$ it is easy to see ${}^{\perp}\mathcal{M}$ is saturated, so we can conclude

Proposition 3.1.1. *A map $p\colon E \to X$ is a (Kan) fibration iff it has the right lifting property (RLP) with respect to all anodyne extensions.*

Definition 3.1.1. *A map $p\colon E \to X$ is said to be a trivial fibration if it has the RLP with respect to the family $\{\partial\Delta[n] \to \Delta[n] | n \geq 0\}$ or, equivalently as above, with respect to the family of all monomorphisms.*

Theorem 3.1.1. *Any map $f\colon X \to Y$ of $\mathbf{S}$ can be factored as*

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "f"'] & & E \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

where i is anodyne and p is a fibration.

Proof. Consider the set L of all commutative diagrams

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta[n] \arrow[r] & Y \end{tikzcd}$$

with $n \geq 1$ and $0 \leq k \leq n$. Summing over L yields a commutative diagram

$$\begin{tikzcd} \sum_L \Lambda^k[n] \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ \sum_L \Delta[n] \arrow[r] & Y \end{tikzcd}$$

with i anodyne. In the pushout

$$\begin{tikzcd} \sum_L \Lambda^k[n] \arrow[r] \arrow[d, "i"'] & X \arrow[d, "i^0"] \\ \sum_L \Delta[n] \arrow[r] & X^1 \end{tikzcd}$$

i^0 is anodyne, and we have a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "i^0"] \arrow[dr, "f"'] & & X^1 \arrow[dl, "f^1"] \\ & Y & \end{tikzcd}$$

Now repeat the process with f^1, obtaining

$$\begin{tikzcd} X \arrow[r, "i^0"] \arrow[dr, "f"'] & X^1 \arrow[r, "i^1"] \arrow[d, "f^1"] & X^2 \arrow[dl, "f^2"] \\ & Y & \end{tikzcd}$$

with i^1 anodyne, etc. Let $X^0 = X$, and $f^0 = f$. Putting $E = \varinjlim_{n \geq 0} X^n$ let $p\colon E \to Y$ be the map induced by f^n on each X^n. Writing i for the map $\mu_0\colon X \to E$, we obtain a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "f"'] & & E \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

where i is anodyne. It remains to show that p is a fibration. So let

$$\begin{tikzcd} \Lambda^k[n] \arrow[r, "h"] \arrow[d] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r] & Y \end{tikzcd}$$

be a commutative diagram with $n \geq 1$ and $0 \leq k \leq n$. $\Lambda^k[n]$ has only finitely many non-degenerate simplices, so h factors through X^n for some $n \geq 0$. But then we have a lifting into X^{n+1}

$$\begin{tikzcd} \Lambda^k[n] \arrow[r, "h"] \arrow[d] & X^n \arrow[r, "i^n"] \arrow[d, "f^n"'] & X^{n+1} \arrow[dl, "f^{n+1}"] \\ \Delta[n] \arrow[r] \arrow[urr, dashed] & Y & \end{tikzcd}$$

and hence a diagonal filler in

$$\begin{tikzcd} \Lambda^k[n] \arrow[r, "h"] \arrow[d] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

□

Corollary 3.1.1. *$i\colon A \to B$ is anodyne if it has the LLP with respect to the class of all fibrations.*

Proof. Factor i in the form

$$\begin{tikzcd} A \arrow[rr, "j"] \arrow[dr, "i"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

where j is anodyne and p is a fibration. Since i has the LLP with respect to p, we can find a diagonal filler in

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ B \arrow[r, "id_B"'] \arrow[ur, dashed, "k"] & B \end{tikzcd}$$

But then i is a retract of j, so i is anodyne. □

Theorem 3.1.2. *Any map $f\colon X \to Y$ of $\mathbf{S}$ can be factored as*

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "f"'] & & E \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

where i is a monomorphism, and p is a trivial fibration.

Proof. Repeat the proof of Theorem 3.1.1 using the family $\{\partial\Delta[n] \to \Delta[n] | n \geq 0\}$ instead of the family $\{\Lambda^k[n] \to \Delta[n] | 0 \leq k \leq n, n \geq 1\}$. □

Remark. Note that the proofs of Theorem 3.1.1 and Theorem 3.1.2 show that the factorizations are functorial in the obvious sense.

Definition 3.1.2. *A simplicial set X is called a Kan complex if $X \to 1$ is a fibration.*

As an example, we have the singular complex sT of any topological space T. Another important example is provided by the following theorem.

Theorem 3.1.3. (Moore) *Any group G in $\mathbf{S}$ is a Kan complex.*

Theorem 3.1.3, together with the following lemma, provides many examples of fibrations.

Lemma 3.1.1. *The property of being a fibration, or trivial fibration, is local. That is, if $p\colon E \to X$ and there exists a surjective map $q\colon Y \to X$ such that in the pullback*

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ Y \arrow[r, "q"'] & X \end{tikzcd}$$

p' is a fibration, or trivial fibration, then p is a fibration, or trivial fibration.

The straightforward proof is left as an exercise.

A *bundle with fiber F* in $\mathbf{S}$ is a mapping $p\colon E \to X$ such that for each n-simplex $\Delta[n] \to X$ of X, there is an isomorphism ϕ in

$$\begin{tikzcd} \Delta[n] \times F \arrow[rr, "\phi"] \arrow[dr, "\pi_1"'] & & \Delta[n] \times_X E \arrow[dl, "\pi_1"] \\ & \Delta[n] & \end{tikzcd}$$

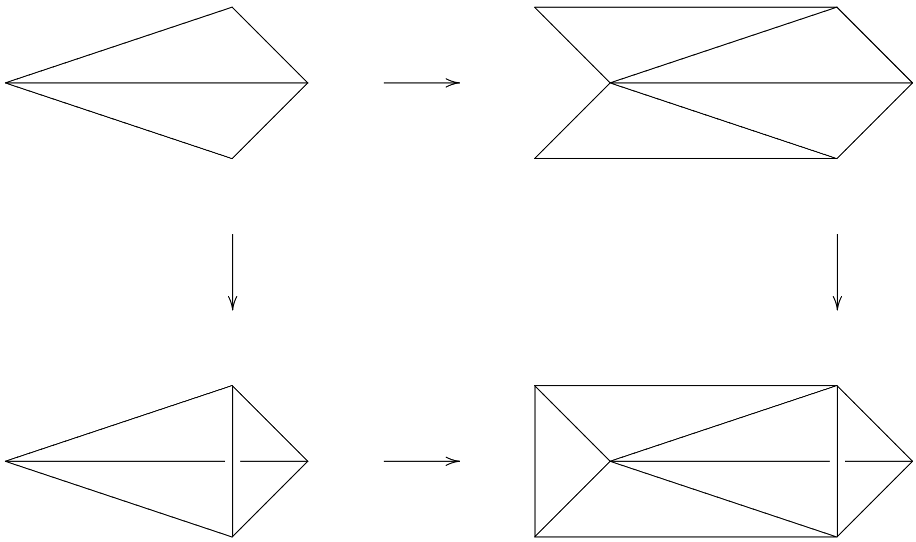
This is the same concept of bundle as given in Definition 2.5.2, except that here we are requiring the fiber to be constant over all of X.

Let $Y = \sum_{\Delta[n]\to X} \Delta[n]$. Then the canonical map $Y \to X$ is surjective. Since $\pi_1\colon Y \times F \to Y$ is clearly a fibration when F is a Kan complex, it follows from Lemma 3.1.1 that a bundle with Kan fiber F is a fibration. In particular, if G is a simplicial group and $p\colon E \to X$ is a principal G-bundle, i.e. a G-torsor over X (see Topic B) then p is a bundle with fiber G and hence a fibration.

Proof of Theorem 3.1.3. We give a new proof of the theorem, which perhaps involves less extensive use of the simplicial identities than the classical one. Thus, let G be a group and let $f\colon \Lambda^k[n] \to G$. We want to extend f to $\Delta[n]$ and we proceed by induction on n. The case $n = 1$ is obvious, since each $\Lambda^k[1]$ is $\Delta[0]$ and a retract of $\Delta[1]$. For the inductive step, let $\Lambda^{k,k-1}[n]$ be $\Lambda^k[n]$ with the $(k-1)$-st face removed (if $k = 0$ use $\Lambda^{0,1}[n]$). Then there is a commutative diagram of inclusions

$$\begin{tikzcd} \Lambda^{k,k-1}[n] \arrow[r] \arrow[d] & \Lambda^{k-1}[n-1] \times \Delta[1] \arrow[d] \\ \Delta[n] \arrow[r] & \Delta[n-1] \times \Delta[1] \end{tikzcd}$$

whose geometric realizations in dimension 3 look like



Now f restricted to $\Lambda^{k,k-1}[n]$ can be extended to $\Lambda^{k-1}[n-1] \times \Delta[1]$, since the inclusion $\Lambda^{k,k-1}[n] \to \Lambda^{k-1}[n-1] \times \Delta[1]$ is an anodyne extension of dimension $n-1$. Moreover, this extension can be further extended to $\Delta[n-1] \times \Delta[1]$ by exponential adjointness and induction, since $G^{\Delta[1]}$ is a group. Restricting this last extension to $\Delta[n]$ we see that f restricted to $\Lambda^{k,k-1}[n]$ can be extended to $\Delta[n]$.

We are thus in the following situation. We have two subcomplexes $\Lambda^{k,k-1}[n]$ and $\Delta[n-1]$ of $\Delta[n]$ where the inclusion $\Delta[n-1] \to \Delta[n]$ is ε^{k-1}. Furthermore, we have a map $f\colon \Lambda^k[n] = \Lambda^{k,k-1}[n] \cup \Delta[n-1] \to G$ whose restriction to $\Lambda^{k,k-1}[n]$ can be extended to $\Delta[n]$. Now put $r = \eta^{k-1}\colon \Delta[n] \to \Delta[n-1]$. Then $r|\Delta[n-1] = id$ and it is easy to see that $r\colon \Delta[n] \to \Delta[n-1] \to \Lambda^k[n]$ maps $\Lambda^{k,k-1}[n]$ into itself. The following lemma then completes the proof. □

Lemma 3.1.2. *Let A and B be subcomplexes of C, and $r\colon C \to B$ a mapping such that $r = id$ on B and $r\colon C \to B \to A \cup B$ maps A into itself. Let $f\colon A \cup B \to G$ where G is a group. Then if $f|A$ can be extended to C, f can be extended to C*

Proof. Extend $f|A$ to $g\colon C \to G$, and define $h\colon C \to G$ by

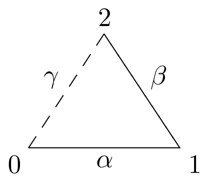
$$h(x) = g(x)g(r(x))^{-1}f(r(x)).$$ □

3.2 Homotopy

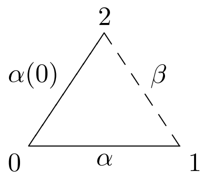
In section 2.4 we defined the set of connected components of a simplicial set X as the coequalizer $\pi_0(X)$ in

$$X_1 \underset{d^1}{\overset{d^0}{\rightrightarrows}} X_0 \longrightarrow \pi_0(X)$$

and we wrote the relation on X_0 determined by d^0 and d^1 as $x \sim y$, saying "x is connected to y by a path". We also remarked there that, for a general X, $x \sim y$ is not an equivalence relation. Here, however, we can say that $x \sim y$ *is* an equivalence relation when X is a Kan complex. In fact, suppose $\alpha\colon I \to X$ and $\beta\colon I \to X$ are such that $\alpha(1) = \beta(0)$. Let $s\colon \Lambda^1[2] \to X$ be the unique map such that $s\varepsilon^0 = \beta$ and $s\varepsilon^2 = \alpha$. A picture is given by



If X is Kan, there is a $t\colon \Delta[2] \to X$ extending s, and $\gamma = t\varepsilon^1$ connects $\alpha(0)$ to $\beta(1)$, showing transitivity. Using $\Lambda^0[2]$ and the constant (degenerate) path $\alpha(0)$ as in yields the symmetry.



A useful fact about $\pi_0(X)$ is the following.

Proposition 3.2.1. *If X and Y are simplicial sets, the canonical mapping*

$$\pi_0(X \times Y) \longrightarrow \pi_0(X) \times \pi_0(Y)$$

is a bijection.

Proof. Denoting the map $X_0 \to \pi_0(X)$ by $x \mapsto \bar{x}$, the canonical mapping above, is given by $\overline{(x,y)} \mapsto (\bar{x}, \bar{y})$. It is clearly surjective. If $(\bar{x_1}, \bar{y_1}) = (\bar{x_2}, \bar{y_2})$ then $\bar{x_1} = \bar{x_2}$, and $\bar{y_1} = \bar{y_2}$. Thus, there is a path of length n connecting x_1 to x_2, and a path of length m connecting y_1 to y_2. By using constant paths, we we may assume $n = m$, so that $\overline{(x_1, y_1)} = \overline{(x_2, y_2)}$ and the map is injective. □

Definition 3.2.1. *If $f, g\colon X \to Y$, we say f is homotopic to g if there is a map $h\colon X \times I \to Y$ such that $h_0 = f$ and $h_1 = g$.*

Clearly, we can interpret a homotopy h as a path in Y^X such that $h(0) = f$ and $h(1) = g$. As above, the relation of homotopy among maps is not an equivalence relation in general, but it is when Y^X is Kan. We will show below that Y^X is Kan when Y is, which, by adjointness, amounts to showing that each $\Lambda^k[n] \times X \to \Delta[n] \times X$ is anodyne.

We denote by $ho(\mathbf{S})$ the category of Kan complexes and homotopy classes of maps. That is, its objects are Kan complexes, and the set of morphisms between two Kan complexes X and Y is $[X, Y] = \pi_0(Y^X)$. Composition is defined as follows. Given X, Y and Z, there is a map $Y^X \times Z^Y \to Z^X$, which is the exponential transpose of the mapping

$$Y^X \times Z^Y \times X \simeq Z^Y \times Y^X \times X \stackrel{id\times ev}{\to} Z^Y \times Y \stackrel{ev}{\to} Z.$$

The composition in $ho(\mathbf{S})$, $[X, Y] \times [Y, Z] \to [X, Z]$, is obtained by applying π_0 to this map, using Proposition 3.2.1.

Notice that an isomorphism of $ho(\mathbf{S})$ is a *homotopy equivalence*, i.e. $X \simeq Y$ in $ho(\mathbf{S})$ iff there are mappings $f\colon X \to Y$ and $f'\colon Y \to X$ such that $ff' \sim id_Y$ and $f'f \sim id_X$.

$ho(\mathbf{S})$ has a number of different descriptions, as we will see. For example, it is equivalent to the category of CW-complexes and homotopy classes of maps.

Returning to the problem of showing that Y^X is Kan when Y is, we will, in fact, prove the following more general result. Let $k\colon Y \to Z$ be a monomorphism and suppose $p\colon E \to X$.

Denote the pullback of X^k and p^Y by

$$\begin{tikzcd} (k,p) \arrow[r] \arrow[d] & E^Y \arrow[d, "p^Y"] \\ X^Z \arrow[r, "X^k"'] & X^Y \end{tikzcd}$$

(k, p) is the "object of diagrams" of the form

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "k"'] & E \arrow[d, "p"] \\ Z \arrow[r] & X \end{tikzcd}$$

The commutative diagram

$$\begin{tikzcd} E^Z \arrow[r, "E^k"] \arrow[d, "p^Z"'] & E^Y \arrow[d, "p^Y"] \\ X^Z \arrow[r, "X^k"'] & X^Y \end{tikzcd}$$

gives rise to a map $k|p\colon E^Z \to (k, p)$, and we have

Theorem 3.2.1. *If $p\colon E \to X$ is a fibration, then $k|p\colon E^Z \to (k,p)$ is a fibration, which is trivial if either k is anodyne, or p is trivial.*

Proof. Let $i\colon A \to B$ be a monomorphism. The problem of finding a diagonal filler in

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & E^Z \arrow[d, "k|p"] \\ B \arrow[r] \arrow[ur, dashed] & (k,p) \end{tikzcd}$$

coincides with the problem of finding a diagonal filler in the adjoint transposed diagram

$$\begin{tikzcd} (A \times Z) \cup (B \times Y) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ B \times Z \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

which exists for any i if p is trivial. By Theorem 3.2.2, below, the left-hand vertical map is anodyne if either i or k is, which will complete the proof. □

Taking k to be the map $0 \to Y$ yields, in particular,

Corollary 3.2.1. *If $p\colon E \to X$ is a fibration, so is $p^Y : E^Y \to X^Y$ for any Y.*

This, of course, generalizes the original statement that X^Y is a Kan complex when X is.

Let $i\colon A \to B$, and $k\colon Y \to Z$ be monomorphisms. Then we have $i \times Z\colon A \times Z \to B \times Z$ and $B \times k\colon B \times Y \to B \times Z$, and we write $i \star k$ for the inclusion $(A \times Z) \cup (B \times Y) \to B \times Z$.

Theorem 3.2.2. *(Gabriel-Zisman) If i is anodyne, so is $i \star k$.*

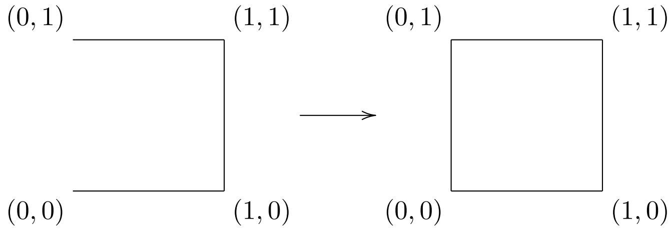
For the proof of Theorem 3.2.2 we need the following auxiliary result. Recall that we denoted by $\mathcal{A}$ the class of anodyne extensions, which is the saturated class generated by all the inclusions $\Lambda^k[n] \to \Delta[n]$ for $n \geq 1$ and $0 \leq k \leq n$. Denote by a the anodyne extension $(e) \to \Delta[1]$ for $e = 0, 1$ and by i_n the inclusion $\partial\Delta[n] \to \Delta[n]$. Now let $\mathcal{B}$ be the saturated class generated by all the inclusions

$$a \star i_n\colon ((e) \times \Delta[n]) \cup (\Delta[1] \times \partial\Delta[n]) \longrightarrow \Delta[1] \times \Delta[n]$$

for $e = 0, 1$ $n \geq 1$. $\Delta[1] \times \Delta[n]$ is called a *prism* and $((e) \times \Delta[n]) \cup (\Delta[1] \times \partial\Delta[n])$ an *open prism.* For example, the geometric realization of the inclusion

$$a \star i_1\colon ((1) \times \Delta[1]) \cup (\Delta[1] \times \partial\Delta[1]) \longrightarrow \Delta[1] \times \Delta[1]$$

is



Finally, we denote by $\mathcal{C}$ the saturated class generated by all inclusions

$$a \star m \colon ((e) \times Y) \cup (\Delta[1] \times X) \longrightarrow \Delta[1] \times Y$$

where $m \colon X \to Y$ is a monomorphism of $\mathbf{S}$ and $e = 0, 1$.

Theorem 3.2.3. *$\mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{A} = \mathcal{C}$*

With Theorem 3.2.3, whose proof we give shortly, we can complete the proof of Theorem 3.2.2.

Proof of Theorem 3.2.2. Let $k \colon Y \to Z$ be an arbitrary monomorphism of $\mathbf{S}$, and denote by $\mathcal{D}$ the class of all monomorphisms $i \colon A \to B$ such that $i \star k \colon (A \times Z) \cup (B \times Y) \to B \times Z$ is anodyne. $\mathcal{D}$ is clearly saturated, so it suffices to show that $\mathcal{C} \subseteq \mathcal{D}$ since $\mathcal{A} = \mathcal{C}$.

Thus, let $m' \colon Y' \to Z'$ be a monomorphism of $\mathbf{S}$, and consider the inclusion $a \star m' \colon ((e) \times Z') \cup (\Delta[1] \times Y') \longrightarrow \Delta[1] \times Z'$ of $\mathcal{C}$. Then

$$(a \star m') \star k \colon (((e) \times Z') \cup (\Delta[1] \times Y')) \times Z \cup (\Delta[1] \times Z') \times Y \longrightarrow (\Delta[1] \times Z') \times Z$$

is isomorphic to

$$a \star (m' \star k) \colon ((e) \times Z' \times Z) \cup \Delta[1] \times (Y' \times Z \cup Z' \times Y) \longrightarrow \Delta[1] \times (Z' \times Z)$$

which is in $\mathcal{C}$, and hence anodyne. It follows that $a \star m'$ is in $\mathcal{D}$, which proves the theorem. □

Proof of Theorem 3.2.3. $\mathcal{B} \subseteq \mathcal{A}$: Taking $e = 1$, we want to show the inclusion

$$a \star i_n \colon ((1) \times \Delta[n]) \cup (\Delta[1] \times \partial\Delta[n]) \longrightarrow \Delta[1] \times \Delta[n]$$

is anodyne. From section 1.4, we know that the top dimensional non-degenerate simplices of $\Delta[1] \times \Delta[n]$ correspond, under the nerve N, to the injective order-preserving maps $\sigma_j \colon [n+1] \to [1] \times [n]$ whose images are the maximal chains

$$((0, 0), \ldots, (0, j), (1, j), \ldots, (1, n))$$

for $0 \leq j \leq n$. As to the faces of the σ_j, we see that $d^{j+1}\sigma_j = d^{j+1}\sigma_{j+1}$ with image

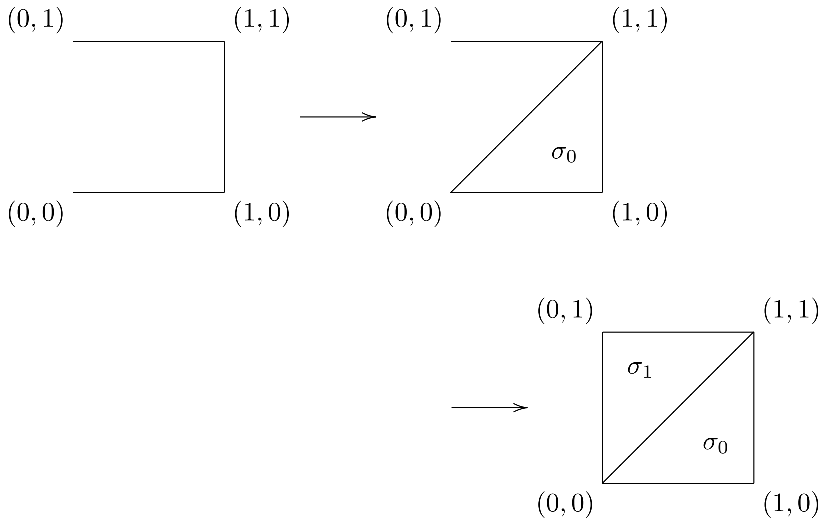
$$((0,0),\ldots,(0,j),(1,j+1),\ldots,(1,n))$$

for $0 \leq j \leq n$. Also, $d^i\sigma_j \in \Delta[1] \times \partial\Delta[n]$ for $i \neq j, j+1$, $d^0\sigma_0 \in (1) \times \Delta[n]$ and $d^{n+1}\sigma_n \in (0) \times \Delta[n]$.

So, we first attach σ_0 to the open prism $((1) \times \Delta[n]) \cup (\Delta[1] \times \partial\Delta[n])$ along $\Lambda^1[n+1]$ since all the faces $d^i\sigma_0$ except $d^1\sigma_0$ are already there. Next we attach σ_1 along $\Lambda^2[n+1]$ since now $d^1\sigma_1 = d^1\sigma_0$ is there, and only $d^2\sigma_1$ is lacking. In general, we attach σ_j along $\Lambda^{j+1}[n+1]$ since $d^j\sigma_j = d^j\sigma_{j-1}$ was attached the step before, and only $d^{j+1}\sigma_j$ is lacking. Thus, we see that the inclusion of the open prism in the prism is a composite of $n+1$ pushouts of horns. For example, the filling of the inclusion

$$a \star i_1 \colon ((1) \times \Delta[1]) \cup (\Delta[1] \times \partial\Delta[1]) \longrightarrow \Delta[1] \times \Delta[1]$$

above proceeds as follows:



When $e = 0$, we first attach σ_n, and work backwards to σ_0.

$\mathcal{C} \subseteq \mathcal{A}$: We proved above that $a \star i_n$ is anodyne. Thus, $a \star i_n$ has the left lifting property (LLP) with respect to any fibration $p \colon E \to Z$. By adjointness, i_n has the LLP with respect to $a|p$. But then any member of the saturated class generated by the i_n, i.e. any monomorphism $m \colon X \to Y$, has the LLP with respect to $a|p$. Thus, again by adjointness, $a \star m$ has the LLP with respect to p, and hence is anodyne.

$\mathcal{A} \subseteq \mathcal{C}$: For $0 \leq k < n$, let $s^k\colon [n] \to [1] \times [n]$ be the injection $s^k(i) = (1, i)$ and $r^k\colon [1] \times [n] \to [n]$ the surjection given by $r^k(1, i) = i$, and $r^k(0, i) = i$ for $i \leq k$, $r^k(0, i) = k$ $i \geq k$. Clearly, $r^k s^k = id_{[n]}$. It is easy to check that $Ns^k\colon \Delta[n] \to \Delta[1] \times \Delta[n]$ induces a map $\Lambda^k[n] \to ((0) \times \Delta[n]) \cup (\Delta[1] \times \Lambda^k[n])$ and $Nr^k\colon \Delta[1] \times \Delta[n] \to \Delta[n]$ a map $((0) \times \Delta[n]) \cup (\Delta[1] \times \Lambda^k[n]) \to \Lambda^k[n]$. It follows that we have a retract

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r] \arrow[d] & ((0) \times \Delta[n]) \cup (\Delta[1] \times \Lambda^k[n]) \arrow[r] \arrow[d] & \Lambda^k[n] \arrow[d] \\
\Delta[n] \arrow[r] & \Delta[1] \times \Delta[n] \arrow[r] & \Delta[n]
\end{tikzcd}$$

The middle vertical map is in $\mathcal{C}$, so the horns $\Lambda^k[n] \to \Delta[n]$ for $k < n$ are in $\mathcal{C}$. For $k = n$, or in general $k > 0$, we use the inclusion $u^k\colon [n] \to [1] \times [n]$ given by $u^k(i) = (0, i)$ and the retraction $v^k\colon [1] \times [n] \to [n]$ given by $v^k(0, i) = i$, $v^k(1, i) = k$ if $i \leq k$ and $v^k(1, i) = i$ for $i \geq k$. □

An important consequence of Theorem 3.2.2 is the *covering homotopy extension property* (CHEP) for fibrations, which is the statement of the following proposition.

Proposition 3.2.2. *Let $p\colon E \to X$ be a fibration, and $h\colon Z \times I \to X$ a homotopy. Suppose that $Y \to Z$ is a monomorphism, and $h'\colon Y \times I \to E$ is a lifting of h to E on $Y \times I$, i.e. the diagram*

$$\begin{tikzcd}
Y \times I \arrow[r, "h'"] \arrow[d] & E \arrow[d, "p"] \\
Z \times I \arrow[r, "h"'] & X
\end{tikzcd}$$

commutes. Suppose further that $f\colon Z \times (e) \to E$ is a lifting of $h_e (e = 0, 1)$ to E, i.e.

$$\begin{tikzcd}
Z \times (e) \arrow[r, "f"] \arrow[d] & E \arrow[d, "p"] \\
Z \times I \arrow[r, "h"'] & X
\end{tikzcd}$$

commutes. Then there is a homotopy $\bar{h}\colon Z \times I \to E$, which lifts h, i.e. $p\bar{h} = h$, agrees with h' on $Y \times I$, and is such that $\bar{h}_e = f$.

Proof. The given data provides a commutative diagram

$$\begin{tikzcd} (Y \times I) \cup (Z \times (e)) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ Z \times I \arrow[r, "h"'] \arrow[ur, dashed, "\bar{h}"] & X \end{tikzcd}$$

which has a dotted lifting $\bar{h}$ by Theorem 3.2.2, since $(e) \to I$ is anodyne. $\square$

We establish now several applications of the CHEP, which will be useful later. To begin, let $i\colon A \to B$ be a monomorphism of $\mathbf{S}$.

Definition 3.2.2. *A is said to be a strong deformation retract of B if there is a retraction $r\colon B \to A$ and a homotopy $h\colon B \times I \to B$ such that $ri = id_A$, $h_0 = id_B$, $h_1 = ir$, and h is "stationary on A", meaning*

$$\begin{tikzcd} A \times I \arrow[r] \arrow[dr, "\pi_1"'] & B \times I \arrow[r, "h"] & B \\ & A \arrow[ur, "i"'] & \end{tikzcd}$$

commutes.

Proposition 3.2.3. *If $i\colon A \to B$ is anodyne, and A and B are Kan complexes, then A is a strong deformation retract of B.*

Proof. We get a retraction as a diagonal filler in

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "id_A"'] & B \arrow[dl, dashed, "r"] \\ A & \end{tikzcd}$$

The required homotopy $h\colon B \times I \to B$ is obtained as the exponential transpose of the diagonal filler in the diagram

$$\begin{tikzcd} A \arrow[r, "\overline{i\pi_1}"] \arrow[d, "i"'] & B^I \arrow[d] \\ B \arrow[r, "(id_B, ir)"'] \arrow[ur, dashed] & B^{((0)+(1))} \end{tikzcd}$$

where the right-hand vertical mapping is a fibration by Theorem 3.2.1. $\square$

Proposition 3.2.4. *If $i\colon A \to B$ is a monomorphism such that A is a strong deformation retract of B, then i is anodyne.*

Proof. Let $r\colon B \to A$ be a retraction, and $h\colon B \times I \to B$ a strong deformation between id_B and ir. If a diagram

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ B \arrow[r, "b"'] \arrow[ur, dashed] & X \end{tikzcd}$$

is given with p a fibration, consider

$$\begin{tikzcd} A \times I \arrow[r, "\pi_1"] \arrow[d, "i\times I"'] & A \arrow[r, "a"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ B \times I \arrow[r, "h"'] & B \arrow[r, "b"'] & X \end{tikzcd}$$

A lifting of bh at 1 is provided by $ar\colon B \to E$, so lift all of bh by the CHEP, and take the value of the lifted homotopy at 0 as a diagonal filler in the original diagram. i is then anodyne by Corollary 3.1.1. $\square$

Proposition 3.2.5. *A fibration $p\colon E \to X$ is trivial iff p is the dual of a strong deformation retraction. That is, iff there is an $s\colon X \to E$ and $h\colon E \times I \to E$ such that $ps = id_X$, $h_0 = id_E$, $h_1 = sp$, and*

$$\begin{tikzcd} E \times I \arrow[r, "h"] \arrow[d, "\pi_1"'] & E \arrow[d, "p"] \\ E \arrow[r, "p"'] & X \end{tikzcd}$$

commutes.

Proof. If p is trivial, construct s and h as diagonal fillers in

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ X \arrow[r, "="'] \arrow[ur, dashed, "s"] & X \end{tikzcd}$$

and

$$\begin{tikzcd} E \times ((0) + (1)) \arrow[r, "(id_E, sp)"] \arrow[d] & E \arrow[d, "p"] \\ E \times I \arrow[r, "p\pi_1"'] \arrow[ur, dashed, "h"] & X \end{tikzcd}$$

On the other hand, if p is the dual of a strong deformation retraction as above, and a diagram

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ B \arrow[r, "b"'] & X \end{tikzcd}$$

is given, with i an arbitrary monomorphism, lift provisionally by sb. Then $psb = b$, and $h' = h(a \times I)$ is a homotopy satisfying $h'_0 = a$ and $h'_1 = spa = sbi$ (h is the homotopy above.) Furthermore, it is easy to check that

$$\begin{tikzcd} A \times I \arrow[r, "h'"] \arrow[d, "\pi_1"'] & E \arrow[d, "p"] \\ A \arrow[r, "bi"'] & X \end{tikzcd}$$

commutes. Thus, the diagram

$$\begin{tikzcd} (A \times I) \cup (B \times (1)) \arrow[r, "(h', sb)"] \arrow[d] & E \arrow[d, "p"] \\ B \times I \arrow[r, "b\pi_1"'] & X \end{tikzcd}$$

commutes. By the CHEP then, lift the lower (constant) homotopy to a homotopy $k\colon B \times I \to X$. k satisfies: $k_1 = sb, pk_0 = b$, and $k_0 i = h'_0 = a$. Thus, k_0 is the desired diagonal filler in the original diagram. $\square$

We can use Proposition 3.2.5 to obtain

Proposition 3.2.6. *Let X be a Kan Complex, then a fibration $p\colon E \to X$ is trivial iff p is a homotopy equivalence.*

Proof. Suppose p is a homotopy equivalence. Since X is Kan E is, so there is a map $s\colon X \to E$ together with one-stage homotopies $k\colon X \times I \to X$ and $h\colon E \times I \to E$ such that $k_0 = ps$, $k_1 = id_X$, $h_0 = sp$, $h_1 = id_E$. First, let k' be a filler in the diagram

$$\begin{tikzcd} X \times (0) \arrow[r, "s"] \arrow[d] & E \arrow[d, "p"] \\ X \times I \arrow[r, "k"'] \arrow[ur, dashed, "k'"] & X \end{tikzcd}$$

Then $s' = k'_1$ is such that $ps' = id_X$, so we may assume from the beginning that $ps = id_X$. Now we have two maps $I \to E^E$. Namely the adjoint transposes of h

and sph, which agree at 0, giving a diagram

$$\begin{tikzcd}
\Lambda^0[2] \arrow[r, "a"] \arrow[d] & E^E \arrow[d, "p^E"] \\
\Delta[2] \arrow[r, "b=ph\eta^1"'] \arrow[ur, dashed, "\alpha"] & X^E
\end{tikzcd}$$

Since p is a fibration, we can find a diagonal filler α. Then, $\alpha\varepsilon^0 = h'$ is a homotopy between id_E and sp, which is fiberwise, i.e.

$$\begin{tikzcd}
E \times I \arrow[r, "h'"] \arrow[d, "\pi_1"'] & E \arrow[d, "p"] \\
E \arrow[r, "p"'] & X
\end{tikzcd}$$

commutes. Thus, p is trivial by Proposition 3.2.5. The converse follows immediately from Proposition 3.2.5. $\square$

Proposition 3.2.7. *Let $p\colon E \to X$ be a fibration, and $i\colon A \to X$ a monomorphism. If A is a strong deformation retract of X, then in*

$$\begin{tikzcd}
p^{-1}(A) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
A \arrow[r, "i"'] & X
\end{tikzcd}$$

$p^{-1}(A)$ is a strong deformation retract of E.

Proof. Let $h\colon X \times I \to X$ denote the deformation of X into A. Then we have two commutative diagrams

$$\begin{tikzcd}
E \times (0) \arrow[rr, "id"] \arrow[d] & & E \arrow[d, "p"] \\
E \times I \arrow[r, "p\times id"'] & X \times I \arrow[r, "h"'] & X
\end{tikzcd}$$

and

$$\begin{tikzcd}
p^{-1}(A) \times I \arrow[r, "\pi_1"] \arrow[d] & p^{-1}(A) \arrow[r] & E \arrow[d, "p"] \\
E \times I \arrow[r, "p\times id"'] & X \times I \arrow[r, "h"'] & X
\end{tikzcd}$$

These provide a diagram

$$\begin{tikzcd}
(p^{-1}(A) \times I) \cup (E \times (0)) \arrow[rr] \arrow[d] & & E \arrow[d, "p"] \\
E \times I \arrow[r, "p \times id"'] \arrow[urr, dashed, "k"] & X \times I \arrow[r, "h"'] & X
\end{tikzcd}$$

A diagonal filler k, which exists by the CHEP, provides a deformation of E into $p^{-1}(A)$. $\square$

As applications of Proposition 3.2.7 we have the following.

Corollary 3.2.2. *Let $p\colon E \to X \times I$ be a fibration. Denote $p^{-1}(X \times (0))$ and $p^{-1}(X \times (1))$ by $p_0\colon E_0 \to X$ and $p_1\colon E_1 \to X$. Then p_0 and p_1 are fiberwise homotopy equivalent. That is, there are mappings*

$$\begin{tikzcd}
E_0 \arrow[rr, "f"] \arrow[dr, "p_0"'] & & E_1 \arrow[dl, "p_1"] \\
& X &
\end{tikzcd}$$

and

$$\begin{tikzcd}
E_1 \arrow[rr, "g"] \arrow[dr, "p_1"'] & & E_0 \arrow[dl, "p_0"] \\
& X &
\end{tikzcd}$$

together with homotopies $h\colon E_0 \times I \to E_0$ and $k\colon E_1 \times I \to E_1$ such that $h_0 = id_{E_0}$, $h_1 = gf$, $k_0 = id_{E_1}$, $k_1 = fg$ and the diagrams

$$\begin{tikzcd}
E_0 \times I \arrow[r, "h"] \arrow[d, "\pi_1"'] & E_0 \arrow[d, "p_0"] \\
E_0 \arrow[r, "p_0"'] & X
\end{tikzcd}$$

and

$$\begin{tikzcd}
E_1 \times I \arrow[r, "k"] \arrow[d, "\pi_1"'] & E_1 \arrow[d, "p_1"] \\
E_1 \arrow[r, "p_1"'] & X
\end{tikzcd}$$

commute.

Proof. $X\times(0)$ and $X\times(1)$ are both strong deformation retracts of $X\times I$. Thus, by Proposition 3.2.7, E_0 and E_1 are strong deformation retracts of E. The inclusions and retractions of E_0 and E_1 yield homotopy equivalences f and g. It follows easily from the proof of Proposition 3.2.7 that the homotopies h and k are fiberwise. □

Corollary 3.2.3. *Let $p\colon E\to Y$ be a fibration and $f,g\colon X\to Y$ two homotopic maps Then the pullbacks $f^*(E)$ and $g^*(E)$ are fiberwise homotopy equivalent.*

Proof. It suffices to consider the case of a homotopy $h\colon X\times I\to Y$ such that $h_0=f$ and $h_1=g$. For this, take the pullback

$$\begin{tikzcd} h^*(E) \arrow[r] \arrow[d, "q"'] & E \arrow[d, "p"] \\ X\times I \arrow[r, "h"'] & Y \end{tikzcd}$$

and apply Corollary 3.2.3 to q. □

Corollary 3.2.4. *Let X be connected and $p\colon E\to X$ a fibration. Then any two fibers of p are homotopy equivalent.*

Proof. It is enough to show that the fibers over the endpoints of any path $\alpha\colon I\to X$ are homotopy equivalent. For this, apply Corollary 3.2.3 to the inclusion of the endpoints of α. □

3.3 Minimal complexes

Let X be a simplicial set and $x,y\colon\Delta[n]\to X$ two n-simplices of X such that $x|\partial\Delta[n]=y|\partial\Delta[n]=a$. We say *$x$ is homotopic to y mod $\partial\Delta[n]$*, written $x\sim y$ mod $\partial\Delta[n]$, if there is a homotopy $h\colon\Delta[n]\times I\to X$ such that $h_0=x$, $h_1=y$ and h is "stationary on $\partial\Delta[n]$", meaning

$$\begin{tikzcd} \partial\Delta[n]\times I \arrow[r, "\pi_1"] \arrow[d] & \partial\Delta[n] \arrow[d, "a"] \\ \Delta[n]\times I \arrow[r, "h"'] & X \end{tikzcd}$$

commutes. It is easy to see that $x\sim y\mod\partial\Delta[n]$ is an equivalence relation when X is a Kan complex.

Definition 3.3.1. *Let X be a Kan complex. X is said to be minimal if $x\sim y$ mod $\partial\Delta[n]$ entails $x=y$.*

Our main goal in this section is to prove the following theorem, and its corresponding version for fibrations.

Theorem 3.3.1. *Let X be a Kan complex. Then there exists a strong deformation retract X' of X which is minimal.*

In the proof of Theorem 3.3.1 we will need a lemma.

Lemma 3.3.1. *Let x and y be two degenerate n-simplices of a simplicial set X. Then $x|\partial\Delta[n] = y|\partial\Delta[n]$ implies $x = y$.*

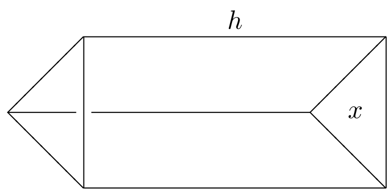
Proof. Let $x = s^i d^i x$ and $y = s^j d^j y$. If $i = j$, we are done. If, say, $i < j$, we have $x = s^i d^i x = s^i d^i y = s^i d^i s^j d^j y = s^i s^{j-1} d^i d^j y = s^j s^i d^i d^j y$. Thus, $x = s^j z$, where $z = s^i d^i d^j y$. Hence, $d^j x = d^j s^j z = z$ and $x = s^j d^j x = s^j d^j y = y$. $\square$

Proof of Theorem 3.3.1. We construct X' skeleton by skeleton. For $Sk^0 X'$ take one representative in each equivalence class of $\pi_0(X)$. Suppose we have defined $Sk^{n-1}X'$. To define $Sk^n X'$ we take one representative in each equivalence class among those n-simplices of X whose restrictions to $\partial\Delta[n]$ are contained in $Sk^{n-1}X'$, choosing a degenerate one wherever possible. Lemma 3.3.1 shows that X'_n contains all degenerate simplices from X'_{n-1}.

For the deformation retraction, suppose we have $h\colon Sk^{n-1}X \times I \to X$. Consider the pushout

$$\begin{tikzcd}
\sum\limits_{e(X)_n} \partial\Delta[n] \times I \arrow[r] \arrow[d] & \sum\limits_{e(X)_n} \Delta[n] \times I \arrow[d] \\
Sk^{n-1}X \times I \arrow[r] & Sk^n X \times I
\end{tikzcd}$$

To extend h to $Sk^n X \times I$ we must define it on each $\Delta[n] \times I$ consistent with its given value on $\partial\Delta[n] \times I$. Thus, let $x \in e(X)_n$. Since h is already defined on the boundary of x, we have an open prism



in X whose $(n-1)$-simplices in the open end belong to X'. Since X is Kan, we can fill the prism obtaining a new n-simplex y at the other end whose boundary is in X'. Now take a homotopy mod $\partial\Delta[n]$ to get into X'. This defines the retraction r and the homotopy h. $\square$

Theorem 3.3.2. *Let X and Y be minimal complexes and $f\colon X \to Y$ a homotopy equivalence. Then f is an isomorphism.*

The proof of Theorem 3.3.2 follows immediately from the following lemma.

Lemma 3.3.2. *Let X be a minimal complex and $f\colon X \to X$ a map homotopic to id_X. Then f is an isomorphism.*

Proof. We show first that $f_n\colon X_n \to X_n$ is injective by induction on n, letting X_{-1} be empty. Thus, let $x, x'\colon \Delta[n] \to X$ be such that $f(x) = f(x')$. By induction, $x|\partial\Delta[n] = x'|\partial\Delta[n] = a$. Let $h\colon X \times I \to X$ satisfy $h_0 = f$ and $h_1 = id_X$. Then the homotopy $h(x \times I)$ is $f(x)$ at 0 and x at 1. Similarly, $h(x' \times I)$ is $f(x')$ at 0 and x' at 1. Since $f(x) = f(x')$, we obtain a map $\Delta[n] \times \Lambda^0[2] \to X$. Let $\partial\Delta[n] \times \Delta[2] \to X$ be the map $h(a \times \eta^1)$. These agree on the intersection of their domains, so we obtain a diagram

$$\begin{tikzcd} (\Delta[n] \times \Lambda^0[2]) \cup (\partial\Delta[n] \times \Delta[2]) \arrow[r] \arrow[d] & X \\ \Delta[n] \times \Delta[2] \arrow[ur, dashed, "k"'] & \end{tikzcd}$$

which has a diagonal filler k by Theorem 3.2.2. $k(id \times \varepsilon^0)$ is a homotopy between x and x' *mod* $\partial\Delta[n]$. Since X is minimal, $x = x'$.

Now assume that $f_m\colon X_m \to X_m$ is surjective for $m < n$, and let $x\colon \Delta[n] \to X$ be an n-simplex of X. By induction, and the first part of the proof, each $x\varepsilon^i$ is uniquely of the form $f(y_i)$ for $y_i\colon \Delta[n-1] \to X$. Hence, we obtain a map $y\colon \partial\Delta[n] \to X$ such that $f(y) = x|\partial\Delta[n]$. The maps $h(y \times I)$ and $x \times (0)$ agree on their intersections giving a diagram

$$\begin{tikzcd} (\Delta[n] \times (0)) \cup (\partial\Delta[n] \times I) \arrow[r] \arrow[d] & X \\ \Delta[n] \times I \arrow[ur, dashed, "k"'] & \end{tikzcd}$$

which has a diagonal filler k. k at 0 is x, and k at 1 is some n-simplex z. The homotopy $h(z \times I)$ is $f(z)$ at 0 and z at 1. Thus, as above, we obtain a diagram

$$\begin{tikzcd} (\Delta[n] \times \Lambda^2[2]) \cup (\partial\Delta[n] \times \Delta[2]) \arrow[r] \arrow[d] & X \\ \Delta[n] \times \Delta[2] \arrow[ur, dashed, "l"'] & \end{tikzcd}$$

which has a diagonal filler l by Theorem 3.2.2. $l(id \times \varepsilon^2)$ is a homotopy between x and $f(z)$ *mod* $\partial\Delta[n]$. Since X is minimal, $x = f(z)$. □

We discuss now the corresponding matters for fibrations. Thus, let $p\colon E \to X$ be a map and $e, e'\colon \Delta[n] \to E$ two n-simplices of E such that $e|\partial\Delta[n] = e'|\partial\Delta[n] = a$, and $pe = pe' = b$. We say *e is fiberwise homotopic to e' mod $\partial\Delta[n]$*, written

$e \sim_f e' \mod \partial\Delta[n]$, if there is a homotopy $h\colon \Delta[n] \times I \to E$ such that $h_0 = e$, $h_1 = e'$, $h = a$ on $\partial\Delta[n]$ as before, meaning

$$\begin{tikzcd} \partial\Delta[n] \times I \arrow[r, "\pi_1"] \arrow[d] & \partial\Delta[n] \arrow[d, "a"] \\ \Delta[n] \times I \arrow[r, "h"'] & E \end{tikzcd}$$

commutes, and h is "fiberwise", meaning

$$\begin{tikzcd} \Delta[n] \times I \arrow[r, "h"] \arrow[d, "\pi_1"'] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r, "b"'] & X \end{tikzcd}$$

commutes. As before, it is easy to see that $e \sim_f e' \mod \partial\Delta[n]$ is an equivalence relation when p is a fibration.

Definition 3.3.2. *A fibration $p\colon E \to X$ is said to be minimal if $e \sim_f e' \mod \partial\Delta[n]$ entails $e = e'$.*

Notice that minimal fibrations are stable under pullback.

Theorem 3.3.3. *Let $p\colon E \to X$ be a fibration. Then there is a subcomplex E' of E such that p restricted to E' is a minimal fibration $p'\colon E' \to X$ which is a strong, fiberwise deformation retract of p.*

Proof. The proof is essentially the same as that for Theorem 3.3.1, with $x \sim y \mod \partial\Delta[n]$ replaced by $e \sim_f e' \mod \partial\Delta[n]$. □

Theorem 3.3.4. *Let $p\colon E \to X$ and $q\colon G \to X$ be minimal fibrations and $f\colon E \to G$ a map such that $qf = p$. Then if f is a fiberwise homotopy equivalence, f is an isomorphism.*

Proof. Again, the proof is essentially the same as that for Theorem 3.3.2, with $x \sim y \mod \partial\Delta[n]$ replaced by $e \sim_f e' \mod \partial\Delta[n]$. □

Theorem 3.3.5. *A minimal fibration is a bundle.*

Proof. Let $p\colon E \to X$ be a minimal fibration, and $x\colon \Delta[n] \to X$ an n-simplex of X. Pulling back p along x yields a minimal fibration over $\Delta[n]$ so it suffices to show that any minimal fibration $p\colon E \to \Delta[n]$ is isomorphic to one of the form $\pi_1\colon \Delta[n] \times F \to \Delta[n]$.

For this, define $c\colon [n] \times [1] \to [n]$ as follows: $c(i, 0) = i$ and $c(i, 1) = n$ for $0 \le i \le n$. $Nc = h$ is a homotopy $\Delta[n] \times I \to \Delta[n]$ between the identity of $\Delta[n]$ and the constant map at the n^{th} vertex of $\Delta[n]$. From Corollary 3.2.3 it follows that $p\colon E \to \Delta[n]$ is fiberwise homotopy equivalent to $\pi_1\colon \Delta[n] \times F \to \Delta[n]$ where F is the fiber of p over the n^{th} vertex of $\Delta[n]$. By Theorem 3.3.4, p is isomorphic to π_1 over $\Delta[n]$. □

3.4 The Quillen homotopy structure

Here we establish Theorem 2.1.2, or rather a modified version thereof.

Definition 3.4.1. *Let $f\colon X \to Y$ be a mapping in* $\mathbf{S}$. *We say f is a weak equivalence if for each Kan complex K, $[f, K]\colon [Y, K] \to [X, K]$ is a bijection.*

Clearly, if $f\colon X \to Y$ is homotopic to $g\colon X \to Y$, then $\pi_0(f) = \pi_0(g)$, so if f is a homotopy equivalence, $\pi_0(f)$ is bijective. Also, if f is homotopic to g, then B^f is homotopic to B^g for any B, so if f is a homotopy equivalence and K is Kan then K^f is a homotopy equivalence and $\pi_0(K^f) = [f, K]$ is bijective, so f is a weak equivalence. Hence a trivial fibration is a weak equivalence by Proposition 3.2.6. Or, if $i\colon A \to B$ is anodyne and K is Kan, then $K^i\colon K^B \to K^A$ is a trivial fibration by Theorem 3.2.1, hence a homotopy equivalence by Proposition 3.2.6. Thus, $\pi_0(K^i) = [i, K]$ is a bijection and i is a weak equivalence. Also, if $f\colon X \to Y$ is a weak equivalence with X and Y Kan, then f is a homotopy equivalence since f becomes an isomorphism in $ho(\mathbf{S})$.

Notice that Definition 3.4.1 is equivalent to saying that if $X \to \overline{X}$ and $Y \to \overline{Y}$ are anodyne extensions with $\overline{X}$ and $\overline{Y}$ Kan, and $\overline{f}$ is any dotted filler in

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & \overline{X} \arrow[d, dashed, "\overline{f}"] \\ Y \arrow[r] & \overline{Y} \end{tikzcd}$$

then $\overline{f}$ is a homotopy equivalence.

Now, in $\mathbf{S}$ we take as *fibrations* the Kan fibrations, as *cofibrations* the monomorphisms, and as *weak equivalences* the ones given in Definition 3.4.1. Then the main theorem of this chapter is the following.

Theorem 3.4.1. *The fibrations, cofibrations and weak equivalences defined above form a proper Quillen homotopy structure on* $\mathbf{S}$.

Remark. We will show in section 4. that the weak equivalences of Definition 3.4.1 coincide with the goemetric homotopy equivalences of section 3, so that Theorem 3.4.1 is, in fact, the same as Theorem 2.1.2.

The proof of Theorem 3.4.1 is based on the following two propositions, which we establish first.

Proposition 3.4.1. *A fibration $p\colon E \to X$ is trivial iff p is a weak equivalence.*

Proposition 3.4.2. *A cofibration $i\colon A \to B$ is anodyne iff i is a weak equivalence.*

Proof of Proposition 3.4.1. Let $p\colon E \to X$ be a fibration and a weak equivalence. By Theorem 3.3.3 there is a minimal fibration $p'\colon E' \to X$ which is a strong deformation retract of p, and hence also a weak equivalence. Let $X \to \overline{X}$ be an anodyne extension with $\overline{X}$ Kan (Theorem 3.1.1). By Theorem 3.3.5, p' is a bundle. Using Lemma 3.4.1 below, we can extend p' uniquely to a bundle $\overline{p}\colon \overline{E} \to \overline{X}$ in such a way as to have a pullback

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "p'"'] & \overline{E} \arrow[d, "\overline{p}"] \\ X \arrow[r] & \overline{X} \end{tikzcd}$$

with $E' \to \overline{E}$ anodyne. Now the fibers of p' are Kan complexes constant on the components of X. Since $\pi_0(X \to \overline{X})$ is bijective, these are also the fibers of $\overline{p}$ which is thus a fibration. So $\overline{E}$ is Kan. Furthermore, $\overline{p}$ is a weak equivalence since all the other maps in the diagram are. Thus, $\overline{p}$ is a homotopy equivalence. So, by Proposition 3.2.6, $\overline{p}$ is a trivial fibration. It follows that p' is also trivial. It is shown in Topic D that the retraction of p on p' is a trivial fibration, so that p is also a trivial fibration. □

Proof of Proposition 3.4.2. Let $i\colon A \to B$ be a cofibration and a weak equivalence. Factor i as

$$\begin{tikzcd} A \arrow[rr, "j"] \arrow[dr, "i"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

where j is anodyne and p is a fibration. p is a weak equivalence since i and j are. Thus, by Proposition 3.4.1, p is a trivial fibration. But then there is a diagonal filler s in

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ B \arrow[r, "id_B"'] \arrow[ur, dashed, "s"] & B \end{tikzcd}$$

so that i is a retract of j and hence anodyne. □

Lemma 3.4.1. *Let $A \to B$ be an anodyne extension and $p\colon E \to A$ a bundle. Then there is a pullback diagram*

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ A \arrow[r] & B \end{tikzcd}$$

such that $p'\colon E' \to B$ is a bundle, and $E \to E'$ is anodyne. Furthermore, such an extension p' of p is unique up to isomorphism.

Proof. Let $\mathcal{E}$ be the class of all monomorphisms having the unique extension property above. We will show that $\mathcal{E}$ contains the horn inclusions and is saturated, hence contains all anodyne extensions.

For the horn inclusions, let C be a simplicial set provided with an anodyne point $c\colon 1 \to C$. Let $p\colon E \to C$ be a principal G-bundle over C. Picking a point $e\colon 1 \to E$ such that $pe = c$, we have a diagonal filler in the diagram

$$\begin{tikzcd} 1 \arrow[r, "e"] \arrow[d, "x"'] & E \arrow[d, "p"] \\ C \arrow[r, "id_C"'] \arrow[ur, dashed] & C \end{tikzcd}$$

so that p has a section and hence is trivial. In case $p\colon E \to C$ is a bundle with fiber F, $Iso(C \times F, E) \to C$ is a principal $Aut(F)$-bundle (see Topic B), and hence has a section. But such a section is a trivialization of p. Thus, any bundle over C is trivial. In particular, any bundle $p\colon E \to \Lambda^k[n]$ is trivial (the kth vertex is an anodyne point), and hence can be extended uniquely as a trivial bundle over $\Lambda^k[n] \to \Delta[n]$.

$\mathcal{E}$ clearly contains all the isomorphisms. Let us see that its maps are stable under pushout. Thus, let $A \to B$ be in $\mathcal{E}$, and let $A \to A'$ be an arbitrary map. Form the pushout

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[d] \\ B \arrow[r] & B' \end{tikzcd}$$

and suppose $p'\colon E' \to A'$ is a bundle. Pull back p' to a bundle $p\colon E \to A$ and extend p as

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] & G \arrow[d, "q"] \\ A \arrow[r] & B \end{tikzcd}$$

with $E \to G$ anodyne since $A \to B$ is in $\mathcal{E}$. Now form the pushout

$$\begin{tikzcd} E \arrow[r] \arrow[d] & E' \arrow[d] \\ G \arrow[r] & G' \end{tikzcd}$$

giving a cube

$$\begin{tikzcd}
& E \arrow[rr] \arrow[dl] \arrow[dd] & & E' \arrow[dl] \arrow[dd] \\
G \arrow[rr, crossing over] \arrow[dd] & & G' & \\
& A \arrow[rr] \arrow[dl] & & A' \arrow[dl] \\
B \arrow[rr] & & B' \arrow[from=uu, crossing over] &
\end{tikzcd}$$

where $q' \colon G' \to B'$ is the natural induced map. In this cube, the left-hand and back faces are pullbacks.Hence, by the lemma following this proof, we can conclude that the front and right-hand faces are also. Thus, the pullback of q' over the surjection $B + A' \to B'$ is the bundle $q + p' \colon G + E' \to B + A'$. It follows that q' is a bundle and unique, for any other bundle which extends p' must pull back over $B \to B'$ to q by the uniqueness of q. Clearly, $E' \to G'$ is anodyne.

Now let

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & A \arrow[d] \\
B \arrow[r] & B' \arrow[r] & B
\end{tikzcd}$$

be a retract with $A' \to B'$ in $\mathcal{E}$. Pushing out $A' \to B'$ along $A' \to A$, and using the stability of $\mathcal{E}$ under pushouts, we see that it is enough to consider retracts of the form

$$\begin{tikzcd}
& A \arrow[dl] \arrow[dr] & \\
B' \arrow[rr, shift left, "r"] & & B \arrow[ll, shift left, "i"]
\end{tikzcd}$$

$ri = id_B$, with $A \to B'$ in $\mathcal{E}$. Thus, let $p \colon E \to A$ be a bundle. Extend p to $p' \colon E' \to B'$, then take the pullback $p'' \colon E'' \to B$ of p' along i, yielding a diagram of pullbacks

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "p"'] & E' \arrow[d, "p'"'] & E'' \arrow[l] \arrow[d, "p''"] \\
A \arrow[r] & B' & B \arrow[l, "i"']
\end{tikzcd}$$

with $E \to E'$ anodyne. Pulling back p'' back along $A \to B$ gives the pullback of p' along $A \to B'$, i.e. p. Thus, p'' is a bundle extending p. But any bundle over B which extends p pulls back along r to a bundle over B' which extends p, so it must be p' by uniqueness. Thus p'' is unique and we have a retract

$$\begin{tikzcd}
& E \arrow[dl] \arrow[dr] & \\
E' \arrow[rr, shift left] & & E'' \arrow[ll, shift left]
\end{tikzcd}$$

so that $E \to E''$ is anodyne.

We leave the straightforward verification of coproducts and countable composites as an exercise for the reader. □

Lemma 3.4.2. *Let*

$$\begin{tikzcd}
& E \arrow[rr] \arrow[dl] \arrow[dd] & & E' \arrow[dl] \arrow[dd] \\
G \arrow[rr, crossing over] \arrow[dd] & & G' & \\
& A \arrow[rr] \arrow[dl] & & A' \arrow[dl] \\
B \arrow[rr] & & B' \arrow[from=uu, crossing over] &
\end{tikzcd}$$

be a commutative cube in $\mathbf{S}$*, whose left-hand and back faces are pullbacks. If* $A \to B$ *is a monomorphism, and the top and bottom faces are pushouts, then the right-hand and front faces are pullbacks.*

Proof. It is enough to prove the lemma in the category of sets. In that case, the right-hand and front faces are

$$\begin{tikzcd}
E' \arrow[r] \arrow[d] & E' + (G - E) \arrow[d] \\
A' \arrow[r] & A' + (B - A)
\end{tikzcd}$$

and

$$\begin{tikzcd}
E + (G - E) \arrow[r] \arrow[d] & E' + (G - E) \arrow[d] \\
A + (B - A) \arrow[r] & A' + (B - A)
\end{tikzcd}$$

In these diagrams, $G - E$ maps to $B - A$ since the left-hand face of the cube is a pullback. Thus, these two faces are the coproducts of

$$\begin{tikzcd}
E' \arrow[r, "id"] \arrow[d] & E' \arrow[d] \\
A' \arrow[r, "id"] & A'
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & (G - E) \arrow[d] \\
0 \arrow[r] & (B - A)
\end{tikzcd}$$

and

$$\begin{tikzcd} E \arrow[r] \arrow[d] & E' \arrow[d] \\ A \arrow[r] & A' \end{tikzcd}$$

and

$$\begin{tikzcd} (G-E) \arrow[r, "id"] \arrow[d] & (G-E) \arrow[d] \\ (B-A) \arrow[r, "id"] & (B-A) \end{tikzcd}$$

respectively. Now use the fact that the coproduct of two pullbacks in Sets is a pullback. □

Remark. Notice that Lemma 3.4.1 shows that weak equivalences are stable under pullback along a bundle. In fact, let $w\colon A \to B$ be a weak equivalence and $p'\colon E' \to B$ a bundle. Factor w as $w = pi$ where i is a cofibration and p is a trivial fibration (Theorem 3.1.2). i is a weak equivalence since w and p are, hence anodyne by Proposition 3.4.2. Trivial fibrations are stable under any pullback, and anodyne extensions are stable under pullback along a bundle by Lemma 3.4.1, so the result follows.

Proof of Theorem 3.4.1. Q1 and Q2 are clear. Q3 follows immediately from Propositions 3.4.1 and 3.4.2, and Q4 follows from Theorems 3.1.1 and 3.1.2.

For Q5, let $w\colon A \to B$ be a weak equivalence and $p\colon E \to B$ a fibration. By Theorem 3.3.3 there is a minimal fibration $p_0\colon E_0 \to B$ which is a strong fiberwise deformation retract of p. Let $p'\colon E' \to A$ be the pullback of p along w and $p_0'\colon E_0' \to A$ the pullback of p_0. Then p_0' is a strong fiberwise deformation retract of p', since these are preserved by pullback. The map $E_0' \to E_0$ is a weak equivalence by the remark following Lemma 3.4.1. Thus $E' \to E$ is a weak equivalence. Dually, the pushout of a weak equivalence along a cofibration is a weak equivalence by Theorem 2.3.1 since every object of **S** is cofibrant. □

Chapter 4

Homotopy groups and Milnor's Theorem

In this chapter we introduce the homotopy groups of a Kan complex, establish the long exact sequence of a fibration, and investigate some of its consequences. Section 1 treats various aspects of the category of pointed simplicial sets. In section 2 we define the homotopy groups and establish their basic properties. Section 3 is devoted to showing that the approach we take in section 2 to the homotopy groups is equivalent to the classical one. Section 4 deals with the long exact sequence, from which we deduce Whitehead's Theorem as a corollary. Milnor's Theorem, which shows that the category of Kan complexes and homotopy classes of maps is equivalent to the category of CW-complexes and homotopy classes of maps, is proved in section 5. In section 6, we show that the weak equivalences used in chapter 3 for the proof of the Quillen structure on $\mathbf{S}$ are the same as the classical ones. Finally, once we have the Quillen structure on $\mathbf{S}$ and Milnor's Theorem, the structure for Top_c is not too hard, and we establish it in section 7.

4.1 Pointed simplicial sets

Let $\mathcal{K}$ be a category with at least finite limits and colimits. Denoting the terminal object of $\mathcal{K}$ by 1, we write $\mathcal{K}_\bullet$ for the category $1/\mathcal{K}$ of *pointed objects* of $\mathcal{K}$. We will often supress mention of the base point, as long as this does not lead to confusion. Thus, if (A, a) is an object of $\mathcal{K}_\bullet$ we will write just A for it, realizing that it is also provided with a base point $a\colon 1 \to A$.

With the obvious base point, 1 becomes initial in $\mathcal{K}_\bullet$, as well as terminal, so $\mathcal{K}_\bullet$ has 1 as a 0-object. Also, any two objects A and B of $\mathcal{K}_\bullet$ have a 0-map $A \to 1 \to B$ between them.

Let $U\colon \mathcal{K}_\bullet \to \mathcal{K}$ denote the underlying functor that forgets the base point. U has a left adjoint $A \mapsto A^+ = A + 1$ with the obvious base point, and U creates limits. That is, limits in $\mathcal{K}_\bullet$ are calculated in $\mathcal{K}$. For example, $(A, a) \times (B, b) = (A \times B, (a, b))$. U also creates coequalizers and pushouts. If $f, g \colon (A, a) \to (B, b)$ are maps in $\mathcal{K}_\bullet$, let

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "q"] & C \end{tikzcd}$$

be a coequalizer in $\mathcal{K}$. Then

$$\begin{tikzcd} (A, a) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (B, b) \arrow[r, "q"] & (C, qb) \end{tikzcd}$$

is a coequalizer in $\mathcal{K}_\bullet$. Pushouts are similar. The sum of (A, a) and (B, b) is the pushout $A \vee B$ in $\mathcal{K}$ with the evident base point.

$$\begin{tikzcd} 1 \arrow[r, "a"] \arrow[d, "b"'] & A \arrow[d] \\ B \arrow[r] & A \vee B \end{tikzcd}$$

Proposition 4.1.1. *Let $(\mathcal{F}, \mathcal{C}, \mathcal{W})$ be a Quillen structure on $\mathcal{K}$. Then $(U^{-1}(\mathcal{F}), U^{-1}(\mathcal{C}), U^{-1}(\mathcal{W}))$ is a Quillen structure on $\mathcal{K}_\bullet$. If the structure on $\mathcal{K}$ is proper, so is the one on $\mathcal{K}_\bullet$*

Proof. Q_1 and Q_2 are clearly satisfied. Suppose

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & E \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

is a commutative diagram in $\mathcal{K}_\bullet$ in which i is a cofibration and p is a fibration. If i or p is in $U^{-1}(\mathcal{W})$, the diagram has a diagonal filler in $\mathcal{K}$. Since i and $A \to E$ preserve base points, so does $B \to E$, giving Q_3.

Let $f \colon X \to Y$ be a map in $\mathcal{K}_\bullet$.Then f can be factored as

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "f"'] & & E \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

in $\mathcal{K}$ where i is a cofibration and p is a fibration in two ways: one in which i is in $\mathcal{W}$, and one in which p is in $\mathcal{W}$. If $x \colon 1 \to X$ is the base point of X, and $y \colon 1 \to Y$

the base point of Y, then

$$\begin{tikzcd}
(X,x) \arrow[rr, "i"] \arrow[dr, "f"'] & & (E,ix) \arrow[dl, "p"] \\
& (Y,y) &
\end{tikzcd}$$

is the desired factorization in $\mathcal{K}_\bullet$, so that Q_4 is satisfied.

From the above discussion, if Q_5 holds in $\mathcal{K}$ it holds in $\mathcal{K}_\bullet$. □

When $\mathcal{K} = \mathbf{S}$ we obtain $\mathbf{S}_\bullet$, the category of *pointed simplicial sets*. It has the above Quillen structure inherited from $\mathbf{S}$. Let X and Y be pointed simplicial sets. There is an obvious inclusion $X \vee Y \to X \times Y$ and we define the *smash product* of X and Y as the pushout $X \wedge Y$ in the diagram

$$\begin{tikzcd}
X \vee Y \arrow[r] \arrow[d] & X \times Y \arrow[d] \\
1 \arrow[r] & X \wedge Y
\end{tikzcd}$$

with its given base point. Let $x\colon 1 \to X$ be the base point of X, and $y\colon 1 \to Y$ the base point of Y. We define $Y_\bullet^X$, the *simplicial set of base point preserving maps* from X to Y as the equalizer in the diagram

$$\begin{tikzcd}
Y_\bullet^X \arrow[r] & Y^X \arrow[rr, "Y^x"] \arrow[dr] & & Y^1 \\
& & 1 \arrow[ur, "y"'] &
\end{tikzcd}$$

with the 0-map as base point. Then the evaluation map $Y^X \times X \to Y$ induces $Y_\bullet^X \wedge X \to Y$ in $\mathbf{S}_\bullet$ which, together with the evident map $X \to (X \wedge Y)_\bullet^Y$ determines an adjunction

$$(\ ) \wedge X \dashv (\ )_\bullet^X$$

in $\mathbf{S}_\bullet$.

Theorem 4.1.1. *Let X be a pointed simplicial set, and $i\colon A \to B$ an anodyne extension of $\mathbf{S}_\bullet$. Then $X \wedge i$ is anodyne.*

Proof. In the following diagram, the top square is a pushout, as is the whole rectangle, so the bottom square is a pushout.

$$\begin{tikzcd}
\{a\} \times X \cup A \times \{x\} \arrow[r] \arrow[d] & A \times X \arrow[d] \\
\{a\} \times X \cup B \times \{x\} \arrow[r] \arrow[d] & A \times X \cup B \times \{x\} \arrow[d] \\
1 \arrow[r] & A \wedge X
\end{tikzcd}$$

Let $k\colon \{x\} \to X$ denote the inclusion of the base point. Then in the diagram below, the two sides are pushouts by the previous remark, so the whole square is a pushout.

$$\begin{tikzcd}
A \times X \cup B \times \{x\} \arrow[rr, "i*k"] \arrow[ddd] & & B \times X \arrow[ddd] \\
& \{a\} \times X \cup B \times \{x\} \arrow[ul] \arrow[ur] \arrow[d] & \\
& 1 \arrow[dl] \arrow[dr] & \\
A \wedge X \arrow[rr, swap, "i \wedge X"] & & B \wedge X
\end{tikzcd}$$

$i * k$ is anodyne by Theorem 3.2.2, so $i \wedge X$ is anodyne. □

Corollary 4.1.1. *Let p be a fibration in $\mathbf{S}_\bullet$ and X a pointed simplicial set. Then $p_\bullet^X$ is a fibration.*

Corollary 4.1.2. *Let X be a pointed simplicial set and let Y in $\mathbf{S}_\bullet$ be Kan. Then $Y_\bullet^X$ is Kan.*

If $X \in \mathbf{S}_\bullet$ we have

$$\mathbf{S}_\bullet(\Delta[n]^+, X) \simeq \mathbf{S}(\Delta[n], X) \simeq X_n.$$

Hence

$$(Y_\bullet^X)_n \simeq \mathbf{S}_\bullet(\Delta[n]^+, Y_\bullet^X) \simeq \mathbf{S}_\bullet(\Delta[n]^+ \wedge X, Y)$$

so that

$$(Z_\bullet^{(X \wedge Y)})_n \simeq \mathbf{S}_\bullet(\Delta[n]^+ \wedge X \wedge Y, Z) \simeq \mathbf{S}_\bullet(\Delta[n]^+ \wedge X, Z_\bullet^Y) \simeq ((Z_\bullet^Y)_\bullet^X)_n$$

and we obtain an isomorphism

$$Z_\bullet^{(X \wedge Y)} \simeq (Z_\bullet^Y)_\bullet^X.$$

The 0-simplices of $Y_\bullet^X$ can be identified with the pointed maps from X to Y. A 1-simplex $h \in (Y_\bullet^X)_1$ such that $d^1h = f$ and $d^0h = g$ can be identified with a homotopy $h\colon \Delta[1] \times X \to Y$ such that $h_0 = f$, $h_1 = g$, and h maps $\Delta[1] \times \{x\}$ into $\{y\}$. Such an h is called a *base point preserving* or *pointed* homotopy from f to g. Thus, in general, f and g are in the same connected component of $Y_\bullet^X$ iff there is a "pointed homotopy of length n" from f to g. When Y is Kan, however, so is $Y_\bullet^X$ and pointed homotopy is an equivalence relation.

Let X and Y be pointed simplicial sets. We write

$$[X, Y]_\bullet = \pi_0(Y_\bullet^X).$$

The mapping $Y^X \times Z^Y \to Z^X$ induces a map $Y_\bullet^X \times Z_\bullet^Y \to Z_\bullet^X$ and hence a composition

$$[X, Y]_\bullet \times [Y, Z]_\bullet \to [X, Z]_\bullet$$

since π_0 preserves finite products. We call the resulting category the category of *pointed simplicial sets and pointed homotopy classes of maps*, and we write it as $\pi(\mathbf{S}_\bullet)$. Note that the canonical functor $\mathbf{S}_\bullet \to \pi(\mathbf{S}_\bullet)$ preserves finite products and sums. In fact, we have

$$\begin{aligned}[X, Y \times Z]_\bullet \simeq \pi_0((Y \times Z)_\bullet^X) \simeq \pi_0((Y_\bullet^X \times Z_\bullet^X) \simeq \\ \simeq \pi_0((Y_\bullet^X) \times \pi_0(Z_\bullet^X) \simeq [X, Y]_\bullet \times [X, Z]_\bullet\end{aligned}$$

and sums are similar. Finally, since $Z_\bullet^{(X \wedge Y)} \simeq (Z_\bullet^Y)_\bullet^X$ we have also

$$[X \wedge Y, Z]_\bullet \simeq [X, Z_\bullet^Y]_\bullet$$

4.2 Homotopy groups

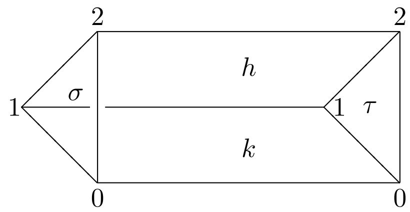
Let X be a Kan complex in $\mathbf{S}_\bullet$. We define a category $\pi(X)$ as follows: its set of objects is X_0 the 0-simplices of X, and its morphisms are equivalence classes of 1-simplices under *endpoint fixed homotopy*. Where, if $s, t \in X_1$ are such that $d^1s = d^1t$ and $d^0s = d^0t$ an endpoint fixed homotopy between s and t is a mapping $h\colon \Delta[1] \times \Delta[1] \to X$ such that $h_0 = s$, $h_1 = t$, and the diagram

$$\begin{tikzcd}
\partial\Delta[1] \times \Delta[1] \arrow[r, "\pi_1"] \arrow[d] & \partial\Delta[1] \arrow[d] \\
\Delta[1] \times \Delta[1] \arrow[r, "h"'] & X
\end{tikzcd}$$

commutes. This is an equivalence relation since X is Kan, and we denote the equivalence class of $s \in X_1$ by $[s]$. Composition is defined as follows: if $\sigma \in X_2$, $[d^1\sigma] = [d^0\sigma] \circ [d^2\sigma]$. If $x \in X_0$, $id_x = [s^0x]$.

Proposition 4.2.1. *Composition in $\pi(X)$ is well-defined.*

Proof. Let $\sigma, \tau \in X_2$, and let h, k be endpoint fixed homotopies between $d^0\sigma$ and $d^0\tau$ and $d^2\sigma$ and $d^2\tau$ respectively. σ, τ and h, k determine an open prism in X.



Since X is Kan, we can fill the prism, and its front face provides an endpoint fixed homotopy between $d^1\sigma$ and $d^1\tau$. □

Let $\pi_1(X)$ be the fundamental groupoid of X-Definition 2.4.3. Recall that $\pi_1(X)$ can be described as follows: its objects are the 0-simplices of X, and its morphisms are generated by the 1-simplices X_1 and their inverses. The relations between these are determined by $id_x = s^0 x$ for $x \in X_0$, and $d^1\sigma = d^0\sigma \circ d^2\sigma$ for $\sigma \in X_2$ Clearly, when X is Kan the canonical mapping of X_1 to the morphism of $\pi_1(X)$ is surjective. That is, both composition and inverses are represented by elements of X_1. It follows that the identity on X_0 and the quotient map on X_1 define a functor $\phi\colon \pi_1(X) \to \pi(X)$.

Proposition 4.2.2. *ϕ is an isomorphism.*

Proof. ϕ is surjective on morphisms. Suppose $s, t \in X_1$, and $\phi(s) = \phi(t)$, i.e. $[s] = [t]$. Then there is an endpoint fixed homotopy $h\colon \Delta[1] \times \Delta[1] \to X$ such that $h_0 = s$, $h_1 = t$. Since h is endpoint fixed, it passes to the pushout

$$\begin{tikzcd}
(0) \times \Delta[1] \arrow[r] \arrow[d] & \Delta[1] \times \Delta[1] \arrow[r, "h"] \arrow[d] & X \\
1 \arrow[r, "v_0"'] & \Delta[2] \arrow[ur, "\overline{h}"'] &
\end{tikzcd}$$

where v_0 is the 0^{th} vertex of $\Delta[2]$. Since h is also constant on $(1) \times \Delta[1]$, $\overline{h}$ is a 2-simplex in X such that $d^1\overline{h} = s$, $d^2\overline{h} = t$ and $d^0\overline{h} = id_{d^0 t}$. It follows that $s = t \circ id_{d^0 t} = t$ in $\pi_1(X)$. ϕ is thus an equivalence which is the identity on objects, i.e. an isomorphism. In particular, $\pi(X)$ is a groupoid. □

Define S^1 by the pushout

$$\begin{tikzcd}
\partial\Delta[1] \arrow[r] \arrow[d] & \Delta[1] \arrow[d] \\
1 \arrow[r] & S^1
\end{tikzcd}$$

with its given base point. Let $X \in \mathbf{S}_\bullet$ be Kan with base point $x \in X_0$. The vertex group of $\pi(X)$ at x is $[S^1, X]_\bullet$, and it follows from the above that $[S^1, X]_\bullet \simeq \pi_1(X, x)$. Write $S^n = S^1 \wedge \cdots \wedge S^1$ n-times for $n \geq 1$, letting $S^0 = \partial\Delta[1]$. Define the n^{th} *homotopy group* of X at $x \in X_0$ by

$$\pi_n(X, x) = [S^n, X]_\bullet \simeq [S^1, X_\bullet^{S^{n-1}}]_\bullet.$$

If $Y \in \mathbf{S}$, let us write (temporarily) $\underline{Y}$ for the pushout

$$\begin{tikzcd} Sk^0 Y \arrow[r] \arrow[d] & Y \arrow[d] \\ 1 \arrow[r] & \underline{Y} \end{tikzcd}$$

provided with its canonical base point. Consider the pushout

$$\begin{tikzcd} \Lambda^1[2] \arrow[r] \arrow[d] & \underline{\Lambda^1[2]} \arrow[d, "a"] \\ \Delta[2] \arrow[r] & \underline{\Delta[2]} \end{tikzcd}.$$

Thus, we have an anodyne map $a\colon S^1 \vee S^1 = \underline{\Lambda^1[2]} \to \underline{\Delta[2]}$, and also a map $\underline{\varepsilon}^1\colon S^1 = \underline{\Delta[1]} \to \underline{\Delta[2]}$. Then, for $X \in \mathbf{S}_\bullet$ Kan we have maps

$$X_\bullet^{S^1 \vee S^1} \xleftarrow{X_\bullet^a} X_\bullet^{\underline{\Delta[2]}} \xrightarrow{X_\bullet^{\underline{\varepsilon}^1}} X_\bullet^{S^1}.$$

Now $X_\bullet^a$ is a trivial fibration, so it has a section, and we get a map

$$X_\bullet^{S^1} \times X_\bullet^{S^1} \simeq X_\bullet^{S^1 \vee S^1} \longrightarrow X_\bullet^{S^1}$$

which is independent, up to homotopy, of the choice of section, since any two sections are homotopic. This map induces a homomorphism

$$*\colon \pi_n(X_\bullet^{S^1} \times X_\bullet^{S^1}, (x, x)) \simeq \pi_n(X_\bullet^{S^1}, x) \times \pi_n(X_\bullet^{S^1}, x) \longrightarrow \pi_n(X_\bullet^{S^1}, x).$$

If we denote the original group structure on $\pi_n(X_\bullet^{S^1}, x)$ by $+$, this means that for $\alpha, \beta, \gamma, \delta \in \pi_n(X_\bullet^{S^1}, x)$ we have

$$(\alpha + \beta) * (\gamma + \delta) = (\alpha * \gamma) + (\beta * \delta).$$

Also, it is easy to see that the two structures share the same unit —the 0-map. But then we have

Theorem 4.2.1 (Eckmann-Hilton)**.** *Let A be a set provided with two binary operations $+, *\colon A \times A \to A$, such that for $\alpha, \beta, \gamma, \delta \in A$ we have*

$$(\alpha + \beta) * (\gamma + \delta) = (\alpha * \gamma) + (\beta * \delta).$$

Moreover suppose that $+$ and $*$ have a common two-sided unit e. Then $+ = *$, and the (single) operation is commutative and associative.

Proof.

$$\alpha * \beta = (\alpha + e) * (e + \beta) = (\alpha * e) + (e * \beta) = \alpha + \beta.$$

Also,

$$\beta * \alpha = (e + \beta) * (\alpha + e) = (e * \alpha) + (\beta * e) = \alpha + \beta$$

and

$$(\alpha * \beta) * \gamma = (\alpha * \beta) + (e * \gamma) = (\alpha + e) * (\beta + \gamma) = \alpha * (\beta * \gamma). \qquad \square$$

It follows that $\pi_n(X_\bullet^{S^1}, x)$ is abelian for $n \geq 1$, so that $\pi_n(X, x)$ is abelian for $n \geq 2$.

4.3 A particular weak equivalence

The classical treatment of the homotopy groups, [ ] etc., uses $\Delta[n]/\partial\Delta[n]$ instead of S^n. Thus, a question could arise in the reader's mind as to whether our treatment is equivalent to the classical one. To allay any such fears, we construct here a weak equivalence $S^n \to \Delta[n]/\partial\Delta[n]$.

We begin by constructing a retraction

$$r\colon \Delta[n] \times \Delta[1] \to \Delta[n+1].$$

On the level of the partially ordered sets involved, define $i\colon [n+1] \to [n] \times [1]$ by $i(k) = (k, 0)$ for $0 \leq k \leq n$ and $i(n+1) = (n, 1)$. We put $r(k, 0) = k$ and $r(k, 1) = n + 1$ for $0 \leq k \leq n$. Clearly, $ri = id$ and $id \leq ir$. Let $\partial^k\Delta[n]$ denote the k^{th} face of $\Delta[n]$. Taking nerves, or remaining in partially ordered sets, we see

$$\begin{aligned} &r(\Delta[n] \times \{0\}) \subseteq \partial^{n+1}\Delta[n+1] \\ &r(\Delta[n] \times \{1\}) = \{n+1\} \\ &r(\partial^k\Delta[n] \times \Delta[1]) \subseteq \partial^k\Delta[n+1] \cup \{n+1\}. \end{aligned}$$

Thus,

$$r(\Delta[n] \times \partial\Delta[1] \cup \partial\Delta[n] \times \Delta[1]) \subseteq \partial\Delta[n+1].$$

So r induces a map, also called r,

$$\begin{aligned} &(\Delta[n]/\partial\Delta[n]) \wedge S^1 = \\ &\qquad = \Delta[n] \times \Delta[1]/(\Delta[n] \times \partial\Delta[1] \cup \partial\Delta[n] \times \Delta[1]) \longrightarrow \Delta[n+1]/\partial\Delta[n+1] \end{aligned}$$

Theorem 4.3.1. *r is a weak equivalence.*

Proof. Recalling the triangulation of $\Delta[n]\times\Delta[1]$ from 1.4, let S be the subcomplex of $\Delta[n]\times\Delta[1]$ which is the union of the first n $(n+1)$-simplices of $\Delta[n]\times\Delta[1]$.

When $n=2$, $[2]\times[1]$ is

$$\begin{tikzcd}
(0,1) \arrow[r] & (1,1) \arrow[r] & (2,1) \\
(0,0) \arrow[r] \arrow[u] & (1,0) \arrow[r] \arrow[u] & (2,0) \arrow[u]
\end{tikzcd}$$

and S is

$$\begin{tikzcd}
(0,1) \arrow[r] & (1,1) \arrow[r] & (2,1) \\
(0,0) \arrow[r] \arrow[u] & (1,0) \arrow[u] &
\end{tikzcd}$$

r collapses S onto the 2-simplex $(0,0)\to(1,0)\to(2,1)$.

Gluing an $(n+1)$-simplex to S along $\Delta[n]$ gives $\Delta[n]\times\Delta[1]$, so we have a pushout

$$\begin{tikzcd}
\Delta[n] \arrow[r] \arrow[d] & S \arrow[d] \\
\Delta[n+1] \arrow[r] & \Delta[n]\times\Delta[1]
\end{tikzcd}$$

and S is contractible. Let

$$\overline{S} = S\cap(\Delta[n]\times\partial\Delta[1]\cup\partial\Delta[n]\times\Delta[1]).$$

Then when we shrink the boundary of $\Delta[n]\times\Delta[1]$ we have

$$\begin{tikzcd}
S \arrow[r] \arrow[d] & \Delta[n]\times\Delta[1] \arrow[d] \\
S/\overline{S} \arrow[r] & (\Delta[n]/\partial\Delta[n])\wedge S^1
\end{tikzcd}$$

where the horizontal arrows are inclusions. Denote $S/\overline{S}$ by C. If we show $\overline{S}$ is contractible, then in the pushout

$$\begin{tikzcd}
\overline{S} \arrow[r] \arrow[d] & S \arrow[d] \\
1 \arrow[r] & S/\overline{S}
\end{tikzcd}$$

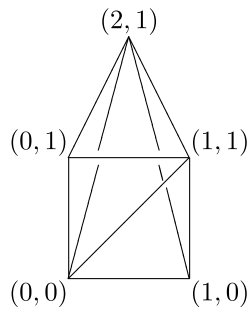
$S \to S/\overline{S}$ is a weak equivalence, so $C = S/\overline{S}$ is contractible. But

$$r\colon (\Delta[n]/\partial\Delta[n]) \wedge S^1 \longrightarrow \Delta[n+1]/\partial\Delta[n+1]$$

collapses C, i.e.

$$\Delta[n+1]/\partial\Delta[n+1] = ((\Delta[n]/\partial\Delta[n]) \wedge S^1)/C$$

so r is a weak equivalence. When $n = 2$, $\overline{S}$ is the 2-dimensional complex



without, of course, the back 2-simplex with vertices $(0,0)$, $(1,0)$ and $(2,1)$. Here we obtain S from $\overline{S}$ by attaching two 3-simplices along horns. So we have pushouts

$$\begin{tikzcd} \Lambda^1[3] \arrow[r] \arrow[d] & \overline{S} \arrow[d] \\ \Delta[3] \arrow[r] & S_1 \end{tikzcd}$$

and

$$\begin{tikzcd} \Lambda^2[3] \arrow[r] \arrow[d] & S_1 \arrow[d] \\ \Delta[3] \arrow[r] & S \end{tikzcd}$$

so $\overline{S}$ is contractible since S is.

In the general case, we obtain S from $\overline{S}$ by attaching n $(n+1)$-simplices along horns $\Lambda^1[n+1], \ldots, \Lambda^n[n+1]$. $\square$

Theorem 4.3.2. *There is a weak equivalence*

$$S^n \longrightarrow \Delta[n]/\partial\Delta[n]$$

Proof. Iterate the weak equivalence above:

$$\overbrace{S^1 \wedge \cdots \wedge S^1}^{n} = \Delta[1]/\partial\Delta[1] \wedge \overbrace{S^1 \wedge \cdots \wedge S^1}^{n-1} \to \Delta[2]/\partial\Delta[2] \wedge \overbrace{S^1 \wedge \cdots \wedge S^1}^{n-2} \to$$
$$\cdots \to \Delta[n-1]/\partial\Delta[n-1] \wedge S^1 \to \Delta[n]/\partial\Delta[n].$$ $\square$

4.4 The long exact sequence and Whitehead's Theorem

Let $p\colon E \to X$ be a fibration with X Kan, and write $i\colon F \to E$ for the inclusion $p^{-1}(x) \to E$, where x is the base point of X. F is called the *fiber* of p. Let $e \in F_0$ be the base point of E. Write $j = \varepsilon^1\colon 1 \to \Delta[1]$ and denote by $p_0\colon X^{\Delta[1]} \to X$ the evaluation of a path at 0.

We have a pullback

$$\begin{tikzcd} (j,p) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ X^{\Delta[1]} \arrow[r, "p_0"'] & X \end{tikzcd}$$

and the commutative diagram

$$\begin{tikzcd} E^{\Delta[1]} \arrow[r, "p_0"] \arrow[d, "p^{\Delta[1]}"'] & E \arrow[d, "p"] \\ X^{\Delta[1]} \arrow[r, "p_0"'] & X \end{tikzcd}$$

determines a mapping $j|p\colon E^{\Delta[1]} \to (j,p)$ which is a trivial fibration. It follows that the map q in the pullback

$$\begin{tikzcd} P(e,F) \arrow[r] \arrow[d, "q"'] & E^{\Delta[1]} \arrow[d, "j|p"] \\ X_\bullet^{S^1} \arrow[r] & (j,p) \end{tikzcd}$$

is also a trivial fibration, where $P(e,F)$ denotes the space of paths in E which begin at e and end in F. Choose a section $s\colon X_\bullet^{S^1} \to P(e,F)$ to q and let $\partial = X_\bullet^{S^1} \xrightarrow{s} P(e,F) \xrightarrow{p_1} F$. ∂ is independent, up to homotopy, of the choice of s, since any two sections of q are homotopic. It is called the *boundary map*.

In this fashion we obtain a sequence

$$\cdots F_\bullet^{S^1} \xrightarrow{i_\bullet^{S^1}} E_\bullet^{S^1} \xrightarrow{p_\bullet^{S^1}} X_\bullet^{S^1} \xrightarrow{\partial} F \xrightarrow{i} E \xrightarrow{p} X.$$

Theorem 4.4.1.

$$\pi_0(E_\bullet^{S^1}) \xrightarrow{\pi_0(p_\bullet^{S^1})} \pi_0(X_\bullet^{S^1}) \xrightarrow{\pi_0(\partial)} \pi_0(F) \xrightarrow{\pi_0(i)} \pi_0(E) \xrightarrow{\pi_0(p)} \pi_0(X)$$

is exact.

Proof. We will establish exactness at $\pi_0(X_\bullet^{S^1})$, leaving the others as exercises. Let $\alpha\colon S^1 \to E$. sq is homotopic to the identity (Proposition 3.2.5), so $sp\alpha$ is homotopic to α, providing a path between $\partial p\alpha$ and e.

On the other hand, suppose $\beta\colon S^1 \to X$ is such that $\partial\beta = (s\beta)(1)$ is in the connected component of e. Let $\gamma\colon \Delta[1] \to F$ be a path such that $\gamma(0) = e$ and $\gamma(1) = (s\beta)(1)$. We obtain a map

$$\theta\colon (0) \times \Delta[1] \cup \Delta[1] \times (1) \cup (1) \times \Delta[1] \longrightarrow E$$

by taking $\theta|(0) \times \Delta[1] = e$, $\theta|\Delta[1] \times (1) = s\beta$ and $\theta|(1) \times \Delta[1] = \gamma$. Since E is Kan, we can extend θ to a map $h\colon \Delta[1] \times \Delta[1] \to E$. We have $h_{0t} = e$, $h_{t1} = s\beta$ and $h_{1s} = \gamma$. Thus, $h_{t0}\colon S^1 \to E$ and ph_{t0} is homotopic to $ph_{t1} = \beta$. □

Notice that we obtain the full long exact sequence in this way, since the next section just repeats the first for the fibration $p_\bullet^{S^1}$, etc.

We now apply the exact sequence to obtain Whitehead's Theorem, which will be needed in the next section. Thus, let $f\colon X \to Y$ be a mapping of Kan complexes. We will call f, just in this chapter, a *homotopy isomorphism* if $\pi_0 f\colon \pi_0(X) \to \pi_0(Y)$ is a bijection, and $\pi_n f\colon \pi_n(X, x) \to \pi_n(Y, fx)$ is an isomorphism for $n \geq 1$ and $x \in X_0$.

Lemma 4.4.1. *Let X be a minimal Kan complex such that $X \to 1$ is a homotopy isomorphism. Then $X \to 1$ is an isomorphism.*

Proof. Choose a base point $x_0\colon 1 \to X$. If x is another vertex of X, there is a path $\alpha\colon I \to X$ joining x and x_0 since X is connected and Kan. $\partial\Delta[0] = 0$ so α can be considered to be a homotopy $x \sim x_0$: mod $\partial\Delta[0]$. Since X is minimal, $x = x_0$ and $X_0 = \{x_0\}$. Suppose $X_m = \{x_0\}$ for $m < n$, and let $\sigma\colon \Delta[n] \to X$ be an n-simplex of X. $\sigma|\partial\Delta[n] = x_0$, so σ represents an element of $\pi_n(X, x_0)$. $\pi_n(X, x_0) = \{x_0\}$, so $\sigma \sim x_0$ mod $\partial\Delta[n]$. But then $\sigma = x_0$. □

Corollary 4.4.1. *Let X be a Kan complex, and $p\colon E \to X$ a minimal fibration. If p is a homotopy isomorphism, then p is an isomorphism.*

Proof. The homotopy exact sequence of p as above has the form

$$\ldots \to \pi_{n+1}(E, e) \xrightarrow{\simeq} \pi_{n+1}(X, pe) \to \pi_n(F, e) \to \pi_n(E, e) \xrightarrow{\simeq} \pi_n(X, pe) \ldots$$

$$\pi_1(E, e) \xrightarrow{\simeq} \pi_1(X, pe) \to \pi_0(F) \to \pi_0(E) \xrightarrow{\simeq} \pi_0(X).$$

It follows that $\pi_0(F) = \{e\}$ and $\pi_n(F, e) = [e]$ for $n \geq 1$. By Lemma 4.4.1, the fiber F over each component of X is a single point, so p is a bijection. □

Theorem 4.4.2 (Whitehead)**.** *Let X and Y be Kan complexes and $f\colon X \to Y$ a homotopy isomorphism. Then f is a homotopy equivalence.*

Proof. Factor f as

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "f"'] & & E \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

where i is anodyne and p is a fibration. Y is Kan, so E is and i is a strong deformation retract by Proposition 3.2.3. By Theorem 3.3.3, let

$$\begin{tikzcd} E' \arrow[rr] \arrow[dr, "p'"'] & & E \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

be a minimal fibration which is a strong, fiberwise deformation retract of p. Since p induces isomorphisms on π_n for $n \geq 0$ so does p'. Thus p' is an isomorphism by Corollary 4.4.1. It follows that p is a homotopy equivalence, so the same is true of f. $\square$

4.5 Milnor's Theorem

Our goal in this section is to prove the following theorem.

Theorem 4.5.1 (Milnor)**.** *Let X be a Kan complex, and let $\eta X\colon X \to s|X|$ be the unit of the adjunction $|\ | \dashv s$. Then ηX is a homotopy equivalence.*

The proof of Theorem 4.5.1 uses some properties of the path space of X, so we establish these first. To begin, since $(0) + (1) \to I$ is a cofibration and X is a Kan complex, $(p_0, p_1)\colon X^I \to X \times X$ is a fibration. Let $x\colon 1 \to X$ be a base point. Define PX as the pullback

$$\begin{tikzcd} PX \arrow[r] \arrow[d, "p_1"'] & X^I \arrow[d, "(p_0,p_1)"] \\ 1 \times X \arrow[r, "x \times id_X"'] & X \times X \end{tikzcd}$$

$(0) \to I$ is anodyne, so $p_0\colon X^I \to X$ is a trivial fibration, again by Theorem 3.2.1. The diagram

$$\begin{tikzcd} PX \arrow[r] \arrow[d] & X^I \arrow[d, "p_0"] \\ 1 \arrow[r, "x"'] & X \end{tikzcd}$$

is a pullback, so $PX \to 1$ is a trivial fibration and hence a homotopy equivalence. Let ΩX denote the fiber of p_1 over x, and write x again for the constant path at x. ΩX is, of course, just $X_\bullet^{S^1}$ from before. Then the homotopy exact sequence of the fibration p_1 has the form

$$\cdots \to \pi_n(PX, x) \to \pi_n(X, x) \xrightarrow{\partial} \pi_{n-1}(\Omega X, x) \to \pi_{n-1}(PX, x) \cdots$$
$$\cdots \pi_1(PX, x) \to \pi_1(X, x) \xrightarrow{\partial} \pi_0(\Omega X) \to \pi_0(PX) \to \pi_0(X)$$

where $\pi_0(PX) = [x]$ and $\pi_n(PX, x) = [x]$ for $n \geq 1$. It follows that the boundary map induces isomorphisms $\pi_1(X, x) \to \pi_0(\Omega X)$ and $\pi_n(X, x) \to \pi_{n-1}(\Omega X, x)$ for $n \geq 2$.

Proof of Theorem 4.5.1. First, let X be connected. Then any two vertices of $|X|$ can be joined by a path. But then any two points of $|X|$ can be joined by a path, since any point is in the image of the realization of a simplex, which is connected. Thus $|X|$ is connected, as is $s|X|$. Otherwise, X is the coproduct of its connected components. Since $s|\ \ |$ preserves coproducts, it follows that $\pi_0\eta X\colon \pi_0(X) \to \pi_0(s|X|)$ is a bijection. Assume by induction that for any Kan complex Y, and any $y \in Y_0$ $\pi_m\eta Y\colon \pi_m(Y, y) \to \pi_m(s|Y|, |y|)$ is an isomorphism for $m \leq n-1$. By naturality we have a diagram

$$\begin{tikzcd}
\Omega X \arrow[r, "\eta\Omega X"] \arrow[d] & s|\Omega X| \arrow[d] \\
PX \arrow[r, "\eta PX"] \arrow[d, "p_1"'] & s|PX| \arrow[d, "s|p_1|"] \\
X \arrow[r, "\eta X"'] & s|X|
\end{tikzcd}$$

By Quillen's theorem (see Topic B), $|p_1|$ is a Serre fibration, so $s|p_1|$ is a Kan fibration. Since $PX \to 1$ is a homotopy equivalence, so is $s|PX| \to 1$. Hence we have a commutative diagram

$$\begin{tikzcd}
\pi_n(X, x) \arrow[r, "\pi_n\eta X"] \arrow[d, "\simeq"'] & \pi_n(s|X|, |x|) \arrow[d, "\simeq"] \\
\pi_{n-1}(\Omega X, x) \arrow[r, "\pi_{n-1}\eta\Omega X"'] & \pi_{n-1}(s|\Omega X|, x)
\end{tikzcd}$$

$\pi_{n-1}\eta\Omega X$ is an isomorphism by induction, so $\pi_n\eta X$ is an isomorphism. By Whitehead's Theorem, ηX is a homotopy equivalence. $\square$

An entirely similar argument, using the topological path space, shows that if T is a topological space, then the counit $\varepsilon T\colon |sT| \to T$ is a topological weak equivalence. Thus, if T is a CW-complex εT is a homotopy equivalence by the topological Whitehead Theorem. Since $|\ |$ and s both clearly preserve the homotopy relation between maps, we see that they induce an equivalence

$$ho(Top_c) \rightleftarrows ho(\mathbf{S})$$

where $ho(Top_c)$ is the category of CW-complexes and homotopy classes of maps.

4.6 Some remarks on weak equivalences

Here we collect all the possible definitions we might have given for weak equivalence, and show they are all the same. We begin with a lemma.

Lemma 4.6.1. *If $j\colon C \to D$ is a mapping in Top_c which has the left lifting property with respect to the Serre fibrations, then C is a strong deformation retract of D.*

Proof. Δ_n is a retract of $\Delta_n \times I$, so every space in Top_c is fibrant. Also, if T is a space and X a simplicial set, we see easily that $s(T^{|X|}) \simeq (sT)^X$. From this it follows that the singular complex of $(p_0, p_1)\colon T^I \to T \times T$ is a Kan fibration, so it itself is a Serre fibration. Now we obtain the retraction r as a lifting in

$$\begin{tikzcd} C \arrow[r, "id_C"] \arrow[d, "j"'] & C \\ D \arrow[ur, dashed, "r"'] & \end{tikzcd}$$

and the strong deformation h as the exponential transpose of a diagonal lifting in

$$\begin{tikzcd} C \arrow[r, "\overline{j\pi_1}"] \arrow[d, "j"'] & D^I \arrow[d, "{(p_0,p_1)}"] \\ D \arrow[ur, dashed, "\overline{h}"'] \arrow[r, "{(id_D,jr)}"'] & D \times D \end{tikzcd}$$

□

As a consequence we obtain immediately

Proposition 4.6.1. *If $i\colon A \to B$ is an anodyne extension then $|i|\colon |A| \to |B|$, and $|A|$ is a strong deformation retract of $|B|$.*

Proposition 4.6.2. *Let X be an arbitrary simplicial set. Then $\eta X\colon X \to s|X|$ is a weak equivalence.*

Proof. Let $i\colon X \to \overline{X}$ be an anodyne extension with $\overline{X}$ Kan. Then in the diagram

$$\begin{tikzcd} X \arrow[r, "\eta X"] \arrow[d, "i"'] & s|X| \arrow[d, "s|i|"] \\ \overline{X} \arrow[r, "\eta\overline{X}"'] & s|\overline{X}| \end{tikzcd}$$

$s|i|$ is a homotopy equivalence by the above, $\eta\overline{X}$ is a homotopy equivalence by Milnor's Theorem, and i is a weak equivalence, so ηX is a weak equivalence. □

Proposition 4.6.3. *$w\colon X \to Y$ is a weak equivalence in the sense of Definition 3.4.1 iff w is a geometric homotopy equivalence.*

Proof. Factor w as

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "w"'] & & E \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

where i is a cofibration and p is a trivial fibration. Since w and p are weak equivalences so is i. i is anodyne by Proposition 3.4.2, so $|i|$ is a homotopy equivalence. p is a homotopy equivalence by Proposition 3.2.6, so $|p|$ is. Thus $|w|$ is a homotopy equivalence.

On the other hand, suppose $w\colon X \to Y$ is a geometric homotopy equivalence. Then in the diagram

$$\begin{tikzcd} X \arrow[r, "\eta X"] \arrow[d, "w"'] & s|X| \arrow[d, "s|w|"] \\ Y \arrow[r, "\eta Y"'] & s|Y| \end{tikzcd}$$

ηX and ηY are weak equivalences as above, and $|w|$ is a homotopy equivalence, so $s|w|$ is. Thus w is a weak equivalence. □

Let $w\colon U \to V$ be a topological weak equivalence. From the diagram

$$\begin{tikzcd} |sU| \arrow[r, "|sw|"] \arrow[d, "\varepsilon U"'] & |sV| \arrow[d, "\varepsilon V"] \\ U \arrow[r, "w"'] & V \end{tikzcd}$$

we see that $|sw|$ is a weak equivalence since εU, εV and w are. By the topological Whitehead Theorem, it follows that $|sw|$ is a homotopy equivalence. Thus sw is a geometric homotopy equivalence, and a weak equivalence by the above. Thus, both s and $|\ |$ preserve weak equivalences. Since $\varepsilon T\colon |sT| \to T$ and $\eta X\colon X \to s|X|$ are weak equivalences, we see that s and $|\ |$ induce an equivalence

$$Top_c[W^{-1}] \rightleftarrows \mathbf{S}[W^{-1}]$$

where W stands for the class of weak equivalences in each case. This is not surprising, of course, since by Appendix E we have that $ho(Top_c)$ is equivalent to $Top_c[W^{-1}]$ and $ho(\mathbf{S})$ is equivalent to $\mathbf{S}[W^{-1}]$.

When X is a Kan complex, the homotopy equivalence $\eta X\colon X \to s|X|$ provides a bijection $\pi_0(X) \to \pi_0(|X|)$ and an isomorphism $\pi_n(X,x) \to \pi_n(|X|,|x|)$ for $n \geq 1$ and $x \in X_0$. Thus, for X an arbitrary simplicial set we can *define* $\pi_n(X,x)$ as $\pi_n(X,x) = \pi_n(|X|,|x|)$ as this is consistent with the case when X is Kan.

Finally, let $f\colon X \to Y$ be a homotopy isomorphism, i.e. $\pi_0 f\colon \pi_0(X) \to \pi_0(Y)$ is a bijection, and $\pi_n f\colon \pi_n(X,x) \to \pi_n(Y,fx)$ is an isomorphism for $n \geq 1$ and $x \in X_0$. Now, if f is a geometric homotopy equivalence, f is a homotopy isomorphism. On the other hand, if f is a homotopy isomorphism then f is a geometric homotopy equivalence by the topological Whitehead Theorem. Thus, the classes of weak equivalences, geometric homotopy equivalences and homotopy isomorphisms all coincide.

4.7 The Quillen model structure on Top_c

Recall that in Top_c the weak equivalences $\mathcal{W}$ are maps $f\colon X \to Y$ such that $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ is a bijection, and for $n \geq 1$ and any $x \in X$, $\pi_n(f)\colon \pi_n(X,x) \to \pi_n(Y,fx)$ is an isomorphism. The fibrations $\mathcal{F}$ are the Serre fibrations, i.e. the maps with the RLP with respect to the maps $\Delta_n \to \Delta_n \times I$ $(t \mapsto (t,0))$ $n \geq 0$. Notice that $\Delta_n \to \Delta_n \times I$ is homeomorphic to $|\Lambda^k[n+1]| \to |\Delta[n+1]|$ so $p\colon X \to Y$ is in $\mathcal{F}$ iff it has the RLP with respect to these, i.e. iff sp is a Kan fibration. Note also that each $|\Lambda^k[n]| \to |\Delta[n]|$ is a strong deformation retract, so every space X is fibrant. $\mathcal{C} =^{\pitchfork} (\mathcal{F} \bigcap \mathcal{W})$. Theorem 2.1.1 then states that $(\mathcal{F}, \mathcal{C}, \mathcal{W})$ is a Quillen model structure on Top_c.

We begin the proof of Theorem 2.1.1: $\mathcal{W}$ is clearly closed under retracts. For "three for two", suppose we have $f\colon X \to Y$ and $g\colon Y \to Z$. If f and g are weak equivalences, certainly gf is, and if gf and g are weak equivalences, f is. The only possible problem is the case when f and gf are weak equivalences. In that case, the point $y \in Y$ might not be in the image of f. But $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ is a bijection, so there is a point $x \in X$ and a path $\alpha\colon I \to Y$ from $f(x)$ to y. This gives a commutative square

$$\begin{tikzcd}
\pi_n(Y,x) \arrow[r, "\pi_n(g)"] \arrow[d] & \pi_n(Z,gy) \arrow[d] \\
\pi_n(Y,fx) \arrow[r, "\pi_n(g)"'] & \pi_n(Z,gfx)
\end{tikzcd}$$

where the left vertical map is conjugation by α and the right vertical map is conjugation by $g\alpha$.The bottom horizontal map is clearly an isomorphism, so the top horizontal map is also. Hence g is a weak equivalence, and $\mathcal{W}$ satisfies "three for two".

We now show $p\colon X \to Y$ is in $\mathcal{F} \bigcap \mathcal{W}$ iff sp is a trivial fibration of simplicial sets. Suppose the latter is true. Then sp is a weak equivalence and has the RLP with respect to any monomorphism $A \to B$ in $\mathbf{S}$. It follows that p has the RLP with respect to $|A| \to |B|$, so, in particular, it is a fibration. From the diagram

$$\begin{tikzcd} {|sX|} \arrow[r, "\epsilon_X"] \arrow[d, "|sp|"'] & X \arrow[d, "p"] \\ {|sY|} \arrow[r, "\epsilon_Y"'] & Y \end{tikzcd}$$

we see that p is a weak equivalence by "three for two" since ϵ_X, ϵ_Y and $|sp|$ are. On the other hand, suppose $p \in \mathcal{F} \cap \mathcal{W}$. Then sp is an acyclic Kan fibration, since s preserves weak equivalences. Thus, $p \in \mathcal{F} \cap \mathcal{W}$ iff p has the RLP with respect to the family $\{|\partial\Delta[n]| \to |\Delta[n]|\ |n \geq 0\}$.

To obtain the factorisations, we need some smallness results. For these we follow Hovey [10].

Say that $f\colon X \to Y$ is a *closed* T_1*-inclusion* if f is a closed inclusion such that every point in $Y - f(X)$ is closed in Y. For example, if $A \to B$ is a monomorphism in $\mathbf{S}$, then $|A| \to |B|$ is a closed T_1-inclusion (see Appendix D).

Lemma 4.7.1. *Closed* T_1*-inclusions are closed under pushouts and transfinite composition.*

Proof. Let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "j"] \\ B \arrow[r, "g"'] & Y \end{tikzcd}$$

be a pushout in which i is a closed T_1-inclusion. Let $D \subseteq X$ be a closed suset of X. Since j is injective, $j(D)$ is closed in Y iff $g^{-1}(j(D))$ is closed in B. But i is injective, so $g^{-1}(j(D)) = i(f^{-1}(D))$ which is closed in B, since i is a closed T_1-inclusion. If $y \in Y - j(X)$ then $g^{-1}(y) \in B - i(A)$ and hence closed. Also, $j^{-1}(y) = 0$, which is closed, so y is closed in Y, and j is a closed T_1-inclusion. The proof that closed T_1-inclusions are closed under transfinite composition is left to the reader. □

Proposition 4.7.1. *Compact spaces are finite with respect to closed* T_1*-inclusions.*

Proof. Let α be a limit ordinal and let $X\colon \alpha \to Top_c$ be an α-sequence of closed T_1-inclusions. Let C be compact, and suppose $f\colon C \to \varinjlim_{i<\alpha} X_i$ is a map. If $f(C)$ is not contained in X_i for any $i < \alpha$, then we can proceed as follows. Let $i_0 = 0$, and choose $x_1 \in f(C)$ such that $x_1 \notin X_0$. Then $x_1 \in X_{i_1}$ for some i_1. Now choose $x_2 \in f(C)$ such that $x_2 \notin X_{i_1}$. Then $x_2 \in X_{i_2}$ for some i_2 etc. In this way we obtain a sequence $\sigma = \{x_n\}_{n=1}^{\infty}$ of points in $f(C)$, and a sequence $\{i_n\}_{n=1}^{\infty}$ of ordinals such that $x_n \in X_{i_n} - X_{i_{n-1}}$. The intersection of any subset of σ with any X_{i_n} is finite —it cannot contain any x_m with $m > n+1$— and it misses X_0, so it is closed in X_{i_n}. Let $\beta = sup i_n$. Then β is a limit ordinal $\leq \alpha$. $X_\beta = \varinjlim_{i_n} X_{i_n}$, so σ is discrete as a subspace of X_β. $X_\beta \to \varinjlim_{i<\alpha} X_i$ is a closed inclusion, so σ has the discrete topology as a subspace of the compact space $f(C)$,which is impossible. Thus, $f(C) \subseteq X_i$ for some i.

Using Proposition 4.7.1, we can apply the small object argument with $\{|\partial\Delta[n]| \to |\Delta[n]| \;|n \geq 0\}$ to factor any map $f\colon X \to Y$ as $f = pi$ with $i \in \mathcal{C}$ and $p \in \mathcal{F} \cap \mathcal{W}$. Similarly, we can use the small object argument with $\{|\Lambda^k[n]| \to |\Delta[n]|| \; 0 \leq k \leq n, \; n \geq 1\}$ to factor any map $f\colon X \to Y$ as $f = pi$ with $i \pitchfork \mathcal{F}$ and $p \in \mathcal{F}$. By Lemma 4.6.1 i is a strong deformation retract and hence a weak equivalence, so we have the two factorisations.

A retract argument shows immediately that $(\mathcal{C} \cap \mathcal{W}) \pitchfork \mathcal{F}$. and $\mathcal{C} \pitchfork (\mathcal{F} \cap \mathcal{W})$ by definition, so the proof of Theorem 2.1.1 is complete. $\square$

The model structure on Top_c is right proper, since every space is fibrant. It is also left proper, though we will not prove this here. See Hirschhorn [9] Theorem 13.1.10 p 242.

Appendix A

Quillen Model Structures

A.1 Preliminaries

An object A of a category $\mathcal{E}$ is said to be a *retract* of an object B of $\mathcal{E}$ if there are maps $i\colon A \to B$ and $r\colon B \to A$ such that $ri = id_A$. A map $u\colon A \to B$ is a retract of a map $v\colon C \to D$ if u is a retract of v in $\mathcal{E}^I$ —the category of arrows of $\mathcal{E}$— i.e. if there is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "u"'] & C \arrow[r, "r"] \arrow[d, "v"] & A \arrow[d, "u"] \\ B \arrow[r, "j"'] & D \arrow[r, "s"'] & B \end{tikzcd}$$

such that $ri = id_A$ and $sj = id_B$.

A class of maps $\mathcal{M}$ in $\mathcal{E}$ is said to be *closed under retracts* if whenever $v \in \mathcal{M}$ and u is a retract of v then $u \in \mathcal{M}$.

A class of maps $\mathcal{M}$ in $\mathcal{E}$ is said to be *closed under pushouts* if whenever $u\colon A \to B$ is in $\mathcal{M}$ and $f\colon A \to C$ is in $\mathcal{E}$ then in the pushout (when it exists)

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & C \arrow[d, "u'"] \\ B \arrow[r] & B +_A C \end{tikzcd}$$

the map u' is in $\mathcal{M}$. There is a dual concept of a class *closed under pullbacks*.

Definition A.1.1. *If $u\colon A \to B$ and $f\colon X \to Y$ are maps in a category $\mathcal{E}$, we say u has the* left lifting property (LLP) *with respect to f or f has the* right lifting property (RLP) *with respect to u, if any commutative square*

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "u"'] & X \arrow[d, "f"] \\ B \arrow[r, "b"'] & Y \end{tikzcd}$$

has a diagonal filler $d\colon B \to X$, *with* $du = a$ *and* $fd = b$. *We write* $u \pitchfork f$ *for this relation*

If $\mathcal{M}$ is a class of maps in a category $\mathcal{E}$ we denote by ${}^{\pitchfork}\mathcal{M}$ (resp. $\mathcal{M}^{\pitchfork}$) the class of all maps which have the LLP (resp. RLP) with respect to each map in $\mathcal{M}$. If $\mathcal{A}$ and $\mathcal{B}$ are two classes of maps in $\mathcal{E}$, we write $\mathcal{A} \pitchfork \mathcal{B}$ to indicate that $u \pitchfork f$ holds for any $u \in \mathcal{A}$ and $f \in \mathcal{B}$. Clearly

$$\mathcal{A} \subseteq^{\pitchfork} \mathcal{B} \Longleftrightarrow \mathcal{A} \pitchfork \mathcal{B} \Longleftrightarrow \mathcal{B} \subseteq \mathcal{A}^{\pitchfork}$$

Proposition A.1.1. $\mathcal{M}^{\pitchfork}$ *contains the isomorphisms and is closed under composition, retracts, and pullbacks. Dually,* ${}^{\pitchfork}\mathcal{M}$ *contains the isomorphisms and is closed under composition, retracts, and pushouts.*

If $\mathcal{E}$ is a category closed under directed colimits, and $\alpha = \{i < \alpha\}$ is a non-zero limit ordinal, a functor

$$C\colon \alpha \to \mathcal{E}$$

is called an *α-sequence* if the canonical map

$$\varinjlim_{i<j} C_i \to C_j$$

is an isomorphism for every limit ordinal $j < \alpha$. The *composite* of an α-sequence C is the canonical map

$$C_0 \to \varinjlim_{i<\alpha} C_i.$$

A subcategory $\mathcal{A}$ of $\mathcal{E}$ is *closed under transfinite composition* if it contains the composite of any α-sequence $C\colon \alpha \to \mathcal{A} \subseteq \mathcal{E}$.

We call a class of maps $\mathcal{M}$ in a cocomplete category $\mathcal{E}$ *saturated* if it satisfies the following conditions:

- $\mathcal{M}$ contains the isomorphisms and is closed under composition;
- $\mathcal{M}$ is closed under pushouts and retracts;
- $\mathcal{M}$ is closed under transfinite composition.

For example, ${}^{\pitchfork}\mathcal{M}$ is saturated for any class $\mathcal{M}$ of maps in a cocomplete category $\mathcal{E}$.

Any class $\mathcal{M}$ of maps in a cocomplete category $\mathcal{E}$ is contained in a smallest saturated class $\overline{\mathcal{M}} \subseteq \mathcal{E}$ called the saturated class *generated* by $\mathcal{M}$.

Proposition A.1.2. *A saturated class $\mathcal{M}$ is closed under arbitrary coproducts.*

Proof. Let I be a set and $A_i \to B_i$ a map in $\mathcal{M}$ for $i \in I$. We want to show that the map

$$\sum_i A_i \to \sum_i B_i$$

is in $\mathcal{M}$. We may assume I is an ordinal since any set is isomorphic to an ordinal. Then define an I-sequence $C\colon I \to \mathcal{E}$ by $C_0 = \sum_i A_i$. C_{i+1} is the pushout

$$\begin{tikzcd}
A_i \arrow[r] \arrow[d] & C_i \arrow[d] \\
B_i \arrow[r] & C_{i+1}
\end{tikzcd}$$

and $C_j = \varinjlim_{i<j} C_i$ for limit ordinals $j < I$. Then the composite $C_0 \to \varinjlim_{i<\alpha} C_i$ is isomorphic to $\sum_i A_i \to \sum_i B_i$. $\square$

A.2 Weak factorisation systems

Definition A.2.1. *A weak factorisation system in a category $\mathcal{E}$ is a pair $(\mathcal{A}, \mathcal{B})$ of classes of maps in $\mathcal{E}$ satisfying*

- *Every map $f\colon X \to Y$ in $\mathcal{E}$ may be factored as $f = pu$ with $u \in \mathcal{A}$ and $p \in \mathcal{B}$;*
- *$\mathcal{A} = {}^{\pitchfork}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\pitchfork}$*

Notice that each class of a weak factorisation system determines the other. Also, $\mathcal{A} \cap \mathcal{B} = Iso(\mathcal{E})$ since a map f such that $f \pitchfork f$ is an isomorphism.

Proposition A.2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two classes of maps in a category $\mathcal{E}$. Suppose the following conditions are satisfied:*

- *Every map $f\colon X \to Y$ in $\mathcal{E}$ may be factored as $f = pu$ with $u \in \mathcal{A}$ and $p \in \mathcal{B}$;*
- *$\mathcal{A} \pitchfork \mathcal{B}$;*
- *$\mathcal{A}$ and $\mathcal{B}$ are closed under retracts.*

Then $(\mathcal{A}, \mathcal{B})$ is a weak factorisation system in $\mathcal{E}$.

Proof. We use an important technique known as "the retract argument" to show, say, ${}^{\pitchfork}\mathcal{B} \subseteq \mathcal{A}$. So let $u \in{}^{\pitchfork} \mathcal{B}$ and factor u as $u = pv$ with $p \in \mathcal{B}$, and $v \in \mathcal{A}$. Then the square

$$\begin{tikzcd} A \arrow[r, "v"] \arrow[d, "u"'] & C \arrow[d, "p"] \\ B \arrow[r, "id"'] \arrow[ur, dashed, "d"] & B \end{tikzcd}$$

has a diagonal filler d. Thus, u is a retract of $v \in \mathcal{A}$, so $u \in \mathcal{A}$. $\mathcal{A}^{\pitchfork} \subseteq \mathcal{B}$ is similar. $\square$

Recall that if S is a set, the *cardinality* of S is the smallest ordinal, $\mid S \mid$, such that there is a bijection $\mid S \mid \rightarrow S$. A *cardinal* is an ordinal κ such that $\kappa = \mid \kappa \mid$.

Definition A.2.2. *Let κ be a cardinal. A limit ordinal α is said to be κ-filtered if $S \subseteq \alpha$ and $\mid S \mid \leq \kappa$ then $sup(S) < \alpha$.*

Definition A.2.3. *Let $\mathcal{E}$ be cocomplete and $A \in \mathcal{E}$. A is called κ-small if for all κ-filtered ordinals α and all α-sequences $X \colon \alpha \rightarrow \mathcal{E}$, the canonical map*

$$\varinjlim_{i<\alpha} \mathcal{E}(A, X_i) \rightarrow \mathcal{E}(A, \varinjlim_{i<\alpha} X_i)$$

is a bijection. A is called small if it is κ-small for some κ, and finite if it is κ-small for a finite cardinal κ, in which case $\mathcal{E}(A, \ \ )$ commutes with colimits over any limit ordinal.

Examples. (i) Any set S is small. In fact, S is $\kappa = \mid S \mid$-small. For, suppose α is a κ-filtered ordinal and X is an α-sequence of sets. If $f \colon S \rightarrow \varinjlim_{i<\alpha} X_i$ then for each $s \in S$ there is an index $i(s)$ such that $f(s)$ is in the image of $X_{i(s)}$. Let j be the supremum of the $i(s)$. Since α is κ-filtered, $j < \alpha$ and f factors through a map $S \rightarrow X_j$. The fact that if two maps $S \rightarrow X_i$ and $S \rightarrow X_k$ are equal in the colimit then they are equal at some stage has a similar proof.

(ii) Let A be a small category. Then any presheaf $X \in [A^{op}, Set]$ is small. Here, let $\kappa = \mid A_1 \mid \times \mid \bigcup_{a \in A_0} X(a) \mid$, where A_0 is the set of objects of A and A_1 is the set of maps. Now suppose α is a κ-filtered ordinal, Y_i is an α-sequence of presheaves, and $f \colon X \rightarrow \varinjlim_{i<\alpha} Y_i$ is a map of presheaves. Each $X(a)$ is κ-small, so there is $i(a) < \alpha$ such that $f(a)$ factors through $Y_{i(a)}(a)$, and then an $i < \alpha$ such that f factors through a map $f' \colon X \rightarrow Y_i$. This map f' may not be a map of presheaves. That is, if $k\colon a \to b$ is a map in A, the diagram

$$\begin{tikzcd} X(b) \arrow[r, "f'(b)"] \arrow[d, "X(k)"'] & Y_i(b) \arrow[d, "Y_i(k)"] \\ X(a) \arrow[r, "f'(a)"'] & Y_i(a) \end{tikzcd}$$

may not commute. However, for each pair (k, x) where $k\colon a \to b$ is a map in A and $x \in X(b)$ there is a $i(k, x) < \alpha$ such that $f'(a)X(k)(x) = Y(k)f'(b)(x)$ in $Y_{i(k,x)}$. Since α is κ-filtered there is, finally, a $j < \alpha$ such that f factors as a map of presheaves through Y_j.

The technique used in the following result is called the "small object argument", so named by Quillen. It is ubiquitous in homotopy theory.

Theorem A.2.1. *Let $\mathcal{E}$ be cocomplete and $\mathcal{M}$ a set of maps in $\mathcal{E}$ with small domains. Then $(\overline{\mathcal{M}}, \mathcal{M}^{\pitchfork})$ is a weak factorisation system in $\mathcal{E}$. Furthermore, the factorisation of a map f can be chosen to be functorial in f.*

Proof. Choose a cardinal κ such that the domain of each element of $\mathcal{M}$ is κ-small, and let α be a κ-filtered ordinal. Let $f\colon X \to Y$ be a map in $\mathcal{E}$. We will define, by transfinite induction, a functorial α-sequence $E\colon \alpha \to \mathcal{E}$ and a natural transformation $p\colon E \to Y$ factoring f. We begin by setting $E_0 = X$, and $p_0 = f$. If we have defined E_i and p_i for all $i < j$ where j is a limit ordinal, set $E_j = \varinjlim_{i<j} E_i$ and let p_j be the map induced by the p_i. Given E_i and p_i, we define E_{i+1} and p_{i+1} as follows. Let S be the set of all commutative squares of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "u"'] & E_i \arrow[d, "p_i"] \\ B \arrow[r] & Y \end{tikzcd}$$

with u in $\mathcal{M}$. For $s \in S$ denote by $u_s\colon A_s \to B_s$ the corresponding map in $\mathcal{M}$ and define E_{i+1} as a pushout

$$\begin{tikzcd} \sum_{s\in S} A_s \arrow[r] \arrow[d] & E_i \arrow[d] \\ \sum_{s\in S} B_s \arrow[r] & E_{i+1} \end{tikzcd}$$

Set $p_{i+1}\colon E_{i+1} \to Y$ equal to the map induced by the maps $B_s \to Y$ and $p_i\colon E_i \to Y$. Let $\overline{E} = \varinjlim_{i<\alpha} E_i$ and let $\overline{p}\colon \overline{E} \to Y$ be the map induced by the p_i. The canonical map $X \to \overline{E}$ is in $\overline{\mathcal{M}}$, and we want to show $p \in \mathcal{M}^{\pitchfork}$, so let

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "u"'] & \overline{E} \arrow[d, "\overline{p}"] \\ B \arrow[r, "b"'] & Y \end{tikzcd}$$

be commutative with u in $\mathcal{M}$. Since A is κ-small, a factors as $A \to E_i \to \overline{E}$ for some $a_i \colon A \to E_i$. By construction then there is a commutative square

$$\begin{tikzcd} A \arrow[r, "a_i"] \arrow[d, "u"'] & E_i \arrow[d] \\ B \arrow[r, "b_{i+1}"'] & E_{i+1} \end{tikzcd}$$

such that $p_{i+1}b_{i+1} = b$. Then the composite of b_{i+1} and the canonical map $E_{i+1} \to \overline{E}$ gives the required diagonal filler. $\square$

Corollary A.2.1. *Let $\mathcal{E}$ be a presheaf category, and $\mathcal{M}$ a set of maps in $\mathcal{E}$. Then $(\overline{\mathcal{M}}, \mathcal{M}^{\pitchfork})$ is a weak factorisation system in $\mathcal{E}$. Furthermore, the factorisation of a map f can be chosen to be functorial in f.*

A.3 Quillen model structures

Let $\mathcal{E}$ be a finitely complete and cocomplete category.

Definition A.3.1. *A Quillen model structure on $\mathcal{E}$ consists of three classes of maps $(\mathcal{F}, \mathcal{C}, \mathcal{W})$ in $\mathcal{E}$ called fibrations, cofibrations, and weak equivalences respectively. These are required to satisfy*

- *$\mathcal{W}$ has the "three for two" property: if any two of f, g or fg is in $\mathcal{W}$, so is the third.*
- *$(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are weak factorisation systems in $\mathcal{E}$.*

A map in $\mathcal{W}$ is also called *acylic*, A map in $\mathcal{C} \cap \mathcal{W}$ is then called an *acylic cofibration*, and a map in $\mathcal{F} \cap \mathcal{W}$ an *acyclic fibration*.

The usual way of expressing the conditions on $(\mathcal{F}, \mathcal{C}, \mathcal{W})$ is:

- $\mathcal{W}$ satisfies "three for two";
- $\mathcal{F}$, $\mathcal{C}$ and $\mathcal{W}$ are closed under retracts;
- $(\mathcal{C} \cap \mathcal{W}) \pitchfork \mathcal{F}$ and $\mathcal{C} \pitchfork (\mathcal{F} \cap \mathcal{W})$;

- Every map f in $\mathcal{E}$ can be factored as $f = pi$ in two ways: one in which $i \in \mathcal{C} \cap \mathcal{W}$ and $p \in \mathcal{F}$ and one in which $i \in \mathcal{C}$ and $p \in \mathcal{F} \cap \mathcal{W}$.

As we have seen, these are the same except for $\mathcal{W}$ being closed under retracts, so

Proposition A.3.1. *The class $\mathcal{W}$ of a model structure as defined above is closed under retracts.*

Proof. Notice first that the class $\mathcal{F} \cap \mathcal{W}$ is closed under retracts since $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ is a weak factorisation system. Suppose now that $f\colon A \to B$ is a retract of $g\colon X \to Y$ in $\mathcal{W}$. We want to show $f \in \mathcal{W}$. We have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "f"'] & X \arrow[r, "t"] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r, "u"'] & Y \arrow[r, "v"'] & B \end{tikzcd}$$

with $ts = 1_A$ and $vu = 1_B$. We suppose first that f is a fibration. In this case, factor g as $g = qj\colon X \to Z \to Y$ with $j \in \mathcal{C} \cap \mathcal{W}$ and $q \in \mathcal{F}$. Then $q \in \mathcal{F} \cap \mathcal{W}$ by "three for two", since $g \in \mathcal{W}$. The square

$$\begin{tikzcd} X \arrow[r, "t"] \arrow[d, "j"'] & A \arrow[d, "f"] \\ Z \arrow[r, "vq"'] & B \end{tikzcd}$$

has a diagonal filler $d\colon Z \to A$, since f is a fibration. We get a commutative diagram

$$\begin{tikzcd} A \arrow[r, "js"] \arrow[d, "f"'] & Z \arrow[r, "d"] \arrow[d, "q"] & A \arrow[d, "f"] \\ B \arrow[r, "u"'] & Y \arrow[r, "v"'] & B \end{tikzcd}$$

Since $d(js) = ts = 1_A$, f is a retract of q, and $f \in \mathcal{W}$, since $q \in \mathcal{F} \cap \mathcal{W}$. In the general case, factor f as $f = pi\colon A \to E \to B$ with $i \in \mathcal{C} \cap \mathcal{W}$ and $p \in \mathcal{F}$. By taking a pushout we obtain a commutative diagram

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "i"'] & X \arrow[r, "t"] \arrow[d, "i_2"] & A \arrow[d, "i"] \\ E \arrow[r, "i_1"] \arrow[d, "p"'] & E +_A X \arrow[r, "r"] \arrow[d, "k"] & E \arrow[d, "p"] \\ B \arrow[r, "u"'] & Y \arrow[r, "v"'] & B \end{tikzcd}$$

where $ki_2 = g$ and $ri_1 = 1_E$. The map i_2 is a pushout of i, so $i_2 \in \mathcal{C} \cap \mathcal{W}$ since i is. Thus, $k \in \mathcal{W}$ by "three for two", since $g = ki_2 \in \mathcal{W}$ by assumption. So $p \in \mathcal{W}$ by the first part, since $p \in \mathcal{F}$. Finally, $f = pi \in \mathcal{W}$ since $i \in \mathcal{W}$. □

If $(\mathcal{F}, \mathcal{C}, \mathcal{W})$ is a model structure on $\mathcal{E}$ then any two of the classes $\mathcal{F}$, $\mathcal{C}$ or $\mathcal{W}$ determines the other. For example, a map $f \in \mathcal{E}$ belongs to $\mathcal{W}$ iff it can be factored as $f = pi$ with $i \in^{\pitchfork} \mathcal{F}$ and $p \in \mathcal{C}^{\pitchfork}$.

An important aspect of the axioms for a Quillen model structure is that they are self dual. That is, if $\mathcal{E}$ carries a Quillen model structure then so does $\mathcal{E}^{op}$. The cofibrations of $\mathcal{E}^{op}$ are the fibrations of $\mathcal{E}$, and the fibrations of $\mathcal{E}^{op}$ are the cofibrations of $\mathcal{E}$. The weak equivalences of $\mathcal{E}^{op}$ are the weak equivalences of $\mathcal{E}$. Thus, we need only prove results about cofibrations, say, since the dual results are automatic.

A model structure is said to be *left proper* if the pushout of aweak equivalence along a cofibration is a weak equivalence. Dually, a model structure is said to be *right proper* if the pullback of a weak equivalence along a fibration is a weak equivalence. A model structure is *proper* if it is both left and right proper.

If $\mathcal{E}$ is a Quillen model category, we say an object$X \in \mathcal{E}$ is *cofibrant* if $0 \to X$ is a cofibration, and *fibrant* if $X \to 1$ is a fibration.

A.4 Examples

(1) There are three trivial examples. Namely, for any $\mathcal{E}$, take $\mathcal{F}$, $\mathcal{C}$ or $\mathcal{W}$ to be the isomorphisms and let the other two classes be all maps.

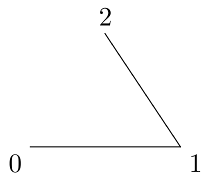
(2) Let *Gpd* denote the category of small groupoids. A functor $w\colon \boldsymbol{G} \to \boldsymbol{H}$ will be said to be a *weak equivalence* if it is a categorical equivalence, i.e. full, faithful and essentially surjective, meaning that for each object h of $\boldsymbol{H}$ there is an object g of $\boldsymbol{G}$ and an arrow $h \to w(g)$. Equivalently, w is a weak equivalence if it has a *quasi-inverse*, i.e. a functor $w'\colon \boldsymbol{H} \to \boldsymbol{G}$ together with isomorphisms $ww' \to id_{\boldsymbol{H}}$ and $id_{\boldsymbol{G}} \to w'w$. We call a functor $i\colon \boldsymbol{A} \to \boldsymbol{B}$ a *cofibration* if it is injective on objects. A *fibration* will be a Grothendieck fibration, which for groupoids just means a functor $p\colon \boldsymbol{E} \to \boldsymbol{C}$ such that if e is an object of $\boldsymbol{E}$ and $\gamma\colon c \to p(e)$ is an arrow of $\boldsymbol{C}$ then there is an arrow $\epsilon\colon e' \to e$ of $\boldsymbol{E}$ such that $p(\epsilon) = \gamma$.

We will show these fibrations, cofibrations and weak equivalences define a proper Quillen model structure on Gpd.

(3) Take $\mathcal{E} = \mathbf{S}$ the category of simplicial sets. The k^{th} *horn* of $\Delta[n]$ is

$$\Lambda^k[n] = \bigcup_{i \neq k} \partial^i \Delta[n].$$

The geometric realization of $\Lambda^k[n]$ is the union of all those (n-1)-dimensional faces of Δ_n that contain the k^{th} vertex of Δ_n. For example,



is the geometric realization of $\Lambda^1[2]$.

$\Delta[n]_k$ contains $\binom{n}{k}$ non-degenerate simplices, corresponding to the order preserving injections $[k] \to [n]$. As a result, $\Delta[n]$, $\partial\Delta[n]$ and $\Lambda^k[n]$ are finite, and any simplicial set is small. We write $\mathcal{S}$ for the set of sphere inclusions $\{\partial\Delta[n] \to \Delta[n] \mid n \geq 0\}$, and $\mathcal{H}$ for the set of horn inclusions $\{\Lambda^k[n] \to \Delta[n] \mid 0 \leq k \leq n, n \geq 1\}$. Using an argument like Proposition 1.2.4 (or see Appendix D), if $m\colon A \to X$ is an arbitrary monomorphism of $\mathbf{S}$, then

$$\begin{tikzcd}
\sum_{e(X-A)_n} \partial\Delta[n] \arrow[r] \arrow[d] & \sum_{e(X-A)_n} \Delta[n] \arrow[d] \\
Sk^{n-1}(X) \cup A \arrow[r] & Sk^n(X) \cup A
\end{tikzcd}$$

is a pushout for $n \geq 0$, where $e(X-A)_n$ is the set of non-degenerate n-simplices of X which are not in A. Furthermore,

$$X = \varinjlim_{n \geq -1} (Sk^n(X) \cup A) \quad \text{and} \quad \mu_{-1}\colon Sk^{-1}(X) \cup A \longrightarrow \varinjlim_{n \geq -1} (Sk^n(X) \cup A)$$

is $m\colon A \to X$. Thus the saturated class $\overline{\mathcal{S}}$ is the class of all monomorphisms of simplicial sets. Furthermore, $(\overline{\mathcal{S}}, \mathcal{S}^{\pitchfork})$ is a weak factorisation system in $\mathbf{S}$. A member of $\mathcal{S}^{\pitchfork}$ is called a *trivial fibration*. $\overline{\mathcal{H}}$ is called the class of *anodyne extensions* and $(\overline{\mathcal{H}}, \mathcal{H}^{\pitchfork})$ is another weak factorisation system in $\mathbf{S}$. The members of $\mathcal{H}^{\pitchfork}$ are called *Kan fibrations*. Notice that the first weak factorisation system is easy to obtain in any topos, since a map which has the RLP with respect to the monomorphisms is just an injective over its base, and any map can be embedded in an injective. It is, however, also the case that the monomorphisms in any Grothendieck topos have a set of generators, so this weak factorisation system can also be obtained in any Grothendieck topos as above.

Now, in $\mathbf{S}$, for the cofibrations $\mathcal{C}$ take the monomorphisms, and for $\mathcal{F}$ take the Kan fibrations. $\mathcal{W}$ is the class of "geometric homotopy equivalences", which are maps $f\colon X \to Y$ such that $|f|\colon |X| \to |Y|$ is a homotopy equivalence, i.e. there is a map $f'\colon |Y| \to |X|$ such that $f'|f|$ is homotopic to $id_{|X|}$ and $|f|f'$ is homotopic to $id_{|Y|}$. Our principal, and most central, example of a proper Quillen model structure will be $(\mathcal{F}, \mathcal{C}, \mathcal{W})$ on $\mathbf{S}$. But it is difficult to establish. The "three for two" property for $\mathcal{W}$ is easy, and we have the two weak factorisation systems $(\overline{\mathcal{H}}, \mathcal{H}^{\pitchfork})$ and $(\overline{\mathcal{S}}, \mathcal{S}^{\pitchfork})$. It is easy to see that $\overline{\mathcal{H}} \subseteq \mathcal{C} \cap \mathcal{W}$ and $\mathcal{S}^{\pitchfork} \subseteq \mathcal{F} \cap \mathcal{W}$ *But*, we do *not* know that $\mathcal{C} \cap \mathcal{W} \subseteq \overline{\mathcal{H}}$, and we do *not* know that $\mathcal{F} \cap \mathcal{W} \subseteq \mathcal{S}^{\pitchfork}$. If we knew, say, the latter, then the former is easy. So that leaves only one thing to show. But this will take a full chapter.

Let us formalize this situation by proving a general proposition to which we can refer later.

Proposition A.4.1. *Let $\mathcal{E}$ be a finitely complete and cocomplete category provided with a class $\mathcal{W}$ of maps which has the "three for two" property. Suppose $(\mathcal{C}_{\mathcal{W}}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F}_{\mathcal{W}})$ are two weak factorisation systems on $\mathcal{E}$ satisfying*

- $\mathcal{C}_{\mathcal{W}} \subseteq \mathcal{C} \cap \mathcal{W}$ *and* $\mathcal{F}_{\mathcal{W}} \subseteq \mathcal{F} \cap \mathcal{W}$
- *Either* $\mathcal{C} \cap \mathcal{W} \subseteq \mathcal{C}_{\mathcal{W}}$ *or* $\mathcal{F} \cap \mathcal{W} \subseteq \mathcal{F}_{\mathcal{W}}$.

Then $(\mathcal{F}, \mathcal{C}, \mathcal{W})$ is a model structure on $\mathcal{E}$.

Proof. Consider the case $\mathcal{C}_{\mathcal{W}} \subseteq \mathcal{C} \cap \mathcal{W}$, $\mathcal{F}_{\mathcal{W}} \subseteq \mathcal{F} \cap \mathcal{W}$ and $\mathcal{F} \cap \mathcal{W} \subseteq \mathcal{F}_{\mathcal{W}}$. We want to show $\mathcal{C} \cap \mathcal{W} \subseteq \mathcal{C}_{\mathcal{W}}$, so let $i \colon A \to B$ be in $\mathcal{C} \cap \mathcal{W}$. Factor i as $i = pj \colon A \to E \to B$, with $j \in \mathcal{C}_{\mathcal{W}}$ and $p \in \mathcal{F}$. Since $i \in \mathcal{W}$ and $j \in \mathcal{C}_{\mathcal{W}} \subseteq \mathcal{C} \cap \mathcal{W}$ it follows that $p \in \mathcal{F} \cap \mathcal{W}$. Since $\mathcal{F} \cap \mathcal{W} \subseteq \mathcal{F}_{\mathcal{W}}$ and $i \in \mathcal{C}$, there is a diagonal filler d in the diagram

$$
\begin{tikzcd}
A \arrow[r, "j"] \arrow[d, "i"'] & E \arrow[d, "p"] \\
B \arrow[r, "1_B"'] \arrow[ur, dashed, "d"] & B
\end{tikzcd}
$$

It follows that i is a retract of j, so $i \in \mathcal{C}_{\mathcal{W}}$. Thus $\mathcal{C}_{\mathcal{W}} = \mathcal{C} \cap \mathcal{W}$, $\mathcal{F}_{\mathcal{W}} = \mathcal{F} \cap \mathcal{W}$, and $(\mathcal{F}, \mathcal{C}, \mathcal{W})$ is a model structure on $\mathcal{E}$. $\square$

(4) Take $\mathcal{E} = Top_c$, the category of compactly generated spaces. The class $\mathcal{W}$ of weak equivalences consists of maps $f \colon X \to Y$ such that $\pi_0(f) \colon \pi_0(X) \to \pi_0(Y)$ is a bijection, and for $n \geq 1$ and $x \in X$, $\pi_n(f) \colon \pi_n(X, x) \to \pi_n(Y, fx)$ is an isomorphism. The class $\mathcal{F}$ is the class of Serre fibrations, i.e. maps $p \colon E \to X$ with the *covering homotopy property* (CHP) for each n-simplex Δ_n, $n \geq 0$. This means that if $h \colon \Delta_n \times I \to X$ is a homotopy ($I = [0, 1]$), and $f \colon \Delta_n \to E$ is such that $pf = h_0$, then there is a "covering homotopy" $\bar{h} \colon \Delta_n \times I \to E$ such that $\bar{h}_0 = f$, and $p\bar{h} = h$. $\mathcal{C} = {}^{\pitchfork}(\mathcal{F} \cap \mathcal{W})$. Then $(\mathcal{F}, \mathcal{C}, \mathcal{W})$ is a proper Quillen model structure on Top_c.

Once example (3) is established, this one is not too hard.

A.5 The homotopy category

Let $\mathcal{E}$ be a finitely complete and cocomplete category equiped with a model srtucture $(\mathcal{F}, \mathcal{C}, \mathcal{W})$. The *homotopy category* of $\mathcal{E}$ is

$$Ho(\mathcal{E}) = \mathcal{E}[\mathcal{W}^{-1}].$$

The trouble with this is that, apriori, $Ho(\mathcal{E})$ may not be locally small, i.e. the maps between two objects may not form a set. In fact, in the presence of a model structure, this problem does not arise. In this section we sketch how this works, following closely the original presentation by Quillen [11], and the subsequent treatment by Hovey [10].

Definition A.5.1. *Let $f, g\colon X \to Y$ be two maps of $\mathcal{E}$.*

- *A* cylinder *for X is a factorisation of the codiagonal $\nabla\colon X + X \to X$ into a cofibration $(i_0, i_1)\colon X+X \to IX$ followed by a weak equivalence $s\colon IX \to X$.*

  A path space *for Y is a factorisation of the diagonal $\Delta\colon Y \to Y \times Y$ as a weak equivalence $t\colon Y \to PY$ followed by a fibration $(p_0, p_1)\colon PY \to Y \times Y$.*

- *A* left homotopy *from f to g is a map $H\colon IX \to Y$ for some cylinder for X such that $Hi_0 = f$ and $Hi_1 = g$. We say f and g are left homotopic if there is a left homotopy from f to g, and we write $f \overset{l}{\sim} g$ for this relation.*

  A right homotopy *from f to g is a map $K\colon X \to PY$ for some path space for Y such that $p_0K = f$ and $p_1K = g$. We say f and g are right homotopic if there is a right homotopy from f to g, and we write $f \overset{r}{\sim} g$ for this relation.*

- *We say f and g are* homotopic *if they are both left homotopic and right homotopic, and we write $f \sim g$ for this relation.*

Proposition A.5.1. *Let $f, g\colon A \to X$ be two maps in a model category $\mathcal{E}$.*

(i) If A is cofibrant, left homotopy is an equivalence relation on $\mathcal{E}(A, X)$. Dually, if X is fibrant, right homotopy is an equivalence relation on $\mathcal{E}(A, X)$.

(ii) If A is cofibrant, $f \overset{l}{\sim} g$ implies $f \overset{r}{\sim} g$. Dually, if X is fibrant, $f \overset{r}{\sim} g$ implies $f \overset{l}{\sim} g$.

Proof. (i) If $f\colon A \to X$ and $(i_0, i_1)\colon A + A \to IA \xrightarrow{s} A$ is a cylinder for A, then fs is a left homotopy from f to f so left homotopy is always reflexive. Similarly, left homotopy is always symmetric: if $IA \to X$ is a left homotopy from f to g, simply interchange i_0 and i_1 to obtain a left homotopy from g to f.

Notice now that if A is cofibrant, and $(i_0, i_1)\colon A+A \to IA \xrightarrow{s} A$ is a cylinder for A, then $i_0, i_1\colon A \to IA$ are acyclic cofibrations. In fact,

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & A \arrow[d, "in_1"] \\ A \arrow[r, "in_0"'] & A+A \end{tikzcd}$$

is a pushout, so in_0, in_1 are cofibrations making i_0, i_1 cofibrations. $si_0 = si_1 = id_A$ so they are acyclic by "three for two". Now let $H\colon IA \to X$ be a left homotopy from f to g, $(i'_0, i'_1)\colon A+A \to I'A \xrightarrow{s'} A$ another cylinder for A, and $H'\colon I'A \to X$ a left homotopy from g to h. Let

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[d, "i'_0"'] & IA \arrow[d] \\ I'A \arrow[r] & P \end{tikzcd}$$

be a pushout. Let $k_0 = A \xrightarrow{i_0} IA \to P$ and $k_1 = A \xrightarrow{i'_1} I'A \to P$. Then s and s' induce a weak equivalence $t : P \to A$, and $(k_0, k_1) : A + A \to P \xrightarrow{t} A$ is a factorisation of the codiagonal. Furthermore, there is a map $K : P \to X$, induced by H and H', such that $Kk_0 = f$, and $Kk_1 = h$. If the diagram

$$\begin{tikzcd} A+A+A+A \arrow[r, "1+\nabla+1"] \arrow[d, "(i_0,i_1)+(i'_0,i'_1)"'] & A+A+A \arrow[d] \\ IA+I'A \arrow[r] & P \end{tikzcd}$$

is a pushout, the composite $A+0+A \to A+A+A \to P$ is $(k_0, k_1) : A+A \to P$, which is thus a cofibration. (We are grateful to Amnon Neeman for this remark.) It follows that $K : P \to X$ is a left homotopy from f to h.

(ii) Suppose A is cofibrant, and $H\colon IA \to X$ is a left homotopy from f to g. Let $X \xrightarrow{w} PX \xrightarrow{(p_0,p_1)} X \times X$ be *any* path space for X. In the diagram

$$\begin{tikzcd} A \arrow[r, "wf"] \arrow[d, "i_0"'] & PX \arrow[d, "(p_0,p_1)"] \\ IA \arrow[r, "(fs,H)"'] & X \times X \end{tikzcd}$$

there is a diagonal filler $K\colon IA \to PX$ since $i_0\colon A \to IA$ is an acyclic cofibration. The map Ki_1 is a right homotopy from f to g. $\square$

Corollary A.5.1. *Let $\mathcal{E}$ be a model category with $A \in \mathcal{E}$ cofibrant and $X \in \mathcal{E}$ fibrant. Then the relations $f \overset{l}{\sim} g$ and $f \overset{r}{\sim} g$ on $\mathcal{E}(A, X)$ coincide, and are equivalence relations. Futhermore, if $f \sim g$ then there is a left homotopy from f to g using any cylinder for A, and a right homotopy from f to g using any path space for X.*

Proposition A.5.2. *Let $f, g \colon A \to X$ be two maps in a model category $\mathcal{E}$.*

(i) Suppose $f \overset{l}{\sim} g$ and $h \colon X \to Y$, then $hf \overset{l}{\sim} hg$. Dually, if $f \overset{r}{\sim} g$ and $h \colon B \to A$, then $fh \overset{r}{\sim} gh$.

(ii) If X is fibrant, $f \overset{l}{\sim} g$ and $h \colon B \to A$, then $fh \overset{l}{\sim} gh$. Dually, if A is cofibrant, $f \overset{r}{\sim} g$ and $h \colon X \to Y$ then $hf \overset{r}{\sim} hg$.

Proof. (i) is easy and left to the reader.
(ii) Suppose $f \overset{l}{\sim} g$ with X fibrant. Let $A + A \xrightarrow{i} IA \xrightarrow{s} A$ be a cylinder for A, and $H \colon IA \to X$ a left homotopy from f to g. Factor s as a trivial cofibration followed by a trivial fibration $s = IA \xrightarrow{k} I'A \xrightarrow{s'} A$. Then $A+A \xrightarrow{ki} I'A \xrightarrow{s'} A$ is also a cylinder for A, and the left homotopy $H \colon IA \to X$ can be extended to a left homotopy $H' \colon I'A \to X$ since X is fibrant. Thus, we may always assume s is a trivial fibration when X is fibrant.

Now suppose $h \colon B \to A$ and let $B + B \xrightarrow{j} IB \xrightarrow{t} B$ be a cylinder for B. The diagram

$$\begin{tikzcd}
B + B \arrow[r, "i(h+h)"] \arrow[d, "j"'] & IA \arrow[d, "s"] \\
IB \arrow[r, "ht"'] & A
\end{tikzcd}$$

has a diagonal filler $k \colon IB \to IA$, and Hk is a left homotopy from fh to gh. $\square$

Let $\mathcal{E}$ be a model category. We denote by $\mathcal{E}_c$ (respectively $\mathcal{E}_f$ and $\mathcal{E}_{cf}$) the full subcategory of cofibrant (respectively fibrant and cofibrant-fibrant) objects of $\mathcal{E}$.

Corollary A.5.2. *The homotopy relation on the maps of $\mathcal{E}_{cf}$ is an equivalence relation which is compatible with composition.*

By the above corollary, we may define a category $\pi\mathcal{E}_{cf}$ as follows: the objects of $\pi\mathcal{E}_{cf}$ are the objects of $\mathcal{E}_{cf}$; if $X, Y \in \mathcal{E}_{cf}$ then $\pi\mathcal{E}_{cf}(X, Y) = \mathcal{E}_{cf}(X, Y)/\sim$, with composition induced by the composition of $\mathcal{E}_{cf}$.

Definition A.5.2. *Let $\mathcal{E}$ be a model category, and $f \colon A \to B$ a map between cofibrant objects of $\mathcal{E}$. A* mapping cylinder *for f is a factorisation of $(f, 1_B) = A + A \xrightarrow{(i_A, i_B)} If \xrightarrow{p_B} B$ as a cofibration followed by a weak equivalence. Then i_A is a cofibration, and i_B is an acyclic cofibration. The factorisation $f = p_B i_A$ is* *called a* mapping cylinder factorisation *of f. Dually, let $f\colon X \to Y$ be a map between fibrant objects of $\mathcal{E}$. A* mapping path space *for f is a factorisation of $(1_X, f) = X \xrightarrow{i_X} Pf \xrightarrow{(p_X, p_Y)} X \times Y$ as a weak equivalence followed by a fibration. Then p_X is an acyclic fibration, and p_Y is a fibration. The factorisation $f = p_Y i_X$ is called a* mapping path space *factorisation of f.*

When considering functors defined on model categories, the following result is often useful.

Lemma A.5.1 (Ken Brown's Lemma)**.** *Let $\mathcal{E}$ be a model category. Suppose $F\colon \mathcal{E} \to \mathcal{D}$ is a functor from $\mathcal{E}$ to a category $\mathcal{D}$ equipped with a concept of weak equivalence satisfying "three for two". Then if F takes acyclic cofibrations between cofibrant objects to weak equivalences, it takes all weak equivalences between cofibrant objects to weak equivalences. Dually, if F takes acyclic fibrations between fibrant objects to weak equivalences, it takes all weak equivalences between fibrant objects to weak equivalences.*

Proof. Let $f\colon A \to B$ be a weak equivalence with A and B cofibrant. Let $f = p_B i_A$ be a mapping cylinder factorisation of f with $p_B i_B = id_B$. Then, by "three for two", both i_A and i_B are acyclic cofibrations. Thus, $F(i_A)$ and $F(i_B)$ are weak equivalences, and $F(p_B)$ is a weak equivalence by "three for two" in $\mathcal{D}$. It follows that $F(f) = F(p_B)F(i_A)$ is a weak equivalence. □

Proposition A.5.3. *Let $\mathcal{E}$ be a model category, and $A \in \mathcal{E}$ a cofibrant object. If $f\colon X \to Y$ is either an acyclic fibration, or a weak equivalence between fibrant objects. then the map*

$$\mathcal{E}(A, X)/ \overset{l}{\sim} \to \mathcal{E}(A, Y)/ \overset{l}{\sim}$$

induced by f is a bijection. Dually, if X is fibrant and $f\colon A \to B$ is either an acyclic cofibration or a weak equivalence between cofibrant objects, then the map

$$\mathcal{E}(B, X)/ \overset{r}{\sim} \to \mathcal{E}(A, X/ \overset{r}{\sim}$$

induced by f is a bijection.

Proof. Let $f\colon X \to Y$ an acyclic fibration, and consider the map

$$\mathcal{E}(A, X)/ \overset{l}{\sim} \to \mathcal{E}(A, Y)/ \overset{l}{\sim}$$

induced by f. Since A is cofibrant, any map $A \to Y$ has a lifting to X, so the map is surjective even before passing to left homotopy. On the other hand, let $(i_0, i_1)\colon A + A \to IA \xrightarrow{s} A$ be a cylinder for A, and $H\colon IA \to Y$ a left homotopy between $fg, fh\colon A \to Y$. Then the diagram

$$\begin{tikzcd}
A + A \arrow[r, "{(g,h)}"] \arrow[d, "{(i_0,i_1)}"'] & X \arrow[d, "f"] \\
IA \arrow[r, "H"'] & Y
\end{tikzcd}$$

has a diagonal filler $K\colon IA \to X$, which is a left homotopy between $g, h\colon A \to X$. Thus the map induced by f is also injective. The case when $f\colon X \to Y$ is a weak equivalence between fibrant objects follows from the first part and Ken Brown's Lemma, using the functor $\mathcal{E}(A, \quad)/ \overset{l}{\sim}$ and bijections of sets as "weak equivalences". ☐

A map $f\colon X \to Y$ in a model category $\mathcal{E}$ is called a *homotopy equivalence* if there is a map $f'\colon Y \to X$ such that $ff' \sim 1_Y$ and $f'f \sim 1_X$. Before beginning the next proposition, notice that the canonical functor $\mathcal{E}_{cf} \to \pi\mathcal{E}_{cf}$ inverts a map f of $\mathcal{E}_{cf}$ iff f is a homotopy equivalence. As a result, the homotopy equivalences of $\mathcal{E}_{cf}$ satisfy "three for two".

Proposition A.5.4. *A map in $\mathcal{E}_{cf}$ is a weak equivalence iff it is a homotopy equivalence.*

Proof. Let X, Y be in $\mathcal{E}_{cf}$ and suppose $f\colon X \to Y$ is a weak equivalence. Let $A \in \mathcal{E}_{cf}$. Then by the above proposition, the map

$$\mathcal{E}_{cf}(A, X)/\sim \to \mathcal{E}_{cf}(A, Y)/\sim$$

induced by f is a bijection. Taking $A = Y$ we find a map $f'\colon Y \to X$ such that $ff' \sim 1_Y$. Since $ff'f \sim f$ it follows that $f'f \sim 1_X$.

For the other direction, suppose first that X, Y are in $\mathcal{E}_{cf}$, and $p\colon X \to Y$ is a fibration which is a homotopy equivalence. We will show p ia a weak equivalence. So let $p'\colon Y \to X$ be a homotopy inverse for p, and $H\colon IY \to Y$ a left homotopy $pp' \sim 1_Y$. Since p is a fibration, there is a diagonal filler $H'\colon IY \to X$ in the diagram

$$\begin{tikzcd} Y \arrow[r, "p'"] \arrow[d, "i_0"'] & X \arrow[d, "p"] \\ IY \arrow[r, "H"'] & Y \end{tikzcd}$$

Put $q = H'i_1$. Then $pq = 1_Y$ and $H'\colon p' \sim q$. We have $1_X \sim p'p \sim qp$, so let $K\colon IX \to X$ be a left homotopy such that $Ki_0 = 1_X$ and $Ki_1 = qp$. By "three for two", K is a weak equivalence, making qp is a weak equivalence. From the diagram

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "p"'] & X \arrow[r, "1_X"] \arrow[d, "qp"] & X \arrow[d, "p"] \\ Y \arrow[r, "q"'] & X \arrow[r, "p"'] & Y \end{tikzcd}$$

we see that p is a retract of pq so p is also a weak equivalence.

Now, for the general case, let $f\colon X \to Y$ be an arbitrary homotopy equivalence in $\mathcal{E}_{cf}$. Factor f as $f = pi\colon X \to Z \to Y$ with i an acyclic cofibration and

p a fibration. Then $Z \in \mathcal{E}_{cf}$, so, by the first part of the theorem, i is a homotopy equivalence. It follows, by "three for two" for the homotopy equivalences of $\mathcal{E}_{cf}$, that p is a homotopy equivalence. But then p is a weak equivalence by the above, so f is a weak equivalence. $\square$

The fact that a weak equivalence $f\colon X \to Y$ of $\mathcal{E}_{cf}$ is a homotopy equivalence is often called Whiteheads's Theorem. The name comes from the classical case where X and Y are CW-complexes, which are cofibrant and fibrant in Top.

Lemma A.5.2. *Let $\mathcal{E}$ be a model category. If $F\colon \mathcal{E} \to \mathcal{D}$ is a functor that inverts weak equivalences, then F identifies left or right homotopic maps.*

Proof. Let $f, g\colon X \to Y$ be maps in $\mathcal{E}$, $(i_0, i_1)\colon X + X \to IX \xrightarrow{s} X$ a cylinder for X, and $H\colon IX \to Y$ a left homotopy between f and g. Since s is a weak equivalence, we have $Fi_0 = Fi_1$. Since $f = Hi_0$ and $g = Hi_1$ it follows that $Ff = FHFi_0 = FHFi_1 = Fg$. Right homotopies are similar. $\square$

Let $\mathcal{E}$ be a model category, and $\mathcal{W}$ its class of weak equivalences. We put $\mathcal{W}_c = \mathcal{E}_c \cap \mathcal{W}$, $\mathcal{W}_f = \mathcal{E}_f \cap \mathcal{W}$, and $\mathcal{W}_{cf} = \mathcal{E}_{cf} \cap \mathcal{W}$. Then

$$Ho(\mathcal{E}_c) = \mathcal{E}_{\rfloor}[\mathcal{W}_c^{-1}] \quad Ho(\mathcal{E}_f) = \mathcal{E}_f[\mathcal{W}_f^{-1}] \quad Ho(\mathcal{E}_{cf}) = \mathcal{E}_{cf}[\mathcal{W}_{cf}^{-1}]$$

Denote by $\delta\colon \mathcal{E}_{cf} \to \pi\mathcal{E}_{cf}$ and $\gamma\colon \mathcal{E}_{cf} \to Ho(\mathcal{E}_{cf})$ the canonical functors. Then we have

Proposition A.5.5. *There is a unique isomorphism of categories*

$$\Phi\colon \pi\mathcal{E}_{cf} \to Ho(\mathcal{E}_{cf})$$

which is the identity on objects, and which satisfies $\Phi\delta = \gamma$.

Proof. We show δ has the the universal property of γ. From Proposition A.5.4 we know δ inverts weak equivalences. Now suppose $F\colon \mathcal{E} \to \mathcal{D}$ is another functor that inverts weak equivalences. As in the above lemma, F then identifies homotopic maps, so there is a unique functor $\overline{F}\colon \pi\mathcal{E}_{cf} \to \mathcal{D}$ such that $\overline{F}\delta = F$. In fact, $\overline{F}X = FX$ on objects, and $\overline{F}([f]) = Ff$ on maps, where $\delta f = [f]$ denotes the homotopy class of f. The result follows. $\square$

In particular, $Ho(\mathcal{E}_{cf})$ is locally small.

Proposition A.5.6. *The diagram of inclusions*

$$\begin{tikzcd}
\mathcal{E}_{cf} \arrow[r] \arrow[d] & \mathcal{E}_f \arrow[d] \\
\mathcal{E}_c \arrow[r] & \mathcal{E}
\end{tikzcd}$$

induces a diagram of equivalences of categories

$$\begin{array}{ccc} Ho(\mathcal{E}_{cf}) & \longrightarrow & Ho(\mathcal{E}_f) \\ \downarrow & & \downarrow \\ Ho(\mathcal{E}_c) & \longrightarrow & Ho(\mathcal{E}) \end{array}$$

Proof. We treat the inclusion $I\colon \mathcal{E}_{cf} \to \mathcal{E}_f$ and leave the others to the reader.

If $X \in \mathcal{E}$ choose an acyclic fibration $LX \to X$ with LX cofibrant, and an acyclic cofibration $X \to RX$ with RX fibrant, choosing $LX = X$ when X is cofibrant and $RX = X$ when X is fibrant. LX is called a *cofibrant replacement* for X, and RX a *fibrant replacement* for X

If X is fibrant, LX is cofibrant-fibrant. If $f\colon X \to Y$ is a map of fibrant objects, there is a map $Lf\colon LX \to LY$ in $\mathcal{E}_{cf}$ making

$$\begin{array}{ccc} LX & \longrightarrow & X \\ {\scriptstyle Lf}\downarrow & & \downarrow{\scriptstyle f} \\ LY & \longrightarrow & Y \end{array}$$

commute. Notice that if f is a weak equivalence, so is Lf. Also, by Proposition A.5.3, Lf is well-defined up to homotopy. It follows that L induces a *functor* $HoL\colon Ho(\mathcal{E}_f) \to Ho(\mathcal{E}_{cf})$. We have $HoLHoI = id$, and the acyclic fibration $LX \to X$ yields an isomorphism $HoIHoL \to id$. □

By the above proposition, if $X, Y \in \mathcal{E}$ we have isomorphisms

$$\mathcal{E}(LRX, LRY)/\sim \quad \simeq \quad Ho(\mathcal{E})(X,Y) \quad \simeq \quad \mathcal{E}(RLX, RLY)/\sim$$

Proposition A.5.3 shows that composing with the acyclic fibration $LRY \to RY$ and the weak equivalence $LX \to LRX$ yields isomorphisms $\mathcal{E}(LRX, LRY)/\sim \; \simeq \; \mathcal{E}(LRX, RY)/\sim \; \simeq \; \mathcal{E}(LX, RY)/\sim$. So we also have an isomorphism

$$Ho(\mathcal{E})(X,Y) \simeq \mathcal{E}(LX, RY)/\sim$$

In particular, $Ho(\mathcal{E})$ is locally small.

The following result is often used in the sequel.

Proposition A.5.7. *Let $\mathcal{E}$ be a model category, and $\gamma\colon \mathcal{E} \to Ho(\mathcal{E})$ the canonical functor. If $f\colon X \to Y$ is a map of $\mathcal{E}$ such that γf is an isomorphism, then f is a weak equivalence.*

Proof. If γf is an isomorphism, then $LRf\colon LRX \to LRY$ is an isomorphism, and LRf is a homotopy equivalence. But then LRf is a weak equivalence, so f is a weak equivalence by "three for two". □

A.6 Quillen functors and equivalences

Definition A.6.1. *We call a cocontinuous functor $F\colon \mathcal{U} \to \mathcal{V}$ between two model categories a* left Quillen functor *if F preserves cofibrations and acyclic cofibrations. Dually, a continuous functor $G\colon \mathcal{V} \to \mathcal{U}$ is called a* right Quillen functor *if G preserves fibrations and acyclic fibrations.*

Notice that by Ken Brown's lemma, a left Quillen functor preserves weak equivalences between cofibrant objects, and a right Quillen functor preserves weak equivalences between fibrant objects.

Proposition A.6.1. *Let $F\colon \mathcal{U} \leftrightarrow \mathcal{V}\colon G$ be an adjoint pair of functors between two model categories. Then F is a left Quillen functor iff G is a right Quillen functor.*

Definition A.6.2. *We call an adjoint pair $F\colon \mathcal{U} \leftrightarrow \mathcal{V}\colon G$ between two model categories a* Quillen pair *if the above equivalent conditions are satisfied.*

Example. Let X be an object of a model category $\mathcal{E}$. Then the the category $\mathcal{E}/X$ inherits a model structure from $\mathcal{E}$ in which a map is a fibration, cofibration or weak equivalence iff it is so in $\mathcal{E}$. A map $f\colon X \to Y$ induces an adjoint pair

$$f_!\colon \mathcal{E}/X \leftrightarrow \mathcal{E}/Y : f^*$$

where $f_!$ is given by composing with f, and f^* is pullback along f. With the model structures described above, this is a Quillen pair.

Proposition A.6.2. *In a model category $\mathcal{E}$, the pullback along a fibration of a weak equivalence between fibrant objects is a weak equivalence. Dually, the pushout along a cofibration of a weak equivalence between cofibrant objects is a weak equivalence.*

Proof. Let $w\colon X \to Y$ be a weak equivalence in $\mathcal{E}$ with $X, Y \in \mathcal{E}_f$. We want to show that in the pullback

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "w'"'] & X \arrow[d, "w"] \\ Y' \arrow[r, "f"'] & Y \end{tikzcd}$$

the map w' is a weak equivalence if f is a fibration. First, consider a mapping path space factorisation of $(1_X, w) = X \xrightarrow{i_X} Pf \xrightarrow{(p_X, p_Y)} X \times Y$ as a weak equivalence followed by a fibration. $p_X i_X = 1_X$ and $w = p_Y i_X$. Since $X, Y \in \mathcal{E}_f$, p_X and p_Y are acyclic fibrations. The pullback of an acyclic fibration is a weak equivalence, so we may suppose $w\colon X \to Y$ is a weak equivalence that has a retraction $r\colon Y \to X$ such that $rw = 1_X$ and r is an acyclic fibration.

Consider the diagram

$$\begin{tikzcd}
X' \arrow[r, "w'"] \arrow[d, "w'"'] & Y' \arrow[d, "u"'] \arrow[dr, "1_{Y'}"] & \\
Y' \arrow[r, "v"] \arrow[d, "rf"'] & Y \times_X Y' \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y' \arrow[d, "rf"] \\
X \arrow[r, "w"'] & Y \arrow[r, "r"'] & X
\end{tikzcd}$$

where $p_1u = f$, $p_2u = 1_{Y'}$, $p_1v = wrf$ and $p_2v = 1_{Y'}$. Now $p_1uw' = fw'$ and $p_1vw' = wrfw' = wrwf' = wf'$. Also, $p_2uw' = w'$ and $p_2vw' = w'$ so the top left-hand square commutes. The lower left-hand square is a pullback, since the lower right-hand square is and the two together are. Similarly, the top left-hand square is a pullback, since the lower left-hand square is and the two together are. It follows that $w' = w^*(u)$, where $w^* \colon \mathcal{E}/Y \to \mathcal{E}/X$ is the pullback functor. But r is an acyclic fibration, hence so is p_2. Since $p_2u = 1_{Y'}$, u is a weak equivalence in $\mathcal{E}/Y$ between fibrant objects. Since w^* is a right Quillen functor, w' is a weak equivalence by Ken Brown's Lemma. □

Corollary A.6.1. *If every object of a model category $\mathcal{E}$ is fibrant, the model structure is right proper. If every object is cofibrant, the model structure is left proper.*

Lemma A.6.1. *A cofibration in $\mathcal{E}$ is acyclic iff it has the left lifting property with respect to any fibration between fibrant objects.*

Proof. The condition is clearly necessary. In the other direction, suppose $i \colon A \to B$ is a cofibration with the LLP with respect to any fibration between fibrant objects. We want to show i is acyclic. For this, let $u \colon B \to B'$ be a weak equivalence with B' fibrant, and $pv \colon A \to B'$ a factorisation of ui into a weak equivalence $v \colon A \to A'$ followed by a fibration $p \colon A' \to B'$. The square

$$\begin{tikzcd}
A \arrow[r, "v"] \arrow[d, "i"'] & A' \arrow[d, "p"] \\
B \arrow[r, "u"'] & Y
\end{tikzcd}$$

has a diagonal filler $d \colon B \to A'$ since p is a fibration between fibrant objects. The arrows u and v are weak equivalences, so they are inverted in the homotopy category. The equations $pd = v$ and $di = u$ say that d has a left and a right inverse in the homotopy category so d is also inverted. It follows that d is a weak equivalence, hence so is i by "three for two". □

Proposition A.6.3. *An adjoint pair of functors $F \colon \mathcal{U} \leftrightarrow \mathcal{V} \colon G$ between two model categories is a Quillen pair iff the following two conditions are satisfied.*

- *F takes a cofibration to a cofibration;*
- *G takes a fibration between fibraant objects to a fibration.*

Proof. The necessity is clear. For the sufficiency, we show F is a left Quillen functor. So let $i\colon A \to B$ be an acyclic cofibration. $F(i)\colon FA \to FB$ is a cofibration, which is acyclic iff if has the LLP with respect to any fibration $f\colon X \to Y$ between fibrant objects. But we have $F(i) \pitchfork f$ iff we have $i \pitchfork G(f)$ by adjointness, and the latter holds since $G(f)$ is a fibration by assumption. □

Notice that both the lemma and the proposition have obvious duals.

Let $\mathcal{E}$ be a model category and $F\colon \mathcal{E} \to \mathcal{D}$ a functor. We consider the problem of extending F along the canonical functor $\gamma\colon \mathcal{E} \to Ho(\mathcal{E})$. Unless F sends weak equivalences to isomorphisms, we cannot do this, of course, but we might get close.

Definition A.6.3. *A* left derived functor *for F is a functor $F^L\colon Ho(\mathcal{E}) \to \mathcal{D}$ together with a natural transformation $\theta\colon F^L\gamma \to F$ which is universal, in the sense that if (G, ϕ) is any other such pair, then there is a unique natural transformation $\phi'\colon G \to F^L$ such that the composite of $\phi'\gamma\colon G\gamma \to F^L\gamma$ with $\theta\colon F^L\gamma \to F$ is ϕ.*

We say F^L, when it exists, is "nearest to F from the left". The universal property above identifies F^L as the *right* Kan extension of F along γ. It is the *right* Kan extension since in the bijection

$$Nat(G\gamma, F) \simeq Nat(G, F^L)$$

it sits on the right, making it the right adjoint to composing with γ when it exists for all F. F^L is thus unique up to unique isomorphism and depends only on F and the weak equivalences in the model category $\mathcal{E}$. Notice also that the Kan extension is *absolute*, i.e. it remains a Kan extension when composed with any functor with domain $\mathcal{D}$.

A *right derived functor* for F is a pair $F^R\colon Ho(\mathcal{E}) \to \mathcal{D}$ together with a natural transformation $\theta\colon F \to F^R\gamma$ making F^R "nearest to F from the right" in the analogous sense. It is a left Kan extension.

Proposition A.6.4. *Let $\mathcal{E}$ be a model category and $F\colon \mathcal{E} \to \mathcal{D}$ a functor which takes weak equivalences between cofibrant objects to isomorphisms. Then the left derived functor (F^L, θ) exists, and has the property that $\theta_X\colon F^L(X) \to F(X)$ is an isomorphism whenever X is cofibrant.*

Proof. Let $LX \to X$ denote a cofibrant replacement for X. As in the proof of Proposition A.5.6 L induces a functor $Ho(\mathcal{E}) \to Ho(\mathcal{E}_c)$. Since F takes weak equivalences between cofibrant objects to isomorphisms it induces a functor $Ho(\mathcal{E}_c) \to \mathcal{D}$, and we let F^L be the composite, i.e. $F^L(X) = FL(X)$. If $p_X : LX \to X$ denotes the acyclic fibration in the definition of L then we put $\theta_X = F(p_X)\colon F^L(X) = F(LX) \to F(X)$. Clearly, when X is cofibrant, $LX = X$ and θ_X is the identity.

To verify that (F^L, θ) has the required universal property, suppose $G\colon Ho(\mathcal{E}) \to \mathcal{D}$ is a functor and $\phi\colon G\gamma \to F$ is a natural transformation. If the required natural transformation $\phi'\colon G \to F^L$ existed, we would obtain, for each X of $\mathcal{E}$, a commutative diagram

$$\begin{tikzcd}
G(LX) \arrow[r, "\phi'_{LX}"] \arrow[d, "G(\gamma(p_X))"'] & F^L(LX) \arrow[r, "\theta_{LX}"] \arrow[d, "F^L(\gamma(p_X))"] & F(LX) \arrow[d, "F(p_X)"] \\
G(X) \arrow[r, "\phi'_X"'] & F^L(X) \arrow[r, "\theta_X"'] & F(X)
\end{tikzcd}$$

in which the rows represent composites of $\phi'\gamma$ with θ. In the diagram we have $\theta_{LX} = id$, $F^L(\gamma(p_X) = id$, and $\theta_X = F(p_X)$. The composite of the top row must be ϕ_{LX}, from which we obtain $\phi'_X = \phi_{LX} G(\gamma(p_X))^{-1}$. It follows that there is at most one such ϕ'. On the other hand, setting $\phi'_X = \phi_{LX} G(\gamma(p_X))^{-1}$ defines a natural transformation $\phi'\colon G\gamma \to F^L\gamma$, and hence a natural transformation $\phi'\colon G \to F^L$ since every map of $Ho(\mathcal{E})$ is of the form $\gamma(f)$ where f is a map of $\mathcal{E}$, or $\gamma(w)^{-1}$ where w is a weak equivalence of $\mathcal{E}$. $\square$

Let $F\colon \mathcal{U} \leftrightarrow \mathcal{V}\colon G$ be a Quillen pair. By the *left derived functor* $F^L\colon Ho(\mathcal{U}) \to Ho(\mathcal{V})$ we mean the left derived functor in the above sense of the functor $\gamma F\colon \mathcal{U} \to Ho(\mathcal{V})$. Similarly, by the *right derived functor* $G^R\colon Ho(\mathcal{V}) \to Ho(\mathcal{U})$ we mean the right derived functor in the above sense of the functor $\gamma G\colon \mathcal{V} \to Ho(\mathcal{U})$. These exist by the above proposition.

Proposition A.6.5. *A Quillen pair $F\colon \mathcal{U} \leftrightarrow \mathcal{V}\colon G$ induces an adjoint pair*

$$F^L\colon Ho(\mathcal{U}) \leftrightarrow Ho(\mathcal{V})\colon G^R$$

between the homotopy categories

Proof. Let A be a cofibrant object of $\mathcal{U}$ and X a fibrant object of $\mathcal{V}$. We show first that the bijection $\mathcal{U}(A, GX) \simeq \mathcal{V}(FA, X)$ respects the homotopy relation $\sim$, and hence passes to a bijection

$$\mathcal{U}(A, GX)/\sim \quad \simeq \quad \mathcal{V}(FA, X)/\sim .$$

Let $(i_0, i_1)\colon A + A \to IA \xrightarrow{s} A$ be a cylinder for A. Since A is cofibrant, and F is a left Quillen functor $(Fi_0, Fi_1)\colon FA + FA \to F(IA) \xrightarrow{Fs} FA$ is a cylinder for FA. If $f\colon A \to GX$ let us denote its adjoint transpose by $f^*\colon FA \to X$. Then if $f, g\colon A \to GX$ are left homotopic by a left homotopy $H\colon IA \to GX$ it follows that $f^*, g^*\colon FA \to X$ are left homotopic by $H^*\colon F(IA) \to X$. A dual argument using path spaces and right homotopies shows that if $f^* \sim_r g^*$ then $f \sim_r g$.

Now, if $A \in \mathcal{U}$ and $X \in \mathcal{V}$, let $p_A\colon LA \to A$ be a cofibrant replacement of A, and $i_X\colon X \to RX$ a fibrant replacement of X. Then we obtain a bijection

$$\begin{aligned} Ho(\mathcal{U})(A, G^R(X)) &\simeq Ho(\mathcal{U})(LA, G(RX)) \simeq \\ &\simeq Ho(\mathcal{V})(F(LA), RX) \simeq Ho(\mathcal{V})(F^L(A), X) \end{aligned}$$

where the first map is composition with $\gamma(p_A)$ and the last is composition with $(\gamma(i_X))^{-1}$. The bijection is clearly natural with respect to maps in $\mathcal{U}$ and $\mathcal{V}$, and the same argument as at the end of Proposition A.6.4 shows it is also natural with respect to maps in the homotopy categories. $\square$

Definition A.6.4. *A Quillen pair (F, G) is called a* Quillen equivalence *if the adjoint pair (F^L, G^R) is an equivalence of categories.*

Proposition A.6.6 (Hovey)**.** *Let $F\colon \mathcal{U} \leftrightarrow \mathcal{V}\colon G$ be a Quillen pair. Then the following are equivalent.*

(i) (F, G) is a Quillen equivalence.

(i) The composite $A \xrightarrow{\eta_A} GFA \xrightarrow{Gi_{FA}} GRFA$ is a weak equivalence for any cofibrant A, and the composite $FLGX \xrightarrow{Fp_{GX}} FGX \xrightarrow{\epsilon_X} X$ is a weak equivalence for any fibrant X.

(i) For any cofibrant A in $\mathcal{U}$ and fibrant X in $\mathcal{V}$, a map $f\colon FA \to X$ is a weak equivalence in $\mathcal{V}$ iff its adjoint transpose $f^*\colon A \to GX$ is a weak equivalence in $\mathcal{U}$.

Proof. To show $(i) \Leftrightarrow (ii)$, we note that the unit of the adjunction $F^L \vdash G^R$ is the composite $A \xrightarrow{p_A^{-1}} LA \xrightarrow{\eta_{LA}} GFLA \xrightarrow{Gi_{FLA}} GRFLA$. This is an isomorphism in $Ho(\mathcal{U})$ iff $LA \xrightarrow{\eta_{LA}} GFLA \xrightarrow{Gi_{FLA}} GRFLA$ is a weak equivalence for any A. But this is the case iff $A \xrightarrow{\eta_A} GFA \xrightarrow{Gi_{FA}} GRFA$ is a weak equivalence for any cofibrant A. Dually, the counit of $F^L \vdash G^R$ is a weak equivalence iff $FLGX \xrightarrow{Fp_{GX}} FGX \xrightarrow{\epsilon_X} X$ is a weak equivalence for any fibrant X.

For $(iii) \Rightarrow (ii)$, note that if (iii) holds and A is cofibrant, then $i_{FA}^*\colon A \to GRFA$ is a weak equivalence since i_{FA} is. But i_{FA}^* is the composite $A \xrightarrow{\eta_A} GFA \xrightarrow{Gi_{FA}} GRFA$. In the same way, if X is fibrant, then $FLGX \xrightarrow{Fp_{GX}} FGX \xrightarrow{\epsilon_X} X$ is a weak equivalence with adjoint transpose $p_{GX}\colon LGX \to GX$.

To show $(ii) \Rightarrow (iii)$, let $f\colon FA \to X$ be a weak equivalence with A cofibrant and X fibrant. Then f^* is the composite $A \xrightarrow{\eta_A} GFA \xrightarrow{Gf} GX$. Consider the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[d, "id"'] & GFA \arrow[r, "Gf"] \arrow[d, "Gi_{FA}"'] & GX \arrow[d, "Gi_X"] \\
A \arrow[r] & GRFA \arrow[r, "GRf"'] & GRX
\end{tikzcd}$$

in which $Gi_X = id$. Since f is a weak equivalence, so is Rf. Since g preserves weak equivalences between fibrant objects, GRf is a weak equivalence. Thus the bottom row is a weak equivalence, so the top row is by "three for two".

On the other hand, suppose $f^* \colon A \to GX$ is a weak equivalence. Consider the commutative diagram

$$\begin{tikzcd}
FLA \arrow[r, "FLf^*"] \arrow[d, "Fp_A"'] & FLGX \arrow[r] \arrow[d, "Fp_{GX}"'] & X \arrow[d, "id"] \\
FA \arrow[r, "F(f^*)"'] & FGX \arrow[r, "\epsilon_X"'] & X
\end{tikzcd}$$

in which $Fp_A = id$. The top row is a weak equivalence, and the bottom row is f, so f is a weak equivalence. $\square$

The most useful criterion for establishing a Quillen equivalence is perhaps the following.

Proposition A.6.7 (Hovey)**.** *Let $F \colon \mathcal{U} \leftrightarrow \mathcal{V} \colon G$ be a Quillen pair. Then the following are equivalent.*

(i) (F, G) is a Quillen equivalence.

(i) G reflects weak equivalences between fibrant objects, and for any cofibrant A, the composite $A \xrightarrow{\eta_A} GFA \xrightarrow{Gi_{FA}} GRFA$ is a weak equivalence.

(i) F reflects weak equivalences between cofibrant objects, and for any fibrant X, the composite $FLGX \xrightarrow{Fp_{GX}} FGX \xrightarrow{\epsilon_X} X$ is a weak equivalence.

Proof. Suppose (F, G) is a Quillen equivalence. Then we have seen above that the composite $A \xrightarrow{l} \eta_A GFA \xrightarrow{Gi_{FA}} GRFA$ is a weak equivalence for any cofibrant A, and the composite $FLGX \xrightarrow{Fp_{GX}} FGX \xrightarrow{\epsilon_X} X$ is a weak equivalence for any fibrant X. Suppose $f \colon A \to B$ is a map between cofibrant objects such that Ff is a weak equivalence. Since $Lf = f$ it follows that $F^L f$ is an isomorphism. Since F^L is an equivalence of categories, f is an isomorphism in $Ho(\mathcal{U})$ so f is a weak equivalence in $\mathcal{U}$. Dually, G reflects weak equivalences between fibrant objects. Hence (i) implies (ii) and (iii).

We show (iii) implies (i). The counit $(F^L)(G^R)X \to X$ is an isomorphism by assumption, so we must show the unit $A \to (G^R)(F^L)A$ is an isomorphism. But F^L of the unit, namely $(F^L)A \to (F^L)(G^R)(F^L)A)$ is inverse to the counit of $(F^L)A$ so it is an isomorphism. Since F reflects weak equivalences between cofibrant objects it follows that $LA \to LGRFLA$ is a weak equivalence for any A. L reflects all weak equivalences, so $A \to GRFLA$ is a weak equivalence. But this is the unit $A \to (G^R)(F^L)A$, which is thus a weak equivalence as required. A dual argument shows (ii) implies (i). □

A.7 Monodial model categories

We say that a functor of two variables

$$\odot \colon \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$$

is *divisible on the left* if the functor $A \odot (-) \colon \mathcal{E}_2 \to \mathcal{E}_3$ admits a right adjoint $A\backslash(-) \colon \mathcal{E}_3 \to \mathcal{E}_2$ for every object $A \in \mathcal{E}_1$. In this case we obtain a functor of two variables $(A, X) \mapsto A\backslash X$,

$$\mathcal{E}_1^o \times \mathcal{E}_3 \to \mathcal{E}_2,$$

called the *left division functor*. Dually, we say that $\odot$ is *divisible on the right* if the functor $(-) \odot B \colon \mathcal{E}_1 \to \mathcal{E}_3$ admits a right adjoint $(-)/B \colon \mathcal{E}_3 \to \mathcal{E}_1$ for every object $B \in \mathcal{E}_2$. In this case we obtain a functor of two variables $(X, B) \mapsto X/B$,

$$\mathcal{E}_3 \times \mathcal{E}_2^o \to \mathcal{E}_1,$$

called the *right division functor*. When $\odot$ is divisible on both sides, there is a bijection between the following three kinds of maps

$$A \odot B \to X, \qquad B \to A\backslash X, \qquad A \to X/B.$$

Hence the contravariant functors $A \mapsto A\backslash X$ and $B \mapsto B\backslash X$ are mutually right adjoint. It follows that we have

$$u \pitchfork (X/v) \quad \Leftrightarrow \quad v \pitchfork (u\backslash X).$$

for $u \in \mathcal{E}_1$ and $v \in \mathcal{E}_2$.

Remark. If a functor of two variables $\odot \colon \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ is divisible on both sides, then so are the left division functor $\mathcal{E}_1^o \times \mathcal{E}_3 \to \mathcal{E}_2$ and the right division functor $\mathcal{E}_3 \times \mathcal{E}_2^o \to \mathcal{E}_1$.

Recall that a monoidal category $\mathcal{E} = (\mathcal{E}, \otimes)$ is said to be *closed* if the tensor product $\otimes$ is divisible on both sides. Let $\mathcal{E} = (\mathcal{E}, \otimes, \sigma)$ be a *symmetric* monoidal closed category, with symmetry $\sigma \colon A \otimes B \simeq B \otimes A$. Then the objects X/A and $A\backslash X$ are canonicaly isomorphic; and we identify them by adopting a common notation, for example $[A, X]$.

Recall that a category with finite products $\mathcal{E}$ is said to be *cartesian closed* if the functor $A \times -\colon \mathcal{E} \to \mathcal{E}$ admits a right adjoint $(-)^A$ for every object $A \in \mathcal{E}$. A cartesian closed category $\mathcal{E}$ is a symmetric monoidal closed category. Every presheaf category and more generally every topos is cartesian closed.

Let $\odot\colon \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ be a functor of two variables with values in a finitely cocomplete category $\mathcal{E}_3$. If $u\colon A \to B$ is map in $\mathcal{E}_1$ and $v\colon S \to T$ is a map in $\mathcal{E}_2$, we denote by $u \odot' v$ the map

$$A \odot T \sqcup_{A \odot S} B \odot S \longrightarrow B \odot T$$

obtained from the commutative square

$$\begin{tikzcd}
A \odot S \arrow[r] \arrow[d] & B \odot S \arrow[d] \\
A \odot T \arrow[r] & B \odot T.
\end{tikzcd}$$

This defines a functor of two variables

$$\odot'\colon \mathcal{E}_1^I \times \mathcal{E}_2^I \to \mathcal{E}_3^I,$$

where $\mathcal{E}^I$ denotes the category of arrows of a category $\mathcal{E}$.

In a topos, if $u\colon A \subseteq B$ and $v\colon S \subseteq T$ are inclusions of subobjects then the map $u \times' v$ is the inclusion

$$A \times T) \cup (B \times S) \subseteq B \times T.$$

Suppose now that a functor $\odot\colon \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ is divisible on both sides, that $\mathcal{E}_1$ and $\mathcal{E}_2$ are finitely complete and that $\mathcal{E}_3$ is finitely cocomplete. Then the functor $\odot'\colon \mathcal{E}_1^I \times \mathcal{E}_2^I \to \mathcal{E}_3^I$ is divisible on both sides. If $u\colon A \to B$ is map in $\mathcal{E}_1$ and $f\colon X \to Y$ is a map in $\mathcal{E}_3$, We denote by $\langle u\backslash\, f\rangle$ the map

$$B\backslash X \to B\backslash Y \times_{A\backslash Y} A\backslash X$$

obtained from the commutative square

$$\begin{tikzcd}
B\backslash X \arrow[r] \arrow[d] & A\backslash X \arrow[d] \\
B\backslash Y \arrow[r] & A\backslash Y.
\end{tikzcd}$$

Then the functor $f \mapsto \langle u\backslash f\rangle$ is right adjoint to the functor $v \mapsto u \odot' v$. Dually, if $v\colon S \to T$ is map in $\mathcal{E}_2$ and $f\colon X \to Y$ is a map in $\mathcal{E}_3$, we denote by $\langle f/v\rangle$ the map

$$X/T \to Y/T \times_{Y/S} X/S$$

obtained from the commutative square

$$\begin{tikzcd} X/T \arrow[r] \arrow[d] & X/S \arrow[d] \\ Y/T \arrow[r] & Y/S. \end{tikzcd}$$

The functor $f \mapsto \langle f\backslash v\rangle$ is right adjoint to the functor $u \mapsto u \odot' v$.

The proof of the following proposition is left to the reader.

Proposition A.7.1. *Let $\odot\colon \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ be a functor of two variables divisible on both sides, where $\mathcal{E}_i$ is a finitely bicomplete category for $i = 1, 2, 3$. If $u \in \mathcal{E}_1$, $v \in \mathcal{E}_2$ and $f \in \mathcal{E}_3$, then*

$$(u \odot' v) \pitchfork f \iff u \pitchfork \langle f/v\rangle \iff v \pitchfork \langle u\backslash f\rangle.$$

Let $\mathcal{E} = (\mathcal{E}, \otimes)$ be a bicomplete symmetric monoidal closed category. Then the category $\mathcal{E}^I$ is symmetric monoidal closed and bicomplete. If u and f are two maps in $\mathcal{E}$, then the maps $\langle f/u\rangle$ and $\langle u\backslash f\rangle$ are canonically isomorphic, and we identify them by adopting a common notation $\langle u, f\rangle$. If $u\colon A \to B$, $v\colon S \to T$ and $f\colon X \to Y$ are three maps in $\mathcal{E}$, then

$$(u \otimes' v) \pitchfork f \iff u \pitchfork \langle v, f\rangle \iff v \pitchfork \langle u, f\rangle.$$

Definition A.7.1 (Hovey)**.** *We say that a functor of two variables between three model categories*

$$\odot\colon \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$$

is a left Quillen functor if it is cocontinuous in each variable and the following conditions are satisfied:

- *$u \odot' v$ is a cofibration if $u \in \mathcal{E}_1$ and $v \in \mathcal{E}_2$ are cofibrations;*
- *$u \odot' v$ is an acyclic cofibration if $u \in \mathcal{E}_1$ and $v \in \mathcal{E}_2$ are cofibrations and if u or v is acyclic.*

Dually, we say that $\odot$ is a right Quillen functor if the opposite functor $\odot^o\colon \mathcal{E}_1^o \times \mathcal{E}_2^o \to \mathcal{E}_3^o$ is a left Quillen functor.

Proposition A.7.2. *Let $\odot\colon \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ be a left Quillen functor of two variables between three model categories. If $A \in \mathcal{E}_1$ is cofibrant, then the functor $B \mapsto A \odot B$ is a left Quillen functor $\mathcal{E}_2 \to \mathcal{E}_3$.*

Proof. If $A \in \mathcal{E}_1$ is cofibrant, then the map $i_A\colon \perp \to A$ is a cofibration, where $\perp$ is the initial object. If $v\colon S \to T$ is a map in $\mathcal{E}_2$, then we have $A \odot v = i_A \odot' v$. Thus, $A \odot v$ is a cofibration if v is a cofibration and $A \odot v$ is acyclic if moreover v is acyclic. $\square$

Proposition A.7.3. *Let $\odot\colon \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ be a functor of two variables, cocontinuous in each, between three model categories. Suppose that the following three conditions are satisfied:*

- *If $u \in \mathcal{E}_1$ and $v \in \mathcal{E}_2$ are cofibrations, then so is $u \odot' v$;*
- *the functor $(-) \odot B$ preserves acyclic cofibrations for every object $B \in \mathcal{E}_2$;*
- *the functor $A \odot (-)$ preserves acyclic cofibrations for every object $A \in \mathcal{E}_1$.*

Then $\odot$ is a left Quillen functor.

Proof. Let $u\colon A \to B$ be a cofibration in $\mathcal{E}_1$ and $v\colon S \to T$ a cofibration in $\mathcal{E}_2$. We show that $u \odot' v$ is acyclic if v is acyclic, the case u acyclic is similar. Consider the commutative diagram

$$\begin{tikzcd}
A \odot S \arrow[r, "u\odot S"] \arrow[d, "A\odot v"'] & B \odot S \arrow[d, "i_2"'] \arrow[dr, "B\odot v"] & \\
A \odot T \arrow[r, "i_1"] & Z \arrow[r, "u\odot' v"] & B \odot T
\end{tikzcd}$$

where $Z = A \odot T \sqcup_{A\odot S} B \odot S$ and where $(u \odot' v)i_1 = u \odot T$. The map $A \odot v$ is an acyclic cofibration since v is an acyclic cofibration. Similarly for the map $B \odot v$. It follows that i_2 is an acyclic cofibration by cobase change. Thus, $u \odot' v$ is acyclic by three-for-two since $(u \odot' v)i_2 = B \odot v$ is acyclic. □

Proposition A.7.4. *Let $\odot\colon \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ be a functor of two variables between three model categories. If the functor $\odot$ is divisible on the left, then it is a left Quillen functor iff the corresponding left division functor $\mathcal{E}_1^o \times \mathcal{E}_3 \to \mathcal{E}_2$ is a right Quillen functor. Dually, if the functor $\odot$ is divisible on the right, then it is a left Quillen functor iff the corresponding right division functor $\mathcal{E}_3 \times \mathcal{E}_2^o \to \mathcal{E}_1$ is a right Quillen functor.*

Definition A.7.2 (Hovey)**.** *A model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on a monoidal closed category $\mathcal{E} = (\mathcal{E}, \otimes)$ is said to be monoidal if the tensor product $\otimes\colon \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ is a left Quillen functor of two variables, and the unit of the tensor product is cofibrant.*

In a monoidal closed model category, if f is a fibration then so are the maps $\langle u\backslash f\rangle$ and $\langle f/u\rangle$ for any cofibration u. Moreover, the fibrations $\langle u\backslash f\rangle$ and $\langle f/u\rangle$ are acyclic if the cofibration u is acyclic or the fibration f is acyclic.

Definition A.7.3. *We say that a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on a cartesian closed category $\mathcal{E}$ is cartesian closed if the cartesian product $\times\colon \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ is a left Quillen functor of two variables and if the terminal object 1 is cofibrant.*

In a cartesian closed model category, if f is a fibration and u is a cofibration, then the map $\langle u, f\rangle$ is a fibration, which is acyclic if u or f is acyclic.

We recall a few notions of enriched category theory. Let $\mathcal{V} = (\mathcal{V}, \otimes, \sigma)$ a bicomplete symmetric monoidal closed category. A category enriched over $\mathcal{V}$ is called a $\mathcal{V}$-*category*. If $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{V}$-categories, there is the notion of a *strong* functor $F\colon \mathcal{A} \to \mathcal{B}$; it is an ordinary functor equipped with a *strength* which is a natural transformation $\mathcal{A}(X, Y) \to \mathcal{B}(FX, FY)$ preserving composition and units. A natural transformation $\alpha\colon F \to G$ between strong functors $F, G\colon \mathcal{A} \to \mathcal{B}$ is said to be *strong* if the following square commutes

$$\begin{tikzcd} \mathcal{A}(X,Y) \arrow[r] \arrow[d] & \mathcal{B}(GX,GY) \arrow[d, "\mathcal{B}(\alpha_X,GY)"] \\ \mathcal{B}(FX,FY) \arrow[r, "\mathcal{B}(FX,\alpha_Y)"] & \mathcal{B}(FX,GY). \end{tikzcd}$$

for every pair of objects $X, Y \in \mathcal{A}$. A *strong adjunction* $\theta\colon F \dashv G$ between strong functors $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$ is a strong natural isomorphism

$$\theta_{XY}\colon \mathcal{A}(FX, Y) \to \mathcal{B}(X, GY).$$

A strong functor $G\colon \mathcal{B} \to \mathcal{A}$ has a strong left adjoint iff it has an ordinary left adjoint $F\colon \mathcal{A} \to \mathcal{B}$ and the map

$$\mathcal{B}(FX,Y) \longrightarrow \mathcal{A}(GFX,GY) \xrightarrow{\mathcal{A}(\eta_X,GY)} \mathcal{A}(X,GY)$$

obtained by composing with the unit η_X of the adjunction is an isomorphism for every pair of objects $X \in \mathcal{A}$ and $Y \in \mathcal{B}$. Recall that a $\mathcal{V}$-category $\mathcal{E}$ is said to admit *tensor products* if the functor $Y \mapsto \mathcal{E}(X, Y)$ admits a strong left adjoint $A \mapsto A \otimes X$ for every object $X \in \mathcal{E}$. A $\mathcal{V}$-category is said to be (strongly) *cocomplete* if it cocomplete as an ordinary category and if it admits tensor products. These notions can be dualised. A $\mathcal{V}$-category $\mathcal{E}$ is said to admit *cotensor products* if the opposite $\mathcal{V}$-category $\mathcal{E}^o$ admits tensor products. This means that the (contravariant) functor $X \mapsto \mathcal{E}(X, Y)$ admits a strong right adjoint $A \mapsto Y^{[A]}$ for every object $Y \in \mathcal{E}$. A $\mathcal{V}$-category is said to be (strongly) *complete* if it complete as an ordinary category and if it admits cotensor products. We shall say that a $\mathcal{V}$-category is (strongly) *bicomplete* if it is both $\mathcal{V}$-complete and cocomplete.

Definition A.7.4. *If $\mathcal{V} = (\mathcal{V}, \otimes, \sigma)$ is a bicomplete symmetric monoidal closed model category, we say that a model structure on a strongly bicomplete $\mathcal{V}$-category $\mathcal{E}$ is a $\mathcal{V}$-enriched model category structure if the functor*

$$\mathcal{E}(-,-)\colon \mathcal{E}^o \times \mathcal{E} \to \mathcal{V}$$

is a right Quillen functor of two variables.

A $\mathcal{V}$-category equipped with a $\mathcal{V}$-enriched model structure is called a *$\mathcal{V}$-enriched model category.*

A *simplicial category* is a category enriched over $\mathbf{S}$.

Definition A.7.5. *Let $\mathcal{E}$ be a strongly bicomplete simplicial category. We say a model structure on $\mathcal{E}$ is* simplicial *if it is enriched with respect to the classical model structure on* $\mathbf{S}$.

A simplicial category equipped with a simplicial model structure is called a *simplicial model category.*

Proposition A.7.5. *Let $\mathcal{E}$ be a simplicial model category. Then a map between cofibrant objects $u\colon A \to B$ is acyclic iff the map of simplicial sets*

$$\mathcal{E}(u, X)\colon \mathcal{E}(B, X) \to \mathcal{E}(A, X)$$

is a weak homotopy equivalence for any fibrant object X.

Proof. The functor $A \mapsto \mathcal{E}(A, X)$ takes an (acyclic) cofibration to an (acyclic) Kan fibration if X is fibrant. It then follows by Proposition A.5.1 that it takes an acyclic map between cofibrant objects to an acyclic map. Conversely, let $u\colon A \to B$ be a map between cofibrant objects in $\mathcal{E}$. If the map $\mathcal{E}(u, X)\colon \mathcal{E}(B, X) \to \mathcal{E}(A, X)$ is a weak homotopy equivalence for any fibrant object X, let us show that u is acyclic. We suppose first that A and B are fibrant. Let $\mathcal{E}_{cf}$ be the full subcategory of fibrant and cofibrant objects of $\mathcal{E}$. We prove that u is acyclic by showing that u is invertible in the homotopy category $Ho(\mathcal{E}_{cf})$. But if $S, X \in \mathcal{E}_{cf}$, then we have $Ho(\mathcal{E}_{cf})(S, X) = \pi_0\mathcal{E}(S, X)$ by [Q]. Hence the map $Ho(\mathcal{E}_{cf})(u, X)\colon Ho(\mathcal{E}_{cf})(B, X) \to Ho(\mathcal{E}_{cf})(A, X)$ is equal to the map $\pi_0\mathcal{E}(u, X)\colon \pi_0\mathcal{E}(B, X) \to \pi_0\mathcal{E}(A, X)$. But the map $\pi_0\mathcal{E}(u, X)$ is bijective since $\mathcal{E}(u, X)$ is a weak homotopy equivalence. This shows that the map $Ho(\mathcal{E}_{cf})(u, X)$ is bijective for every $X \in \mathcal{E}_{cf}$. It follows by the Yoneda lemma that u is invertible in $Ho(\mathcal{E}_{cf})$. Thus, u is acyclic by. In the general case, choose a fibrant replacement $i_A\colon A \to A'$ with i_A an acyclic cofibration. Similarly, choose a fibrant replacement $i_B\colon B \to B'$ with i_B an acyclic cofibration. Then there exists a map $u'\colon A' \to B'$ such that $u'i_A = i_B u$. We then have a commutative square

$$\begin{tikzcd}
\mathcal{E}(B', X) \arrow[r] \arrow[d] & \mathcal{E}(A', X) \arrow[d] \\
\mathcal{E}(B, X) \arrow[r] & \mathcal{E}(A, X)
\end{tikzcd}$$

for any object X. If X is fibrant, the vertical maps of the square are weak homotopy equivalences by the first part of the proof. Hence also the map $\mathcal{E}(u', X)\colon \mathcal{E}(B', X) \to \mathcal{E}(A', X)$ by "three for two". This shows u' is acyclic since A' and B' are fibrant. It follows by "three for two" that u is acyclic. □

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