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Coalgebraic algebra

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Abstract

We investigate module objects in categories of coalgebras, setting up tensor products and initiating the study of the resulting homological algebra. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

The object of this paper is to set up tensor products and homological algebra for ring and module objects in categories of coalgebras. Our interest in and applications of these results are motivated by homotopy theory and we begin in this section by sketching the algebraic topology that lies behind our work. We hope this will aid the reader in two ways: not only will it supply motivation for this paper but it will also provide what we believe are useful examples to keep in mind while discussing the pure algebra in later sections.

Consider $H_*(X;\mathbb{F}_p)$, the ordinary (Eilenberg–MacLane) homology of a space X with coefficients in the integers mod p. Just as the cohomology $H^*(X;\mathbb{F}_p)$ of the space has cup products making it into a graded, commutative, associative $\mathbb{F}_p$ algebra, so the homology $H_*(X;\mathbb{F}_p)$ has a coaction giving it a natural structure of a graded, cocommutative, coassociative $\mathbb{F}_p$ coalgebra. Similarly, if $E_*(-)$ is any generalised homology theory with a suitable Künneth theorem $E_*(X\times X)\cong E_*(X)\otimes_{E_*}E_*(X)$, for example, $E=K(n)$ the nth Morava K-theory, then $E_*(X)$ comes with the structure of a graded, cocommutative, coassociative E_* coalgebra.

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It now follows that an algebraic structure on the space X, or at least such a structure up to homotopy, will give rise under $H_*(-;\mathbb{F}_p)$ to an algebraic object of the same type in the category of graded coalgebras. Thus, for example, if X is a topological group $H_*(X;\mathbb{F}_p)$ is a group object in the category of coalgebras – better known as a *Hopf algebra* after Hopf who first considered this process; the coaction and counit arise from the theory $H_*(-;\mathbb{F}_p)$ as before while the product, conjugation and unit arise, respectively, from the product, inverse and unit maps in X. The same holds when X is merely a group up to homotopy.

Another classical example is that of the n-fold loops on the n sphere, $\Omega^n S^n$, $n>0$. This space has the structure of an abelian group up to homotopy: addition arises from the loop sum operation (concatenation of loops) and inverses from the reversal of loops. However, $\Omega^n S^n$ has the further structure of a ring up to homotopy: if we interpret a point in $\Omega^n S^n$ as a map $S^n \to S^n$, composition of these maps gives a product which, up to homotopy, is associative and distributes over the addition.[1] When we take the mod p homology of $\Omega^n S^n$ we consequently get a ring object in the category of graded $\mathbb{F}_p$ coalgebras, an object that has come to be termed a *coalgebraic ring* or *Hopf ring*.

Other examples of spaces with algebraic structures also arise naturally. If F is a ring spectrum [1] it has a corresponding Ω spectrum $\{\underline{F}_n; n \in \mathbb{Z}\}$. These are the spaces representing the multiplicative generalised cohomology theory $F^*(-)$ associated to F and are related to each other by the based loop space construction $\underline{F}_n = \Omega\underline{F}_{n+1}$. Thus, $F^n(X) = [X, \underline{F}_n]$, the homotopy classes of maps from the space X to $\underline{F}_n$, and the addition in $F^n(X)$ arises from the loop sum operation $\underline{F}_n \times \underline{F}_n \to \underline{F}_n$ in $\underline{F}_n = \Omega\underline{F}_{n+1}$. The inverse operation in $F^n(X)$ corresponds to the map $\underline{F}_n \to \underline{F}_n$ reversing loops and the unit corresponds to the inclusion of the point $* \to \underline{F}_n$ as the constant loop. Thus, as in the previous example, each $\underline{F}_n$ is a group up to homotopy and so each $H_*(\underline{F}_n;\mathbb{F}_p)$ is a graded Hopf algebra.

However, the ring spectrum F gives rise to a product in the cohomology theory $F^*(-)$. This gives us homomorphisms

$$F^n(X) \otimes F^m(X) \to F^{n+m}(X)$$

corresponding to maps of the representing spaces

$$\underline{F}_n \times \underline{F}_m \to \underline{F}_{n+m}.$$

Hence, the set of all the spaces, $\underline{F}_* = \{\underline{F}_n; n \in \mathbb{Z}\}$ is a graded ring object in the homotopy category. Applying $H_*(-;\mathbb{F}_p)$ gives us products

$$\circ : H_*(\underline{F}_n;\mathbb{F}_p) \otimes H_*(\underline{F}_m;\mathbb{F}_p) \to H_*(\underline{F}_{n+m};\mathbb{F}_p)$$

and this makes the bigraded object $H_*(\underline{F}_*;\mathbb{F}_p)$ into a graded ring object in the category of graded $\mathbb{F}_p$ coalgebras, i.e. into a (bigraded) coalgebraic ring or Hopf ring.

[1] Paolo Salvatore has pointed out to us that distributivity fails to $\Omega^n S^n$ does not in fact provide an example of a ring up to homotopy. The limits $QS^0 = \lim_n \Omega^n S^n$ does however give an example. The same remark also applies to Example 2.4.

In the same way, if F is a ring spectrum and G is an F module spectrum [1], the object $H_*(\underline{G}_*; \mathbb{F}_p)$ has the structure of a module object in the category of coalgebras over the ring object $H_*(\underline{F}_*; \mathbb{F}_p)$. Such objects can appropriately be called *coalgebraic modules* or *Hopf modules*.

Our final, and somewhat simpler example of a Hopf ring arises as follows. Suppose that R and S are commutative rings. Forgetting the product on S for the moment we may form the group ring $R[S]$. Imposing a coproduct on $R[S]$ by declaring $\psi([s]) = [s] \otimes [s]$ and a counit $R[S] \to R$ given by the usual augmentation map makes $R[S]$ into a Hopf algebra: recall the product in $R[S]$ comes from the operation $[s_1] * [s_2] = [s_1 + s_2]$. If we define a second product in $R[S]$ by linearly extending the operation

$$[s_1] \circ [s_2] = [s_1 s_2]$$

given by the multiplication in S, we have given the 'ring–ring' object $R[S]$ the structure of a Hopf ring. This construction arises also from the study of Ω spectra. If F is a ring spectrum as above, the (homological) zero-dimensional part of $H_*(\underline{F}_*; \mathbb{F}_p)$, i.e., $H_0(\underline{F}_*; \mathbb{F}_p)$, can be identified as the ring–ring $\mathbb{F}_p[F^*]$ where F^* is the ring of coefficients of F, i.e., the F cohomology of a point.

Hopf rings first appeared explicitly in the work of Milgram [7] and were later used by Ravenel and Wilson [9] on the homology of spaces in the Ω spectra for complex cobordism and the Brown Peterson theories and in further papers discussing a wide variety of Ω spectra. They are the technical ingredient needed to set up and run unstable Adams spectral sequences [3] and of course are the definitive language with which to speak about unstable cohomology operations [4]. Hopf modules were used in [2] to discuss the homology of pro-spaces in the pro-Ω spectrum representing the I_n-adic completion of $E(n)$. The ideas of Hopf rings and their like have now been used extensively in topology and have proved to be a powerful tool. The additional algebraic structure inherent in a Hopf ring, over say that contained in the underlying Hopf algebra (or Hopf algebras), allows much complicated topological detail to be encapsulated in relatively simple algebraic terms; see, for example, Ravenel and Wilson's description of the homology of spaces in the Ω spectrum for complex cobordism as a certain free Hopf ring modulo a single family of relations derived from the associated formal group laws. In all, Hopf rings have proved to be a natural and an invaluble part of the tool kit of a topologist.

It has been noted in several papers that though already established the use of the term *Hopf ring*, *module*, etc., to indicate a ring, module, etc., object in a category of coalgebras has a serious drawback: the traditional Hopf algebra in this framework ought to be denoted a Hopf group. In this paper we shall endeavour to avoid these difficulties by using the term "coalgebraic –" for a – object in a category of coalgebras. Thus the traditional Hopf algebra is a coalgebraic group and Hopf rings and modules are coalgebraic rings and modules.

One of our main topological interests in coalgebraic rings and coalgebraic modules arises from the study of the homology of Ω spectra, as just described. In particular, given an F module spectrum G we are interested in the relation between a coalgebraic module $E_*(\underline{G}_*)$ and its 'ground' coalgebraic ring $E_*(\underline{F}_*)$. Examination of the analagous stable case suggests that we should aim to set up notions of homological algebra for these coalgebraic objects.

In this paper we establish two main results. These are

Theorem 3.4. *The category of finite-type coalgebraic modules over a connective coalgebraic ring $A_{*,*}$ is abelian.*

Theorem 5.3. *For a given coalgebraic ring A, the category of A coalgebraic modules is a symmetric monoidal category under $\overline{\otimes}_A$ with special object A.*

The important part of this latter result is the existence of a tensor product $\overline{\otimes}$ of coalgebraic modules. This we construct explicitly and in such a way as to form one of the major ingredients to the following theorem which will appear elsewhere in a more topological article: the current paper may be considered in part as providing the necessary algebraic background for this result.

Theorem. *Suppose G is a module spectrum over the ring spectrum F and the homology theory associated to G is exact over that for F, i.e., for any space X we have $G_*(X) = G_* \otimes_{F_*} F_*(X)$. Suppose also that E is a ring spectrum equipped with a suitable Künneth theorem. Then $E_*(\underline{G}_*)$ and $E_*(\underline{F}_*)$ are related by the coalgebraic module tensor product*

$$E_*(\underline{G}_*) = E_*(\underline{F}_*) \overline{\otimes}_{E_*[F^*]} E_*[G^*].$$

The tensor product of coalgebraic rings has occurred in two previous papers, [9, 5], neither of which considered either its explicit construction or its basic properties. We hope this paper will add further insight to the contents of those articles. We also hope that the coalgebraic homological algebra we set up will be of independent interest, in particular to recent results of Kashiwabara and Wilson on the relations in Morava K-homology between the spaces in the Ω spectra for p local Eilenberg–MacLane theories, $BP\langle n\rangle$ and BP theory.

The arrangement of this paper is as follows. In Section 2 we set up coalgebraic rings and coalgebraic modules introducing notation and examples. We also discuss quotients of these objects. In Section 3 we prove Theorem 3.4 and in the process show that the category of general $A_{*,*}$ coalgebraic modules is always additive. In Section 4 we introduce the notion of a coalgebraic pairing and relate it to the idea of a coalgebraic ring. In Section 5 we construct tensor products by way of representing sets of coalgebraic pairings and we prove Theorem 5.3. In Section 6 we show the coalgebraic tensor product to be half exact and in the final section we examine the functor $R[-]$ in greater detail, in particular studying some of the related homological algebra.

2. Coalgebraic rings and coalgebraic modules

In this section we shall more formally define the concepts of (abelian) coalgebraic groups, coalgebraic rings, coalgebraic modules and coalgebraic ideals. We shall establish some basic notation and describe the free coalgebraic module functor. Throughout this section and the rest of the paper R is a commutative ring with unit.

Definition 2.1. A graded *abelian coalgebraic group over R* is an abelian group object in the category of graded cocommutative, coassociative coalgebras with counit over R.

In other words, we have a graded, bicommutative, biassociative Hopf algebra (in the traditional sense) over R with unit, counit and conjugation.

Definition 2.2. A bigraded *commutative coalgebraic ring over R* is a graded commutative ring object in the category of graded cocommutative, coassociative coalgebras with counit over R. Denote the category of such objects with obvious (bigrading preserving) morphisms by $\mathscr{CR}$.

To establish notation we will be a bit more explicit. We will not dwell on the details too much but refer the reader to the standard references [9] and now also [4]. A bigraded coalgebraic ring over R is a graded abelian coalgebraic group $H_{*,*}$ over R with structure maps

$\psi : H_{*,a} \to H_{*,a} \otimes_R H_{*,a}$	coproduct,
$\varepsilon : H_{*,*} \to R$	counit,
$\chi : H_{a,b} \to H_{a,b}$	conjugation,
$* : H_{a,c} \otimes_R H_{b,c} \to H_{a+b,c}$	product,
$\eta : R \to H_{0,0}$	unit

together with additional structure maps

$\circ : H_{a,b} \otimes_R H_{c,d} \to H_{a+c,b+d}$	$\circ$ product
$u : R \to H_{0,0}$	$\circ$ unit

subject to certain axioms.

For a general element x we use the notation $\psi(x) = \sum x' \otimes_R x''$ to indicate its coproduct.

We write [0] for $\eta(1)$ and [1] for $u(1)$; these names are consistent with the elements of the same name in the coalgebraic ring $R[S]$ introduced in Section 1.

Here are the examples mentioned in the last section.

Example 2.3. Let S be a commutative ring with unit. The group algebra $R[S]$ is a coalgebraic ring with coproduct $\psi([s]) = [s] \otimes [s]$ and circle product $[s_1] \circ [s_2] = [s_1 s_2]$.

Example 2.4. The mod p homology of the n-fold loop space $\Omega^n S^n$ is a monograded coalgebraic ring with the $\circ$ product induced from the composition of loops. In the same way, so is the mod p homology of the infinite loop space $QS^0 = \lim_n \Omega^n S^n$. At the prime 2 a description of this has been given in [11] and at odd primes in [6].

Example 2.5. Let $\{\underline{F}_k\}_{k\in\mathbb{Z}}$ be a ring Ω spectrum; its mod p homology is a bigraded coalgebraic ring.

A further comment on our use of gradings is in order. If the rings R and S are ungraded, the ring–ring $R[S]$ is ungraded; the topological example of $H_*(\Omega^n S^n; \mathbb{F}_p)$ is monograded while those involving Ω spectra above are bigraded. We shall draw on the topology by naming the gradings of an abstract bigraded coalgebraic ring the *homological* and *space* gradings, and they shall be denoted in that order. Thus, in a bigraded coalgebraic ring $M_{*,*}$, each $M_{*,n}$ is itself a graded coalgebraic group, while the $\circ$ product passes from $M_{*,n} \otimes M_{*,m}$ to $M_{*,n+m}$. When we wish to consider monograded objects the grading in question will be the homological grading; in particular, these correspond to a bigraded object $M_{*,*}$ with all but $M_{*,0}$ trivial. Similarly, an ungraded object corresponds to a bigraded object $M_{*,*}$ with all but $M_{0,0}$ trivial. We shall denote the homology grading of an element $m \in M_{*,*}$ by $|m|$ and its space grading by $\|m\|$.

It is possible to define a product in the category of coalgebraic rings. For $H_{*,*}$ and $K_{*,*}$ in $\mathscr{CR}$ define $H_{*,*} \boxtimes_R K_{*,*}$ to be the sum of all modules of the form $H_{a,c} \otimes_R K_{b,c}$ and is bigraded by declaring that the elements of bidegree (r,s) are those in the direct sum $\oplus_{a+b=r} H_{a,s} \otimes_R K_{b,s}$. It is well known that this can be given the structure of a coalgebraic group and the circle product is given by

$$(a\otimes b)\circ(c\otimes d) = (-1)^{|c||b|}(a\circ c)\otimes(b\circ d).$$

The $\circ$ unit is $[1]\otimes[1]$.

Proposition 2.6. *The operation $\boxtimes_R$ is a product in $\mathscr{CR}$. The projections are given by*

$$1_H \otimes \varepsilon_K : H_{*,*} \boxtimes_R K_{*,*} \to H_{*,*} \boxtimes_R R \cong H_{*,*}$$

and

$$\varepsilon_H \otimes 1_K : H_{*,*} \boxtimes_R K_{*,*} \to R \boxtimes_R K_{*,*} \cong K_{*,*}.$$

For morphisms $f: B_{*,*} \to H_{*,*}$ and $g: B_{*,*} \to K_{*,*}$ the unique morphism $h: B_{*,*} \to H_{*,*} \boxtimes_R K_{*,*}$ is given by $h(b) = \sum f(b') \otimes g(b'')$.

Now let $A_{*,*}$ be a commutative coalgebraic ring over R. We shall consider module objects over $A_{*,*}$, providing more explicit details than before.

Definition 2.7. An $A_{*,*}$ *coalgebraic module* over R is a graded abelian coalgebraic group $M_{*,*}$ over R equipped with a coalgebra map

$$\circ : A_{*,*} \otimes_R M_{*,*} \to M_{*,*}$$

satisfying the following five axioms for all $a, b \in A_{*,*}$ and $x, y \in M_{*,*}$.

(1) $(a\circ b)\circ x = a\circ(b\circ x)$,

(2) $[1] \circ x = x$,
(3) $[0] \circ x = \eta\varepsilon(x)$,
(4) $(a * b) \circ x = \sum(-1)^{|x'||b|}(a \circ x') * (b \circ x'')$,
(5) $a \circ (x * y) = \sum(-1)^{|a''||x|}(a' \circ x) * (a'' \circ y)$.

We can turn this left action into a right one by defining

$$x \circ a = (-1)^{|x||a|}[-1]^{\circ\|x\|\|a\|} \circ a \circ x,$$

where $[-1] = \chi[1]$. We will do this without further comment.

Denote by $\mathscr{CM}_{A_{*,*}}$ the category of $A_{*,*}$ coalgebraic modules with structure and bigrading preserving morphisms.

Example 2.8. Let T be an S module. Then the group algebra $R[T]$ is an $R[S]$ coalgebraic module.

Example 2.9. Any commutative coalgebraic group $H_{*,*}$ over R is an $R[\mathbb{Z}]$ coalgebraic module with structure map $\circ: R[\mathbb{Z}] \otimes_R H_{*,*} \to H_{*,*}$ given by $[n] \otimes x \mapsto \sum x' * x'' * \cdots * x^{(n)}$ where $\sum x' \otimes x'' \otimes \cdots \otimes x^{(n)}$ is the iterated coproduct acting on x.

Example 2.10. Let X be an infinite loop space and write QS^0 for $\lim \Omega^n S^n$ as usual. Then $H_*(X; \mathbb{F}_p)$ is a $H_*(QS^0; \mathbb{F}_p)$ coalgebraic module.

Example 2.11. Let F be a ring spectrum and G an F module spectrum. Then $H_*(\underline{G}_*; \mathbb{F}_p)$ is an $H_*(\underline{F}_*; \mathbb{F}_p)$ coalgebraic module.

Remark 2.12. If $M_{*,*}$ is an $A_{*,*}$ coalgebraic module the $*$ indecomposables $QM_{*,*}$ can be given the structure of a $QA_{*,*}$ module using the $\circ$ action; this follows from the fifth axiom, above.

Remark 2.13. For $H_{*,*}$ and $K_{*,*}$ coalgebraic modules over $A_{*,*}$, the coalgebra $H_{*,*} \boxtimes_R K_{*,*} = \oplus H_{*,c} \otimes_R K_{*,c}$ also has the structure of an $A_{*,*}$ coalgebraic module. The coalgebraic group structure is as usual and the $A_{*,*}$ action is generated by

$$a \circ (h \otimes k) = \sum(-1)^{|a''||h|}(a' \circ h) \otimes (a'' \circ k).$$

We shall want to know when we can take a quotient and still get a coalgebraic module. This is easy and similar to the case of coalgebraic groups.

Definition 2.14. An $A_{*,*}$ coalgebraic ideal in $H_{*,*}$ is an ideal I in $H_{*,*}$ such that
(1) $\psi(I) \subset I \otimes_R H_{*,*} + H_{*,*} \otimes_R I$,
(2) $\varepsilon(I) = 0$,
(3) $A_{*,*} \circ I \subset I$.

An ideal I satisfying the first two of these properties is a *Hopf ideal* in the sense of Sweedler [10]. It is shown there that a Hopf algebra quotiented by a Hopf ideal again carries the structure of a Hopf algebra.

Proposition 2.15. *Let I be an $A_{*,*}$ coalgebraic ideal in $H_{*,*}$. Then $H_{*,*}/I$ is an $A_{*,*}$ coalgebraic module over R.*

Proof. (1) and (2) ensure that $H_{*,*}/I$ is a coalgebraic group over R and (3) ensures that $\circ$ is defined. □

We shall need to consider free $A_{*,*}$ coalgebraic modules constructed from given supplemented coalgebras. Recall from [9] that a supplemented coalgebra $C_{*,*}$ is a graded, cocommutative, coassociative coalgebra with counit over R equipped with a map $\eta: R \to C_{*,*}$ with $\varepsilon\eta$ the identity on R. Define [0] to be the element $\eta(1)$. Write $\mathscr{SC}$ for the category of such objects and let $C_{*,*}$ be an object in $\mathscr{SC}$. Construct the *free $A_{*,*}$ coalgebraic module* $\mathscr{F}(C_{*,*})$ by taking all sums of all $*$ products of all $\circ$ products of elements in $A_{*,*}$ with elements in $C_{*,*}$ subject to the coalgebraic module axioms and identifying $[0] \in A_{*,*}$ with $[0] \in C_{*,*}$. This gives a functor

$$\mathscr{F}: \mathscr{SC} \to \mathscr{CM}_{A_{*,*}}.$$

There is a canonical supplemented coalgebra map $\theta_{C_{*,*}}: C_{*,*} \to \mathscr{F}(C_{*,*})$ and we have the following standard universal property.

Proposition 2.16. *Given an $A_{*,*}$ coalgebraic module $M_{*,*}$ and a map of supplemented coalgebras $\sigma: C_{*,*} \to M_{*,*}$ there is a unique $A_{*,*}$ coalgebraic module map $\phi: \mathscr{F}(C_{*,*}) \to M_{*,*}$ with*

$$\sigma = \phi\theta_{C_{*,*}}: C_{*,*} \to \mathscr{F}(C_{*,*}) \to M_{*,*}.$$

Proposition 2.17. *Any coalgebraic module $M_{*,*}$ is the quotient of a free coalgebraic module.*

Proof. Let us write $\mathscr{D}: \mathscr{CM}_{A_{*,*}} \to \mathscr{SC}$ for the forgetful functor. Then we can form $\mathscr{F}\mathscr{D}(M_{*,*})$, a free coalgebraic module. By the universal property of $\mathscr{F}$ the identity map $\mathscr{D}(M_{*,*}) \to M_{*,*}$ lifts to a map $\mathscr{F}\mathscr{D}(M_{*,*}) \to M_{*,*}$ which, by construction, is onto. □

3. The abelian category $\mathscr{CM}_{A_{*,*}}$

In this section we show that under suitable restrictions the category of $A_{*,*}$ coalgebraic modules is abelian. Much of our work rests on previous results which show that suitable categories of Hopf algebras are abelian (see in particular [10]). To use these foundations we need to make the global assumption for this section that all coalgebraic modules $M_{*,*}$ are of finite type, by which we mean that they are $\mathbb{N} \times \mathbb{Z}$ graded and that each $M_{a,b}$ is of finite rank as an R module. We also need to make the second assumption that the underlying coalgebraic ring $A_{*,*}$ is *connective*.

Definition 3.1. We say the coalgebraic ring $A_{*,*}$ over R is connective if it is $\mathbb{N}\times\mathbb{Z}$ graded and $A_{0,*}$ is isomorphic (as coalgebraic rings over R) to $R[S]$ for some (graded) ring S.

We shall see that the point of this definition is that it puts significant restrictions on the coproduct. The following lemma highlights the nature of the coproduct in a connective coalgebraic ring.

Lemma 3.2. *If $A_{*,*}$ is a connective coalgebraic ring and $a\in A_{d,*}$ for some $d>0$ then $\psi(a)=[0]\otimes a+a\otimes[0]+\sum b_i\otimes c_i$ where the b_i and c_i are all of homological dimension bigger than zero. The homological dimension zero elements are generated as a free R module by group like elements, i.e., by elements a with $\psi(a)=a\otimes a$.*

Proof. The first part follows from [8] and the second is implicit in the definition of a connective coalgebraic ring and the structure of a ring–ring $R[S]$, as given in Section 7. □

Our first result however uses no additional assumptions on the category of coalgebraic modules.

Theorem 3.3. *The category of $A_{*,*}$ coalgebraic modules is additive.*

Proof. We first establish that the category has products. Given maps of $A_{*,*}$ coalgebraic modules $f:B_{*,*}\to H_{*,*}$ and $g:B_{*,*}\to K_{*,*}$ it suffices to show that the map $h:B_{*,*}\to H_{*,*}\boxtimes_R K_{*,*}$ of Proposition 2.6 defined by $h(b)=\sum f(b')\otimes g(b'')$ is also a map of $A_{*,*}$ coalgebraic modules. Thus we must show that $a\circ h(b)=h(a\circ b)$ for any $a\in A_{*,*}$ and $b\in B_{*,*}$. We compute as follows, remembering the $A_{*,*}$ action on $H_{*,*}\boxtimes_R K_{*,*}$ from Remark 2.13:

$$\begin{aligned}
a\circ h(b) &= a\circ\textstyle\sum f(b')\otimes g(b'')\\
&= \textstyle\sum a\circ(f(b')\otimes g(b''))\\
&= \textstyle\sum(-1)^{|a''''||b'|}(a'''\circ f(b'))\otimes(a''''\circ g(b''))\\
&= \textstyle\sum(-1)^{|a''''||b'|}f(a'''\circ b')\otimes g(a''''\circ b'')\\
&= (f\otimes g)\textstyle\sum(-1)^{|a''''||b'|}(a'''\circ b')\otimes(a''''\circ b'')\\
&= (f\otimes g)\psi(a\circ b)\\
&= h(a\circ b).
\end{aligned}$$

Now consider the morphism sets. Suppose $M_{*,*}$ and $N_{*,*}$ are two $A_{*,*}$ coalgebraic modules. The morphisms $f,g:M_{*,*}\to N_{*,*}$, etc., the $A_{*,*}$ coalgebraic module maps from $M_{*,*}$ to $N_{*,*}$, have a sum given by convolution

$$(f\bowtie g)(m)=\textstyle\sum f(m')*g(m'').$$

It is easy to check that convolution does give an $A_{*,*}$ coalgebraic module map. Moreover, χf is the convolution inverse of f and so this operation makes the morphism sets into groups. In fact, as the underlying coalgebras are cocommutative, they are always abelian groups. The composition distributes over $\bowtie$ and so the category in question is additive with zero object R (regarded as the free R module on the element [0]). $\square$

Theorem 3.4. *The category of finite type coalgebraic modules over a connective coalgebraic ring $A_{*,*}$ is abelian.*

Proof. The last result tells us that the category is additive. To prove that it is also abelian we need to establish the existence of kernels and cokernels. In fact, the coalgebraic module kernels and cokernels coincide under the forgetful functor with the notions of kernel and cokernel for coalgebraic groups. We recall their definition from [10].

Definition 3.5. Let $f : M_{*,*} \to N_{*,*}$ be a map of coalgebraic groups. Define HKer(f), the *Hopf kernel* of f, by

$$\mathrm{HKer}(f) = \{m \in M_{*,*} \mid (1 \otimes f)\psi(m) = m \otimes [0]\}.$$

By cocommutativity this is the same as

$$\{m \in M_{*,*} \mid (f \otimes 1)\psi(m) = [0] \otimes m\}.$$

The *Hopf cokernel*, HCoker(f), is $N_{*,*}/\langle f(M^+_{*,*})\rangle$ where $\langle f(M^+_{*,*})\rangle$ is the left Hopf ideal in $N_{*,*}$ generated by the image of f and $M^+_{*,*}$ is the augmentation ideal in $M_{*,*}$. By commutativity of $N_{*,*}$ this is the same as the right ideal generated by $f(M^+_{*,*})$. The work of [10] shows that HKer and HCoker are indeed kernels and cokernels in the category of coalgebraic groups.

Definition 3.6. Let $f : M_{*,*} \to N_{*,*}$ be a map of coalgebraic modules. Define CKer(f) as HKer(f), the Hopf kernel of the underlying map of coalgebraic groups. Similarly, define CCoker(f) as the Hopf cokernel of the underlying coalgebraic group map.

To prove Theorem 3.4 it suffices to show that if $f : M_{*,*} \to N_{*,*}$ is in fact a map of $A_{*,*}$ coalgebraic modules, then HKer(f) is closed under $\circ$ action by elements of $A_{*,*}$ and that $\langle f(M^+_{*,*})\rangle$ is also a coalgebraic ideal in the sense of Definition 2.14.

For the first of these, suppose $m \in$HKer(f) and $a \in A_{*,*}$:

$$\begin{aligned}(1 \otimes f)\psi(a \circ m) &= \textstyle\sum(-1)^{|a''''||m'|}(a''' \circ m') \otimes (a'''' \circ f(m''))\\ &= \textstyle\sum a \circ (m' \otimes f(m''))\\ &= a \circ (\textstyle\sum m' \otimes f(m''))\\ &= a \circ (m \otimes [0])\\ &= \textstyle\sum(-1)^{|a''||m|}(a' \circ m) \otimes (a'' \circ [0])\\ &= (a \circ m) \otimes [0].\end{aligned}$$

The final equality needs further explanation. We may assume a to be of homogeneous homological dimension and we split the argument into two cases. First, if $|a| = 0$ we know that a is an R linear sum of elements of the form $[s]$ and $\psi([s]) = [s] \otimes [s]$. In this case the equality follows as $[s] \circ [0] = [0]$. On the other hand, if $|a| > 0$, by Lemma 3.2, we have

$$\sum (-1)^{|a''||m|}(a' \circ m) \otimes (a'' \circ [0]) = (a \circ m) \otimes [0] + T,$$

where T is a sum of terms involving products of the form $a'' \circ [0]$ with $|a''| > 0$. However, $\circ$ multiplication by $[0]$ annihilates all elements of positive homological degree.

For the cokernels we need to check that

$$A_{*,*} \circ \langle f(M^+_{*,*})\rangle \subset \langle f(M^+_{*,*})\rangle.$$

However, this is easy. An element of $\langle f(M^+_{*,*})\rangle$ can be written as a sum of elements of the form $f(m) * n$ for $m \in M^+_{*,*}$ and $n \in N_{*,*}$. Now

$$\begin{aligned} a \circ (f(m) * n) &= \sum (-1)^{|a''||m|}(a' \circ f(m)) * (a'' \circ n) \\ &= \sum (-1)^{|a''||m|}(f(a' \circ m)) * (a'' \circ n). \end{aligned}$$

As $\varepsilon(a' \circ m) = \varepsilon(a') \cdot \varepsilon(m) = 0$ we see that $a' \circ m \in M^+_{*,*}$, thus HCoker(f) is closed under the $A_{*,*}$ action as required.

The universal properties of kernel and cokernel (justifying such names in the category in question) now follow from their same properties as constructions in the category of coalgebraic groups. This concludes the proof of Theorem 3.4. □

4. Coalgebraic pairings and coalgebraic rings

Let $L_{*,*}$, $M_{*,*}$ and $N_{*,*}$ be $A_{*,*}$ coalgebraic modules over R. The next definition extends the idea of a bilinear map to the setting of coalgebraic modules.

Definition 4.1. A coalgebra (over R) map $f : L_{*,*} \otimes_R M_{*,*} \to N_{*,*}$ is said to be an $A_{*,*}$ *coalgebraic pairing* when the following conditions hold for all $a \in A_{*,*}$, $u, v \in L_{*,*}$, $x, y \in M_{*,*}$,

1. $f(u \otimes (x * y)) = \sum (-1)^{|u''||x|} f(u' \otimes x) * f(u'' \otimes y)$,
2. $f((u * v) \otimes x) = \sum (-1)^{|v||x'|} f(u \otimes x') * f(v \otimes x'')$,
3. $f((a \circ u) \otimes x) = a \circ f(u \otimes x)$,
4. $f(u \otimes (x \circ a)) = f(u \otimes x) \circ a$.

Of course, if we start with a coalgebraic ring (considered now as an $R[\mathbb{Z}]$ coalgebraic module) then the circle product is a coalgebraic pairing. So we can make a new definition.

Definition 4.2. An $A_{*,*}$ *coalgebraic ring* is an $A_{*,*}$ coalgebraic module $H_{*,*}$ equipped with an associative coalgebraic pairing $\circ : H_{*,*} \otimes_R H_{*,*} \to H_{*,*}$.

Equivalently, an $A_{*,*}$ *coalgebraic ring* is a coalgebraic ring $H_{*,*}$ equipped with a map of coalgebraic rings $A_{*,*} \to H_{*,*}$.

We call an $A_{*,*}$ coalgebraic ring *unital* if there is an element $[1] \in H_{0,0}$ (the $\circ$ unit) such that $[1] \circ x = x$ for all $x \in H_{*,*}$. Note that for a unital $A_{*,*}$ coalgebraic ring there is a canonical map $A_{*,*} \to H_{*,*}$ given by $a \mapsto a \circ [1]$. An $A_{*,*}$ coalgebraic ring is *commutative* if $a \circ b = (-1)^{|a||b|}[-1]^{\circ\|a\|\|b\|} \circ b \circ a$.

Denote by $\mathscr{CR}_{A_{*,*}}$ the category of commutative unital $A_{*,*}$ coalgebraic rings with the obvious structure-preserving morphisms.

Example 4.3. Let T be an S algebra. Then the group algebra $R[T]$ is an $R[S]$ coalgebraic ring.

Example 4.4. Any commutative coalgebraic ring $H_{*,*}$ over R is an $R[\mathbb{Z}]$ coalgebraic ring.

Example 4.5. Let E be a ring spectrum. Then the homology of the infinite loop space $\Omega^\infty E$ is a $H_*(QS^0;\mathbb{F}_p)$ coalgebraic ring.

5. Tensor products of $A_{*,*}$ coalgebraic modules

Let $L_{*,*}$ and $M_{*,*}$ be $A_{*,*}$ coalgebraic modules. We will construct an $A_{*,*}$ coalgebraic module $L_{*,*} \overline{\otimes}_{A_{*,*}} M_{*,*}$ with the following characteristic property. For all $A_{*,*}$ coalgebraic modules $N_{*,*}$ there is a bijection of sets

$$\mathcal{CP}(L_{*,*}, M_{*,*}; N_{*,*}) \simeq \mathscr{CM}_{A_{*,*}}(L_{*,*} \overline{\otimes}_{A_{*,*}} M_{*,*}, N_{*,*}),$$

where $\mathcal{CP}(L_{*,*}, M_{*,*}; N_{*,*})$ is the set of coalgebraic pairings $L_{*,*} \otimes_R M_{*,*} \to N_{*,*}$ and the right hand side is a set of morphisms in the category $\mathscr{CM}_{A_{*,*}}$. In fact this correspondence is natural in $N_{*,*}$ so the more precise statement is that the functor $\mathcal{CP}(L_{*,*}, M_{*,*}; -) : \mathscr{CM}_{A_{*,*}} \to \mathscr{S}ets$ taking the $A_{*,*}$ coalgebraic module $N_{*,*}$ to the set $\mathcal{CP}(L_{*,*}, M_{*,*}; N_{*,*})$ is represented by $L_{*,*} \overline{\otimes}_{A_{*,*}} M_{*,*}$.

To construct this tensor product we start with the underlying supplemented coalgebras of $L_{*,*}$ and $M_{*,*}$, which we were denoting earlier by $\mathscr{D}(L_{*,*})$ and $\mathscr{D}(M_{*,*})$ and form $\mathscr{D}(L_{*,*}) \otimes_R \mathscr{D}(M_{*,*})$. Now form the free $A_{*,*}$ coalgebraic module $\mathscr{F}(\mathscr{D}(L_{*,*}) \otimes_R \mathscr{D}(M_{*,*}))$ and let I be the $A_{*,*}$ coalgebraic ideal generated by all elements of the form

$$u \otimes (x * y) - \sum (-1)^{|u''||x|}(u' \otimes x)\overline{*}(u'' \otimes y),$$

$$(u * v) \otimes x - \sum (-1)^{|v||x'|}(u \otimes x')\overline{*}(v \otimes x''),$$

$$(a \circ u) \otimes x - a\overline{\circ}(u \otimes x),$$

$$u \otimes (x \circ a) - (u \otimes x)\overline{\circ} a,$$

where $u, v \in L_{*,*}$, $x, y \in M_{*,*}$ and $a \in A_{*,*}$. Here the $*$'s on the left indicate $*$ products in $L_{*,*}$ and $M_{*,*}$, while the $\overline{*}$'s on the right are the star products in the free $A_{*,*}$ coalgebraic module $\mathscr{F}(\mathscr{D}(L_{*,*}) \otimes_R \mathscr{D}(M_{*,*}))$. We distinguish between the corresponding $\circ$ products likewise in the second two sets of generators. Let $L_{*,*} \overline{\otimes}_{A_{*,*}} M_{*,*} = \mathscr{F}(\mathscr{D}(L_{*,*}) \otimes_R \mathscr{D}(M_{*,*}))/I$.

Theorem 5.1. *The functor $\mathcal{CP}(L_{*,*}, M_{*,*}; -)$ is represented by the $A_{*,*}$ coalgebraic module $L_{*,*} \overline{\otimes}_{A_{*,*}} M_{*,*}$.*

Proof. The proof is standard: given a coalgebraic pairing $f : L_{*,*} \otimes_R M_{*,*} \to N_{*,*}$, the universal property of the functor $\mathscr{F}$ gives an $A_{*,*}$ coalgebraic module map from $\mathscr{F}(\mathscr{D}(L_{*,*}) \otimes_R \mathscr{D}(M_{*,*}))$ to $N_{*,*}$. The axioms for a coalgebraic pairing show that this map must factor through $L_{*,*} \overline{\otimes}_{A_{*,*}} M_{*,*}$, giving a map, α say, from $\mathcal{CP}(L_{*,*}, M_{*,*}; N_{*,*})$ to $\mathscr{CM}_{A_{*,*}}(L_{*,*} \overline{\otimes}_{A_{*,*}} M_{*,*}, N_{*,*})$.

Conversely, given g, an $A_{*,*}$ coalgebraic module map from $L_{*,*} \overline{\otimes}_{A_{*,*}} M_{*,*}$ to $N_{*,*}$, construct a coalgebraic pairing $\beta(g) : L_{*,*} \otimes_R M_{*,*} \to N_{*,*}$ as the composite

$$L_{*,*} \otimes_R M_{*,*} \to \mathscr{F}(\mathscr{D}(L_{*,*}) \otimes_R \mathscr{D}(M_{*,*})) \to L_{*,*} \overline{\otimes}_{A_{*,*}} M_{*,*} \to N_{*,*},$$

where the left-hand map is the natural inclusion θ, the middle the quotient by I and the right-hand map is g. It is then easy to show that α and β are mutually inverse. □

The construction $\overline{\otimes}_{A_{*,*}}$ has all the usual properties that a tensor product should have. We leave the proof of the following proposition to the reader.

Proposition 5.2. *There are isomorphisms*
1. $L_{*,*} \to L_{*,*} \overline{\otimes}_{A_{*,*}} A_{*,*}$ *and* $A_{*,*} \overline{\otimes}_{A_{*,*}} L_{*,*} \to L_{*,*}$,
2. $L_{*,*} \overline{\otimes}_{A_{*,*}} (M_{*,*} \overline{\otimes}_{A_{*,*}} N_{*,*}) \to (L_{*,*} \overline{\otimes}_{A_{*,*}} M_{*,*}) \overline{\otimes}_{A_{*,*}} N_{*,*}$,
3. $L_{*,*} \overline{\otimes}_{A_{*,*}} M_{*,*} \to M_{*,*} \overline{\otimes}_{A_{*,*}} L_{*,*}$.

Theorem 5.3. *$\mathscr{CM}_{A_{*,*}}$ is a symmetric monoidal category under $\overline{\otimes}_{A_{*,*}}$ with special object $A_{*,*}$.*

Proof. This follows easily from Proposition 5.2. □

Further properties of $\overline{\otimes}_{A_{*,*}}$ are developed in detail in the next two sections.

Let $H_{*,*}$ and $K_{*,*}$ be $A_{*,*}$ coalgebraic rings. We can give the structure of an $A_{*,*}$ coalgebraic ring to the tensor product $H_{*,*} \overline{\otimes}_{A_{*,*}} K_{*,*}$ by extending the map

$$(w \overline{\otimes} x) \otimes (y \overline{\otimes} z) \mapsto (-1)^{|y||x|}(w \circ y) \overline{\otimes} (x \circ z)$$

to a coalgebraic pairing

$$\circ : (H_{*,*} \overline{\otimes}_{A_{*,*}} K_{*,*}) \otimes_R (H_{*,*} \overline{\otimes}_{A_{*,*}} K_{*,*}) \to (H_{*,*} \overline{\otimes}_{A_{*,*}} K_{*,*}).$$

Proposition 5.4. *If $H_{*,*}$ and $K_{*,*}$ are unital coalgebraic rings then the tensor product $H_{*,*} \overline{\otimes}_{A_{*,*}} K_{*,*}$ has a $\circ$ unit $[1]\overline{\otimes}[1]$. If $H_{*,*}$ and $K_{*,*}$ are commutative coalgebraic rings then so is $H_{*,*} \overline{\otimes}_{A_{*,*}} K_{*,*}$.*

Proposition 5.5. *$\mathscr{CR}_{A_{*,*}}$ is a symmetric monoidal category under $\overline{\otimes}_{A_{*,*}}$ with special object $A_{*,*}$.*

6. Half exactness of the tensor product

The results of Section 3 show that, when we have a class of $A_{*,*}$ coalgebraic modules with enough projectives, we can perform the usual constructions of homological algebra; we shall see an example in the next section. In particular, we may wish to form the left derived functors of the tensor product operation taking $N_{*,*}$ to $N_{*,*} \overline{\otimes}_{A_{*,*}} M_{*,*}$. The resulting kth derived functor we shall name $\mathrm{CTor}_k^{A_{*,*}}(N_{*,*}, M_{*,*})$. In this section, we shall prove that the coalgebraic module tensor product is half exact; the usual machinary of homological algebra then allows us to identify $\mathrm{CTor}_0^{A_{*,*}}(N_{*,*}, M_{*,*})$ with $N_{*,*} \overline{\otimes}_{A_{*,*}} M_{*,*}$.

Proposition 6.1. *The functor sending $N_{*,*}$ to $N_{*,*} \overline{\otimes}_{A_{*,*}} M_{*,*}$ is right exact.*

Proof. Our proof is just an adaptation of the proof of the analagous result about the tensor product in the category of R modules. We begin with a short exact sequence in $\mathscr{CM}_{A_{*,*}}$,

$$0 \to J_{*,*} \xrightarrow{f} K_{*,*} \xrightarrow{g} L_{*,*} \to 0.$$

Recall that this means that $J_{*,*}$ is the kernel of g and $L_{*,*}$ is the cokernel of f in $\mathscr{CM}_{A_{*,*}}$. We must show that

$$J_{*,*} \overline{\otimes}_{A_{*,*}} M_{*,*} \xrightarrow{f\overline{\otimes}1} K_{*,*} \overline{\otimes}_{A_{*,*}} M_{*,*} \xrightarrow{g\overline{\otimes}1} L_{*,*} \overline{\otimes}_{A_{*,*}} M_{*,*} \to 0$$

is exact also.

That $g\overline{\otimes}1$ is onto and that the composite $(g\overline{\otimes}1)(f\overline{\otimes}1) = (gf\overline{\otimes}1)$ is trivial are both immediate. The part to consider is the claim that the image of $f\overline{\otimes}1$ contains the kernel of $g\overline{\otimes}1$.

Let D be the coalgebraic ideal in $K_{*,*} \overline{\otimes}_{A_{*,*}} M_{*,*}$ generated by elements of the form $k\overline{\otimes}m$ with $g(k) = 0$. Clearly, D is contained in the image of $f\overline{\otimes}1$ by exactness of the original sequence and it suffices to show that the kernel of $g\overline{\otimes}1$ is contained in D.

Construct the R coalgebra homomorphism

$$\gamma\colon L_{*,*} \otimes_R M_{*,*} \to (K_{*,*} \overline{\otimes}_{A_{*,*}} M_{*,*})/D$$

by setting $\gamma(l\otimes m) = [k\overline{\otimes}m]$ and extending linearly, where the square brackets indicate the equivalence class modulo D and k is any element of $K_{*,*}$ satisfying $g(k) = l$ (note that γ exists and is well defined by exactness and the definition of D). However, as D is an $A_{*,*}$ coalgebraic ideal, γ is a coalgebraic pairing and so extends to a homomorphism of $A_{*,*}$ coalgebraic modules

$$\Gamma\colon L_{*,*}\,\overline{\otimes}_{A_{*,*}}\,M_{*,*} \to (K_{*,*}\,\overline{\otimes}_{A_{*,*}}\,M_{*,*})/D.$$

Moreover, the composite

$$\Gamma(g\,\overline{\otimes}\,1)\colon (K_{*,*}\,\overline{\otimes}_{A_{*,*}}\,M_{*,*}) \to (K_{*,*}\,\overline{\otimes}_{A_{*,*}}\,M_{*,*})/D$$

is just the quotient by D and so the kernel of $g\,\overline{\otimes}\,1$ is contained in D as required. □

7. The functor $R[-]$

Suppose S is a ring. In this section we examine in detail some of the properties of the functor from S modules to $R[S]$ coalgebraic modules given by the construction $R[-]$ outlined in Sections 1 and 2.

Proposition 7.1. *The functor $R[-]$ takes*
(1) *kernels to kernels,*
(2) *cokernels to cokernels,*
(3) *tensor products to tensor products,*
(4) *projectives to projectives.*

Proof. We begin with (1) and (2). We suppose $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ to be a short exact sequence of S modules. Let us write $\overline{f}$ and $\overline{g}$ for $R[f]$ and $R[g]$, respectively. It will suffice to show that

$$0 \to R[A] \xrightarrow{\overline{f}} R[B] \xrightarrow{\overline{g}} R[C] \to 0$$

is also short exact. Now,

$$\begin{aligned}
\mathrm{CKer}(\overline{g}) &= \{x \in R[B] \mid (1\otimes\overline{g})\psi(x) = x\otimes[0]\}\\
&= \{\textstyle\sum r_b[b] \mid (1\otimes\overline{g})(\sum r_b[b]\otimes[b]) = \sum(r_b[b]\otimes[0])\}\\
&= \{\textstyle\sum r_b[b] \mid (\sum(r_b[b]\otimes[g(b)])) = \sum(r_b[b]\otimes[0])\}\\
&= \{\textstyle\sum r_b[b] \mid [g(b)] = [0]\}\\
&= \{\textstyle\sum r_b[b] \mid b \in \mathrm{Ker}(g)\}\\
&= R[\mathrm{Ker}(g)]\\
&= R[A].
\end{aligned}$$

This proves (1) and (2) is similar. Recall that $\mathrm{CCoker}(\overline{f})$ is $R[B]/\langle \overline{f}(R[A]^+)\rangle$ where $R[A]^+$ is the augmentation ideal in $R[A]$. We can describe $R[A]^+$ as the free R module on basis elements $[a]-[0]$ as a runs over the non-zero elements of A. Then $\bar{f}(R[A]^+)$ is the free R module on the elements $[fa]-[0]$ in $R[B]$ and is already a coalgebraic ideal in $R[B]$. Thus $\langle \bar{f}(R[A]^+)\rangle = \bar{f}(R[A]^+)$ and

$$\mathrm{CCoker}(\bar{f}) = R[B]/\bar{f}(R[A]^+) = R[B/A] = R[C] = R[\mathrm{Coker}(f)].$$

The proof of part 3 is a useful exercise in unwinding the definition of the coalgebraic tensor product. We wish to show

$$R[A]\,\bar{\otimes}_{R[S]}\,R[B] \cong R[A \otimes_S B]$$

for S modules A and B. Recall the definition of the left-hand side: this is

$$\mathscr{F}(\mathscr{D}(R[A]) \otimes_R \mathscr{D}(R[B]))/I = \mathscr{F}\mathscr{D}(R[A \times B])/I$$

where $\mathscr{F}$ and $\mathscr{D}$ are the functors defined in Section 2 and I is the coalgebraic ideal specified in Section 5. Now $\mathscr{F}\mathscr{D}(R[A \times B])$ is the set of all R linear sums of all $*$ products of all elements of the form $[s]\,\bar{\circ}\,[(a,b)]$ with $s \in S$ and $(a,b) \in A \times B$ and so a typical R basis element is of the form

$$[s_1]\,\bar{\circ}\,[(a_1,b_1)]\,\bar{*}\cdots\bar{*}\,[s_n]\,\bar{\circ}\,[(a_n,b_n)].$$

Given the nature of $*$ product, $\circ$ action and coproduct on a coalgebraic module of the form $R[X]$, the ideal I becomes that generated by elements

$$[(a,b_1+b_2)] - [(a,b_1)]\,\bar{*}\,[(a,b_2)],$$

$$[(a_1+a_2,b)] - [(a_1,b)]\,\bar{*}\,[(a_2,b)],$$

$$[(s\cdot a,b)] - [s]\,\bar{\circ}\,[(a,b)],$$

$$[(a,b\cdot s)] - [(a,b)]\,\bar{\circ}\,[s].$$

But these are just a straight translation of the usual relations introduced in the free S module $A \times B$ to form the tensor product $A \otimes_S B$. This proves (3).

Finally, we turn to part 4 of the proposition. For an $R[S]$ coalgebraic module Y define $\mathscr{G}(Y)$ to be the set of group like elements of Y. Then $\mathscr{G}(Y)$ is an S module with addition given by $*$ and S action by $s(g) = [s] \circ g$ for $s \in S$ and $g \in \mathscr{G}(Y)$. In fact, $\mathscr{G}(-)$ is the right adjoint to the functor $R[-]$.

Now suppose P is a projective S module; we shall show that $R[P]$ is a projective $R[S]$ coalgebraic module. To do this, let $\varepsilon: X \to Y$ be a surjection in $\mathscr{CM}_{R[S]}$ and suppose $\phi: R[P] \to Y$ is the homomorphism in $\mathscr{CM}_{R[S]}$ which we shall try to lift up ε.

Define $\phi': P \to \mathscr{G}(Y)$ by $\phi'(p) = \phi([p])$; this is an S module map. By projectivity of P we can lift ϕ' up $\mathscr{G}(\varepsilon): \mathscr{G}(X) \to \mathscr{G}(Y)$, say to $\bar{\phi}: P \to \mathscr{G}(X)$. Now define the required lift of ϕ by sending the R basis element $[p]$ to $\bar{\phi}(p) \in \mathscr{G}(X) \subset X$ and extending linearly. □

Restating our results, we find we have proved

Corollary 7.2. *The functor* $R[-]$ *is exact and preserves projective resolutions.*

Remark 7.3. As the category of S modules has enough projectives, we see that the image of this category under $R[-]$ also has enough projectives. We can thus form the derived functors $\mathrm{CTor}_k^{R[S]}(R[A], M_{*,*})$ for a (graded) S module A of finite type and an $R[S]$ coalgebraic module $M_{*,*}$.

It is well known that if U is an exact functor then the left derived functors L^kF of a functor F satisfy the relation $U(L^kF) = L^k(UF)$. In the light of Proposition 7.1(3) we now have

Corollary 7.4. *For S modules A and B,*

$$\mathrm{CTor}_k^{R[S]}(R[A], R[B]) = R[\mathrm{Tor}_k^S(A, B)].$$

Note added in proof.

Kashiwabara has pointed out that the connectivity assumption in Section 3 and in particular Theorem 3.4 is superfluous. Without this assumption we still have $a \circ (m \otimes [0]) = (a \circ m) \otimes [0]$ as required in the proof. From the properties of Hopf algebras we have that $\psi(a) - a \otimes 1 \in \mathrm{Ker}(1 \otimes \varepsilon) = A \otimes I$ where I is the augmentation ideal. It follows that $\psi(a) = a \otimes 1 + \sum b' \otimes b''$ for $b'' \in I$. Note that $b'' \circ [0] = 0$. Thus $a \circ (m \otimes [0]) = (a \circ m) \otimes (1 \circ [0]) + \sum (-1)^{|b''||m|}(b \circ m) \otimes (b'' \circ [0]) = (a \circ m) \otimes [0]$.

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