REAL COBORDISM AND GREEK LETTER ELEMENTS IN THE GEOMETRIC CHROMATIC SPECTRAL SEQUENCE

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1. Introduction

In a previous paper [5], the authors investigated extensively the Landweber-Araki Real cobordism spectrum $M\mathbb{R}$ ([1, 7]), and its stable summand at $p = 2$ called $BP\mathbb{R}$. An Adams-like spectral sequence based on $BP\mathbb{R}$ was constructed, and homotopy groups of many spectra related to $BP\mathbb{R}$ were calculated.

The goal of the present note is to begin investigating the question as to how $BP\mathbb{R}$-theory contributes to the known information about stable homotopy groups of spheres. This question is not easy. The dimension to which stable stems have been calculated to date is, (if somewhat hazy), certainly high enough to make calculating past that point from scratch a substantial challenge for any new method. At this point, the authors did not get far enough in calculating with $BP\mathbb{R}$ to get any new information that way. On the other hand, any effort to compare the $BP\mathbb{R}$-based spectral sequence with other known spectral sequences is frought with the usual difficulty: elements can get renamed.

Nevertheless, there is one basic case when a rigorous comparison can be made, namely on the *edge* of a spectral sequence. To illustrate this, assume we have a series of cofibrations

$$X_{n-1} \to Y_{n-1} \to X_n \tag{1.1}$$

where $X_{-1} = S^0$. Then we have a spectral sequence

$$E_1 = \pi_* Y_{n-1} \Rightarrow \pi_* S^0. \tag{1.2}$$

An element $\omega \in \pi_* Y_{n-1}$ is a permanent cycle if it lifts to $\pi_* X_{n-1}$. Now however it may not even be easy to know $\pi_* Y_{n-1}$ explicitly, and suppose we use the Adams-Novikov spectral sequence to calculate it. Suppose further that in this spectral sequence, ω has filtration degree 0, i.e.

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has non-trivial Hurewicz homomorphism image $\omega_0 \in BP_* Y_n$. Suppose further that in BP-homology, (1.1) induce short exact sequences, so we cannot use BP to obtain a necessary condition for ω to lift. But now we have a factorization of the unit $1 \in \pi_* BP$ of the form

$$S^0 \longrightarrow BP\mathbb{R} \xrightarrow{\kappa} BP$$

(actually, $BP\mathbb{R}$ is a $\mathbb{Z}/2$-equivariant spectrum, so we should really write $BP\mathbb{R}^{\mathbb{Z}/2}$, but that is a detail for now). Suppose we can completely calculate the map

$$BP\mathbb{R}_* X_n \to BP\mathbb{R}_* Y_n \tag{1.3}$$

and suppose we know that no element $\omega_1 \in BP\mathbb{R}_* Y_n$ with $\kappa_* \omega_1 = \omega_0$ is in the image of (1.3). Then ω cannot be a permanent cycle in the spectral sequence (1.2).

On the other hand, suppose we know that there is a lift $\omega_2 \in BP\mathbb{R}_* X_n$, and that further any such lift (of any choice of ω_1) has the property that

$$\delta^* \omega_2 \neq 0$$

where $\delta^* : BP\mathbb{R}_* X_{n-1} \to \Sigma^n BPR\mathbb{R}_* S^0$ is the connecting map. Then the in particular the same must hold for any ω_2 which lifts to stable homotopy, and we conclude that ω cannot be a target of a differential in (1.2).

In this note, we shall apply this simple method to the geometric chromatic spectral sequence, where we shall see that strikingly, it does give new information on possible permanent cycles in all families of Greek-letter elements, thus generalizing, and in fact even somewhat explaining, the numerology of the known permanent cycles in α, β, γ. This signals that however difficult to extract it may be, $BP\mathbb{R}$ does contain new useful information about the stable stems. Throughout this note, we work locally at the prime 2, as $BP\mathbb{R}$ is just BP at odd primes. Nevertheless, algebraic cobordism (cf. [6]) gives tantalizing hints of some possible analogues of the present method at odd primes. Such extension, however, at present is unknown, as is the equally interesting question of possible connections of $BP\mathbb{R}$ with the root invariant.

To state our results, we must recall the chromatic spectral sequence of Miller-Ravenel-Wilson [8]:
(1.4)

$$E_1 = \bigoplus_n Ext_{BP_*BP}(BP_*, v_n^{-1} BP_*/(v_0^\infty, ..., v_{n-1}^\infty)) \Rightarrow Ext_{BP_*BP}(BP_*, BP_*).$$

In the E_1-term (1.4), an element of $Ext^0_{BP_*BP}(BP_*, v_n^{-1}BP_*/(v_0, ..., v_{n-1}))$ represented by

$$\frac{v_n^k}{v_{n-1}^{i_{n-1}} v_{n-2}^{i_{n-2}} \cdot ... \cdot v_0^{i_0}} \mod (v_0, ..., v_{n-1}) \tag{1.5}$$

is denoted by

$$\alpha^{(n)}_{k/i_{n-1}, i_{n-2}, ..., i_0} \tag{1.6}$$

where $\alpha^{(n)}$ is the n-th Greek letter. If the numbers $i_0, ... i_{\ell-1}$ are equal to 1, they are omitted.

In [9], Ravenel constructed BP-local spectra Y_n such that

$$BP_*Y_n = v_n^{-1}BP_*/(v_0^\infty, ..., v_{n-1}^\infty),$$

and a *geometric chromatic spectral sequence*

$$E_1 = \bigoplus_n \pi_*Y_n \Rightarrow \pi_*(S^0)^\wedge_2. \tag{1.7}$$

At this point, relatively little is known about the spectral sequence (1.7). However, call the element (1.5) a *geometric Greek letter element* if it is the image of an element of π_*Y_n via the Hurewicz map $\pi_*Y_n \to BP_*Y_n$, i.e. an element of the E_1-term of the geometric chromatic spectral sequence (1.7). In that case, we also use the notation (1.6) for that element. Then the main results of this paper are:

Theorem 1.8. *If* $\alpha^{(n)}_{k/i_{n-1}, i_{n-2}, ..., i_\ell}$ *is a geometric Greek letter element which is a permanent cycle in the geometric chromatic spectral sequence, then there exists a* j, $0 \leq j \leq \ell$ *(we can have* $\ell = n$*) such that*

$$k(2^n - 1) \equiv \sum_{m=0}^{n-1} i_m(2^m - 1) + 2^j - 1 \mod 2^{j+1}.$$

Theorem 1.9. *Let*

$$k_n = \begin{array}{ll} n+2 & \text{if } n \text{ is even} \\ 2^n + n + 2 & \text{if } n \text{ is odd.} \end{array}$$

Then a geometric Greek letter element $\alpha^{(n)}_k$ *is not a target of a differential in the geometric chromatic spectral sequence if* $k \equiv k_n \mod 2^{n+1}$, *and supports a differential in the geometric chromatic spectral sequence if* $k \equiv k_n + 2^n \mod 2^{n+1}$.

For $n = \ell = 1$, Theorem 1.8 says that if α_k is a permanent cycle then k is not congruent to 3 mod 4, which is well known.

For $n = \ell = 2$, Theorems 1.8, 1.9 imply that β_k is not a permanent cycle when k is divisible by 8. For $n = 2$ and $\ell = 1$, Theorem 1.8 says that if β_{k/i_1} is a permanent cycle, then $k + i_1$ is not congruent to $1 \mod 4$. Doug Ravenel points out that the first examples of these elements are β_8 and $\beta_{8/5}$, and conjectures that we have $d_3(\beta_8) = \eta^3\beta_{8/3}$ and $d_3(\beta_{8/5}) = \eta^3\eta_5$ in the Adams-Novikov spectral sequence.

For $n = \ell = 3$, Theorem 1.8 implies that if γ_k is a permanent cycle, then k is not congruent to $5 \mod 16$. The first element excluded is γ_5, in dimension 59.

Theorems 1.8, 1.9 should not be thought of as results directly about the stable 2-stems $(\pi_* S^0)_{(2)}$, since they do not give any sufficient condition when the said elements are permanent cycles in the geometric chromatic spectral sequence. However, the nilpotence and periodicity theorems of Devinatz, Hopkins and Smith [2], [3] give a general existence theorem for geometric Greek letter elements as elements of stable 2-stems. There was, as far as we know, no detection theorem for such elements beyond the γ family, and this was a well known problem (although it is usually phrased in terms of the Adams-Novikov spectral sequence). The present paper does contribute to answering this question. The difficulty is that the existence theorems [2],[3] are phrased globally, and as far as we know, concrete names of elements which these theorems construct have not been worked out. Therefore we still do not know what, if any, is the intersection between the existence and uniqueness theorems.

The present paper is organized as follows: Preliminaries, including the construction of the spectra Y_n, are recalled in Section 2. The necessary $BP\mathbb{R}$-homology calculations are presented in Section 3, and proofs of Theorems 1.8, 1.9 following the method we outlined are deduced. The $BP\mathbb{R}$-calculations are proved in Section 4.

2. Preliminaries

First we shall recall the construction of the spectra Y_n in (1.7) ([9]). We refer the reader to [3] for the definition of v_{n-1}-spectra and v_n-self maps.

Lemma 2.1. *Let $f : V \to V'$ be any map of v_{n-1}-spectra (see [3]). Let $v : \Sigma^k V \to V$, $v' : \Sigma^k V' \to V'$ be v_n-self maps. Then there exists an N*

such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} \Sigma^{Nk}V & \xrightarrow{f} & \Sigma^{Nk}V' \\ {\scriptstyle v^N}\downarrow & & \downarrow{\scriptstyle v'^N} \\ V & \xrightarrow{f} & V'. \end{array} \tag{2.2}$$

Proof: Consider the two maps

$$Dv \wedge 1, 1 \wedge v' : \Sigma^k DV \wedge V' \to DV \wedge V'.$$

They are both v_n-self maps, so by Hopkins-Smith [3], there is an N such that

$$(Dv \wedge 1)^N \simeq (1 \wedge v')^N.$$

Therefore, considering f as an element of $\pi_*(DV \wedge V')$, we have

$$(Dv \wedge 1)^N_*(f) = (1 \wedge v')^N_*(f). \tag{2.3}$$

But the two sides of (2.3), considered as maps $\Sigma^{Nk}V \to V'$, are the two ways around the diagram (2.2). □

Now Ravenel's construction is essentially as follows: One constructs a spectrum X'_{n-1} as a telescope of v_{n-1}-spectra

$$V_{n-1,1} \xrightarrow{f_{n-1,1}} V_{n-1,2} \xrightarrow{f_{n-1,2}} V_{n-1,3} \xrightarrow{f_{n-1,3}} \dots \tag{2.4}$$

For $n = 0$, let $X'_{-1} = S^0 = V_{-1,i}$ for all i, where the maps $f_{-1,i}$ are equal to the identity. Provided (2.4) is constructed, we next construct a spectrum Y'_{n-1} and a cofibration sequence of the form

$$X'_{n-1} \to Y'_{n-1} \to X'_n. \tag{2.5}$$

Concretely, by Lemma 2.1, we can inductively find v_n-self maps

$$v_{n,i} : \Sigma^{k_{n-1,i}} V_{n-1,i} \to V_{n-1,i}$$

and numbers $N_{n-1,i} > 1$ such that

$$k_{n-1,i+1} = N_{n-1,i} k_{n-1,i}$$

and the following diagrams commute:

$$\begin{array}{ccc} \Sigma^{k_{n-1,i+1}} V_{n-1,i} & \xrightarrow{f_{n-1,i}} & \Sigma^{k_{n-1,i+1}} V_{n-1,i+1} \\ {\scriptstyle (v_{n,i})^{N_{n-1,i}}}\downarrow & & \downarrow{\scriptstyle v_{n,i+1}} \\ V_{n-1,i} & \xrightarrow{f_{n-1,i}} & V_{n-1,i+1}. \end{array}$$

Then consider the diagram
(2.6)

$$\begin{array}{ccccccc}
V_{n-1,1} & \xrightarrow{f_{n-1,1}} & V_{n-1,2} & \xrightarrow{f_{n-1,2}} & V_{n-1,3} & \xrightarrow{f_{n-1,3}} & \cdots \\
\downarrow{\scriptstyle v_{n,1}} & & \downarrow{\scriptstyle v_{n,2}} & & \downarrow{\scriptstyle v_{n,3}} & & \\
\Sigma^{-k_{n-1,1}}V_{n-1,1} & \xrightarrow[f'_{n-1,1}]{} & \Sigma^{-k_{n-1,2}}V_{n-1,2} & \xrightarrow[f'_{n-1,2}]{} & \Sigma^{-k_{n-1,3}}V_{n-1,3} & \xrightarrow[f'_{n-1,3}]{} & \cdots .
\end{array}$$

where

$$f'_{n-1,i} = f_{n-1,i}(v_{n,i})^{N_{n-1,i}-1}.$$

The cofiber of the vertical rows of (2.6) is, by definition,

$$V_{n,1} \xrightarrow{f_{n,1}} V_{n,2} \xrightarrow{f_{n,2}} V_{n,3} \longrightarrow \cdots . \tag{2.7}$$

Now let Y'_{n-1} be the telescope of the bottow row (2.6), and let X'_n be the telescope of (2.7). Thus, we have (2.5). One easily proves by induction that when smashing (2.5) with BP, one obtains the cofiber sequence of MU-modules

$$BP/(v_0^\infty, ...v_{n-1}^\infty) \to v_n^{-1}BP/(v_0^\infty, ..., v_{n-1}^\infty) \to BP/(v_0^\infty, ..., v_n^\infty). \tag{2.8}$$

Now absent a proof of the telescope conjecture, we do not know that the Adams-Novikov spectral sequence converges for X'_n, Y'_n, $n > 1$. However, Ravenel [9] proved that if we denote by X_n, Y_n the Bousfield localizations of X'_n, Y'_n at BP, then the Adams-Novikov spectral sequence

$$Ext_{BP_*BP}(BP_*, v_n^{-1}BP_*Y_{n-1}/(v_0^\infty, ..., v_{n-1}^\infty)) \Rightarrow \pi_*(Y_{n-1}) \tag{2.9}$$

converges. We call (2.9) the *Chromatic Adams-Novikov spectral sequence*). Also, since stable Bousfield localization preserves cofibration sequences, we have cofibrations

$$X_{n-1} \xrightarrow{\alpha_{n-1}} Y_{n-1} \xrightarrow{\gamma_{n-1}} X_n. \tag{2.10}$$

Applying π_* to (2.10), we obtain an exact couple, which leads to the spectral sequence (1.7). Ravenel [9] proved that this spectral sequence converges. Note that we now have the following diagram of spectral sequences

(2.11)

$$
\begin{array}{ccccc}
 & & \oplus Ext_{BP_*BP}(v_n^{-1}BP_*/(v_0^\infty, ...v_{n-1}^\infty)) & & \\
 & \swarrow^{CANSS, E_2} & & \searrow^{CSS, E_1} & \\
\oplus \pi_* Y_{n-1} & & & & Ext_{BP_*BP}(BP_*) \\
 & \searrow_{GCSS, E_1} & & \swarrow_{ANSS, E_2} & \\
 & & (\pi_* S^0)^\wedge_2. & &
\end{array}
$$

The superscript of each arrow indicates the abbreviated name of the spectral sequence, and its initial term. The Greek letter elements (1.6) are native in the top corner of (2.11). Geometric Greek letter elements are those which are permanent cycles in the CANSS, and therefore live in the left corner of (2.11). Theorems 1.8, 1.9 concern aspects of the behaviour of these elements in the GCSS, caused by Real cobordism. This will be discussed in the next section. Note, however, that the statements of the Theorems would be stronger if we could phrase them in terms of Greek letter elements of the Adams-Novikov spectral sequence, i.e. those which are permanent cycles of the CSS, and therefore live in the right corner of (2.11). The reason this would be better is that the CSS is purely algebraic, and hence in principle completely computable. Unfortunately, we were unable to prove such stronger results using the present methods. Without referring to the methods, we can say that is possible for renaming to occur in the ANSS, so that elements labelled as Greek letters do not correspond to such elements in the GCSS.

3. Real cobordism and proof of the main results

We begin by recalling some facts about Real cobordism, proved in [5]. The Real cobordism spectrum $M\mathbb{R}$ is a $\mathbb{Z}/2$-equivariant spectrum obtained by considering the $\mathbb{Z}/2$-action on MU by complex conjugation. This can be taken almost literally, if we consider the usual prespectrum defining MU, consisting of Thom spaces of n-dimensional universal complex bundles. Then complex conjugation acts non-trivially on both the spaces and structure maps of the prespectrum: denoting by α the 1-dimensional real sign representation of $\mathbb{Z}/2$, then the 1-point compactification of $\mathbb{C}$ with respect to complex conjugation is $S^{1+\alpha}$. As a result, $M\mathbb{R}$ is an $RO(\mathbb{Z}/2)$-graded spectrum, or spectrum indexed by a complete $\mathbb{Z}/2$-universe. The reader is referred to [5] for details and other relevant references.

Before proceeding further, we will make certain crucial conventions: First of all, all $\mathbb{Z}/2$-equivariant spectra will be $RO(\mathbb{Z}/2)$-graded, and we will use the subscript $\star$ to denote $RO(\mathbb{Z}/2)$-graded coefficients; therefore, the possible dimensions represented by $\star$ are $k+\ell\alpha,\ k,\ell\in\mathbb{Z}$. We will use the subscript $*$ if we are referring only to the "twist 0" dimensions, i.e. $k+0\alpha$.

Next, we will make notational conventions of certain $RO(\mathbb{Z}/2)$-graded homotopy and homology elements. First, let a be the element of $\pi_{-\alpha}S^0_{\mathbb{Z}/2}$ represented by the non-trivial unstable map $S^0\to S^\alpha$. Next, recall from [5] that the $\mathbb{Z}/2$-equivariant Borel cohomology ring with coefficients in $\mathbb{Z}/2$ is

$$\mathbb{Z}/2[\sigma,\sigma^{-1}][a].$$

The element σ has dimension $\alpha-1$. This element does not lift to the coefficients of $S^0_{\mathbb{Z}/2}$ (or $M\mathbb{R}$), but as we shall see, its powers survive as multipliers of certain elements of $M\mathbb{R}_\star$. Also, using the fact that Milnor manifolds are defined over $\mathbb{R}$, we get a map

$$BP_\star\to M\mathbb{R}_\star$$

where by $BP_\star$ we mean $\mathbb{Z}[v_1,v_2,...]$, but where v_i is in dimension $(2^i-1)(1+\alpha)$. We will also denote by I_n the ideal $(v_0,v_1,...v_{n-1})$ in $BP_\star$. We will rely on the $\star$ to indicate the fact that we are working in the $RO(\mathbb{Z}/2)$-graded dimensions (this notation was also used in [4]).

Araki [1], [5] has developed a theory of Real-oriented $\mathbb{Z}/2$-spectra very parallel with the classical theory of complex-oriented spectra. In particular, 2-locally, there is a Quillen-idempotent $e: M\mathbb{R}\to M\mathbb{R}$. The spectrum $eM\mathbb{R}$ is denoted by $BP\mathbb{R}$, and is easier to work with. When forgetting $\mathbb{Z}/2$-equivariant structure, $BP\mathbb{R}$ becomes just BP, but when applying the geometric fixed point functor to $BP\mathbb{R}$, one obtains $H\mathbb{Z}/2$. Recall that a G-spectrum E is called *complete with respect to G-action* if the canonical map $E\to F(EG_+,E)$ is an equivalence. A crucial result stated in [5] is

Theorem 3.1. *The spectrum $BP\mathbb{R}$ is complete with respect to $\mathbb{Z}/2$-action and we have*

$$BP\mathbb{R}_\star =$$

$$\bigoplus_{\ell=(2s+1)2^n\in\mathbb{Z}} Ker(BP_\star[a]/(a^{2^{i+1}-1}v_i|i\geq 0)\to$$
$$BP_\star[a]/(v_0,...,v_n,a^{2^{i+1}-1}v_i|i\geq n+1))\cdot\sigma^{2\ell}$$

In the summand for $\ell = 0$, we count n as ∞. Moreover, the multiplicative structure is the obvious one, i.e. as a subring of

$$BP_\star[\sigma, \sigma^{-1}, a]/(a^{2^{i+1}-1}v_i | i \geq 0).$$

□

We now recall from [5] that $M\mathbb{R}$ is also a $\mathbb{Z}/2$-equivariant E_∞ ring spectrum, and therefore we can construct cofiber sequences of $M\mathbb{R}$-modules

(3.2)
$$BP\mathbb{R}/(v_0^\infty, ..., v_{n-1}^\infty) \to v_n^{-1}BP\mathbb{R}/(v_0^\infty, ..., v_{n-1}^\infty) \to BP\mathbb{R}/(v_0^\infty, ..., v_n^\infty).$$

Here we consider $v_i \in BP_\star$. The significance of these $\mathbb{Z}/2$-equivariant spectra for our purposes is in the following

Proposition 3.3. *When smashing (2.10) with $BP\mathbb{R}$, we obtain the cofibration sequence of $\mathbb{Z}/2$-equivariant spectra (3.2).*

Our main calculational result on (3.2) is contained in the following

Proposition 3.4. *The spectrum $BP\mathbb{R}/(v_0^\infty, v_1^\infty, ..., v_{n-1}^\infty)$ is complete with respect to $\mathbb{Z}/2$-action and we have*

(3.5)
$$\begin{array}{l}(BP\mathbb{R}/(v_0^\infty, ..., v_{n-1}^\infty))_\star = \\ \bigoplus_{k=0}^{n-1} BP_\star/(v_0, ..., v_{k-1}, v_k^\infty, ...v_{n-1}^\infty)[\sigma^{\pm 2^{k+1}}][a]/(a^{2^{k+1}-1}) \\ \cdot\{v_0^{-1} \cdot ... \cdot v_{k-1}^{-1}\sigma^{-2^k+1}\} \\ \oplus BP\mathbb{R}_\star/(\sigma^{2\ell_0}v_0, \sigma^{4\ell_1}v_1, ..., \sigma^{2^n\ell_{n-1}}v_{n-1})\{v_0^{-1} \cdot ... \cdot v_{n-1}^{-1}\sigma^{-2^n+1}\}.\end{array}$$

We should explain that in our notation, when we are writing algebra, elements enclosed in the braces {} indicate additive generators, while elements enclosed in the brackets [] indicate multiplicative (polynomial) generators. While the notation for the generators in (3.5) indicates their origin in the computation, at the moment the significance of introducing additive generators is just suspension by their dimension.

Corollary 3.6. *The coimage of the forgetful map*

$$(BP\mathbb{R}/(v_0^\infty, ..., v_{n-1}^\infty))_\star \xrightarrow{\lambda} (BP/(v_0^\infty, ...v_{n-1}^\infty))_\star \tag{3.7}$$

is spanned by the following elements:

(3.8)

$$\bigoplus_{j=0}^{n-1} BP_\star/(v_0,...,v_{j-1},v_j^\infty,...,v_{n-1}^\infty)\{v_0^{-1}\cdot ...\cdot v_{j-1}^{-1}\cdot \sigma^{-2^j+1}\}[\sigma^{\pm 2^{j+1}}]$$

$$\bigoplus_{j=n}^{\infty} BP_\star/(v_0,...,v_{j-1})\{v_0^{-1}\cdot ...\cdot v_{n-1}^{-1}\cdot v_j\cdot \sigma^{-2^n+1}\}[\sigma^{\pm 2^{n+1}}]$$

where, as before, $dim(v_k)=(2^k-1)(1+\alpha)$ *and* $dim(\sigma)=\alpha-1$.

Proof: It will be obvious from the proof of Proposition 3.4 via the Borel cohomology spectral sequence (see next section) that the elements (3.8) map non-trivially, while all other elements are multiples of a. □

Finally, we shall need information on the connecting maps of the cofibration sequences (3.2). This is given by

Proposition 3.9. *The connecting map*

$$\partial_n : (BP\mathbb{R}/(v_0^\infty,...,v_n^\infty))_\star \to \Sigma(BP\mathbb{R}/(v_0^\infty,...,v_{n-1}^\infty))_\star \tag{3.10}$$

associated with (3.2) is a map of $BP\mathbb{R}_\star$*-modules given by*

$$\partial_n : \sigma^{-2^n}v_n^{-1}\mapsto a^{2^{n+1}-1} \tag{3.11}$$

on the last summand (3.5), and by 0 *on the other summands.*

Propositions 3.4, 3.9, 3.3 will be proved in the next section. We will now apply these propositions to prove our main results.

Proof of Theorem 1.8: Consider the diagram

$$\begin{array}{ccc} \pi_* X_{n-1} & \xrightarrow{\iota} & BP_*X_{n-1} = BP_*/(v_0^\infty,...,v_{n-1}^\infty) \\ \downarrow{\scriptstyle \pi_*\alpha_{n-1}} & & \downarrow{\scriptstyle BP_*\alpha_{n-1}} \\ \pi_*Y_{n-1} & \xrightarrow{\kappa} & BP_*Y_{n-1} = v_n^{-1}BP_*/(v_0^\infty,...,v_{n-1}^\infty) \end{array} \tag{3.12}$$

where ι,κ are Hurewicz maps. The Greek letter element (1.5) is an element of BP_*Y_{n-1}. Now assuming that (1.5) is a geometric Greek letter element is equivalent to the existence of an element $y\in\pi_*Y_{n-1}$ such that (1.5) is equal to $\kappa(y)$. Assuming further that y is a permanent cycle in the GCSS is equivalent to the existence of an element $x\in\pi_*X_{n-1}$ such that

$$\pi_*\alpha_{n-1}(x)=y.$$

Since $BP_*\alpha_{n-1}$ is injective, this implies that

$$\iota(x) = \frac{v_n^k}{v_{n-1}^{i_{n-1}} v_{n-2}^{i_{n-2}} \cdot \ldots \cdot v_\ell^{i_\ell} v_{\ell-1} \cdot \ldots \cdot v_0} \tag{3.13}$$

(provided we are considering the Greek letter element figuring in the statement of the Theorem).

But now the map ι factors as

$$\pi_* X_{n-1} \xrightarrow{\nu} BP\mathbb{R}_* X_{n-1} \xrightarrow{\lambda} BP_* X_{n-1}. \tag{3.14}$$

Therefore, (3.13) must be in the image of λ in twist 0, i.e. of the form

$$\lambda(z),\ z \in BP\mathbb{R}_{*+0\alpha} X_{n-1}. \tag{3.15}$$

Inspecting (3.8), we find that by Corollary 3.6, the leading term of z must be of the form

$$\frac{v_n^k}{v_{n-1}^{i_{n-1}} v_{n-2}^{i_{n-2}} \cdot \ldots \cdot v_\ell^{i_\ell} v_{\ell-1} \cdot \ldots \cdot v_0} \sigma^{-2^j+1+m\cdot 2^{j+1}} \tag{3.16}$$

with dimensional conventions as in Corollary 3.6. The requirement that (3.16) be of twist 0 then gives the condition of the Theorem. □

Lemma 3.17. *Let $z \in BP\mathbb{R}_* X_{n-1}$ be such that*

$$\lambda(z) = \frac{v_n^k}{v_{n-1} v_{n-2} \cdot \ldots \cdot v_0} \tag{3.18}$$

where

$$2^{n+1} \mid k(2^n - 1) - \sum_{m=0}^{n} (2^m - 1). \tag{3.19}$$

Then the image of z under the connecting map

$$\delta^* : BP\mathbb{R}_\star X_{n-1} \to \Sigma BP_\star X_{n-2} \to \ldots \to \Sigma^n BP\mathbb{R}_\star S^0 \tag{3.20}$$

is non-zero.

Proof: Corollary 3.6 implies that under the condition (3.19),

$$z = \frac{v_n^k}{v_{n-1} v_{n-2} \ldots v_0} \sigma^{-2^n+1+m2^{n+1}} \mod (a) \tag{3.21}$$

for some $m \in \mathbb{Z}$. Note that the element on the right hand side of (3.21) is in the summand of (3.8) which has $j = n$.

But by Proposition 3.9,

$$\delta^*\left(\frac{v_n^k}{v_{n-1}v_{n-2}\dots v_0}\sigma^{-2^n+1+m2^{n+1}}\right)=v_n^k\sigma^{m2^{n+1}}a^{\sum\limits_{m=0}^{n-1}(2^{m+1}-1)},\tag{3.22}$$

and, moreover,

$$Im(\delta^*)\subset(a)^{\sum\limits_{m=0}^{n-1}(2^{m+1}-1)}.\tag{3.23}$$

We conclude that the a-multiples in (3.21) map to

$$(a)^{1+\sum\limits_{m=0}^{n-1}(2^{m+1}-1)}$$

by δ^*, and hence cannot cancel the non-zero element (3.22). The Lemma follows. □

Proof of Theorem 1.9: We begin with the second statement, which is a consequence of Theorem 1.8. To this end, note that for $i_0=\dots=i_{m-1}=1$, the condition of Theorem 1.8 reads

$$2^{j+1}\mid k(2^n-1)-\sum_{m=0}^{n-1}(2^m-1)-2^j+1\tag{3.24}$$

for some $0\leq j\leq n$. Processing (3.24) further gives

$$2^{j+1}\mid(k-1)(2^n-1)+(n+1)-2^j.\tag{3.25}$$

Note that the subsets of $\mathbb{Z}/2^{n+1}\mathbb{Z}$ satisfying (3.25) for different $j=0,\dots,n$ are disjoint, and the class for j has 2^{n-j} elements, which form a congruence class mod 2^{j+1}. It follows that there is precisely one class $q+2^{n+1}\mathbb{Z}$ which does not satisfy (3.25) for any $j=0,\dots,n$, and it is

$$q=2^n+k\tag{3.26}$$

where k satisfies (3.25) with $j=n$. To determine k, note that

$$-(2^n+1)=(2^n-1)^{-1}\in(\mathbb{Z}/2^{n+1}\mathbb{Z})^\times,$$

so (3.25) with $j=n$ gives

(3.27)

$$\begin{aligned}k-1\equiv((n+1)-2^n)(2^n+1)\quad&\equiv n+1\mod 2^{n+1}\text{ for } n \text{ even}\\&\equiv 2^n+n+1\mod 2^{n+1}\text{ for } n\text{ odd.}\end{aligned}$$

It follows from (3.26) that $q\equiv k_n+2^n\mod 2^{n+1}$, as claimed.

To prove the first statement of Theorem 1.9, we turn to Lemma 3.17. The condition 3.19 is clearly equivalent to 3.25 with $j = n$, and hence to $k \equiv k_n \mod 2^{n+1}$. But under this condition, Lemma 3.17 implies

$$\delta^*\nu(x) = \delta^*(z) \neq 0 \in BP\mathbb{R}_\star S^0. \tag{3.28}$$

Therefore, if we denote also by δ^* the connecting map

$$\delta^* : \pi_* X_{n-1} \to \Sigma\pi_* X_{n-2} \to ... \to \Sigma^n \pi_* S^0,$$

we conclude from (3.28) that

$$\delta^*(x) \neq 0 \in \pi_* S^0 \tag{3.29}$$

(since δ^* obviously commutes with Hurewicz maps). But (3.29) occurring for *every* lift $x$of y is equivalent to y not being hit by a differential in the GCSS. □

4. $BP\mathbb{R}$-homology calculations

The purpose of this section is to prove Propositions 3.4, 3.9, 3.3. We begin with Proposition 3.4.

Lemma 4.1. *We have*

$$\begin{aligned} &BP\mathbb{R}/(v_0, ..., v_{n-1})_\star = \\ &BP\mathbb{R}_\star/(\sigma^{2\ell_0}v_0, \sigma^{4\ell_1}v_1, ..., \sigma^{2^n\ell_{n-1}}v_{n-1}) \cdot \{1, \sigma^{-1}, ..., \sigma^{-2^n+1}\}. \end{aligned} \tag{4.2}$$

On the right hand side of (4.2), we quotient out over all values of $\ell_i \in \mathbb{Z}$. Moreover, $BP\mathbb{R}/(v_0, ..., v_{n-1})$ is a complete spectrum with respect to $\mathbb{Z}/2$-action, and the differentials of its Borel cohomology spectral sequence have the form

$$d_{2^{m+1}-1}\sigma^{-2^m} \cdot q \cdot \sigma^{-i} = v_m a^{2^{m+1}-1} q \cdot \sigma^{-i} \tag{4.3}$$

where $0 \leq i \leq 2^n - 1$, $m \geq n$ and $q = \sigma^{\ell 2^{m+1}} v_R$ where $\ell \in \mathbb{Z}$ and $R = (0, 0, ..., 0, r_m, r_{m+1}, ...)$.

Here we denote

$$v_{(r_0, r_1, ...)} = v_0^{r_0} v_1^{r_1} ...$$

(of course, only finitely many of the r_i's are allowed to be non-zero).

Proof: To establish completeness, recall from [5] that $BP\mathbb{R}$ is complete with respect to $\mathbb{Z}/2$-action, and that we have cofibrations of $M\mathbb{R}$-modules

$$\begin{array}{c} \Sigma^{(2^n-1)(1+\alpha)}BP\mathbb{R}/(v_0,...,v_{n-1}) \\ \downarrow{\scriptstyle v_n} \\ BP\mathbb{R}/(v_0,...,v_{n-1}) \\ \downarrow \\ BP\mathbb{R}/(v_0,...,v_n). \end{array} \tag{4.4}$$

The differentials (4.3) are established exactly in the same way as in the case $n = 0$, which is done in [5]. □

Proof of Proposition 3.4: First consider the cofibration sequence of $M\mathbb{R}$-modules (3.2). Recall from [5] that

$$\Phi^{\mathbb{Z}/2}BP\mathbb{R} = H\mathbb{Z}/2. \tag{4.5}$$

Since multiplication by v_i is 0 on (4.5), we conclude inductively that

$$\Phi^{\mathbb{Z}/2}BP\mathbb{R}/(v_0^\infty,...,v_{n-1}^\infty) = H\mathbb{Z}/2 \tag{4.6}$$

and

$$\Phi^{\mathbb{Z}/2}v_n^{-1}BP\mathbb{R}/(v_0^\infty,...,v_{n-1}^\infty) = 0. \tag{4.7}$$

Therefore, the completeness statement of the Proposition will follow if we can show that

$$(v_n^{-1}BP\mathbb{R}/(v_0^\infty,...,v_{n-1}^\infty))^\wedge = 0 \tag{4.8}$$

where the hat indicates the Tate spectrum. We will show this by induction on n, jointly with the following

Claim 4.9. *The differentials in the Borel cohomology spectral sequence of*

$$BP\mathbb{R}/(v_0^\infty,...,v_{n-1}^\infty)$$

are as follows:

(4.10)

$$d_{2^{k+1}-1}(v_0^{-1}v_1^{-1}\cdot ...\cdot v_{k-1}^{-1}v_k^{-i_k}\cdot .. \cdot v_{n-1}^{-i_{n-1}}v_R\sigma^{-2^{k+1}+1+\ell 2^{k+1}}) =$$
$$v_0^{-1}v_1^{-1}\cdot ...\cdot v_{k-1}^{-1}v_k^{-i_k+1}v_{k+1}^{-i_{k+1}}\cdot ...\cdot v_{n-1}^{-i_{n-1}}v_R\sigma^{-2^k+1+\ell 2^{k+1}}a^{2^{k+1}-1}$$

where $0 \leq k \leq n-1$, $i_k > 1$, $i_{k+1},...,i_{n-1} \geq 1$, $\ell \in \mathbb{Z}$, $R = (0,...,0,r_n,r_{n+1},...)$ *and*

$$\begin{aligned} &d_{2^{m+1}-1}(v_0^{-1}\cdot ...\cdot v_{n-1}^{-1}v_R\sigma^{-2^n+1-2^m+\ell 2^{m+1}}) = \\ &v_0^{-1}\cdot ...\cdot v_{n-1}^{-1}v_m v_R\sigma^{-2^n+1+\ell 2^{m+1}}a^{2^{m+1}-1} \end{aligned} \tag{4.11}$$

where $m \geq n$, $\ell \in \mathbb{Z}$, $R = (0, ..., 0, r_m, r_{m+1}, ...)$.

More precisely, we will show that the Claim implies (4.8), and also the Claim with n replaced by $n+1$. Note that these statements jointly imply the Proposition, by computing the Borel cohomology spectral sequence E_∞ term via (4.10), (4.11).

Thus, assume the Claim is valid for a fixed n. Then we can compute the Tate spectral sequence for $v_n^{-1}BP\mathbb{R}/(v_0^\infty, ..., v_{n-1}^\infty)$ by inverting v_n and a (for v_n, use the $M\mathbb{R}$-module structure). Note, however, that then the differential (4.10) wipes out the k-th summand of (3.5), and (4.11) with $m = n$ wipes out the last summand. Consequently, the Tate spectral sequence for $v_n^{-1}BP\mathbb{R}/(v_0^\infty, ..., v_{n-1}^\infty)$ collapses to $E_{2^n} = 0$, and thus (4.8) follows.

To prove the Claim with n replaced by $n+1$, we will construct a map of spectral sequences

$$E' \oplus E'' \to E \tag{4.12}$$

which will be onto each E_r-term. To this end, let E' be the Borel cohomology spectral sequence for $v_n^{-1}BP\mathbb{R}/(v_0^\infty, ..., v_{n-1}^\infty)$ where the map $E' \to E$ is induced by the second map (3.2). Let, on the other hand, E'' be the Borel cohomology spectral sequence for $BP\mathbb{R}/(v_0, ..., v_n)$ with $E'' \to E$ induced by the obvious map of $M\mathbb{R}$-modules

$$\{v_0^{-1} \cdot ... \cdot v_n^{-1}\}BP\mathbb{R}/(v_0, ..., v_n) \to BP\mathbb{R}/(v_0^\infty, ..., v_n^\infty).$$

To examine the map (4.12), write, as usual ([4], [5]), the Borel cohomology spectral sequence for $BP\mathbb{R}/(v_0^\infty, ..., v_n^\infty)_\star$ in the form

$$E_1 = BP_\star/(v_0^\infty, ..., v_n^\infty)[a][\sigma^{\pm 1}] \Rightarrow BP\mathbb{R}/(v_0^\infty, ..., v_n^\infty)_\star. \tag{4.13}$$

In this notation, let, for $i = 0, ..., n$, ${}_iE_1$ be spanned by those monomials in E_1 which involve σ^s where the exponents of $v_0, ..., v_{i-1}$ are -1, and in addition either $s \equiv -2^i + 1 \mod 2^{i+1}$, or $s \equiv -2^i + 1 \equiv 1 \mod 2^i$ and the exponent of v_i is < -1.

Let, further, ${}_\infty E_1$ be the summand of E_1 spanned by the remaining monomials, i.e. those involving σ^s where $s \equiv -2^{n+1} + 1 \equiv 1 \mod 2^{n+1}$, and the exponents of $v_0, ..., v_n$ are -1.

Now we know inductively that the Borel cohomology spectral sequence for

$$BP\mathbb{R}/(v_0^\infty, ..., v_{n-1}^\infty)$$

splits as a sum of $n+1$ summands corresponding to the $n+1$ summands on the right hand side of (3.5). Inverting v_n, $v_n^{-1}BP\mathbb{R}/(v_0^\infty, ..., v_{n-1}^\infty)$ correspondingly splits into $n+1$ summands, which we will denote by

${}_0E'_r$, ..., ${}_nE'_r$. Then we find that

$${}_iE'_1 \twoheadrightarrow {}_iE_1,$$

where $i = 0, ..., n$, while

$$(4.14) \qquad {}_iE'_{2^{i+1}} \twoheadrightarrow H({}_iE_1, d_{2^{i+1}-1}).$$

On the other hand, recalling Lemma 4.1,

$$E'' = {}_0E'' \oplus ... \oplus {}_{2^n-1}E''$$

where ${}_iE''$ is the sub-spectral sequence of E'' involving the factor σ^{-i} in (4.3). We then see that

$$(4.15) \qquad {}_{2^n-1}E'' \xrightarrow{\simeq} {}_\infty E.$$

Now (4.14), (4.15) imply that (4.12) is onto every E_r term, completing the induction step. □

Proof of Proposition 3.9: As noted in the preceding proof, the last summand of (3.5) (with n replaced by $n+1$) is in the image of the map

$$(4.16) \qquad \{v_0^{-1} \cdot ... \cdot v_n^{-1}\}(BP\mathbb{R}/(v_0, ..., v_n))_\star \to (BP\mathbb{R}/(v_0^\infty, ..., v_n^\infty))_\star,$$

so the connecting map on these summands can be figured out from the connecting map of the cofibration sequence

(4.17)

$$\{v_n\}BP\mathbb{R}/(v_0, ..., v_{n-1}) \xrightarrow{v_n} BP\mathbb{R}/(v_0, ..., v_{n-1}) \longrightarrow BP\mathbb{R}/(v_0, ..., v_n).$$

The Borel cohomology spectral sequence gives

$$(4.18) \qquad \begin{array}{l}(BP\mathbb{R}/(v_0, ..., v_{n-1}))_\star = \\ BP\mathbb{R}_\star/(\sigma^{2\ell_0}v_0, \sigma^{4\ell_1}v_1, ..., \sigma^{2^n\ell_{n-1}}v_{n-1})\{1, \sigma^{-1}, ..., \sigma^{-2^{n-1}+1}\}.\end{array}$$

The target of the connecting map δ_n of (4.17) is the kernel of the self map v_n in (4.18), which clearly consists of multiples of $a^{2^{n+1}-1}$. Differentials in the Borel cohomology spectral sequence of $BP\mathbb{R}/(v_0, ..., v_{n-1})$ then give the formula

$$\delta_n : \sigma^{-2^n} \mapsto a^{2^{n+1}-1}(v_n)$$

which remains valid when multiplied by σ^{-i} with $0 \leq i < 2^n$, and hence implies (3.11).

To show that the target of ∂_n is in the last summand of (3.5), note that this target is the direct limit of the targets of connecting maps of the form

$$(4.19) \qquad \{v_n^{-k}\}BP\mathbb{R}/(v_0^\infty, v_1^\infty, ..., v_{n-1}^\infty, v_n^k)_\star \to \Sigma BP\mathbb{R}/(v_0^\infty, ..., v_{n-1}^\infty)_\star.$$

But elements in the image of (4.19) must be in the kernel of v_n^k, which is clearly injective on all but the last summand of (3.5). □

Proof of Proposition 3.3: Induction on n. Suppose the statement is true with n replaced by $n-1$. Consider a Hopkins-Smith v_{n-1}-spectrum V such that

$$BP_*V = BP_*/(v_0^{k_0}, ..., v_{n-1}^{k_{n-1}}),$$

and the Hurewicz map

$$\eta : V \to BP\mathbb{R} \wedge V. \tag{4.20}$$

We shall prove that for a v_n-self map $v : V \to V$ (ignoring suspensions in the notation), we have a commutative diagram of the form

$$\begin{array}{ccc} V & \xrightarrow{\eta} & BP\mathbb{R} \wedge V \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle v_n^N \sigma^K} \\ V & \xrightarrow[\eta]{} & BP\mathbb{R} \wedge V, \end{array} \tag{4.21}$$

for some N and K, at least when v is replaced by its suitable power. Similarly as above, this can be done by considering η as an element of the $BP\mathbb{R}$ cohomology of $V \wedge DV$. The question then becomes what map in $BP\mathbb{R}$-cohomology the map $w = v \wedge Id : V \wedge DV \to V \wedge DV$ induces. We can assume by Hopkins-Smith [3] that w induces a power of v_n in BP_*. Thus, by the Borel cohomology spectral sequence,

$$BP\mathbb{R}^\star w = v_n^N \sigma^K \mod (a).$$

However, considering the structure of $BP\mathbb{R}_\star$ (Theorem 3.1), we see that for any finite fixed spectrum X, a is nilpotent on elements of $BP\mathbb{R}_{k+\ell\alpha}X$ with $k >> 0$ (use the Atiyah-Hirzebruch-type spectral sequence associated with a finite cell-decomposition of X). Now assume

$$BP\mathbb{R}^\star w = v_n^N \sigma^K + at.$$

Then

$$BP\mathbb{R}^\star w^{2^M} = (v_n^N \sigma^K)^{2^M} + a^M t^{2^M}$$

(since, in $BP\mathbb{R}_\star$, $2a = 0$). Now for $M >> 0$, the second term disappears, thus proving (4.21).

Now passing to homotopy colimit and BP-localization, we get a commutative diagram

$$\begin{array}{ccc} X_{n-1} & \longrightarrow & BP\mathbb{R}/(v_0^\infty, ..., v_{n-1}^\infty) \\ \downarrow & & \downarrow \\ Y_{n-1} & \longrightarrow & v_n^{-1} BP\mathbb{R}/(v_0^\infty, ..., v_{n-1}^\infty). \end{array} \tag{4.22}$$

(Note that

$$v_n^{-1}BP\mathbb{R}/(v_0^\infty, ..., v_{n-1}^\infty) \simeq (v_n^N\sigma^K)^{-1}BP\mathbb{R}/(v_0^\infty, ..., v_{n-1}^\infty),$$

since each of the elements v_n, $v_n^N\sigma^K$ divides a power of the other.) Using the ring structure of $BP\mathbb{R}$, we get a diagram

$$\begin{array}{ccc} BP\mathbb{R}\wedge X_{n-1} & \xrightarrow{\phi} & BP\mathbb{R}/(v_0^\infty, ..., v_{n-1}^\infty) \\ \downarrow & & \downarrow \\ BP\mathbb{R}\wedge Y_{n-1} & \xrightarrow{\psi} & v_n^{-1}BP\mathbb{R}/(v_0^\infty, ..., v_{n-1}^\infty) \end{array} \tag{4.23}$$

where the top horizontal arrow ϕ is an equivalence, and the bottom horizontal arrow ψ is an equivalence non-equivariantly. But both $BP\mathbb{R}\wedge Y_{n-1}$, $v_n^{-1}BP\mathbb{R}/(v_0^\infty, ..., v_{n-1}^\infty)$ are free spectra (v induces 0 in homology), so ψ is an equivalence. Passing to cofibers of the vertical arrows of (4.23) gives the induction step. □

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