

Quasi-categories vs Segal spaces

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To Ross Street on the occasion of his 60th birthday

ABSTRACT. We show that complete Segal spaces and Segal categories are Quillen equivalent to quasi-categories.

Introduction

Quasi-categories were introduced by Boardman and Vogt in their work on homotopy invariant algebraic structures [BV]. They are often called weak Kan complexes in the literature. The category of simplicial sets \mathbf{S} admits a Quillen model structure in which the fibrant objects are the quasi-categories by a result of the first author in [J2]. We call this model structure, the *model structure for quasi-categories*. The theory of quasi-categories has applications to homotopical algebra and higher category theory, see [J3], [Lu1] and [Lu2]. Complete Segal spaces were introduced by Rezk in his work on the homotopy theory of homotopy theories [Rez]. The category of bisimplicial sets $\mathbf{S}^{(2)}$ admits a Quillen model structure in which the fibrant objects are the complete Segal spaces. We call this model structure the *model structure for complete Segal spaces*. The main result of this paper is to establish a Quillen equivalence between the model category for quasi-categories and that for complete Segal spaces:

$$p_1^* : \mathbf{S} \leftrightarrow \mathbf{S}^{(2)} : i_1^*.$$

The functor i_1^* associates to a bisimplicial set X its first row X_{*0} . The theorem implies that the first row of a complete Segal space contains all the homotopy information about the space. We also describe a Quillen equivalence in the opposite direction,

$$t_1 : \mathbf{S}^{(2)} \leftrightarrow \mathbf{S} : t^!,$$

where the functor t_1 associates to a bisimplicial set X a total simplicial set $t_1(X)$. Segal categories were first introduced by Dwyer, Kan and Smith [DKS], where they are called special Δ^o -diagrams of simplicial sets. The theory of Segal categories was extensively developed by Hirschowitz and Simpson for application to algebraic geometry. They show that the category of precategories admits a model structure

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in which the fibrant objects are Segal categories. We call this model structure the *model structure for Segal categories*. By a theorem of Bergner in [B2], the inclusion functor $\pi^* : \mathbf{PCat} \subset \mathbf{S}^{(2)}$ induces a Quillen equivalence between the model structure for Segal categories and the model structure for complete Segal spaces,

$$\pi^* : \mathbf{PCat} \leftrightarrow \mathbf{S}^{(2)} : \pi_*$$

By combining these equivalences, we obtain an equivalence between the model category for quasi-categories and that for Segal categories:

$$q^* : \mathbf{S} \leftrightarrow \mathbf{PCat} : j^*$$

The functor j^* associates to a precategory X its first row X_{*0} . The theorem implies that the first row of a fibrant Segal category contains all the homotopy information about the Segal category. We also describe a Quillen equivalence in the opposite direction,

$$d^* : \mathbf{PCat} \leftrightarrow \mathbf{S} : d_*$$

where the functor d^* associates to a precategory X its diagonal simplicial set.

The paper has five sections, one addendum, one appendix and one epilogue. In the first section we give a brief description of the relevant aspects of the model structure for quasi-categories. The fibrant objects are the quasi-categories and the acyclic maps are called weak categorical equivalences. In the second section we introduce two Reedy model structures on the category of bisimplicial sets, respectively called the vertical and the horizontal model structures. The acyclic maps are the column-wise weak homotopy equivalences in the vertical structure but they are the row-wise weak categorical equivalences in the horizontal. The interaction between the two model structures is one of the main technical tools of the paper. We define a total space functor from bisimplicial sets to simplicial sets and show that it is a left Quillen functor with values in the model category for quasi-categories for both the vertical and the horizontal model structures. In the third section we show that every row of a Segal space is a quasi-category. In section four we show that the functor which associates to a bisimplicial set its first row induces a Quillen equivalence between complete Segal spaces and quasi-categories. We also show that the total space functor induces a Quillen equivalence in the opposite direction. In section five we show that the functor which associates to a precategory its first row induces a Quillen equivalence between Segal categories and quasi-categories. We also prove that the functor which associates to a precategory its diagonal induces a Quillen equivalence in the opposite direction. In the addendum we show that the model category for complete Segal spaces can be obtained from the model category for quasi-categories by a method developed by Rezk, Schwede and Shipley in their paper on model categories and functors [RSS]. In the epilogue, we discuss other models of homotopy theories and other Quillen equivalences.

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1. The model structure for quasi-categories

The results of this section are taken from [J1] and [J2]. Quasi-categories were introduced by Boardman and Vogt in their work on homotopy invariant algebraic structures [BV]. The category of simplicial sets **S** admits a model structure in which the fibrant objects are the quasi-categories [J2]. We call it the *model structure for quasi-categories*.

See the appendix for notation, and basic results. We denote by **Set** the category of sets, by **S** the category of simplicial sets and by **Cat** the category of small categories.

Recall that an arrow $u : A \rightarrow B$ in a category is said to have the *left lifting property* with respect to another arrow $f : X \rightarrow Y$, or that f has the *right lifting property* with respect to u , if every commutative square

$$\begin{array}{ccc}
 A & \xrightarrow{x} & X \\
 u \downarrow & \nearrow d & \downarrow f \\
 B & \xrightarrow{y} & Y
 \end{array}$$

has a diagonal filler $d : B \rightarrow X$ (that is, $du = x$ and $fd = y$). We shall denote this relation by $u \pitchfork f$.

We first recall the classical model structure on the category **S**. The notion of weak homotopy equivalence between simplicial sets is usually defined by using the geometric realisation functor $\mathbf{S} \rightarrow \mathbf{Top}$. An alternative definition uses Kan complexes and the homotopy category \mathbf{S}^{π_0} introduced by Gabriel and Zisman [GZ]. Recall that a simplicial set X is called a *Kan complex* if every horn $\Lambda^k[n] \rightarrow X$ has a filler $\Delta[n] \rightarrow X$. If $A, B \in \mathbf{S}$ let us put

$$\pi_0(A, B) = \pi_0(B^A).$$

If we apply the functor π_0 to the composition map $C^B \times B^A \rightarrow C^A$ we obtain a composition law

$$\pi_0(B, C) \times \pi_0(A, B) \rightarrow \pi_0(A, C)$$

for a category \mathbf{S}^{π_0} , where $\mathbf{S}^{\pi_0}(A, B) = \pi_0(A, B)$. A map of simplicial sets is called a *homotopy equivalence* if it is invertible in the category \mathbf{S}^{π_0} . A map $u : A \rightarrow B$ is called a *weak homotopy equivalence* the map

$$\pi_0(u, X) : \pi_0(B, X) \rightarrow \pi_0(A, X)$$

is bijective for every Kan complex X . This notion of weak equivalence is equivalent to the usual notion which is defined via the geometric realisation functor.

A map of simplicial sets is called a *Kan fibration* if it has the right lifting property with respect to every horn inclusion $\Lambda^k[n] \subset \Delta[n]$. The following theorem describes the *classical model structure* on the category \mathbf{S} .

THEOREM 1.1 (Quillen, see [Q]). *The category of simplicial sets \mathbf{S} admits a model structure $(\mathcal{C}_0, \mathcal{W}_0, \mathcal{F}_0)$ in which a cofibration is a monomorphism, a weak equivalence is a weak homotopy equivalence and a fibration is a Kan fibration. The model structure is cartesian closed and proper.*

See Definition 7.7 in the appendix for the notion of model structure. See [JT2] for a purely combinatorial proof of the theorem. We call a map of simplicial sets a *trivial fibration* if it has the right lifting property with respect to every monomorphism. We note that this notion can be defined in any topos, see Definition 7.2. The acyclic fibrations of the classical model structure are the trivial fibrations. For every $n \geq 0$, we denote by δ_n the map $\partial\Delta[n] \rightarrow \Delta[n]$ defined by the inclusion $\partial\Delta[n] \subset \Delta[n]$.

PROPOSITION 1.2. [GZ] *The class of monomorphisms in the category \mathbf{S} is generated as a saturated class by the set of inclusions*

$$\delta_n : \partial\Delta[n] \subset \Delta[n], \quad \text{for } n \geq 0.$$

A map of simplicial sets is a trivial fibration iff it has the right lifting property with respect to δ_n for every $n \geq 0$.

For the notion of a saturated class, see 7.4.

DEFINITION 1.3. [GZ] A map of simplicial sets is said to be *anodyne* if it belongs to the saturated class generated by the horns $\Lambda^k[n] \subset \Delta[n]$ for $n \geq 1$ and $0 \leq k \leq n$.

A map of simplicial set is anodyne iff it is an acyclic cofibration.

Before describing the model structure for quasi-categories it is good to describe a related model structure on \mathbf{Cat} . We call a functor $p : X \rightarrow Y$ a *quasi-fibration* if for every object $a \in X$ and every isomorphism $g \in Y$ with source $p(a)$ there exists an isomorphism $f \in X$ with source a such that $p(f) = g$. A functor $p : X \rightarrow Y$ is a quasi-fibration iff it has the right lifting property with respect to the inclusion $\{0\} \subset J$, where J is the groupoid generated by one isomorphism $0 \rightarrow 1$. We say that a functor $A \rightarrow B$ is *monic on objects* (resp. *surjective on objects*) if the induced map $Ob(A) \rightarrow Ob(B)$ is monic (resp. surjective).

THEOREM 1.4. [JT1] [Rez] *The category \mathbf{Cat} admits a model structure in which a cofibration is a functor monic on objects, a weak equivalence is an equivalence of categories and a fibration is a quasi-fibration. The acyclic fibrations are the equivalences surjective on objects. The model structure is proper and cartesian closed. Every object is fibrant and cofibrant.*

We call this model structure the *natural model structure* on \mathbf{Cat} . The notion of cartesian closed model category is defined in 7.29.

The category Δ is a full subcategory of \mathbf{Cat} . The *nerve* of a category $C \in \mathbf{Cat}$ is the simplicial set NC obtained by putting $(NC)_n = \mathbf{Cat}([n], C)$ for every $n \geq 0$. The nerve functor $N : \mathbf{Cat} \rightarrow \mathbf{S}$ is full and faithful and we shall regard it as an

inclusion $N : \mathbf{Cat} \subset \mathbf{S}$ by adopting the same notation for a category and its nerve. The functor N has a left adjoint

$$\tau_1 : \mathbf{S} \rightarrow \mathbf{Cat}.$$

We say that $\tau_1 X$ is the *fundamental category* of a simplicial set X . The fundamental groupoid $\pi_1 X$ is obtained by inverting the arrows of $\tau_1 X$.

We shall say that a horn $\Lambda^k[n] \subset \Delta[n]$ is *inner* if $0 < k < n$. The following definition is due to Boardman and Vogt.

DEFINITION 1.5. [BV] A simplicial set X is called a *quasi-category* if every inner horn $\Lambda^k[n] \rightarrow X$ has a filler $\Delta[n] \rightarrow X$. A *map* of quasi-categories is a map of simplicial sets.

Quasi-categories are often called *weak Kan complexes* in the literature. The nerve of a category and a Kan complex are examples. We shall denote by \mathbf{QCat} the category of quasi-categories; it is a full subcategory of \mathbf{S} .

The next step is to introduce an appropriate notion of equivalence for quasi-categories. If A is a simplicial set, we shall denote by $\tau_0 A$ the set of isomorphism classes of objects of the fundamental category $\tau_1 A$. The functor $\tau_1 : \mathbf{S} \rightarrow \mathbf{Cat}$ preserves finite products [GZ], hence also the functor $\tau_0 : \mathbf{S} \rightarrow \mathbf{Set}$. If $A, B \in \mathbf{S}$ let us put

$$\tau_0(A, B) = \tau_0(B^A).$$

If we apply the functor τ_0 to the composition map $C^B \times B^A \rightarrow C^A$ we obtain a composition law

$$\tau_0(B, C) \times \tau_0(A, B) \rightarrow \tau_0(A, C)$$

for a category \mathbf{S}^{τ_0} , where $\mathbf{S}^{\tau_0}(A, B) = \tau_0(A, B)$. We shall say that a map of simplicial sets is a *categorical equivalence* if it is invertible in the category \mathbf{S}^{τ_0} . If X and Y are quasi-categories, a categorical equivalence $X \rightarrow Y$ is called an *equivalence of quasi-categories*. We shall say that a map of simplicial sets $u : A \rightarrow B$ is a *weak categorical equivalence* if the map

$$\tau_0(u, X) : \tau_0(B, X) \rightarrow \tau_0(A, X)$$

is bijective for every quasi-category X .

The notion of equivalence between quasi-categories has another description which we don't need but which is good to know. If X be a simplicial set, we shall denote by $X(a, b)$ the fiber at $(a, b) \in X_0 \times X_0$ of the projection $(p_0, p_1) : X^I \rightarrow X \times X$ defined from the inclusion $\{0, 1\} \subset I$. The simplicial set $X(a, b)$ is a Kan complex if X is a quasi-category.

DEFINITION 1.6. [J2] We shall say that a map between simplicial sets $u : A \rightarrow B$ is *essentially surjective* if the map $\tau_0(u) : \tau_0 A \rightarrow \tau_0 B$ is surjective. We shall say that a map between quasi-categories $f : X \rightarrow Y$ is *fully faithful* if the map

$$X(a, b) \rightarrow Y(fa, fb)$$

induced by f is a weak homotopy equivalence for every pair $a, b \in X_0$.

THEOREM 1.7. [J2] *A map between quasi-categories is an equivalence iff it is fully faithful and essentially surjective.*

DEFINITION 1.8. [J2] We shall say that a map of simplicial sets is a *quasi-fibration* if it has the right lifting property with respect to every monic weak categorical equivalence.

THEOREM 1.9. [J2] *The category of simplicial sets \mathbf{S} admits a model structure $(\mathcal{C}_1, \mathcal{W}_1, \mathcal{F}_1)$ in which a cofibration is a monomorphism, a weak equivalence is a weak categorical equivalence and a fibration is a quasi-fibration. The acyclic fibrations are the trivial fibrations. The fibrant objects are the quasi-categories. The model structure is cartesian closed and left proper.*

We shall say that it is the *model structure for quasi-categories*. The notion of cartesian closed model category is defined in 7.29. The quasi-fibrations between quasi-categories have a simple description. It is based on the following notion:

DEFINITION 1.10. [J2] We shall say that a map of simplicial sets is *mid anodyne* if it belongs to the saturated class generated by the inner horns $\Lambda^k[n] \subset \Delta[n]$. We shall say that a map is a *mid fibration* if it has the right lifting property with respect to every inner horn $\Lambda^k[n] \subset \Delta[n]$.

For the notion of a saturated class, see 7.4.

PROPOSITION 1.11. [J2] *Every mid anodyne map is a weak categorical equivalence bijective on vertices. The functor $\tau_1 : \mathbf{S} \rightarrow \mathbf{Cat}$ takes a mid anodyne map to an isomorphism of categories.*

PROPOSITION 1.12. [J2] *If \mathcal{B} is the class of mid fibrations and \mathcal{A} is the class of mid anodyne maps then the pair $(\mathcal{A}, \mathcal{B})$ is a weak factorisation system in the category \mathbf{S}*

For the notion of weak factorisation system, see 7.1.

Let us regard the groupoid J as a simplicial set via the nerve functor.

PROPOSITION 1.13. [J2] *Every quasi-fibration is a mid fibration. Conversely, a mid fibration between quasi-categories $p : X \rightarrow Y$ is a quasi-fibration iff the following equivalent conditions are satisfied:*

- *p has the right lifting property with respect to the inclusion $\{0\} \subset J$*
- *the functor $\tau_1 p : \tau_1 X \rightarrow \tau_1 Y$ is a quasi-fibration.*

PROPOSITION 1.14. [J2] *The pair of adjoint functors*

$$\tau_1 : \mathbf{S} \leftrightarrow \mathbf{Cat} : N$$

is a Quillen pair between the model structure for quasi-categories and the natural model structure on \mathbf{Cat} . A functor $u : A \rightarrow B$ in \mathbf{Cat} is a quasi-fibration (resp. an equivalence) iff the map $Nu : NA \rightarrow NB$ is a quasi-fibration (a weak categorical equivalence) in \mathbf{S} .

It follows that the functor $\tau_1 : \mathbf{S} \rightarrow \mathbf{Cat}$ takes a weak categorical equivalence to an equivalence of categories.

PROPOSITION 1.15. [J2] *The classical model structure on \mathbf{S} is a Bousfield localisation of the model structure for quasi-categories.*

For the notion of Bousfield localisation, see Definition 7.20. Thus, every weak categorical equivalence is a weak homotopy equivalence and every Kan fibration is a quasi-fibration. Conversely, a map between Kan complexes is a weak homotopy equivalence (resp, a Kan fibration) iff it is a weak categorical equivalence (resp. a quasi-fibration).

Let **Kan** be the category of Kan complexes.

PROPOSITION 1.16. [J2] *The inclusion functor $\mathbf{Kan} \subset \mathbf{QCat}$ has a right adjoint*

$$J : \mathbf{QCat} \rightarrow \mathbf{Kan}.$$

The Kan complex $J(X)$ is the largest sub-Kan complex of a quasi-category X . The functor J takes an equivalence of quasi-categories to a homotopy equivalence and a quasi-fibration to a Kan fibration.

If X is a simplicial set, then the contravariant functor $A \mapsto X^A$ is right adjoint to itself. If X is a quasi-category, then so is the simplicial set X^A . We shall denote by $X^{(A)}$ be the full simplicial subset of X^A whose vertices are the maps $f : A \rightarrow X$ such that $f(A) \subseteq J(X)$. The contravariant functor $A \mapsto J(X^A)$ is a subfunctor of the contravariant functor $A \mapsto X^A$.

PROPOSITION 1.17. [J2] *Let X be a quasi-category. Then the contravariant functors $A \mapsto J(X^A)$ and $A \mapsto X^{(A)}$ are mutually right adjoint. The contravariant functor $A \mapsto X^{(A)}$ takes a weak homotopy equivalence to an equivalence of quasi-categories.*

Consider the functor $k : \Delta \rightarrow \mathbf{S}$. defined by putting $k[n] = \Delta'[n]$ for every $n \geq 0$, where $\Delta'[n]$ denotes the (nerve of the) groupoid freely generated by the category $[n]$. We denote by $k^! : \mathbf{S} \rightarrow \mathbf{S}$ the functor defined by putting

$$k^!(X)_n = \mathbf{S}(\Delta'[n], X)$$

for every simplicial set X and for every $n \geq 0$. The functor $k^!$ has a left adjoint $k_! : \mathbf{S} \rightarrow \mathbf{S}$ which is the left Kan extension of the functor $k : \Delta \rightarrow \mathbf{S}$ along the Yoneda functor $y : \Delta \rightarrow \mathbf{S}$.

PROPOSITION 1.18. *For every $X \in \mathbf{S}$, we have $\tau_1 k_! X = \pi_1 X$.*

Proof: The result is obvious if $X = \Delta[n]$. The general case follows, since the functors $X \mapsto \tau_1 k_! X$ and $X \mapsto \pi_1 X$ are cocontinuous. ■

THEOREM 1.19. [J2] *The pair of adjoint functors*

$$k_! : \mathbf{S} \leftrightarrow \mathbf{S} : k^!$$

is Quillen pair $(\mathcal{C}_0, \mathcal{W}_0, \mathcal{F}_0) \leftrightarrow (\mathcal{C}_1, \mathcal{W}_1, \mathcal{F}_1)$.

The pair $(k_!, k^!)$ is actually a homotopy colocalisation in the sense of 7.16.

The inclusion $\Delta[n] \subseteq \Delta'[n]$ is natural in $[n] \in \Delta$. Hence it defines a natural transformation $y \rightarrow k$, where y is the Yoneda functor $\Delta \rightarrow \mathbf{S}$. The transformation has a unique extension $Id \rightarrow k_!$, where Id is the identity functor of \mathbf{S} . There is a corresponding adjoint transformation $k^! \rightarrow Id$. It is easy to verify that the map $X \rightarrow k_!(X)$ is monic and a weak homotopy equivalence for every $X \in \mathbf{S}$. Dually, the map $k^!(X) \rightarrow X$ is a trivial fibration for every Kan complex X . If X is a quasi-category, then the map $k^!(X) \rightarrow X$ induces a map $k^!(X) \rightarrow J(X)$.

PROPOSITION 1.20. **[J2]** *The natural map $k^!(X) \rightarrow J(X)$ is a trivial fibration for every quasi-category X .*

Let X be a quasi-category. Then the canonical map $X^{k_!(A)} \rightarrow X^{(A)}$ defined from the inclusion $A \subseteq k_!(A)$ is a categorical equivalence for every simplicial set A .

The following combinatorial result will be used in section 4.

For $n > 0$, the n -chain $I_n \subseteq \Delta[n]$ is defined to be the union of the edges $(i, i+1) \subseteq \Delta[n]$ for $0 \leq i \leq n-1$. Let us put $I_0 = 1$.

LEMMA 1.21. **[J2]** *The inclusion $I_n \subseteq \Delta[n]$ is mid anodyne.*

The following notion will be used in section 5.

Recall that a functor $u : A \rightarrow B$ is said to be *conservative* if the implication

$$u(f) \text{ invertible} \Rightarrow f \text{ invertible}$$

is true for every arrow $f \in A$.

DEFINITION 1.22. **[J2]** We shall say that a map of simplicial sets $u : A \rightarrow B$ is *conservative* if the functor $\tau_1(u) : \tau_1 A \rightarrow \tau_1 B$ is conservative.

THEOREM 1.23. **[J2]** *If X is a quasi-category and A is a simplicial set, then the projection*

$$X^A \rightarrow X^{A_0}$$

is conservative.

PROPOSITION 1.24. **[J2]** *Let $p : X \rightarrow Y$ be a conservative quasi-fibration between quasi-categories. If Y is a Kan complex, then X is a Kan complex and p is a Kan fibration.*

COROLLARY 1.25. **[J2]** *The base change in \mathbf{QCat} of a conservative quasi-fibration between quasi-categories is conservative.*

2. The vertical and horizontal Reedy model structures

In this section we introduce two model structures which play an important role in the paper. They are called the vertical and the horizontal model structures. Each is a Reedy model structure on the category of bisimplicial sets associated to a model structure on the category of simplicial sets. In the vertical model structure the acyclic maps are the column-wise weak homotopy equivalences but in the horizontal they are the row-wise weak categorical equivalences. The vertical model structure is associated to the classical model structure on the the category of simplicial sets while the horizontal is associated to the quasi-category model structure. We define a “total space” functor $t_! : \mathbf{S}^{(2)} \rightarrow \mathbf{S}$ and show that it is a left Quillen functor with respect to both model structures.

A *bisimplicial set* is a contravariant functor $\Delta \times \Delta \rightarrow \mathbf{Set}$. We shall denote the category of bisimplicial sets by $\mathbf{S}^{(2)}$. A *simplicial space* is a contravariant functor $\Delta \rightarrow \mathbf{S}$. We regard a simplicial space X as a bisimplicial set by putting $X_{mn} = (X_m)_n$ for every $m, n \geq 0$. Conversely, we regard a bisimplicial set X as a

simplicial space by putting $X_m = X_{m*}$ for every $m \geq 0$. The *box product* $A \square B$ of two simplicial sets A and B is the bisimplicial set obtained by putting

$$(A \square B)_{mn} = A_m \times B_n$$

for every $m, n \geq 0$. This defines a functor of two variables $\square : \mathbf{S} \times \mathbf{S} \rightarrow \mathbf{S}^{(2)}$. The functor is divisible on both sides. This means that the functor $A \square (-) : \mathbf{S} \rightarrow \mathbf{S}^{(2)}$ admits a right adjoint $A \setminus (-) : \mathbf{S}^{(2)} \rightarrow \mathbf{S}$ for every simplicial set A . If $X \in \mathbf{S}^{(2)}$, then a simplex $\Delta[n] \rightarrow A \setminus X$ is a map $A \square \Delta[n] \rightarrow X$. The simplicial set $\Delta[m] \setminus X$ is the m th column X_{m*} of X . Dually, the functor $(-) \square B : \mathbf{S} \rightarrow \mathbf{S}^{(2)}$ admits a right adjoint $(-)/B : \mathbf{S}^{(2)} \rightarrow \mathbf{S}$ for every simplicial set B . If $X \in \mathbf{S}^{(2)}$, then a simplex $\Delta[m] \rightarrow X/B$ is a map $\Delta[m] \square B \rightarrow X$. The simplicial set $X/\Delta[n]$ is the n th row X_{*n} of X . If $X \in \mathbf{S}^{(2)}$ and $A, B \in \mathbf{S}$, there is a bijection between the following three kinds of maps

$$A \square B \rightarrow X, \quad B \rightarrow A \setminus X, \quad A \rightarrow X/B.$$

Hence the contravariant functors $A \mapsto A \setminus X$ and $B \mapsto B \setminus X$ are mutually right adjoint.

If $u : A \rightarrow B$ and $v : S \rightarrow T$ are maps of simplicial sets we shall denote by $u \square' v$ the map

$$A \square T \sqcup_{A \square S} B \square S \rightarrow B \square T$$

obtained from the commutative square

$$\begin{array}{ccc} A \square S & \longrightarrow & B \square S \\ \downarrow & & \downarrow \\ A \square T & \longrightarrow & B \square T. \end{array}$$

This defines a functor of two variables

$$\square' : \mathbf{S}^I \times \mathbf{S}^I \rightarrow (\mathbf{S}^{(2)})^I,$$

where \mathcal{E}^I denotes the category of arrows of a category \mathcal{E} . The functor \square' is divisible on both sides. If $u : A \rightarrow B$ is map in \mathbf{S} and $f : X \rightarrow Y$ is a map in $\mathbf{S}^{(2)}$, we denote by $\langle u \setminus f \rangle$ the map

$$B \setminus X \rightarrow B \setminus Y \times_{A \setminus Y} A \setminus X$$

obtained from the square

$$\begin{array}{ccc} B \setminus X & \longrightarrow & A \setminus X \\ \downarrow & & \downarrow \\ B \setminus Y & \longrightarrow & A \setminus Y. \end{array}$$

The functor $f \mapsto \langle u \setminus f \rangle$ is right adjoint to the functor $v \mapsto u \square' v$. Dually, if $v : S \rightarrow T$ is map in \mathbf{S} , we shall denote by $\langle f/v \rangle$ the map

$$X/T \rightarrow Y/T \times_{Y/S} X/S$$

obtained from the commutative square

$$\begin{array}{ccc} X/T & \longrightarrow & X/S \\ \downarrow & & \downarrow \\ Y/T & \longrightarrow & Y/S. \end{array}$$

The functor $f \mapsto \langle f \setminus v \rangle$ is right adjoint to the functor $u \mapsto u \square' v$.

PROPOSITION 2.1. *For any triple of maps $u \in \mathbf{S}$, $v \in \mathbf{S}$ and $f \in \mathbf{S}^{(2)}$ we have*

$$(u \square' v) \pitchfork f \iff u \pitchfork \langle f/v \rangle \iff v \pitchfork \langle u \setminus f \rangle.$$

This follows from 7.6.

Let us denote by δ_n the inclusion $\partial\Delta[n] \subset \Delta[n]$. Recall that a map of bisimplicial sets is said to be a *trivial fibration* if it has the right lifting property with respect to every monorphism.

PROPOSITION 2.2. [JT2] *The class of monomorphisms in the category $\mathbf{S}^{(2)}$ is generated as a saturated class by the set of inclusions*

$$\delta_m \square' \delta_n : (\partial\Delta[m] \square \Delta[n]) \cup (\Delta[m] \square \partial\Delta[n]) \subset \Delta[m] \square \Delta[n], \quad \text{for } m, n \geq 0.$$

A map of bisimplicial sets is a trivial fibration iff it has the right lifting property with respect to the map $\delta_m \square' \delta_n$ for every $m, n \geq 0$.

PROPOSITION 2.3. *A map of bisimplicial sets $f : X \rightarrow Y$ is a trivial fibration iff the following equivalent conditions are satisfied:*

- the map $\langle \delta_m \setminus f \rangle$ is a trivial fibration for every $m \geq 0$;
- the map $\langle u \setminus f \rangle$ is a trivial fibration for every monomorphism $u \in \mathbf{S}$;
- the map $\langle f/\delta_n \rangle$ is a trivial fibration for every $n \geq 0$;
- the map $\langle f/v \rangle$ is a trivial fibration for every monomorphism $v \in \mathbf{S}$.

Proof: Let us show that f is a trivial fibration iff the map $\langle \delta_m \setminus f \rangle$ is a trivial fibration for every $m \geq 0$. But the map $\langle \delta_m \setminus f \rangle$ is a trivial fibration iff we have $\delta_n \pitchfork \langle \delta_m \setminus f \rangle$ for every $n \geq 0$ by 1.2. But the condition $\delta_n \pitchfork \langle \delta_m \setminus f \rangle$ is equivalent to the condition $(\delta_m \square' \delta_n) \pitchfork f$ by 2.1. Hence the result follows from 2.2. Let us now show that first condition implies the fourth. But the map $\langle f/v \rangle$ is a trivial fibration iff we have $\delta_m \pitchfork \langle f/v \rangle$ for every $m \geq 0$ by 1.2. But the condition $\delta_m \pitchfork \langle f/v \rangle$ is equivalent to the condition $v \pitchfork \langle \delta_m \setminus f \rangle$ by 2.1. This proves the result since $\langle \delta_m \setminus f \rangle$ is a trivial fibration by hypothesis and since v is monic. The rest of the equivalences are proved similarly. ■

We shall use the following simplicial enrichment of the category of bisimplicial sets $\mathbf{S}^{(2)}$. The functor $i_2 : \Delta \rightarrow \Delta \times \Delta$ defined by putting $i_2([n]) = ([0], [n])$ is right adjoint to the second projection $p_2 : \Delta \times \Delta \rightarrow \Delta$. Hence the functor $i_2^* : \mathbf{S}^{(2)} \rightarrow \mathbf{S}$ is right adjoint to the functor p_2^* . If X is a bisimplicial set, then $i_2^*(X)$ is the first column of X . If A is a simplicial set, then $p_2^*(A) = 1 \square A$. For any pair of bisimplicial sets X and Y , let us put

$$Hom_2(X, Y) = i_2^*(Y^X).$$

This defines an enrichment of the category $\mathbf{S}^{(2)}$ over the category \mathbf{S} .

PROPOSITION 2.4. *The enriched category $(\mathbf{S}^{(2)}, Hom_2)$ admits tensor and cotensor products. The tensor product of a simplicial space X by a simplicial set A is the simplicial space $X \times_2 A = X \times p_2^*(A)$ and the cotensor product is the simplicial space $X p_2^*(A) = X 1 \square A$.*

We say that a map of bisimplicial sets $f : X \rightarrow Y$ is a *column-wise weak homotopy equivalence* if the map $\Delta[m] \setminus f = f_m : X_m \rightarrow Y_m$ is a weak homotopy equivalence for every $m \geq 0$. We say that $f : X \rightarrow Y$ is a *vertical fibration*, or

a *v-fibration*, if the map $\langle \delta_m \setminus f \rangle$ is a Kan fibration for every $m \geq 0$ (a notion of horizontal fibration will be considered later). We say that a bisimplicial set X is *v-fibrant* if the map $X \rightarrow 1$ is a *v-fibration*.

Let h_n^k denotes the inclusion $\Lambda^k[n] \subset \Delta[n]$.

PROPOSITION 2.5. *A map of bisimplicial sets $f : X \rightarrow Y$ is a v-fibration iff it satisfies the following equivalent conditions:*

- the map $\langle \delta_m \setminus f \rangle$ is a Kan fibration for every $m \geq 0$;
- the map $\langle u \setminus f \rangle$ is a Kan fibration for every monomorphism u ;
- the map $\langle f/h_n^k \rangle$ is a trivial fibration for every $n > 0$ and $0 \leq k \leq n$;
- the map $\langle f/v \rangle$ is a trivial fibration for every anodyne map $v \in \mathcal{S}$.

Proof: The equivalences (i) \Leftrightarrow (ii) \Leftrightarrow (iv) follow from 7.33. The implication (iv) \Rightarrow (iii) is obvious. Let us prove the implication (iii) \Rightarrow (ii). Let us show that the map $\langle \delta_m \setminus f \rangle$ is a Kan fibration for every $m \geq 0$. For this, it suffices to show that we have $h_n^k \pitchfork \langle \delta_m \setminus f \rangle$ for every $n > 0$ and $0 \leq k \leq n$. But the condition $h_n^k \pitchfork \langle \delta_m \setminus f \rangle$ is equivalent to the condition $\delta_m \pitchfork \langle f/h_n^k \rangle$ by 2.1. But we have $\delta_m \pitchfork \langle f/h_n^k \rangle$ since $\langle f/h_n^k \rangle$ is a trivial fibration by assumption. The implication (iii) \Rightarrow (ii) is proved. ■

The following theorem describes the *vertical model structure* on $\mathbf{S}^{(2)}$.

THEOREM 2.6. [Ree] *The category $(\mathbf{S}^{(2)}, Hom_2)$ admits a simplicial model structure $(\mathcal{C}_0^v, \mathcal{W}_0^v, \mathcal{F}_0^v)$ in which the cofibrations are the monomorphisms, the weak equivalence are the column-wise weak homotopy equivalences and the fibrations are the vertical fibrations. The model structure is proper and cartesian closed. The acyclic fibrations are the trivial fibrations.*

Proof: The model structure $(\mathcal{C}_0^v, \mathcal{W}_0^v, \mathcal{F}_0^v)$ is the Reedy model structure $(\mathcal{C}'_0, \mathcal{W}'_0, \mathcal{F}_0)$ associated to the classical model structure $(\mathcal{C}_0, \mathcal{W}_0, \mathcal{F}_0)$ of 1.1 on \mathbf{S} . The existence of the model structure follows directly from Theorem 7.35 except for the identification of the Reedy cofibrations with the monomorphisms. For this, it suffices to show that the acyclic *v-fibrations* are the trivial fibrations. But a map f is an acyclic *v-fibration* iff the map $\langle \delta_m \setminus f \rangle$ is trivial fibration for every $m \geq 0$ by 7.35 and by 1.1. It then follows by 2.3 that f is an acyclic *v-fibration* iff it is a trivial fibration. This completes the proof that the Reedy cofibrations are the monomorphisms. The model structure is left proper by 7.13 since every object is cofibrant. Let us show that it is right proper. If $f : X \rightarrow Y$ is a *v-fibration*, then the map $f_m : X_m \rightarrow Y_m$ is a Kan fibration for every $m \geq 0$ by 7.34. Hence the base change of a column-wise weak homotopy equivalence along a *v-fibration* is a column-wise weak homotopy equivalence since the model structure $(\mathcal{C}_0, \mathcal{W}_0, \mathcal{F}_0)$ is proper. Let us show that the model structure $(\mathcal{C}'_0, \mathcal{W}'_0, \mathcal{F}'_0)$ is cartesian. See 7.29 for this notion. By 7.27, it suffices to show that the functor $A \times (-) : \mathbf{S}^{(2)} \rightarrow \mathbf{S}^{(2)}$ takes an acyclic map to an acyclic map. But this is clear since since the model structure $(\mathcal{C}_0, \mathcal{W}_0, \mathcal{F}_0)$ is cartesian closed. Let us show that the model structure is simplicial. The internal hom functor $(X, Y) \mapsto Y^X$ is a right Quillen functor of two variables (contravariant in the first) since the model structure $(\mathcal{C}'_0, \mathcal{W}'_0, \mathcal{F}'_0)$ is cartesian closed. Hence the composite $(X, Y) \mapsto Hom_2(X, Y) = i_2^*(Y^X)$ is also a right Quillen functor of two variables, since the functor $i_2^* : \mathbf{S}^{(2)} \rightarrow \mathbf{S}$ is a right Quillen functor by 7.37. ■

Let X be a bisimplicial set. From the map $[n] \rightarrow [0]$ we obtain a canonical map between the rows, $X_{\star 0} \rightarrow X_{\star n}$

DEFINITION 2.7. We shall say that a bisimplicial set X *categorically constant* if the canonical map $X_{\star 0} \rightarrow X_{\star n}$ is a weak categorical equivalence for every $n \geq 0$.

PROPOSITION 2.8. *A v -fibrant bisimplicial set is categorically constant.*

Proof: If i denotes the inclusion $\Delta[0] \subseteq \Delta[n]$, then the map $X/i : X/\Delta[n] \rightarrow X/\Delta[0]$ is a trivial fibration by 2.5, since i is a monic weak homotopy equivalence. But we have $(X/i)(X/t) = id$ since we have $ti = id$, where t is the map $\Delta[n] \rightarrow \Delta[0]$. This shows by three-for-two that the map X/t is a weak categorical equivalence. ■

We call a map of of bisimplicial sets $f : X \rightarrow Y$ a *row-wise weak categorical equivalence* if the map $f_{\star n} : X_{\star n} \rightarrow Y_{\star n}$ is a weak categorical equivalence for every $n \geq 0$.

PROPOSITION 2.9. *A map between v -fibrant simplicial sets $f : X \rightarrow Y$ is a row-wise weak categorical equivalence iff it induces a weak categorical equivalence between the first rows.*

Proof: If $f_{\star 0} : X_{\star 0} \rightarrow Y_{\star 0}$ is a weak categorical equivalence, let us show that the map $f_{\star n} : X_{\star n} \rightarrow Y_{\star n}$ is a weak categorical equivalence for every $n \geq 0$. Consider the commutative square

$$\begin{array}{ccc} X_{\star 0} & \xrightarrow{f_{\star 0}} & Y_{\star 0} \\ \downarrow & & \downarrow \\ X_{\star n} & \xrightarrow{f_{\star n}} & Y_{\star n}, \end{array}$$

where the vertical maps are obtained from the map $\Delta[n] \rightarrow \Delta[0]$. The vertical maps are weak categorical equivalence by 2.8. It follows by three-for-two that the map $f_{\star n}$ is a weak categorical equivalence. ■

We shall say that $f : X \rightarrow Y$ is a *horizontal fibration* or an *h-fibration* if the map $\langle f/\delta_n \rangle$ is a quasi-fibration for every $n \geq 0$. We shall say that a bisimplicial set X is *h-fibrant* if the map $X \rightarrow 1$ is an h-fibration.

PROPOSITION 2.10. *The category $\mathbf{S}^{(2)}$ admits a model structure $(\mathcal{C}_1^h, \mathcal{W}_1^h, \mathcal{F}_1^h)$ in which a cofibration is a monomorphism and a weak equivalence is a row-wise weak categorical equivalence. A fibration is an h-fibration. The model structure is left proper and cartesian closed. The acyclic fibrations are the trivial fibrations.*

We call the model structure, the *horizontal model structure* on $\mathbf{S}^{(2)}$. For the notion of trivial fibration, see Definition 7.2.

Proof: The model structure $(\mathcal{C}_1^h, \mathcal{W}_1^h, \mathcal{F}_1^h)$ is the Reedy model structure $(\mathcal{C}'_1, \mathcal{W}'_1, \mathcal{F}'_1)$ associated to the model structure $(\mathcal{C}_1, \mathcal{W}_1, \mathcal{F}_1)$ of Theorem 1.9. It is similar to the vertical model structure of Theorem 2.6, except that the weak equivalences are now defined row-wise and by using weak categorical equivalences. The existence of the model structure follows from Theorem 7.35. Let us show that \mathcal{C}'_1 is the class of monomorphisms. For this, it suffices to show that the acyclic h-fibrations are the trivial fibrations. But a map f is an acyclic h-fibration iff the map $\langle f/\delta_n \rangle$ is trivial fibration for every $m \geq 0$ by 7.35 and by 1.9. It then follows by 2.3 that f is an

acyclic h-fibration iff it is a trivial fibration. We have proved that \mathcal{C}'_1 is the class of monomorphisms. It follows that every object is cofibrant. Hence the model structure is left proper by 7.13. It remains to show that the model structure is cartesian closed. By 7.27, it suffices to show that the functor $A \times (-) : \mathbf{S}^{(2)} \rightarrow \mathbf{S}^{(2)}$ takes an acyclic map to an acyclic map. But this is clear since since the model structure $(\mathcal{C}_1, \mathcal{W}_1, \mathcal{F}_1)$ is cartesian closed. ■

Recall the pair of adjoint functors $k_! : \mathbf{S} \leftrightarrow \mathbf{S} : k^!$ of Proposition 1.19. By definition, we have $k_!(\Delta[n]) = \Delta'[n]$ for every $n \geq 0$, where $\Delta'[n]$ denotes the (nerve of the) groupoid freely generated by the category $[n]$. Consider the functor $t : \Delta \times \Delta \rightarrow \mathbf{S}$ defined by putting $t([m], [n]) = \Delta[m] \times \Delta'[n]$ for every $m, n \geq 0$ and let $t_! : \mathbf{S}^{(2)} \rightarrow \mathbf{S}$ be the left Kan extension of the functor t along the Yoneda functor $\Delta^2 \subset \mathbf{S}^{(2)}$. By definition, we have

$$t_!(\Delta[m] \square \Delta[n]) = \Delta[m] \times \Delta'[n].$$

The functor $t_!$ has a right adjoint $t^! : \mathbf{S} \rightarrow \mathbf{S}^{(2)}$. If $X \in \mathbf{S}$, then

$$t^!(X)_{mn} = \mathbf{S}(\Delta[m] \times \Delta'[n], X)$$

for every $m, n \geq 0$.

LEMMA 2.11. *There are natural isomorphisms*

$$t_!(A \square B) = A \times k_!(B), \quad A \setminus t^!(X) = k^!(X^A) \quad \text{and} \quad t^!(X)/B = X^{k_!(B)}$$

for $A, B \in \mathbf{S}$ and $X \in \mathbf{S}^{(2)}$.

Proof: The functor $(A, B) \mapsto t_!(A \square B)$ is cocontinuous in each variable and it extends the functor $([m], [n]) \mapsto \Delta[m] \times \Delta'[n]$ along the Yoneda functor $\Delta \times \Delta \rightarrow \mathbf{S}^{(2)}$. Similarly for the functor $(A, B) \mapsto A \times k_!(B)$, since $k_!(\Delta[n]) = \Delta'[n]$. It follows that there is a natural isomorphism $t_!(A \square B) = A \times k_!(B)$. The first statement of the proposition is proved. Let us prove the second. The functor $X \mapsto A \setminus t^!(X)$ is right adjoint to the functor $B \mapsto t_!(A \square B)$, since a composite of right adjoints is right adjoint to the composite in reverse order. Similarly, the functor $X \mapsto k^!(X^A)$ is right adjoint to the functor $B \mapsto A \times k_!(B)$. This proves the result by uniqueness of adjoint since $t_!(A \square B) = A \times k_!(B)$. Let us prove the third. The functor $X \mapsto t^!(X)/B$ is right adjoint to the functor $A \mapsto t_!(A \square B)$, since a composite of right adjoints is right adjoint to the composite in reverse order. Similarly, the functor $X \mapsto X^{k_!(B)}$ is right adjoint to the functor $A \mapsto A \times k_!(B)$. This proves the result by uniqueness of adjoints since $t_!(A \square B) = A \times k_!(B)$. ■

THEOREM 2.12. *The adjoint pair of functors*

$$t_! : \mathbf{S}^{(2)} \leftrightarrow \mathbf{S} : t^!$$

is a Quillen pair between the horizontal (resp. vertical) model structure on $\mathbf{S}^{(2)}$ and the model structure for quasi-categories on \mathbf{S} .

Proof: Let us first show that $(t_!, t^!)$ is a Quillen pair between the horizontal model structure and the model structure for quasi-categories. We shall use the criteria of 7.15. We first verify that $t_!$ takes a cofibration to a cofibration. Let \mathcal{A} be the class of maps $u \in \mathbf{S}^{(2)}$ such that $t_!(u)$ is monic. The class \mathcal{A} is saturated since the functor t is cocontinuous. Let us show that the map $\delta_m \square' \delta_n$ belongs to \mathcal{A} for every $m, n \geq 0$. We have $t_!(\delta_m \square' \delta_n) = \delta_m \times' k_!(\delta_n)$. by Lemma 2.11. But $k_!(\delta_n)$ is monic since the functor $k_!$ takes a monomorphism to a monomorphism by 1.19. Hence

the map $\delta_m \times' k_1(\delta_n)$ is monic and this shows that the map $\delta_m \square' \delta_n$ belongs to \mathcal{A} . It follows by 2.2 that every monomorphism belongs to \mathcal{A} . We have proved that t_1 takes a cofibration to a cofibration. Let us now show that t^1 takes a fibration to a fibration. For this we have to show that if $f : X \rightarrow Y$ is a quasi-fibration then the map $t^1(f) : t^1(X) \rightarrow t^1(Y)$ is an h-fibration. For this it suffices to show that the map $\langle t^1(f)/u \rangle$ is a quasi-fibration for every monomorphism of simplicial sets $u : A \rightarrow B$. But the square

$$\begin{array}{ccc} t^1(X)/B & \longrightarrow & t^1(X)/A \\ \downarrow & & \downarrow \\ t^1(Y)/B & \longrightarrow & t^1(Y)/A \end{array}$$

is isomorphic to the square

$$\begin{array}{ccc} X^{k_1(B)} & \longrightarrow & X^{k_1(A)} \\ \downarrow & & \downarrow \\ Y^{k_1(B)} & \longrightarrow & Y^{k_1(A)} \end{array}$$

by Lemma 2.11. Hence the map $\langle t^1(f)/u \rangle$ is isomorphic to the map $\langle k_1(u), f \rangle$. But the map $k_1(u)$ is monic. Hence the $\langle k_1(u), f \rangle$ is a quasi-fibration by 1.9 since f is a quasi-fibration. We have proved (t_1, t^1) is a Quillen pair between the horizontal model structure and the model structure for quasi-categories. Let us now show that it is a Quillen pair between the vertical model structure and the model structure for quasi-categories. We saw that t_1 takes a cofibration to a cofibration. Let us now show that t^1 takes a fibration to a fibration. For this we have to show that if $f : X \rightarrow Y$ is a quasi-fibration then the map $t^1(f) : t^1(X) \rightarrow t^1(Y)$ is a v-fibration. For this it suffices to show that the map $\langle u \setminus t^1(X) \rangle$ is a Kan fibration for every monomorphism of simplicial sets $u : A \rightarrow B$. But the square

$$\begin{array}{ccc} B \setminus t^1(X) & \longrightarrow & A \setminus t^1(X) \\ \downarrow & & \downarrow \\ B \setminus t^1(Y) & \longrightarrow & A \setminus t^1(Y) \end{array}$$

is isomorphic to the square

$$\begin{array}{ccc} k^1(X^B) & \longrightarrow & k^1(X^A) \\ \downarrow & & \downarrow \\ k^1(Y^B) & \longrightarrow & k^1(Y^A) \end{array}$$

by Lemma 2.11. Hence the map $\langle u \setminus t^1(X) \rangle$ is isomorphic to the map $k^1 \langle u, f \rangle$. But $\langle u, f \rangle$ is a quasi-fibration by Theorem 1.9. Thus, $k^1 \langle u, f \rangle$ is a Kan fibration by 1.19. We have proved that (t_1, t^1) is a Quillen pair between the vertical model structure and the model structure for quasi-categories. ■

3. Segal spaces

Segal spaces were introduced by Rezk in [Rez]. They are the fibrant objects of a model structure on the category of bisimplicial sets. The model structure is a Bousfield localisation of the (vertical) Reedy model structure introduced in the previous section.

Let $I_n \subseteq \Delta[n]$ be the n -chain. For any simplicial space X we have a canonical bijection

$$I_n \setminus X = X_1 \times_{\partial_0, \partial_1} X_1 \times \cdots \times_{\partial_0, \partial_1} X_1,$$

where the successive fiber products are calculated by using the face maps $\partial_0, \partial_1 : X_1 \rightarrow X_0$.

DEFINITION 3.1. [Rez] We shall say that a simplicial space X satisfies the *Segal condition* if the map

$$i_n \setminus X : \Delta[n] \setminus X \longrightarrow I_n \setminus X$$

obtained from the inclusion $i_n : I_n \subseteq \Delta[n]$ is a weak homotopy equivalence for every $n \geq 2$. A *Segal space* is a v -fibrant simplicial space which satisfies the Segal condition,

We shall say that a map of simplicial spaces $u : A \rightarrow B$ is a *Segal weak equivalence* if the map

$$Hom_2(u, X) : Hom_2(B, X) \rightarrow Hom_2(A, X)$$

is a weak homotopy equivalence for every Segal space X .

THEOREM 3.2. [Rez] *The category $(\mathbf{S}^{(2)}, Hom_2)$ admits a simplicial model structure $(\mathcal{C}_S, \mathcal{W}_S, \mathcal{F}_S)$ in which \mathcal{C}_S is the class of monomorphisms and \mathcal{W}_S is the class of Segal weak equivalences. The model structure is left proper and cartesian closed. The acyclic fibrations are the trivial fibrations. The fibrant objects are the Segal spaces.*

We shall say that $(\mathcal{C}_S, \mathcal{W}_S, \mathcal{F}_S)$ is the *model structure for Segal spaces*. The model structure is a Bousfield localisation of the vertical model structure $(\mathcal{C}_0^v, \mathcal{W}_0^v, \mathcal{F}_0^v)$ of Theorem 2.6. For the notion of Bousfield localisation, see Definition 7.20. In particular, a map between Segal spaces is acyclic (resp. a fibration) iff it is a column-wise weak homotopy equivalence (resp. a v -fibration).

Recall the total space functor $t_! : \mathbf{S}^{(2)} \rightarrow \mathbf{S}$ of Theorem 2.12.

THEOREM 3.3. *The adjoint pair of functors*

$$t_! : \mathbf{S}^{(2)} \leftrightarrow \mathbf{S} : t^!$$

is a Quillen pair between the model category for Segal spaces and the model category for quasi-categories.

Proof: We saw in 2.12 that the pair of adjoint functors $(t_!, t^!)$ is a Quillen pair between the vertical model category and the model category for quasi-categories. Hence it suffices to show by 7.15. that the functor $t^!$ takes a quasi-category to a Segal space. If X is a quasi-category then $t^!(X)$ is v -fibrant by 2.12. Let us show that $t^!(X)$ satisfies the Segal condition. For this, it suffices to show that the map $i_n \setminus t^!(X)$ is a trivial fibration for every $n \geq 0$, where i_n denotes the inclusion

$I_n \subseteq \Delta[n]$. But the map $i_n \setminus t^!(X)$ is isomorphic to the map $k^!(X^{i_n})$ by Lemma 2.11. The map i_n is weak categorical equivalence by 1.11 since it is mid anodyne by 1.21. Hence the map X^{i_n} is a trivial fibration by 1.9 since i_n is monic. It follows that the map $k^!(X^{i_n})$ is a trivial fibration since k^* is a right Quillen functor by 1.19. We have proved that the map $i_n \setminus t^!(X)$ is a trivial fibration. This shows that $t^!(X)$ is a Segal space. ■

PROPOSITION 3.4. *Let X be a v -fibrant simplicial space. Then the following conditions are equivalent:*

- (i) X is a Segal space
- (ii) The map $h_n^k \setminus X$ is a trivial fibration for every $0 < k < n$;
- (iii) the map $u \setminus X$ is a trivial fibration for every mid anodyne map $u \in \mathbf{S}$;
- (iv) the map X/δ_n is a mid fibration for every $n \geq 0$;
- (v) the map X/v is a mid fibration for every monomorphism $v \in \mathbf{S}$

The notion of mid anodyne map was defined in 1.10. The proof depends on the following lemma.

We shall say that a class of monomorphisms $\mathcal{A} \subseteq \mathbf{S}$ has the *right cancellation property* if the implication

$$vu \in \mathcal{A} \quad \text{and} \quad u \in \mathcal{A} \quad \Rightarrow \quad v \in \mathcal{A}$$

is true for any pair of monomorphisms $u : A \rightarrow B$ and $v : B \rightarrow C$.

Let $I_n \subseteq \Delta[n]$ be the n -chain.

LEMMA 3.5. *Let $\mathcal{A} \subseteq \mathbf{S}$ be a saturated class of monomorphisms having the right cancellation property. If the inclusion $I_n \subseteq \Delta[n]$ belongs to \mathcal{A} for every $n \geq 2$, then every mid anodyne map belongs to \mathcal{A} .*

Proof: We shall first prove that the inclusion $I_n \subseteq \partial_0 \Delta[n] \cup \partial_n \Delta[n]$ belongs to \mathcal{A} for every $n > 1$. This is obvious if $n = 2$ since we have $I_n = \partial_0 \Delta[n] \cup \partial_n \Delta[n]$ in this case. Let us suppose $n > 2$. It suffices to show that each inclusion

$$I_n \xrightarrow{i_n} I_n \cup \partial_n \Delta[n] \xrightarrow{j_n} \partial_0 \Delta[n] \cup \partial_n \Delta[n]$$

belongs to \mathcal{A} , since \mathcal{A} is closed under composition. The square of inclusions

$$\begin{array}{ccc} I_{n-1} & \longrightarrow & I_n \\ \downarrow & & \downarrow i_n \\ \partial_n \Delta[n] & \longrightarrow & \partial_n \Delta[n] \cup I_n \end{array} .$$

is a pushout since $\partial_n \Delta[n] \cap I_n = I_{n-1}$. Thus, $i_n \in \mathcal{A}$ since \mathcal{A} is closed under cobase change and since the inclusion $I_{n-1} \subseteq \Delta[n-1] = \partial_n \Delta[n]$ belongs to \mathcal{A} . It remains to show that the inclusion j_n belongs to \mathcal{A} . If $d_0 : \Delta[n-1] \rightarrow \Delta[n]$, then we have $d_0^{-1}(I_n) = I_{n-1}$ and $d_0^{-1}(\partial_n \Delta[n]) = \partial_{n-1} \Delta[n-1]$. It follows from this observation that the following square is a pushout,

$$\begin{array}{ccc} I_{n-1} \cup \partial_{n-1} \Delta[n-1] & \longrightarrow & I_n \cup \partial_n \Delta[n] \\ \downarrow k_{n-1} & & \downarrow j_n \\ \Delta[n-1] & \xrightarrow{d'_0} & \partial_0 \Delta[n] \cup \partial_n \Delta[n], \end{array}$$

where k_{n-1} is the inclusion and where d'_0 is induced by d_0 . Let us show that $k_{n-1} \in \mathcal{A}$. The composite

$$I_{n-1} \xrightarrow{i_{n-1}} I_{n-1} \cup \partial_{n-1}\Delta[n-1] \xrightarrow{k_{n-1}} \Delta[n-1]$$

belongs to \mathcal{A} by assumption. We have $i_{n-1} \in \mathcal{A}$ by the same argument as above since the inclusion $I_{n-2} \subset \Delta[n-2]$ belongs to \mathcal{A} . It follows by the right cancellation property of the class \mathcal{A} that k_{n-1} belongs to \mathcal{A} . Thus, $j_n \in \mathcal{A}$ since the class \mathcal{A} is closed under cobase change. This completes the proof that the inclusion $I_n \subset \partial_0\Delta[n] \cup \partial_n\Delta[n]$ belongs to \mathcal{A} for $n > 2$, hence also for $n > 1$. We can now prove the lemma. For this it suffices to show that the inclusion $\Lambda^k[n] \subset \Delta[n]$ belongs to the class \mathcal{A} for every $0 < k < n$, since the class \mathcal{A} is saturated. This is obvious if $n = 2$ since $\Lambda^1[2] = I_2$. We shall suppose $n > 2$. By the cancellation property of the class \mathcal{A} , it suffices to show that the inclusion $I_n \subseteq \Lambda^k[n]$ belongs to \mathcal{A} , since the inclusion $I_n \subseteq \Delta[n]$ belongs to \mathcal{A} . If S is a subset of $[n]$, let us put

$$\Lambda^S[n] = \bigcup_{i \notin S} \partial_i \Delta[n].$$

We shall prove that if $n > 2$ and S is a non-empty subset of the interval $[1, n-1]$, then the inclusion $I_n \subset \Lambda^S[n]$ belongs to \mathcal{A} . We argue by induction on $n > 2$ and $s = n - \text{Card}(S) > 0$. If $s = 1$, then $S = [1, n-1]$ and $\Lambda^S[n] = \partial_0\Delta[n] \cup \partial_n\Delta[n]$. The result was proved above in this case. If $s > 1$ let us choose an element $b \in [1, n-1] \setminus S$ and put $T = S \cup \{b\}$. The inclusion $I_n \subset \Lambda^T[n]$ belongs to \mathcal{A} by the induction hypothesis since $n - \text{Card}(T) < s$. Let us show that the inclusion $\Lambda^T[n] \subset \Lambda^S[n]$ belongs to \mathcal{A} . The square

$$\begin{array}{ccc} \partial_b \Delta[n] \cap \Lambda^T[n] & \longrightarrow & \Lambda^T[n] \\ \downarrow & & \downarrow \\ \partial_b \Delta[n] & \longrightarrow & \Lambda^S[n] \end{array}$$

is a pushout since $\Lambda^S[n] = \partial_b \Delta[n] \cup \Lambda^T[n]$. Hence it suffices to show that the inclusion $\partial_b \Delta[n] \cap \Lambda^T[n] \subset \partial_b \Delta[n]$ belongs to \mathcal{A} . Let $U \subseteq [n-1]$ be the inverse image of the subset T by the map $d_b : [n-1] \rightarrow [n]$. The inclusion $\partial_b \Delta[n] \cap \Lambda^T[n] \subset \partial_b \Delta[n]$ is isomorphic to the inclusion $\Lambda^U[n-1] \subset \Delta[n-1]$. Hence it suffices to show that the latter belongs to \mathcal{A} . The subset U is non-empty since it is in bijection with S . Moreover, $U \subseteq [1, n-2]$ since $S \subseteq [1, n-1]$ and $0 < b < n$. Hence the inclusion $I_{n-1} \subset \Lambda^U[n-1]$ belongs to \mathcal{A} by the induction hypothesis on n . It follows that the inclusion $\Lambda^U[n-1] \subset \Delta[n-1]$ belongs to \mathcal{A} by the cancellation property of the class \mathcal{A} . Hence the inclusion $\Lambda^T[n] \subset \Lambda^S[n]$ belongs to \mathcal{A} , since \mathcal{A} is closed under cobase change. It then follows by composing that the inclusion $I_n \subset \Lambda^S[n]$ belongs to \mathcal{A} , since the inclusion $I_n \subset \Lambda^T[n]$ belongs to \mathcal{A} by the induction hypothesis. We have proved that if S is a non-empty subset of the interval $[1, n-1]$, then the inclusion $I_n \subset \Lambda^S[n]$ belongs to \mathcal{A} . In particular, this shows that the inclusion $I_n \subseteq \Lambda^k[n]$ belongs to \mathcal{A} . ■

Proof of proposition 3.4: Let us prove the implication (i) \Rightarrow (iii). Let \mathcal{A} be the class of monomorphisms $u \in \mathbf{S}$ such that the map $u \setminus X$ is a weak homotopy equivalence. It is obvious from this definition that \mathcal{A} has the right cancellation property. Let us verify that \mathcal{A} is saturated. A monomorphism $u \in \mathbf{S}$ belongs to

\mathcal{A} iff the map $u \setminus X$ is a trivial fibration since it is a Kan fibration by 2.5. But $u \setminus X$ is a trivial fibration iff we have $\delta_n \pitchfork (u \setminus X)$ for every $n \geq 0$ by 1.2. But the condition $\delta_n \pitchfork (u \setminus X)$ is equivalent to the condition $u \pitchfork (X/\delta_n)$ by 2.1. Thus, a monomorphism u belongs to \mathcal{A} iff we have $u \pitchfork (X/\delta_n)$ for every $n \geq 0$. This shows that the class \mathcal{A} is saturated. If X is a Segal space then the inclusion $I_n \subseteq \Delta[n]$ belongs to \mathcal{A} for every $n \geq 2$. It then follows from 3.5 that every mid anodyne map belongs to \mathcal{A} . The implication (i) \Rightarrow (iii) is proved. The converse follows from the fact that the inclusion $I_n \subseteq \Delta[n]$ is mid anodyne by 1.21. The implication (iii) \Rightarrow (ii) is obvious. Let us prove the implication (ii) \Rightarrow (v). If $v \in \mathbf{S}$ is monic, let us show that X/v is a mid fibration. But the condition $h_n^k \pitchfork (X/v)$ is equivalent to the condition $v \pitchfork (h_n^k \setminus X)$ by 2.1. This shows that we have $h_n^k \pitchfork (X/v)$ for every $0 < k < n$, since the map $h_n^k \setminus X$ is a trivial fibration in this case. The implication (ii) \Rightarrow (v) is proved. The implication (v) \Rightarrow (iv) is obvious. The implication (iv) \Rightarrow (iii) follows from 7.33. ■

COROLLARY 3.6. *If X is a Segal space, then the simplicial set X/A is a quasi-category for any simplicial set A . In particular, every row of X is a quasi-category.*

Proof: If i_A denotes the inclusion $\emptyset \subset A$, then the map $X/i_A : X/A \rightarrow X/\emptyset$ is a mid fibration by (v). This shows that X/A is a quasi-category since $X/\emptyset = 1$. In particular, the n th row $X_{\star n} = X/\Delta[n]$ is a quasi-category. ■

LEMMA 3.7. *Let $\mathcal{A} \subseteq \mathbf{S}$ be a saturated class of monomorphisms having the right cancellation property. If every face map $d_i : \Delta[n - 1] \subset \Delta[n]$ belongs to \mathcal{A} , then every anodyne map belongs to \mathcal{A} .*

See 1.3 for the notion of anodyne map.

Proof: It suffices to show that every horn $h_n^k : \Lambda^k[n] \subset \Delta[n]$ belongs to \mathcal{A} , since \mathcal{A} is saturated. More generally, if S is a proper non-empty subset of $[n]$, let us put

$$\Lambda^S[n] = \bigcup_{i \notin S} \partial_i \Delta[n].$$

We shall prove by induction on $n \geq 1$ that the inclusion $\Lambda^S[n] \subset \Delta[n]$ belongs to \mathcal{A} . The result is clear if $n = 1$, since $h_1^0 = d_1$ and $h_1^1 = d_0$. Let us suppose $n > 1$. It suffices to show that the inclusion $\partial_i \Delta[n] \subset \Lambda^S[n]$ belongs to \mathcal{A} for some $i \notin S$, since the class \mathcal{A} has the right cancellation property. We have $\Lambda^S[n] = \partial_i \Delta[n]$ if $S = [n] \setminus \{i\}$. Hence it suffices to show that the inclusion $\Lambda^T[n] \subseteq \Lambda^S[n]$ belongs to \mathcal{A} for any pair of proper non-empty subsets $S \subset T \subset [n]$. For this it suffices to consider the case where $T = S \cup \{t\}$ with $t \notin S$, since the class \mathcal{A} is closed under composition. The square

$$\begin{array}{ccc} \partial_t \Delta[n] \cap \Lambda^T[n] & \longrightarrow & \Lambda^T[n] \\ \downarrow & & \downarrow \\ \partial_t \Delta[n] & \longrightarrow & \Lambda^S[n] \end{array}$$

is a pushout since $\Lambda^S[n] = \partial_t \Delta[n] \cup \Lambda^T[n]$. Hence it suffices to show that the inclusion $\partial_t \Delta[n] \cap \Lambda^T[n] \subset \partial_t \Delta[n]$ belongs to \mathcal{A} . Let $U \subseteq [n - 1]$ be the inverse image of the subset T by the map $d_t : [n - 1] \rightarrow [n]$. The inclusion $\partial_t \Delta[n] \cap \Lambda^T[n] \subset \partial_t \Delta[n]$ is isomorphic to the inclusion $\Lambda^U[n - 1] \subset \Delta[n - 1]$. Hence it suffices to show that

the latter belongs to \mathcal{A} . The subset U is non-empty since it is in bijection with S . Moreover U is proper, since $\text{Card}(U) = \text{Card}(S) < \text{Card}(T)$ and T is a proper subset of $[n]$. Thus, the inclusion $\Lambda^U[n-1] \subset \Delta[n-1]$ belongs to \mathcal{A} by the induction hypothesis. This proves that the inclusion $\Lambda^T[n] \subseteq \Lambda^S[n]$ belongs to \mathcal{A} . ■

LEMMA 3.8. *A mid fibration between quasi-categories is a trivial fibration iff it is a weak categorical equivalence surjective on vertices.*

Proof: The necessity is clear. Conversely, if a mid fibration between quasi-categories $f : X \rightarrow Y$ is a weak categorical equivalence surjective on vertices, let us show that it is a trivial fibration. For this it suffices to show that f is a quasi-fibration. But for this, it suffices to show that the functor $\tau_1(f)$ is a quasi-fibration by 1.13, since f is a map between quasi-categories. But the functor $\tau_1(f)$ is an equivalence of categories, since f is a weak categorical equivalence by assumption. Moreover, the functor $\tau_1(f)$ is surjective on objects, since the map f is surjective on vertices by assumption. Thus, $\tau_1(f)$ is an acyclic fibration for the natural model structure on \mathbf{Cat} . It follows that $\tau_1(f)$ is a quasi-fibration, and hence that f is a trivial fibration. ■

PROPOSITION 3.9. *A bisimplicial set X is a Segal space iff the following three conditions are satisfied:*

- (i) *the map X/δ_n is a mid fibration for every $n \geq 0$;*
- (ii) *X_0 is a Kan complex;*
- (iii) *X is categorically constant.*

Proof: (\Rightarrow) The simplicial set X_0 is a Kan complex by 7.34, since X is v -fibrant by assumption. Moreover, X is categorically constant by 2.8. The map X/δ_n is a mid fibration for every $n \geq 0$ by Proposition 3.4 since X is a Segal space. (\Leftarrow) We shall prove that X is vertically fibrant. by showing that the map X/v is a trivial fibration for every anodyne map $v \in \mathbf{S}$. Observe first that the map X/v is a mid fibration for every monomorphism $v : S \rightarrow T$ by 7.33, since the map X/δ_n is a mid fibration for every $n \geq 0$ by assumption. In particular, X/S is a quasi-category for any simplicial set S , since the map $X/S \rightarrow X/\emptyset = 1$ is a mid fibration. Hence the map X/v is a mid fibration between quasi-categories for any monomorphism $v : S \rightarrow T$. We claim that if v is anodyne and X/v is a weak categorical equivalence, then X/v is actually a trivial fibration. For this, it suffices to show that X/v is surjective on vertices by Lemma 3.8. But every map $S \rightarrow X_0$ can be extended along v to a map $T \rightarrow X_0$, since v is anodyne and X_0 is a Kan complex by assumption. Hence the map $(X/v)_0 : (X/T)_0 \rightarrow (X/S)_0$ is surjective on vertices. This shows that X/v is a trivial fibration if v is anodyne and X/v is a weak categorical equivalence. Let us now prove that X is vertically fibrant. For this it suffices to show that the map X/v is a trivial fibration for every anodyne map $v \in \mathbf{S}$ by 2.5. Let \mathcal{A} be the class of anodyne maps $v \in \mathbf{S}$ such that the map X/v is a trivial fibration. We shall prove that every anodyne map belongs to \mathcal{A} by using Lemma 3.7. An anodyne map belongs to \mathcal{A} iff the map X/v is a weak categorical equivalence by the above. It follows by three-for-two that \mathcal{A} has the right cancellation property. Let us show that \mathcal{A} is saturated. The condition $\delta_n \pitchfork (X/v)$ is equivalent to the condition $v \pitchfork (\delta_n \setminus X)$ by 2.1. Thus, an anodyne map v belongs to \mathcal{A} iff we have $v \pitchfork (\delta_n \setminus X)$ for every $n \geq 0$. This description implies that the class \mathcal{A} is saturated. Let us show that every face map $d_i : \Delta[n-1] \rightarrow \Delta[n]$ belongs to \mathcal{A} . The canonical maps $X_{\star 0} \rightarrow X_{\star n}$ and

$X_{*0} \rightarrow X_{*,n-1}$ are weak categorical equivalences, since X is categorically constant. It follows by three-for-two that the map $X/d_i : X_{*n} \rightarrow X_{*,n-1}$ is a weak categorical equivalence. This proves that $d_i \in \mathcal{A}$, since d_i is anodyne. It then follows by Lemma 3.7 that every anodyne map belongs to \mathcal{A} . Thus, X is v -fibrant by 2.5. Let us now show that X satisfies the Segal condition. For this, it suffices to show that the map $i_m \setminus X$ is a trivial fibration for every $m \geq 0$, where i_m denotes the inclusion of the m -chain $I_m \subseteq \Delta[m]$. But for this it suffices to show that we have $\delta_n \pitchfork (i_m \setminus X)$ for every $n \geq 0$. But i_m is mid anodyne by Lemma 1.21. Hence we have $i_m \pitchfork (X/\delta_n)$ by (i). It follows that we have $\delta_n \pitchfork (i_m \setminus X)$ by 2.1. This shows that X is a Segal space. ■

PROPOSITION 3.10. *Let $f : X \rightarrow Y$ be a v -fibration between Segal spaces. Then the map*

$$\langle u \setminus f \rangle : B \setminus X \longrightarrow B \setminus Y \times_{A \setminus Y} A \setminus X$$

is a trivial fibration for any mid anodyne map $u : A \rightarrow B$. Moreover, the map

$$\langle f/v \rangle : X/T \longrightarrow Y/T \times_{Y/S} X/S$$

is a mid fibration between quasi-categories for any monomorphism of simplicial sets $v : S \rightarrow T$.

Proof: Let us prove the first statement. The map $\langle u \setminus f \rangle$ is a Kan fibration by 2.5. Let us show that it is a weak homotopy equivalence. The horizontal maps in the commutative square

$$\begin{array}{ccc} B \setminus X & \longrightarrow & A \setminus X \\ \downarrow & & \downarrow \\ B \setminus Y & \longrightarrow & A \setminus Y \end{array}$$

are trivial fibrations by 3.4. It follows that $\langle u \setminus f \rangle$ is a weak homotopy equivalence. It is thus a trivial fibration since it is a Kan fibration. Let us prove the second statement. The domain of $\langle f/v \rangle$ is a quasi-category by 3.4. Let us show that its codomain is a quasi-category. The projection p_2 in the pullback square

$$\begin{array}{ccc} Y/S \times_{Y/S} X/T & \xrightarrow{p_2} & X/S \\ \downarrow & & \downarrow \\ Y/T & \longrightarrow & Y/S \end{array}$$

is a mid fibration since the bottom map is a mid fibration by 3.4. It follows that the domain of p_2 is a quasi-category since its codomain is a quasi-category by 3.4. Let us now show that $\langle f/v \rangle$ is a mid fibration. By 1.12 it suffices to show that we have $u \pitchfork \langle f/v \rangle$ for every mid anodyne map $u : A \rightarrow B$. But the condition $u \pitchfork \langle f/v \rangle$ is equivalent to the condition $v \pitchfork \langle u \setminus f \rangle$ by 2.1. But we have $v \pitchfork \langle u \setminus f \rangle$ since $\langle u \setminus f \rangle$ is a trivial fibration by the first part of the proof. ■

4. Two equivalences with complete Segal spaces

Complete Segal spaces were introduced by Charles Rezk in [Rez]. They are the fibrant objects of a Quillen model structure on the category of simplicial spaces $[\Delta^\circ, \mathbf{S}] = \mathbf{S}^{(2)}$. We call this model structure the *model structure for complete Segal spaces*. The goal of the section is to establish two Quillen equivalences

$$p_1^* : \mathbf{S} \leftrightarrow \mathbf{S}^{(2)} : i_1^* \quad \text{and} \quad t_! : \mathbf{S}^{(2)} \leftrightarrow \mathbf{S} : t^!$$

between the model category for quasi-categories and the model category for complete Segal spaces. The functor i_1^* associates to a bisimplicial set X its first row X_{*0} . It shows that a complete Segal space is determined up to equivalence by its first row. The functor $t_!$ associates to a bisimplicial set X a total simplicial set $t_!X$.

Let J be the groupoid generated by one isomorphism $0 \rightarrow 1$. We shall regard J as a simplicial set via the nerve functor. A Segal space X is said to be *complete* if the map

$$1 \backslash X \longrightarrow J \backslash X$$

obtained from the map $J \rightarrow 1$ is a weak homotopy equivalence.

We shall say that a map of simplicial spaces $u : A \rightarrow B$ is a *Rezk weak equivalence* if the map

$$Hom_2(u, X) : Hom_2(B, X) \rightarrow Hom_2(A, X)$$

is a weak homotopy equivalence for every complete Segal space X .

THEOREM 4.1 (Rezk, see [Rez]). *The simplicial category $(\mathbf{S}^{(2)}, Hom_2)$ admits a simplicial model structure $(\mathcal{C}_R, \mathcal{W}_R, \mathcal{F}_R)$ in which \mathcal{C}_R is the class of monomorphisms and \mathcal{W}_R is the class of Rezk weak equivalences. The model structure is left proper and cartesian closed. The acyclic fibrations are the trivial fibrations. The fibrant objects are the complete Segal spaces.*

We call $(\mathcal{C}_R, \mathcal{W}_R, \mathcal{F}_R)$ the *Rezk model structure* or the *model structure for complete Segal spaces*. For the notion of trivial fibration, see Definition 7.2. The model structure is a Bousfield localisation of the Segal space model structure of Theorem 3.2. Hence it is also a Bousfield localisation of the vertical model structure $(\mathcal{C}_0^v, \mathcal{W}_0^v, \mathcal{F}_0^v)$ of Theorem 2.6. For the notion of Bousfield localisation, see Definition 7.20. In particular, a map between complete Segal spaces is acyclic (resp. a fibration) iff it is a column-wise weak homotopy equivalence (resp. a v-fibration).

The first projection $p_1 : \Delta \times \Delta \rightarrow \Delta$ is left adjoint to the functor $i_1 : \Delta \rightarrow \Delta \times \Delta$ defined by putting $i_1([n]) = ([n], [0])$ for every $n \geq 0$. We thus obtain a pair of adjoint functors

$$p_1^* : \mathbf{S} \leftrightarrow \mathbf{S}^{(2)} : i_1^*$$

If X is a bisimplicial set, then $i_1^*(X)$ is the first row of X . Notice that we have $p_1^*(A) = A \square 1$ for every simplicial set A . In 4.11 we shall prove that

Theorem *The adjoint pair of functors*

$$p_1^* : \mathbf{S} \leftrightarrow \mathbf{S}^{(2)} : i_1^*$$

is a Quillen equivalence between the model category for quasi-categories and the model category for complete Segal spaces.

Recall the “total space” functor $t_! : \mathbf{S}^{(2)} \rightarrow \mathbf{S}$ of Theorem 3.3. In 4.12, we shall prove that

Theorem *The adjoint pair of functors*

$$t_! : \mathbf{S}^{(2)} \leftrightarrow \mathbf{S} : t^!$$

is a Quillen equivalence between the model category for complete Segal spaces and the model category for quasi-categories.

We have stated the main results of the section. We now proceed to the proofs.

Let u_0 be the inclusion $\{0\} \subset J$.

LEMMA 4.2. [Rez] *A Segal space X is complete iff the map*

$$u_0 \backslash X : J \backslash X \longrightarrow 1 \backslash X$$

is a trivial fibration.

Proof: By definition, a Segal space X is complete if the map

$$t \backslash X : 1 \backslash X \longrightarrow J \backslash X$$

obtained from the map $t : J \rightarrow 1$ is a weak homotopy equivalence. But we have $(u_0 \backslash X)(t \backslash X) = id$ since we have $tu_0 = id$. Hence the map $t \backslash X$ is a weak homotopy equivalence iff the map $u_0 \backslash X$ is a weak homotopy equivalence by three-for-two. But the map $u_0 \backslash X$ is a Kan fibration by 2.5. Hence the map $u_0 \backslash X$ is a weak homotopy equivalence iff it is a trivial fibration. ■

LEMMA 4.3. *Let $f : X \rightarrow Y$ be a Rezk fibration between complete Segal spaces. then the map*

$$\langle f/v \rangle : X/T \longrightarrow Y/T \times_{Y/S} X/S$$

is a quasi-fibration for any monomorphism of simplicial sets $v : S \rightarrow T$.

Proof: The map $\langle f/v \rangle$ is a mid fibration between quasi-categories by 3.10. Hence it suffices to show that it has the right lifting property with respect to the inclusion $u_0 : \{0\} \subset J$ by 1.13. But the condition $u_0 \pitchfork \langle f/v \rangle$ is equivalent to the condition $v \pitchfork \langle u_0 \backslash f \rangle$ by 2.1. Hence it suffices to show that the map

$$\langle u_0 \backslash f \rangle : J \backslash X \longrightarrow J \backslash Y \times_{1 \backslash Y} 1 \backslash X$$

is a trivial fibration. But it is a Kan fibration by 2.5. Hence it suffices to show that it is a weak homotopy equivalence. But the horizontal maps in the commutative square

$$\begin{array}{ccc} J \backslash X & \longrightarrow & 1 \backslash X \\ \downarrow & & \downarrow \\ J \backslash Y & \longrightarrow & 1 \backslash Y \end{array}$$

are trivial fibrations by 4.2. It follows that $\langle u_0 \backslash f \rangle$ is a weak homotopy equivalence. It is thus a trivial fibration. We have proved that $\langle f/v \rangle$ is a quasi-fibration. ■

PROPOSITION 4.4. *A bisimplicial set X is a complete Segal space iff the following two conditions are satisfied:*

- (i) *the map X/δ_n is a quasi-fibration for every $n \geq 0$;*
- (ii) *X is categorically constant.*

Proof: (\Rightarrow) A complete Segal space X is categorically constant by 3.9. Moreover, the map X/δ_n is a quasi-fibration for every $n \geq 0$ by 4.3, since the map $X \rightarrow 1$ is a Rezk fibration. (\Leftarrow) The bisimplicial set X is h-fibrant by condition (i). Let us show that it is v-fibrant. By 2.5 it suffices to show that the map X/v is a trivial fibration for every anodyne map $v \in \mathbf{S}$. Let \mathcal{A} be the class of monomorphisms $v \in \mathbf{S}$ such that the map X/v is a trivial fibration. Let us show that \mathcal{A} is saturated. The condition $\delta_n \pitchfork (X/v)$ is equivalent to the condition $v \pitchfork (\delta_n \backslash X)$ by 2.1. Thus, a monomorphism v belongs to \mathcal{A} iff we have $v \pitchfork (\delta_n \backslash X)$ for every $n \geq 0$. It follows that the class \mathcal{A} is saturated. Let us show that \mathcal{A} has the right cancellation property. The map X/v is a quasi-fibration for any monomorphism v by 7.33 since X is h-fibrant. Thus, X/v is a trivial fibration iff it is a weak categorical equivalence. It follows by three-for-two that \mathcal{A} has the right cancellation property. Let us show that every face map $d_i : \Delta[n-1] \rightarrow \Delta[n]$ belongs to \mathcal{A} . The canonical maps $X_{\star 0} \rightarrow X_{\star n}$ and $X_{\star 0} \rightarrow X_{\star, n-1}$ are weak categorical equivalences, since X is categorically constant. It follows by three-for-two that the map $X/d_i : X_{\star n} \rightarrow X_{\star, n-1}$ is a weak categorical equivalence. This proves that $d_i \in \mathcal{A}$. It then follows by Lemma 3.7 that every anodyne map belongs to \mathcal{A} . This shows that X is v-fibrant. Thus, X_0 is a Kan complex by 7.34. It follows that X is a Segal space by 3.4. Let us show that the Segal space X is complete. The inclusion $u_0 : \{0\} \subset J$ is an equivalence of categories. It is thus a weak categorical equivalence. Hence the map $u_0 \backslash X : J \backslash X \rightarrow 1 \backslash X$ is a trivial fibration by 7.33 since X is h-fibrant. This shows that X is a complete Segal space by 4.2. ■

THEOREM 4.5. *The Rezk model structure $(\mathcal{C}_R, \mathcal{W}_R, \mathcal{F}_R)$ is a Bousfield localisation of the horizontal model structure $(\mathcal{C}_1^h, \mathcal{W}_1^h, \mathcal{F}_1^h)$. An h-fibrant bisimplicial set is a complete Segal space iff it is categorically constant. A row-wise weak categorical equivalence is a Rezk weak equivalence.*

For the notion of Bousfield localisation, see Definition 7.20.

Proof: Let us prove the first statement. We have $\mathcal{C}_1^h = \mathcal{C}_R$, since \mathcal{C}_1^h is the class of monomorphisms by 2.10. If $f : X \rightarrow Y$ is a Rezk fibration between complete Segal space, then the map $\langle f/\delta_n \rangle \in \mathbf{S}$ is a quasi-fibration for every $n \geq 0$ by 4.3. This means that a Rezk fibration between complete Segal space is an h-fibration. This proves the first statement by 7.15. The second statement follows from 4.4. The third statement is a consequence of the first and of Proposition 7.11. ■

In particular, a map between complete Segal spaces is a Rezk weak equivalence (resp. a Rezk fibration) iff it is a row-wise weak categorical equivalence (resp. an h-fibration).

PROPOSITION 4.6. *The box product functor $\square : \mathbf{S} \times \mathbf{S} \rightarrow \mathbf{S}^{(2)}$ is a left Quillen functor of two variables*

$$(\mathcal{C}_1, \mathcal{W}_1, \mathcal{F}_1) \times (\mathcal{C}_0, \mathcal{W}_0, \mathcal{F}_0) \rightarrow (\mathcal{C}_R, \mathcal{W}_R, \mathcal{F}_R),$$

where $(\mathcal{C}_1, \mathcal{W}_1, \mathcal{F}_1)$ is the model structure for quasi-categories on \mathbf{S} , $(\mathcal{C}_0, \mathcal{W}_0, \mathcal{F}_0)$ is the classical model structure on \mathbf{S} and $(\mathcal{C}_R, \mathcal{W}_R, \mathcal{F}_R)$ is the Rezk model structure on $\mathbf{S}^{(2)}$.

Proof: Observe that the cofibrations are the monomorphisms in the three model structures, Let $u : A \rightarrow B$ and $v : S \rightarrow T$ be a pair of monomorphisms in \mathbf{S} . The map $u \square' v$ is a cofibration since it is monic. If $v \in \mathcal{W}_0$, let us show that $u \square' v \in \mathcal{W}_R$. But we have $u \square' v \in \mathcal{W}'_0$ by 7.36. The result follows since a column-wise weak homotopy equivalence is a Rezk weak equivalence. If $u \in \mathcal{W}_1$, let us show that $u \square' v \in \mathcal{W}_R$. For this it suffices to show that we have $(u \square' v) \pitchfork f$ for every Rezk fibration between complete Segal spaces $f : X \rightarrow Y$ by 7.14. But the condition $(u \square' v) \pitchfork f$ is equivalent to the condition $u \pitchfork \langle f/v \rangle$ by 2.1. The map $\langle f/v \rangle$ is a quasi-fibration by 7.33. Hence we have $u \pitchfork \langle f/v \rangle$, since $u \in \mathcal{C}_1 \cap \mathcal{W}_1$. ■

We recall that the first projection $p_1 : \Delta \times \Delta \rightarrow \Delta$ is left adjoint to the functor $i_1 : \Delta \rightarrow \Delta \times \Delta$ defined by putting $i_1([n]) = ([n], [0])$ for every $n \geq 0$.

PROPOSITION 4.7. *The adjoint pair of functors*

$$p_1^* : \mathbf{S} \rightarrow \mathbf{S}^{(2)} : i_1^*$$

is a homotopy localisation between the model category for quasi-categories and the model category for complete Segal spaces.

See Definition 7.16 for the notion of homotopy localisation.

Proof: We have $p_1^*(A) = A \square 1$ for every $A \in \mathbf{S}$. The functor $A \mapsto A \square 1$ is a left Quillen functor since the box product functor $(A, B) \mapsto A \square B$ is a left Quillen functor by 4.6 and since 1 is cofibrant. This proves that the pair (p_1^*, i_1^*) is a Quillen pair. Let us show that it is a homotopy localisation. For this, we shall use Proposition 7.17. It suffices to show that the adjunction counit $\epsilon : p_1^* i_1^* X \rightarrow X$ is a Rezk weak equivalence for every complete Segal space X . For this, it suffices to show by Theorem 4.5 that the map $\epsilon_{*n} : X_{*0} \rightarrow X_{*n}$ is a weak categorical equivalence for every $n \geq 0$. But ϵ_n is equal to the map $X_{*0} \rightarrow X_{*n}$ obtained from the map $[n] \rightarrow [0]$. This proves the result since X is categorically constant by 4.4. ■

If X is a bisimplicial set, then for any pair of simplicial sets A and B there is a natural bijection between the maps $A \rightarrow X/B$ and the maps $B \rightarrow A \setminus X$. This means that the contravariant functors

$$A \mapsto A \setminus X \quad \text{and} \quad B \mapsto X/B$$

(from \mathbf{S} to itself) are mutually right adjoint.

LEMMA 4.8. *Every continuous functor $G : \mathbf{S}^o \rightarrow \mathbf{S}$ is of the form $G(A) = A \setminus X$ for a simplicial space $X \in \mathbf{S}^{(2)}$. We have $X_m = G(\Delta[m])$ for every $m \geq 0$.*

Proof: Let X be the simplicial space defined by putting $X_m = G(\Delta[m])$ for every $m \geq 0$. Then we have $\Delta[m] \setminus X = X_m = G(\Delta[m])$ for every $m \geq 0$. It follows that we have $A \setminus X = G(A)$ for every simplicial set A , since the functors $A \mapsto A \setminus X$ and $A \mapsto G(A)$ are both continuous and every simplicial set is a colimit of simplices. ■

If X is a quasi-category, we shall denote by $\Gamma(X)$ the simplicial space obtained by putting

$$\Gamma(X)_m = J(X^{\Delta[m]})$$

for every $m \geq 0$.

PROPOSITION 4.9. *There are canonical isomorphisms*

$$A \setminus \Gamma(X) = J(X^A) \quad \text{and} \quad \Gamma(X)/A = X^{(A)},$$

natural in $A \in \mathbf{S}$.

Proof: The contravariant functors $A \mapsto J(X^A)$ and $A \mapsto X^{(A)}$ are mutually right adjoint by 1.16. Hence the contravariant functor $A \mapsto J(X^A)$ is continuous. It follows by 4.8 that we have $A \setminus \Gamma(X) = J(X^A)$ for every simplicial set A . But the contravariant functors $A \mapsto A \setminus \Gamma(X)$ and $A \mapsto \Gamma(X)/A$ are mutually right adjoint. Hence we have $\Gamma(X)/A = X^{(A)}$ for every simplicial set A by the uniqueness of a right adjoint. ■

If X is a simplicial set, then we have

$$i_1^* \Gamma(X) = \Gamma(X)_{*0} = \Gamma(X)/1 = X^{(1)} = X$$

by 4.9. By the adjointness $p_1^* \dashv i_1^*$ we obtain a natural map $p_1^*(X) \rightarrow \Gamma(X)$.

PROPOSITION 4.10. *If X is a quasi-category, then $\Gamma(X)$ is a complete Segal space and the natural map $p_1^*(X) \rightarrow \Gamma(X)$ is a Rezk weak equivalence. Hence $\Gamma(X)$ is a fibrant replacement of the simplicial space $p_1^*(X) = X \square 1$.*

Proof: Let X be a quasi-category. Let us first show that $\Gamma(X)$ is vertically Reedy fibrant. Let δ_n be the inclusion $\partial \Delta[n] \subset \Delta[n]$. The map $X^{\delta_n} : X^{\Delta[n]} \rightarrow X^{\partial \Delta[n]}$ is a quasi-fibration since the model structure for quasi-categories is cartesian closed by 1.9. Hence the map $J(X^{\delta_n}) : J(X^{\Delta[n]}) \rightarrow J(X^{\partial \Delta[n]})$ is a Kan fibration by 1.16. But $J(X^{\delta_n})$ is isomorphic to the map $\delta_n \setminus \Gamma(X) : \Delta[n] \setminus \Gamma(X) \rightarrow \partial \Delta[n] \setminus \Gamma(X)$ by 4.9. This shows that the map $\delta_n \setminus \Gamma(X)$ is a Kan fibration. We have proved that $\Gamma(X)$ is vertically Reedy fibrant. Let us now show that $\Gamma(X)$ is a Segal space. The inclusion $i_n : I_n \subseteq \Delta[n]$ is mid anodyne by 1.21. It is thus a weak categorical equivalence by 1.11. Hence the map X^{i_n} is a trivial fibration by 1.9. It follows that the map $J(X^{i_n})$ is a trivial fibration by 1.16. But $J(X^{i_n})$ is isomorphic to the map $i_n \setminus \Gamma(X)$ by 4.9. This shows that the map $i_n \setminus \Gamma(X)$ is a trivial fibration. We have proved that $\Gamma(X)$ is a Segal space. It remains to show that $\Gamma(X)$ is a complete Segal space. The map $p : J \rightarrow 1$ is an equivalence of categories. Hence the map X^p is an equivalence of quasi-categories by 1.9. It follows that the map $J(X^p)$ is a homotopy equivalence by 1.16. But $J(X^p)$ is isomorphic to the map $p \setminus \Gamma(X)$ by 4.9. This shows that the map $p \setminus \Gamma(X)$ is a homotopy equivalence. We have proved that $\Gamma(X)$ is a complete Segal space. Let us now show that the natural map $p_1^*(X) \rightarrow \Gamma(X)$ is a Rezk weak equivalence. For this, it suffices to show that the natural map $p_1^*(X) \rightarrow \Gamma(X)$ is a row-wise weak categorical equivalence by Lemma 4.5. But the map $p_1^*(X)_{*n} \rightarrow \Gamma(X)_{*n}$ is equal to the map $X^{(t_n)} : X^{(1)} \rightarrow X^{(\Delta[n])}$, where $t_n : \Delta[n] \rightarrow 1$. But $X^{(t_n)}$ is an equivalence of quasi-categories by 1.17, since t_n is a weak homotopy equivalence. ■

THEOREM 4.11. *The adjoint pair of functors*

$$p_1^* : \mathbf{S} \rightarrow \mathbf{S}^{(2)} : i_1^*$$

is a Quillen equivalence between the model category for quasi-categories and the model category for complete Segal spaces.

Proof: We shall use proposition 7.22. We saw in 4.7 that the pair (p_1^*, i_1^*) is a homotopy localisation. Hence it suffices to show that the pair (p_1^*, i_1^*) is a homotopy colocalisation by 7.22. By 7.17, for this it suffices to show that the map $A \rightarrow i_1^* R p_1^* A$ is a weak equivalence for every fibrant-cofibrant object $A \in \mathbf{S}$, where $p_1^* A \rightarrow R p_1^* A$ denotes a fibrant replacement of $p_1^* A$. But we can take $R p_1^* A = \Gamma(A)$ by 4.10. In this case we have

$$i_1^* R p_1^* A = i_1^* \Gamma(A) = A$$

and the canonical map $A \rightarrow i_1^* R p_1^* A$ is the identity. The result is proved. ■

Recall the “total space” functor $t_1 : \mathbf{S}^{(2)} \rightarrow \mathbf{S}$ of Theorem 3.3.

THEOREM 4.12. *The adjoint pair of functors*

$$t_1 : \mathbf{S}^{(2)} \leftrightarrow \mathbf{S} : t^!$$

is a Quillen equivalence between the model category for complete Segal spaces and the model category for quasi-categories.

Proof: Let us first show that $(t_1, t^!)$ is a Quillen pair. We saw in 3.3 that it is a Quillen pair between the model category for Segal spaces and the model category for quasi-categories. Hence it suffices to show by 7.15. that the functor $t^!$ takes a quasi-category to a complete Segal space. If X is a quasi-category then $t^!(X)$ is a Segal space by 2.12. Let us show that the Segal space $t^!(X)$ is complete. For this, it suffices to show that the map $u_0 \backslash t^!(X)$ is a trivial fibration by 4.2, where u_0 denotes the inclusion $\{0\} \subset J$. But the map $u_0 \backslash t^!(X)$ is isomorphic to the map $k^!(X^{u_0})$ by Lemma 2.11. Hence it suffices to show that the map $k^!(X^{u_0})$ is a trivial fibration. But u_0 is a weak categorical equivalence since it is an equivalence of categories. It follows that the map X^{u_0} is a trivial fibration by 1.9 since u_0 is monic. This shows that the map $u_0 \backslash t^!(X)$ is a trivial fibration. We have proved that $t^!(X)$ is a complete Segal space. We have proved that $(t_1, t^!)$ is a Quillen pair between the model category for complete Segal spaces and the model category for quasi-categories. It remains to show that it is a Quillen equivalence. The composite $t_1 p_1^* : \mathbf{S} \rightarrow \mathbf{S}$ is isomorphic to the identity functor since we have

$$t_1 p_1^*(A) = t_1(A \square 1) = A \times k_1(1) = A$$

for every simplicial set A by 2.11 and since $k_1(1) = 1$. Hence the composite $i_1^* t^! : \mathbf{S} \rightarrow \mathbf{S}$ is also isomorphic to the identity functor by adjointness. We saw in 4.11 that the pair (p_1^*, i_1^*) is a Quillen equivalence. It follows by three-for-two in 7.23 that the pair $(t_1, t^!)$ is a Quillen equivalence. ■

5. Two equivalences with Segal categories

Segal categories were first introduced by Dwyer, Kan and Smith [DKS], where they are called special Δ° -diagrams of simplicial sets . The theory of Segal categories was extensively developed by Hirschowitz and Simpson for application to algebraic geometry. A simplicial space X is called a precategory if its first column X_0 is discrete. There is a model structure on the category of precategories in which the fibrant objects are Segal categories. The goal of the section is to establish two Quillen equivalences

$$j^* : \mathbf{S} \leftrightarrow \mathbf{PCat} : j_* \quad \text{and} \quad d^* : \mathbf{PCat} \leftrightarrow \mathbf{S} : d_*$$

between the model category for quasi-categories and the model category for Segal categories. The functor j^* associates to a precategory X its first row X_{*0} . The functor d^* associates to a precategory its diagonal.

We recall that a simplicial space $X : \Delta^\circ \rightarrow \mathbf{S}$ is called a *precategory* if X_0 is discrete. We shall denote by \mathbf{PCat} the full subcategory of $\mathbf{S}^{(2)}$ spanned by the precategories.

Consider the functor $i_2 : \Delta \rightarrow \Delta \times \Delta$ defined by putting $i_2([n]) = ([0], [n])$. The functor $i_2^* : \mathbf{S}^{(2)} \rightarrow \mathbf{S}$ associates to a simplicial space X its first column X_0 . The functor i_2^* has a right adjoint $\text{Cosk} = (i_2)_* : \mathbf{S} \rightarrow \mathbf{S}^{(2)}$. If A is a simplicial set, then we have

$$\text{Cosk}(A)_n = A^{[n]_0}$$

for every $n \geq 0$, where $[n]_0$ denotes the set of vertices on $\Delta[n]$. If X is a simplicial space, the unit of the adjunction $i_2^* \dashv (i_2)_*$ is a canonical map

$$v_X : X \rightarrow \text{Cosk}(X_0)$$

called the *vertex map*. Let us suppose that X is a precategory. Then the map $(v_X)_n : X_n \rightarrow \text{Cosk}(X_0)_n$ takes its values in a discrete simplicial set $X_0^{[n]_0}$ for each $n \geq 0$. We thus have a decomposition

$$X_n = \bigsqcup_{a \in X_0^{[n]_0}} X(a),$$

where $X(a) = X(a_0, a_1, \dots, a_n)$ denotes the fiber of the vertex map $(v_X)_n : X_n \rightarrow X_0^{[n]_0}$ at $a = (a_0, a_1, \dots, a_n)$. If $u : [m] \rightarrow [n]$ is a map in Δ , then the map $X(u) : X_m \rightarrow X_n$ induces a map

$$X(a_0, a_1, \dots, a_m) \rightarrow X(a_{u(0)}, a_{u(1)}, \dots, a_{u(m)})$$

for every $a \in X_0^{[n]_0}$.

A precategory X is called a *Segal category* in [HS] if it satisfies the Segal condition 3.1. It is easy to verify that we have a decomposition

$$I_n \setminus X = \bigsqcup_{a \in X_0^{[n]_0}} X(a_0, a_1) \times \dots \times X(a_{n-1}, a_n).$$

It follows that a precategory X is a Segal category iff the canonical map

$$X(a_0, a_1, \dots, a_n) \rightarrow X(a_0, a_1) \times \dots \times X(a_{n-1}, a_n)$$

is a weak homotopy equivalence for every $n \geq 2$ and every $a \in X_0^{[n]_0}$ (the condition is trivially satisfied if $n < 2$).

DEFINITION 5.1. [HS] A map of Segal categories $f : X \rightarrow Y$ is said to be *fully faithful* if the map

$$X(a, b) \rightarrow Y(fa, fb)$$

is a weak homotopy equivalence for every pair $a, b \in X_0$.

If C is a small category, then the bisimplicial set $N(C) = C \square 1$ is a Segal category. The functor $N : \mathbf{Cat} \rightarrow \mathbf{PCat}$ has a left adjoint

$$\tau_1 : \mathbf{PCat} \rightarrow \mathbf{Cat}.$$

We say that $\tau_1 X$ is the *fundamental category* of a precategory X . We shall say that a map of precategories $f : X \rightarrow Y$ is *essentially surjective* if the functor $\tau_1(f) : \tau_1 X \rightarrow \tau_1 Y$ is essentially surjective.

DEFINITION 5.2. [HS] A map between Segal categories $f : X \rightarrow Y$ is called an *equivalence* if it is fully faithful and essentially surjective.

Hirschowitz and Simpson construct a completion functor $Seg : PCat \rightarrow PCat$ which associates to a precategory X a Segal category $Seg(X)$ “generated” by X . A map of precategories $f : X \rightarrow Y$ is called a *weak categorical equivalence* if the map $Seg(f) : Seg(X) \rightarrow Seg(Y)$ is an equivalence of Segal categories.

THEOREM 5.3 (Hirschowitz-Simpson, see [HS]). *The category of precategories \mathbf{PCat} admits a model structure in which a cofibration is a monomorphism and a weak equivalence is a weak categorical equivalence. The model structure is left proper and cartesian closed.*

We call the model structure, the *Hirschowitz-Simpson model structure* or the *model structure for Segal categories*. The model structure is cartesian closed by a result of Pellisier in [P]. A precategory is fibrant iff it is a Segal space by [B3] and [J4].

A simplicial set $X : \Delta^\circ \rightarrow \mathbf{Set}$ is discrete iff it takes every map in Δ to a bijection. It follows that a bisimplicial set $X : (\Delta^2)^\circ \rightarrow \mathbf{Set}$ is a precategory iff it takes every map in $[0] \times \Delta$ to a bijection. Let us put

$$\Delta^{|\!|2} = ([0] \times \Delta)^{-1}(\Delta \times \Delta)$$

and let π be the canonical functor $\Delta^2 \rightarrow \Delta^{|\!|2}$.

PROPOSITION 5.4. *The functor π^* induces an isomorphism between the presheaf category $[\Delta^{|\!|2}, \mathbf{Set}]$ and the subcategory $\mathbf{PCat} \subset \mathbf{S}^{(2)}$.*

We shall regard the functor π^* as an inclusion by adopting the same notation for a contravariant functor $X : \Delta^{|\!|2} \rightarrow \mathbf{Set}$ and the precategory $\pi^*(X)$. The functor $\pi^* : \mathbf{PCat} \subset \mathbf{S}^{(2)}$ has a left adjoint $\pi_!$ and a right adjoint π_* .

THEOREM 5.5 (Bergner, see[B2]). *The pair of adjoint functors*

$$\pi^* : \mathbf{PCat} \leftrightarrow \mathbf{S}^{(2)} : \pi_*$$

is a Quillen equivalence between the model structure for Segal categories and the Rezk model structure on $\mathbf{S}^{(2)}$. A map of precategories $X \rightarrow Y$ is a weak categorical equivalence iff the map $\pi^(u)$ is a Rezk weak equivalence.*

The functor $i_i : \Delta \rightarrow \Delta \times \Delta$ defined by putting $i_1([n]) = ([n], 0)$ is right adjoint to the projection $p_1 : \Delta \times \Delta \rightarrow \Delta$. The projection p_1 inverts every arrow in $[0] \times \Delta$. Hence there is a unique functor $q : \Delta^{[2]} \rightarrow \Delta$ such that $q\pi = p_1$. The functor $j = \pi i_1 : \Delta \rightarrow \Delta^{[2]}$ is then right adjoint to the functor q . Hence the functor $j^* : \mathbf{PCat} \rightarrow \mathbf{S}$ is right adjoint to the functor q^* . If X is a precategory, then $j^*(X)$ is the first row of X . If $A \in \mathbf{S}$, then $q^*(A) = A\Box 1$. The following result was conjectured by Bertrand Töen in [T1]:

THEOREM 5.6. *The adjoint pair of functors*

$$q^* : \mathbf{S} \leftrightarrow \mathbf{PCat} : j^*$$

is a Quillen equivalence between the model category for quasi-categories and the model category for Segal categories.

Proof: Let us show that q^* is a left Quillen functor. Obviously, q^* preserves monomorphisms. Let us show that it takes a weak categorical equivalence to a weak categorical equivalence. The functor p_1^* takes a weak categorical equivalence to a Rezk weak equivalence by 4.11 (and by Proposition 7.11). Thus, if $u \in \mathbf{S}$ is a weak categorical equivalence, then $p_1^*(u)$ is a Rezk weak equivalence. But we have $p_1^* = \pi^* q^*$, since we have $q\pi = p_1$. Thus, $\pi^* q^*(u)$ is a Rezk weak equivalence. It follows by 5.5 that $q^*(u)$ is a weak categorical equivalence. We have proved that q^* is a left Quillen functor. It remains to show that the pair (q^*, j^*) is a Quillen equivalence. But the pair (p_1^*, i_1^*) is a Quillen equivalence by 4.11. Hence also the pair (q^*, j^*) by three-for-two in 7.23, since the pair (π^*, π_*) is a Quillen equivalence by 5.5 and since we have $\pi^* q^* = p_1^*$. ■

Let us put $d = \pi\delta : \Delta \rightarrow \Delta^{[2]}$, where δ is the diagonal functor $\Delta \rightarrow \Delta \times \Delta$. The simplicial set $d^*(X)$ is the diagonal of a precategory X . The functor

$$d^* : \mathbf{PCat} \rightarrow \mathbf{S}$$

admits a left adjoint $d_!$ and a right adjoint d_* .

THEOREM 5.7. *The adjoint pair of functors*

$$d^* : \mathbf{PCat} \leftrightarrow \mathbf{S} : d_*$$

is a Quillen equivalence between the model category for Segal categories and the model category for quasi-categories.

The proof is given after Lemma 5.11. Let δ_* be the right adjoint of the functor $\delta^* : \mathbf{S} \rightarrow \mathbf{S}^{(2)}$.

LEMMA 5.8. *For every $A, B, X \in \mathbf{S}$ we have*

$$\delta^*(A\Box B) = A \times B, \quad A \setminus \delta_*(X) = X^A \quad \text{and} \quad \delta_*(X)/B = X^B.$$

Proof: We have

$$\delta^*(A\Box 1) = \delta^* p_1^*(A) = (p_1\delta)^*(A) = A$$

since $p_1\delta = id$. Similarly, we have $\delta^*(1\Box B) = B$ since $p_2\delta = id$. But

$$A\Box B = (A\Box 1) \times (1\Box B).$$

Thus, $\delta^*(A \square B) = \delta^*(A \square 1) \times \delta^*(1 \square B) = A \times B$, since the functor δ^* preserves products. Let us show that $A \setminus \delta_*(X) = X^A$. If B is a simplicial set, there is a natural bijection between the maps $B \rightarrow A \setminus \delta_*(X)$, the maps $A \square B \rightarrow \delta^*(X)$, the maps $\delta^*(A \square B) \rightarrow X$, the maps $A \times B \rightarrow X$ and the maps $B \rightarrow X^A$. This shows by Yoneda lemma that $A \setminus \delta_*(X) = X^A$. The formula $\delta_*(X)/B = X^B$ is proved similarly. ■

PROPOSITION 5.9. *If X is a simplicial space, then we have a pullback square of bisimplicial sets,*

$$\begin{array}{ccc} \pi_* X & \longrightarrow & X \\ \downarrow & & \downarrow v \\ \text{Cosk}(X_{00}) & \longrightarrow & \text{Cosk}(X_0), \end{array}$$

where $v = v_X : X \rightarrow \text{Cosk}(X_0)$ is the vertex map.

Proof: Let P be the bisimplicial set defined by the pullback square

$$\begin{array}{ccc} P & \xrightarrow{i} & X \\ \downarrow & & \downarrow v \\ \text{Cosk}(X_{00}) & \longrightarrow & \text{Cosk}(X_0). \end{array}$$

The map v induces an isomorphism on the first columns, hence also the map $P \rightarrow \text{Cosk}(X_{00})$. Thus, $P_0 = X_{00}$. This shows that P is a precategory. Let us show that the map $i : P \rightarrow X$ coreflects X in the subcategory \mathbf{PCat} . For this, we have to show that if Z is a precategory, then every map $f : Z \rightarrow X$ factors uniquely through i . But the map $f_0 : Z_0 \rightarrow X_0$ factors through the inclusion $X_{00} \subseteq X_0$ since Z_0 is discrete. The result then follows by using the adjunction $(i_2)^* \dashv (i_2)_*$. ■

Recall from 4.10 the functor $\Gamma : \mathbf{QCat} \rightarrow \mathbf{S}^{(2)}$ which associates to a quasi-category X a complete Segal space $\Gamma(X)$. We have

$$\Gamma(X)_{*n} = \Gamma(X)/\Delta[n] = X^{(\Delta[n])}$$

for every $n \geq 0$ by 4.9.

LEMMA 5.10. *If X is a quasi-category, then $d_* X = \pi_* \Gamma(X)$.*

Proof: It suffices to show that we have $(\pi_* \Gamma X)_n = (\pi_* \delta_* X)_n$ for every $n \geq 0$. If Y is a simplicial space, then we have a pullback square of simplicial sets

$$\begin{array}{ccc} (\pi_* Y)_n & \longrightarrow & Y_n \\ \downarrow & & \downarrow \\ Y_{00}^{[n]_0} & \longrightarrow & Y_0^{[n]_0} \end{array}$$

by Lemma 5.9. In particular, if $Y = \delta_* X$, we have a pullback square of simplicial sets

$$\begin{array}{ccc} (\pi_* \delta_* X)_n & \longrightarrow & X^{\Delta[n]} \\ \downarrow & & \downarrow \\ X_0^{[n]_0} & \longrightarrow & X^{[n]_0}. \end{array}$$

since $Y_n = \Delta[n] \backslash \delta_* X = X^{\Delta[n]}$ by 5.8. The projection $X^{\Delta[n]} \rightarrow X_0^{[n]_0}$ is a conservative quasi-fibration by 1.23. Hence also the projection $(\pi_* \delta_* X)_n \rightarrow X_0^{[n]_0}$ by 1.25. It follows that the simplicial set $(\pi_* \delta_* X)_n$ is a Kan complex by 1.24, since $X_0^{[n]_0}$ is a Kan complex. If we apply the functor J to the pullback square above we obtain a pullback square

$$\begin{array}{ccc} (\pi_* \delta_* X)_n & \longrightarrow & J(X^{\Delta[n]}) \\ \downarrow & & \downarrow \\ X_0^{[n]_0} & \longrightarrow & J(X)^{[n]_0}, \end{array}$$

since the functor J preserves pullbacks by 1.16 and since the vertical map on the left hand side is a map between Kan complexes. But we have $J(X^{\Delta[n]}) = (\Gamma X)_n$, $J(X) = (\Gamma X)_0$ and $X_0 = (\Gamma X)_{00}$ by 4.9. We thus obtain a pullback square

$$\begin{array}{ccc} (\pi_* \delta_* X)_n & \longrightarrow & (\Gamma X)_n \\ \downarrow & & \downarrow \\ (\Gamma X)_{00}^{[n]_0} & \longrightarrow & (\Gamma X)^{[n]_0} \end{array} .$$

This shows that $(\pi_* \delta_* X)_n = (\pi_* \Gamma X)_n$. ■

LEMMA 5.11. *If $f : X \rightarrow Y$ is a quasi-fibration between quasi-categories, then the map $\Gamma(f) : \Gamma X \rightarrow \Gamma Y$ is a Rezk fibration between complete Segal spaces.*

Proof: The bisimplicial sets ΓX and ΓY are complete Segal spaces by 4.10. Hence it suffices to show that $\Gamma(f)$ is a v-fibration. For this it suffices to show that the map $u \backslash \Gamma(f)$ is a Kan fibration for every monomorphism $u : A \rightarrow B$. But the square

$$\begin{array}{ccc} B \backslash \Gamma X & \longrightarrow & A \backslash \Gamma X \\ \downarrow & & \downarrow \\ B \backslash \Gamma Y & \longrightarrow & A \backslash \Gamma Y \end{array}$$

is isomorphic to the square

$$\begin{array}{ccc} J(X^B) & \longrightarrow & J(X^A) \\ \downarrow & & \downarrow \\ J(Y^B) & \longrightarrow & J(Y^A) \end{array}$$

by Lemma 4.9 . Hence the map $u \backslash \Gamma(f)$ is isomorphic to the image by the functor J of the map

$$\langle u, f \rangle : X^B \rightarrow Y^B \times_{Y^A} X^A$$

But $\langle u, f \rangle$ is a quasi-fibration by Theorem 1.9. It follows that $J \langle u, f \rangle$ is a Kan fibration by 1.16. We have proved that $\Gamma(f)$ is a Rezk fibration. ■

Proof of Theorem 5.7: Let us first show that the pair (d^*, d_*) is a Quillen pair. For this, we shall use the criteria of 7.15. Obviously, the functor d^* takes a monomorphism to a monomorphism. Let us show that its right adjoint d_* takes a fibration between fibrant objects to a fibration. If $f : X \rightarrow Y$ is a quasi-fibration between quasi-categories, let us show that the map $d_*(f) : d_*(X) \rightarrow d_*(Y)$ is a fibration. But we have $d_*(f) = \pi_*\Gamma(f)$ by 5.10. The map $\Gamma(f)$ is a Rezk fibration by 5.11. Thus, $\pi_*\Gamma(f)$ is a fibration since π_* is a right Quillen functor by 5.5. We have proved that the pair (d^*, d_*) is a Quillen pair. It remains to show that it is a Quillen equivalence. The composite $d^*q^* : \mathbf{S} \rightarrow \mathbf{S}$ is isomorphic to the identity functor since $qd = id$. We saw in 5.6 that the pair (q^*, j^*) is a Quillen equivalence. It follows by three-for-two in 7.23 that the pair (d^*, d_*) is a Quillen equivalence. ■

6. Addendum

The model structure for quasi-categories in Theorem 1.9 is not simplicial. However, it is Quillen equivalent to the model category for complete Segal spaces, which is simplicial by Theorem 4.1. In their paper *Simplicial structures on model categories and functors* [RSS] Rezk, Schwede and Shipley study the problem of associating to a model category \mathcal{E} a Quillen equivalent simplicial model category. A simplicial object $X : \Delta^o \rightarrow \mathcal{E}$ is said to be *homotopically constant* if it takes every map in Δ to a weak equivalence in \mathcal{E} . For any model category \mathcal{E} there is at most one model structure $M^c = (C^c, \mathcal{W}^c, \mathcal{F}^c)$ on the category $[\Delta^o, \mathcal{E}]$ such that

- The model structure M^c is a Bousfield localisation of the Reedy model structure on $[\Delta^o, \mathcal{E}]$;
- The fibrant objects are the homotopically constant Reedy fibrant objects.

For the notion of Bousfield localisation, see Definition 7.20. When it exists, M^c is called the *canonical model structure* on $[\Delta^o, \mathcal{E}]$. Under certain conditions, the model category M^c is shown to be simplicial and the adjoint pair

$$p^* : \mathcal{E} \leftrightarrow [\Delta^o, \mathcal{E}] : i^*$$

to be a Quillen equivalence, where $p^*(A) = 1 \square A$ and $j^*(X) = X_0$.

THEOREM 6.1. *The Rezk model structure $(C_R, \mathcal{W}_R, \mathcal{F}_R)$ on $\mathbf{S}^{(2)} = [\Delta^o, \mathbf{S}]$ is the canonical model structure associated to the model structure for quasi-categories $(C_1, \mathcal{W}_1, \mathcal{F}_1)$ on \mathbf{S} .*

Proof: The model structure $(C_R, \mathcal{W}_R, \mathcal{F}_R)$ is a Bousfield localisation of the horizontal model structure $(C_1^h, \mathcal{W}_1^h, \mathcal{F}_1^h)$ by 4.5. Moreover, an h-fibrant simplicial space X is a complete Segal space iff it is categorically constant. ■

7. Appendix

The goal of this appendix is to review the basic homotopical algebra needed in the paper and to introduce some notation.

We shall denote by $Ob\mathcal{C}$ the class of objects of a category \mathcal{C} and by $\mathcal{C}(A, B)$ the set of arrows between two objects of \mathcal{C} . If $F : \mathcal{A} \rightarrow \mathcal{B}$ and $G : \mathcal{B} \rightarrow \mathcal{A}$ are two functors, we shall write $F \dashv G$, or write

$$F : \mathcal{A} \leftrightarrow \mathcal{B} : G,$$

to indicate that the functor F is left adjoint to the functor G .

We shall denote by **Set** the category of sets and by **Cat** the category of small categories. If A is a small category and \mathcal{E} is a category (possibly large) we shall denote the category of functors $A \rightarrow \mathcal{E}$ by $[A, \mathcal{E}]$ or by \mathcal{E}^A . Recall that a *presheaf* on a small category A is a contravariant functor $A \rightarrow \mathbf{Set}$. We shall denote by \hat{A} the category $[A^o, \mathbf{Set}]$ of presheaves on A . We shall regard the Yoneda functor $y : A \rightarrow \hat{A}$ as an inclusion by adopting the same notation for an object $a \in A$ and the representable functor $y(a) = A(-, a)$. For every functor $u : A \rightarrow \mathcal{E}$, we shall denote by $u^! : \mathcal{E} \rightarrow \hat{A}$ the functor obtained by putting $u^!(X)(a) = \mathcal{E}(u(a), X)$ for every object $X \in \mathcal{E}$ and every object $a \in A$. The functor $u^!$ has a left adjoint $u_!$ when the category \mathcal{E} is cocomplete. The functor $u_! : \hat{A} \rightarrow \mathcal{E}$ is the left Kan extension of the functor u along the Yoneda functor $A \rightarrow \hat{A}$. If B is a small category and $u : A \rightarrow B$, we shall denote by $u^* : \hat{B} \rightarrow \hat{A}$ the functor obtained by putting $u^*(X) = Xu$ for every presheaf $X \in \hat{B}$. The functor u^* has a left adjoint denoted $u_!$ and a right adjoint denoted u_* .

We denote by Δ the category whose objects are the finite non-empty ordinals and whose arrows are the order preserving maps. The ordinal $n + 1$ is represented by the ordered set $[n] = \{0, \dots, n\}$, so that $Ob\Delta = \{[n] : n \geq 0\}$. A *simplicial set* is a presheaf on Δ . If X is a simplicial set, the set $X([n])$ is denoted by X_n for every $n \geq 0$. We denote the category of simplicial sets $\hat{\Delta}$ by **S**. Recall that the simplex $\Delta[n]$ is defined to be the representable functor $\Delta(-, [n])$. We shall denote its boundary by $\partial\Delta[n]$. A category enriched over **S** is called a *simplicial category*.

Let $c : \mathbf{Set} \rightarrow \mathbf{S}$ be the functor which associate to a set S the constant simplicial set cS obtained by putting $(cS)_n = S$ for every $n \geq 0$. The functor c is full and faithful and we shall regard it as an inclusion $\mathbf{Set} \subset \mathbf{S}$ by adopting the same notation for S and cS . The functor c has a left adjoint

$$\pi_0 : \mathbf{S} \rightarrow \mathbf{Set},$$

where $\pi_0(X)$ is the set of connected components of a simplicial set X .

If $u : A \rightarrow B$ and $f : X \rightarrow Y$ are two maps in a category \mathcal{E} , we write $u \pitchfork f$ to indicate that f has the right lifting property with respect to u . If S is an object of \mathcal{E} , we write $u \pitchfork S$ to indicate that the map $\mathcal{E}(u, S) : \mathcal{E}(B, S) \rightarrow \mathcal{E}(A, S)$ is surjective and we write $S \pitchfork f$ to indicate that the map $\mathcal{E}(S, f) : \mathcal{E}(S, A) \rightarrow \mathcal{E}(S, B)$ is surjective. If \mathcal{E} has a terminal object \top , the condition $u \pitchfork S$ is equivalent to the condition $u \pitchfork t_S$, where t_S is the map $S \rightarrow \top$. If \mathcal{E} has an initial object \perp , the condition $S \pitchfork f$ is equivalent to the condition $i_S \pitchfork f$, where i_S is the map $\perp \rightarrow S$.

For any class of maps $\mathcal{M} \subseteq \mathcal{E}$, we denote by ${}^{\pitchfork}\mathcal{M}$ (resp. \mathcal{M}^{\pitchfork}) the class of maps having the left (resp. right) lifting property with respect to every map in \mathcal{M} . If \mathcal{A} and \mathcal{B} are two classes of maps in \mathcal{E} , we write $\mathcal{A} \pitchfork \mathcal{B}$ to indicate that we have $a \pitchfork b$ for every $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Then

$$\mathcal{A} \subseteq {}^{\pitchfork}\mathcal{B} \iff \mathcal{A} \pitchfork \mathcal{B} \iff \mathcal{B} \subseteq \mathcal{A}^{\pitchfork}.$$

If $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ is a pair of adjoint functors, then for an arrow $f \in \mathcal{U}$ and an arrow $g \in \mathcal{V}$ we have

$$f \pitchfork G(g) \iff F(f) \pitchfork g.$$

DEFINITION 7.1. We shall say that a pair $(\mathcal{A}, \mathcal{B})$ of classes of maps in a category \mathcal{E} is a *weak factorisation system* if the following conditions are satisfied:

- every map $f \in \mathcal{E}$ admits a factorisation $f = pi$ with $i \in \mathcal{A}$ and $p \in \mathcal{B}$;
- $\mathcal{B} = \mathcal{A}^{\frown}$ and $\mathcal{A} = \mathcal{B}^{\frown}$.

We call \mathcal{A} is the *left class* and \mathcal{B} the *right class* of the weak factorisation system.

DEFINITION 7.2. We shall say that a map in a topos is a *trivial fibration* if it has the right lifting property with respect to every monomorphism.

This terminology is non-standard but it is useful. The trivial fibrations often coincide with the acyclic fibrations (which can be defined in any model category).

PROPOSITION 7.3. [J2] *If \mathcal{A} is the class of monomorphisms in a topos and \mathcal{B} is the class of trivial fibrations, then the pair $(\mathcal{A}, \mathcal{B})$ is a weak factorisation system.*

Recall that a map $u : A \rightarrow B$ in a category \mathcal{E} is said to be a *retract* of a map $f : X \rightarrow Y$ if u is a retract of f as objects of the category of arrows \mathcal{E}^I . Recall that a map $u : A \rightarrow B$ is called a *domain retract* of a map $v : C \rightarrow B$, if u is a retract of v as objects of the category \mathcal{E}/B . There is a dual notion of codomain retract. The two classes of a weak factorisation system are closed under retracts.

DEFINITION 7.4. We shall say that a class \mathcal{A} of maps in a cocomplete category \mathcal{E} is *saturated* if it contains the isomorphisms and is closed under composition, transfinite composition, cobase change and codomain retracts.

The class \mathcal{A}^{\frown} is saturated for any class $\mathcal{M} \subseteq \mathcal{E}$. In particular, the class \mathcal{A} of a weak factorisation system $(\mathcal{A}, \mathcal{B})$ in \mathcal{E} is saturated. Every class of maps $\mathcal{M} \subseteq \mathcal{E}$ is contained in a smallest saturated class $\overline{\mathcal{M}} \subseteq \mathcal{E}$ called the *saturated class generated* by \mathcal{M} .

The following proposition is a special case of a more general result, see [J2]:

PROPOSITION 7.5. *If Σ is a set of maps in a presheaf category, then the pair $(\overline{\Sigma}, \Sigma^{\frown})$ is a weak factorisation system.*

We shall say that a functor of two variables

$$\odot : \mathcal{E}_1 \times \mathcal{E}_2 \rightarrow \mathcal{E}_3$$

is *divisible on the left* if the functor $A \odot (-) : \mathcal{E}_2 \rightarrow \mathcal{E}_3$ admits a right adjoint $A \backslash (-) : \mathcal{E}_3 \rightarrow \mathcal{E}_2$ for every object $A \in \mathcal{E}_1$. In this case we obtain a functor of two variables $(A, X) \mapsto A \backslash X$,

$$\mathcal{E}_1^{\circ} \times \mathcal{E}_3 \rightarrow \mathcal{E}_2,$$

called the *left division functor*. Dually, we shall say that \odot is *divisible on the right* if the functor $(-) \odot B : \mathcal{E}_1 \rightarrow \mathcal{E}_3$ admits a right adjoint $(-)/B : \mathcal{E}_3 \rightarrow \mathcal{E}_1$ for every object $B \in \mathcal{E}_2$. In this case we obtain a functor of two variables $(X, B) \mapsto X/B$,

$$\mathcal{E}_3 \times \mathcal{E}_2^{\circ} \rightarrow \mathcal{E}_1,$$

called the *right division functor*. When the functor \odot is divisible on both sides, there is a bijection between the following three kinds of maps

$$A \odot B \rightarrow X, \quad B \rightarrow A \backslash X, \quad A \rightarrow X/B.$$

Hence the contravariant functors $A \mapsto A \backslash X$ and $B \mapsto B \backslash X$ are mutually right adjoint. It follows that we have

$$u \pitchfork (X/v) \iff v \pitchfork (u \backslash X).$$

for every map $u \in \mathcal{E}_1$ and every map $v \in \mathcal{E}_2$.

Remark: If a functor of two variables $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \rightarrow \mathcal{E}_3$ is divisible on both sides, then so are the left division functor $\mathcal{E}_1^o \times \mathcal{E}_3 \rightarrow \mathcal{E}_2$ and the right division functor $\mathcal{E}_3 \times \mathcal{E}_2^o \rightarrow \mathcal{E}_1$. This is called a tensor-hom-cotensor situation in [G].

Recall that a monoidal category $\mathcal{E} = (\mathcal{E}, \otimes)$ is said to be *closed* if the tensor product \otimes is divisible on each side. Let $\mathcal{E} = (\mathcal{E}, \otimes, \sigma)$ be a *symmetric* monoidal closed category, with symmetry $\sigma : A \otimes B \simeq B \otimes A$. Then the objects X/A and $A \backslash X$ are canonically isomorphic; we can identify them by adopting a common notation, for example $[A, X]$.

Recall that a category with finite products \mathcal{E} is said to be *cartesian closed* if the functor $A \times - : \mathcal{E} \rightarrow \mathcal{E}$ admits a right adjoint $(-)^A$ for every object $A \in \mathcal{E}$. A cartesian closed category \mathcal{E} is symmetric monoidal closed. Every presheaf category and more generally every topos is cartesian closed.

Let $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \rightarrow \mathcal{E}_3$ be a functor of two variables with values in a finitely cocomplete category \mathcal{E}_3 . If $u : A \rightarrow B$ is map in \mathcal{E}_1 and $v : S \rightarrow T$ is a map in \mathcal{E}_2 , we shall denote by $u \odot' v$ the map

$$A \odot T \sqcup_{A \odot S} B \odot S \longrightarrow B \odot T$$

obtained from the commutative square

$$\begin{array}{ccc} A \odot S & \longrightarrow & B \odot S \\ \downarrow & & \downarrow \\ A \odot T & \longrightarrow & B \odot T. \end{array}$$

This defines a functor of two variables

$$\odot' : \mathcal{E}_1^I \times \mathcal{E}_2^I \rightarrow \mathcal{E}_3^I,$$

where \mathcal{E}^I denotes the category of arrows of a category \mathcal{E} .

In a topos, if $u : A \subseteq B$ and $v : S \subseteq T$ are inclusions of sub-objects then the map $u \times' v$ is the inclusion of sub-objects

$$(A \times T) \cup (B \times S) \subseteq B \times T.$$

Suppose now that a functor $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \rightarrow \mathcal{E}_3$ is divisible on both sides, that \mathcal{E}_1 and \mathcal{E}_2 are finitely complete and that \mathcal{E}_3 is finitely cocomplete. Then the functor $\odot' : \mathcal{E}_1^I \times \mathcal{E}_2^I \rightarrow \mathcal{E}_3^I$ is divisible on both sides. If $u : A \rightarrow B$ is map in \mathcal{E}_1 and $f : X \rightarrow Y$ is a map in \mathcal{E}_3 , let us denote by $\langle u \backslash f \rangle$ the map

$$B \backslash X \rightarrow B \backslash Y \times_{A \backslash Y} A \backslash X$$

obtained from the commutative square

$$\begin{array}{ccc} B \backslash X & \longrightarrow & A \backslash X \\ \downarrow & & \downarrow \\ B \backslash Y & \longrightarrow & A \backslash Y. \end{array}$$

Then the functor $f \mapsto \langle u \setminus f \rangle$ is right adjoint to the functor $v \mapsto u \odot' v$. Dually, if $v : S \rightarrow T$ is map in \mathcal{E}_2 and $f : X \rightarrow Y$ is a map in \mathcal{E}_3 , we shall denote by $\langle f / v \rangle$ the map

$$X/T \rightarrow Y/T \times_{Y/S} X/S$$

obtained from the commutative square

$$\begin{array}{ccc} X/T & \longrightarrow & X/S \\ \downarrow & & \downarrow \\ Y/T & \longrightarrow & Y/S. \end{array}$$

The functor $f \mapsto \langle f \setminus v \rangle$ is right adjoint to the functor $u \mapsto u \odot' v$.

The verification of the following result is left to the reader. See [J2].

PROPOSITION 7.6. *Let $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \rightarrow \mathcal{E}_3$ be a functor of two variables divisible on both sides, where \mathcal{E}_i is a finitely bicomplete category for $i = 1, 2, 3$. If $u \in \mathcal{E}_1$, $v \in \mathcal{E}_2$ and $f \in \mathcal{E}_3$, then*

$$(u \odot' v) \pitchfork f \iff u \pitchfork \langle f / v \rangle \iff v \pitchfork \langle u \setminus f \rangle.$$

Let $\mathcal{E} = (\mathcal{E}, \otimes)$ be a bicomplete symmetric monoidal closed category. Then the operation \otimes' gives the category \mathcal{E}^I the structure of a symmetric monoidal closed category. If u and f are two maps in \mathcal{E} , then the maps $\langle f / u \rangle$ and $\langle u \setminus f \rangle$ are canonically isomorphic. We shall identify them by adopting a common notation $\langle u, f \rangle$. If $u : A \rightarrow B$, $v : S \rightarrow T$ and $f : X \rightarrow Y$ are three maps in \mathcal{E} , then

$$(u \otimes' v) \pitchfork f \iff u \pitchfork \langle v, f \rangle \iff v \pitchfork \langle u, f \rangle.$$

We now recall the notion of a Quillen model category:

DEFINITION 7.7. [Q] Let \mathcal{E} be a finitely bicomplete category. A *model structure* on \mathcal{E} is a triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ of classes of maps in \mathcal{E} satisfying the following conditions:

- (“three-for-two”) if two of the three maps $u : A \rightarrow B$, $v : B \rightarrow C$ and $vu : A \rightarrow C$ belong to \mathcal{W} , then so does the third;
- the pair $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ is a weak factorisation system;
- the pair $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ is a weak factorisation system.

These conditions imply that \mathcal{W} is closed under retracts by 7.8 below. A category \mathcal{E} equipped with a model structure is called a *model category*. A map in \mathcal{C} is called a *cofibration*, a map in \mathcal{F} a *fibration* and a map in \mathcal{W} a *weak equivalence*. A map in \mathcal{W} is also said to be *acyclic*. An object $X \in \mathcal{E}$ is *fibrant* if the map $X \rightarrow 1$ is a fibration, where 1 is the terminal object of \mathcal{E} . Dually, an object $A \in \mathcal{E}$ is *cofibrant* if the map $0 \rightarrow A$ is a cofibration, where 0 is the initial object of \mathcal{E} .

Any two of the three classes of a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ determine the third.

A model structure is said to be *left proper* if the cobase change of an acyclic map along a cofibration is acyclic. Dually, a model structure is said to be *right proper* if the base change of an acyclic map along a fibration is acyclic. A model structure is *proper* if it is both left and right proper.

PROPOSITION 7.8. [JT2] *The class \mathcal{W} of a model structure is closed under retracts.*

Proof: Observe first that the class $\mathcal{F} \cap \mathcal{W}$ is closed under retracts since the pair $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ is a weak factorisation system. Suppose now that a map $f : A \rightarrow B$ is a retract of a map $g : X \rightarrow Y$ in \mathcal{W} . Let us show that $f \in \mathcal{W}$. By definition, we have a commutative diagram,

$$\begin{array}{ccccc} A & \xrightarrow{s} & X & \xrightarrow{t} & A \\ f \downarrow & & \downarrow g & & \downarrow f \\ B & \xrightarrow{u} & Y & \xrightarrow{v} & B \end{array}$$

where $gf = 1_X$ and $vu = 1_B$. Let us first consider the case where f is a fibration. In this case, let us choose a factorisation $g = qj : X \rightarrow Z \rightarrow Y$ with $j \in \mathcal{C} \cap \mathcal{W}$ and $q \in \mathcal{F}$. We have $q \in \mathcal{F} \cap \mathcal{W}$ by three-for-two, since $g \in \mathcal{W}$. The square

$$\begin{array}{ccc} X & \xrightarrow{t} & A \\ j \downarrow & & \downarrow f \\ Z & \xrightarrow{vq} & B \end{array}$$

has a diagonal filler $d : Z \rightarrow A$, since f is a fibration. We then have a commutative diagram,

$$\begin{array}{ccccc} A & \xrightarrow{js} & Z & \xrightarrow{d} & A \\ f \downarrow & & \downarrow q & & \downarrow f \\ B & \xrightarrow{u} & Y & \xrightarrow{v} & B. \end{array}$$

Thus, f is a retract of q , since $d(js) = ts = 1_A$. This shows that $f \in \mathcal{W}$ since $q \in \mathcal{F} \cap \mathcal{W}$. In the general case, let us choose a factorisation $f = pi : A \rightarrow E \rightarrow B$ with $i \in \mathcal{C} \cap \mathcal{W}$ and $p \in \mathcal{F}$. By taking a pushout, we obtain a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{s} & X & \xrightarrow{t} & A \\ i \downarrow & & \downarrow i_2 & & \downarrow i \\ E & \xrightarrow{i_1} & E \sqcup_A X & \xrightarrow{r} & E \\ p \downarrow & & \downarrow k & & \downarrow p \\ B & \xrightarrow{u} & Y & \xrightarrow{v} & B, \end{array}$$

where $ki_2 = g$ and $ri_1 = 1_E$. The map i_2 is a cobase change of the map i . Thus, $i_2 \in \mathcal{C} \cap \mathcal{W}$ since $i \in \mathcal{C} \cap \mathcal{W}$. Thus, $k \in \mathcal{W}$ by three-for-two since $g = ki_2 \in \mathcal{W}$ by hypothesis. Thus, $p \in \mathcal{W}$ by the first part since $p \in \mathcal{F}$. Thus $f = pi \in \mathcal{W}$ since $i \in \mathcal{W}$. ■

The *homotopy category* of a model category \mathcal{E} is defined to be the category of fractions $Ho(\mathcal{E}) = \mathcal{W}^{-1}\mathcal{E}$. We shall denote by $[u]$ the image of an arrow $u \in \mathcal{E}$ by the canonical functor $\mathcal{E} \rightarrow Ho(\mathcal{E})$. The arrows $[u]$ is invertible iff u is a weak equivalence by a result in [Q].

Let \mathcal{E}_f (resp. \mathcal{E}_c) be the full sub-category of fibrant (resp. cofibrant) objects of \mathcal{E} and let us put $\mathcal{E}_{fc} = \mathcal{E}_f \cap \mathcal{E}_c$. Let us put $Ho(\mathcal{E}_f) = \mathcal{W}_f^{-1}\mathcal{E}_f$ where $\mathcal{W}_f = \mathcal{W} \cap \mathcal{E}_f$

and similarly for $Ho(\mathcal{E}_c)$ and $Ho(\mathcal{E}_{f_c})$. Then the diagram of inclusions

$$\begin{array}{ccc} \mathcal{E}_{f_c} & \longrightarrow & \mathcal{E}_f \\ \downarrow & & \downarrow \\ \mathcal{E}_c & \longrightarrow & \mathcal{E} \end{array}$$

induces a diagram of equivalences of categories

$$\begin{array}{ccc} Ho(\mathcal{E}_{f_c}) & \longrightarrow & Ho(\mathcal{E}_f) \\ \downarrow & & \downarrow \\ Ho(\mathcal{E}_c) & \longrightarrow & Ho(\mathcal{E}). \end{array}$$

A *fibrant replacement* of an object $X \in \mathcal{E}$ is a weak equivalence $X \rightarrow RX$ with codomain a fibrant object. Dually, a *cofibrant replacement* of X is a weak equivalence $LX \rightarrow X$ with domain a cofibrant object.

Recall from [Ho] that a cocontinuous functor $F : \mathcal{U} \rightarrow \mathcal{V}$ between two model categories is said to be a *left Quillen functor* if it takes a cofibration to a cofibration and an acyclic cofibration to an acyclic cofibration. Dually, a continuous functor $G : \mathcal{V} \rightarrow \mathcal{U}$ between two model categories is said to be a *right Quillen functor* if it takes a fibration to a fibration and an acyclic fibration to an acyclic fibration.

PROPOSITION 7.9. [Q] *Let $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ be an adjoint pair of functors between two model categories. Then F is a left Quillen functor iff G is a right Quillen functor.*

The adjoint pair (F, G) is said to be a *Quillen pair* if the conditions of 7.9 are satisfied.

The following lemma is due to Ken Brown, see [Ho] and [JT2].

LEMMA 7.10. *Let \mathcal{E} be a model category and $F : \mathcal{E} \rightarrow \mathcal{D}$ be a functor with values in a category equipped with a class of weak equivalences \mathcal{W}' which satisfies three-for-two. If F takes an acyclic cofibration between cofibrant objects to a weak equivalence, then it takes a weak equivalence between cofibrant objects to a weak equivalence.*

COROLLARY 7.11. *A left Quillen functor takes a weak equivalence between cofibrant objects to a weak equivalence.*

The following result is due to Reedy [Ree]. See [Hi] and [JT2].

PROPOSITION 7.12. *The cobase change along a cofibration of a weak equivalence between cofibrant objects is a weak equivalence.*

COROLLARY 7.13. *If every object of model category is cofibrant then the model structure is left proper.*

LEMMA 7.14. *In a model category, a cofibration is acyclic iff it has the left lifting property with respect to every fibration between fibrant objects.*

Proof: The necessity is clear. Conversely, let us suppose that a cofibration $u : A \rightarrow B$ has the left lifting property with respect to every fibration between fibrant objects. We shall prove that u is acyclic. For this, let us choose a fibrant replacement $j : B \rightarrow B'$ of the object B together with a factorisation of the composite $ju : A \rightarrow B'$ as a weak equivalence $i : A \rightarrow A'$ followed by a fibration $p : A' \rightarrow B'$. The square

$$\begin{array}{ccc} A & \xrightarrow{i} & A' \\ u \downarrow & & \downarrow p \\ B & \xrightarrow{j} & B' \end{array}$$

has a diagonal filler $d : B \rightarrow A'$ since p is a fibration between fibrant objects. The arrows i and j are invertible in the homotopy category since they are acyclic. The relations $pd = j$ and $du = i$ then implies that d is invertible in the homotopy category. It thus acyclic [Q]. It follows by three-for-two that u is acyclic. ■

PROPOSITION 7.15. *An adjoint pair of functors $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ between two model categories is a Quillen pair iff the following two conditions are satisfied:*

- F takes a cofibration to a cofibration;
- G takes a fibration between fibrant objects to a fibration.

Proof: The necessity is obvious. Let us prove the sufficiency. For this it suffices to show that F is a left Quillen functor by 7.9. Thus we show that F takes an acyclic cofibration $u : A \rightarrow B$ to an acyclic cofibration $F(u) : F(A) \rightarrow F(B)$. But $F(u)$ is acyclic iff it has the left lifting property with respect to every fibration between fibrant objects $f : X \rightarrow Y$ by Lemma 7.14. But the condition $F(u) \pitchfork f$ is equivalent to the condition $u \pitchfork G(f)$ by the adjointness $F \dashv G$. We have $u \pitchfork G(f)$ since $G(f)$ is a fibration by (ii). This proves that we have $F(u) \pitchfork f$. Thus, $F(u)$ is acyclic. ■

A left Quillen functor $F : \mathcal{U} \rightarrow \mathcal{V}$ induces a functor $F_c : \mathcal{U}_c \rightarrow \mathcal{V}_c$ hence also a functor $Ho(F_c) : Ho(\mathcal{U}_c) \rightarrow Ho(\mathcal{V}_c)$ by Proposition 7.11. A *left derived functor* is a functor

$$F^L : Ho(\mathcal{U}) \rightarrow Ho(\mathcal{V})$$

for which the following diagram of functors commutes up to isomorphism,

$$\begin{array}{ccc} Ho(\mathcal{U}_c) & \xrightarrow{Ho(F_c)} & Ho(\mathcal{V}_c) \\ \downarrow & & \downarrow \\ Ho(\mathcal{U}) & \xrightarrow{F^L} & Ho(\mathcal{V}), \end{array}$$

The functor F^L is unique up to a canonical isomorphism. It can be computed as follows. For each object $A \in \mathcal{U}$, we can choose a cofibrant replacement $\lambda_A : LA \rightarrow A$, with λ_A an acyclic fibration. We can then choose for each arrow $u : A \rightarrow B$ an arrow $L(u) : LA \rightarrow LB$ such that $u\lambda_A = \lambda_B L(u)$,

$$\begin{array}{ccc} LA & \xrightarrow{\lambda_A} & A \\ L(u) \downarrow & & \downarrow u \\ LB & \xrightarrow{\lambda_B} & B. \end{array}$$

Then

$$F^L([u]) = [F(L(u))] : FLA \rightarrow FLB.$$

A right Quillen functor $G : \mathcal{V} \rightarrow \mathcal{U}$ induces a functor $G_f : \mathcal{V}_f \rightarrow \mathcal{U}_f$ hence also a functor $Ho(G_f) : Ho(\mathcal{V}_f) \rightarrow Ho(\mathcal{U}_f)$ by Proposition 7.11. The *right derived functor* is a functor

$$G^R : Ho(\mathcal{V}) \rightarrow Ho(\mathcal{U})$$

for which the following diagram of functors commutes up to a canonical isomorphism,

$$\begin{array}{ccc} Ho(\mathcal{V}_f) & \xrightarrow{Ho(G_f)} & Ho(\mathcal{U}_f) \\ \downarrow & & \downarrow \\ Ho(\mathcal{V}) & \xrightarrow{G^R} & Ho(\mathcal{U}). \end{array}$$

The functor G^R is unique up to a canonical isomorphism. It can be computed as follows. For each object $X \in \mathcal{V}$ let us choose a fibrant replacement $\rho_X : X \rightarrow RX$, with ρ_X an acyclic cofibration. We can then choose for each arrow $u : X \rightarrow Y$ an arrow $R(u) : RX \rightarrow RY$ such that $R(u)\rho_X = \rho_Y u$,

$$\begin{array}{ccc} X & \xrightarrow{\rho_X} & RX \\ u \downarrow & & \downarrow R(u) \\ Y & \xrightarrow{\rho_Y} & RY. \end{array}$$

Then

$$G^R([u]) = [G(R(u))] : GRX \rightarrow GRY.$$

A Quillen pair of adjoint functors $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ induces a pair of adjoint functors

$$F^L : Ho(\mathcal{U}) \leftrightarrow Ho(\mathcal{V}) : G^R.$$

If $A \in \mathcal{U}$ is cofibrant, the adjunction unit $A \rightarrow G^R F^L(A)$ is obtained by composing the maps $A \rightarrow GFA \rightarrow GRFA$, where $FA \rightarrow RFA$ is a fibrant replacement of FA . If $X \in \mathcal{V}$ is fibrant, the adjunction counit $F^L G^R(X) \rightarrow X$ is obtained by composing the maps $FLGX \rightarrow FGX \rightarrow X$, where $LGX \rightarrow GX$ is a cofibrant replacement of GX .

DEFINITION 7.16. We shall say that a Quillen pair $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ is a *homotopy localisation* $\mathcal{U} \rightarrow \mathcal{V}$ if the right derived functor G^R is full and faithful. Dually, we shall say that the pair (F, G) is a *homotopy colocalisation* $\mathcal{V} \rightarrow \mathcal{U}$ if the left derived functor F^L is full and faithful.

PROPOSITION 7.17. *The following conditions on a Quillen pair $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ are equivalent:*

- *The pair (F, G) is a homotopy localisation $\mathcal{U} \rightarrow \mathcal{V}$;*
- *The map $FLGX \rightarrow X$ is a weak equivalence for every fibrant object $X \in \mathcal{V}$, where $LGX \rightarrow GX$ denotes a cofibrant replacement of GX ;*
- *The map $FLGX \rightarrow X$ is a weak equivalence for every fibrant-cofibrant object $X \in \mathcal{V}$, where $LGX \rightarrow GX$ denotes a cofibrant replacement of GX .*

Proof: The functor G^R is full and faithful iff the counit of the adjunction $F^L \dashv G^R$ is an isomorphism. But if $X \in \mathcal{V}$ is fibrant, this counit is obtained by composing the maps $FLGX \rightarrow FGX \rightarrow X$, where $LGX \rightarrow GX$ is a cofibrant replacement of GX . This proves the equivalence (i) \Leftrightarrow (ii). The implication (ii) \Rightarrow (iii) is obvious. Let us prove the implication (iii) \Rightarrow (ii). For every fibrant object X , there is a an acyclic fibration $p : Y \rightarrow X$ with domain a cofibrant object Y . The map $Gp : GY \rightarrow GX$ is an acyclic fibration, since G is a right Quillen functor. Let $q : LGY \rightarrow GY$ be a cofibrant replacement of GY . Then the map $FLGY \rightarrow FGY \rightarrow Y$ is a weak equivalence by assumption, since Y is fibrant-cofibrant. But the composite $G(p)q : LGY \rightarrow GY \rightarrow GX$ is a cofibrant replacement of GX , since $G(p)$ is a weak equivalence. Moreover, the composite $FLGY \rightarrow FGX \rightarrow X$ is a weak equivalence, since p is a weak equivalence and the following diagram commutes

$$\begin{array}{ccccc}
 FLGY & \longrightarrow & FGY & \longrightarrow & Y \\
 & \searrow & \downarrow & & \downarrow p \\
 & & FGX & \longrightarrow & X.
 \end{array}$$

This proves that condition (ii) is satisfied for a cofibrant replacement of GX . ■

PROPOSITION 7.18. *If $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ is a homotopy localisation, then the right adjoint G preserves and reflects weak equivalences between fibrant objects.*

Proof: The functor G^R is equivalent to the functor $Ho(G_f) : Ho(\mathcal{V}_f) \rightarrow Ho(\mathcal{U}_f)$ induced by the functor G . Thus, $Ho(G_f)$ is full and faithful since G^R is full and faithful. This proves the result since a full and faithful functor is conservative. ■

PROPOSITION 7.19. *Let $M_i = (C_i, \mathcal{W}_i, \mathcal{F}_i)$ ($i = 1, 2$) be two model structures on a category \mathcal{E} . Suppose that $C_1 \subseteq C_2$ and $\mathcal{W}_1 \subseteq \mathcal{W}_2$. Then the identity functor $\mathcal{E} \rightarrow \mathcal{E}$ is a homotopy localisation $M_1 \rightarrow M_2$.*

DEFINITION 7.20. Let $M_i = (C_i, \mathcal{W}_i, \mathcal{F}_i)$ ($i = 1, 2$) be two model structures on a category \mathcal{E} . If $C_1 = C_2$ and $\mathcal{W}_1 \subseteq \mathcal{W}_2$, we shall say that M_2 is a *Bousfield localisation* of M_1 .

PROPOSITION 7.21. *Let $M_2 = (C_2, \mathcal{W}_2, \mathcal{F}_2)$ be a Bousfield localisation of a model structure $M_1 = (C_1, \mathcal{W}_1, \mathcal{F}_1)$ on a category \mathcal{E} . Then a map between M_2 -fibrant objects is a M_2 -fibration iff it is a M_1 -fibration.*

Proof: By hypothesis, we have $C_1 = C_2$ and $\mathcal{W}_1 \subseteq \mathcal{W}_2$. It follows that we have $\mathcal{F}_2 \cap \mathcal{W}_2 = \mathcal{F}_1 \cap \mathcal{W}_1$ and $\mathcal{F}_2 \subseteq \mathcal{F}_1$. Let $f : X \rightarrow Y$ be a map between two M_2 -fibrant objects. Let us show that f is a M_2 -fibration iff it is a M_1 -fibration. The implication (\Rightarrow) is clear, since $\mathcal{F}_2 \subseteq \mathcal{F}_1$. Conversely, if $f \in \mathcal{F}_1$, let us show that $f \in \mathcal{F}_2$. Let us choose a factorisation $f = pi : X \rightarrow Z \rightarrow Y$ with $i \in C_2 \cap \mathcal{W}_2$ and $p \in \mathcal{F}_2$. We have $i \in \mathcal{W}_1$ by Proposition 7.11, since the identity functor is a right Quillen functor $M_2 \rightarrow M_1$ and since i is a map between M_2 -fibrant objects. Thus, $i \in \mathcal{W}_1 \cap C_1$, since $C_1 = C_2$. Hence the square

$$\begin{array}{ccc}
 X & \xrightarrow{id} & X \\
 i \downarrow & & \downarrow f \\
 E & \xrightarrow{p} & Y
 \end{array}$$

has a diagonal filler, making f a retract of p and therefore $f \in \mathcal{F}_2$. ■

A Quillen pair (F, G) is said to be a *Quillen equivalence* if the adjoint pair (F^L, G^R) is an equivalence of categories.

PROPOSITION 7.22. *A Quillen pair $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ is a Quillen equivalence iff the following equivalent conditions are satisfied:*

- *The pair (F, G) is both a homotopy localisation and colocalisation;*
- *The pair (F, G) is a homotopy localisation and the functor F reflects weak equivalences between cofibrant objects;*
- *The pair (F, G) is a homotopy colocalisation and the functor G reflects weak equivalences between fibrant objects;*

The composite of two adjoint pairs

$$F_1 : \mathcal{E}_1 \leftrightarrow \mathcal{E}_2 : G_1 \quad \text{and} \quad F_2 : \mathcal{E}_2 \leftrightarrow \mathcal{E}_3 : G_2$$

is an adjoint pair $F_2F_1 : \mathcal{E}_1 \leftrightarrow \mathcal{E}_3 : G_1G_2$.

PROPOSITION 7.23 (Three-for-two, [Ho]). *The composite of two Quillen pairs (F_1, G_1) and (F_2, G_2) is a Quillen pair (F_2F_1, G_1G_2) . Moreover, if two of the pairs (F_1, G_1) , (F_2, G_2) and (F_2F_1, G_1G_2) are Quillen equivalences, then so is the third.*

DEFINITION 7.24. [Ho] We shall say that a functor of two variables between three model categories

$$\odot : \mathcal{E}_1 \times \mathcal{E}_2 \rightarrow \mathcal{E}_3$$

is a *left Quillen functor* if it is cocontinuous in each variable and the following conditions are satisfied:

- $u \odot' v$ is a cofibration if $u \in \mathcal{E}_1$ and $v \in \mathcal{E}_2$ are cofibrations;
- $u \odot' v$ is an acyclic cofibration if $u \in \mathcal{E}_1$ and $v \in \mathcal{E}_2$ are cofibrations and if u or v is acyclic.

Dually, we shall say that \odot is a *right Quillen functor* if the opposite functor $\odot^\circ : \mathcal{E}_1^\circ \times \mathcal{E}_2^\circ \rightarrow \mathcal{E}_3^\circ$ is a left Quillen functor.

PROPOSITION 7.25. *Let $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \rightarrow \mathcal{E}_3$ be a left Quillen functor of two variables between three model categories. If $A \in \mathcal{E}_1$ is cofibrant, then the functor $B \mapsto A \odot B$ is a left Quillen functor $\mathcal{E}_2 \rightarrow \mathcal{E}_3$.*

Proof: If $A \in \mathcal{E}_1$ is cofibrant, then the map $i_A : \perp \rightarrow A$ is a cofibration, where \perp is the initial object. If $v : S \rightarrow T$ is a map in \mathcal{E}_2 , then we have $A \odot v = i_A \odot' v$. Thus, $A \odot v$ is a cofibration if v is a cofibration and $A \odot v$ is acyclic if moreover v is acyclic. ■

PROPOSITION 7.26. *Let $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \rightarrow \mathcal{E}_3$ be a functor of two variables between three model categories. If the functor \odot is divisible on the left, then it is a left Quillen functor iff the corresponding left division functor $\mathcal{E}_1^\circ \times \mathcal{E}_3 \rightarrow \mathcal{E}_2$ is a right Quillen functor. Dually, if the functor \odot is divisible on the right, then it is a left Quillen functor iff the corresponding right division functor $\mathcal{E}_3 \times \mathcal{E}_2^\circ \rightarrow \mathcal{E}_1$ is a right Quillen functor.*

PROPOSITION 7.27. Let $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \rightarrow \mathcal{E}_3$ be a functor of two variables, cocontinuous in each, between three model categories. Suppose that the following three conditions are satisfied:

- If $u \in \mathcal{E}_1$ and $v \in \mathcal{E}_2$ are cofibrations, then so is $u \odot' v$;
- the functor $(-)\odot B$ preserves acyclic cofibrations for every object $B \in \mathcal{E}_2$;
- the functor $A \odot (-)$ preserves acyclic cofibrations for every object $A \in \mathcal{E}_1$.

Then \odot is a left Quillen functor.

Proof: Let $u : A \rightarrow B$ be a cofibration in \mathcal{E}_1 and $v : S \rightarrow T$ be a cofibration in \mathcal{E}_2 . Let us show that $u \odot' v$ is acyclic if u or v is acyclic. We only consider the case where v is acyclic. Consider the commutative diagram

$$\begin{array}{ccccc}
 A \odot S & \xrightarrow{u \odot S} & B \odot S & & \\
 A \odot v \downarrow & & i_2 \downarrow & \searrow B \odot v & \\
 A \odot T & \xrightarrow{i_1} & Z & \xrightarrow{u \odot' v} & B \odot T
 \end{array}$$

where $Z = A \odot T \sqcup_{A \odot S} B \odot S$ and where $(u \odot' v)i_1 = u \odot T$. The map $A \odot v$ is an acyclic cofibration since v is an acyclic cofibration. Similarly for the map $B \odot v$. It follows that i_2 is an acyclic cofibration by cobase change. Thus, $u \odot' v$ is acyclic by three-for-two since $(u \odot' v)i_2 = B \odot v$ is acyclic.

DEFINITION 7.28. [Ho] A model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on a monoidal closed category $\mathcal{E} = (\mathcal{E}, \otimes)$ is said to be *monoidal* if the tensor product $\otimes : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E}$ is a left Quillen functor of two variables and if the unit object of the tensor product is cofibrant.

In a monoidal closed model category, if f is a fibration then so are the maps $\langle u \setminus f \rangle$ and $\langle f / u \rangle$ for any cofibration u . Moreover, the fibrations $\langle u \setminus f \rangle$ and $\langle f / u \rangle$ are acyclic if the cofibration u is acyclic or the fibration f is acyclic.

DEFINITION 7.29. We shall say that a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on a cartesian closed category \mathcal{E} is *cartesian closed* if the cartesian product $\times : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E}$ is a left Quillen functor of two variables and if the terminal object 1 is cofibrant.

In a cartesian closed model category, if f is a fibration and u is a cofibration, then the map $\langle u, f \rangle$ is a fibration, which is acyclic if u or f is acyclic.

We recall a few notions of enriched category theory [K]. Let $\mathcal{V} = (\mathcal{V}, \otimes, \sigma)$ a bicomplete symmetric monoidal closed category. A category enriched over \mathcal{V} is called a \mathcal{V} -category. If \mathcal{A} and \mathcal{B} are \mathcal{V} -categories, there is a notion of a *strong* functor $F : \mathcal{A} \rightarrow \mathcal{B}$; it is an ordinary functor equipped with a *strength* which is a natural transformation $\mathcal{A}(X, Y) \rightarrow \mathcal{B}(FX, FY)$ preserving composition and units. A natural transformation $\alpha : F \rightarrow G$ between strong functors $F, G : \mathcal{A} \rightarrow \mathcal{B}$ is said to be *strong* if the following square commutes

$$\begin{array}{ccc}
 \mathcal{A}(X, Y) & \longrightarrow & \mathcal{B}(GX, GY) \\
 \downarrow & & \downarrow \mathcal{B}(\alpha_X, GY) \\
 \mathcal{B}(FX, FY) & \xrightarrow{\mathcal{B}(FX, \alpha_Y)} & \mathcal{B}(FX, GY).
 \end{array}$$

for every pair of objects $X, Y \in \mathcal{A}$. A *strong adjunction* $\theta : F \dashv G$ between strong functors $F : \mathcal{A} \rightarrow \mathcal{B}$ and $G : \mathcal{B} \rightarrow \mathcal{A}$ is a strong natural isomorphism

$$\theta_{XY} : \mathcal{A}(FX, Y) \rightarrow \mathcal{B}(X, GY).$$

A strong functor $G : \mathcal{B} \rightarrow \mathcal{A}$ has a strong left adjoint iff it has an ordinary left adjoint $F : \mathcal{A} \rightarrow \mathcal{B}$ and the map

$$\mathcal{B}(FX, Y) \longrightarrow \mathcal{A}(GFX, GY) \xrightarrow{\mathcal{A}(\eta_X, GY)} \mathcal{A}(X, GY)$$

obtained by composing with the unit η_X of the adjunction is an isomorphism for every pair of objects $X \in \mathcal{A}$ and $Y \in \mathcal{B}$. Recall that a \mathcal{V} -category \mathcal{E} is said to admit *tensor products* if the functor $Y \mapsto \mathcal{E}(X, Y)$ admits a strong left adjoint $A \mapsto A \otimes X$ for every object $X \in \mathcal{E}$. A \mathcal{V} -category is said to be (strongly) *cocomplete* if it cocomplete as an ordinary category and if it admits tensor products. These notions can be dualised. A \mathcal{V} -category \mathcal{E} is said to admit *cotensor products* if the opposite \mathcal{V} -category \mathcal{E}^o admits tensor products. This means that the (contravariant) functor $X \mapsto \mathcal{E}(X, Y)$ admits a strong right adjoint $A \mapsto Y^{[A]}$ for every object $Y \in \mathcal{E}$. A \mathcal{V} -category is said to be (strongly) *complete* if it complete as an ordinary category and if it admits cotensor products. We shall say that a \mathcal{V} -category is (strongly) *bicomplete* if it is both \mathcal{V} -complete and cocomplete.

DEFINITION 7.30. [**Q**] Let \mathcal{E} be a strongly bicomplete simplicial category. We shall say that a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on \mathcal{E} is *simplicial* if the tensor product

$$\otimes : \mathbf{S} \times \mathcal{E} \rightarrow \mathcal{E}$$

is a left Quillen functor of two variables, where \mathbf{S} is equipped with the model structure $(\mathcal{C}_0, \mathcal{W}_0, \mathcal{F}_0)$ of 1.1.

A simplicial category equipped with a simplicial model structure is called a *simplicial model category*.

PROPOSITION 7.31. *Let \mathcal{E} be a simplicial model category. Then a map between cofibrant objects $u : A \rightarrow B$ is acyclic iff the map of simplicial sets*

$$\mathcal{E}(u, X) : \mathcal{E}(B, X) \rightarrow \mathcal{E}(A, X)$$

is a weak homotopy equivalence for every fibrant object X .

Proof: The functor $A \mapsto \mathcal{E}(A, X)$ takes an (acyclic) cofibration to an (acyclic) Kan fibration if X is fibrant. It then follows by Proposition 7.11 that it takes an acyclic map between cofibrant objects to an acyclic map. Conversely, let $u : A \rightarrow B$ be a map between cofibrant objects in \mathcal{E} . If the map $\mathcal{E}(u, X) : \mathcal{E}(B, X) \rightarrow \mathcal{E}(A, X)$ is a weak homotopy equivalence for every fibrant object X , let us show that u is acyclic. Let us first suppose that A and B are fibrant. Let \mathcal{E}_{cf} be the full subcategory of fibrant and cofibrant objects of \mathcal{E} . We shall prove that u is acyclic by showing that u is invertible in the homotopy category $Ho(\mathcal{E}_{cf})$. But if $S, X \in \mathcal{E}_{cf}$, then we have $Ho(\mathcal{E}_{cf})(S, X) = \pi_0 \mathcal{E}(S, X)$ by [**Q**]. Hence the map $Ho(\mathcal{E}_{cf})(u, X) : Ho(\mathcal{E}_{cf})(B, X) \rightarrow Ho(\mathcal{E}_{cf})(A, X)$ is equal to the map $\pi_0 \mathcal{E}(u, X) : \pi_0 \mathcal{E}(B, X) \rightarrow \pi_0 \mathcal{E}(A, X)$. But the map $\pi_0 \mathcal{E}(u, X)$ is bijective since the map $\mathcal{E}(u, X)$ is a weak homotopy equivalence. This shows that the map $Ho(\mathcal{E}_{cf})(u, X)$ is bijective for every $X \in \mathcal{E}_{cf}$. It follows by the Yoneda lemma that u is invertible in $Ho(\mathcal{E}_{cf})$. Thus, u is acyclic by [**Q**]. In the general case, let us choose a fibrant replacement $i_A : A \rightarrow A'$ with i_A an acyclic cofibration. Similarly, let us choose a fibrant

replacement $i_B : B \rightarrow B'$ with i_B an acyclic cofibration. Then there exists a map $u' : A' \rightarrow B'$ such that $u'i_A = i_B u$. We then have a commutative square

$$\begin{CD} \mathcal{E}(B', X) @>>> \mathcal{E}(A', X) \\ @VVV @VVV \\ \mathcal{E}(B, X) @>>> \mathcal{E}(A, X) \end{CD}$$

for every object X . If X is fibrant, then the vertical maps of the square are weak homotopy equivalences by the first part of the proof. Hence also the map $\mathcal{E}(u', X) : \mathcal{E}(B', X) \rightarrow \mathcal{E}(A', X)$ by three-for-two. This shows that u' is an acyclic map since A' and B' are fibrant. It follows by three-for-two that u is acyclic. ■

Let \mathcal{E} be a bicomplete category. The *box product* of a simplicial set A by an object $B \in \mathcal{E}$ is defined to be the simplicial object $A \square B \in [\Delta^\circ, \mathcal{E}]$ obtained by putting

$$(A \square B)_n = A_n \times B$$

for every $n \geq 0$, where $A_n \times B$ denotes the coproduct of A_n copies of the object B . The functor $\square : \mathbf{S} \times \mathcal{E} \rightarrow [\Delta^\circ, \mathcal{E}]$ is divisible on both sides. If $X \in [\Delta^\circ, \mathcal{E}]$ and $A \in \mathbf{S}$, then

$$A \setminus X = \int_{[n] \in \Delta} X_n^{A_n}.$$

If $B \in \mathcal{E}$, then $(X/B)_n = \mathcal{E}(B, X_n)$ for every $n \geq 0$.

To a map $u : A \rightarrow B$ in \mathbf{S} and a map $v : S \rightarrow T$ in \mathcal{E} , we can associate the map

$$u \square' v : A \square T \sqcup_{A \square S} B \square S \longrightarrow B \square T$$

in $[\Delta^\circ, \mathcal{E}]$. If $f : X \rightarrow Y$ is a map in $[\Delta^\circ, \mathcal{E}]$ we then have a map

$$\langle u \setminus f \rangle : B \setminus X \longrightarrow B \setminus Y \times_{A \setminus Y} A \setminus X$$

in \mathcal{E} and a map

$$\langle f / v \rangle : X / T \longrightarrow Y / T \times_{Y / S} X / S$$

in \mathbf{S} .

In Reedy theory [Ho], the object $\partial \Delta[n] \setminus X$ is the *matching space* $M_n X$. If δ_n denotes the inclusion $\partial \Delta[n] \subset \Delta[n]$, then $\delta_n \setminus X$ is the canonical map $X_n \rightarrow M_n X$. If $f : X \rightarrow Y$ is a map in $[\Delta^\circ, \mathcal{E}]$ then $\langle \delta_n \setminus f \rangle$ is the *matching map*

$$X_n \longrightarrow Y_n \times_{M_n Y} M_n X$$

obtained from the square

$$\begin{CD} X_n @>>> M_n X \\ @VVV @VVV \\ Y_n @>>> M_n Y. \end{CD}$$

These constructions can be dualised. Recall that a *cosimplicial set* is a covariant functor $\Delta \rightarrow \mathbf{Set}$. If A is a cosimplicial set and $X \in [\Delta^\circ, \mathcal{E}]$ we shall put

$$A \otimes X = \int^{[n] \in \Delta} A_n \times X_n.$$

Let us denote by $\Delta^c[n]$ the cosimplicial object $\Delta([n], -) : \Delta \rightarrow \mathbf{Set}$. Its boundary is defined to be the maximal proper sub-object $\partial\Delta^c[n] \subset \Delta^c[n]$. In Reedy theory [Ho], the object $\partial\Delta^c[n] \otimes X$ is the *latching space* $L_n X$. If δ_n^c denotes the inclusion $\partial\Delta^c[n] \subset \Delta^c[n]$, then $\delta_n^c \otimes X$ is the canonical map $L_n X \rightarrow X_n$. If $f : X \rightarrow Y$ is a map in $[\Delta^o, \mathcal{E}]$, then the map $\delta_n^c \otimes' f$ is the *latching map*

$$L_n Y \sqcup_{L_n X} X_n \longrightarrow Y_n$$

obtained from the square

$$\begin{array}{ccc} L_n X & \longrightarrow & X_n \\ \downarrow & & \downarrow \\ L_n Y & \longrightarrow & Y_n. \end{array}$$

Let $(\mathcal{A}, \mathcal{B})$ be a weak factorisation system in a finitely bicomplete category \mathcal{E} . We shall say that a map $f : X \rightarrow Y$ in $[\Delta^o, \mathcal{E}]$ is a *Reedy \mathcal{A} -cofibration* if the latching map $\delta_n^c \otimes' f$ belongs to \mathcal{A} for every $n \geq 0$. We shall say that f is a *Reedy \mathcal{B} -fibration* if the matching map $\langle \delta_n \setminus f \rangle$ belongs to \mathcal{B} for every $n \geq 0$.

THEOREM 7.32. *Let $(\mathcal{A}, \mathcal{B})$ be a weak factorisation system in a finitely bicomplete category \mathcal{E} . If \mathcal{A}' is the class of Reedy \mathcal{A} -cofibrations in $[\Delta^o, \mathcal{E}]$ and \mathcal{B}' is the class of Reedy \mathcal{B} -fibrations, then the pair $(\mathcal{A}', \mathcal{B}')$ is a weak factorisation system.*

PROPOSITION 7.33. *Let $(\mathcal{A}, \mathcal{B})$ be a weak factorisation system in a bicomplete category \mathcal{E} . Then the following conditions on a map $f \in [\Delta^o, \mathcal{E}]$ are equivalent:*

- *f is a Reedy \mathcal{B} -fibration;*
- *the map $\langle u \setminus f \rangle$ belongs to \mathcal{B} for every monomorphism $u \in \mathbf{S}$;*
- *the map $\langle f/v \rangle$ is a trivial fibration for every $v \in \mathcal{A}$.*

Proof: The implication (ii) \Rightarrow (i) is obvious. Let us prove the implication (i) \Rightarrow (iii). By 1.2 it suffices to show that we have $\delta_m \pitchfork \langle f/v \rangle$ for every $m \geq 0$. But the condition $\delta_m \pitchfork \langle f/v \rangle$ is equivalent to the condition $v \pitchfork \langle \delta_m \setminus f \rangle$ by 2.1. We have $\langle \delta_m \setminus f \rangle \in \mathcal{B}$ since f is a Reedy \mathcal{B} -fibration by assumption. Thus, $v \pitchfork \langle \delta_m \setminus f \rangle$ since $v \in \mathcal{A}$. This proves that $\langle f/v \rangle$ is a trivial fibration. Let us prove the implication (iii) \Rightarrow (ii). It suffices to show that we have $v \pitchfork \langle u \setminus f \rangle$ for every $v \in \mathcal{A}$. But the condition $v \pitchfork \langle u \setminus f \rangle$ is equivalent to the condition $u \pitchfork \langle f/v \rangle$ by 2.1. This proves the result since $\langle f/v \rangle$ is a trivial fibration by hypothesis. ■

COROLLARY 7.34. *Let $(\mathcal{A}, \mathcal{B})$ be a weak factorisation system in a finitely bicomplete category \mathcal{E} . If a map $f : X \rightarrow Y$ is a Reedy \mathcal{B} -fibration, then the map $f_n : X_n \rightarrow Y_n$ belongs to \mathcal{B} for every $n \geq 0$*

Proof: For any simplicial set A , we have $A \setminus f = \langle i_A \setminus f \rangle$, where i_A denotes the inclusion $\emptyset \subseteq A$. Thus, $A \setminus f \in \mathcal{B}$ by 7.33. This proves the result if we take $A = \Delta[n]$. ■

Let \mathcal{E} be a model category with model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$. We shall say that a map $f : X \rightarrow Y$ in $[\Delta^\circ, \mathcal{E}]$ is *level-wise acyclic* if the map $f_n : X_n \rightarrow Y_n$ is acyclic for every $n \geq 0$. We shall say that f is a *Reedy cofibration* if it is a Reedy \mathcal{C} -cofibration. We shall say that f is a *Reedy fibration* if it is a Reedy \mathcal{F} -fibration.

THEOREM 7.35 (Reedy, see [Ree]). *Let $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ be a model structure on a category \mathcal{E} . Then the category $[\Delta^\circ, \mathcal{E}]$ admits a model structure $(\mathcal{C}', \mathcal{W}', \mathcal{F}')$ in which \mathcal{C}' is the class of Reedy cofibrations, \mathcal{W}' is the class of level-wise acyclic maps and \mathcal{F}' is the class of Reedy fibrations. A map $u : A \rightarrow B$ in $[\Delta^\circ, \mathcal{E}]$ is an acyclic Reedy cofibration iff the latching map $\delta_n^c \otimes u$ is an acyclic cofibration for every $n \geq 0$. Dually, a map $f : X \rightarrow Y$ is an acyclic Reedy fibration iff the map $\langle \delta_n \setminus f \rangle$ is an acyclic fibration for every $n \geq 0$.*

We call $(\mathcal{C}', \mathcal{W}', \mathcal{F}')$ the *Reedy model structure* associated to $(\mathcal{C}, \mathcal{W}, \mathcal{F})$.

PROPOSITION 7.36. *Let \mathcal{E} be a bicomplete model category. If $u \in \mathbf{S}$ is monic and $v \in \mathcal{E}$ is a cofibration, then $u \square v$ is a Reedy cofibration which is acyclic if v is acyclic.*

Proof: If u is monic and v is a cofibration, let us show that the map $u \square v$ is a Reedy cofibration. For this, it suffices to show that we have $(u \square v) \pitchfork f$ for every acyclic Reedy fibration f . Let $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ be the model structure on \mathcal{E} . The map f is a Reedy $\mathcal{F} \cap \mathcal{W}$ -fibration by 7.35. Hence the map $\langle f/v \rangle$ is a trivial fibration by 7.33, since $v \in \mathcal{C}$ and since the pair $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ is a weak factorisation system. Hence we have $u \pitchfork \langle f/v \rangle$, since u is monic. It follows that we have $(u \square v) \pitchfork f$ by 2.1. Let us now show that $u \square v$ is acyclic if moreover v is acyclic. For this, it suffices to show that we have $(u \square v) \pitchfork f$ for every Reedy fibration f . The map $\langle f/v \rangle$ is a trivial fibration by 7.33, since $v \in \mathcal{C} \cap \mathcal{W}$ and since the pair $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ is a weak factorisation system. Hence we have $u \pitchfork \langle f/v \rangle$, since u is monic. It follows that we have $(u \square v) \pitchfork f$ by 2.1. ■

The object $[0]$ is terminal in the category Δ . Hence the functor $i : 1 \rightarrow \Delta$ defined by putting $i(1) = [0]$ is right adjoint to the constant functor $p : \Delta \rightarrow 1$. It follows that the functor $i^* : [\Delta^\circ, \mathcal{E}] \rightarrow \mathcal{E}$ is right adjoint to the functor $p^* : \mathcal{E} \rightarrow [\Delta^\circ, \mathcal{E}]$. If $X \in [\Delta^\circ, \mathcal{E}]$, then $i^*(X) = X_0$; if $A \in \mathcal{E}$, then $p^*(A) = 1 \square A$ is the constant simplicial object with value A .

PROPOSITION 7.37. *Let \mathcal{E} be a model category. Then the pair of adjoint functors*

$$p^* : \mathcal{E} \leftrightarrow [\Delta^\circ, \mathcal{E}] : i^*$$

is a Quillen pair, where $[\Delta^\circ, \mathcal{E}]$ is given the Reedy model structure.

Proof: It suffices to show that the functor p^* is a left Quillen functor by 7.9. This follows from 7.36 since $p^*(A) = 1 \square A$.

8. Epilogue

A Quillen model category is the primary example of an “abstract homotopy theory” [Q]. It permits the construction of homotopy pullbacks and pushouts, of fiber and cofiber sequences, etc. Dwyer and Kan have proposed using general simplicial categories to model general homotopy theories, see [DK1] and [DK2].

Recently, Bergner [B1] has established a model structure on the category of simplicial categories. In [B2], she introduces a new model structure *Segal cat'* on the category of Segal precategories and she obtains a chain of Quillen equivalences:

$$\text{Simp.cat} \rightarrow \text{Segal cat}' \leftarrow \text{Segal cat} \leftarrow \text{Comp. Segal sp}$$

(in this chain, a Quillen equivalence is represented by the right adjoint functor). See [B4] for a survey. It follows from her results combined the result proved in the present paper that the model structure for simplicial categories is indirectly Quillen equivalent to the model structure for quasi-categories. We shall see in [J4] that the coherent nerve functor defined by Cordier [C] and studied in Cordier-Porter [CP] defines a direct Quillen equivalence

$$\text{Simp.cat} \rightarrow \text{Quasicat}$$

between the model structure for simplicial categories and the model structure for quasi-categories. See Lurie [Lu1] for another proof. An axiomatic approach to proving all the equivalences above was proposed by Töen in [T2].

There are other important notions of homotopy theories, for example the A_∞ -spaces of Stasheff [MSS]. See also Batanin for A_∞ -categories [Ba]. A theory of homotopical categories was developed by Dwyer, Hirschhorn, Kan and Smith [DHKS]. Dugger studies universal homotopy theories in [D]. The model structure for quasi-categories belongs to a class of model structures in presheaf categories studied by Cisinski [Ci]. A quite different notion of homotopy theory was introduced by Heller [He] based on the idea of hyperdoctrine of Lawvere [La]. A similar notion called *dérivateur* was later introduced by Grothendieck and studied by Maltsoniotis [M1]. See [M2] for an extension.

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