NUMBER 156

Sören Illman

# Equivariant singular homology and cohomology I

MEMOIRS
OF THE AMERICAN
MATHEMATICAL SOCIETY

VOLUME 1 · ISSUE 2 · NUMBER 156 (first of 3 numbers) · MARCH 1975 · CODEN: MAMCAU

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**Published bimonthly** beginning in January, by the American Mathematical Society. Subscriptions for journals published by the American Mathematical Society should be addressed to American Mathematical Society, P.O. Box 1571, Annex Station, Providence, Rhode Island 02901.

Second-class postage permit pending at Providence, Rhode Island, and additional mailing offices.

Memoirs of the American Mathematical Society

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Published by the

AMERICAN MATHEMATICAL SOCIETY

Providence, Rhode Island

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Abstract

Let G be a topological group. We construct an equivariant homology and equivariant cohomology theory, defined on the category of all G-pairs and G-maps, which both satisfy all seven equivariant Eilenberg-Steenrod axioms and have a given covariant and contravariant, respectively, coefficient system as coefficients. We also establish some further properties of these equivariant singular homology and cohomology theories, such as, a naturality property in the transformation group, transfer homomorphisms and a cup-product in equivariant singular cohomology with coefficients in a commutative ring coefficient system.

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AMS (MOS) Subject Classifications (1970). Primary 55B99, 55B25, 57E99.

ISBN 0-8218-1856-2.

# EQUIVARIANT SINGULAR HOMOLOGY AND COHOMOLOGY I

By Sören Illman

Let $G$ be a topological group. By a G-space $X$ we mean a topological space $X$ together with a left action of $G$ on $X$. A G-pair $(X,A)$ consists of a G-space $X$ and a G-subspace $A$ of $X$. The notions G-map, G-homotopy, etc. have the usual meaning. In Chapter I of this paper we construct an equivariant homology and cohomology theory, defined on the category of all G-pairs and G-maps, which both satisfy all seven equivariant Eilenberg-Steenrod axioms and which have a given covariant coefficient system $k$ and contravariant coefficient system $m$, respectively, as coefficients. See Definition 1.2. and Theorems 2.1. and 2.2. in Chapter I for precise statements. We call these equivariant homology and cohomology theories for "equivariant singular homology with coefficients in $k$" and "equivariant singular cohomology with coefficients in $m$". The construction of equivariant singular homology and cohomology is very much analogous to the construction of ordinary singular homology and cohomology. The ordinary singular theory in its present form is due to S. Eilenberg [5]. We have chosen the exposition in Eilenberg-Steenrod [6] as our model. This applies especially to the proof of the excision axiom.

For actions of discrete groups equivariant cohomology and homology theories which satisfy all seven equivariant Eilenberg-Steenrod axioms and have predescribed coefficients exist before, see G. Bredon [2], [3] and Th. Bröcker [4].

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Received by the editors March 6, 1974.

In Chapter II we establish some further properties of equivariant singular homology and cohomology. We prove a naturality property in the transformation group and construct transfer homomorphisms in both equivariant singular homology and cohomology. Moreover we define a Kronecker index and also a cup-product in equivariant singular cohomology with coefficients in a commutative ring coefficient system. We conclude by proving that this cup-product is commutative.

This paper is a slightly extended and simplified version of chapter III and part of chapter IV of my thesis [8]. A geometrically more intuitive but technically more complicated construction of equivariant homology and cohomology theories which satisfy all seven equivariant Eilenberg-Steenrod axioms and have predescribed coefficients is given in [7], where also some other results from [8] can be found. This paper gives the details for everything stated in [9].

# I. EQUIVARIANT SINGULAR THEORY

## 1. COEFFICIENT SYSTEMS

In the following G denotes an arbitrary topological group. Let R be a ring with identity element. All R-modules will be unitary.

DEFINITION 1.1. A family $\mathfrak{F}$ of subgroups of G is called an orbit type family for G if the following is true: if $H \in \mathfrak{F}$ and H' is conjugate to H, then $H \in \mathfrak{F}$.

Thus the family of all closed subgroups of G, and the family of all finite subgroups of G are examples of orbit type families for G. A more special example is the following. Let G = O(n), and let $\mathfrak{F}$ be the family of all subgroups conjugate to O(m) (standard inbedding) for some m, where $0 \leq m \leq n$.

DEFINITION 1.2. Let $\mathfrak{F}$ be an orbit type family for G. A covariant coefficient system k for $\mathfrak{F}$ over the ring R is a covariant functor from the category of G-spaces of the form G/H, where $H \in \mathfrak{F}$, and G-homotopy classes of G-maps to the category of <u>left</u> R-modules.

A contravariant coefficient system m for $\mathfrak{F}$ over the ring R is a contravariant functor from the category of G-spaces of the form G/H, where $H \in \mathfrak{F}$, and G-homotopy classes of G-maps to the category of <u>right</u> R-modules.

If $\alpha: G/H \to G/K$ is a G-map, and $H,K \in \mathfrak{F}$, we denote

$$k(\alpha) = \alpha_* : k(G/H) \to k(G/K),$$

and

$$m(\alpha) = \alpha^* : m(G/K) \to m(G/H).$$

Let k and k' be covariant coefficient systems for $\mathfrak{F}$. A natural transformation

$$\theta : k \to k'$$

will be called a homomorphism of covariant coefficient systems. If $\theta$ is natural equivalence, we vall $\theta$ an isomorphism. Similarly for contravariant coefficient systems.

## 2. THE EXISTENCE THEOREMS FOR EQUIVARIANT SINGULAR HOMOLOGY AND COHOMOLOGY

THEOREM 2.1. Let $G$ be a topological group. Let $\mathfrak{F}$ be an orbit type family for $G$, and let $k$ be a covariant coefficient system for $\mathfrak{F}$ over the ring $R$.

Then there exists an equivariant homology theory $H_*^G(\ ;k)$, defined on the category of all G-pairs and all G-maps, and with values in the category of left R-modules, which satisfies all seven equivariant Eilenberg-Steenrod axioms and which has the given coefficient system $k$ as coefficients.

This means:

For each G-pair $(X,A)$ we have a left R-module $H_n^G(X,A;k)$ for every integer n.

Each G-map $f : (X,A) \to (Y,B)$ induces a homomorphism

$$f_* : H_n^G(X,A;k) \to H_n^G(Y,B;k)$$

for every integer n.

Each G-pair $(X,A)$ determines a boundary homomorphism

$$\partial : H_n^G(X,A;k) \to H_{n-1}^G(A;k)$$

for every integer $n$.

In addition the following axioms are satisfied.

A.1. If f = identity, then $f_*$ = identity.

A.2. $f : (X,A) \to (Y,B)$ and $f' = (Y,B) \to (Z,C)$ are G-maps, then

$$(f'f)_* = f'_* f_*.$$

A.3. For any G-map $f : (X,A) \to (Y,B)$ we have

$$\partial f_* = (f|A)_* \, \partial.$$

A.4. (Exactness axiom). Any G-pair $(X,A)$ gives rise to an exact homology sequence

$$\cdots \xleftarrow{i_*} H^G_{n-1}(A;k) \xleftarrow{\partial} H^G_n(X,A;k) \xleftarrow{j_*} H^G_n(X;k) \xleftarrow{i_*} H^G_n(A;k) \xleftarrow{\partial} \cdots$$

A.5. (Homotopy axiom). If $f_0, f_1 : (X,A) \to (Y,B)$ are G-homotopic, then

$$(f_0)_* = (f_1)_*.$$

A.6. (Excision axiom). An inclusion of the form

$$i : (X - U, A - U) \to (X,A)$$

where $\overline{U} \subset A^\circ$ (U and A are G-subsets) induces an isomorphism

$$i_* : H^G_n(X - U, A - U; k) \xrightarrow{\cong} H^G_n(X,A; k)$$

for every integer $n$.

A.7. (Dimension axiom). If $H \in \mathfrak{F}$, then

$$H^G_m(G/H;k) = 0 \quad \text{for all } m \neq 0.$$

Moreover, for every $H \in \mathfrak{F}$ we have an isomorphism

$$\gamma : H^G_0(G/H;k) \xrightarrow{\cong} k(G/H)$$

such that if also $K \in \mathfrak{F}$ and $\alpha : G/H \to G/K$ is a G-map, then the diagram

$$\begin{array}{ccc} H_0^G(G/H;k) & \xrightarrow{\gamma} & k(G/H) \\ \alpha_* \downarrow & & \downarrow \alpha_* \\ H_0^G(G/K;k) & \xrightarrow{\gamma} & k(G/H) \end{array}$$

commutes.

Moreover, this equivariant homology theory has no "negative homology", that is, for any G-pair $(X,A)$ we have

$$H_m^G(X,A;k) = 0 \quad \text{if } m < 0.$$

We call this equivariant homology theory $H_*^G(\ ;k)$ for "equivariant singular homology with coefficients in k".

THEOREM 2.2. Let $G$ be a topological group. Let $\mathfrak{F}$ be an orbit type family for $G$, and let $m$ be a contravariant coefficient system for $\mathfrak{F}$ over the ring $R$.

Then there exists an equivariant cohomology theory $H_G^*(\ ;m)$ defined on the category of all G-maps, and with values in the category of right R-modules, which satisfies all seven equivariant Eilenberg-Steenrod axioms, and which has the given coefficient system $m$ as coefficients.

That $H_G^*(\ ;m)$ satisfies the dimension axiom means the following. If $H \in \mathfrak{F}$, then

$$H_G^p(G/H;m) = 0 \quad \text{for all } p \neq 0,$$

and there is an isomorphism

$$\xi : H_G^0(G/H;m) \xrightarrow{\cong} m\,(G/H),$$

such that if also $K \varepsilon \mathfrak{F}$ and $\alpha : G/H \to G/K$ is a G-map, then the diagram

$$\begin{array}{ccc} H_G^0(G/K;m) & \xrightarrow{\ \xi\ } & m(G/K) \\ \alpha^* \downarrow & & \alpha^* \downarrow \\ H_G^0(G/H;m) & \xrightarrow{\ \xi\ } & m(G/H) \end{array}$$

commutes. The meaning of the rest of Theorem 2.2. is clear. Let us point out that also here the excision axiom is satisfied in the strong form that the G-subset $U$ need not be open. We call the equivariant cohomology theory $H_G^*(\ ;m)$ for "equivariant singular cohomology with coefficients in m". We have $H_G^p(X,A;m) = 0$ if $p < 0$, for every G-pair $(X,A)$.

EXAMPLE. As a simple illustration we determine the equivariant singular homology and cohomology of the following example. Let $G = S^1$ the circle group and $X = S^2$ the two-sphere. Assume that $S^1$ acts on $S^2$ by the standard rotation leaving the north and south poles fixed, and acting freely elsewhere. Let $X_1$ and $X_2$ denote the northern and southern hemispheres, respectively, and $X_0 = X_1 \cap X_2$ the equator.

Now assume that $\mathfrak{F}$ is an orbit type family, such that, both $G \varepsilon \mathfrak{F}$ and $\{e\} \varepsilon \mathfrak{F}$. Let, as before, $R$ be a ring with identity element and $k$ a covariant coefficient system for $\mathfrak{F}$ over $R$. It is a formal consequence of the axioms that we in this situation have the following exact Mayer-Vietoris sequence

$$0 \longleftarrow H_0^G(X;k) \xleftarrow{j_{1*}+j_{2*}} H_0^G(X_1;k) \oplus H_0^G(X_2;k) \xleftarrow{(i_{1*},-i_{2*})} H_0^G(X_0;k) \xleftarrow{\partial} H_1^G(X;k) \leftarrow 0.$$

Since both $X_1$ and $X_2$ are G-homotopy equivalent to a point and $X_0 \cong G$ as G-spaces, it follows that the above exact sequence equals

$$0 \leftarrow H_0^G(X;k) \leftarrow k(G/G) \oplus k(G/G) \xleftarrow{(p_*,-p_*)} k(G/\{e\}) \leftarrow H_1^G(X;k) \leftarrow 0$$

where $p_* : k(G/\{e\}) \to k(G/G)$ is induced by the G-map $p : G \to G/G$. Thus

$$H_0^G(X;k) \cong (k(G/G) \oplus k(G/G))/\{(p_*(a), - p_*(a)) | a \in k(G)\}$$

$$H_1^G(X;k) \cong \ker (p_* : k(G) \to k(G/G))$$

$$H_m^G(X;k) = 0 \qquad \text{for } m \neq 0,1.$$

Let us now consider this result for some specific convariant coefficient systems. Let the orbit type family $\mathcal{F}$ be, for example, the family of all closed subgroups of $G = S^1$, and let $R$ be the ring of integers $Z$.

1. Define a convariant coefficient system $k_1$ as follows. Let $k_1(G/H) = Z$ if $H \neq G$ and $k(G/G) = Z_2$, and let $p : G/H \to G/G$, where $H \neq G$, induce the natural projection $Z \to Z_2$ and let all other induced homomorphisms on $k_1$ be the identity on $Z$. Then

$$H_0^G(X;k_1) \cong Z_2$$

$$H_1^G(X;k_1) \cong Z$$

$$H_m^G(X;k_1) = 0 \quad \text{for } m \neq 0,1.$$

2. Define $k_2$ by : $k_2(G/\{e\}) = Z$, and $k_2(G/H) = 0$ for $H \neq \{e\}$. Then

$$H_0^G(X;k_2) = 0$$

$$H_1^G(X;k_2) \cong Z$$

$$H_m^G(X;k_2) = 0 \quad \text{for } m \neq 0,1.$$

3. Define $k_3$ by : $k_3(G/H) = 0$ for $H = G$, and $k_3(G/G) = Z$. Then

$$H_0^G(X;l_3) = Z \oplus Z$$

$$H_p^G(X;k_3) = 0 \qquad \text{for } p \neq 0.$$

Observe that this equals the ordinary singular homology of the fixed point set $X^G$.

4. Define $k_4$ by : $k_4(G/H) = Z$ for every closed subgroup $H$ of $G$ and all induced homomorphisms are the identity on $Z$. Then

$$H_0^G(X;k_4) \cong Z$$

$$H_p^G(X;k_4) = 0 \qquad \text{for } p \neq 0.$$

Observe that this equals the ordinary singular homology of the orbit space $G\backslash X$.

5. Define $k_5$ by : $k_5(G/H) = Z$ for every closed subgroup $H$ of $G$, and every G-map $\alpha : G/H \to G/K$, where $H \subset K$ but $H \neq K$, induces the zero homomorphism, and every G-map $\beta : G/H \to G/H$ induces the identity on $Z$. Then

$$H_0^G(X;k_5) \cong Z \oplus Z$$

$$H_1^G(X;k_5) \cong Z$$

$$H_m^G(X;k_5) = 0 \qquad \text{for } m \neq 0,1.$$

To determine the equivariant singular cohomology of the G-space $X$ we use the analogous exact Mayer-Vietoris sequence for cohomology

$$0 \to H_G^0(X;m) \xrightarrow{(j_1^*,j_2^*)} H_G^0(X_1;m) \oplus H_G^0(X_2;m) \xrightarrow{i_1^*-i_2^*} H_G^0(X_0;m) \xrightarrow{\partial} H_G^1(X;m) \to 0.$$

In the same way as above we see that this exact sequence equals

$$0 \to H^0_G(X;m) \to m(G/G) \oplus m(G/G) \xrightarrow{p^*(\pi_1 - \pi_2)} m(G/\{e\}) \to H^1_G(X;m) \to 0,$$

where $\pi_i : m(G/G) \oplus m(G/G) \to m(G/G)$ denotes the projection onto the i:th factor, $i = 1,2$. Thus

$$H^0_G(X;m) \cong \ker \ (p^* (\pi_1 - \pi_2) : m(G/G) \oplus m(G/G) \to m(G/\{e\})$$

$$H^1_G(X;m) \cong m(G/\{e\})/\mathrm{im}\ p^*(\pi_1 - \pi_2)$$

$$H^q_G(X;m) = 0 \quad \text{for } q \neq 0,1.$$

## 3. CONSTRUCTION OF EQUIVARIANT SINGULAR HOMOLOGY

Let $\Delta_n$ be the standard n-simplex, that is, $\Delta_n = \{(x_0,\ldots,x_n) \varepsilon R^{n+1} \mid \sum_{i=0}^{n} x_i = 1, x_i \geq 0\}$. We have the face maps

$$e^i_n : \Delta_{n-1} \to \Delta_n \qquad i = 0,\ldots,n,$$

defined by $e^i_n(x_0,\ldots,x_{n-1}) = (x_0,\ldots,x_{i-1},0,x_i,\ldots,x_{n-1})$. The identity

$$e^i_n e^j_{n-1} = e^j_n e^{i-1}_{n-1}, \quad \text{where } 0 \leq j < i \leq n, \quad \text{is valid.}$$

Now let $K$ be a subgroup of $G$. We call the G-space $\Delta_n \times G/K$ for the standard equivariant n-simplex of type K. We have the face maps

$$e^i_n \times id : \Delta_{n-1} \times G/K \to \Delta_n \times G/K \qquad i = 0,\ldots,n.$$

DEFINITION 3.1. A G-map

$$T : \Delta_n \times G/K \to X$$

is called an equivariant singular n-simplex in X. We call K for the type of T and denote

$$t(T) = K.$$

The equivariant singular (n-1)-simplex

$$T^{(i)} = T(e_n^i \times id) : \Delta_{n-1} \times G/K \to X$$

is called the i:th face of T, i = 0,...,n.

DEFINITION 3.2. Let $\mathfrak{F}$ be an orbit type family for G. We say that the equivariant n-simplex $T : \Delta_n \times G/K \to X$ belongs to $\mathfrak{F}$ if $K \in \mathfrak{F}$.

Given an equivariant singular n-simplex $T : \Delta_n \times G/K \to X$ belongs to $\mathfrak{F}$, we form

$$Z_T \otimes k(G/t(T)) = Z_T \otimes k(G/K).$$

Here $Z_T$ denotes the infinite cyclic group on the generator T, and the tensor product is over the integers. The left R-module structure on k(G/t(T)) makes $Z_T \otimes k(G/t(T))$ into a left R-module such that the map $i : k(G/t(T)) \to Z_T \otimes k(G/t(T))$ defined by $i(a) = T \otimes a$ is an isomorphism of left R-modules.

DEFINITION 3.3. We define

$$\hat{C}_n^G(X;k) = \sum_{t(T)\in\mathfrak{F}} \oplus (Z_T \otimes k(G/t(T))$$

where the direct sum is over all equivariant singular n-simplexes in X, which belong to $\mathfrak{F}$. Thus for n < 0 we have $\hat{C}_n^G(X;k) = 0$.

The boundary homomorphism

$$\hat{\partial}_n : \hat{C}_n^G(X;k) \to \hat{C}_{n-1}^G(X;k)$$

is defined in the usual way, that is, for $n \leq 0$ we define $\hat{\partial}_n = 0$, and if $n > 0$ and $T$ is an equivariant singular n-simplex in $X$ and $a \in k(G/t(T))$, we define

$$\hat{\partial}_n(T \otimes a) = \sum_{i=0}^{n} (-1)^{(i)} T^{(i)} \otimes a.$$

The standard calculation then shows that $\hat{\partial}_{n-1}\hat{\partial}_n = 0$.

Thus we get the chain complex

$$\hat{S}^G(X;k) = \{\hat{C}^G_n(X;k), \hat{\partial}_n\}.$$

Our main interest is not in the chain complex $\hat{S}^G(X;k)$, but in a quotient of it. We now proceed to define this quotient.

Let

$$h : \Delta_n \times G/K \to \Delta_n \times G/K'$$

be a G-map which covers $id : \Delta_n \to \Delta_n$. Every $x \in \Delta_n$ gives rise to a G-map

$$h_x : G/K \to G/K'$$

defined by $h_x(gK) = pr_2\ h(x,gK)$, where $pr_2 : \Delta_n \times G/K' \to G/K'$ is projection onto the second factor.

LEMMA 3.4. Let the notation be as above and let $x,y \in \Delta_n$. Then the G-maps $h_x, h_y : G/K \to G/K'$ are G-homotopic.

PROOF. Define $F: I \times G/K \to G/K'$ by $F(t,gK) = pr_2\ h((1-t)x + ty, gK)$. Then $F$ is a G-homotopy from $h_x$ to $h_y$. q.e.d.

Thus, if $K,K' \in \mathcal{F}$, it follows that $(h_x)_* = (h_y)_* : k(G/K) \to k(G/K')$, that is, the G-map $h$ induces in this way a unique homomorphism from $k(G/K)$ to $k(G/K')$. We denote this homomorphism by

$h_* : k(G/K) \to k(G/K')$.

Let for the moment $\mathcal{G}_n \subset \hat{C}^G_n(X;k)$ denote the set of all elements in $\hat{C}^G_n(X;k)$ that have at most one coordinate $\neq 0$. Every element in $\mathcal{G}_n$ has a unique expression of the form $T \otimes a$, where $T$ is some equivariant singular n-simplex belonging to $\mathcal{F}$ in X, and $a \in k(G/t(T))$.

We define a relation $\sim$ in $\mathcal{G}_n$ in the following way. Let $T \otimes a$ and $T' \otimes a'$ be two arbitrary elements in $\mathcal{G}_n$, where $T : \Delta_n \times G/K \to X$ and $T' : \Delta_n \times \Delta_n \times G/K' \to X$ are equivariant singular n-simplexes belonging to $\mathcal{F}$ in X, and $a \in k(G/K_n)$, $a' \in k(G/K')$. We how define

$$T \otimes a \sim T' \otimes a' \Leftrightarrow \begin{cases} \text{there exists a G-map} \\ h : \Delta_n \times G/K \to \Delta_n \times G/K' \text{ covering} \\ id : \Delta_n \to \Delta_n \text{ such that } T = T'h \text{ and} \\ h_*(a) = a'. \end{cases}$$

DEFINITION 3.5. Let the notation be as above. We define

$$\overline{C}^G_n(X;k) \subset \hat{C}^G_n(X;k)$$

to be the submodule of $\hat{C}^G_n(X;k)$ consisting of all elements of the form

$$\sum_{i=1}^{s} (T_i \otimes a_i \sim T'_i \otimes a'_i)$$

where $T_i \otimes a_i \sim T'_i \otimes a'_i$ or $T'_i \otimes a'_i \sim T_i \otimes a_i$, for $i = 1,\ldots,s$.

DEFINITION 3.6. We define the left R-module $C^G_n(X;k)$ by

$$C^G_n(X;k) = \hat{C}^G_n(X;k)/\overline{C}^G_n(X;k).$$

Now observe that if $T \otimes a \sim T' \otimes a'$, then also $T^{(i)} \otimes a \sim (T')^{(i)} \otimes a'$, $i = 0,\ldots,n$. It follows that the boundary homomorphism $\hat{\partial}_n : \hat{C}^G_n(X;k) \to \hat{C}^G_{n-1}(X;k)$

restricts to $\bar{\partial}_n : \bar{C}^G_n(X;k) \to \bar{C}^G_{n-1}(X;k)$, and thus induces a boundary homomorphism

$$\partial_n : C^G_n(X;k) \to C^G_{n-1}(X;k).$$

Since $\hat{\partial}_{n-1}\hat{\partial}_n = 0$ it follows that $\bar{\partial}_{n-1}\bar{\partial}_n = 0$ and $\partial_{n-1}\partial_n = 0$. Thus we have the chain complexes

$$\bar{S}^G(X;k) = \{\bar{C}^G_n(X;k), \bar{\partial}_n\}$$

$$S^G(X;k) = \{C^G_n(X;k), \partial_n\}.$$

It is the chain complex $S^G(X;k)$ that gives us the equivariant singular homology groups with coefficients in $k$ of $X$. We shall now consider the relative case.

Let $(X,A)$ be a G-pair. The inclusion $i : A \to X$ induces a monomorphism of chain complexes

$$\hat{i} : \hat{S}^G(A;k) \to \hat{S}^G(X;k).$$

Moreover, the image $\hat{i}(\hat{C}^G_n(A;k))$ is a direct summand in $\hat{C}^G_n(X;k)$, for each $n$. We identify $\hat{C}^G_n(A;k)$ with $\hat{i}(\hat{C}^G_n(A;k))$, that is, we consider $\hat{S}^G(A;k)$ as a subcomplex of $\hat{S}^G(X;k)$ through the monomorphism $\hat{i}$. We define

$$\hat{C}^G_n(X,A;k) = \hat{C}^G_n(X;k)/\hat{C}^G_n(A;k)$$

and denote the corresponding chain complex by $\hat{S}^G(X,A;k)$. We have the short exact sequence of chain complexes

$$0 \to \hat{S}^G(A;k) \to \hat{S}^G(X;k) \to \hat{S}^G(X,A;k) \to 0.$$

Clearly $\hat{i}$ restricts to $\bar{i}$: $\bar{S}^G(A;k) \to \bar{S}^G(X;k)$ and hence $\hat{i}$ induces

$i : S^G(A;k) \to S^G(X;k)$.

LEMMA 3.7. The homomorphism $i : S^G(A;k) \to S^G(X;k)$ induced by $\hat{i}$ is a monomorphism. Moreover, $i(C_n^G(A;k))$ is a direct summand in $C_n^G(X;k)$ for each n.

PROOF. Define a homomorphism

$$\hat{\alpha} : \hat{C}_n^G(X;k) \to \hat{C}_n^G(A;k)$$

by

$$\hat{\alpha}(T \otimes a) = \begin{cases} T \otimes a & \text{if } \operatorname{Im}(T) \subset A \\ 0 & \text{if } \operatorname{Im}(T) \cap (X-A) \neq \emptyset. \end{cases}$$

Thus $\hat{\alpha}$ is a left inverse to $\hat{i}$. If $T \otimes a \sim T' \otimes a'$ it follows that $\operatorname{Im}(T) = \operatorname{Im}(T')$. Therefore $\hat{\alpha}$ restricts to $\bar{\alpha} : \bar{C}_n^G(X;k) \to \bar{C}_n^G(A;k)$, and hence $\hat{\alpha}$ induces a homomorphism

$$\alpha : C_n^G(X;k) \to C_n^G(A;k)$$

which is a left inverse to i. q.e.d.

We define

$$C_n^G(X,A;k) = C_n^G(X;k)/C_n^G(A;k),$$

and denote the corresponding chain complex by

$$S^G(X,A;k) = \{C_n^G(X,A;k),\partial_n\}.$$

DEFINITION 3.8. We define

$$H_n^G(X,A;k)$$

to be the n:th homology module of the chain complex $S^G(X,A;k)$.

By Lemma 3.7 and by definition we have the short exact sequence

$$0 \to S^G(A;k) \to S^G(X;k) \to S^G(X,A;k) \to 0.$$

This gives us the boundary homomorphism

$$\partial : H_n^G(X,A;k) \to H_{n-1}^G(A;k)$$

and the exact homology sequence of a G-pair $(X,A)$ in the standard way.

More or less as a side remark let us point out the following. Define the chain complex $\overline{S}^G(X,A;k)$ to be the quotient of $\overline{S}^G(X;k)$ by $\overline{S}^G(A;k)$. Then the sequence

$$0 \to \overline{S}^G(X,A;k) \to \hat{S}^G(X,A;k) \to S^G(X,A;k) \to 0$$

is exact. This can be seen "directly" or by drawing the obvious commutative $3 \times 3$ diagram and applying the $3 \times 3$ lemma.

We denote the homology groups of the chain complexes $\overline{S}^G(X,A;k)$ and $\hat{S}^G(X,A;k)$ by $\overline{H}_*^G(X,A;k)$ and $\hat{H}_*^G(X,A;k)$, respectively. Thus we get a long exact sequence

$$\ldots \leftarrow \overline{H}_{n-1}^G(X,A;k) \overset{\partial}{\leftarrow} H_n^G(X,A;k) \leftarrow \hat{H}_n^G(X,A;k) \leftarrow \overline{H}_n^G(X,A;k) \leftarrow \ldots$$

Our main interest is in $H_*^G(\ ;k)$. But in the process of the proof of the fact that $H_*^G(\ ;k)$ satisfies all seven equivariant Eilenberg-Steenrod axioms it will be shown that both $\overline{H}_*^G(\ ;k)$ and $\hat{H}_*^G(\ ;k)$ satisfy the first six axioms.

Let $(X,A)$ and $(Y,B)$ be G-pairs and let $f : (X,A) \to (Y,B)$ be a G-map. If $T : \Delta_n \times G/K \to X$ is an equivariant singular n-simplex belonging to $\mathfrak{F}$ in $X$, then $fT : \Delta_n \times G/K \to Y$ is an equivariant singular n-simplex belonging to $\mathfrak{F}$ in $Y$. Thus we get a homomorphism

$$\hat{f}_{\#} : \hat{C}^G_n(X,A;k) \to \hat{C}^G_n(Y,B;k)$$

by defining $\hat{f}_{\#}(T \otimes a) = (fT) \otimes a$. Since $(fT)^{(i)} = fT^{(i)}$, for $i = 0,\ldots,n$, it follows that the homomorphisms $\hat{f}_{\#}$ form a chain homomorphism. If $T \otimes a \sim T' \otimes a'$, then $(fT) \otimes a \sim (fT') \otimes a'$, and hence $f_{\#}$ restricts to $\overline{f}_{\#} : \overline{S}^G(X,A;k) \to \overline{S}^G(Y,B;k)$ and hence $\hat{f}_{\#}$ induces a chain homomorphism

$$f_{\#} : S^G(X,A;k) \to S^G(Y,B;k).$$

Now $f_{\#}$ induces a homomorphism $f_* : H^G_n(X,A;k) \to H^G_n(Y,B;k)$ for every $n$. It is now clear that we have proved everything up to the exactness axiom in the statement of Theorem 2.1.

In the next section we construct the equivariant singular cohomology theory and establish at the same time everything up to the exactness axiom in the statement of Theorem 2.2. The homotopy, excision, and dimension axioms will be proved simultaneously for homology and cohomology in sections 5, 6, and 7.

## 4. CONSTRUCTION OF EQUIVARIANT SINGULAR COHOMOLOGY

Let $G$, $\mathfrak{F}$ and $R$ be as before. Let $m$ be a contravariant coefficient system for $\mathfrak{F}$ over the ring $R$. Recall that each $m(G/K)$, where $K \in \mathfrak{F}$, is a <u>right</u> R-module.

Let $X$ be a G-space. We denote

$$\hat{C}^G_n(X) = \sum_{t(T)\in\mathfrak{F}} \oplus\, Z_T$$

where the direct sum is over all equivariant singular n-simplexes belonging to $\mathfrak{F}$ in X. That is, $\hat{C}^G_n(X)$ is the free abelian group on all equivariant singular n-simplexes belonging to $\mathfrak{F}$ in X. The boundary homomorphism

$$\hat{\partial}_n : \hat{C}^G_n(X) \to \hat{C}^G_{n-1}(X)$$

is defined by

$$\hat{\partial}_n(T) = \sum_{i=0}^{n} (-1)^i T^{(i)}.$$

Then $\hat{\partial}_{n-1}\hat{\partial} = 0$, and we have the chain complex

$$\hat{S}^G(X) = \{\hat{C}^G_n(X), \hat{\partial}_n\}.$$

That is

$$\hat{S}^G(X) = \hat{S}^G( \ ;Z)$$

where Z denotes the covariant coefficient for which Z(G/K) = Z for every $K \varepsilon \mathfrak{F}$, (and all the induced homomorphisms are the identity on Z).

Denote

$$M = \sum_{K\varepsilon\mathfrak{F}} \oplus\, m(G/K),$$

where the direct sum is over all subgroups belonging to $\mathfrak{F}$. By $\mathrm{Hom}_Z(\hat{C}^G_n(X),M)$ we denote the set of all homomorphisms of abelian groups from $\hat{C}^G_n(X)$ to M. The right R-module structure on M makes $\mathrm{Hom}_Z(\hat{C}^G_n(X),M)$ into a right R-module.

DEFINITION 4.1. We define the right R-module $\hat{C}^n_G(X;m)$ by

$$\hat{C}^n_G(X;m) = \mathrm{Hom}_t(\hat{C}^G_n(X),M).$$

Here $\mathrm{Hom}_t(\hat{C}^G_n(X),M)$ consists of all homomorphisms of abelian groups $c : \hat{C}^G_n(X) \to M$ which satisfy the condition

$$c(T) \in m(G/t(T))$$

for every equivariant singular n-simplex $T$ belonging to $\mathcal{F}$ in $X$. Thus $\hat{C}^n_G(X;m)$ is a submodule of the right R-module $\mathrm{Hom}_Z(\hat{C}^G_n(X),M)$.

For any homomorphism $\hat{\alpha} : \hat{C}^G_n(Y)$ we have the dual homomorphism

$$\hat{\alpha}^* : \mathrm{Hom}_Z(\hat{C}^G_m(Y),M) \to \mathrm{Hom}_Z(\hat{C}^G_n(X),M)$$

defined by $\hat{\alpha}^*(c) = c\,\hat{\alpha}$, $c \in \mathrm{Hom}_Z(\hat{C}^G_m(Y),M)$. Observe that $\hat{\alpha}^*$ is a homomorphism of right R-modules.

DEFINITION 4.2. We call a homomorphism

$$\hat{\alpha} : \hat{C}^G_n(X) \to \hat{C}^G_m(Y)$$

for "type preserving" if the following condition is satisfied. For every equivariant singular n-simplex $T$, belonging to $\mathcal{F}$, in $X$ we have

$$\hat{\alpha}(T) = \sum_{j=0}^{q} r_j S_j, \quad r_j \in Z,$$

with $t(S_j) = t(T)$, for $j = 0,\ldots,q$. (Each $S_j$ is an equivariant singular m-simplex, belonging to $\mathcal{F}$, in $Y$).

Clearly the dual $\hat{\alpha}^*$ of a "type preserving" homomorphism $\hat{\alpha} : \hat{C}^G_n(X) \to \hat{C}^G_m(Y)$ restricts to give a homomorphism

$$\hat{\alpha}^{\#} : \hat{C}^m_G(Y;m) \to \hat{C}^n_G(X;m)$$

which we again call the dual of $\hat{\alpha}$.

The boundary homomorphism $\hat{\partial}_n$ is "type preserving". We denote its dual by

$$\hat{\delta}^{n-1} : \hat{C}^{n-1}_G(X;m) \to \hat{C}^n_G(X;m)$$

and call it the coboundary homomorphism. Then $\hat{\delta}^n\hat{\delta}^{n-1} = 0$, and we have the cochain complex

$$\hat{S}_G(X;m) = \{\hat{C}_G^n(X;m), \hat{\delta}^n\}.$$

Let $(X,A)$ be a G-pair, and let $i : A \to X$ be the inclusion. Both the monomorphism $\hat{i}_\# : \hat{C}_n^G(A) \to \hat{C}_n^G(X)$ and the homomorphism $\hat{\alpha} : \hat{C}_n^G(X) \to \hat{C}^G(A)$, which is a left inverse to $\hat{i}_\#$, are "type preserving" (see the proof of Lemma 3.7). We denote the dual of $\hat{i}_\#$ by

$$\hat{i}^\# : \hat{C}_G^n(X;m) \to \hat{C}_G^n(A;m).$$

Then $\hat{\alpha}^\#$ is a right inverse to $\hat{i}^\#$, and it follows in particular that $\hat{i}^\#$ is onto.

Define $\hat{C}_G^n(X,A;m)$ to be the submodule of $\hat{C}_G^n(X;m) = \mathrm{Hom}_t(\hat{C}_n^G(X),M)$ consisting of all the homomorphisms that vanish on $\hat{C}_n^G(A)$. That is, we have a short exact sequence

$$0 \to \hat{C}_G^n(X,A;m) \to \hat{C}_G^n(X;m) \overset{\hat{i}^\#}{\to} \hat{C}_G^n(A;m) \to 0.$$

Since $\hat{i}^\#$ has a right inverse $\hat{\alpha}^\#$ it follows that the above sequence splits.

We have the corresponding short exact sequence of cochain complexes

$$0 \to \hat{S}_G(X,A;m) \to \hat{S}_G(X;m) \overset{\hat{i}^\#}{\to} \hat{S}_G(A;m) \to 0.$$

Let $f : (X,A) \to (Y,B)$ be a G-map. The induced homomorphism $\hat{f}_\# : \hat{C}_n^G(X) \to \hat{C}_n^G(Y)$ is "type preserving". We denote its dual by $\hat{f}^\# : \hat{C}_G^n(Y;m) \to \hat{C}_G^n(X;m)$. These homomorphisms commute with the coboundary homomorphisms and also restrict to the corresponding relative cochain groups. Thus we have a homomorphism of cochain complexes

$$\hat{f}^\# : \hat{S}_G(Y,B;m) \to \hat{S}_G(X,A;m).$$

In constructing equivariant singular homology we took a quotient of the "roof" chain complex. Here, dually, in constructing equivariant singular cohomology we shall consider an appropriate subcomplex of $\hat{S}_G(X;m)$. We now define this one.

DEFINITION 4.3. We define $C^n_G(X;m)$ to be the submodule of $\hat{C}^n_G(X;m) = \mathrm{Hom}_t(\hat{C}^n_G(X),M)$ consisting of all $c \in \mathrm{Hom}_t(\hat{C}^n_G(X),M)$ which satisfy the following condition. If $T : \Delta_n \times G/K \to X$ and $T' : \Delta_n \times G/K' \to X$ are equivariant singular n-simplexes belonging to $\mathcal{F}$ in $X$, and $h : \Delta_n \times G/K \to \Delta_n \times G/K'$ is a G-map, which covers $id : \Delta_n \to \Delta_n$, such that $T = T'h$, then

$$c(T) = h^* c(T') \in m(G/K).$$

Here $h^* : m(G/K') \to m(G/K)$ is the homomorphism induced by $h$, (see Lemma 3.4.).

DEFINITION 4.4. We say that a "type preserving" homomorphism $\hat{\alpha} : \hat{C}^G_n(X) \to \hat{C}^G_m(Y)$ "preserves the relation $\sim$" if $\hat{\alpha}$ besides being a homomorphism also determines the following extra structure. First, there exists a natural number $q$, and $q + 1$ integers $r_j$, $j = 0,\ldots,q$, such that, for any equivariant singular n-simplexes $T$ and $T'$ belonging to $\mathcal{F}$ in $X$ we have

$$\hat{\alpha}(T) = \sum_{j=0}^{q} r_j S_j, \quad \text{and} \quad \hat{\alpha}(T') = \sum_{j=0}^{q} r_j S'_j,$$

where $S_j$ and $S'_j$ denote equivariant singular m-simplexes belonging to $\mathcal{F}$ in $Y$. Secondly, if $h : \Delta_n \times G/K \to \Delta_n \times G/K'$ is a G-map, which covers $id : \Delta_n \to \Delta_n$, then $\hat{\alpha}$ determines $q + 1$ G-maps

$$h_j : \Delta_m \to G/K \to \Delta_m \times G/K', \; j = 0,\ldots,q,$$

which cover $id : \Delta_m \to \Delta_m$, such that

$$[h_j] = [h] : G/K \to G/K', \quad j = 0,\ldots,q,$$

where $[h_j]$ and $[h]$ denote the G-homotopy classes determined by $h_j$ and $h$, respectively (see Lemma 3.4), and such that if $T = T'h$ then

$$S_j = S_j'h_j \qquad j = 0,\dots,q.$$

LEMMA 4.5. Assume that $\hat{\alpha} : \hat{C}_n^G(X) \to \hat{C}_m^G(Y)$ "preserves the relation ~". Then its dual $\hat{\alpha}^{\#} : \hat{C}_G^m(Y;m) \to \hat{C}_G^n(X;m)$ restricts to a homomorphism

$$\alpha^{\#} : C_G^m(Y;m) \to C_G^n(X;m).$$

PROOF. Let $c \in C_G^m(Y;m)$. We claim that then $\hat{\alpha}^{\#}(c) \in C_G^n(X;m)$. Let $T : \Delta_n \times G/K \to X$, and $T' : \Delta_n \times G/K' \to X$, and let $h : \Delta_n \times G/K \to \Delta_n \times G/K'$ be a G-map, which covers $id : \Delta_n \to \Delta_n$, such that $T = T'h$. Preserving the same notation as in Definition 4.4 we then have

$$(\hat{\alpha}^{\#}(c))(T) = c(\hat{\alpha}(T)) = \sum_{j=0}^{q} r_j\, c\,(S_j) = \sum_{j=0}^{q} r_j\, c\,(S_j'h_j) =$$

$$h^*(\sum_{j=0}^{q} r_j\, c\,(S_j')) = h^*(c(\hat{\alpha}(T'))) = h^*(\hat{\alpha}^{\#}(c)(T')). \qquad \text{q.e.d.}$$

We also call $\alpha^{\#}: C_G^m(Y;m) \to C_G^n(X;m)$ for the dual of $\hat{\alpha} : \hat{C}_n^G(X) \to \hat{C}_m^G(Y)$.

The boundary homomorphism $\hat{\partial}_n : \hat{C}_n^G(X) \to \hat{C}_{n-1}^G(X)$ "preserves the relation ~". To see that the conditions of Definition 4.4 are satisfied we simply take $q = n$, and $r_j = (-1)^j$, $j = 0,\dots,n$, and given $h$ the G-map $h_j$ is the restriction of $h$ to the j:th face, that is $h_j = h| : e_n^j(\Delta_{n-1}) \times G/K \to e_n^j(\Delta_{n-1}) \times G/K'$, $j = 0,\dots,n$. Thus the coboundary $\hat{\delta}^{n-1} : C_G^{n-1}(X;m) \to \hat{C}_G^n(X;m)$ restricts to

$$\delta^{n-1} : C_G^{n-1}(X;m) \to C_G^n(X;m).$$

Then $\delta^n\delta^{n-1} = 0$, and we have the cochain complex

$$0 \to S_G(X,A;m) \to S_G(X;m) \xrightarrow{i^{\#}} S_G(A;m) \to 0,$$

where by definition $S_G(X,A;m) = \ker i^{\#}$. In each degree the above short exact sequence splits as a sequence of right R-modules. We now define the equivariant singular cohomology groups.

DEFINITION 4.6. We define

$$H^n_G(X,A;m)$$

to be the n:th homology module of the cochain complex $S_G(X,A;m)$.

Let $f : (X,A) \to (Y,B)$ be a G-map. The induced homomorphism $\hat{f}_{\#} : \hat{C}^G_n(X) \to \hat{C}^G_n(Y)$ clearly "preserves the relation $\sim$". It follows that the dual $f^{\#}$ of $\hat{f}_{\#}$ gives us a homomorphism of cochain complexes

$$f^{\#} : S_G(Y,B;m) \to S_G(X,A;m)$$

and hence the induced homomorphisms $f^* : H^n_G(Y,B;m) \to H^n_G(X,A;m)$. It is now clear that so far we have proved everything up to the exactness axiom in the statement of Theorem 2.2.

We can also define a cochain complex $\overline{S}_G(X;m)$ by

$$\overline{S}_G(X;m) = \hat{S}_G(X;m)/S_G(X;m).$$

Both $\hat{i}^{\#} : \hat{C}^n_G(X;m) \to \hat{C}^n_G(A;m)$ and $\hat{\alpha}^{\#} : \hat{C}^n_G(A;m) \to \hat{C}^n_G(X;m)$ induce homomorphisms $\overline{i}^{\#}: \overline{C}^n_G(X;m) \to \overline{C}^n_G(A;m)$ and $\overline{\alpha}^{\#}: C^n_G(A;m) \to C^n_G(X;m)$, and $\overline{\alpha}^{\#}$ is a right inverse to $\overline{i}^{\#}$. Thus we have a short exact sequence of cochain complexes

$$0 \to \overline{S}_G(X,A;m) \to \overline{S}_G(X;m) \xrightarrow{\overline{i}^{\#}} \overline{S}_G(A;m) \to 0$$

where by definition $\overline{S}_G(X,A;m) = \ker \overline{i}^{\#}$, which in each degree splits as a sequence of right R-modules.

Applying the 3 × 3-lemma we now see that we have the short exact sequence of cochain complexes

$$0 \to S_G(X,A;m) \to \hat{S}_G(X,A;m) \to \overline{S}_G(X,A;m) \to 0.$$

Define $\hat{H}^n_G(X,A;m)$ and $\overline{H}^n_G(X,A;m)$ to be the n:th homology modules of the cochain complexes $\hat{S}_G(X,A;m)$ and $\overline{S}_G(X,A;m)$, respectively. Thus we get the long exact sequence

$$\cdots \to H^n_G(X,A;m) \to \hat{H}^n_G(X,A;m) \to \overline{H}^n_G(X,A;m) \xrightarrow{\delta} H^{n+1}_G(X,A;m) \to \cdots$$

In the process of showing that $H^*_G(\ ;m)$ satisfies all seven equivariant Eilenberg-Steenrod axioms it will be shown that both $\hat{H}^*_G(\ ;m)$ and $\overline{H}^*_G(\ ;m)$ satisfy the first six axioms.

## 5. THE HOMOTOPY AXIOM

In this section we prove the homotopy axiom for both equivariant singular homology and cohomology.

Let $V$ be a convex set in some euclidean space $R^q$, and let $v^0,\dots,v^n$ be $n + 1$ points in $V$. Denote $d^i = (0,\dots,1,\dots,0) \in \Delta_n$, $0 \le i \le n$, where the 1 occurs in the i-coordinate (recall that we index the coordinates such that a point in $\Delta_n$ is denoted by $(x_0,\dots,x_n)$). We use the notation

$$v^0 \dots v^n : \Delta_n \to V$$

to denote the linear map from $\Delta_n$ into $V$, which is uniquely determined by the condition that it takes $d^i$ into $v^i$, $i = 0,\dots,n$. We have

$(x_0,\dots,x_n) = \sum_{i=0}^{n} x_i d^i$, and thus $v^0\dots v^n(x_0,\dots,x_n) = \sum_{i=0}^{n} x_i v^i \varepsilon V$, where $(x_0,\dots,x_n) \in \Delta_n$. The map $v^0\dots v^n$ is a singular n-simplex in $V$, and its j:th face is the map $v^0\dots\hat{v}^j\dots v^n : \Delta_{n-1} \to V$. Naturally $v^0\dots v^n$ is called a linear n-simplex in $V$. We are now ready to begin the proof of the homotopy axiom.

Let $f_0,f_1 : (X,A) \to (Y,B)$ be two G-homotopic G-maps, and let $F : I \times (X,A) \to (Y,B)$ be a G-homotopy such that $F(0,x) = f_0(x)$ and $F(1,x) = f_1(x)$, for every $x \varepsilon X$. Using this specific G-homotopy $F$ we now construct homomorphisms

$$\hat{D}_n : \hat{C}_n^G(X;k) \to \hat{C}_{n+1}^G(Y;k)$$

for all $n$, which form a chain homotopy from $(f_1)_{\#}$ to $(f_0)_{\#}$. We shall also show that this chain homotopy induces the other chain homotopies we need.

Let $T : \Delta_n \times G/K \to X$ be an equivariant singular n-simplex belonging to $\mathcal{F}$ in $X$. Composing $id_I \times T$ with the G-homotopy $F$ we get the G-map $F(id_I \times T) : I \times \Delta_n \times G/K \to Y$. Now consider linear (n+1)-simplexes in $I \times \Delta_n$ of the form

$$(0,d^0)\cdots(0,d^i)(1,d^i)\cdots(1,d^n) : \Delta_{n+1} \to I \times \Delta_n, \quad 0 \leq i \leq n.$$

To shorten our notation we denote

$$\tau_{n+1}^i = (0,d^0)\cdots(0,d^i)(1,d^i)\cdots(1,d^n), \quad 0 \leq i \leq n.$$

We shall also have use of the following notation

$$\omega_n^i = (0,d^0)\cdots(0,d^{i-1})(1,d^i)\cdots(1,d^n) : \Delta_n \to I \times \Delta_n, \quad 0 \leq i \leq n+1.$$

Observe that $\omega_n^0 = (1,d^0)\cdots(1,d^n)$ and $\omega_n^{n+1} = (0,d^0)\cdots(0,d^n)$. Now consider G-maps of the form

$$F(id_I \times T)(\tau^i_{n+1} \times id_{G/K}) : \Delta_{n+1} \times G/K \to Y.$$

We define

$$\hat{D}_n(T \otimes a) = \sum_{i=0}^{n} (-1)^n[F(id \times T)(\tau^i_{n+1} \times id)] \otimes a,$$

where $a \in k(G/K)$. (From now on we omit the subscripts on the identity maps. The symbol id denotes the identity map on the unit interval $I$ if it appears immediately to the left of a product sign $\times$, and id denotes the identity map on a space of the form $G/t(T)$ whenever it appears immediately to the right of a product sign $\times$.) This defines the homomorphism $\hat{D}_n$ : $\hat{C}^G_n(X;k) \to \hat{C}^G_{n+1}(X;k)$. We shall show that

$$\hat{\partial}_{n+1}\hat{D}_n + \hat{D}_{n-1}\hat{\partial}_n = (\hat{f}_1)_{\#} - (\hat{f}_0)_{\#}.$$

It is immediately seen that the identities

$$\tau^i_{n+1}e^j_{n+1} = \begin{cases} (id \times e^j_n)\tau^{i-1}_{n-1}, & 0 \leq j \leq i-1 \leq n-1 \\ \omega^i_n, & 0 \leq j = i \leq n \\ \omega^{i+1}_n, & 1 \leq i+1 = j \leq n+1 \\ (id \times e^{j-1}_n)\tau^i_n, & 2 \leq i+2 \leq j \leq n+1 \end{cases}$$

are valid. Using these identities and the fact that, by definition, $T^{(j)} = T(e^j_n \times id)$, $0 \leq j \leq n$, we have (we omit the coefficient element $a \varepsilon k(G/t(T)$ in the calculation below).

$$\hat{\partial}_{n+1}\hat{D}_n(T) = \sum_{0\leq j\leq i-1\leq n-1} (-1)^{i+j}F(id \times T^{(j)})(\tau^{i-1}_n \times id) +$$
$$\sum_{0\leq i\leq n} F(id \times T)\omega^i_n - \sum_{0\leq i\leq n} F(id \times T)\omega^{i+1}_n$$
$$\sum_{2\leq i+2\leq j\leq n+1} (-1)^{i+j}F(id \times T^{(j-1)})(\tau^i_n \times id).$$

The second line of the above sum equals

$$F(\mathrm{id} \times T)\omega_n^0 - F(\mathrm{id} \times T)\omega_n^{n+1} = f_1 T - f_0 T.$$

Changing the index $i$ to $i + 1$ on the first line of the sum and the index $j$ to $j + 1$ on the third line of the sum and summing over the index $i$, we see that the sum of the first and third line in the above sum equals

$$- \sum_{0 \leq j \leq n} \hat{D}_{n-1}((-1)^{(j)} T^{(j)}) = - \hat{D}_{n-1}\hat{\partial}_n(T).$$

Thus $\hat{\partial}_{n+1}\hat{D}_n(T) = (\hat{f}_1)_{\#}(T) - (\hat{f}_0)_{\#}(T) - \hat{D}_{n-1}\hat{\partial}_n(T)$, which is exactly what we wanted to prove.

PROPOSITION 5.1. Let the notation be as above. The homomorphism $\hat{D}_n : \hat{C}_n^G(X;k) \to \hat{C}_{n+1}^G(Y;k)$ restricts to $\bar{D}_n : \bar{C}_n^G(X;k) \to \bar{C}_{n+1}^G(Y;k)$ and hence induces a homomorphism $D_n : C_n^G(X;k) \to C_{n+1}^G(Y;k)$. In fact the homomorphism $\hat{D}_n : \hat{C}_n^G(X) \to \hat{C}_{n+1}^G(Y)$ "preserves the relation $\sim$" and hence its dual $\hat{D}^{n+1}$: $\hat{C}_G^{n+1}(Y;m) \to \hat{C}_G^n(X;m)$ restricts to $D^{n+1} : C_G^{n+1}(Y;m) \to C_G^n(X;m)$ and thus also induces $\bar{D}^{n+1} : \bar{C}_G^{n+1}(Y;m) \to \bar{C}_G^n(X;m)$. All these homomorphisms induce homomorphisms on the corresponding relative versions.

PROOF. We shall show that the homomorphisms $\hat{D}_n : \hat{C}_n^G(X) \to \hat{C}_{n+1}^G(Y)$ "preverses the relation $\sim$". The conditions for this, given in Definition 4.4, are seen to be satisfied as follows. First, choose the integers $r_i$ by $r_i = (-1)^i$, $i = 0,\dots,n$. Secondly, if $h : \Delta_n \times G/K \to \Delta_n \times G/K'$ is a G-map, which covers $\mathrm{id} : \Delta_n \to \Delta_n$, we define the G-map $h_i : \Delta_{n+1} \times G/K \to \Delta_{n+1} \times G/K'$, $i = 0,\dots,n$, by the condition that the diagram

$$\begin{array}{ccc}
\Delta_{n+1} \times G/K & \xrightarrow{\tau_{n+1}^i \times \mathrm{id}} & I \times \Delta_n \times G/K \\
{\scriptstyle h_i}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \times h} \\
\Delta_{n+1} \times G/K' & \xrightarrow{\tau_{n+1}^i \times \mathrm{id}} & I \times \Delta_n \times G/K'
\end{array}$$

commutes. That such a G-map $h_i$ exists and is unique follows immediately from the fact that the linear map $\tau^i_{n+1} : \Delta_{n+1} \to I \times \Delta_n$ is an imbedding. Also observe that each $h_i : \Delta_{n+1} \times G/K \to \Delta_{n+1} \times G/K'$ covers $id : \Delta_{n+1} \to \Delta_{n+1}$, and that each $h_i$ determines the same G-homotopy class from $G/K$ to $G/K'$ as $h$ does.

If now $T = T'h$, where $T$ and $T'$ are equivariant singular n-simplexes in $X$, then $F(id \times T)(\tau^i_{n+1} \times id) = F(id \times T')(\tau^i_{n+1} \times id)h_i$, $i = 0,\dots,n$. We have shown that the homomorphism $\hat{D}_n : \hat{C}^G_n(X) \to \hat{C}^G_{n+1}(Y)$ "preserves the relation $\sim$". At the same time we have shown that if $T \otimes a \sim T' \otimes a'$ then $\hat{D}_n(T \otimes a - T' \otimes a') \in \overline{C}^G_{n+1}(Y;k)$ and thus $\hat{D}_n : \hat{C}^G_n(X;k) \to \hat{C}^G_{n+1}(Y;k)$ restricts to $\overline{C}^G_n(X;k) \to \overline{C}^G_{n+1}(Y;k)$ and hence induces $D_n : C^G_n(X;k) \to C^G_{n+1}(Y;k)$.

Since $F : I \times (X,A) \to (Y,B)$ it follows immediately from the definition of the homomorphism $\hat{D}_n$ that all the homomorphisms $D_n$, $\overline{D}_n$, etc. induce homomorphisms on the corresponding relative versions. q.e.d.

COROLLARY 5.2. Let $f_0,f_1 : (X,A) \to (Y,B)$ be two G-homotopic G-maps. Then the induced homomorphisms

$$(f_0)_{\#},(f_1)_{\#} : S^G(X,A;k) \to S^G(Y,B;k)$$

are chain homotopic, and the same is true for the homomorphisms

$$(f_0)^{\#},(f_1)^{\#} : S_G(Y,B;m) \to S_G(X,A;m).$$

Moreover, the same conclusion holds for both the "roof" and "bar" versions.

q.e.d.

This result establishes the homotopy axiom for equivariant singular homology $H^G_*(\ ;k)$, and equivariant singular cohomology $H^*_G(\ ;m)$, as well as for the theories $\hat{H}^G_*(\ ;k)$, $\overline{H}^G_*(\ ;k)$, $\hat{H}^*_G(\ ;m)$ and $\overline{H}^*_G(\ ;m)$.

## 6. THE EXCISION AXIOM

In this section we prove the excision axiom for both equivariant singular homology and cohomology.

Consider the G-space $\Delta_n \times G/K$. An equivariant linear q-simplex in $\Delta_n \times G/K$ is a map of the form

$$v^0 \dots v^q \times id : \Delta_q \times G/K \to \Delta_n \times G/K$$

where $v^0 \dots v^q$ is a linear q-simplex in $\Delta_n$, and id denotes the identity mapping on G/K. Now assume that $K \varepsilon \mathcal{F}$, the fixed orbit type family under consideration. The equivariant linear q-simplexes in $\Delta_n \times G/K$ "generate" a submodule, which we denote by $\hat{C}^G_q Q(\Delta_n \times G/K;k)$, of $\hat{C}^G_q(\Delta_n \times G/K;k)$. Here "generate" means that $\hat{C}^G_q Q(\Delta_n \times G/K;k)$ is the submodule of $\hat{C}^G_q(\Delta_n \times G/K;k)$ generated by all elements of the form $T \boxtimes a$, where $a \varepsilon k(G/K)$ and $T = v^0 \dots v^q \times id$ is an equivariant linear q-simplex in $\Delta_n \times G/K$. The boundary $\hat{\partial}_q$ maps $\hat{C}^G_q Q(\Delta_n \times G/K;k)$ into $\hat{C}^G_{q-1} Q(\Delta_n \times G/K;k)$ and we have the corresponding subcomplex $\hat{S}^G Q(\Delta_n \times G/K;k)$ of $\hat{S}^G(\Delta_n \times G/K;k)$.

Let $v \varepsilon \Delta_n$. Define homomorphisms

$$v\cdot : \hat{C}^G_q Q(\Delta_n \times G/K;k) \to \hat{C}^G_{q+1} Q(\Delta_n \times G/K;k)$$

by $v\cdot [(v^0 \dots v^q \times id) \boxtimes a] = (vv^0 \dots v^q \times id) \boxtimes a$, where $a \varepsilon k(G/K)$. Direct calculation shows that

$$\hat{\partial}_{q+1}(v\cdot\sigma) = \sigma - v\cdot(\hat{\partial}_q(\sigma)), \qquad , \qquad q \geq 1,$$

$$\hat{\partial}_1(v\cdot\sigma) = \sigma - (v \times id) \boxtimes In(\sigma), \qquad q = 0,$$

where $\sigma \varepsilon \hat{C}^G_q Q(\Delta_n \times G/K;k)$ and $v \times id : \Delta_0 \times G/K \to \Delta_n \times G/K$ is the equivariant linear 0-simplex in $\Delta_n \times G/K$ determined by the point $v \varepsilon \Delta_n$, and

In : $\hat{C}_0^G Q(\Delta_n \times G/K;k) \to k(G/K)$ is the homomorphism defined by $\mathrm{In}[(v^0 \times id) \otimes a] = a$.

We now inductively define homomorphisms

$$\hat{Sd}_q : \hat{C}_q^G Q(\Delta_n \times G/K;k) \to \hat{C}_q^G Q(\Delta_n \times G/K;k)$$

$$\hat{R}_q : \hat{C}_q^G Q(\Delta_n \times G/K;k) \to \hat{C}_{q+1}^G Q(\Delta_n \times G/K;k)$$

in the following way. We set $\hat{Sd}_0 = id$ and $\hat{R}_0 = 0$. If $\omega = v^0 \ldots v^q \times id$ is an equivariant linear q-simplex in $\Delta_n \times G/K$ and $a \in k(G/K)$ we define

$$\hat{Sd}_q(\omega \otimes a) = b_\omega \cdot \hat{Sd}_{q-1}(\hat{\partial}_q(\omega \otimes a))$$

$$\hat{R}_q(\omega \otimes a) = b_\omega \cdot (\omega \otimes a - \hat{Sd}_q(\omega \otimes a) - \hat{R}_{q-1}(\hat{\partial}_q(\omega \otimes a))).$$

Here $b_\omega \in \Delta_n$ denotes the barycenter of $\omega$, that is, the point $b_\omega = \frac{1}{q+1} v^0 + \ldots + \frac{1}{q+1} v^q$.

By induction with respect to $q$ it is easy to prove that

$$\hat{\partial}_q \hat{Sd}_q = \hat{Sd}_{q-1} \hat{\partial}_q$$

$$\hat{\partial}_{q+1} \hat{R}_q + \hat{R}_{q-1} \hat{\partial}_q = id - \hat{Sd}_q,$$

that is, the homomorphisms $\hat{Sd}_q$ form a chain map $\hat{Sd}$, and the homomorphisms $\hat{R}_q$ form a chain homotopy $\hat{R}$ from id to $\hat{Sd}$.

Let $X$ be an arbitrary G-space. We now define homomorphisms

$$\hat{Sd}_n : \hat{C}_n^G(X;k) \to \hat{C}_n^G(X;k)$$

$$\hat{R}_n : \hat{C}_n^G(X;k) \to \hat{C}_{n+1}^G(X;k)$$

in the following way. Let $T : \Delta_n \times G/K \to X$ be an equivariant singular n-simplex belonging to $\mathfrak{F}$ in $X$, and let $a \in k(G/K)$. We define

$$\hat{Sd}_n(T \otimes a) = \hat{T}_{\#}\hat{Sd}_n((d^0 \ldots d^n \times id) \otimes a),$$

$$\hat{R}_n(T \otimes a) = \hat{T}_{\#}\hat{R}_n((d^0 \ldots d^n \times id) \otimes a).$$

It is easy to see that these homomorphisms $\hat{Sd}_n$ again form a chain map $\hat{Sd}$ and that these homomorphisms $\hat{R}_n$ form a chain homotopy from id to $\hat{Sd}$. The proof of this is a formal computation using the fact that both $\hat{Sd}_q$ and $\hat{R}_q$, when defined on the equivariant linear chain complexes, commute with the homomorphisms induced by the face maps $e^i_n \times id : \Delta_{n-1} \times G/K \to \Delta_n \times G/K$, and the fact that the homomorphisms $\hat{Sd}_q$ and $\hat{R}_q$ defined on the equivariant linear chain complexes already have the desired properties.

PROPOSITION 6.1. The homomorphism $Sd_n : \hat{C}^G_n(X;k) \to \hat{C}^G_n(X;k)$ restricts to $\overline{Sd} : \overline{C}^G_n(X;k) \to \overline{C}^G_n(X;k)$ and hence induces a homomorphism $Sd_n : C^G_n(X;k) \to C^G_n(X;k)$. In fact the homomorphism $\hat{Sd}_n : \hat{C}^G_n(X) \to \hat{C}^G_n(X)$ "preverses the relation $\sim$" and hence its dual $\hat{Sd}^n : \hat{C}^n_G(X;m) \to \hat{C}^n_G(X;m)$ restricts to $Sd^n : C^n_G(X;m) \to C^n_G(X;m)$ and thus also induces $\overline{Sd} : \overline{C}^n_G(X;m) \to \overline{C}^n_G(X;m)$. All the corresponding statements hold for the homomorphism $\hat{R}_n$.

PROOF. We first prove that the homomorphism $\hat{Sd}_n : \hat{C}^G_n(X) \to \hat{C}^G_n(X)$ "preserves the relation $\sim$". Consider $\hat{Sd}_n : \hat{C}^G_n Q(\Delta_n \times G/K) \to \hat{C}^G_n Q(\Delta_n \times G/K)$. We have $Sd_q(d^0 \ldots d^n \times id) = \sum_{j=1}^{N} \pm \sigma_j \times id$, where each $\sigma_j$ is a linear $n$-simplex in $\Delta_n$. Also observe that each expression $\pm \sigma_j$ and the integer $N$ (in fact $N = (n + 1)!$) are independent of the subgroup $K$. Moreover it follows immediately from the recursive definition of $Sd_q$ that each $\sigma_j : \Delta_n \to \Delta_n$ is an imbedding. If now $h : \Delta_n \times G/K \to \Delta_n \times G/K'$ is a G-map, which covers $id : \Delta_n \to \Delta_n$, we define the G-map $h_j : \Delta_n \times G/K \to \Delta_n \times G/K'$ by the condition that the diagram

$$\begin{array}{ccc} \Delta_n \times G/K & \xrightarrow{\sigma_j \times id} & \Delta_n \times G/K \\ h_j \downarrow & & \downarrow h \\ \Delta_n \times G/K' & \xrightarrow{\sigma_j \times id} & \Delta_n \times G/K' \end{array}$$

commutes. Observe that $h_j$ covers $id : \Delta_n \to \Delta_n$ and that $h_j$ determines the same G-homotopy class of G-maps from $G/K$ to $G/K'$ as $h$ does.

Now recall that for any equivariant singular n-simplex, belonging to , $T : \Delta_n \times G/K \to X$ we have by definition, $\hat{Sd}_n(T) = \hat{T}_{\#}Sd_n(d^0 \ldots d^n \times id) = \sum_{j=1}^{N} \pm T(\sigma_j \times id)$. If now $T = T'h$, it follows that $T(\sigma_j \times id) = T'h(\sigma_j \times id) = T'h(\sigma_j \times id) = T'(\sigma_j \times id)h_j$. We have shown that $\hat{Sd}_n : \hat{C}_n^G(X) \to \hat{C}_n^G(X)$ "preserves the relation $\sim$".

To prove that the homomorphism $\hat{R}_n : \hat{C}_n^G(X) \to \hat{C}_{n+1}^G(X)$ "preserves the relation $\sim$" we proceed in a completely analogous way as above. First consider the homomorphism $\hat{R}_n : \hat{C}_n^G Q(\Delta_n \times G/K) \to \hat{C}_{n+1}^G Q(\Delta_n \times G/K)$, and observe that $\hat{R}_n(d^0 \ldots d^n \times id) = \sum_{i=1}^{M} \pm \tau_j \times id$, where each $\tau_j$ is a linear $(n + 1)$-simplex in $\Delta_n$. Moreover, the expression $\pm \tau_j$ and the integer $M$ are independent of the subgroup $K$. If now $\bar{h} : \Delta_n \times G/K \to \Delta_n \times G/K'$ is a G-map, which covers $id : \Delta_n \to \Delta_n$, we define the G-map $\bar{h}_j : \Delta_{n+1} \times G/K \to \Delta_{n+1} \times G/K'$ by requiring that $\bar{h}_j$ covers $id : \Delta_{n+1} \to \Delta_{n+1}$ and that the diagram

$$\begin{array}{ccc} \Delta_{n+1} \times G/K & \xrightarrow{\tau_j \times id} & \Delta_n \times G/K \\ h_j \downarrow & & \downarrow \bar{h} \\ \Delta_{n+1} \times G/K' & \xrightarrow{\tau_j \times id} & \Delta_n \times G/K' \end{array}$$

commutes. (Thus $\overline{h}_j(x,gK) = (x,pr_2\overline{h}(\tau_j(x),gK))$, where $pr_2$ denotes projection onto the second factor).

If $T = T'\overline{h}$, it follows that $T(\tau_j \times id) = T'(\tau_j \times id)\overline{h}_j$. Hence $\hat{R}_n(T) = \sum_{j=1}^{M} \pm T(\tau_j \times id) = \sum_{j=1}^{M} \pm T'(\tau_j \times id)\overline{h}_j$. This shows that $\hat{R}_n : \hat{C}^G_n(X) \to \hat{C}^G_{n+1}(X)$ "preserves the relation $\sim$". q.e.d.

Exactly as in the case of ordinary singular theory the subdivision chain map $Sd : S^G(X;k) \to S^G(X;k)$ and the chain homotopy $R : S^G(X;k) \to S^G(X;k)$ are the crucial ingredients for the proof of the excision axiom. We proceed to give the remaining details.

DEFINITION 6.2. Let $V$ be a family of G-subsets of the G-space $X$. An equivariant singular n-simplex $T : \Delta_n \times G/K \to X$ is said to be in $V$ if $T(\Delta_n \times G/K)$ is contained in at least one of the sets in $V$.

Clearly all equivariant singular n-simplexes belonging to $\mathfrak{F}$ in $X$ which are in $V$ "generate" a submodule $\hat{C}^G_n(X;k;V)$ of $\hat{C}^G_n(X;k)$. For the case $k = Z$, i.e. the coefficient system defined by $Z(G/K) = Z$, for every $K \in \mathfrak{F}$, and all the induced homomorphisms are the identity on $Z$, we use the simplified notation $\hat{C}^G_n(X;V)$, in complete analogy with our earlier notation. We denote the inclusion by

$$\hat{\eta} : \hat{C}^G_n(X;k,V) \to \hat{C}^G_n(X;k).$$

Now observe that if $T \otimes a \in \hat{C}^G_n(X;k;V)$ and $T \otimes a \sim T' \otimes a'$, where $T' \otimes a' \in \hat{C}^G_n(X;k)$, then it follows that also $T' \otimes a' \in \hat{C}^G_n(X;k;V)$. That is, the relation $\sim$ restricts to $\hat{C}^G_n(X;k;V)$ in this way. This fact allows us to use these new modules $\hat{C}^G_n(X;k;V)$ in a way completely analogous to the way we have used the modules $\hat{C}^G_n(X;k)$. To be more specific we mean the following. We define $\overline{C}^G_n(X;k;V)$ and $C^G_n(X;k;V)$ by complete analogy to the definitions of $\overline{C}^G_n(X;k)$

and $C_n^G(X;k)$. We can use the notion of a homomorphism $\hat{\alpha}$ which "preserves the relation $\sim$" (see Definition 4.4.) also when the range or domain (or both) of the homomorphisms $\hat{\alpha}$ is one of the modules $\hat{C}_n^G(X;V)$. We can use the same kind of duals of $\hat{C}_n^G(X;V)$ as of $\hat{C}_n^G(X)$, that is, we define $\hat{C}_G^n(X;m;V) = \mathrm{Hom}_t(\hat{C}_n^G(X;V),M)$, the right R-module consisting of all homomorphisms of abelian groups $c : \hat{C}_n^G(X;V) \to M$, which satisfy the condition $c(T) \in m(G/t(T))$, for every equivariant singular n-simplex T, belonging to $\mathcal{F}$ in X, which is in V. (Recall that M denotes the right R-module $M = \sum_{K\in\mathcal{F}} \oplus\, m(G/K)$). Then $C_G^n(X;m;V)$ is defined to be the submodule of $\hat{C}_G^n(X;m;V)$ consisting of all homomorphisms $c : \hat{C}_n^G(X;V) \to M$ which satisfy the condition in Definition 4.3. If now for example $\hat{\alpha} : \hat{C}_n^G(X) \to \hat{C}_n^G(X;V)$ is a homomorphism which "preserves the relation $\sim$" it follows that $\hat{\alpha}$ has a dual $\hat{\alpha}^{\#} : \hat{C}_G^n(X;m;V) \to \hat{C}_G^n(X;m)$ which restricts to $\alpha^{\#} : C_G^n(X;m;V) \to C_G^n(X;m)$, (and $\alpha^{\#}$ is again called the dual of $\hat{\alpha}$).

LEMMA 6.3. Let V be a family of G-subsets of X such that $X = \bigcup_{B\in V} B^{\circ}$. Let $T : \Delta_n \times G/K \to X$ be an equivariant singular n-simplex belonging to $\mathcal{F}$ in X, and $a \in k(G/K)$. Then there exists an integer m such that $\hat{Sd}^m(T \otimes a) \in \hat{S}^G(X;k;V)$.

PROOF. Consider the (ordinary) singular n-simplex $T| : \Delta_n \to X$, where $(T|)(x) = T(x,eK)$, $x \in \Delta_n$. From the corresponding result in ordinary singular theory we know that there exists m such that $Sd^m(T|) \in S(X;V)$ (see Eilenberg-Steenrod [6], p. 198-199.) Here $Sd : S(X) \to S(X)$ is the subdivision chain map on the ordinary singular chain complex of X. But since V is a family of G-subsets, it now follows from the way our Sd is defined that we have $\hat{Sd}^m(T \otimes a) \in \hat{S}^G(X;k;V)$. q.e.d.

For any equivariant singular simplex T belonging to $\mathcal{F}$ in X we denote by $m(T)$ the smallest integer for which $\hat{Sd}^{m(T)}(T \otimes a) \in \hat{S}^G(X;k;V)$. The coefficient element $a \in k(G/t(T))$ does not affect this situation at all. Clearly we have $m(T^{(i)}) \leq m(T)$. If $T \otimes a \in \hat{S}^G(X;k;V)$ then $m(T) = 0$, and if $T \otimes a \in \hat{S}^G(A;k)$ then also $\hat{Sd}^{m(T)}(T \otimes a) \in \hat{S}^G(A;k;V)$. The following proposition corresponds to Theorem 8.2. on page 197 in Eilenberg-Steenrod [6]. The proof we give follows the proof they give in the Notes at the end of Chapter III, and not the proof they give in the text. Note the remark on page 207 in Eilenberg-Steenrod [6].

PROPOSITION 6.4. Let $V$ be a family of G-subsets of $X$ such that $X = \bigcup_{B \in V} B^{\circ}$. Then, for any G-subset $A$ of $X$, the inclusion

$$\eta : S^G(X,A;k;V) \to S^G(X,A;k)$$

is a homotopy equivalence, and the same is true for the corresponding $\hat{\eta}$ and $\bar{\eta}$. The dual

$$\eta^{\#} : S_G(X,A;m) \to S_G(X,A;m;V)$$

is also a homotopy equivalence, and the same is true for the corresponding $\hat{\eta}^{\#}$ and $\bar{\eta}^{\#}$.

PROOF. Let $T$ be an equivariant singular n-simplex belonging to $\mathcal{F}$ in $X$, and $a \in k(G/t(T))$.

Define

$$\hat{\tau}(T \otimes a) = \hat{Sd}^{m(T)}(T \otimes a) + \sum_{i=0}^{n} (-1)^i \sum_{j=m(T^{(i)})}^{m(T)-1} \hat{R}\, \hat{Sd}^j(T^{(i)} \otimes a)$$

$$\hat{D}(T \otimes a) = \sum_{j=0}^{m(T)-1} \hat{R}\, \hat{Sd}^j(T \otimes a).$$

Observe that $\hat{\tau}(T \otimes a) \in \hat{C}_n^G(X;k;V)$ and that $\hat{D}(T \otimes a) \in \hat{C}_{n+1}^G(X;k)$.

This defines homomorphisms

$$\hat{\tau}_n : \hat{C}^G_n(X,A;k) \to \hat{C}^G_n(X,A;k;V)$$

$$\hat{D}_n : \hat{C}^G_n(X;A;k) \to \hat{C}^G_{n+1}(X,A;k).$$

A formal computation shows that

$$\hat{\partial}_{n+1}\hat{D}_n + \hat{D}_{n-1}\hat{\partial}_n = id - \hat{\eta}_n\hat{\tau}_n$$

for all $n$. From this it follows that $\hat{\partial}_n\hat{\eta}_n\hat{\tau}_n = \hat{\eta}_{n-1}\hat{\tau}_{n-1}\hat{\partial}_n$. Since $\hat{\eta}$ : $\hat{S}^G(X,A;k;V) \to \hat{S}^G(X,A;k)$ is a chain map and an inclusion it follows that $\hat{\partial}_n\hat{\tau}_n = \hat{\tau}_{n-1}\hat{\partial}_n$, that is, the homomorphism $\hat{\tau}_n$ form a chain map $\hat{\tau} : \hat{S}(X,A;k;V) \to S^G(X,A;k)$. Since $\hat{\tau}\hat{\eta} = id$ and the above formula tells us that $\hat{\eta}\hat{\tau}$ is chain homotopic to the identity map, it follows that $\hat{\eta}$ is a homotopy equivalence, and in fact $\hat{\tau}$ is a homotopy inverse to $\hat{\eta}$.

By Proposition 6.1. the maps $\hat{Sd}$ and $\hat{R}$ restrict to maps $\overline{Sd}$ and $\overline{R}$, and hence induce $Sd$ and $R$. Since $m(T) = m(T')$, if $T \otimes a \sim T' \otimes a'$, it follows that the maps $\hat{\tau}$ and $\hat{D}$ restrict to corresponding maps $\overline{\tau}$ and $\overline{D}$ and therefore induce $\tau$ and $D$. Thus $\tau$ and $\overline{\tau}$ are homotopy inverses to $\eta$ and $\overline{\eta}$, respectively. In the same way it follows from Proposition 6.1. that the homomorphisms $\hat{\tau}_n : \hat{C}^G_n(X) \to \hat{C}^G_n(X;V)$ and $\hat{D}_n : \hat{C}^G_n(X) \to \hat{C}^G_{n+1}(X)$ "preverse the relation $\sim$" and thus the duals $\hat{\tau}^{\#}$, $\tau^{\#}$, and $\overline{\tau}^{\#}$are homotopy inverses to $\hat{\eta}^{\#}$, $\eta^{\#}$, and $\overline{\eta}^{\#}$, respectively. q.e.d.

COROLLARY 6.5. Let $(X,A)$ be a G-pair and let $U$ a G-subset of $X$ such that $\overline{U} \subset A^{o}$. Then the inclusion

$$i : (X - U, A - U) \to (X,A)$$

induces a homotopy equivalence

$$i_{\#} : S^G(X - U, A - U;k) \to S^G(X,A;k).$$

The corresponding $\hat{i}_{\#}$ and $\bar{i}_{\#}$ are also homotopy equivalences, and the same is true for the duals $\hat{i}^{\#}$, $i^{\#}$, and $\bar{i}^{\#}$.

PROOF. Let $V$ denote the family consisting of the G-subsets $A$ and $X - U$. Since $\overline{U} \subset A^{\circ}$ it follows that $X = (X - U)^{\circ} \cup A^{\circ}$, that is, the family $V$ satisfies the dondition of Proposition 6.4. Since

$$\hat{S}^G(X;k;V) = \hat{S}^G(X - U;k) + \hat{S}^G(A;k)$$

$$\hat{S}^G(A;k;V) = \hat{S}^G(A;k) \quad \text{and}$$

$$\hat{S}^G(X - U;k) \cap \hat{S}^G(A;k) = \hat{S}^G(A - U;k)$$

it follows (by the Noether isomorphism theorem) that

$$\hat{j} : \hat{S}^G(X - U, A - U;k) \to \hat{S}^G(X,A;k;V)$$

is an isomorphism. Since $\hat{i}_{\#} = \hat{\eta}\hat{j}$, and $\hat{\eta} : \hat{S}^G(X,A;k;V) \to \hat{S}^G(X,A;k)$ is a homotopy equivalence by Proposition 6.4., it follows that $\hat{i}_{\#}$ is a homotopy equivalence. Since also the maps $\bar{j}$ and $j$ corresponding to $\hat{j}$, and the duals $\hat{j}^{\#}$, $j^{\#}$, and $\bar{j}^{\#}$ are isomorphisms, it follows that the maps $\bar{i}_{\#}$ and $i_{\#}$, as well as the maps $\hat{i}^{\#}$, $i^{\#}$, and $\bar{i}^{\#}$ are homotopy equivalences. q.e.d.

This result proves the excision axiom for equivariant singular homology $H^G_*(\ ;k)$, and equivariant singular cohomology $H^*_G(\ ;m)$, as well as for the theories $H^G_*(\ ;k)$, $\overline{H}^G_*(\ ;k)$, $\hat{H}^*_G(\ ;m)$, and $\overline{H}^*_G(\ ;m)$,

## 7. THE DIMENSION AXIOM

Let $H \varepsilon \mathcal{F}$. We shall determine the R-modules $H_n^G(G/H;k)$, for every $n$. Let $\pi_n : \Delta_n \times G/H \to G/H$ be the projection onto the second factor. The map $\pi_n$ is an equivariant singular n-simplex belonging to $\mathcal{F}$ in X. We have $\partial_n(\pi_n \otimes b) = \sum_{i=0}^{n} (-1)^i \pi_{n-1} \otimes b$, that is,

$$\hat{\partial}_n(\pi_n \otimes b) = \begin{cases} \pi_{n-1} \otimes b & , \text{ for } n \text{ even, and } n \geq 2 \\ 0 & , \text{ for } n \text{ odd,} \end{cases}$$

where $b \varepsilon k(G/H)$. Let $S^G spe.(G/H;k)$ denote the chain complex which in degree $n$ is $C_n^G spe.(G/H;k) = Z_{\pi_n} \otimes k(G/H)$ and the boundary homomorphism is the standard one i.e. the one given above. Clearly the homology of the chain complex $S^G spe.(G/H;k)$ is given by

$$H_0(S^G spe.(G/H;k)) = C_0^G spe.(G/H;k) \cong k(G/H),$$

$$H_m(S^G spe.(G/H;k)) = 0, \quad \text{for } m \neq 0.$$

We shall now establish an isomorphism of chain complexes

$$\beta : S^G(G/H;k) \xrightarrow{\cong} S^G spe.(G/H;k).$$

First we define a chain map

$$\hat{\beta} : \hat{S}^G(G/H;k) \to S^G spe.(G/H;k)$$

as follows. Let $T : \Delta_n \times G/K \to G/H$ be any equivariant singular n-simplex belonging to $\mathcal{F}$ in G/H. Then define $\overline{T} : \Delta_n \times G/K \to \Delta_n \to G/H$, by $\overline{T}(x,gK) = (x,T(x,gK))$. Observe that $T = \pi_n \overline{T}$, and $T \otimes a \sim \pi_n \otimes (\overline{T})_*(a)$, where

$a \in k(G/K)$. Now define

$$\hat{\beta}(T \otimes a) = \pi_n \otimes (\overline{T})_*(a) \in C_n^G spe.(G/H;k).$$

Since $\hat{\beta}((-1)^i T^{(i)} \otimes a) = (-1)^i \pi_{n-1} \otimes \overline{(T^{(i)})}_*(a) = (-1)^i \pi_{n-1} \otimes (\overline{T})_*(a)$, $i = 0,\ldots,n$, it follows that $\hat{\beta}$ is a chain map. Now assume that $T \otimes a \sim T' \otimes a'$. Let $h : \Delta_n \times G/t(T) \to \Delta_n \times G/t(T')$ be a G-map which covers $id : \Delta_n \to \Delta_n$, such that $T = T'h$ and $h_*(a) = a'$. Then $\overline{T} = \overline{T}'h$, and hence $(\overline{T})_*(a) = (\overline{T'})_* h_*(a) = (\overline{T'})_*(a')$. Thus $\hat{\beta}(T \otimes a) = (T' \otimes a')$, and it follows that $\hat{\beta}$ induces the chain map $\beta$. An inverse $\alpha : S^G spe.(G/H;k) \to S^G(G/H;k)$ to $\beta$ is defined as follows. Let $\pi_n \otimes b \in C_n^G spe.(G/H;k)$ and define $\alpha(\pi_n \otimes b) = \{\pi_n \otimes b\}$, where $\{\pi_n \otimes b\}$ denotes the image of the element $\pi_n \otimes b \in \hat{C}_n^G(G/H;k)$ under the natural projection $\hat{C}_n^G(G/H;k) \to C_n^G(G/H;k)$. Then $\beta\alpha(\pi_n \otimes b) = \beta\{\pi_n \otimes b\} = \pi_n \otimes (\overline{\pi}_n)_*(b) = \pi_n \otimes b$, since $\overline{\pi}_n = id : \Delta_n \times G/H \to \Delta_n \times G/H$. Also $\alpha\beta(\{T \otimes a\}) = \alpha(\pi_n \otimes (\overline{T})_*(a)) = \{\pi_n \otimes (\overline{T})_*(a)\} = \{T \otimes a\}$ since, as we already noted before, $T \otimes a \sim \pi_n \otimes (\overline{T})_*(a)$. Thus $\alpha$ is an inverse to $\beta$ and we have shown that $\beta : S^G(G/H;k) \to S^G spe.(G/H;k)$ is an isomorphism of chain complexes. It follows that the homology R-modules of the chain complex $S^G(G/H;k)$, that is, the equivariant singular homology R-modules $H_p^G(G/H;k)$ are given by

$$H_0^G(G/H;k) \cong k(G/H) \quad ,$$

$$H_m^G(G/H;k) = 0 \quad , \text{ for } m \neq 0.$$

The explicit isomorphism

$$\alpha : H_0^G(G/H;k) \xrightarrow{\cong} k(G/H)$$

is described as follows. Since every element in $C_G^0(G/H;k)$ is a cycle and only the zero element is a boundary it follows that $H_0^G(G/H;k) = C_0^G(G/H;k)$.

Then $\alpha : H_0^G(G/H;k) = C_0^G(G/H;k) \to k(G/H)$ is defined by the following. If $T : \Delta_0 \times G/K = G/K \to G/H$ is any equivariant singular 0-simplex belonging to $\mathfrak{F}$ in $G/H$, we have $\alpha(\{T \otimes a\}) = T_*(a) \in k(G/H)$, where $a \in k(G/K)$ and $T_* : k(G/K) \to k(G/H)$ is the homomorphism induced by the G-map $T : G/K \to G/H$. From this description of the isomorphism $\alpha$ it also follows immediately that $\alpha$ has the naturality property described in the statement of the dimension axiom in Theorem 2.1. This concludes the proof of the dimension axiom for equivariant singular homology.

Let us now prove the dimension axiom for equivariant singular cohomology. Denote $C_G^n\text{spe.}(G/H;m) = \text{Hom}_Z(Z_{\pi_n}, m(G/H))$.

We have

$$(\hat{\delta}^n(d))(\pi_{n+1}) = d(\hat{\partial}_{n+1}(\pi_{n+1})) = \begin{cases} d(\pi_n), & \text{for } n \text{ odd, } n \geq 1 \\ 0 & , \text{ for } n \text{ even,} \end{cases}$$

where $d \in C_G^n\text{spe.}(G/H;m)$. Thus $\hat{\delta}^n : C_G^n\text{spe.}(G/H;m) \to C_G^{n+1}\text{spe.}(G/H;m)$ is the zero homomorphism if $n$ is even, and $\hat{\delta}^n$ is an isomorphism if $n$ is odd and $n \geq 1$. Let $S_G\text{spe.}(G/H;m)$ denote the corresponding cochain complex. Clearly the homology of the cochain complex $S_G\text{spe.}(G/H;m)$ is given by

$$H_0(S_G\text{spe.}(G/H;m)) = C_G^0\text{spe.}(G/H;m) = \text{Hom}_Z(Z_{\pi_n}, m(G/H)) \cong m(G/H)$$

$$H_m(S_G\text{spe.}(G/H;m)) = 0.$$

We shall now define an isomorphism of cochain complexes

$$\alpha' : S_G(G/H;m) \to S_G\text{spe.}(G/H;m).$$

Let $c \in C_G^n(G/H;m)$, we then define $\alpha'(c) \in C_G^n\text{spe.}(G/H;m)$ by $(\alpha'(c)(\pi_n) =$

$c(\pi_n)$. Clearly $\alpha'$ is a cochain map. To see that $\alpha'$ is an isomorphism we define a cochain map

$$\beta' : S_G spe.(G/H;m) \to S_G(G/H;m),$$

and show that $\beta'$ is the inverse to $\alpha'$. We first define $\hat{\beta}' : S_G spe.(G/H;m) \to \hat{S}_G(G/H;m)$ as follows. If $d \in C^n_G spe.(G/H;m)$ we define $\hat{\beta}'(d)$ by $(\hat{\beta}'(d))(T) = (\bar{T})^* d(\pi_n) \in m(G/t(T))$, for any equivariant singular n-simplex $T$ belonging to $\mathcal{F}$ in $G/H$. It is easy to see that $\hat{\beta}'$ is a cochain map. If now $T = T'h$, where $h : \Delta_n \times G/t(T) \to \Delta_n \times G/t(T')$ covers $id : \Delta_n \to \Delta_n$, then $(\hat{\beta}'(d))(T) = (\bar{T})^* d(\pi_n) = h^*(\bar{T}')^* d(\pi_n) = h^*(\hat{\beta}'(d))(T')$. Thus the values of $\hat{\beta}'$ in fact lie in $S_G(G/H;m)$. This defines the cochain map $\beta'$. For any equivariant singular n-simplex $T$ belonging to in $G/H$ we have $T = \pi_n \bar{T}$, and hence $c(T) = (\bar{T})^* c(\pi_n)$, for every $c \in C^n_G(G/H;m)$. Thus we have $(\beta'\alpha'(c)(T) = (\bar{T})^* \alpha'(C)(\pi_n) = (\bar{T})^* c(\pi_n) = c(T)$, that is $\beta'\alpha' = id$. Moreover $(\alpha'\beta'(d)(\pi_n) = (\beta'(d)(\pi_n) = (\bar{\pi}_n)^* d(\pi_n) = d(\pi_n)$, since $\bar{\pi}_n = id$. We have shown that $\beta'$ is the inverse to $\alpha'$ and hence that $\alpha'$ is an isomorphism. It follows that the homology R-modules of the cochain complex $S_G(G/H;m)$, that is, the equivariant singular cohomology R-modules $H^P_G(G/H;m)$ are given by

$$H^0_G(G/H;m) \cong m(G/H),$$

$$H^q_G(G/H;m) = 0 \quad , \text{ for } q \neq 0.$$

The explicit isomorphism

$$\xi : H^0_G(G/H;m) \xrightarrow{\cong} m(G/H)$$

is described as follows. First we have $H^0_G(G/H;m) = C^0_G(G/H;m)$, and then $\xi$ is defined in the following way. Let $c \in C^0_G(G/H;m)$, then $\xi(c) = c(\pi_0)$, where $\pi_0 : \Delta_0 \times G/H = G/H \to G/H$ equals the identity map. Using this description of the isomorphism one easily shows that $\xi$ has the naturality property described in the statement of the dimension axiom in Theorem 2.2. This concludes the proof of the dimension axiom for equivariant singular cohomology.

We have now completed the proofs of both Theorem 2.1. and Theorem 2.2.

## 8. ADDITIVITY PROPERTIES

Assume that the G-space $X$ is the topological sum of the G-spaces $X_j$, $j \in J$. We denote this by $X = \biguplus_{j\in J} X_j$. Let $A$ be a G-subset of $X$ and denote $A_j = A \cap X_j$. Then also $A = \biguplus_{j\in J} A_j$. By $i_j : (X_j, A_j) \to (X,A)$ we denote the natural inclusion.

PROPOSITION 8.1. The homomorphisms

$$\sum_{j\in J} \oplus (i_j)_* : \sum_{j\in J} \oplus H_n^G(X_j, A_j; k) \to H_n^G(X,A;k)$$

$$\prod_{j\in J} (i_j)^* : H_G^n(X,A;m) \to \prod_{j\in J} H_G^n(X_j, A_j; m)$$

are isomorphisms for every $n$.

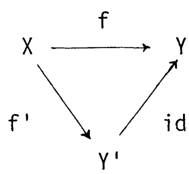
PROOF. Follows immediately using standard properties of direct sums and products from the way we have defined the equivariant singular homology and cohomology modules. q.e.d.

# II. FURTHER PROPERTIES OF EQUIVARIANT SINGULAR HOMOLOGY AND COHOMOLOGY

## 1. NATURALITY IN THE GROUP

Let $P$ and $G$ be topological groups, and let $\mathfrak{F}$ and $\mathfrak{F}'$ be orbit type families for $P$ and $G$, respectively. Assume that $\phi : P \to G$ is a continuous homomorphism, such that, if $Q \in \mathfrak{F}$ then $\phi(Q) \in \mathfrak{F}$ .

Let $X$ be a P-space and $Y$ a G-space, and let $f : X \to Y$ be a $\phi$-map. Thus $f(px) = \phi(p)f(x)$, for every $p \in P$ and $x \in X$. Make $Y$ into a P-space through the homomorphism $\phi : P \to G$. That is, $P$ acts on $Y$ by $py = \phi(p)y$. We denote this P-space by $Y'$. Now observe that we have the commutative diagram




where the map $f'$ equals $f$ as a map of topological spaces, and $id$ is the identity on the underlying topological spaces. The map $f' : X \to Y'$ is a P-map, and $id : Y' \to Y$ is a $\phi$-map.

We shall define induced homomorphisms on equivariant singular homology and cohomology of the $\phi$-map $f : X \to Y$. Due to the above commutative diagram it is enough to consider the $\phi$-map

$$id : Y' \to Y,$$

and define the homomorphisms it induces on equivariant singular homology and cohomology.

Let

$$\alpha : P/Q \to P/N$$

be a P-map ($Q$ and $N$ are subgroups of $P$). Denote $\alpha(eQ) = p_0N$. Thus $\alpha(pQ) = pp_0N$. We have $Q \subset p_0Np_0^{-1}$, and hence $\phi(Q) \subset \phi(p_0)\phi(N)\phi(p_0)^{-1}$. Therefore we can define a G-map

$$\phi(\alpha) : G/\phi(Q) \to G/\phi(N)$$

by the condition $\phi(\alpha)(e\phi(Q)) = \phi(p_0)\phi(N)$. We then have $\phi(\alpha)(g\phi(Q)) = g\phi(p_0)\phi(N)$, $g \in G$.

Now let $k'$ and $k$ be covariant coefficient systems for $\mathfrak{F}'$ and $\mathfrak{F}$, respectively, over the ring $R$. Let

$$\Phi : k' \to k$$

be a natural transformation with respect to the homomorphism $\phi : P \to G$. By this we mean that for each $Q \in \mathfrak{F}$ we have a homomorphism of left R-modules

$$\Phi : k'(P/Q) \to k(G/\phi(Q)),$$

such that if $\alpha : P/Q \to P/N$ is a P-map, where also $N \in \mathfrak{F}'$, then the diagram

$$\begin{array}{ccc} k'(P/Q) & \xrightarrow{\Phi} & k(G/\phi(Q)) \\ {\scriptstyle\alpha_*}\downarrow & & \downarrow{\scriptstyle(\phi(\alpha))_*} \\ k'(P/N) & \xrightarrow{\Phi} & k(G/\phi(N)) \end{array}$$

commutes.

PROPOSITION 1.1. Let the homomorphism $\phi : P \to G$, and the natural transformation $\Phi : k' \to k$ be as above. Let $(Y,B)$ be a G-pair and make it into a P-pair $(Y',B')$ through the homomorphism $\phi$. Then the $\phi$-map $id : (Y',B') \to (Y,B)$ induces homomorphisms

$$(\phi,\Phi)_* : H_n^P(Y',B';k') \to H_n^G(Y,B;k),$$

for all $n$, with the following properties.

1. If $\phi = \mathrm{id}$ and $\Phi = \mathrm{id}$, then $(\phi,\Phi)_* = \mathrm{id}$.
2. $(\phi,\Phi)_*$ commutes with the boundary homomorphisms.
3. If $s : (Y,B) \to (\tilde{Y},\tilde{B})$ is a G-map, and $s' = s : (Y',B') \to (\tilde{Y}',\tilde{B}')$ is the corresponding P-map, we have $s_*(\phi,\Phi)_* = (\phi,\Phi)_* s'_*$.

PROOF. We define a chain map

$$\widehat{(\phi,\Phi)}_\# = \hat{S}^P(Y',B';k') \to \hat{S}^G(Y,B;k)$$

as follows. Let

$$T_P : \Delta_n \times P/Q \to Y'$$

be any P-equivariant singular n-simplex belonging to $\mathfrak{F}'$ in $Y'$. We define the corresponding G-equivaraint singular n-simplex in $Y$

$$T_G : \Delta_n \times G/\phi(Q) \to Y$$

by $T_G(x,g\phi(Q)) = g\, T_P(x,eQ)$. Since the point $T_P(x,eQ) \in Y'$ is fixed under the subgroup $Q$, it follows that the same point $T_P(x,eQ) \in Y$, when considered as a point in the G-space $Y$, is fixed under $\phi(Q)$. It follows that $T_G$ is a well-defined G-map. Since we assumed that $Q \in \mathfrak{F}'$, it follows that $\phi(Q) \in \mathfrak{F}$, that is, that $T_G$ belongs to $\mathfrak{F}$. Now put

$$\widehat{(\phi,\Phi)}_\#(T_P \otimes b) = T_G \otimes \Phi(b),$$

where $b \in k'(P/t(T_P))$, and $\Phi : k'(P/t(T_P)) \to k(G/t(T_G))$ (we have $t(T_G) = \phi(t(T_P))$). This defines the homomorphism $\widehat{(\phi,\Phi)}_n : \hat{C}_n^P(Y';k') \to \hat{C}_n^G(Y;k)$. It is clear that $\widehat{(\phi,\Phi)}_n$ maps $\hat{C}_n^P(B';k')$ into $\hat{C}_n^G(B;k)$, and that the homomorphisms

$\widehat{(\phi,\Phi)}_n$ commute with the boundary. We have constructed the chain map $\widehat{(\phi,\Phi)}_{\#}$.

It remains to show that $\widehat{(\phi,\Phi)}_n$ restricts to $\overline{(\phi,\Phi)}_n : \overline{C}^P_n(Y';k') \to \overline{C}^G_n(Y;k)$, and therefore induces $(\phi,\Phi)_n : C^P_n(Y';k') \to C^G_n(Y;k)$. Assume that $T_P \boxtimes b \sim T'_P \boxtimes b'$, where $T_P : \Delta_n \times P/Q \to Y'$, $T'_P : \Delta_n \times P/Q' \to Y'$, and $b \varepsilon k'(P/Q)$, $b' \varepsilon k'(P/Q')$. Let $h_P : \Delta_n \times P/Q \to \Delta_n \times P/Q'$ be a P-map, which covers $id : \Delta_n \to \Delta_n$, such that $T_P = T'_P h_P$ and $(h_P)_*(b) = b'$. Consider the diagram

$$\begin{array}{ccc} \Delta_n \times P/Q & \xrightarrow{\Delta(\phi)} & \Delta_n \times G/\phi(Q) \\ {\scriptstyle h_P}\downarrow & & \downarrow{\scriptstyle h_G} \\ \Delta_n \times P/Q' & \xrightarrow{\Delta(\phi)'} & \Delta_n \times G/\phi(Q') \end{array}$$

where $\Delta(\phi)$ is defined by $\Delta(\phi)(x,pQ) = (x,\phi(p)\phi(Q))$, $x \varepsilon \Delta_n$, $p \varepsilon P$, and $\Delta(\phi)'$ is defined analogously. It is immediately seen that $\Delta(\phi)$ and $\Delta(\phi)'$ are well-defined $\phi$-maps. Define the map $h_G$ by $h_G(x,g\phi(Q)) = g(\Delta(\phi)'h_P(x,eQ))$, $g \varepsilon G$. The point $\Delta(\phi)'h_P(x,eQ)$ is fixed under the subgroup $\phi(Q)$ of $G$, since the composite map $\Delta(\phi)'h_P$ is a $\phi$-map. It follows that $h_G$ is a well-defined G-map. It is now easy to see that $T'_G h_G = T_G$.

We claim that $(h_G)_*\Phi(b) = \Phi(b')$. This is seen as follows. Restricting the maps $h_P$ and $h_G$ to, for example, the orbit over $d^0 \varepsilon \Delta_n$ gives us the P-map $(h_P)_0 : P/Q \to P/Q'$ and the G-map $(h_G)_0 : G/\phi(Q) \to G/\phi(Q')$. It is easily seen that $(h_G)_0 = \phi((h_P)_0)$. It follows that $(h_G)_*\Phi = \Phi(h_P)_*$. Thus we have $(h_G)_*\Phi(b) = \Phi(h_P)_*(b) = \Phi(b')$. We have now shown that $T_G \boxtimes \Phi(b) \sim T'_G \boxtimes \Phi(b')$.

Thus the chain map $\widehat{(\phi,\Phi)}_{\#}$ restricts to $\overline{(\phi,\Phi)}_{\#}$, and hence induces a chain map $(\phi,\Phi)_{\#}$. This chain map induces homomorphisms

$$(\phi,\Phi)_* : H_n^P(Y',B';k') \to H_n^G(Y,B;k),$$

for all $n$. It is clear that the properties 1-3 are satisfied. q.e.d.

Let $P_1$ be another topological group and let $\mathcal{F}_1$ be an orbit type family for $P_1$, and let $k_1$ be a covariant coefficient system for $\mathcal{F}_1$ over $R$. Assume that $\phi_1 : P_1 \to P$ is a continuous homomorphism, such that, $\phi_1(Q_1) \in \mathcal{F}'$ if $Q_1 \in \mathcal{F}_1$, and let $\Phi_1 : k_1 \to k'$ be a natural transformation with respect to $\phi_1$. It is immediately seen that for any $P_1$-map $\alpha_1 : P_1/Q_1 \to P_1/N_1$, we have $\phi((\phi_1)(\alpha)) = (\phi\phi_1)(\alpha) : G/\phi\phi_1(Q_1) \to G/\phi\phi_1(N_1)$ and that $\Phi\Phi_1 : k_1 \to k$ is a natural transformation with respect to the homomorphism $\phi\phi_1 : P_1 \to G$.

PROPOSITION 1.2. Let the notation be as above. We then have $(\phi\phi_1,\Phi\Phi_1)_* = (\phi,\Phi)_*(\phi_1,\Phi_1)_*$. q.e.d.

Let us now return to the situation where we are given a $\phi$-map $f : (X,A) \to (Y,B)$ from the P-pair $(X,A)$ to the G-pair $(Y,B)$. (All notation and terminology will be as above). The $\phi$-map $f$ induces homomorphisms

$$(f,\phi,\Phi)_* : H_n^P(X,A;k') \to H_n^G(Y,B;k)$$

which by definition, are given by $(f,\phi,\Phi)_* = (\phi,\Phi)_* f'_*$, where $f'_* : H_n^P(X,A;k') \to H^P(Y',B';k')$ is the homomorphism induced by the P-map $f' : (X,A) \to (Y',B')$.

COROLLARY 1.3. Let all the notation be as above. The $\phi$-map $f : (X,A) \to (Y,B)$ induces homomorphism

$$(f,\phi,\Phi) : H_n^P(X,A;k') \to H_n^G(Y,B;k),$$

for all $n$, with the following properties.

1. If $f = id$, $\phi = id$ and $\Phi = id$, then $(f,\phi,\Phi)_* = id$.
2. $(f,\phi,\Phi)_*$ commutes with the boundary homomorphism.
3. Let $s : (Y,B) \to (\tilde{Y},\tilde{B})$ be a G-map, then $(sf,\phi,\Phi)_* = s_*(f,\phi,\Phi)_*$.

4. Let $r : (\tilde{X},\tilde{A}) \to (X,A)$ be a P-map, then $(fr,\phi,\Phi)_* = (f,\phi,\Phi)_* r_*$.

5. Let

$$\begin{array}{ccc} (X,A) & \xrightarrow{\ \ f\ \ } & (Y,B) \\ r \downarrow & & \downarrow s \\ (\tilde{X},\tilde{A}) & \xrightarrow{\ \ \tilde{f}\ \ } & (\tilde{Y},\tilde{B}) \end{array}$$

be a commutative diagram where $s$ is a G-map, $r$ is a P-map, and both $f$ and $\tilde{f}$ are $\phi$-maps. Then we have $(\tilde{f},\phi,\Phi)_* r_* = s_*(f,\phi,\Phi)_*$

6. If $h : (U,C) \to (X,A)$ is a $\phi_1$-map, we have

$$(fh,\phi\phi_1,\Phi\Phi_1)_* = (f,\phi,\Phi)_*(h,\phi_1,\Phi_1)_*.$$

PROOF. The definition of the induced homomorphism $(f,\phi,\Phi)_*$ was already given above, and the properties 1. and 2. are then immediate consequences of the corresponding properties in Proposition 1.1. We shall next show that property 3. is valid. Let $s : (Y,B) \to (\tilde{Y},\tilde{B})$ be a G-map and $s' : (Y',B') \to (\tilde{Y}',\tilde{B}')$ the corresponding P-map. Since $s_*(\phi,\Phi)_* = (\phi,\Phi)_* s'_*$, by property 3. in Proposition 1.1., it follows that $(sf,\phi,\Phi)_* = (\phi,\Phi)_* (sf)'_* = (\phi,\Phi)_* s'_* f'_* = s_*(\phi,\Phi)_* f'_* = s_*(f,\phi,\Phi)_*$. The property 4. is an immediate consequence of the definition, since $(fr,\phi,\Phi)_* = (\phi,\Phi)_*(fr)'_* = (\phi,\Phi)_* f' r_* = (f,\phi,\Phi)_* r_*$. The property 5. is a consequence of properties 3. and 4.

Finally we prove property 6. The notation will be as follows. Let $h'' : (U,C) \to (X'',A'')$ be the $P_1$-map corresponding to the $\phi_1$-map $h : (U,C) \to (X,A)$. Let $f' : (X,A) \to (Y',B')$ be the P-map corresponding to the $\phi$-map $f : (X,A) \to (Y,B)$, and let then $f'' : (X'',A'') \to (Y'',B'')$ be the $P_1$-map corresponding to the P-map $f'$. By property 3. in Proposition 1.1. we have $f'_*(\phi_1,\Phi_1)_* = (\phi_1,\Phi_1)_* f''_*$. By Proposition 1.2. we have $(\phi\phi_1,\Phi\Phi_1)_* = (\phi,\Phi)_*(\phi_1\Phi_1)_*$.

Thus, using these two results, we have $(fh,\phi\phi_1,\Phi\Phi_1)_* = (\phi\phi_1,\Phi\Phi_1)_* f''_* h''_* =$ $(\phi,\Phi)_*(\phi_1,\Phi_1)_* f''_* h''_* = (\phi,\Phi)_* f'_*(\phi_1,\Phi_1)_* h''_* = (f,\phi,\Phi)_*(h,\phi_1,\Phi_1)_*$.

Let us now consider the cohomology version of Proposition 1.1. Let the continuous homomorphism $\phi : P \to G$ and the orbit type families $\mathcal{F}'$ and $\mathcal{F}$ for $P$ and $G$, respectively, be as before. Let $m'$ and $m$ be contravariant coefficient systems for $\mathcal{F}'$ and $\mathcal{F}$, respectively, over the ring. Let

$$\Psi : m \to m'$$

be a natural transformation with respect to the homomorphism $\phi : P \to G$. By this we mean that for any $Q \in \mathcal{F}'$ we have a homomorphism of R-modules

$$\Psi : m(G/\phi(Q)) \to m'(P/Q),$$

such that if $\alpha : P/Q \to G/N$ is a P-map, where also $N \in \mathcal{F}'$, then the diagram

$$\begin{array}{ccc} m(G/\phi(N)) & \xrightarrow{\Psi} & m'(P/N) \\ \phi(\alpha)^* \downarrow & & \downarrow \alpha^* \\ m(G/\phi(Q)) & \xrightarrow{\Psi} & m'(P/Q) \end{array}$$

commutes.

PROPOSITION 1.4. Let the homomorphism $\phi : P \to G$, and the natural transformation $\Psi : m \to m'$ be as above. Let $(Y,B)$ be a G-pair and make it into a P-pair $(Y',B')$ through the homomorphism $\phi$. Then the $\phi$-map $\mathrm{id} : (Y',B') \to (Y,B)$ induces homomorphisms

$$(\phi,\Psi)^* : H^n_G(Y,B;m) \to H^n_P(Y',B';m'),$$

for all $n$, and the contravariant versions of the properties 1.-3.

in Proposition 1.1. are valid.

PROOF. Define a cochain map

$$\widehat{(\phi,\Psi)}^{\#} : \hat{S}_G(Y,B;m) \to \hat{S}_P(Y',B';m')$$

as follows. If $T_P : \Delta_n \times P/Q \to Y'$ is a P-equivariant singular n-simplex belonging to $\mathfrak{F}'$ in $Y'$ we let $T_G : \Delta_n \times G/\phi(Q) \to Y$ be the corresponding G-equivariant singular n-simplex belonging to $\mathfrak{F}$ in Y, as defined in the proof of Proposition 1.1. Now let $\hat{c} \in \hat{C}^n_G(Y,B;m)$ and define $\widehat{(\phi,\Psi)}^{\#}(\hat{c})$ by,

$$(\widehat{(\phi,\Psi)}^{\#}(\hat{c}))(T_P) = \Psi(\hat{c}(T_G)) \in m'(P/Q),$$

where $\Psi : m(G/\phi(Q)) \to m'(P/Q)$. This defines the homomorphism $\widehat{(\phi,\Psi)}^{\#}$, and it is immediately seen that $\widehat{(\phi,\Psi)}^{\#}$ is a cochain map.

It remains to show that $\widehat{(\phi,\Psi)}^{\#}$ restricts to $(\phi,\Psi)^{\#} : S_G(Y,B;m) \to S_P(Y',B';m')$. Let $c \in C^n_G(Y,B;m)$. We shall use the same notation as in the proof of Proposition 1.1. Assume that $T_P = T_P'h_P$, and recall that then $T_G = T_G'h_G$. Moreover $(h_G)_0 = \phi((h_P)_0)$, and hence $\Psi(h_G)^* = (h_P)^*\Psi$. Thus we have $(\widehat{(\phi,\Psi)}^{\#}(c))(T_P) = \Psi(c(T_G)) = \Psi((h_G)^*c(T_G')) = (h_P)^*\Psi(c(T_G')) = (h_P)^*(\widehat{(\phi,\Phi)}^{\#}(c)(T_P')$. Hence $\widehat{(\phi,\Psi)}^{\#}(c) \in C^n_P(Y',B';m)$. This completes the proof. q.e.d.

REMARK. Observe that the expression $(\phi,\Psi)^*$ is contravariant in $\phi$ but covariant in $\Psi$. Thus, if $\Psi_1 : m' \to m_1$ is a natural transformation (between contravariant coefficient systems $m'$ and $m_1$) with respect to the continuous homomorphism $\phi_1 : P_1 \to P$, the cohomology analogue of Proposition 1.2. reads

$$(\phi\phi_1,\Psi_1\Psi)^* = (\phi_1,\Psi_1)^*(\phi,\Psi)^*.$$

Returning to the $\phi$-map $f : (X,A) \to (Y,B)$, from the P-pair $(X,A)$ to the G-pair $(Y,B)$, we now define its induced homomorphisms

$$(f,\phi,\Psi)^* : H^n_G(Y,B;m) \to H^n_P(X,A;m')$$

by $(f,\phi,\Psi)^* = (f')^*(\phi,\Psi)^*$ , where $(f')^* : H^n_P(Y',B';m') \to H^n_P(X,A;m')$ is the homomorphism induced by the P-map $f' : (X,A) \to (Y',B')$. The following corollary follows from Proposition 1.4. and the above remark is exactly the same way as Corollary 1.3. followed from Propositions 1.1. and 1.2.

COROLLARY 1.5. The $\phi$-map $f : (X,A) \to (Y,B)$ induces homomorphisms

$$(f,\phi,\Psi)^* : H^n_G(Y,B;m) \to H^n_P(X,A;m')$$

for all $n$, and the cohomology versions of the properties 1.-6. in Corollary 1.3. are valid. q.e.d.

We conclude this section by determining the induced homomorphisms of the $\phi$-map $f_\phi : P/Q \to G/\phi(Q)$, where $Q \in \mathcal{F}'$ and $f_\phi(pQ) = \phi(p)\phi(Q)$. Since $Q \in \mathcal{F}'$ and $\phi(Q) \in \mathcal{F}$, it follows by the dimension axiom that the P-space $P/Q$ has only 0-dimensional P-equivariant singular homology and cohomology, and that the G-space $G/\phi(Q)$ has only 0-dimensional G-equivariant singular homology and cohomology. The following proposition thus gives the complete answer.

PROPOSITION 1.6. Consider the $\phi$-map $f_\phi : P/Q \to G/\phi(Q)$, where $Q \in \mathcal{F}'$ and $f_\phi(pQ) = \phi(p)\phi(Q)$. We have the commutative diagrams

$$\begin{array}{ccc} H^P_0(P/Q;k') & \xrightarrow{(f_\phi,\phi,\Phi)^*} & H^G_0(G/\phi(Q);k) \\ \gamma_P \downarrow \cong & & \cong \downarrow \gamma_G \\ k'(P/Q) & \xrightarrow{\Phi} & k(G/\phi(Q)) \end{array}$$

$$\begin{array}{ccc} H^0_G(G/\phi(G);m) & \xrightarrow{(f_\phi,\phi,\Psi)^*} & H^0_P(P/Q;m') \\ \xi_G \big\downarrow \cong & & \cong \big\downarrow \xi_P \\ m(G/\phi(Q)) & \xrightarrow{\Psi} & m'(P/Q) \end{array}$$

Here the vertical arrows denote isomorphisms given by the dimension axiom.

PROOF. Let $T \otimes b \in C^P_0(P/Q;k') = H^P(P/Q;k')$, where $T : \Delta_0 \times P/N = P/N \to P/Q$ is a P-equivariant singular 0-simplex belonging to $\mathfrak{F}'$ in $P/Q$, and $b \in k'(P/N)$. We have $(f_\phi,\phi,\Phi)_*(T \otimes b) = (\phi,\Phi)_*(f'_\phi)_*(T \otimes b) = (\phi,\Phi)_*((f'_\phi T) \otimes b) = (f'_\phi T)_G \otimes \Phi(b)$. Here $(f'_\phi T)_G : G/\phi(N) \to G/\ (Q)$ denotes the G-equivariant singular 0-simplex in $G/\phi(Q)$ corresponding to the P-equivariant singular 0-simplex $f'_\phi T : P/N \to G/\phi(N)'$, where $(G/\phi(Q))'$ denotes the P-space obtained by making the G-space $G/\phi(Q)$ into a P-space through the homomorphism $\phi : P \to G$. It is immediately seen that $(f'_\phi T)_G = \phi(T) : G/\phi(N) \to G/\phi(Q)$ the G-map corresponding to the P-map $T : P/N \to P/Q$. Since $\phi(T)_*\Phi = \Phi T_*$ it follows that $\gamma_G(f_\phi,\phi,\Phi)_*(T \otimes b) = \phi(T)_*\Phi(b) = \Phi T_*(b) = \Phi\gamma_P(T \otimes b)$. We have proved that the first diagram commutes.

Recall that $H^0_P(P/Q;m') = C^0_P(P/Q;m')$, and that if $c' \in C^0_P(P/Q;m') = \mathrm{Hom}_t(\hat{C}^P_0(P/Q),M')$ then $\xi_P(c') = c'(\mathrm{id}_{P/Q})$. Also observe that $f'_\phi : P/Q \to (G/\phi(Q))'$ can be considered as a P-equivariant singular 0-simplex in $(G/\phi(Q))'$, and that $(f'_\phi)_G$, the corresponding G-equivariant singular 0-simplex in $G/\phi(Q)$, equals $\mathrm{id}_{G/\phi(Q)}$. It follows that if $c \in C^0_G(G/\phi(Q);m)$ then $\xi_P((f_\phi,\phi,\Psi)^*(c)) = ((f'_\phi)^*(\phi,\Phi)^*(c))(\mathrm{id}_{P/Q}) = ((\phi,\Psi)^*(c))(f'_\phi) = \Psi(c((f'_\phi)_G)) = \Psi(c(\mathrm{id}_{G/\phi(Q)})) = \Psi\xi_G(c)$. This shows that the second diagram commutes. q.e.d.

## 2. TRANSFER HOMOMORPHISMS

In this section $P$ denotes a fixed closed subgroup of $G$ such that the space of right cosets $P\backslash G$ consists of $s$ elements, that is,

$$P\backslash G = \{Pg_1, \ldots, Pg_s\}.$$

Since $P$ is assumed to be closed in $G$ it follows that each point in $P\backslash G$ ($P\backslash G$ has the quotient topology from the projection $\pi: G \to P\backslash G$) is closed in $P\backslash G$. It follows that $P\backslash G$ has the discrete topology.

We say that a G-map

$$\beta : G/H \to G/H'$$

($H$ and $H'$ denote arbitrary subgroups of $G$) is of "type $P$" if $\beta(eH) = p_0H$, where $p_0 \varepsilon P$. In this case we have $H \subset p_0H'p_0^{-1}$, and hence $P \cap H \subset p_0(P \cap H)p_0^{-1}$. Thus we can define a P-map

$$\beta^{\dot{}} : P/P \cap H \to P/P \cap H'$$

by the condition $\beta^{\dot{}}(e(P \cap H)) = p_0(P \cap H')$. We have $\beta^{\dot{}}(p(P \cap H) = pp_0(P \cap H')$, $p \varepsilon P$. Moreover, the P-map $\beta^{\dot{}}$ depends only on the G-map $\beta$ of "type $P$", and not on the specific choice of the element $p_0 \varepsilon P$. For if $\beta(eH) = p_1H'$, where $p_1 \varepsilon P$, then $p_1^{-1}p_0 \varepsilon P \cap H'$ and hence $p_0(P \cap H') = p_1(P \cap H')$. We have shown how any G-map $\beta : G/H \to G/H'$ of "type $P$" determines a P-map $\beta^{\dot{}} : P/P \cap H \to P/P \cap H'$.

Let as before $\mathfrak{F}$ be an orbit type family for $G$, and let $\mathfrak{F}'$ be an orbit type family for $P$, such that, if $H \varepsilon \mathfrak{F}$ then $P \cap H \varepsilon \mathfrak{F}'$. Now let $k'$ and $k$ be convariant coefficient systems for $\mathfrak{F}'$ and $\mathfrak{F}$, respectively, over the ring $R$. Let

$$\Lambda : k \to k'$$

be a natural transformation of transfer type with respect to the inclusion $P \hookrightarrow G$. By this we mean that for every $H \in \mathfrak{F}$ we have a homomorphism of left R-modules

$$\Lambda : k(G/H) \to k'(P/P \cap H)$$

such that if $\beta : G/H \to G/K$, where also $K \in \mathfrak{F}$, is a G-map of "type P", then the diagram

$$\begin{array}{ccc} k(G/H) & \xrightarrow{\Lambda} & k'(P/P \cap H) \\ \beta_* \downarrow & & \downarrow (\beta^!)_* \\ k(G/K) & \xrightarrow{\Lambda} & k'(P/P \cap K) \end{array}$$

commutes.

Let $(Y,B)$ be a G-pair. We denote by $(Y',B')$ the P-pair obtained by restricting the G-action to the subgroup P. We shall construct transfer homomorphisms

$$(\tau^!,\Lambda) : H_n^G(Y,B;k) \to H_n^P(Y',B';k')$$

for every $n$.

We begin by defining for each element $Pg \in P\backslash G$ an induced chain map

$$(Pg)_{\#} : \hat{S}^G(Y,B;k) \to S^P(Y',B';k').$$

(Observe that $(Pg)_{\#}$ has as domain the "roof" chain complex $\hat{S}^G(Y,B;k)$ and as range the chain complex $S^P(Y',B';k')$ which gives the equivariant singular homology modules of the P-pair $(Y',B')$). Let for the moment $g \in G$ be a fixed element of G. We then define

$$(g)_{\#} : \hat{C}^G_n(Y,B;k) \to \hat{C}^P_n(Y',B';k)$$

as follows. Let $T : \Delta_n \times G/K \to Y$ be an equivariant singular n-simplex belonging to $\mathcal{F}$ in Y. Consider the composite map

$$\Delta_n \times P/P \cap gKg^{-1} \xrightarrow{\eta} \Delta_n \times G/gKg^{-1} \xrightarrow{[g]} \Delta_n \times G/K \xrightarrow{T} Y$$

where $\eta(x,p(P \cap gKg^{-1})) = (x,p(gKg^{-1}))$, $\eta \in P \subset G$, and $[g]$ is the G-map, in fact G-homeomorphism, determined by the condition $[g](x,e(gKg^{-1})) = (x,gK)$. The map $T[g]\eta : \Delta_n \times P/P \cap gKg^{-1} \to Y'$, when considered as a map into the P-space $Y'$, is a P-equivariant singular n-simplex belonging to $\mathcal{F}'$ in Y'. Now set

$$(g)_{\#}(T \otimes a) = T[g]\eta \otimes \Lambda[g]_*^{-1}(a),$$

where $a \in k(G/K)$ and $[g]_* : k(G/gKg^{-1}) \to k(G/K)$ is the isomorphism determined by the G-homeomorphism $[g]$, and $\Lambda : k(G/gKg^{-1}) \to k'(P/P \cap gKg^{-1})$. This defines the homomorphism $(g)_{\#} : \hat{C}^G_n(Y,B;k) \to \hat{C}^P_n(Y',B';k')$. Clearly $(g)_{\#}$ commutes with the boundary homomorphism.

Now let $p \in P$. We shall show that $(g)_{\#}(T \otimes a) - (pg)_{\#}(T \otimes a) \in \bar{C}^P_n(Y',B';k')$. Consider the diagram

$$\begin{array}{ccccc}
\Delta_n \times P/P \cap gKg^{-1} & \longrightarrow & \Delta_n \times G/gKg^{-1} & & \\
\downarrow \{p^{-1}\} & & \downarrow [p^{-1}] & \searrow [p] & \\
& & & & \Delta_n \times G/K \\
& & & \nearrow [pg] & \\
\Delta_n \times P/P \cap (pg)K(pg)^{-1} & \xrightarrow{\eta} & \Delta_n \times G/(pg)K(pg)^{-1} & &
\end{array}$$

where $[p^{-1}]$ and $[pg]$ denote the G-homeomorphisms determined by the conditions $[p^{-1}](x,gKg^{-1}) = (x,p^{-1}(pg)K(pg)^{-1}))$ and $[pg](x,e((pg)K(pg)^{-1})) = (x,pgK)$, and $\{p^{-1}\}$ is the P-homeomorphism determined by the condition $\{p^{-1}\}(x,e(P\ gKg^{-1})) = (x,p^{-1}(P \cap (pg)K(pg)^{-1}))$. The diagram commutes. We have $(g)_{\#}(T \otimes a) = T[g]\eta \otimes \Lambda[g]_*^{-1}(a)$, and $(pg)_{\#}(T \otimes a) = T[pg]\eta \otimes \Lambda[pg]_*^{-1}(a)$. We claim that $T[g]\eta \otimes \Lambda[g]_*^{-1}(a) \sim T[pg]\eta \otimes \Lambda[pg]_*^{-1}(a)$. Since $\{p^{-1}\}$ is a P-map, which covers $id : \Delta_n \to \Delta_n$, and $T[g]\eta = T[pg]\eta\{p^{-1}\}$, it only remains to show that $\{p^{-1}\}_*(\Lambda[g]_*^{-1}(a)) = \Lambda[pg]_*^{-1}(a)$. This is seen as follows. Let $\{p^{-1}\}_0 : P/P \cap gKg^{-1} \to P/P \cap (pg)K(pg)^{-1}$ and $[p^{-1}]_0 : G/gKg^{-1} \to G/(pg)K(pg)^{-1}$ be the maps obtained by restricting $\{p^{-1}\}$ and $[p^{-1}]$ to the orbit $d^0 \in \Delta_n$. Then the G-map $[p^{-1}]_0$ is of "type P" and the corresponding P-map is $\{p^{-1}\}_0$, that is, $([p^{-1}]_0)^! = \{p^{-1}\}$. It follows that $\{p^{-1}\}_*\Lambda = \Lambda[p^{-1}]_*$, and hence $\{p^{-1}\}_*(\Lambda[g]_*^{-1}(a)) = \Lambda[p^{-1}]_*[g]_*^{-1}(a) = \Lambda[pg]_*^{-1}(a)$.

We have shown that if $p \in P$ then $\pi(pg)_{\#} = \pi(g)_{\#} : \hat{C}_n^G(Y,B;k) \to C^P(Y',B';k')$, where $\pi$ denotes the natural projection from $\hat{C}_n^P(Y',B';k')$ onto $C_n^P(Y',B';k')$. Thus we can for each element $Pg \in P\backslash G$ define an induced homomorphism $(Pg)_{\#}: \hat{C}_n^G(Y,B;k) \to C^P(Y',B';k')$ by defining $(Pg)_{\#} = \pi(\bar{g})_{\#}$, where $\bar{g}$ is any representative for the right conset $Pg$, that is, $\bar{g} \in Pg$. Since $(g)_{\#}$ commutes with the boundary it follows that the homomorphisms $(Pg)_{\#}$ form a chain map $(Pg)_{\#} : \hat{S}^G(Y,B;k) \to S^P(Y',B';k')$.

We now define

$$\hat{\tau}_{\#} : S^G(Y,B;k) \to S^P(Y',B';k')$$

to be the chain map

$$\hat{\tau}_{\#} = \sum_{i=1}^{s} (Pg_i)_{\#}.$$

Thus $\hat{\tau}_{\#}(T \otimes a) = \pi(\sum_{i=1}^{s} (g_i)_{\#}(T \otimes a))$, where $g_1,\ldots,g_s \in G$ form any complete set of representatives for the set of right cosets $P\backslash G$. We shall now show that $\hat{\tau}_{\#}$ induces a chain map

$$\tau_{\#} : S^G(Y,B;k) \to S^P(Y',B';k').$$

Assume that $T \otimes a \sim T' \otimes a'$, where $T : \Delta_n \times G/K \to Y$, $T' : \Delta_n \times G/K' \to Y'$, and $a \in k(G/K)$, $a' \in k(G/K')$. Let $h : \Delta_n \times G/K \to \Delta_n \times G/K'$ be a G-map which covers $id : \Delta_n \to \Delta_n$, such that $T = T'h$ and $h_*(a) = a'$. Let $g_0 \in G$ be such that $h(d^0,eK) = (d^0,g_0K')$. Now let $g \in G$ be any element of $G$ and consider the diagram

$$\begin{array}{ccccc}
\Delta_n \times P/P \cap gKg^{-1} & \xrightarrow{\eta} & \Delta_n \times G/gKg^{-1} & \xrightarrow{[g]} & \Delta_n \times G/K \\
\downarrow r & & \downarrow \bar{h} & & \downarrow \\
\Delta_n \times P/P \cap (gg_0)K'(gg_0)^{-1} & \xrightarrow{\eta} & \Delta_n \times G/(gg_0)K'(gg_0)^{-1} & \xrightarrow{[gg_0]} & \Delta_n \times G/K'
\end{array}$$

We claim that the image of the map $h[g]\eta$ is the P-subset $P(\Delta_n \times \{gg_0K'\}) = \{(x,p(gg_0)K') \in \Delta_n \times G/K' | p \in P\}$ of $\Delta_n \times G/K'$. This is seen as follows. Let $\bar{\pi} : \Delta_n \times G/K' \to \Delta_n \times P\backslash G/K'$ be the natural projection. The set of double cosets $P\backslash G/K'$ is discrete since $P\backslash G$ is discrete. Denote for convenience $Q = P \cap gKg^{-1}$. Since the subset $\bar{\pi}h[g]\eta(\Delta_n \times \{eQ\}) \subset \Delta_n \times P\backslash G/K'$ is connected and the map $\bar{\pi}h[g]\eta$ covers $id : \Delta_n \to \Delta_n$, and since moreover $(d^0,P(gg_0)K') \in \bar{\pi}h[g]\eta(\Delta_n \times \{eQ\})$, it follows that $\bar{\pi}h[g]\eta(\Delta_n \times \{eQ\}) = \Delta_n \times \{P(gg_0)K'\} \subset \Delta_n \times P\backslash G/K'$. From this and the fact that $Im(h[g]\eta)$ is a P-subset it follows that $Im(h[g]\eta) = P(\Delta_n \times \{(gg_0)K'\})$. Now observe that the map $[gg_0]\eta$ is a P-homeomorphism onto the P-subset $P(\Delta_n \times \{gg_0K\})$. Thus we define the map $r$ in the diagram by $r = ([gg_0]\eta)^{-1}h[g]\eta$. Then $r$ is a P-map, which covers $id : \Delta_n \to \Delta_n$. We have $T[g]\eta = T'[gg_0]\eta r$.

We now claim that $(r)_*\Lambda[g]_*^{-1}(a) = \Lambda[gg_0]_*^{-1}(a')$. This is seen as follows. Define the G-map $\bar{h}$ in the diagram by $\bar{h} = [gg_0]^{-1}h[g]$. The whole diagram now commutes. Restricting the maps $\bar{h}$ and $r$ to the orbit over $d^0 \in \Delta_n$ we get the G-map $\bar{h}_0 : G/gKg^{-1} \to G/(gg_0)^{-1}K'(gg_0)^{-1}$ and the P-map $r_0 : P/P \cap gKg^{-1} \to P/P \cap (gg_0)K'(gg_0)^{-1}$. Observe that $gKg^{-1} \subset (gg_0)K'(gg_0)^{-1}$ and that $\bar{h}_0$ and $r_0$ in fact are the natural projections. Thus in particular the G-map $\bar{h}$ is of "type P" and its corresponding P-map is $r_0$. It follows that $r_*\Lambda = \Lambda\bar{h}_*$. Thus $r_*\Lambda[g]_*^{-1}(a) = \Lambda\bar{h}_*[g]_*^{-1}(a) = \Lambda[gg_0]_*^{-1}h_*(a) = \Lambda[gg_0]_*^{-1}(a')$. We have now shown that $\pi(g)_\#(T \otimes a) = \pi(gg_0)_\#(T' \otimes a')$.

If $g_1,\ldots,g_s \in G$ is any complete set of representatives for the set of right cosets $P\backslash G$ then also $g_1g_0,\ldots,g_sg_0 \in G$ is a complete set of representatives. Thus $\hat{\tau}_\#(T \otimes a) = \pi(\sum_{i=1}^{s} (g_i)_\#(T \otimes a)) = $
$= \pi(\sum_{i=1}^{s} (g_ig_0)_\#(T' \otimes a')) = \hat{\tau}_\#(T' \otimes a')$. We have proved that $\hat{\tau}_\#$ induces a chain map $\tau_\#: S^G(Y,B;k) \to S^P(Y',B';k')$. Moreover it is clear from the way the chain map $\tau_\#$ is defined that $\tau_\#$ commutes with the chain maps induced by a G-map $f : (X,A) \to (Y,B)$ and its corresponding P-map $f' : (X',Y') \to (Y',B')$. We use the notation $(\tau^!,\Lambda) : H_n^G(Y,B;k) \to H_n^P(Y',B';k')$ for the homomorphism induced by the chain map $\tau_\#$. We have proved.

THEOREM 2.1. Assume that P is a closed subgroup of G such that $P\backslash G$ is a finite set. Let the covariant coefficient systems $k'$ and $k$ for P and G, respectively, be as above, and let $\Lambda : k \to k'$ be a natural transformation of transfer type with respect to $P \hookrightarrow G$. For any G-pair $(Y,B)$ we have transfer homomorphisms

$$(\tau^!,\Lambda) : H_n^G(Y,B;k) \to H_n^P(Y',B';k')$$

for every $n$. The homomorphisms $(\tau^!,\Lambda)$ commute with the boundary homomorphism and with homomorphisms induced by G-maps. q.e.d.

We shall now study the composite of the transfer homomorphism $(\tau^!,\Lambda)$ followed by the homomorphism $(i,\Phi)_*$ induced by the inclusion $i : P \hookrightarrow G$. Let $\mathfrak{F}'$ and $\mathfrak{F}$ be orbit type families for $P$ and $G$, respectively, and let $k'$ and $k$ be covariant coefficient system for $\mathfrak{F}'$ and $\mathfrak{F}$, respectively, over the ring $R$. We assume, that if $H \in \mathfrak{F}$ then $P \cap H \in \mathfrak{F}'$, and if $Q \in \mathfrak{F}'$, then also $Q \in \mathfrak{F}$. Let $\Lambda : k \to k'$ be a natural transformation of transfer type with respect to $P \hookrightarrow G$, and let $\Phi : k' \to k$ be a natural transformation with respect to $i : P \hookrightarrow G$. Moreover let $\Theta : k \to k$ be a homomorphism from $k$ to itself, that is, a natural transformation from $k$ to itself with respect to the identity homomorphism $id : G \to G$. We assume that the following condition is satisfied. For every $H \in \mathfrak{F}$ the diagram

$$\begin{array}{ccccc} k(G/H) & \xrightarrow{\Lambda} & k'(P/P \cap H) & \xrightarrow{\Phi} & k(G/P \cap H) \\ & \searrow^{\Theta} & & & \downarrow \rho_* \\ & & & & k(G/H) \end{array}$$

commutes. Here $\rho : G/P \cap H \to G/H$ denotes the natural projection.

THEOREM 2.2. Assume that $P$ is a closed subgroup of $G$ such that $P\backslash G$ consists of $s$ elements. Let $\Lambda : k \to k'$, $\Phi : k' \to k$, and $\Theta : k \to k$ be as above, and assume that the above condition is satisfied. Then, for any $G$-pair $(Y,B)$ and every integer $n$, the composite homomorphism

$$H_n^G(Y,B;k) \xrightarrow{(\tau^!,\Lambda)} H_n^P(Y',B';k') \xrightarrow{(i,\Phi)_*} H_n^G(Y,B;k)$$

equals $s\Theta_*$. In particular if $\Theta = id$ this composite equals multiplication by $s$.

PROOF. Let $T : \Delta_n \times G/K \to Y$ be a $G$-equivariant singular $n$-simplex belonging to $\mathfrak{F}$ in $Y$, and let $a \in k(G/K)$. Let $Pg \in P\backslash G$ and consider the composite homomorphism $(i,\Phi)_{\#}(Pg)_{\#} : \hat{C}_n^P(Y,B;k) \to C_n^P(Y',B';k') \to C_n^G(Y,B;k)$. We have

$$(i,\Phi)_{\#}(Pg)_{\#}(T \otimes a) = \pi((T[g]\eta)_G \otimes \Phi\Lambda[g]_*^{-1}(a)).$$

Here $T[g]\eta$ is the P-equivariant singular n-simplex belonging to ' in Y' as in (2.1), and $(T[g]\eta)_G$ is its corresponding G-equivariant singular n-simplex belonging to $\mathfrak{F}$ in Y, as defined in section 1. of this chapter. It is immediately seen that $(T[g]\eta)_G = T[g]\rho$, where $\rho : \Delta_n \times G/P \cap gKg^{-1} \to \Delta_n \times G/gKg^{-1}$ denotes the natural projection. We have

$$T[g]\rho \otimes \Phi\Lambda[g]_*^{-1}(a) \sim T \otimes [g]_*\rho_*\Phi\Lambda[g]_*^{-1}(a) = T \otimes [g]_*\Theta[g]_*^{-1}(a) = T \otimes \Theta(a).$$

Thus

$$(i,\Phi)_{\#}(Pg)_{\#}(T \otimes a) = \pi\hat{\Theta}_{\#}(T \otimes a).$$

It follows that already on the chain level we have $(i,\Phi)_{\#}\tau_{\#} = s\Theta_{\#} : C_n^G(Y,B;k) \to C_n^G(Y,B;k)$. q.e.d.

We shall show that the transfer homomorphisms compose in a natural way. Assume, in addition to the assumptions made in establishing Theorem 2.1., that N is a closed subgroup of P such that $N\backslash P$ is a finite set, and that $\mathfrak{F}_1$ is an orbit type family for N, such that, if $Q \in \mathfrak{F}'$ then $N \cap Q \in \mathfrak{F}_1$. Moreover let $k_1$ be a covariant coefficient system for $\mathfrak{F}_1$ and $\Lambda_1 : k' \to k_1$ a natural transformation of transfer type with respect to $N \hookrightarrow P$. Observe that $\Lambda_1\Lambda : k \to k_1$ is then a natural transformation of transfer type with respect to $N \hookrightarrow G$.

PROPOSITION 2.3. Let the assumptions and notation be as above. Then

$$(\tau^!,\Lambda_1)(\tau^!,\Lambda) = (\tau^!,\Lambda_1\Lambda).$$

PROOF. Let $g \in G$, $p \in P$ and consider the homomorphisms $(g)$ : $\hat{C}^G_n(Y,B;k) \to \hat{C}^P_n(Y',B';k')$ and $(p)_{\#}$ : $\hat{C}^P_n(Y',B';k') \to \hat{C}^N_n(Y_1,B_1;k_1)$ (here $(Y_1,B_1)$ denotes the N-pair obtained from the G-pair $(Y,B)$ by restricting the action to the subgroup N). It is easy to see that $(p)_{\#}(g)_{\#} = (pg)_{\#}$. The proposition now follows from this fact for if $g_1,\ldots,g_s \in G$ is a complete set of representatives for $P\backslash G$ and $p_1,\ldots,p_r \in P$ is a complete set of representatives for $N\backslash P$ then the elements $p_i g_j \in G$, $1 \leq i \leq s$, form a complete set of representatives for $N\backslash G$. q.e.d.

The construction of the transfer homomorphism in cohomology is dual to the construction in homology. We give the necessary details below, using the same diagrams and constructions we already used in constructing the transfer homomorphism in homology.

Assume that $\mathfrak{F}'$ and $\mathfrak{F}$ are orbit type families for P and G, respectively, such that if $H \in \mathfrak{F}$ the $P \cap H \in \mathfrak{F}'$. Let m' and m be contravariant coefficient systems for $\mathfrak{F}'$ and $\mathfrak{F}$, respectively, over the ring R. Let

$$\Omega : m' \to m$$

be a natural transformation of transfer type with respect to the inclusion $P \hookrightarrow G$. By this we mean that for any $H \in \mathfrak{F}$ we have a homomorphism of right R-modules

$$\Omega : m'(P/P \cap H) \to m(G/H)$$

such that if $\beta : G/H \to G/K$, where also $K \in \mathfrak{F}$, is a G-map of "type P", then the diagram

$$\begin{array}{ccc} m'(P/P \cap K) & \xrightarrow{\Omega} & m(G/K) \\ (\beta')^* \downarrow & & \downarrow \beta^* \\ m'(P/P \cap H) & \xrightarrow{\Omega} & m(G/H) \end{array}$$

commutes.

THEOREM 2.4. Assume that $P$ is a closed subgroup of $G$ such that $P\backslash G$ is a finite set. Let the contravariant coefficient systems $m'$ and $m$ for $P$ and $G$, respectively, be as above, and let $\Omega : m' \to m$ be a natural transformation of transfer type with respect to $P \hookrightarrow G$. For any G-pair $(Y,B)$ we have transfer homomorphisms

$$(\tau_!,\Omega) : H_P^n(Y',B';m') \to H_G^n(Y,B;m)$$

for every $n$. The homomorphisms $(\tau_!,\Omega)$ commute with the boundary homomorphism and with homomorphisms induced by G-maps.

PROOF. Let $g \in G$ and define

$$(g)^{\#} : C_P^n(Y',B';m') \to \hat{C}_G^n(Y,B;m)$$

as follows. Let $c \in C_P^n(Y',B';k')$ and define $(g)^{\#}(c)$ by the following. If $T : \Delta_n \times G/K \to Y$ is a G-equivariant singular n-simplex belonging to $\mathfrak{F}$ in $Y$ we define the value of $(g)^{\#}(c)$ on $T$ by (see (2.1))

$$((g)^{\#}(c))(T) = ([g]^*)^{-1}\Omega c(T[g]_n).$$

Here $c(T[g]_n) \in m'(P/P \cap gKg^{-1})$ and $\Omega : m'(P/P \cap gKg^{-1}) \to m(G/gKg^{-1})$, and $[g]^*$ is an isomorphism $[g]^* : m(G/K) \to m(G/gKg^{-1})$. This defines the homomorphism $(g)^{\#}$ and it is clear that is commutes with the coboundary.

Now let $p \in P$. We claim that $(pg)^{\#} = (g)^{\#}$. Consider the diagram (2.2.). Since $c \in C_P^n(Y',B';k')$ (no roof!) it follows that $c(T[g]\eta) = c(T[pg]\eta\{p^{-1}\}) = \{p^{-1}\}^*c(T[pg]\eta)$. Thus we have

$$((g)^{\#}(c)(T) = ([g]^*)^{-1}\Omega c(T[g]\eta) = ([g]^*)^{-1}\Omega\{p^{-1}\}^*c(T[pg]\eta) =$$

$$([g]^*)^{-1}[p^{-1}]^*\Omega c(T[pg]\eta) = ((pg)^{\#}(c))(T).$$

It follows that each element $Pg \in P\diagdown G$ gives rise to homomorphisms

$$(Pg)^{\#} : C_P^n(Y',B';m') \to \hat{C}_G^n(Y,B;m)$$

for all $n$, defined by $(Pg)^{\#} = (\overline{g})^{\#}$, where $\overline{g}$ is any representative for the right coset $Pg$.

We now define

$$\hat{\tau}^{\#} : C_P^n(Y',B';m') \to \hat{C}_G^n(Y,B;m)$$

to be the homomorphism

$$\hat{\tau}^{\#} = \sum_{i=1}^{s} (Pg_i)^{\#}.$$

Thus $\tau^{\#}(c) = \sum_{i=1}^{s} (g_i)^{\#}(c)$, where $g_1,\ldots,g_s \in G$ form any complete set of representatives for the set of right cosets $P\diagdown G$. We claim that the image of $\hat{\tau}^{\#}$ lies $C_G^n(Y,B;m)$ and $\hat{\tau}^{\#}$ thus induces

$$\tau^{\#} : C_P^n(Y',B';m') \to C_G^n(Y,B;m).$$

This is seen as follows. Let $T : \Delta_n \times G/K \to Y$ and $T' : \Delta_n \times G/K \to Y$ be equivariant singular n-simplexes belonging to in $Y$, and assume that $h : \Delta_n \times G/K \to \Delta_n \times G/K'$ is a G-map, which covers $id : \Delta_n \to \Delta_n$, such that $T = T'h$. We must show that $(\hat{\tau}^{\#}(c))(T) = h^*(\hat{\tau}^{\#}(c))(T')$. Let $g_0 \in G$ be such that $h(d^0,eK) = (d^0,g_0K)$. Now let $g \in G$ and consider the diagram (2.3.).

We then have

$$((g)^{\#}(c))(T) = ([g]^*)^{-1}\Omega c(T([g]\eta) = ([g]^*)^{-1}\Omega c(T'[gg_0]\eta r) =$$

$$([g]^*)^{-1}\Omega r^* c(T'[gg_0]\eta) = ([g]^*)^{-1}\overline{h}^*\Omega c(T'[gg_0]\eta) =$$

$$h^*([gg_0]^*)^{-1}\Omega c(T'[gg_0]\eta) = h^*(gg_0)^{\#}(c)(T').$$

Now if $g_1,\dots,g_s \in G$ is any complete set representatives for $P\backslash G$ the same is true for $g_1g_0,\dots,g_sg_0 \in G$. Thus by what we just showed it follows that $(\hat{\tau}^{\#}(c))(T) = \sum_{i=1}^{s} ((g_i)^{\#}(c))(T) = \sum_{i=1}^{s} h^*((g_ig_0)^{\#}(c))(T') = h^*(\hat{\tau}^{\#}(c)(T'))$. This proves our claim and thus $\hat{\tau}^{\#}$ induces $\tau^{\#}$. The homomorphisms $\tau^{\#}$ form a cochain map $\tau^{\#} : S_P(Y',B';m') \to S_G(Y,B;m)$. We denote the induced homomorphism on cohomology by $(\tau_!,\Omega) : H_P^n(Y',B';m') \to H_G^n(Y,B;m)$ and call it the transfer homomorphism. q.e.d.

We shall now give the cohomology version ot Theorem 2.2. For this let $\mathfrak{F}'$ and $\mathfrak{F}$ be orbit type families for $P$ and $G$, respectively, such that if $H \in \mathfrak{F}$ then $P \cap H \in \mathfrak{F}'$ and if $Q \in \mathfrak{F}$ then also $Q \in \mathfrak{F}$. Let $m'$ and $m$ be contravariant coefficient systems for $\mathfrak{F}'$ and $\mathfrak{F}$, respectively. Let $\Omega : m' \to m$ be a natural transformation of transfer type with respect to $P \hookrightarrow G$, and let $\Psi : m \to m'$ be a natural transformation with respect to $i : P \hookrightarrow G$. Moreover let $\Theta : m \to m$ be a homomorphism from the contravariant coefficient system $m$ to itself. (The homomorphism induced by $\Theta$ on equivariant singular cohomology is denoted by $\Theta_*$.) We assume that the following condition is satisfied. For every $H \in \mathfrak{F}$ the diagram

$$\begin{array}{ccccc} m(G/P\cap H) & \xrightarrow{\Psi} & m'(P/P\cap H) & \xrightarrow{\Omega} & m(G/H) \\ \uparrow{\scriptstyle \rho_*} & & & \nearrow{\scriptstyle \Theta} & \\ m(G/H) & & & & \end{array}$$

commutes. Here $\rho : G/P \cap H \to G/H$ denotes the natural projection.

THEOREM 2.5. Assume that $P$ is a closed subgroup of $G$ such that $P\backslash G$ consists of $s$ elements. Let $\Omega : m' \to m$, $\Psi : m \to m'$, and $\Theta : m \to m$ be as above, and assume that the above condition is satisfied. Then for any G-pair $(Y,B)$ and every integer $n$, the composite homomorphism

$$H^n_G(Y,B;m) \xrightarrow{(i,\Psi)^*} H^n_P(Y',B';m) \xrightarrow{(\tau_!,\Omega)} H^n_G(Y,B;m)$$

equals $s\Theta_*$. In particular if $\Theta = id$ this composite equals multiplication by $s$.

PROOF. Let $Pg \in P\backslash G$, and consider the composite homomorphism $(Pg)^{\#}(i,\Psi)^{\#} : C^n_G(Y,B;m) \to C^n_P(Y',B';m') \to \hat{C}^n_G(Y,B;m)$. Let $c \in C^n_G(Y,B;m)$. The value of $(Pg)^{\#}(i,\Psi)^{\#}(c)$ on an equivariant singular n-simplex $T : \Delta_n \times G/K \to Y$ belonging to $\mathfrak{F}$ equals $((Pg)^{\#}(i,\Psi)^{\#}(c))(T) = ([g]^*)^{-1}\Omega(i,\Psi)^{\#}(c))(T[g]) = ([g]^*)^{-1}\Omega\Psi c((T[g]\eta)_G)$. (The notation is the same as in (2.1) and the proof of Theorem 2.2.). We have $(T[g]\eta)_G = T[g]\rho$, where $\rho : \Delta_n \times G/P \cap gKg^{-1} \to \Delta_n \times G/gKg^{-1}$ denotes the natural projection, and $c(T[g]\eta) = \rho^*[g]^*c(T)$. Hence

$$((Pg)^{\#}(i,\Psi)^{\#}(c))(T) = ([g]^*)^{-1}\Omega\Psi\rho^*[g]^*c(T) = \Theta(c(T)) = \Theta_{\#}(c))(T).$$

The result follows. q.e.d.

REMARK. The transfer homomorphisms in cohomology also compose in a natural way. The cohomology version of Proposition 2.3. reads

$$(\tau_!,\Omega)(\tau_!,\Omega_1) = (\tau_!\Omega\Omega_1)$$

where $\Omega_1 : m_1 \to m'$ is a natural transformation of transfer type with respect to $N \hookrightarrow P$.

## 3. THE KRONECKER INDEX AND THE CUP-PRODUCT

In this section we assume that $R$ is a commutative ring. By $\mathfrak{F}$ we denote an orbit type family for $G$.

DEFINITION 3.1. Let $k$ and $m$ be a covariant and a contravariant, respectively, coefficient system for $\mathfrak{F}$ over $R$. A pairing $\omega$ of $k$ and $m$ consists of the following. For each $H \in \mathfrak{F}$ we have a homomorphism of $R$-modules

$$\omega : m(G/H) \otimes_R k(G/H) \to R,$$

such that if $\alpha : G/H \to G/K$, where also $K \in \mathfrak{F}$, is a $G$-map and $b \in m(G/K)$, $a \in k(G/K)$, then

$$\omega(b \otimes_R \alpha_*(a)) = \omega(\alpha^*(b) \otimes_R a).$$

Let $X$ be a $G$-space, and let $\hat{c} \in \hat{C}^n_G(X;m)$ and $\hat{\sigma} \in \hat{C}^G_n(X;k)$. Assume that we are given a pairing $\omega$ of $k$ and $m$. We then define the Kronecker index of $\hat{c}$ and $\hat{\sigma}$, denoted $\langle \hat{c},\hat{\sigma}\rangle \in R$, as follows. If $\hat{\sigma} = \sum_{i=1}^{q} T_i \otimes_R a_i$, we set

$$\langle \hat{c},\hat{\sigma}\rangle = \omega\Big(\sum_{i=1}^{q} \hat{c}(T_i) \otimes_R a_i\Big).$$

It is immediately seen that this gives us a well-defined homomorphism of $R$-modules

$$\langle\ ,\ \rangle : \hat{C}^n_G(X;m) \otimes_R \hat{C}^G_n(X;k) \to R.$$

Let $T$ be an equivariant singular $(n+1)$-simplex belonging to $\mathfrak{F}$ in $X$ and $a \in k(G/t(T))$. We then have

$$\langle \hat{c},\hat{\partial}_{n+1}(T \otimes a)\rangle = \Big\langle \hat{c}, \sum_{i=1}^{n+1} (-1)^i T^{(i)} \otimes a \Big\rangle =$$

$$\omega\Big(\sum_{i=0}^{n+1} (-1)^i \hat{c}(T^{(i)}) \otimes_R a\Big) = \omega(\hat{\delta}_n \hat{c}(T) \otimes_R a) = \Big\langle \hat{\delta}_n \hat{c},\ T \otimes a \Big\rangle .$$

It follows that

$$\Big\langle \hat{c}, \hat{\partial}_{n+1}(\hat{\sigma}) \Big\rangle = \Big\langle \hat{\delta}_n \hat{c}, \hat{\sigma} \Big\rangle$$

for any $\hat{c} \in \hat{C}^n_G(X;m)$ and $\hat{\sigma} \in \hat{C}^G_{n+1}(X;k)$.

Now assume that $c \in C^n_G(X;m)$ and $\sigma \in C^G_n(X;k)$. We claim that the definition

$$\langle c, \sigma \rangle = \langle c, \hat{\sigma} \rangle ,$$

where $\hat{\sigma} \in \hat{C}^G_n(X;k)$ is any representative for $\sigma$, gives a well-defined homomorphism

$$\langle\ ,\ \rangle : C^n_G(X;m) \otimes_R C^G_n(X;k) \to R.$$

This is seen as follows. Assume that $T \otimes a \sim T' \otimes a'$, and let $h : \Delta_n \times G/t(T) \to \Delta_n \times G/t(T')$ be a G-map, which covers $id : \Delta_n \to \Delta_n$, such that $T = T'h$ and $h_*(a) = a'$. Since $c \in C^n_G(X,m)$ it follows that $c(T) = h^*c(T')$, and hence we have

$$\langle c, T \otimes a \rangle = \omega(c(T) \otimes_R a) = \omega(h^*c(T') \otimes_R a) =$$

$$\omega(c(T') \otimes_R h_*(a)) = \omega(c(T') \otimes_R a') = \langle c, T' \otimes a' \rangle .$$

This proves our claim.

Let $(X,A)$ be a G-pair. It follows directly from the definitions that the already established pairing $\langle\ ,\ \rangle$ for the absolute case induces a pairing

$$\langle\ ,\ \rangle : C^n_G(X,A;m) \otimes_R C^G_n(X,A;k) \to R.$$

Since now

$$\left\langle c, \partial_{n+1}\sigma \right\rangle = \left\langle \delta_n c, \sigma \right\rangle ,$$

where $c \varepsilon C_G^n(X,A;m)$ and $\sigma \varepsilon C_{n+1}^G(X,A;k)$, it follows that we have an induced pairing

$$\langle\ ,\ \rangle : H_G^n(X,A;m) \otimes_R H_n^G(X,A;k) \to R.$$

This map, $\langle\ ,\ \rangle$ is a homomorphism of R-modules, and we call it the Kronecker index. The Kronecker index gives rise to the homomorphism of R-modules

$$v : H_G^n(X,A;m) \to \mathrm{Hom}_R(H_n^G(X,A),R)$$

defined by $v(\eta)(\xi) = \langle \eta,\xi \rangle$, where $\eta \varepsilon H_G^n(X,A;m)$ and $\xi \varepsilon H_n^G(X,A;k)$.

For a G-space of the form $G/H$, where $H \varepsilon \mathfrak{F}$, the Kronecker index agrees with the given pairing $\omega$. To be more precise we have the following proposition.

PROPOSITION 3.2. Let $H \varepsilon \mathfrak{F}$. Then the diagram

$$\begin{array}{ccc}
H_G^0(G/H;m) \otimes_R H_0^G(G/H;k) & \xrightarrow{\langle\ ,\ \rangle} & R \\
\xi \otimes \gamma \cong \big\downarrow & & \big\downarrow \mathrm{id} \\
m(G/H) \otimes_R k(G/H) & \xrightarrow{\ \ \omega\ \ } & R
\end{array}$$

commutes. Here $\gamma$ and $\xi$ are the isomorphisms given by the dimension axiom.

PROOF. Let $c \varepsilon C_G^0(G/H;m) = H_G^0(G/H;m)$ and $T \otimes a \varepsilon C_0^G(G/H;k) = H_0^G(G/H;k)$ where $T : G/K \to G/H$ is an equivariant singular 0-simplex in $G/H$ and $a \varepsilon k(G/H)$. We have $c(T) = T^*c(\mathrm{id}_{G/H}) \varepsilon m(G/K)$. Thus

$$\omega(\xi \otimes \gamma)(c \otimes_R (T \otimes a)) = \omega(\xi(c) \otimes_R \gamma(T \otimes a)) =$$

$$\omega(c(id_{G/H}) \otimes_R T_*(a)) = \omega(T^*c(id_{G/H}) \otimes_R a) =$$

$$\omega(c(T) \otimes_R a) = \left\langle c, T \otimes a \right\rangle \quad . \qquad \text{q.e.d.}$$

We shall now construct a cup-product in equivariant singular cohomology. We assume in the following that the orbit type family $\mathcal{F}$ is such that $G \varepsilon \mathcal{F}$ .

DEFINITION 3.3. A contravariant coefficient system $m$ for $\mathcal{F}$ , over the ring $R$, is called a commutative ring coefficient sysstem if the following condition is satisfied. Each $m(G/H)$, $H \varepsilon \mathcal{F}$ , is a commutative ring and all induced homomorphisms are ring homomorphisms, and moreover $m(G/G) = R$ and the R-module structure on each $m(G/H)$ is the same as the one induced by the ring homomorphism $\pi^* : R = m(G/G) \to m(G/H)$.

Assume from now that $m$ is a commutative ring coefficient system. Let $\hat{c} \varepsilon \hat{C}^n_G(X;m)$ and $\hat{c}_1 \varepsilon \hat{C}^p_G(X;m)$. We define the cup-product $\hat{c} \cup \hat{c}_1 \varepsilon \hat{C}^{n+p}_G(X;m)$ by the following. Let $T : \Delta_{n+p} \times G/K \to X$ be an equivariant singular (n+p)-simplex belonging to $\mathcal{F}$ in $X$. We use the notation

$$\alpha_n : \Delta_n \times G/K \to \Delta_{n+p} \times G/K, \quad \text{and}$$

$$\beta_p : \Delta_p \times G/K \to \Delta_{n+p} \times G/K$$

for the front n-face and back p-face, respectively, of $\Delta_{n+p} \times G/K$, that is, $\alpha_n((x_0,\dots,x_n),gK) = ((x_0,\dots,x_n,0,\dots,0),gK)$ and $\beta_p((x_0,\dots,x_p),gK) = ((0,\dots,0,x_0,\dots,x_p),gK)$. We now define the value of $\hat{c} \cup \hat{c}_1$ on $T$ to be

$$(\hat{c} \cup \hat{c}_1) = (\hat{c}(T\alpha_n))(\hat{c}_1(T\beta_n)) \ \varepsilon \ m(G/K).$$

This defines a homomorphism of R-modules

$$\cup : \hat{C}^n_G(X;m) \otimes_R \hat{C}^p_G(X;m) \to \hat{C}^{n+p}_G(X;m).$$

The formula

$$\hat{\delta}(\hat{c} \cup \hat{c}_1) = (\hat{\delta}\hat{c}) \cup \hat{c}_1 + (-1)^n \hat{c} \cup (\hat{\delta}\hat{c}_1)$$

is established by the standard calculation.

We now claim that if $c \in C_G^n(X;m)$ and $c_1 \in C_G^p(X;m)$ then also $c \cup c_1 \in C_G^{n+p}(X;m)$. This is seen as follows. Let $T : \Delta_{n+p} \times G/K \to X$ and $T' : \Delta_{n+p} \times G/K' \to X$ and assume that $h : \Delta_{n+p} \times G/K \to \Delta_{n+p} \times G/K'$ is a G-map, which covers $id : \Delta_{n+p} \to \Delta_{n+p}$, such that $T = T'h$. We have to show that

$$(c \cup c_1)(T) = h^*(c \cup c_1)(T').$$

The G-map $h$ determines G-maps $h_\alpha : \Delta_n \times G/K \to \Delta_n \times G/K'$ and $h_\beta : \Delta_p \times G/K \to \Delta_p \times G/K'$, which cover the identity, such that $h\alpha_n = \alpha_n h_\alpha$ and $h\beta_p = \beta_p h_\beta$. Moreover $h_\beta^* = h_\beta^* = h^* : m(G/K') \to m(G/K)$. Since now $T\alpha_n = T'\alpha h_\alpha$ and $T\beta_p = T'\beta_p h_\beta$ it follows that

$$(c \cup c_1)(T) = (c(T'\alpha_n h_\alpha))(c_1(T'\beta_p h)) =$$

$$(h^*c(T'\alpha_n))(h^*c_1(T'\beta_p)) = h^*((c(T'\alpha_n))(c_1(T'\beta_p))) =$$

$$h^*(c \cup c_1)(T'),$$

where we used the fact that $h^*$ is a ring homomorphism. This proves our claim.

If $(X,A)$ is a G-pair and, for example, $c \in C_G^n(X,A;m)$ then also $c \cup c_1 = C_G^{n+p}(X,A;m)$. In particular we have

$$\cup : C_G^n(X,A;m) \otimes_R C_G^p(X,A;m) \to C_G^{n+p}(X,A;m).$$

Since now $\delta(c \cup c_1) = (\delta c) \cup c_1 + (-1)^n c \cup (\delta c_1)$, that is, the homomorphisms $\cup$ form a cochain map, it follows that we get a cup-product on the cohomology level

$$\cup : H_G^n(X,A;m) \otimes_R C_G^p(X,A;m) \to C_G^{n+p}(X,A;m).$$

We shall conclude by showing that the cup-product is commutative. The proof in Artin-Braun [1] for the commutativity (also sometimes called anticommutativity) of the cup-product in ordinary singular cohomology carries over to our situation without any difficulties. (See section 22 in Artin-Braun [1] for more details than we give below.)

The reader should recall the notion of an equivariant linear q-simplex $v^0 \ldots v^q \times id : \Delta_q \times G/K \to \Delta_n \times G/K$ in $\Delta_n \times G/K$ and the definition of the linear chain groups $\hat{C}_q^G Q(\Delta_n \times G/K)$ and the corresponding chain complex $\hat{S}^G Q(\Delta_n \times G/K)$, as defined in Section 6. of Chapter I. We shall also use the join homomorphism $v\cdot : C_{q+1}^G Q(\Delta_n \times G/K) \to C_{q+1}^G Q(\Delta_n \times G/K)$ and its properties with respect to the boundary homomorphism. For this we again refer to the beginning of Section 6. of Chapter I.

Define the homomorphism

$$\hat{\rho}_q : \hat{C}_q^G Q(\Delta_n \times G/K) \to \hat{C}_q^G Q(\Delta_n \times G/K)$$

by

$$\hat{\rho}_q(v^0 \ldots v_q^q \times id) = (-1)^{q(q+1)/2}(v^q \ldots v^0 \times id).$$

It is easily seen that the homomorphisms $\hat{\rho}_q$ commute with the boundary and thus form a chain map $\hat{\rho}_\#$. We now inductively define homomorphisms

$$\hat{D}_q : \hat{C}_q^G Q(\Delta_n \times G/K) \to \hat{C}_{q+1}^G Q(\Delta_n \times G/K)$$

by setting $\hat{D}_0 = 0$, and

$$\hat{D}_q(\sigma) = v^0\cdot(\sigma - \hat{\rho}_q(\sigma) - \hat{D}_{q-1}(\hat{\partial}_q(\sigma)), \quad q \geq 1$$

where $\sigma = v^0 \dots v^q \times \mathrm{id}$. A formal computation and induction shows that the homomorphisms $\hat{D}_q$ form a chain homotopy from the identity map to $\hat{\rho}_{\#}$.

Let $X$ be a G-space. Define the homomorphisms

$$\hat{\rho}_n : \hat{C}_n^G(X) \to \hat{C}_n^G(X)$$

$$\hat{D}_n : \hat{C}_n^G(X) \to \hat{C}_{n+1}^G(X)$$

as follows. If $T : \Delta_n \times G/K \to X$ is an equivariant singular n-simplex belonging to $\mathfrak{F}$ in $X$ we set

$$\hat{\rho}_n(T) = \hat{T}_{\#}\hat{\rho}_n(d^0 \dots d^n \times \mathrm{id}),$$

$$\hat{D}_n(T) = \hat{T}_{\#}\hat{D}_n(d^0 \dots d^n \times \mathrm{id}).$$

(Recall that $d^0 \dots d^n \times \mathrm{id} : \Delta_n \times G/K \to \Delta_n \times G/K$ is the identity map.) It is easy to see that these "new" homomorphisms $\hat{\rho}_n$ form a chain map $\hat{\rho} : S^G(X) \to S^G(X)$, and that the new homomorphisms $\hat{D}_n$ form a chain homotopy from the identity map to $\hat{\rho}_{\#}$.

We now claim that the homomorphisms $\hat{\rho}_n : \hat{C}_n^G(X) \to \hat{C}_n^G(X)$ and $\hat{D}_n : \hat{C}_n^G(X) \to \hat{C}_{n+1}^G(X)$ both "preserve the relation $\sim$" (see Def. 4.4. in Chapter I). This is easily proved in exactly the same way as Proposition 6.1. in Chapter I. It follows that $\hat{\rho}_n$ and $\hat{D}_n$ have duals $\hat{\rho}^n : \hat{C}_G^n(X,A;m) \to \hat{C}_G^n(X,A;m)$ and $\hat{D}^{n+1} : \hat{C}_G^{n+1}(X,A;m) \to \hat{C}_G^n(X,A;m)$ which restrict to give

$$\rho^n : C_G^n(X,A;m) \to C_G^n(X,A;m)$$

$$D^{n+1} : C_G^{n+1}(X,A;m) \to C_G^n(X,A;m).$$

The homomorphisms $\rho^n$ form a cochain map $\rho^{\#}$ and the homomorphisms $D^n$ form a cochain homotopy from the identity map to $\rho^{\#}$.

We can now show that the cup-product is commutative. Let $y \in H^n_G(X,A;m)$, $y_1 \in H^p_G(X,A;m)$ and let $c \in C^n_G(X,A;m)$ and let $c_1 \in C^p_G(X,A;m)$ be cocycles representing $y$ and $y_1$, respectively. The cohomology class $y \cup y_1$ is represented by the cocycle $c \cup c_1$. It now follows from what we showed above that the cocycle $\rho^{n+p}((\rho^n c) \cup (\rho^p c_1))$ also represents $y \cup y_1$. Let $T : \Delta_{n+p} \times G/K \to X$ be an equivariant singular $(n+p)$-simplex belonging to $\mathcal{F}$ in $X$. Since we have $T(d^{n+p}\ldots d^0)\alpha_n(d^n\ldots d^0) = T(d^p\ldots d^{n+p}) = T\beta_n$ and $T(d^{n+p}\ldots d^0)\beta_p(d^p\ldots d^0) = T(d^0\ldots d^p) = T\alpha_p$ it follows that

$$\rho^{n+p}((\rho^n c) \cup (\rho^p c_1))(T) = (-1)^{np}(c(T\beta_n))(c_1(T\alpha_n)) =$$

$$(-1)^{np}(c_1(T\alpha_n))(c(T\beta_n)) = (-1)^{np}(c_1 \cup c)(T).$$

(The sign is as stated since $((n+p)(n+p+1) + n(n+1) + p(p+1))/2 \equiv np \pmod 2$.) This proves that

$$y \cup y_1 = (-1)^{np}(y_1 \cup y),$$

for $y \in H^n_G(X,A;m)$ and $y_1 \in H^p_P(X,A;m)$.

## BIBLIOGRAPHY


[1] E. Artin and H. Braun, Introduction to Algebraic Topology, Merrill Publishing Co., Columbus, Ohio, 1969.

[2] G. Bredon, Equivariant cohomology theories, Bull. Amer. Math. Soc. 73 (1967), 266-268.

[3] ______, Equivariant cohomology theories, Lecture Notes in Math., volume 34, Springer-Verlag, Berlin and New York, 1967.

[4] Th. Bröker, Singuläre Definition der äquivarianten Bredon Homologie, Manuscripta Math. 5 (1971), 91-102.

[5] S. Eilenberg, Singular homology theory, Ann. of Math. 45 (1944), 407-447.

[6] S. Eilenberg and N.E. Steenrod, Foundations of Algebraic Topology, Princeton Univ. Press, Princeton, N.J., 1952.

[7] S. Illman, Equivariant singular homology and cohomology for actions of compact Lie groups, Proc. Conference on Transformation Groups (University of Massachusetts, Amherst, 1971) Lecture Notes in Math. Springer-Verlag volume 298, Berlin and New York, 1972.

[8] - , Equivariant Algebraic Topology, Thesis, Princeton University, Princeton, N.J., 1972.

[9] - , Equivariant singular homology and cohomology, Bull. Amer. Math. Soc. 79 (1973), 188-192.


University of Helsinki

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