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ABSTRACT

We give a definition of a norm functor from H-Mackey functors to G-Mackey functors for G a finite group and H a subgroup of G. We check that this agrees with the construction of Mazur in the case G cyclic of prime power order and also with the topological definintion of norm, which has an algebraic presentation due to Ullman. We then use this norm functor to give a characterization of Tambara functors as monoids of an appropriate flavor.

The second chapter is part of a joint project with Andrew Baker. We consider what happens when we take the sphere spectrum, and kill elements of homotopy in an E_∞ fashion. This process starts with the element 2 and is repeated in order to kill all higher homotopy groups. We provide methods for identifying spherical classes and for understanding the Dyer-Lashof action at each step of the construction. We outline how this construction might be used to compute the André-Quillen homology of Eilenberg-MacLane spectra considered as algebras over the sphere spectrum.

CHAPTER 1

INTRODUCTION

1.1 Tambara functors and algebraic norms of Mackey functors

Equivariant topology considers spaces with actions of a group G. (Throughout this paper, we restrict attention to the case G finite.) In order to comprehensively account for the action of the group, one must be careful when defining invariants. The most basic example is that the notion of a weak homotopy equivalence of G-spaces requires not just an equivariant map that is a weak homotopy equivalence of underlying spaces, but must also induce weak homotopy equivalences on all fixed-point spaces over all subgroups of G.

In a similar vein, the correct generalization of an abelian group is often not just a G-module, but rather a G-Mackey functor. This consists of a network of modules $\underline{M}(G/H)$ indexed on the subgroups of G. These modules are related to each other by structure maps satisfying certain compatibility conditions, including restriction maps of the form $\underline{M}(G/K) \to \underline{M}(G/H)$ for inclusions $H \leqslant K$ and transfer maps in the other direction. In particular, given some G-spectrum E the natural structure on $\underline{\pi}_0 E$ is a Mackey functor.

For G finite, the notion of G-Tambara functor was introduced in [21] to axiomatize the structure gained from the multiplicative transfers arising in representation theory and in the cohomology of finite groups (for instance see Chapter 5 of [10]). Such multiplicative transfers were introduced to stable homotopy theory in [12].

Tambara functors became of interest to stable homotopy theorists when Brun [7] demonstrated that $\underline{\pi}_0 R$ is a Tambara functor for any E_∞ ring G-spectrum R. In the other direction, Ullman [23] has recently proved that any Tambara functor can be realized by a ring structure on the corresponding Eilenberg-MacLane spectrum. Furthermore, the category of Tambara functors is shown to be equivalent to the homotopy category of commutative ring Eilenberg-MacLane G-spectra.

We now consider the previous attempts to build multiplicative norm functors

$$N_H^G \colon \mathrm{Mack}_H \to \mathrm{Mack}_G$$

in a way consistent with the notion of Tambara functor. In particular, this consistency means that when restricted to Tambara functors, this functor is essentially the free functor $\mathrm{Tamb}_H \to \mathrm{Tamb}_G$. In the case of G cyclic of prime power, there is an algebraic construction due to Mazur [18].

For arbitrary finite groups G, there is also a topological construction using the Hill-Hopkins-Ravenel norm [13] in a good model of the stable equivariant category. Ullman [22] gives an algebraic description for this functor, but it would be preferable to have a definition not going through topology.

We provide a new definition of N_H^G for all finite groups G as a left Kan extension, and in particular check that this agrees with both definitions as above. This definition is purely algebraic, and is categorically universal in a way that contrasts with any description built purely from generators and relations.

We use this functor to give an extrinsic characterization of Tambara functors as commutative monoids in an appropriate sense. We give an explicit account of the coherence conditions necessary for this structure.

1.2 Eilenberg-MacLane spectra as relative E_∞ cell complexes

André-Quillen cohomology is a natural invariant theory on the category of commutative rings. It is a relative theory in the sense that groups $\mathrm{AQ}^*(B/A)$ are defined for a pair of commutative rings A, B with a given map $A \to B$. In the world of stable homotopy theory, Basterra [6] defined the corresponding theory TAQ^* of topological André-Quillen cohomology. This construction is analogous to the construction in ordinary algebra, and relies on a stable category with symmetric monoidal smash product.

The theory TAQ^* is the natural obstruction theory for commutative S-algebras (which are exactly the commutative monoids over the smash product) in the sense that there is a decomposition similar to the traditional Postnikov tower, with the invariants lying in appropriate topological André-Quillen cohomology groups.

As of yet, there are very few explicit calculations of TAQ^* in the literature. An example of what is known is Richter's calculation [19] of $\mathrm{TAQ}_*(HA/H\mathbb{F})$, where A is a smooth commutative algebra over a field $\mathbb{F}$. Here HA and $H\mathbb{F}$ are Eilenberg-MacLane spectra. While a field is a reasonable base to use in the algebraic context, in stable topological algebra the universal base is instead the sphere spectrum S. In particular, the invariants in the aforementioned Postnikov tower lie in $\mathrm{TAQ}^*(R_i/S)$ for appropriate commutative S-algebras R_i. In this case, little is known calculationally.

The basic building block to consider is when R is an Eilenberg-MacLane spectrum. We consider the case of $R = H\mathbb{F}_2$, for which a sketch of a calculation exists in the literature [16]. We proceed by methods analogous to the computation of ordinary cohomology of a space with the structure of a cell complex. More specifically, we form a relative cell complex in the category of commutative S-algebras. This consists of taking the sphere spectrum S, and killing off elements in homotopy through attaching E_∞ cells. Attaching such cells creates new elements in homotopy, which are then killed by the next stage of attached cells.

In order to understand this decomposition, we need algebraic control over the set of cells and the attaching maps for each stage of our construction. In order to do this, we apply a theorem of Steinberger [8] to observe that at each stage we have a spectrum that splits as a wedge of Eilenberg-MacLane spectra. We strengthen this observation to give algebraic control over this splitting. We verify that there is a correspondence between the set of cells and the proposed basis for $\mathrm{TAQ}^*(H\mathbb{F}_2/S)$.

We develop an explicit understanding of the action of the Dyer-Lashof algebra on the homology of each stage of our construction. We also produce formulae to replace polynomial generators with spherical classes. These spherical classes then produce the E_∞ attaching maps to obtain the next stage of the construction. We give a similar description of how to obtain spherical classes in the first stage of a complex intended to compute $\mathrm{TAQ}^*(H\mathbb{Z}_{(2)}/S)$.

CHAPTER 2

TAMBARA FUNCTORS AND ALGEBRAIC NORMS OF MACKEY FUNCTORS

2.1 Introduction

In this chapter we give a new algebraically-motivated construction for the norm functor $N_H^G \underline{M}\colon \mathrm{Mack}_H \to \mathrm{Mack}_G$, for H some subgroup of a finite group G. We then give comparisons with previously-existing constructions and use this structure to give an alternate characterization of Tambara functors.

In Section 2.2, we give some background on Mackey and Tambara functors, and establish our notational conventions.

In Section 2.3, we give our definition for the norm $N_H^G \underline{M}$, and prove the following:

Theorem 2.1.1. *The norm functor* $N_H^G\colon \mathrm{Mack}_H \to \mathrm{Mack}_G$ *agrees with the free functor* $\mathrm{Tamb}_H \to \mathrm{Tamb}_G$ *when the input is an* H*-Tambara functor. Furthermore,* N_H^G *is strong symmetric monoidal with respect to to the box product of Mackey functors.*

This result allows us to construct a commutative diagram of the following form, where the horizontal maps are forgetful functors and the squares commute up to natural isomorphism:

$$\begin{tikzcd}
\mathrm{Mack}_H \arrow[d, "N_H^G"'] & \mathrm{Green}_H \arrow[l] \arrow[d, "N_H^G"'] & \mathrm{Tamb}_H \arrow[l] \arrow[d, "N_H^G"'] \\
\mathrm{Mack}_G & \mathrm{Green}_G \arrow[l] & \mathrm{Tamb}_G \arrow[l]
\end{tikzcd}.$$

We then check that our construction agrees with Ullman's algebraic presentation. We also construct a natural unit map $\underline{M} \to i_H N_H^G \underline{M}$ for any H-Mackey functor $\underline{M}$ equipped with a map $\underline{A} \to \underline{M}$. Here $\underline{A}$ is the Burnside Mackey functor, which is the initial object in the category of Green functors.

In Section 2.4, we construct part of what we call a G-symmetric monoidal structure on the category Mack_G. We define $\underline{M}^{\otimes T}$ for any finite G-set T together with natural isomorphisms corresponding to G-isomorphisms $T \cong T'$. In the case of $T = T' = G/H$, this gives an action of the Weyl group $W_G H$ through isomorphisms of G-Mackey functors.

In Section 2.5 we consider the underlying K-Mackey functor of the norm of an H-Mackey functor, where $H, K \leqslant G$. We obtain a decomposition analogous to the usual double coset formula. In particular, if $H = K$ is the maximal subgroup for G a cyclic group of prime power order, we have an isomorphism between $i_H N_H^G \underline{M}$ and the $|G/H|$-fold box product $\underline{M}^{\boxtimes |G/H|}$ for any H-Mackey functor $\underline{M}$. This is used in Section 2.6 to construct an isomorphism between our norm functor and the construction of Mazur.

In Section 2.7, we define G-commutative monoids and coherent G-commutative monoids. We then prove that the categories of G-Tambara functors and coherent G-commutative monoids are equivalent. In the latter case the structure is defined extrinsically in terms of our norm functor, and in the former case the structure is defined through internal norm maps $\underline{M}(G/H) \to \underline{M}(G/K)$ for any chain of subgroup inclusions $H \leqslant K \leqslant G$.

2.2 Background and conventions on Tambara functors and Mackey functors

Let G be a finite group. In order to set our notation, we briefly describe the definition of Tambara functor as given in [21] or [20].

Let $\mathcal{A}_G$ denote the Burnside category of spans of finite G-sets. This category has finite G-sets for objects. Any map of G-sets $f\colon X \to Y$ induces a map $R_f \in \mathcal{A}_G(Y, X)$ and a map $T_f \in \mathcal{A}_G(X, Y)$. The morphisms in $\mathcal{A}_G(X, Y)$ are expressible as spans (up to isomorphisms of spans) of the following form:

$$X \xleftarrow{f} A \xrightarrow{g} Y\,.$$

The above span should be interpreted as the composite $T_g R_f$ in $\mathcal{A}_G$. Composition is then defined via pullback, meaning concretely that if we have the following pullback square, then $R_f T_g = T_\rho R_\pi$:

$$\begin{tikzcd} X \underset{A}{\times} Y \arrow[r, "\rho"] \arrow[d, "\pi"'] & Y \arrow[d, "f"] \\ X \arrow[r, "g"'] & A \end{tikzcd}.$$

Similarly, let $\mathcal{U}_G$ denote the category of polynomial bispans. (The general picture here is outlined in [11]). The objects are the same, but $f\colon X \to Y$ induces a map $N_f \in \mathcal{U}_G(X,Y)$, in addition to the maps R_f, T_f. The morphisms in $\mathcal{U}_G(X,Y)$ are expressible as bispans (up to isomorphisms of bispans) of the following form:

$$X \xleftarrow{f} A \xrightarrow{g} B \xrightarrow{h} Y .$$

Again, the above bispan can be written as the composite $T_h N_g R_f$ in $\mathcal{U}_G$.

In order to compose such bispans, we use pullback as in the span case to define compositions of the form $R_f N_g$ and $R_f T_h$. For maps $h\colon X \to Y$, $g\colon Y \to Z$, we define the composition $N_g T_h$ by using the following diagram, referred to as the corresponding *exponential diagram*:

$$\begin{tikzcd} & & Y \arrow[rrd, "g"] & & \\ X \arrow[rru, "h"] & & Y \underset{Z}{\times} \Pi_g X \arrow[ll, "e"] \arrow[u] \arrow[r, "\pi"'] & \Pi_g X \arrow[r, "\Pi_g h"'] & Z \end{tikzcd}.$$

In the category $\mathcal{U}_G$, the composite $N_g T_h$ is given by $T_{\Pi_g h} N_\pi R_e$.

Here Π_g, referred to as the dependent product, is part of the local cartesian structure on G-Set, although we only use it in the full subcategory G-Set^F of finite G-sets. It is the right adjoint to the pullback functor $g^*\colon G\text{-Set}/Z \to G\text{-Set}/Y$. The counit of this adjunction yields the map e above. On the set level we have

$$\Pi_g X = \{(z,s)\colon z \in Z, s\colon g^{-1}\{z\} \to X, h \circ s = \mathrm{id}_{g^{-1}\{z\}}\}.$$

The G action is given by the left action on elements of Z and by conjugation of the maps s. The map $\Pi_g h\colon \Pi_g X \to Z$ is the evident projection $(z,s) \mapsto z$, and the evaluation map $e\colon Y \times_Z \Pi_g X \to X$ is given by $e(y,(z,s)) = s(y)$.

The bottom row of the exponential diagram is called the *distributor* associated to g and h, referring to the fact that it tells us how to interpret the composition of a multiplication (the norm N_g) with an addition (the transfer T_h).

Product-preserving functors $\underline{M}\colon \mathcal{A}_G \to \mathrm{Set}$ define G-semi-Mackey functors, and product-preserving functors $\underline{R}\colon \mathcal{U}_G \to \mathrm{Set}$ define G-semi-Tambara functors. Note that the products in $\mathcal{A}_G$ and $\mathcal{U}_G$ are given by disjoint union of G-sets.

If $\underline{M}$ is a semi-Mackey functor, then $\underline{M}(X)$ naturally inherits the structure of a commutative monoid, and the category of Mackey functors Mack_G is the full subcategory of those $\underline{M}$ such that $\underline{M}(X)$ is an abelian group for all X.

This definition is equivalent to viewing Mackey functors as product-preserving functors $\mathcal{A}_G^+ \to \mathrm{Set}$, where $\mathcal{A}_G$ is the category with the same objects as $\mathcal{A}_G$ and with $\mathcal{A}_G^+(X,Y)$ the Grothendieck completion of $\mathcal{A}_G(X,Y)$, where the sum of two spans $X \leftarrow A \to Y$ and $X \leftarrow B \to Y$ is the disjoint union $X \leftarrow A \amalg B \to Y$.

Similarly, if $\underline{R}$ is a semi-Tambara functor, then $\underline{R}(X)$ inherits the structure of a commutative semiring, and the category of Tambara functors Tamb_G is the full subcategory of those $\underline{R}$ such that $\underline{R}(X)$ is a commutative ring for all X. Equivalently, Tambara functors are product-preserving functors $\mathcal{U}_G^+ \to \mathrm{Set}$ analogous to the above. Here $\mathcal{U}_G^+(X,Y)$ is the Grothendieck completion of $\mathcal{U}_G(X,Y)$, where the sum of two bispans $X \leftarrow A \to B \to Y$ and $X \leftarrow C \to D \to Y$ is given by $X \leftarrow A \amalg C \to B \amalg D \to Y$.

In the Mackey functor case, we could equivalently use additive product-preserving functors $\mathcal{A}_G \to \mathrm{Ab}$, but this is not possible in the Tambara functor case since composition does not preserve the additive structure on morphisms, i.e. the composition of polynomials is not linear.

2.2.1 Alternate characterizations of Mackey and Tambara functors

Since Mackey and Tambara functors are product-preserving, and the products in the categories $\mathcal{A}_G, \mathcal{U}_G$ are given by disjoint unions of G-sets, the combinatorial data of such functors can be reduced by breaking arbitrary finite G-sets into orbits.

A Mackey functor $\underline{M}$ is determined by an abelian group $\underline{M}(G/H)$ for each subgroup $H \leqslant G$. The maps in $\mathcal{A}_G$ can be written in terms of three kinds of maps. Given $H \leqslant K \leqslant G$, the quotient map $q(H, K)\colon G/H \to G/K$ induces a transfer map

$$\mathrm{tr}_H^K = \underline{M}(T_{q(H,K)})\colon \underline{M}(G/H) \to \underline{M}(G/K)$$

and a restriction map

$$\mathrm{res}_H^K = \underline{M}(R_{q(H,K)})\colon \underline{M}(G/K) \to \underline{M}(G/H)\,.$$

These maps must be compatible with the composition in $\mathcal{A}_G$, which yields the traditional double coset formula. Next, any conjugacy relation between subgroups H and H' induces an isomorphism $\underline{M}(G/H) \cong \underline{M}(G/H')$. In particular, maps induced by the isomorphisms $G/H \to G/H$ yield an action of the Weyl group $W_G H$ on $\underline{M}(G/H)$. These isomorphisms must be compatible under composition and also appropriately compatible with the transfer and restriction maps.

A Tambara functor can be reduced similarly to the data of a commutative ring $\underline{R}(G/H)$

for each subgroup H. There are now norm maps

$$\mathrm{norm}_H^K = \underline{M}(N_{q(H,K)})\colon \underline{M}(G/H) \to \underline{M}(G/K)\,,$$

compatible with multiplication but not addition. The restrictions and conjugation isomorphism are compatible with both multiplication and addition, and the transfers are purely additive. The compatibility between norms and the restrictions gives a multiplicative semi-Mackey functor structure on $\underline{M}$ in addition to the additive Mackey structure given by transfers and restrictions. The exponential diagram must now encode the necessary compatibility between norms and transfers, as well as norms and addition and transfers and multiplication.

2.3 Multiplicative norms of Mackey functors

Now set $H \leqslant G$. We use the following adjunctions, where $i\colon G\text{-Set} \to H\text{-Set}$ is the forgetful functor:

$$\begin{aligned} G\text{-Set}[G \times_H X, Y] &\cong H\text{-Set}[X, iY] \\ H\text{-Set}[iX, Y] &\cong G\text{-Set}[X, \mathrm{Map}_H(G, Y)]\,. \end{aligned}$$

Here we note that the equivariance condition for maps in $\mathrm{Map}_H(G, Y)$ relies on the left action of H on G and Y, and the G-action on such a function is given by precomposition with the right action of G on G, i.e. given an H-map f, then gf sends an element g' to $f(g'g)$.

We note that $G \times_H (-)$ preserves pullbacks and distributors (i.e., it preserves the local cartesian structure), so it induces maps $G \times_H (-)\colon \mathcal{A}_H \to \mathcal{A}_G$ and $G \times_H (-)\colon \mathcal{U}_H \to \mathcal{U}_G$. The functor $\mathrm{Map}_H(G, -)$ preserves pullbacks (but not the local cartesian structure), so it induces a map $\mathcal{A}_H \to \mathcal{A}_G$.

The forgetful functor $i_H \colon \mathrm{Tamb}_G \to \mathrm{Tamb}_H$ is induced by precomposition with the functor $G \times_H (-) \colon \mathcal{U}_H \to \mathcal{U}_G$. This might be counterintuitive but reflects the fact that given a G-Tambara functor $\underline{R}$, and an H-orbit H/K, the natural choice for $i_H\underline{R}(H/K)$ is the ring $\underline{R}(G/K) \cong \underline{R}(G \times_H (H/K))$.

We give a categorical description of the left adjoint to i_H.

Definition 2.3.1. *The functor $N_H^G \colon \mathrm{Tamb}_H \to \mathrm{Tamb}_G$ is given by the left Kan extension along the functor $G \times_H (-)$. This can be computed by the following coend, where $\underline{R}$ is a given H-Tambara functor:*

$$N_H^G\underline{R}(Y) := \mathrm{Lan}_{G \underset{H}{\times} (-)}\underline{R}(Y) = \int^{X \in \mathcal{U}_H} \mathcal{U}_G(G \underset{H}{\times} X, Y) \times \underline{R}(X)\,.$$

The coend formula and the fact that N_H^G is the left adjoint to i_H follow from 4.25 and 4.39 of [14], respectively. It is formal that any such left Kan extension of a product-preserving functor results in a product-preserving functor (Proposition 2.5 of [15]), so this gives us a new G-Tambara functor. It is also true that it preserves the property of having additive inverses, but we defer this proof. We could have instead used the categories $\mathcal{U}_H^+$, $\mathcal{U}_G^+$ as domains.

Informally, $N_H^G\underline{R}$ builds in any additional norms, transfers, and Weyl actions that arise from passage to the larger group as freely as possible. In particular, one can see from the coend that $N_H^G\underline{R}(G/e)$ is isomorphic to the indexed tensor product $\bigotimes\limits_{G/H} \underline{R}(H/e)$. As a set this is the tensor product of $|G/H|$ copies of $\underline{R}(H/e)$, with G acting by simultaneously permuting coordinates and acting on each coordinate in an appropriate fashion. (This is an example of the construction of indexed monoidal products introduced in Section A.3 of [13].)

We now define an analogous functor on the Mackey functor level.

Definition 2.3.2. *The functor $N_H^G \colon \mathrm{Mack}_H \to \mathrm{Mack}_G$ is given by left Kan extension along $\mathrm{Map}_H(G, -) \colon \mathcal{A}_H \to \mathcal{A}_G$. Concretely, we again have a coend formula, where $\underline{M}$ is a given*

H-Mackey functor:

$$N_H^G \underline{M}(Y) := \mathrm{Lan}_{\mathrm{Map}_H(G,-)} \underline{M}(Y) = \int^{X \in \mathcal{A}_H} \mathcal{A}_G(\mathrm{Map}_H(G,X), Y) \times \underline{M}(X)\,.$$

Some comments about this construction are in order. It is not immediately clear that this functor is analogous to the Tambara-level functor N_H^G. The two Kan extensions are along different functors with different domain and target categories. The point is that the use of the indexed product functor $\mathrm{Map}_H(G,-)$ on the Mackey functor level is somehow building in some of the extra structure present in the more complicated Tambara diagram categories $\mathcal{U}_G$ and $\mathcal{U}_H$.

Note that while $\mathrm{Map}_H(G,-)$ is a right adjoint when considered as a functor H-Set $\to$ G-Set, it is not a right adjoint when considered as a map $\mathcal{A}_H \to \mathcal{A}_G$. In particular, $\mathrm{Map}_H(G,-)\colon \mathcal{A}_H \to \mathcal{A}_G$ does not preserve products, so it does not induce a functor $\mathrm{Mack}_G \to \mathrm{Mack}_H$ via pullback. Thus, while N_H^G is a left adjoint as a map between the functor categories $\mathrm{Set}^{\mathcal{A}_H} \to \mathrm{Set}^{\mathcal{A}_G}$, it is not a left adjoint as a functor $\mathrm{Mack}_H \to \mathrm{Mack}_G$.

The existence of some compatible functor N_H^G on the Mackey functor level is an appropriately natural extension of the observation that the underlying Mackey functor of $N_H^G \underline{R} \in \mathrm{Tamb}_G$ depends only on the underlying Mackey functor structure of $\underline{R} \in \mathrm{Tamb}_H$. Here compatibility means the following result.

Theorem 2.3.3. *There is a natural isomorphism $U_G N_H^G \cong N_H^G U_H$, where*

$$U_G\colon \mathrm{Tamb}_G \to \mathrm{Mack}_G \text{ and}$$

$$U_H\colon \mathrm{Tamb}_H \to \mathrm{Mack}_H$$

are the forgetful functors.

A direct construction of N_H^G in the case G cyclic of prime power order, as well as the isomorphism $U_G N_H^G \cong N_H^G U_H$ is given by Mazur [18]. On the Mackey functor level, $N_H^G \underline{M}$

is intended to be the universal home for norms $\underline{M}(G/H) \to \underline{M}(G/G)$.

To define the isomorphism of Theorem 2.3.3, we construct a map directly on coend representatives. We use the following result to reduce arbitrary bispans in $\mathcal{U}_G(G \times_H X, Y)$ to ones of a specific form.

Lemma 2.3.4. *Take an arbitrary element in $\mathcal{U}_G(G \times_H X, Y)$, which is represented by some bispan of finite G-sets:*

$$T_h N_g R_f = G \underset{H}{\times} X \xleftarrow{f} A \xrightarrow{g} B \xrightarrow{h} Y\,.$$

Then there exists an H-set D satisfying $G \times_H D \cong A$ such that the above bispan is equivalent in $\mathcal{U}_G$ to a bispan of the following form:

$$T_h N_\varepsilon N_{G \underset{H}{\times} g'} R_{G \underset{H}{\times} f'} = G \underset{H}{\times} X \xleftarrow{G \underset{H}{\times} f'} G \underset{H}{\times} D \xrightarrow{\varepsilon(G \underset{H}{\times} g')} B \xrightarrow{h} Y\,.$$

Here $f' \colon D \to X, g' \colon D \to iB$ are maps of H-sets and $\varepsilon \colon G \times_H iB \to B$ is the counit of the adjunction $G \times_H (-) \dashv i$.

Proof. The G-set A must be a disjoint union of orbits G/K_i, and the existence of a G-map $f \colon A \to G \times_H X$ says that the i-th component of f is a map $G/K_i \to G/L_i$ where the target is of form $G/L_i \cong G \times_H (H/L_i)$ for some $L_i \leqslant H$. Thus K_i is itself conjugate to some $K_i' \leqslant L_i \leqslant H$, and the map $G/K_i \to G/L_i$ factors as a map $G/K_i \cong G/K_i' \to G/L_i$, where the second map is the quotient map. The desired H-set D is the disjoint union of the H/K_i', and then f factors as $A \cong G \times_H D \to G \times_H X$, where the latter map is of the desired form $G \times_H f'$.

Next, we examine the composite map $G \times_H D \cong A \to B$, which has adjoint $g' \colon D \to iB$. We get a factorization of the form $G \times_H D \to G \times_H iB \to B$ using the counit of the adjunction. Here the first map is $G \times_H g'$, as desired.

The vertical arrows in the claimed isomorphism of bispans are the identity map of $G \times_H X$, the constructed isomorphism $A \cong G \times_H D$, and the identity maps of B and Y.

$\square$

With this lemma in hand, we return to the proof of Theorem 2.3.3. Examining our coend for $\mathrm{Lan}_{G \underset{H}{\times}(-)} \underline{R}(Y)$, we see that for any $x \in \underline{R}(X)$, the pair

$$\begin{aligned}(T_h N_g R_f, x) &= \Big(G \underset{H}{\times} X \xleftarrow{f} A \xrightarrow{g} B \xrightarrow{h} Y,\, x\Big) \\ &= \Big(G \underset{H}{\times} X \xleftarrow{G \underset{H}{\times} f'} G \underset{H}{\times} D \xrightarrow{G \underset{H}{\times} g'} G \underset{H}{\times} iB \xrightarrow{\varepsilon} B \xrightarrow{h} Y,\, x\Big)\end{aligned}$$

is identified with the pair

$$(T_h N_\varepsilon, N_{g'} R_{f'} x) = \Big(G \underset{H}{\times} iB = G \underset{H}{\times} iB \xrightarrow{\varepsilon} B \xrightarrow{h} Y,\, N_{g'} R_{f'} x\Big).$$

Similarly, in the coend for $\mathrm{Lan}_{\mathrm{Map}_H(G,-)} \underline{R}(Y)$, if we are given an arbitrary pair

$$(T_h R_k, x) = \Big(\mathrm{Map}_H(G, X) \xleftarrow{k} B \xrightarrow{h} Y,\, x\Big),$$

we can apply the adjunction $i \dashv \mathrm{Map}_H(G, -)$ to identify it with the pair

$$(T_h R_\eta, x) = \Big(\mathrm{Map}_H(G, iB) \xleftarrow{\eta} B \xrightarrow{h} Y,\, R_{\hat{k}} x\Big)\ ,$$

where η is the unit of the adjunction and $\hat{k} \colon iB \to X$ is the adjoint to k.

We have now shown that every element in the domain has a representative of the form

$$(T_h N_\varepsilon, b) = \Big(G \underset{H}{\times} iB = G \underset{H}{\times} iB \xrightarrow{\varepsilon} B \xrightarrow{h} Y,\, b\Big)\ ,$$

and every element in the target has a representative of the form

$$(T_h R_\eta, b) = \big(\mathrm{Map}_H(G, iB) \xleftarrow{\eta} B \xrightarrow{h} Y,\, b\big)\ .$$

Such elements in the two coends are said to be in *standard form*. Our isomorphism takes the former element to the latter, and we must check that this is well-defined.

First, we must check that this does not depend on our factorization into standard form, which is not unique. We simultaneously check that the equivalence class of the target does not depend on the choice of isomorphism class of bispans. Given two isomorphic bispans and two corresponding standard form expressions, it is easy to construct a diagram of the following form, which shows that the two images in the target are equivalent:

$$\begin{tikzcd}
G \underset{H}{\times} X \arrow[d, equal] & G \underset{H}{\times} D \arrow[l, "G \underset{H}{\times} f'"'] \arrow[r, "G \underset{H}{\times} g'"] \arrow[d, "\cong"'] & G \underset{H}{\times} iB \arrow[r, "\varepsilon_B"] \arrow[d, "\cong"', "G \underset{H}{\times} i\beta"] & B \arrow[r, "h"] \arrow[d, "\cong"', "\beta"] & Y \arrow[d, equal] \\
G \underset{H}{\times} X & G \underset{H}{\times} D' \arrow[l, "G \underset{H}{\times} f''"] \arrow[r, "G \underset{H}{\times} g''"'] & G \underset{H}{\times} iB' \arrow[r, "\varepsilon_{B'}"'] & B' \arrow[r, "h'"'] & Y
\end{tikzcd}\ .$$

We must still check that our isomorphism is compatible with the coend identifications. Compatibility with identifications induced by restriction and norm maps is essentially automatic from our construction. Compatibility with identifications induced by transfer maps requires the following lemma, whose proof is deferred until the end of the section.

Lemma 2.3.5. *For maps $f\colon X \to iB$, we have a natural isomorphim*

$$\alpha\colon \Pi_\varepsilon(G \times_H X) \cong \eta^* \mathrm{Map}_H(G, X)$$

in G-Set^F/B, where η is the unit for the $i \dashv \mathrm{Map}_H(-)$ adjunction and ε is the counit for the $G \times_H (-) \dashv i$ adjunction. Furthermore, we have a natural isomorphism β making the diagram

$$\begin{tikzcd}
& (G \underset{H}{\times} iB) \underset{B}{\times} \Pi_\varepsilon(G \underset{H}{\times} X) \arrow[r] \arrow[dl] \arrow[d, "\beta", "\cong"'] & \Pi_\varepsilon(G \underset{H}{\times} X) \arrow[d, "\alpha", "\cong"'] \arrow[dr] & \\
G \underset{H}{\times} X & G \underset{H}{\times} i(\eta^*\mathrm{Map}_H(G,X)) \arrow[l, "G \underset{H}{\times} \hat{\pi}"] \arrow[r, "\varepsilon"'] & \eta^*\mathrm{Map}_H(G,X) \arrow[r, "\rho"'] & B
\end{tikzcd}.$$

commute. Here $\hat{\pi}$ is the adjoint to the projection $\pi\colon \eta^*\mathrm{Map}_H(G,X) \to \mathrm{Map}_H(G,X)$ and the bispan on the top is the distributor for the composition $G \times_H X \xrightarrow{G \times_H f} G \times_H iB \xrightarrow{\epsilon} B$.

We now show that the images of the elements $(T_h N_\varepsilon T_{G\times_H f}, x)$ and $(T_h R_\eta, T_f x)$ are the same under our proposed map. We check this using the result and notation of Lemma 2.3.5 to rewrite the composite $T_h N_\varepsilon$ in such a way as to yield an element in standard form:

$$\begin{aligned}
(T_h N_\varepsilon T_{G\times_H f}, x) &= (T_h T_\rho N_\varepsilon R_{G\times_H \hat{\pi}}, x) \\
&= (T_h T_\rho N_\varepsilon, R_{\hat{\pi}} x)\,.
\end{aligned}$$

Now our proposed isomorphism sends $(T_h T_\rho N_\varepsilon, R_{\hat{\pi}} x)$ to the standard form element $(T_h T_\rho R_\eta, R_{\hat{\pi}} x)$, which we must manipulate further:

$$\begin{aligned}
(T_h T_\rho R_\eta, R_{\hat{\pi}} x) &= (T_h T_\rho R_\eta R_{\mathrm{Map}_H(G,\hat{\pi})}, x) \\
&= (T_h R_\eta T_{\mathrm{Map}_H(G,f)}, x) \\
&= (T_h R_\eta, T_f x)\,.
\end{aligned}$$

The second relation we used above is the fact that the following square is a pullback:

$$\begin{tikzcd}
\eta^*\mathrm{Map}_H(G,X) \arrow[rr, "\rho"] \arrow[d, "\pi"'] & & B \arrow[d, "\eta"] \\
\mathrm{Map}_H(G,X) \arrow[rr, "\mathrm{Map}_H(G,f)"'] & & \mathrm{Map}_H(G,iB)
\end{tikzcd}.$$

The element $(T_h R_\eta, T_f x)$ is exactly the image of the standard form element $(T_h N_\varepsilon, T_f x)$, as desired.

This shows that we have defined an isomorphism of sets. For it to be an isomorphism of Mackey functors, we must check that our map preserves the structure induced by postcomposition of restrictions and transfers. Compatibility with transfers is essentially automatic, and allows us to reduce the task of checking compatibility with restrictions to the case where h is the identity map. In this case the compatibility is easy and reduces to the naturality of η and ε.

The proof of Theorem 2.3.3 is now complete. $\square$

Proof of Lemma 2.3.5: Lacking a more categorical proof, we describe the sets $\Pi_\varepsilon(G \times_H X)$ and $\eta^*\mathrm{Map}_H(G,X)$ directly.

We choose a complete set $t_1, \ldots, t_n$ of left coset representatives of H, where $n = |G/H|$. Thus for any H-set C any element of $G \times_H C$ has a unique representative pair (t_i, c). This choice also gives $t_1^{-1}, \ldots, t_n^{-1}$ as a complete set of right coset representatives.

An element of $\Pi_\varepsilon(G \times_H X)$ consists of a point b and a map $s\colon \varepsilon^{-1}\{b\} \to G \times_H X$ such that $(G \times_H f)s$ is the identity on $\varepsilon^{-1}\{b\}$. We see that $\varepsilon^{-1}\{b\}$ consists precisely of the pairs $(t_i, t_i^{-1}b)$. Thus, a suitable map s consists of compatible choices for $s(t_i, t_i^{-1}b) = (t_i, x_i)$ for $1 \leqslant i \leqslant n$, where compatibility requires $f(x_i) = t_i^{-1}b$.

An element of $\eta^*\mathrm{Map}_H(G,X)$ is given by a pair b, ψ, where ψ is an H-map $G \to X$. The identity $\psi(ht_i^{-1}) = h\psi(t_i^{-1})$ tells us that ψ is uniquely determined by the choice of elements

$x_i = \psi(t_i^{-1})$. Being in the pullback implies $\phi_b = f \circ \psi$, where ϕ_b is the map $G \to iB$ given by $\phi_b(g) = gb$. Thus our compatibility condition on the x_i is again $t_i^{-1}b = f(x_i)$.

A set isomorphism is now given by noting that the data in both cases is a choice of x_i for $1 \leqslant i \leqslant n$ satisfying $t_i^{-1}b = f(x_i)$. We must now check that the G-actions on the two sets agree. First, we see that $g(b, \psi) = (gb, g\psi)$, where $g\psi(g_1) = \psi(g_1 g)$. We define functions $h_i(g)$ and $\sigma_g(i)$ via the equation $t_i^{-1}g = h_i(g)t_{\sigma_g(i)}^{-1}$. This lets us evaluate $\psi(t_i^{-1})g = h_i(g)x_{\sigma_g(i)}$.

Next, we have $g(b, s) = (gb, gsg^{-1})$. We evaluate directly:

$$\begin{aligned} gsg^{-1}(t_i, t_i^{-1}gb) &= gs(g^{-1}t_i, t_i^{-1}gb) \\ &= gs(t_{\sigma_g(i)}h_i(g)^{-1}, t_i^{-1}gb) \\ &= gs(t_{\sigma_g(i)}, t_{\sigma_g(i)}^{-1}b) \\ &= g(t_{\sigma_g(i)}, x_{\sigma_g(i)}) \\ &= \left(t_i, h_i(g)x_{\sigma_g(i)}\right). \end{aligned}$$

Therefore our isomorphism α is equivariant.

Our isomorphism β takes elements of the form $(t_m, t_m^{-1}b), (b, s)$ (which is the general form for an element mapping to (b, s)) to $(t_m, (t_m^{-1}b, t_m^{-1}\psi_s))$, where $\alpha(b, s) = (b, \psi_s)$ as above. This makes the diagram commute, as $G \times_H \hat{\pi}$ takes $(t_m, (t_m^{-1}b, t_m^{-1}\psi_s))$ to $(t_m, t_m^{-1}\psi_s(e))$, which is evidently (t_m, x_m). □

2.3.1 Additive inverses

In light of Theorem 2.3.3, to check that the constructions N_H^G preserve the existence of additive inverses (i.e. that we can work in the categories of Mackey and Tambara functors instead of semi-Mackey and semi-Tambara functors), it suffices to check in the case of the coend $\mathrm{Lan}_{\mathrm{Map}_H(G,-)}(-)$.

Proposition 2.3.6. *The coend* $\mathrm{Lan}_{\mathrm{Map}_H(G,-)}(-)$ *takes Mackey functors to Mackey functors. That is, if* $\underline{M}(X)$ *is an abelian group for all finite* H*-sets* X *then* $\mathrm{Lan}_{\mathrm{Map}_H(G,-)}\underline{M}(Y)$ *is an abelian group for all finite* G*-sets* Y*.*

Proof. It suffices to check that inverses exist in the case $Y = G/K$, and we assume inductively that this is true for $Y = G/L$ for all $L \lneq K$. We note that transfers are additive, so any element transferred from a smaller subgroup inductively has inverses. Conceptually, we know that $N(b) + N(-b)$ should be equal to zero modulo transfers for any element b, so even though an exact formula for an inverse might be messy, we can use our inductive hypothesis to verify the existence of an inverse.

Our simplifications leave us needing to demonstrate that inverses exist for elements with representatives of the form

$$\big(R_\eta, y \in \underline{M}(iG/K)\big) = \big(\mathrm{Map}_H(G, i(G/K)) \leftarrow G/K = G/K, y\big)\,.$$

We then decompose iG/K into $H/(H \cap K) \amalg X$ for some H-set X, and use this to write y as a pair (b, x). This decomposition is necessary since our intended inverse is built around the element $(-b, x)$ instead of the element $-y = (-b, -x)$, just as the inverse of a tensor $a_1 \otimes a_2$ is given by $(-a_1) \otimes a_2$ instead of $(-a_1) \otimes (-a_2)$.

In the case $b = 0$ it is easy to see that $(R_\eta, (0, x))$ represents zero. We perform the following coend operation. Here χ is the map $iG/K \amalg H/(H \cap K) \to iG/K$ given by the identity on the first component and the inclusion on the second:

$$0 = (R_\eta, (0, x)) = (R_\eta, (b, x) + (-b, 0)) = (R_\eta T_{\mathrm{Map}_H(G,\chi)}, (b, x, -b))\,.$$

We now consider the following pullback:

$$\begin{tikzcd}
Z \arrow[r, "r"] \arrow[d] & \mathrm{Map}_H(G, iG/K \amalg H/(H \cap K)) \arrow[d, "\mathrm{Map}_H(G,\chi)"] \\
G/K \arrow[r, "\eta"'] & \mathrm{Map}_H(G, iG/K)
\end{tikzcd}$$

Every element of Z has a subgroup of (a conjugate of) K for its stabilizer. Inductively, we can ignore all nontrivial transfers in our answer, so we need only identify the orbits of Z of the form G/K.

We claim that there are exactly two of these orbits, and moreover that the restriction of the map r to these orbits is adjoint to the map $\hat{r}\colon iG/K \amalg iG/K \to iG/K \amalg H/(H \cap K)$ that takes iG/K identically to itself on the first coordinate, and breaks the second coordinate into the map

$$iG/K \cong X \amalg H/(H \cap K) \hookrightarrow iG/K \amalg H/(H \cap K)\,.$$

Given these claims, we see that zero is equal to the sum of $(R_\eta, (b, x))$ and $(R_\eta, (-b, x))$, modulo transfers, completing the argument. This follows since the above sum has the following representative, where ∇ is the fold map $G/K \amalg G/K \xrightarrow{\nabla} G/K$:

$$\begin{aligned}
&\big(\mathrm{Map}_H(G, i(G/K \amalg G/K)) \xleftarrow{\eta} G/K \amalg G/K \xrightarrow{\nabla} G/K, (b, x, -b, x)\big) \\
&= \big(T_\nabla R_\eta, (b, x, -b, x)\big) \\
&= \big(T_\nabla R_\eta, R_{\hat{r}}(b, x, -b)\big) \\
&= \big(T_\nabla R_\eta R_{\mathrm{Map}_H(G,\hat{r})}, (b, x, -b)\big)\,.
\end{aligned}$$

To check the claims, we again fix some set t_i of left coset representatives. An element of the pullback consists of an element $gK \in G/K$, and an H-map $\beta\colon G \to iG/K \amalg H/(H \cap K)$ such that $\chi\beta(g') = g'gK$. We choose the orbit representative with $g = e$, and note that β is determined by $\beta(t_i^{-1})$. If $\beta(t_i^{-1}) \in iG/K$, then it is forced to be $t_i^{-1}K$. Otherwise, it is some $h_i(H \cap K)$ such that $h_i K = t_i^{-1} K$, or equivalently $t_i h_i \in K$. Note that different such choices of elements h_i yield the same element of $H/(H \cap K)$.

Thus, β is completely determined by choosing the set of those i such that $\beta(t_i^{-1})$ lies in $H/H \cap K$ instead of iG/K. We note that this set consists only of those i such that for some h_i we have $t_i h_i \in K$. Thus, the set of all possible β corresponds to the subsets of the set I of those i such that a suitable h_i exists. The map β_S corresponding to a given subset $S \subset I$ is then given by requiring $\beta_S(t_i^{-1})$ to lie in $H/H \cap K$ if $i \in S$, and lie in iG/K otherwise.

The only choices of S such that the isotropy subgroup of β_S is all of K are the cases $S = \varnothing$ or $S = I$. To see that $k\beta_S = \beta_S$ in these cases, we note that $k\beta_S(t_i^{-1})$ is given by writing $t_i^{-1} k = h t_j^{-1}$ for some $h \in H$ and some j, and then we have

$$k\beta_S(t_i^{-1}) = \beta_S(t_i^{-1} k) = \beta_S(h t_j^{-1}) = h\beta_S(t_j^{-1})\,.$$

From the relation $k^{-1} t_i = t_j h^{-1}$, we see that $i \in I$ if and only if $j \in I$. In the case $S = \varnothing$, we see that $k\beta_S = \beta_S$ since the image in both cases lies entirely within iG/K. In the case $S = I$, we see that $k\beta_S = \beta_S$, since we have checked in this case that $k\beta_S(t_i^{-1})$ lies in $H/H \cap K$ for all $i \in I$.

For all other subsets S, choose some $i \in S$ and some $j \in I - S$. We choose suitable h_i, h_j, and then have for some $k \in K$ that $t_i h_i h_j^{-1} t_j^{-1} = k$. Now we have

$$k\beta_S(t_i^{-1}) = \beta_S(t_i^{-1} k) = \beta_S(t_i^{-1} t_i h_i h_j^{-1} t_j^{-1}) = h_i h_j^{-1} \beta_S(t_j^{-1})\,.$$

We now see that $\beta_S(t_i^{-1})$ lies within $H/(H \cap K)$, since $i \in S$. However, $k\beta_S(t_i^{-1})$ lies in iG/K, as $j \notin S$. Thus, k does not fix β.

The statement about restrictions follows immediately. $\square$

2.3.2 Comparison with Ullman's model

Here we write out Definition 5.18 of [22].

For any map $f\colon V \to W$ of finite H-sets, define a G-set $D_H(W, f, V)$ and maps e and p so that the bispan

$$G \underset{H}{\times} W \xleftarrow{e} D_H(W, f, V) \times G/H \xrightarrow{\pi_1} D_H(W, f, V) \xrightarrow{p} V$$

is the distributor for the composition $G \times_H W \to V \times G/H \to V$. Restricting e to the set $D_H(W, f, V) \times \{eH\}$ yields an H-map $e_H\colon D_H(W, f, V) \to i(G \times_H W)$.

Then for any H-Mackey functor $\underline{M}$ define a G-Mackey functor $N^{G,H}\underline{M}$ by letting $N^{G,H}\underline{M}(X)$ be the quotient of the free abelian group of pairs $(j\colon V \to X, u \in \underline{M}(i_H V))$ modulo the relations

1. $(j\colon V \to X, u) = (j'\colon V' \to X, u')$ when there is a commutative diagram

$$\begin{tikzcd} V \arrow[r, "j"] & X \\ V' \arrow[u, "f"', "\cong"] \arrow[ur, "j'"'] & \end{tikzcd}$$

   and $R_{i_H f} u = u'$,

2. $(j_1 \amalg j_2\colon V_1 \amalg V_2 \to X, (u_1, u_2)) = (j_1\colon V_1 \to X, u_1) + (j_2\colon V_2 \to X, u_2)$, and

3. $(j\colon V \to X, T_f(w)) = (j \circ p\colon D_H(W, f, V) \to X, R_{e_H}(w))$ for any finite H-set W and any H-map $f\colon W \to i_H V$.

A Mackey functor structure on $N^{G,H}\underline{M}$ is now given. The transfer $T_f(j\colon V \to X, u)$ corresponding to a map $f\colon X \to Y$ is given by the postcomposition $(f \circ j\colon V \to Y, u)$. The restriction associated to a map $f\colon Y \to X$ is defined by $R_f(j\colon V \to X, u) = (k\colon P \to Y, R_{i_H q}(u))$, where the following diagram is a pullback:

$$\begin{tikzcd} V \arrow[r, "j"] & X \\ P \arrow[u, "q"] \arrow[r, "k"'] & Y \arrow[u, "f"'] \end{tikzcd}.$$

In [22], Ullman proves that this yields a presentation of $\underline{\pi}_0(N_H^G \mathcal{H}\underline{M})$, where $\mathcal{H}\underline{M}$ is an appropriate model for the Eilenberg-Maclane spectrum corresponding to the Mackey functor $\underline{M}$ and N_H^G is the Hill-Hopkins-Ravenel norm [13]. We check that this topologically-motivated norm agrees with our construction.

Proposition 2.3.7. *There is an isomorphism of G-Mackey functors $N^{G,H}\underline{M} \cong N_H^G \underline{M}$.*

Proof. Our identification takes the generator $(j \colon V \to X, u)$ in $N^{H,G}\underline{M}(X)$ to the coend representative $(T_j R_\eta, u)$ in $N_H^G \underline{M}(X)$. Note that all elements in the target have a representative in this standard form.

To check that this yields a well-defined isomorphism, we need to see how the relations in $N^{G,H}\underline{M}$ correspond to relations in $N_H^G \underline{M}$. Relation (1) corresponds to the identifications in the coend coming from identifying spans in $\mathcal{A}_G$. Relation (2) corresponds to how addition is defined in $N_H^G \underline{M}$. Relation (3) corresponds to coend identifications coming from transfer maps as an application of Lemma 2.3.5.

The transfer and restriction maps are clearly compatible with these identifications. □

2.3.3 Multiplicative norms of Green functors

We now consider Green functors, which have two equivalent formulations. Note that we use the term "Green functor" where others might use the term "commutative Green functor".

The traditional definition uses the symmetric monoidal structure on the category of Mackey functors given by the box product.

For our purposes, we note that the box product can be explicitly constructed as a left Kan extension. The functor $\prod\colon G\text{-Set}^F \times G\text{-Set}^F \to G\text{-Set}^F$, which takes two G sets X, Y to their product $X \times Y$, preserves pullbacks, and thus induces a functor $\prod\colon \mathcal{A}_G \times \mathcal{A}_G \to \mathcal{A}_G$. Given any Mackey functors $\underline{P}, \underline{Q}$, the box product $\underline{P} \boxtimes \underline{Q}$ is formed as the left Kan extension of $\underline{P} \times \underline{Q}$ along the functor $\prod$. This is given explicitly by the coend formula

$$(\underline{P} \boxtimes \underline{Q})(Y) = \int^{(U,V) \in \mathcal{A}_G \times \mathcal{A}_G} \mathcal{A}_G(U \times V, Y) \times \underline{P}(U) \times \underline{Q}(V)\,.$$

The box product has a unit $\underline{A}$, whose value on orbits is given by the Burnside ring A_H. Since $\underline{A}$ is the initial object in the category of Tambara functors, it is preserved by the left adjoint N_H^G.

A Green functor is then a commutative monoid under the box product. Given a Green functor $\underline{R}$, the abelian group $\underline{R}(X)$ inherits the structure of a commutative ring, with the unit element being the image of the element $1 \in \underline{A}(X)$ under the unit map $\underline{A}(X) \to \underline{R}(X)$. There is a natural map $\underline{R}(X) \otimes \underline{R}(X) \to (\underline{R} \boxtimes \underline{R})(X)$, which upon postcomposition with the monoid multiplication $\underline{R} \boxtimes \underline{R} \to \underline{R}$ yields the ring multiplication of $\underline{R}(X)$. This natural map sends $x \otimes y$ to the representative $(X \times X \xleftarrow{\Delta} X = X,\, x, y)$ in the coend for $\underline{R} \boxtimes \underline{R}$.

The structure of the box product tells us that the multiplication, transfers and restriction must be compatible via a Frobenius reciprocity relation. Given a quotient map $f\colon G/K \to G/H$, the formula $a \cdot T_f(b) = T_f(R_f(a) \cdot b)$ must be satisfied for any $a \in \underline{R}(G/H), b \in \underline{R}(G/K)$.

We can also describe a Green functor via a diagram category, similar to our definition of Mackey and Tambara functors. Let $\mathcal{G}_G$ be the subcategory of $\mathcal{U}_G$ consisting of those bispans such that the middle map N_g is restricted to only allow G-maps g that are injective on orbits (alternatively formulated as g being a map that "preserves isotropy" in [20]). Note that g being injective on orbits does not mean that g is actually injective. One example is the fold map $G/H \amalg G/H \to G/H$. (For this to actually be a subcategory, one must check that this property is closed under composition, which is Proposition 12.4 of [20].)

A semi-Green functor is then a product-preserving functor $\mathcal{G}_G \to \mathrm{Set}$ (which naturally takes on values in commutative semirings), and a Green functor is one taking values in commutative rings. The equivalence of the two descriptions of Green functors is Proposition 12.11 of [20].

Since the Green functors are the commutative monoids under the box product, the existence of a norm functor N_H^G from H-Green functors to G-Green functors requires only the following:

Proposition 2.3.8. *The functor* $N_H^G \colon \mathrm{Mack}_H \to \mathrm{Mack}_G$ *is strong symmetric monoidal with respect to the box product.*

Proof. We first note that we have isomorphisms

$$\mathrm{Map}_H(G, A) \times \mathrm{Map}_H(G, B) \cong \mathrm{Map}_H(G, A \times B)$$

for all pairs of finite H-sets A, B. This induces a natural isomorphism

$$\prod \circ (\mathrm{Map}_H(G, -) \times \mathrm{Map}_H(G, -)) \cong \mathrm{Map}_H(G, -) \circ \prod$$

of functors $\mathcal{A}_H \times \mathcal{A}_H \to \mathcal{A}_G$. This yields a natural isomorphism between the corresponding left Kan extensions

$$N_H^G \underline{P} \boxtimes N_H^G \underline{Q} \cong N_H^G(\underline{P} \boxtimes \underline{Q})\,.$$

Note that the Burnside functor $\underline{A}$ is the initial object in the category of Green functors and the category of Tambara functors. The unit map $1 : \underline{A} \to N_H^G \underline{A}$ is then an isomorphism by Yoneda's Lemma.

We must check the compatibility diagram

$$\begin{tikzcd}
N_H^G \underline{M} \arrow[r, "\cong"] \arrow[d, "\cong"'] & N_H^G(\underline{A} \boxtimes \underline{M}) \arrow[d, "\cong"] \\
\underline{A} \boxtimes N_H^G \underline{M} \arrow[r, "1 \boxtimes \text{ id}"'] & N_H^G \underline{A} \boxtimes N_H^G \underline{M}
\end{tikzcd},$$

which commutes since given any $x \in N_H^G \underline{M}(X)$, its image in $N_H^G \underline{M} N_H^G \underline{A} \boxtimes N_H^G \underline{M}(X)$ is $1_X \otimes x$.

There are also corresponding associativity and symmetry diagrams, but they are trivial to verify.

□

An application of Proposition 2.3.8 and Theorem 2.3.3 shows us that our Tambara functor level norm N_H^G and our Green functor level norm N_H^G are compatible. This compatibility takes the form of a natural isomorphism $U_G N_H^G \cong N_H^G U_H$, where

$$U_G \colon \mathrm{Tamb}_G \to \mathrm{Green}_G \text{ and}$$
$$U_H \colon \mathrm{Tamb}_H \to \mathrm{Green}_H$$

are the forgetful functors.

2.3.4 A unit map for unital Mackey functors

For our characterization of Tambara functors in Section 2.7, we need to examine the unit map $\underline{R} \to i_H N_H^G \underline{R}$ given by the unit of the $N_H^G \dashv i_H$ adjunction in the category of Tambara functors. We check that this map can be extended to the category Mack_G^1 of unital G-Mackey functors, which are Mackey functors $\underline{M}$ equipped with a map $\underline{A} \to \underline{M}$. In particular, all

G-Green functors are unital G-Mackey functors.

As an aside, we note that we can define a subcategory $\mathcal{A}_G^1 \subset \mathcal{U}_G$ consisting of those bispans where the middle map N_g is required to be injective. We observe that injective maps are in particular injective on orbits, so we have the following inclusions of categories

$$\mathcal{A}_G \subset \mathcal{A}_G^1 \subset \mathcal{G}_G \subset \mathcal{U}_G \,,$$

corresponding to the chain of forgetful functors

$$\mathrm{Tamb}_G \to \mathrm{Green}_G \to \mathrm{Mack}_G^1 \to \mathrm{Mack}_G \,.$$

Here the inclusion $\mathcal{A}_G \subset \mathcal{A}_G^1$ takes a span $X \leftarrow A \to Y$ to the bispan $X \leftarrow A = A \to Y$. To see that $\mathcal{A}_G^1$ is a subcategory as claimed, we observe that pullbacks of inclusions are inclusions, as well as the following easy result.

Lemma 2.3.9. *The distributor for a composition of the form $X \to Y \hookrightarrow Z$ is given by*

$$X = X \hookrightarrow X \amalg (Z - Y) \to Z \,.$$

This allows us to redefine unital semi-Mackey functors as product-preserving functors $\mathcal{A}_G^1 \to \mathrm{Set}$, and unital Mackey functors as those such functors taking on values in abelian groups. This definition is equivalent to the notion of a Mackey functor $\underline{M}$ equipped with a map $\underline{A} \to \underline{M}$ by defining the norm map induced by an inclusion $X \to X \amalg Y$ to be given by sending any $x \in \underline{M}(X)$ to

$$(x, 1_Y) \in \underline{M}(X) \times \underline{M}(Y) \cong \underline{M}(X \amalg Y) \,,$$

where 1_Y is the image of $1_Y \in \underline{A}(Y)$ under the unit map $1\colon \underline{A} \to \underline{M}$.

Proposition 2.3.10. *There is a natural transformation $\tilde{\eta}\colon \mathrm{Id} \to i_H N_H^G$ of endofunctors of Mack_G^1. When restricted to Tambara functors, $\tilde{\eta}$ extends the unit map in the sense that the diagram*

$$\begin{tikzcd}
U_H\underline{R} \arrow[d, "U_H\eta"'] \arrow[dr, "\tilde{\eta}"] & \\
U_H i_H N_H^G \underline{R} \arrow[r, "\cong", "\alpha"'] & i_H N_H^G U_H \underline{R}
\end{tikzcd}$$

commutes for any Tambara functor $\underline{R}$, where α is the composite of the natural isomorphism $U_H i_H \cong i_H U_G$ with the natural isomorphism of Theorem 2.3.3. Here U_H and U_G are the corresponding forgetful functors.

Proof. To determine what the map $\tilde{\eta}$ should be, we first examine what the image is when $\underline{R}$ is Tambara. An element $x \in \underline{R}(X)$ is sent to the following coend representative under the unit η of the $i_H \dashv \mathrm{Lan}_{i_H}(-)$ adjunction:

$$\eta(x) = \left(G \underset{H}{\times} X = G \underset{H}{\times} X = G \underset{H}{\times} X = G \underset{H}{\times} X, x\right).$$

We factorize the identity map of $G \times_H X$ as the composition

$$G \times_H X \xrightarrow{G \underset{H}{\times} \hat{\eta}} G \underset{H}{\times} i(G \underset{H}{\times} X) \xrightarrow{\varepsilon} G \underset{H}{\times} X\,,$$

using the triangle identity for the $G \times_H (-) \dashv i$ adjunction. Here we write $\hat{\eta}$ for the unit map of the $G \times_H (-) \dashv i$ adjunction, since we are already using η to denote the unit maps of the $i \dashv \mathrm{Map}_H(G, -)$ and $i_H \dashv \mathrm{Lan}_{i_H}(-)$ adjunctions.

Examining the isomorphism of Theorem 2.3.3 then shows us that

$$\alpha(\eta(x)) = (R_\eta, N_{\hat{\eta}}x) = (\mathrm{Map}_H(G, i(G \underset{H}{\times} X)) \xleftarrow{\eta} G \underset{H}{\times} X = G \underset{H}{\times} X, N_{\hat{\eta}}x)\,.$$

in the Mackey-level coend for $N_H^G U_H \underline{R}$.

For any unital Mackey functor $\underline{M}$, this formula gives well-defined elements of $i_H N_H^G \underline{M}$, since the map $\hat{\eta}\colon X \to i(G \times_H X)$ is an inclusion. Thus, we define $\tilde{\eta}$ to send an element $x \in \underline{M}(X)$ to the element $(R_\eta, N_{\hat{\eta}} x)$ of $i_H N_H^G \underline{M}(X)$.

We must now check that $\tilde{\eta}$ preserves the structure maps of a unital Mackey functor. The unit map is just the composition $\underline{A} \to \underline{M} \to i_H N_H^G \underline{M}$.

Given a map $f\colon Y \to X$, we check that the restriction $R_f x$ in the domain maps to the restriction of the image of x in the target. We observe that the following square is a pullback:

$$\begin{tikzcd} Y \arrow[r, "\hat{\eta}"] \arrow[d, "f"'] & i(G \underset{H}{\times} Y) \arrow[d, "i(G \underset{H}{\times} f)"] \\ X \arrow[r, "\hat{\eta}"'] & i(G \underset{H}{\times} X) \end{tikzcd} .$$

This allows us the following manipulations, which give us compatibility with restrictions:

$$\begin{aligned} &(R_{G \times_H f} R_\eta, N_{\hat{\eta}} x) \\ &= (R_\eta R_{\mathrm{Map}_H(G, i(G \times_H f))}, N_{\hat{\eta}} x) \\ &= (R_\eta, R_{i(G \times_H f)} N_{\hat{\eta}} x) \\ &= (R_\eta, N_{\hat{\eta}} R_f x)\,. \end{aligned}$$

Next, we take a transfer map $f\colon X \to Y$, and see by Lemma 2.3.9 that

$$X \xrightarrow{j} X \amalg \big(i(G \underset{H}{\times} Y) - Y\big) \xrightarrow{f_0} i(G \underset{H}{\times} Y) \cong Y \amalg \big(i(G \underset{H}{\times} Y) - Y\big)\,.$$

is a distributor corresponding to the composition $N_{\hat{\eta}} T_f$. Here j is the obvious inclusion and f_0 is given by f on X and the identity on $i(G \times_H Y)$.

Let $\phi\colon i(G \times_H X) \cong X \amalg (i(G \times_H X) - X) \to X \amalg (i(G \times_H Y) - Y)$ be given by the identity on the first component and the map $(g, x) \mapsto (g, f(x))$ on the second component. Then

$$\begin{tikzcd}
X \arrow[r, "\hat{\eta}"] \arrow[d, equal] & i(G \underset{H}{\times} X) \arrow[d, "\psi"] \\
X \arrow[r, "j"'] & X \amalg \big(i(G \underset{H}{\times} Y) - Y\big)
\end{tikzcd}$$

is clearly a pullback square.

We claim that

$$\begin{tikzcd}
G \underset{H}{\times} X \arrow[r, "\hat{\psi}"] \arrow[d, "G \underset{H}{\times} f"'] & \mathrm{Map}_H\big(G, X \amalg \big(i(G \underset{H}{\times} Y) - Y\big)\big) \arrow[d, "\mathrm{Map}_H(G, f_0)"] \\
G \underset{H}{\times} Y \arrow[r, "\eta"'] & \mathrm{Map}_H(G, i(G \underset{H}{\times} Y))
\end{tikzcd} \tag{2.3.1}$$

is also a pullback square, where $\hat{\psi}$ is the adjoint to ψ.

Assuming this claim, we check compatibility with transfers:

$$\begin{aligned}
&(R_\eta, N_{\hat{\eta}} T_f x) \\
&= (R_\eta, T_{f_0} N_j x) \\
&= (R_\eta T_{\mathrm{Map}_H(G, f_0)}, N_j x) \\
&= (T_{G \times_H f} R_{\hat{\psi}}, N_j x) \\
&= (T_{G \times_H f} R_\eta R_{\mathrm{Map}_H G, \psi}, N_j x) \\
&= (T_{G \times_H f} R_\eta, R_\psi N_j x) \\
&= (T_{G \times_H f} R_\eta, N_{\hat{\eta}} x)\,.
\end{aligned}$$

We now check that the diagram (2.3.1) is a pullback square, completing the proof.

An element of the pullback is given by a triple (g, y, β), where $\beta\colon G \to X \amalg (i(G \times_H Y) - Y)$ is an H-map. The compatibility conditions require $f(\beta(g')) = g'gx \in X$ if $g'g \in H$ and $\beta(g') = (g'g, y) \in i(G \times_H Y) - Y$ if $g'g \notin H$. This data can then be reduced to g and the value $x = \beta(g^{-1})$, and this data determines an element of $G \times_H X$, as desired. It does not matter which element of the left coset gH is chosen. The projection to $G \times_H Y$ takes (g, x) to $(g, f(x))$ and the projection to $\mathrm{Map}_H(G, X \amalg (i(G \underset{H}{\times} Y) - Y))$ takes (g, x) to the function

$$g' \mapsto \psi(g'g, x) = \begin{cases} g'gx \in X & \text{if } g'g \in H \\ (g'g, f(x)) \in i(G \underset{H}{\times} Y) - Y & \text{if } g'g \notin H\,. \end{cases}$$

□

2.4 Weyl actions and a G-symmetric monoidal structure on $\mathbf{Mack}_G$

The notion of "G-symmetric monoidal category" has been outlined in the literature [18, 22] to the extent required for the definition of a G-commutative monoid, which generalizes the notion of a commutative monoid in a symmetric monoidal category. The essential point is to define $\underline{M}^{\otimes T}$ for any finite G-set T and G-Mackey functor $\underline{M}$, as well as structural isomorphisms $\underline{M}^{\otimes T} \to \underline{M}^{\otimes T'}$ for any G-isomorphism $T \cong T'$. In particular, the Weyl group $W_G H = N_G H/H$ acts on $\underline{M}^{\otimes G/H}$.

For any G-sets T, S, there is a G-action by conjugation on the set $\mathrm{Map}(T, S)$ of all maps $T \to S$. For any G-set T, we see that the functor $\mathrm{Map}(T, -)\colon G\text{-Set} \to G\text{-Set}$ preserves pullbacks, so it induces a functor $\mathcal{A}_G \to \mathcal{A}_G$.

Definition 2.4.1. *The functor* $(-)^{\otimes T}\colon \mathrm{Mack}_G \to \mathrm{Mack}_G$ *is given by left Kan extension along* $\mathrm{Map}(T, -)\colon \mathcal{A}_G \to \mathcal{A}_G$. *Concretely, for any G-Mackey functor $\underline{M}$ we have a coend* *formula for $\underline{M}^{\otimes T}$:*

$$\underline{M}^{\otimes T}(Y) := \mathrm{Lan}_{\mathrm{Map}(T,-)}\underline{M}(Y) = \int^{X \in \mathcal{A}_G} \mathcal{A}_G(\mathrm{Map}(T,X),Y) \times \underline{M}(X)\,.$$

The set-level natural isomorphisms

$$\begin{aligned} \mathrm{Map}(T_1 \amalg T_2, S) &\cong \mathrm{Map}(T_1,S) \times \mathrm{Map}(T_2,S) \\ \mathrm{Map}(T, S_1 \times S_2) &\cong \mathrm{Map}(T,S_1) \times \mathrm{Map}(T,S_2) \\ \mathrm{Map}(T_2, \mathrm{Map}(T_1,S)) &\cong \mathrm{Map}(T_1 \times T_2, S) \end{aligned}$$

immediately induce corresponding natural isomorphisms of Mackey functors:

$$\begin{aligned} \underline{M}^{\otimes (T_1 \amalg T_2)} &\cong \underline{M}^{\otimes T_1} \boxtimes \underline{M}^{\otimes T_2} \\ (\underline{M}_1 \boxtimes \underline{M}_2)^{\otimes T} &\cong \underline{M}_1^{\otimes T} \boxtimes \underline{M}_2^{\otimes T} \\ (\underline{M}^{\otimes T_1})^{\otimes T_2} &\cong \underline{M}^{\otimes (T_1 \times T_2)}\,. \end{aligned}$$

Since $\mathrm{Map}(T,-)$ does not preserve products in $\mathcal{A}_G$, the left Kan extension does not yield a left adjoint.

We now concretely relate this structure to our norm construction.

Proposition 2.4.2. *There is a natural isomorphism $\beta\colon N_H^G i_H \underline{M} \cong \underline{M}^{\otimes G/H}$ of G-Mackey functors.*

Now we note that the construction of $\underline{M}^{\otimes T}$ is a priori only a semi-Mackey functor. However, Propositions 2.4.2 and 2.3.6 imply that this construction preserves additive inverses and thus takes on values in Mackey functors.

Definition 5.2 of Ullman [22] gives an explicit algebraic presentation of $\underline{M}^{\otimes T}$. Again, as in Proposition 2.3.7 we see that this presentation agrees with our construction. The topological analogue of Proposition 2.4.2 is obvious since both sides are defined via the same indexed smash product in a nice category of G-spectra.

If T is a trivial G-set with n elements, then $\underline{M}^{\otimes T}$ is isomorphic to the n-fold box product $\underline{M}^{\boxtimes n}$ and the corresponding structural isomorphisms are the action of the symmetric group Σ_n given by permutation of factors.

More generally, if we have a decomposition $T \cong \coprod_i G/H_i$, then we get a corresponding decomposition of $\underline{M}^{\otimes T}$:

$$\underline{M}^{\otimes T} \cong \mathop{\boxtimes}_i N_{H_i}^G(i_{H_i}\underline{M}) .$$

Any map of finite G-sets $f\colon T \to T'$ induces a natural transformation

$$f^*\colon \mathrm{Map}(T', -) \to \mathrm{Map}(T, -)$$

of endofunctors of G-Set. However, these maps do not induce natural tranformations $\mathrm{Map}(T', -) \to \mathrm{Map}(T', -)$ of endofunctors of $\mathcal{A}_G$ except in the case of isomorphisms, in which case the naturality squares in G-Set are pullbacks. This case induces natural isomorphisms $\underline{M}^{\otimes T} \cong \underline{M}^{\otimes T'}$ for any isomorphism $T \cong T'$.

Proof of Proposition 2.4.2: We first let ζ denote the natural isomorphism

$$\zeta : G/H \times (-) \cong G \times_H i(-) .$$

Fix a G-set Y and start with an arbitrary element of $N_H^G i_H \underline{M}(Y)$. This element has a standard form representative

$$(T_h R_\eta, b) = (\mathrm{Map}_H(G, iB) \xleftarrow{\eta} B \xrightarrow{h} Y, b \in \underline{M}(G \underset{H}{\times} iB)) ,$$

which we send to the element in the coend for $\underline{M}^{\otimes T}$ with the following representative, where

η' is the unit of the adjunction $G/H \times (-) \dashv \mathrm{Map}(G/H, -)$:

$$\beta(T_h R_\eta, b) := (T_h R'_\eta, R_\zeta b) = (\mathrm{Map}(G/H, G/H \times B) \xleftarrow{\eta'} B \xrightarrow{h} Y, R_\zeta b \in \underline{M}(G/H \times B)).$$

We first observe that this target element is in a similar universal form for elements of the target coend, so that β is an isomorphism provided that it is well-defined.

As in the proof of Theorem 2.3.3, the only real work remaining is to ensure that β respects the coend identifications coming from transfer maps. We fix an H-map $f\colon X \to iB$ and some $x \in \underline{M}(G \times_H X)$, and show that the images under β of elements of the form $(T_h R_\eta, T_{G \times_H f} x)$ and $(T_h R_\eta T_{\mathrm{Map}_H(G,f)}, x)$ are identified in the target.

We recall that we denote the unit of the $G \times_H (-) \dashv i$ adjunction by $\hat{\eta}$, and also introduce the notation $\hat{f}\colon G \times_H X \to G/H \times B$ for the composite $\zeta^{-1} \circ (G \times_H f)$.

We now define the G-set Z and maps π_1, π_2 such that the left square in the following commutative diagram is a pullback. Here $\hat{\pi}_2\colon iZ \to X$ is the adjoint to π_2:

$$\begin{tikzcd}
 & & \mathrm{Map}(G/H, G/H \times Z) \arrow[drr, "\mathrm{Map}(G/H, (G \underset{H}{\times} \hat{\pi}_2) \circ \zeta)"] & & \\
Z \arrow[urr, "\eta'"] \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"'] & \mathrm{Map}_H(G, X) \arrow[r, "\mathrm{Map}(G, \hat{\eta})"'] \arrow[d] & \mathrm{Map}_H(G, i(G \underset{H}{\times} X)) \arrow[rr, "\cong"'] \arrow[d] & & \mathrm{Map}(G/H, G \underset{H}{\times} X) \arrow[d, "\mathrm{Map}(G/H, \hat{f})"] \\
B \arrow[r, "\eta"'] & \mathrm{Map}_H(G, iB) \arrow[r, "\mathrm{Map}(G, \hat{\eta})"'] & \mathrm{Map}_H(G, i(G \underset{H}{\times} iB)) \arrow[rr, "\cong"'] & & \mathrm{Map}(G/H, G/H \times B)
\end{tikzcd}.$$

The naturality squares for η' are pullbacks, and the functor $\mathrm{Map}_H(G, -)$ preserves pullbacks. Thus, the bottom rectangle is also a pullback. We also observe that the lower composite is the unit map $\eta'\colon B \to \mathrm{Map}(G/H, G/H \times B)$.

We now perform the following coend manipulations using the above diagram:

$$\begin{aligned}
\beta(T_h R_\eta, T_{G\times_H f}x) &= (T_h R_{\eta'}, R_\zeta T_{G\times_H f}x) \\
&= (T_h R_{\eta'}, T_{\zeta^{-1}} T_{G\times_H f}x) \\
&= (T_h R_{\eta'}, T_{\hat{f}}x) \\
&= (T_h R_{\eta'} T_{\mathrm{Map}(G/H,\hat{f})}, x) \\
&= (T_h T_{\pi_1} R_{\eta'} R_{\mathrm{Map}(G/H,(G\times_H \hat{\pi}_2)\circ\zeta)}, x) \\
&= (T_{h\pi_1} R_{\eta'}, R_\zeta R_{G\times_H \hat{\pi}_2}, x)
\end{aligned}$$

To evaluate β on the coend representative $(T_h R_\eta T_{\mathrm{Map}_H(G,f)}, x)$, we must first convert this representative to one in standard form:

$$\begin{aligned}
(T_h R_\eta T_{\mathrm{Map}_H(G,f)}, x) &= (T_h T_{\pi_1} R_{\pi_2}, x) \\
&= (T_{h\pi_1} R_\eta R_{\mathrm{Map}_H(G,\hat{\pi}_2)}, x) \\
&= (T_{h\pi_1} R_\eta, R_{G\times_H \hat{\pi}_2}x)
\end{aligned}$$

The map β takes this element to the representative $(T_{h\pi_1} R_{\eta'}, R_\zeta R_{G\times_H \hat{\pi}_2}, x)$, as desired. $\square$

2.4.1 Conjugation functors and an alternate form of the Weyl action

As a result of Proposition 2.4.2, for any isomorphism $\gamma\colon G/H \to G/H'$, we get a natural isomorphism of the form

$$w_\gamma\colon N_H^G i_H \underline{M} \cong N_{H'}^G i_{H'} \underline{M}\,.$$

In particular, for $H = H'$ we get an action of the Weyl group $W_G H$ on $N_H^G i_H \underline{M}$ through isomorphisms of Mackey functors. This Weyl action is analogous to the corresponding action on the indexed sum $G \times_H (i_H M)$ and the indexed product $\mathrm{Map}_H(G, i_H M)$ of any G-set M.

A similar Weyl action on $N_H^G i_H R$ for R a commutative orthogonal ring G-spectrum follows from Corollary 2.14 of [13].

To specify these conjugation isomorphisms w_γ concretely, we start by choosing some G-isomorphism $\gamma\colon G/H' \to G/H$. This sends any coset gH' to the coset $g\gamma H$, for some conjugacy relation $\gamma H \gamma^{-1} = H'$. Such a map exists for any element γ in the normal transporter $N_G(H', H)$, although the map depends only on the double coset in $H' \backslash N_G(H', H)/H$.

We now introduce the notation ${}^\gamma H = \gamma H \gamma^{-1}, H^\gamma = \gamma^{-1} H \gamma$ for conjugate subgroups to a subgroup $H \leqslant G$ and an element $\gamma \in G$.

Definition 2.4.3. *Given any G-set B let $\hat{\gamma}\colon G \times_{{}^\gamma H} B \to G \times_H B$ denote the isomorphism given elementwise by taking (g, b) to $(g\gamma, \gamma^{-1} b)$.*

This is a "twisted" version of the map $G/{}^\gamma H \times B \to G/H \times B$. Given a G-Mackey functor M, we then get a restriction map $R_{\hat{\gamma}}\colon \underline{M}(G \times_H B) \to \underline{M}(G \times_{{}^\gamma H} B)$. This can also be viewed as a map $i_H \underline{M}(i_H B) \to i_{{}^\gamma H} \underline{M}(i_{{}^\gamma H} B)$. Note that in the case $B = G/G$, this map induces the standard conjugation map $c_\gamma\colon \underline{M}(G/H) \to \underline{M}(G/{}^\gamma H)$, which yields the usual Weyl action when $\gamma \in N_G(H)$.

We now introduce a general construction to further decompose the isomorphism $\hat{\gamma}$.

Definition 2.4.4. *For any $\gamma \in G$, there is a equivalence of categories $\gamma \cdot (-)\colon H\text{-Set} \to {}^\gamma H\text{-Set}$ given on objects by taking an H-set X to the set $\gamma \cdot X$ of formal symbols $\gamma \cdot x$, with the ${}^\gamma H$-action given by $\gamma h \gamma^{-1}(\gamma \cdot x) = \gamma \cdot hx$. On morphisms $\gamma \cdot f$ takes $\gamma \cdot x$ to $\gamma \cdot f(x)$. There exist natural isomorphisms as follows.*

- *In the case $H = G$, there is a natural isomorphism $X \cong \gamma \cdot X$ given by sending x to $\gamma \cdot \gamma^{-1} x$.*

- *Given $H \leqslant K \leqslant G$, there is a natural isomorphism $\gamma \cdot (K \times_H X) \cong {}^\gamma K \times_{{}^\gamma H} \gamma \cdot X$ given by taking $\gamma \cdot (k, x)$ to $(\gamma k \gamma^{-1}, \gamma \cdot x)$.*

- *Given $H \leqslant K \leqslant G$, there are natural isomorphisms of functors of the following form, where $i_H\colon K\text{-Set} \to H\text{-Set}$ is the forgetful functor:*

$$\gamma \cdot i_H(-) \cong i_{\gamma H}(\gamma \cdot (-))$$
$$\gamma \cdot \mathrm{Map}_H(K, -) \cong \mathrm{Map}_{\gamma H}({}^{\gamma}K, \gamma \cdot (-))\,.$$

We observe that the map $\hat{\gamma}\colon G \times_{\gamma H} B \to G \times_H B$ of Definition 2.4.3 is given by the composition

$$G \underset{\gamma H}{\times} i_{\gamma H} B \cong G \underset{\gamma H}{\times} i_{\gamma H}(g \cdot B) \cong G \underset{\gamma H}{\times} \gamma \cdot (i_H B) \cong g \cdot (G \underset{H}{\times} i_H B) \cong G \underset{H}{\times} i_H B\,.$$

We now consider the context of Mackey functors.

Proposition 2.4.5. *There is an equivalence of categories $c_\gamma\colon \mathrm{Mack}_H \to \mathrm{Mack}_{\gamma H}$ given by setting $c_\gamma \underline{M}(X) = \underline{M}(\gamma^{-1} \cdot X)$ for any finite ${}^{\gamma}H$-set X. There then exist natural isomorphisms of the following form.*

- *In the case $H = G$, there is a natural isomorphism $c_\gamma \cong \mathrm{Id}$.*
- *Given $H \leqslant K \leqslant G$, there is a natural isomorphism $c_\gamma i_H(-) \cong i_{\gamma H} c_\gamma(-)$ of functors $\mathrm{Mack}_K \to \mathrm{Mack}_{\gamma H}$.*
- *Given $H \leqslant K \leqslant G$, there is a natural isomorphism $c_\gamma N_H^K(-) \cong N_{\gamma H}^{\gamma K} c_\gamma(-)$ of functors $\mathrm{Mack}_H \to \mathrm{Mack}_{\gamma K}$.*

Proof. Since the functor $\gamma^{-1} \cdot (-)$ preserves pushouts and disjoint unions, it induces a functor $\mathcal{A}_{\gamma H} \to \mathcal{A}_H$. Precomposition along this functor gives a well-defined functor at the Mackey level, and the first two isomorphisms follow from the first two isomorphisms of Definition 2.4.4.

To define the third isomorphism, we fix a finite ${}^{\gamma}H$-set Y and an H-Mackey functor $\underline{M}$ and write an element of $c_\gamma N_H^K \underline{M}(Y) \cong N_H^K \underline{M}(\gamma^{-1} \cdot Y)$ in standard form as

$$\big(T_h R_\eta, x \in \underline{M}(i_H B)\big) \ ,$$

where B is some finite K-set and $h\colon B \to \gamma^{-1} \cdot Y$ is some K-map.

The image of this element $N_{\gamma H}^{\gamma K} c_\gamma \underline{M}(X)$ under our proposed isomorphism is then the element

$$\Big(T_{\gamma \cdot h} R_\eta, x \in \underline{M}(\gamma^{-1} \cdot i_{\gamma H}(\gamma \cdot B)) \cong \underline{M}(i_H B)\Big) \ .$$

We must demonstrate that this respects coend identifications induced by transfer maps. This boils down to the following isomorphism for any H-map of the form $Y \to i_H B$:

$$\gamma \cdot \left(\mathrm{Map}_H(K, Y) \underset{\mathrm{Map}_H(K, i_H B)}{\times} B \right) \cong \mathrm{Map}_{\gamma H}({}^{\gamma}K, \gamma \cdot Y) \underset{\mathrm{Map}_{\gamma H}({}^{\gamma}K, i_{\gamma H}(\gamma \cdot B))}{\times} \gamma \cdot B \, .$$

This isomorphism follows from composing the isomorphisms of Definition 2.4.4 with the fact that $\gamma \cdot (-)$ preserves pullbacks:

Checking that our proposed isomorphism is a map of Mackey functors is straightforward.

$\square$

Chasing definitions now gives us a concrete characterization of the maps w_γ.

Corollary 2.4.6. *The map $w_\gamma\colon N_H^G i_H \underline{M} \cong N_{\gamma H}^G i_{\gamma H} \underline{M}$ has the following decomposition:*

$$N_H^G i_H \underline{M} \cong c_\gamma N_H^G i_H \underline{M} \cong N_{\gamma H}^G c_\gamma i_H \underline{M} \cong N_{\gamma H}^G i_{\gamma H} c_\gamma \underline{M} \cong N_{\gamma H}^G i_{\gamma H} \underline{M} \, .$$

An element in $N_H^G i_H \underline{M}(Y)$ with standard form representative

$$(T_h R_\eta, b) = \big(\mathrm{Map}_H(G, i_H B) \xleftarrow{\eta} B \xrightarrow{h} Y, \, b \in i_H \underline{M}(i_H B)\big)$$

is taken by w_γ to the element of $N^G_{\gamma H} i_{\gamma H}\underline{M}(Y)$ with standard form representative

$$(T_h R_\eta, R_{\hat\gamma} b) = \left(\mathrm{Map}_{\gamma H}(G, i_{\gamma H}B) \xleftarrow{\eta} B \xrightarrow{h} Y,\ R_{\hat\gamma} b \in i_{\gamma H}\underline{M}(i_{\gamma H}B)\right).$$

Here $\hat\gamma$ is the isomorphism of Definition 2.4.3.

2.5 A double coset formula for norms of Mackey functors

As motivation, we examine the structural isomorphisms $(\underline{M}^{\otimes T})^{\otimes T'} \cong \underline{M}^{\otimes(T\times T')}$ in the case of $T = G/H$, $T' = G/K$ for $H, K \leqslant G$. In light of Proposition 2.4.2 these can be rewritten to be of the form

$$N^G_K i_K N^G_H i_H \underline{M} \cong \underset{\gamma}{\boxtimes} N^G_{K^\gamma \cap H}(i_{K^\gamma\cap H}\underline{M}).$$

Here $K^\gamma = \gamma^{-1}K\gamma$, and γ ranges over a set of double coset representatives for $K\backslash G/H$. This reflects the decomposition of $G/K \times G/H$ into orbits.

These isomorphisms reflect a more general decomposition.

Theorem 2.5.1. *Let $H, K \leqslant G$ and $\underline{M}$ be an H-Mackey functor. Then there is a natural isomorphism*

$$i_K N^G_H \underline{M} \cong \underset{\gamma}{\boxtimes} N^K_{K\cap {}^\gamma H}(i_{K\cap {}^\gamma H}(c_\gamma \underline{M})).$$

Here γ ranges over a complete set of double coset representatives for $K\backslash G/H$ and c_γ are the conjugation functors of Proposition 2.4.5.

Combining this theorem with the structural isomorphisms of Proposition 2.4.5 allows us to construct the following decompositions that do not involve the functors c_γ:

$$i_K N^G_H i_H \underline{M} \cong \underset{\gamma}{\boxtimes} N^K_{K\cap {}^\gamma H}(i_{K\cap {}^\gamma H}\underline{M}) \text{ for } \underline{M} \in \mathrm{Mack}_G \text{ , and}$$

$$N^G_K i_K N^G_H \underline{M} \cong \underset{\gamma}{\boxtimes} N^G_{K^\gamma \cap H}(i_{K^\gamma\cap H}\underline{M}) \text{ for } \underline{M} \in \mathrm{Mack}_H\,.$$

We remark that the unit map of Proposition 2.3.10 can be redescribed as the map

$$\underline{M} \boxtimes \underline{A}^{\boxtimes n-1} \to \boxed{\times}_{\gamma} N_{H \cap {}^{\gamma}H}^{H}(i_{H \cap {}^{\gamma}H}(c_{\gamma}\underline{M}))$$

given by the identity on the first coordinate (corresponding to the double coset HeH) and the unit on the rest, where $n = |H \backslash G/H|$.

Proof of Theorem 2.5.1: Fix $H, K \leqslant G$. We set $\gamma_1, \ldots, \gamma_n$ to be a complete set of double coset representatives for $K \backslash G/H$, where $n = |K \backslash G/H|$. We first describe an analogous natural isomorphism on the level of finite H-sets, for B any finite K-set:

$$\hat{\theta}\colon \coprod_{j=1}^{n} \gamma_j^{-1} \cdot \left({}^{\gamma_j}H \underset{K \cap {}^{\gamma_j}H}{\times} B \right) \cong i_H \left(G \underset{K}{\times} B \right).$$

In the j-th component, the map $\hat{\theta}$ takes a pair $\gamma_j^{-1} \cdot (\gamma_j h \gamma_j^{-1}, x)$ to the pair $(h\gamma_j^{-1}, x)$. For any element $g \in G$, we first write $g = h\gamma_j^{-1}k$ for some j, and then the inverse map $\hat{\theta}^{-1}$ takes a pair of the form $(g, x) = (h\gamma_j^{-1}k, x)$ to the element $\gamma_j^{-1} \cdot (\gamma_j h \gamma_j^{-1}, kx)$ in the j-th component.

We now fix a K-set Y and take an arbirary element in the coend for $i_K(N_H^G\underline{M})$. This element has a standard form representative

$$(T_{G \times_K h} R_{\eta}, b) = \left(\mathrm{Map}_H(G, i_H(G \underset{K}{\times} B)) \xleftarrow{\eta} G \underset{K}{\times} B \xrightarrow{G \times_K h} G \underset{K}{\times} Y, b \in \underline{M}(i_H(G \underset{K}{\times} B)) \right),$$

which is taken by our proposed map θ to the following element in the composite coend for $\boxtimes_j N_{K \cap {}^{\gamma_j}H}^{K}(i_{K \cap {}^{\gamma_j}H}(c_{\gamma_j}\underline{M}))$:

$$\begin{aligned} \theta(T_{G \times_K h} R_{\eta}, b) &:= (T_h R_{\Pi_j \eta_j}, R_{\hat{\theta}} b) \\ &= \left(\prod_{j=1}^{n} \mathrm{Map}_{K \cap {}^{\gamma_j}H}(K, i_{K \cap {}^{\gamma_j}H} B) \xleftarrow{\eta_1 \times \cdots \times \eta_n} B \xrightarrow{h} Y, R_{\hat{\theta}} b \right). \end{aligned}$$

We observe that this element is in a similarly universal form for elements in the target, so that θ is an isomorphism provided that it is well-defined. We implicitly using the canonical isomorphism to view $R_{\hat{\theta}} b$ as an element of the correct product of modules:

$$R_{\hat{\theta}} b \in \underline{M}\left(\coprod_{j=1}^{n} \gamma_j^{-1} \cdot \left({}^{\gamma_j}H \underset{K\cap {}^{\gamma_j}H}{\times} B\right)\right) \cong \prod_{j=1}^{n} i_{K\cap {}^{\gamma_j}H}(c_{\gamma_j}\underline{M}\,(B)) \,.$$

As in the proof of Theorem 2.3.3, the only real work remaining is to check that θ respects the coend identifications induced by transfer maps. We fix a K-map $f\colon X \to i_H G \times_K B$, and define $K \cap {}^{\gamma_j}H$-sets X_j and maps $f_j\colon X_j \to i_{K\cap {}^{\gamma_j}H} B$ such that a commutative diagram exists of the form

$$\begin{tikzcd}
X \arrow[rr, "f"] & & i_H(G \underset{K}{\times} B) \\
\coprod_j \gamma_j^{-1} \cdot \left({}^{\gamma_j}H \underset{K\cap {}^{\gamma_j}H}{\times} X_j\right) \arrow[u, "\cong"] \arrow[rr, "\coprod_j \gamma_j^{-1} \cdot \left({}^{\gamma_j}H \underset{K\cap {}^{\gamma_j}H}{\times} f_j\right)"'] & & \coprod_j \gamma_j^{-1} \cdot \left({}^{\gamma_j}H \underset{K\cap {}^{\gamma_j}H}{\times} B\right) \arrow[u, "\cong"', "\hat{\theta}"]
\end{tikzcd}.$$

We define a K-set $\hat{X}$ and maps π, ψ by the following pullback square:

$$\begin{tikzcd}
\hat{X} \arrow[r, "\psi"] \arrow[d, "\pi"'] & \prod_j \mathrm{Map}_{K\cap {}^{\gamma_j}H}(K, X_j) \arrow[d, "\prod_j \mathrm{Map}_{K\cap {}^{\gamma_j}H}(K, f_j)"] \\
B \arrow[r, "\eta_1 \times \cdots \times \eta_n"'] & \prod_j \mathrm{Map}_{K\cap {}^{\gamma_j}H}(K, B)
\end{tikzcd}.$$

Now let $\psi_j\colon \hat{X} \to X_j$ denote the adjoint to the j-th component of ψ. Define a map $\hat{\psi}\colon G \underset{K}{\times} \hat{X} \to \mathrm{Map}_H(G, X)$ as the adjoint to the map $\bar{\psi}\colon i_H(G \times_K \hat{X}) \to X$ defined by the composition

$$i_H\left(G\underset{K}{\times}\hat{X}\right) \cong \coprod_{j=1}^{n} \gamma_j^{-1}\cdot\left({}^{\gamma_j}H \underset{K\cap{}^{\gamma_j}H}{\times} \hat{X}\right) \xrightarrow{\psi_j} \coprod_{j=1}^{n} \gamma_j^{-1}\cdot\left({}^{\gamma_j}H \underset{K\cap{}^{\gamma_j}H}{\times} X_j\right) \cong X\,.$$

We claim that

$$\begin{tikzcd} G\underset{K}{\times}\hat{X} \arrow[r, "\hat{\psi}"] \arrow[d, "G\underset{K}{\times}\pi"'] & \mathrm{Map}_H(G,X) \arrow[d, "\mathrm{Map}_H(G,f)"] \\ G\underset{K}{\times}B \arrow[r, "\eta"'] & \mathrm{Map}_H(G,G\underset{K}{\times}B) \end{tikzcd} \tag{2.5.1}$$

is a pullback diagram.

Assuming this claim, we show that $\theta(T_{G\times_K h}R_\eta, T_f x)$ and $\theta(T_{G\times_K h}R_\eta T_{\mathrm{Map}_H(G,f)}, x)$ are identified in the target. We start with the computation

$$\begin{aligned}\theta(T_{G\times_K h}R_\eta T_{\mathrm{Map}_H(G,f)}, x) &= \theta(T_{G\times_K h}T_{G\times_K \pi}R_{\hat{\psi}}, x)\\ &= \theta(T_{G\times_K (h\pi)}R_\eta R_{\mathrm{Map}_H(G,\bar{\psi})}, x)\\ &= \theta(T_{G\times_K (h\pi)}R_\eta, R_{\bar{\psi}}x)\\ &= (T_{h\pi}R_{\Pi_j \eta_j}, R_{\hat{\theta}}R_{\bar{\psi}}x)\,.\end{aligned}$$

To compute the other image, we let $x_j \in i_{K\cap{}^{\gamma_j}H}(c_{\gamma_j}\underline{M}\left(X_j\right))$ denote the j-th component of the following decomposition:

$$x \in \underline{M}(X) \cong \prod_{j=1}^{n} i_{K\cap{}^{\gamma_j}H}(c_{\gamma_j}\underline{M}\left(X_j\right))\,.$$

We have that $R_{\hat{\theta}}T_f$ is given by $T_{f_j}x_j$ in the j-th component, and that $R_{\hat{\theta}}R_{\bar{\psi}}x$ is given by

$R_{\psi_j} x_j$ in the j-th component. We evaluate

$$\begin{aligned}
\theta(T_{G\times_K h} R_\eta, T_f x) &= (T_h R_{\Pi_j \eta_j}, R_{\hat{\theta}} T_f x) \\
&= (T_h R_{\Pi_j \eta_j}, (T_{f_j} x_j)_{j=1}^n) \\
&= (T_h R_{\Pi_j \eta_j} T_{\Pi_j \mathrm{Map}_{K\cap^{\gamma_j} H}(K, f_j)}, (x_j)_{j=1}^n) \\
&= (T_h T_\pi R_\psi, (x_j)_{j=1}^n) \\
&= (T_{h\pi} R_{\Pi_j \eta_j}, (R_{\psi_j} x_j)_{j=1}^n) \\
&= (T_{h\pi} R_{\Pi_j \eta_j}, R_{\hat{\theta}} R_{\bar{\psi}} x)\,.
\end{aligned}$$

We now check that (2.5.1) is a pullback square, which completes the proof.

An element of $\hat{X}$ is a $(n+1)$-tuple $(b, \beta_1, \ldots, \beta_n)$, where $b \in B$ and β_j is a $K \cap^{\gamma_j} H$-map $K \to X_j$ satisfying the compatibility condition $f(\beta_j(k)) = kb$ for all j and all $k \in K$.

An element in the pullback of $\mathrm{Map}_H(G, f)$ and η consists of a triple (g, b, β), where $g \in G, b \in B$, and $\beta\colon G \to X$ is an H-map. Here the compatibility condition is that for all $g' \in G$ we have $f(\beta(g')) = (g'g, b) \in G \times_K B$. We use the decomposition

$$i_H\left(G \underset{K}{\times} B\right) \cong \coprod_{j=1}^n \gamma_j^{-1} \cdot \left({}^{\gamma_j}H \underset{K\cap^{\gamma_j}H}{\times} B\right)$$

to further describe $f(\beta(g'))$. If we choose $h \in H, k \in K$ and j such that $g'g = h\gamma_j^{-1}k$, then we see that $(g'g, b)$ is taken to $\gamma_j^{-1} \cdot (\gamma_j h \gamma_j^{-1}, kb)$ on the j-th component. This implies that $\beta(g')$ lies in the j-th component of the isomorphism

$$X \cong \coprod_j \gamma_j^{-1} \cdot \left({}^{\gamma_j}H \underset{K\cap^{\gamma_j}H}{\times} X_j\right)\,.$$

Considering the case $g' = \gamma_j^{-1}kg^{-1}$, we define a map $\beta_j \colon K \to X_j$ by the formula

$$\beta(\gamma_j^{-1}kg^{-1}) = \gamma_j^{-1} \cdot (e, \beta_j(k)) \in \gamma_j^{-1} \cdot \left({}^{\gamma_j}H \underset{K \cap {}^{\gamma_j}H}{\times} X_j \right) .$$

This then forces $\beta(h\gamma_j^{-1}kg^{-1}) = \gamma_j^{-1} \cdot (h, \beta_j(k))$, and this is well-defined exactly when all of the maps β_j are $K \cap {}^{\gamma_j}H$-equivariant. This shows us that the data $(g, b, \beta_1, \ldots, \beta_n)$ of an element of $G \times_K \hat{X}$ consists of the same data as an element of the pullback. These identifications are compatible with the identifications coming from different choices of element from the left coset gK. It is easy to check that this identification respects the group action and that the two projection maps are $G \times_K \pi$ and $\hat{\psi}$. $\square$

2.6 Comparisons with Mazur's model

Because $N_K^G N_H^K$ is naturally isomorphic to N_H^G for any subgroup chain $H \leqslant K \leqslant G$, we can check that our model for the norm functor N_H^G is isomorphic to the model constructed by Mazur for groups of cyclic prime power order in [18] by restricting our attention to the case N_H^G where H is maximal in G. In this case, Mazur's construction is given by starting with $\underline{M}^{\boxtimes |G/H|}$ and adding in norm elements in an appropriately free manner to the module $N_H^G \underline{M}(G/G)$. In particular, this yields a natural isomorphism $i_H N_H^G \underline{M} \cong (\underline{M})^{\boxtimes |G/H|}$. In our case such an isomorphism is a special case of Theorem 2.5.1.

Corollary 2.6.1. *Let G be a finite abelian group. Let $H, K \leqslant G$ and $\underline{M}$ be an H-Mackey functor. Then there is a natural isomorphism*

$$i_K N_H^G \underline{M} \cong \underset{|G/HK|}{\boxtimes} N_{K \cap H}^K (i_{K \cap H} \underline{M}) .$$

In the case $K \leqslant H$ this isomorphism takes the form $i_K N_H^G \underline{M} \cong (i_K \underline{M})^{\boxtimes |G/H|}$.

We now recall Mazur's Definition 2.2.2. We fix H to be the maximal subgroup of G and

$\underline{M}$ to be an H-Mackey functor. For any $K \leqslant H$, we set $N_H^G\underline{M}(G/K)$ to be $\underline{M}^{\boxtimes|G/H|}(H/K)$. The Weyl action is given symbolically in the form

$$c_\gamma(m_e \otimes m_\gamma \otimes \cdots \otimes m_{\gamma^{p-1}}) = (c_{\gamma^p} m_{\gamma^{p-1}} \otimes m_e \otimes \cdots \otimes m_{\gamma^{p-2}})\,.$$

Then the top-level module $N_H^G\underline{M}(G/G)$ is given by the explicit presentation

$$N_H^G\underline{M}(G/G) := \Big(\mathbb{Z}\{\underline{M}(H/H)\} \oplus \underline{M}^{\boxtimes|G/H|}(H/H)/_{W_G(H)}\Big)/_{TR}\,.$$

The additive generators of $\mathbb{Z}\{\underline{M}(H/H)\}$ are denoted $N(x)$, as these elements are designed to encode the norm map $\underline{M}(G/H) \to \underline{M}(G/G)$. The submodule $\underline{M}^{\boxtimes|G/H|}(H/H)/_{W_G(H)}$ is referred to as $\mathrm{Im}(\mathrm{tr}_H^G)$, as this is the image of the transfer map $N_H^G\underline{M}(G/H) \to N_H^G\underline{M}(G/G)$, which is the composite

$$\underline{M}^{\boxtimes|G/H|}(H/H) \to \underline{M}^{\boxtimes|G/H|}(H/H)/_{W_G(H)} \hookrightarrow N_H^G\underline{M}(G/G)\,.$$

The submodule TR is generated by elements of the following two forms:

$$\begin{aligned}
&N(a+b) - N(a) - N(b) - \mathrm{tr}_H^G(g(a,b)) \\
&N(\mathrm{tr}_K^H) - \mathrm{tr}_H^G(\mathrm{tr}_K^H(F(x)))\,.
\end{aligned}$$

The polynomials g, F are universally defined from the compatibility conditions defining a G-Tambara functor in the sense explained in Remark 2.2.2 of [18]. Restriction maps are defined on the elements of $\mathrm{Im}(\mathrm{tr}_H^G)$ and on elements of the form $N(a)$ by the usual double coset formulae.

Proposition 2.6.2. *There is an isomorphism of Mackey functors between our construction* $\mathrm{Lan}_{\mathrm{Map}_H(G,-)}\underline{M}$ *and Mazur's definition of* $N_H^G\underline{M}$.

Proof. The two constructions have underlying H-Mackey functors isomorphic to $\underline{M}^{\boxtimes|G/H|}$.

This is built into Mazur's construction directly, and in our construction the isomorphism follows from Corollary 2.6.1. One can check directly via the description of the isomorphism of Theorem 2.5.1 that the Weyl actions agree on $\underline{M}(G/K)$ for any $K \leqslant H$.

What remains is to construct an isomorphism of modules for the two definitions of $N_H^G(G/G)$, which must be compatible with restrictions and transfers.

The elements $N(a)$ are sent to the coend representatives $(R_{\mathrm{Id}_*}, a \in \underline{M}(i(G/G)))$. It is straightforward to then check that is compatible with the restriction to $N_H^G \underline{M}(G/H)$. Note that the unit map of the terminal object can be given by the trival map $\mathrm{Id}_* \colon * \to *$.

The elements of $\underline{M}^{\boxtimes |G/H|}(H/H)$ can be written via the isomorphism of Theorem 2.5.1 as coend representatives of the form $(T_t R_\eta, x \in i_H \underline{M}(i(G \times_H B)))$, where we use the notation $t \colon G \times_H B \to G/H$ for the map induced by the trivial H-map $B \to *$. The transfer of such an element is then given by the coend representative $(T_{\mathrm{triv}} R_\eta, x)$, where $\mathrm{triv} \colon G \times_H B \to *$ is the trivial G-map. It is straightforward to see that this then maps the image of the transfer $N_H^G \underline{M}(G/H) \to N_H^G \underline{M}(G/G)$ isomorphically onto the submodule $\mathrm{Im}(\mathrm{tr}_H^G)$ of Mazur's construction in a way compatible with the restriction map to $N_H^G \underline{M}(G/H)$.

The only remaining task is to demonstrate that the generators in TR correspond precisely to the coend identifications coming from transfer maps. This is a reflection of the fact that both constructions build in the corresponding compatibility conditions in a universal manner.

For the relations of the form $N(a+b) - N(a) - N(b) - \mathrm{tr}_H^G(g(a,b))$ for $a, b \in \underline{M}(H/H)$, we compute the element $N(a+b)$ in our coend, where $\nabla \colon * \amalg * \to *$ is the fold map:

$$\begin{aligned}
N(a+b) = (R_{\mathrm{Id}_*}, T_\nabla(a,b)) &= (R_{\mathrm{Id}_*} T_{\mathrm{Map}_H(G,\nabla)}, (a,b)) \\
&= (T_{\mathrm{triv}}, (a,b)) \\
&= (T_{\mathrm{triv}} R_\eta R_{\mathrm{Map}_H(G,\hat{\varepsilon})}, (a,b)) \\
&= (T_{\mathrm{triv}} R_\eta, R_{\hat{\varepsilon}}(a,b))\,.
\end{aligned}$$

Here $\hat{\varepsilon}\colon i_H \mathrm{Map}_H(G, * \amalg *) \to * \amalg *$ is the counit map. This yields the correct universal formula by inspection of the following bispan, which by Lemma 2.3.5 is the corresponding distributor:

$$G/H \amalg G/H \xleftarrow{G \underset{H}{\times} \hat{\varepsilon}} G \underset{H}{\times} \mathrm{Map}_H(G, * \amalg *) \xrightarrow{\varepsilon} \mathrm{Map}_H(G, * \amalg *) \xrightarrow{\mathrm{triv}} * \quad .$$

Note that $\mathrm{Map}_H(G, * \amalg *)$ has exactly two orbits of isotropy G corresponding to the terms $N(a)$ and $N(b)$. This follows from setting $K = G$ in the proof of Proposition 2.3.6.

For the relations of the form $N(\mathrm{tr}_K^H) - \mathrm{tr}_H^G(\mathrm{tr}_K^H(F(x)))$, we let q denote the quotient map $H/K \to H/H$ and $x \in \underline{M}(H/K)$. We now compute an expression for $N(T_q(x))$ in our coend:

$$\begin{aligned} N(\mathrm{tr}_K^H) = (R_{\mathrm{Id}_*}, T_q x) &= (R_{\mathrm{Id}_*} T_{\mathrm{Map}_H(G,q)}, x) \\ &= (T_{\mathrm{triv}}, x) \\ &= (T_{\mathrm{triv}} R_\eta R_{\mathrm{Map}_H(G,\hat{\varepsilon})}, x) \\ &= (T_{\mathrm{triv}} R_\eta, R_{\hat{\varepsilon}} x)\,. \end{aligned}$$

Again $\hat{\varepsilon}\colon i_H \mathrm{Map}_H(G, H/K) \to H/K$ is the counit map. This yields the correct universal formula by inspection of the following bispan, which by Lemma 2.3.5 is the corresponding distributor:

$$G/K \xleftarrow{G \underset{H}{\times} \hat{\varepsilon}} G \underset{H}{\times} \mathrm{Map}_H(G, H/K) \xrightarrow{\varepsilon} \mathrm{Map}_H(G, H/K) \xrightarrow{\mathrm{triv}} * \quad .$$

$\square$

2.7 A characterization of Tambara functors

As motivation, we first give an unbiased characterization of commutative monoids in a symmetric monoidal category $\mathcal{C}$. For any object $C \in \mathcal{C}$ and any finite set X we have the object $C^{\otimes X}$ given by the monoidal product of n copies of C, where n is the cardinality of X. Given any isomorphism $\gamma\colon X \cong X'$, we get a corresponding structural isomorphism $C^{\otimes X} \cong C^{\otimes X'}$. When $X = X'$, this is the corresponding permutation of factors. When $X = \varnothing$, $C^{\otimes \varnothing} = I_{\mathcal{C}}$, the unit of the symmetric monoidal structure.

A commutative monoid is then an object C with an assignment of maps $\mu_f\colon C^{\otimes X} \to C^{\otimes Y}$ for all set maps $f\colon X \to Y$ that agrees with the structural isomorphism when f is an isomorphism. This assignment must be functorial in the sense that $\mu_{fg} = \mu_f \mu_g$.

The maps are required to be compatible with the symmetric monoidal structures of $(\mathcal{C}, \otimes, I_{\mathcal{C}})$ and $(\mathrm{FinSet}, \amalg, \varnothing)$ in the sense that given $f_i\colon X_i \to Y_i$ for $i = 1, 2$, then we have the following commutative diagram:

$$\begin{tikzcd}
C^{\otimes(X_1 \amalg X_2)} \arrow[r, "\mu_{f_1 \amalg f_2}"] \arrow[d, "\cong"'] & C^{\otimes(Y_1 \amalg Y_2)} \arrow[d, "\cong"] \\
C^{\otimes X_1} \otimes C^{\otimes X_2} \arrow[r, "\mu_{f_1} \otimes \mu_{f_2}"'] & C^{\otimes Y_1} \otimes C^{\otimes Y_2}
\end{tikzcd}.$$

Using this compatibility we can recover the biased definition, under which the data of a commutative monoid is determined by the multiplication map $\mu_\nabla\colon C^{\otimes 2} \to C^{\otimes 1}$ and the unit map $\mu_\varnothing\colon C^{\otimes \varnothing} \to C^{\otimes 1}$, and must satisfy the usual unit, associativity, and commutativity axioms.

We have an equivariant generalization of the structure of a symmetric monoidal category on the category of G-Mackey functors due to the discussion and results in Section 2.4. This entails a description of $\underline{M}^{\otimes T}$ for any Mackey functor $\underline{M}$ and any finite G-set T, as well as structural isomorphisms $\underline{M}^{\otimes T} \cong \underline{M}^{\otimes T'}$ corresponding to any isomorphism of finite G-

sets $T \cong T'$. This leads to the following generalization of the above definition of monoid. Compare with Section 2.1 of [18].

Definition 2.7.1. *A G-commutative monoid is an object $\underline{M}$ along with action maps*

$$\mu_f \colon \underline{M}^{\otimes T} \to \underline{M}^{\otimes S}$$

for every G-map $f \colon T \to S$ satisfying the following identities:

- *The actions are functorial in the sense that $\mu_{fg} = \mu_f \mu_g$.*
- *When f is an isomorphism, the action maps μ_f agree with the structural isomorphisms of Corollary 2.4.6.*
- *For $f_i \colon T_i \to S_i$ for $i = 1, 2$ we have the following commutative diagram:*

$$\begin{tikzcd}
\underline{M}^{\otimes(T_1 \amalg T_2)} \arrow[r, "\mu_{f_1 \amalg f_2}"] \arrow[d, "\cong"'] & \underline{M}^{\otimes(S_1 \amalg S_2)} \arrow[d, "\cong"] \\
\underline{M}^{\otimes T_1} \boxtimes \underline{M}^{\otimes T_2} \arrow[r, "\mu_{f_1} \boxtimes \mu_{f_2}"'] & \underline{M}^{\otimes S_1} \boxtimes \underline{M}^{\otimes S_2}
\end{tikzcd}.$$

- *Define a map $\mu_H^G \colon N_H^G i_H \underline{M} \to \underline{M}$ by the composition*

$$N_H^G i_H \underline{M} \cong \underline{M}^{\otimes G/H} \xrightarrow{\mu_{\mathrm{triv}}} \underline{M}^{\otimes G/G} \cong \underline{M}\,.$$

  Then the diagram

$$\begin{tikzcd}
\underline{M} \arrow[r, "\tilde{\eta}"] \arrow[dr, equal] & N_H^G i_H \underline{M} \arrow[d, "\mu_H^G"] \\
& \underline{M}
\end{tikzcd}$$

  must commute for any $H \leqslant G$, where $\tilde{\eta}$ is the map of Proposition 2.3.10.

We will henceforth refer to the last condition as the triangular axiom.

Conceptually, if we have a Tambara functor $\underline{R}$, then $\underline{R}$ inherits the structure of a G-commutative monoid. When f is the quotient map $G/H \to G/G$, the action map μ_f is induced by the counit $\varepsilon\colon N_H^G i_H \underline{R} \to \underline{R}$. As $N_H^G i_H \underline{R}$ is the universal home for norm maps $\underline{R}(G/H) \to \underline{R}(G/G)$ (in a way that we make explicit in the proof of Theorem 2.7.4), we can similarly recover the norm map $\underline{R}(G/H) \to \underline{R}(G/G)$ from the data of the action map.

Note that in this context, the triangular axiom is the corresponding triangle identity for the $N_H^K \dashv i_H$ adjunction.

To encode the information of norm maps $\underline{R}(G/H) \to \underline{R}(G/K)$ for arbitrary K, we instead use the counit $\varepsilon\colon N_H^K i_H \underline{R} \to i_K \underline{R}$, which is part of the data of viewing $i_K \underline{R}$ as a K-commutative monoid. This map induces the action map $\underline{R}^{\otimes G/H} \to \underline{R}^{\otimes G/K}$ after applying the functor N_K^G. Our goal to characterize Tambara functors as Mackey functors $\underline{M}$ with extra structure now leads to the following definition. Recall our notation ${}^{\gamma}H = \gamma H \gamma^{-1}$ and $H^{\gamma} = \gamma^{-1} H \gamma$ for conjugate subgroups.

Definition 2.7.2. *A coherent G-commutative monoid is a Mackey functor $\underline{M}$ along with compatible H-commutative monoid structures (given by actions which we denote μ_f^H) on $i_H \underline{M}$ for each subgroup $H \leqslant G$.*

There are two compatibility conditions that must be satisfied. First, the diagram

$$\begin{tikzcd}
(i_K \underline{M})^{\otimes (K \times_H T)} \arrow[r, "\mu^K_{(K \times_H f)}"] \arrow[d, "\cong"'] & (i_K \underline{M})^{\otimes (K \times_H S)} \arrow[d, "\cong"] \\
N_H^K (i_H \underline{M})^{\otimes T} \arrow[r, "N_H^K \mu_f^H"'] & N_H^K (i_H \underline{M})^{\otimes S}
\end{tikzcd}$$

must commute for any $H \leqslant K \leqslant G$ and $f\colon T \to S$ any map of finite H-sets.

Second, we require compatibility with conjugations in the sense that the diagram

$$\begin{tikzcd}
c_\gamma((i_H\underline{M})^{\otimes T}) \arrow[r, "c_\gamma(\mu^H_f)"] \arrow[d, "\cong"'] & c_\gamma((i_H\underline{M})^{\otimes S}) \arrow[d, "\cong"] \\
(i_{\gamma H}\underline{M})^{\otimes \gamma\cdot T} \arrow[r, "\mu^{\gamma H}_{\gamma\cdot f}"'] & (i_{\gamma H}\underline{M})^{\otimes \gamma\cdot S}
\end{tikzcd}.$$

must commute for any H-map $f\colon T \to S$ and any $\gamma \in G$. Here the vertical isomorphisms are derived from chaining together the structural isomorphisms of Proposition 2.4.5.

The data of a coherent G-commutative monoid can be reduced as follows by breaking up H-sets into their orbits and applying the commutativity diagrams of Definition 2.7.1.

Lemma 2.7.3. *A coherent G-commutative monoid is a G-Green functor $\underline{M}$ along with maps of K-Green functors $\mu^K_H\colon N^K_H i_H\underline{M} \to i_K\underline{M}$ for each chain of subgroups $H \leqslant K \leqslant G$ satisfying the following compatibility conditions.*

- *If $L \leqslant H \leqslant K$ is a chain of subgroups, then μ^K_L is given by the following composition:*

$$N^K_L i_L\underline{M} \cong N^K_H N^H_L i_L\underline{M} \xrightarrow{N^K_H\mu^H_L} N^K_H i_H\underline{M} \xrightarrow{\mu^K_H} i_K\underline{M}\,.$$

- *Given any $H \leqslant K$, we have the following commutative square, where the vertical isomorphisms are derived from chaining together the structural isomorphisms of Proposition 2.4.5 :*

$$\begin{tikzcd}
c_\gamma N^K_H i_H\underline{M} \arrow[r, "c_\gamma(\mu^K_H)"] \arrow[d, "\cong"'] & c_\gamma i_K\underline{M} \arrow[d, "\cong"] \\
N^{\gamma K}_{\gamma H} i_{\gamma H}\underline{M} \arrow[r, "\mu^{\gamma K}_{\gamma H}"'] & i_{\gamma K}\underline{M}
\end{tikzcd}.$$

In particular, when $\gamma = k \in K$, this means that $\mu_H^K = \mu_{kH}^K \circ w_k$ in the notation of Corollary 2.4.6.

- *Given any $H \leqslant K$ the following diagram commutes, where $\tilde{\eta}$ is the map of Proposition 2.3.10:*

$$\begin{tikzcd} i_K\underline{M} \arrow[r, "\tilde{\eta}"] \arrow[dr, equal] & N_H^K i_H \underline{M} \arrow[d, "\mu_H^K"] \\ & i_K\underline{M} \end{tikzcd}.$$

The following result tells us we have the correct notion of monoid, giving an extrinsic characterization of when a Mackey functor has internal norms. The crux of the matter is that the internal structure of our norm functors N_H^K universally encode the necessary compatibility conditions between norms and restrictions, and between norms and transfers.

Theorem 2.7.4. *For any finite group G, the G-Tambara functors are precisely the coherent G-commutative monoids. In other words, there is an equivalence of categories between* Tamb_G *and the category of coherent G-commutative monoids.*

Here we define a map of coherent G-commutative monoids to be a map of G-Green functors compatible with the action maps.

Proof. For one direction, if $\underline{R}$ is a Tambara functor, then the action maps

$$\mu_H^K \colon N_H^K i_H \underline{M} \to i_K \underline{M}$$

for subgroups $H \leqslant K$ are given by the counit of the $N_H^K \dashv i_K$ adjunction. It is then straightforward to see that the resulting action maps are appropriately compatible.

For the other direction, we assume we have action maps μ_H^K, and use them to define internal norm maps. We define a set map $N \colon \underline{M}(G/H) \to N_H^K i_H \underline{M}(K/K)$ by taking an element $x \in \underline{M}(G/H)$ to the coend representative

$$N(x) = \big(R_\eta, x \in i_H\underline{M}(i_H(K/K)) \cong \underline{M}(G/H)\big) \in N_H^K i_H \underline{M}(K/K)\,.$$

This map realizes our notion of $N_H^K i_H \underline{M}$ as the universal home for suitable norm maps $\underline{M}(G/H) \to \underline{M}(G/K)$.

We can now define the map $\mathrm{norm}_H^K \colon \underline{M}(G/H) \to \underline{M}(G/K)$ as the composition

$$\underline{M}(G/H) \xrightarrow{N} N_H^K i_H \underline{M}(K/K) \xrightarrow{\mu_H^K} i_K\underline{M}(K/K) \cong \underline{M}(G/K)\,.$$

We must now check that if the μ_H^K are the structure maps of a coherent G-commutative monoid, then the above maps norm_H^K satisfy the compatibility conditions necessary to view $\underline{M}$ as a Tambara functor.

We immediately see that our internal norm maps are multiplicative and unital, by the fact that the maps μ_H^K were maps of K-Green functors. The identity $\mathrm{norm}_H^K \mathrm{norm}_L^H = \mathrm{norm}_L^K$ for chains of subgroups $L \leqslant H \leqslant K$ also follows immediately from unpacking the identity $\mu_L^K = \mu_H^K \circ N_H^K \mu_L^H$.

For the condition $c_g \circ \mathrm{norm}_H^K = \mathrm{norm}_{{}^gH}^{{}^gK} \circ c_g$, we note the following commutative diagram:

$$\begin{tikzcd}
\underline{M}(G/H) \arrow[r, "N"'] \arrow[d, "\cong"] & N_H^K i_K \underline{M}(K/K) \arrow[r, "\mu_H^K"'] \arrow[d, "\cong"] & \underline{M}(G/K) \arrow[d, "\cong"] \\
c_g\underline{M}(G/{}^gH) \arrow[r, "c_gN"'] \arrow[d, "\cong"] & c_g(N_H^K i_K \underline{M})({}^gK/{}^gK) \arrow[r, "c_g(\mu_H^K)"'] \arrow[d, "\cong"] & c_g\underline{M}(G/{}^gK) \arrow[d, "\cong"] \\
\underline{M}(G/{}^gH) \arrow[r, "N"'] & N_{{}^gH}^{{}^gK} i_{{}^gK} \underline{M}({}^gK/{}^gK) \arrow[r, "\mu_{{}^gH}^{{}^gK}"'] & \underline{M}(G/{}^gK)
\end{tikzcd}.$$

The left and right vertical composites give the conjugation isomorphisms c_g inherent in the

Mackey structure of $\underline{M}$, as desired.

In order to check the remaining compatibility conditions, we need the following result.

Lemma 2.7.5. *Let $K \leqslant G$, and $H, L \leqslant K$. Fix elements $k \in K$ and $y \in \underline{M}(G/(H \cap L^k)) \cong i_H\underline{M}(H/(H \cap L^k))$. Let $j_k\colon H/(H \cap L^k) \to i_H(K/L)$ be the H-equivariant inclusion defined by $h(H \cap L^k) \mapsto hk^{-1}L$. Then the image under $\mu_H^K\colon N_H^K i_H\underline{M} \to i_K\underline{M}$ of the element with coend representative*

$$\left(R_\eta, N_{j_k}y \in i_H\underline{M}(i_H(K/L))\right)$$

is the element $\operatorname{norm}^L_{{}^kH \cap L}(c_k y) \in \underline{M}(G/L)$.

This lemma, whose proof is deferred to the end of the section, conceptually says that the elements of the coend $N_H^K i_H\underline{M}$ are universal formulae that can be used to determine their image in $i_K\underline{M}$ under the action μ_H^K.

We first prove the compatibility formula for commuting restrictions past norms. Let our norm be norm_H^K and our restriction be induced by the quotient map $f\colon G/L \to G/K$, which is induced by the K-quotient map $\bar{f}\colon K/L \to K/K$.

We fix an element $x \in \underline{M}(G/H) \cong i_H\underline{M}(i_H(K/K))$, and seek to compute the composition $R_f\operatorname{norm}_H^K(x) = \operatorname{res}_L^K\operatorname{norm}_H^K(x)$. We get the following expansion using our definition of norm_H^K and the fact that μ is a map of Mackey functors:

$$\begin{aligned}
\operatorname{res}_L^K\operatorname{norm}_H^K(x) &= R_{\bar{f}}\mu_H^K\left(R_\eta, x\right) \in i_K\underline{M}(K/L) \cong \underline{M}(G/L) \\
&= \mu_H^K\left(R_{\bar{f}}R_\eta, x\right) \\
&= \mu_H^K\left(R_\eta R_{\operatorname{Map}_H(K, i_H\bar{f})}, x\right) \\
&= \mu_H^K\left(R_\eta, R_{i_H\bar{f}}x\right) .
\end{aligned}$$

We now need a better description of $R_{i_H\bar{f}}x \in i_H\underline{M}(i_H(K/L))$. Let $n = |L\backslash K/H|$, and set $k_1, \ldots, k_n$ to be a complete set of double coset representatives for $L\backslash K/H$. Now $i_H(K/L)$

breaks into the union of the sets $H/(H \cap L^{k_i})$ for $1 \leqslant i \leqslant n$.

Using this decomposition we can write $R_{i_H \bar{f}} x$ as the ordered n-tuple $(y_1, \ldots, y_n)$, where $y_i = \mathrm{res}^H_{H \cap L^{k_i}} x \in i_H \underline{M}(H/(H \cap L^{k_i})) \cong \underline{M}(G/(H \cap L^{k_i}))$.

We now write $x_i = (1, \ldots, 1, y_i, 1, \ldots, 1)$ and note that $R_{i_H \bar{f}} x$ is the product of the x_i's. Furthermore, we get that $x_i = N_{j_{k_i}} y_i$, so we can apply Lemma 2.7.5 to get the following computation:

$$
\begin{aligned}
\mathrm{res}^K_L \mathrm{norm}^K_H(x) &= \mu^K_H \left(R_\eta, R_{i_H \bar{f}} x \right) \\
&= \prod_{i=1}^n \mu^K_H \left(R_\eta, x_i \right) \\
&= \prod_{i=1}^n \mu^K_H \left(R_\eta, N_{j_{k_i}} y_i \right) \\
&= \prod_{i=1}^n \mathrm{norm}^L_{{}^{k_i}H \cap L}(c_{k_i} y_i) \\
&= \prod_{i=1}^n \mathrm{norm}^L_{{}^{k_i}H \cap L}(c_{k_i}(\mathrm{res}^H_{H \cap L^{k_i}} x))\,.
\end{aligned}
$$

This yields the standard double coset formula, as desired.

In order to check compatibility with transfers, we fix $H \leqslant L \leqslant K$, and decompose $\mathrm{Map}_L(K, L/H)$ into orbits of the form K/L_j. We construct a diagram

$$
\begin{tikzcd}
 & K \underset{L}{\times} \mathrm{Map}_L(K, L/H) \arrow[r, "\varepsilon"] \arrow[dl, "K \underset{H}{\times} \hat{\varepsilon}"'] \arrow[d, "\cong"'] & \mathrm{Map}_L(K, L/H) \arrow[d, "\cong"'] \arrow[dr] & \\
K/H & \coprod_{i \in I} K/L \cap L^{k_i}_{j(i)} \arrow[l, "f = \coprod_{i \in I} f_i"] \arrow[r, "g = \coprod_{i \in I} g_i"'] & \coprod_j K/L_j \arrow[r, "\mathrm{triv}"'] & K/K
\end{tikzcd}
$$
,

where the top composite is the distributor for the composition $K/H \to K/L \to K/K$ by Lemma 2.3.5.

Here $\hat{\varepsilon}\colon i_L\mathrm{Map}_L(K, L/H) \to L/H$ is the counit of the $i_L \dashv \mathrm{Map}_L(K, -)$ adjunction. The indexing set I is given by the union $\coprod_j L_j\backslash K/L$. For each element $i \in I$ we have an index $j(i)$ and a corresponding double coset representative $k_i \in L_{j(i)}\backslash K/L$.

The map f_i is observed to be given by some subconjugacy $L \cap L_{j(i)}^{k_i} \leqslant {}^{k_i'}H$. The map g_i is observed to be given by the subconjugacy $L \cap L_{j(i)}^{k_i} \leqslant L_{j(i)}^{k_i}$.

We now fix some element $x \in \underline{M}(G/H) \cong i_H\underline{M}(i_H(K/K))$. Here we use $h\colon L/H \to L/L$ to denote the quotient map. Note that $\mathrm{Map}_L(K, i_L(K/K)) \cong K/K$:

$$\begin{aligned}
\mathrm{norm}_L^K \mathrm{transfer}_H^L(x) &= \mu_L^K(R_\eta, T_h x) \in i_K\underline{M}(K/K) \cong \underline{M}(G/K)\\
&= \mu_L^K(R_\eta T_{\mathrm{Map}_L(K,h)}, x)\\
&= \mu_L^K(T_{\mathrm{triv}}, x)\\
&= T_{\mathrm{triv}}\mu_L^K(R_\eta R_{\mathrm{Map}_L(K,\hat{\varepsilon})}, x)\\
&= T_{\mathrm{triv}}\mu_L^K(R_\eta, R_{\hat{\varepsilon}}x)\,.
\end{aligned}$$

We see that $R_{\hat{\varepsilon}}x$ can be viewed as

$$R_{\coprod f_{i,j}} x \in \underline{M}(i_L\mathrm{Map}_L(K, L/H)) \cong i_K\underline{M}(K \times_L \mathrm{Map}_L(K, L/H))\,.$$

As before we can decompose this into components as $(y_1, \dots y_n)$, where

$$y_i = R_{f_i}(x) = \mathrm{res}_{L \cap L_j^{k_i}}^{k_i' H}(c_{k_i'} x)$$

and $n = |I|$.

We again write $x_i = (1, \dots, 1, y_i, 1, \dots, 1)$ and get $x_i = N_{i_L(\iota_j)} N_{j_{k_{i,j}}} y_i$. Here ι_j is the inclusion of the j-th summand K/L_j into the coproduct $\coprod K/L_j$. We now compute via

Lemma 2.7.5, noting again that μ is a map of unital Mackey functors:

$$\begin{aligned}
\mu_L^K(R_\eta, R_{\hat{\varepsilon}}x) &= \prod_i \mu_L^K\left(R_\eta, x_i\right) \\
&= \prod_i \mu_L^K\left(R_\eta, N_{i_L(\iota_j)} N_{j_{k_{i,j}}} y_i\right) \\
&= \prod_i \mu_L^K N_{\iota_j}\left(R_\eta, N_{j_{k_{i,j}}} y_i\right) \\
&= \prod_i N_{\iota_j}\mu_L^K\left(R_\eta, N_{j_{k_{i,j}}} y_i\right) \\
&= \prod_i N_{\iota_j}\mathrm{norm}_{k_i L\cap L_j}^{L_j}(c_{k_{i,j}} y_i)\,.
\end{aligned}$$

We need the following easy lemma for the identification

$$\left(R_\eta, N_{i_L(\iota_j)} N_{j_{k_{i,j}}} y_i\right) = N_{\iota_j}\left(R_\eta, N_{j_{k_{i,j}}} y_i\right)\,.$$

Lemma 2.7.6. *Let $L \leqslant K$ and let $\underline{M}$ be a unital L-Mackey functor. The norm maps N_ι corresponding to inclusion maps of K-sets take on the following form for coend representatives without transfers:*

$$N_\iota(R_\eta, x) = (R_\eta, N_{i_L(\iota)}x)\,.$$

Now if we piece things together we get that $\mathrm{norm}_L^K \mathrm{transfer}_H^L(x) = T_{\mathrm{triv}} N_g R_f(x)$, as desired.

Checking the compability conditions for $\mathrm{norm}_H^K(x_1 + x_2)$ is virtually the same argument as above, using an orbit decomposition of the following bispan, which is the corresponding distributor by Lemma 2.3.5:

$$\frac{K}{H} \amalg \frac{K}{H} \longleftarrow K \underset{H}{\times} \mathrm{Map}_H\left(K, \frac{K}{H} \amalg \frac{K}{H}\right) \xrightarrow{\varepsilon} \mathrm{Map}_H\left(K, \frac{K}{H} \amalg \frac{K}{H}\right) \xrightarrow{\mathrm{triv}} \frac{K}{K}\,.$$

We have finished showing that we have well-defined constructions taking Tambara functors to commutative G-commutative monoids and vice-versa.

For functoriality, we now note that a map of G-Green functors $f\colon \underline{M}_1 \to \underline{M}_2$ between G-Tambara functors is a map of G-Tambara functors precisely when the following diagram commutes for all subgroup chains $H \leqslant K \leqslant G$:

$$\begin{tikzcd} N_H^K i_H \underline{M}_1 \arrow[r, "\varepsilon"] \arrow[d, "N_H^K i_H f"'] & i_K \underline{M}_1 \arrow[d, "i_K f"] \\ N_H^K i_H \underline{M}_2 \arrow[r, "\varepsilon"'] & i_K \underline{M}_2 \end{tikzcd} .$$

If f is a map of G-Tambara functors then the above diagram commutes by naturality of ε, and if the above diagram commutes, we have that $f(\mathrm{norm}_H^K(x)) = \mathrm{norm}_H^K(f(x))$ for all $x \in \underline{M}_1(G/H)$. We also need to note that our two constructions are inverse to each other. Assume that $\underline{M}$ is a Tambara functor and the action map is given by the counit ε of the $N_H^K \dashv i_K$ adjunction. For this we use the formula

$$\mu_H^K(T_h R_\eta, x \in i_H \underline{M}(i_H B)) = T_h N_\varepsilon x \in i_K \underline{M}(K \underset{H}{\times} B)\,,$$

which is derived from the isomorphism of Theorem 2.3.3 and the properties of ε.

When h is identity map and B the trivial K-set K/K, we get that the image is $\mathrm{norm}_H^K x$. This tells us that if we start with a G-Tambara functor, we recover the same norm maps going through both constructions.

Lemma 2.7.5 can be used similarly to show that the action map μ_H^K is determined by the collection of maps $\mathrm{norm}_{H'}^L$ over all subgroup chains $H' \leqslant L \leqslant K$. This tells us that if we start with a coherent G-commutative monoid and apply both constructions we get the same action maps.

Thus, our two functors form an equivalence of categories. $\square$

Proof of Lemma 2.7.5: Here we have the following computation:

$$
\begin{aligned}
\mu_H^K(R_\eta, N_{j_k}y) &= \mu_{H\cap L^k}^K(R_\eta, N_{\hat{j}_k}y) \\
&= \mu_{{}^kH\cap L}^K(R_\eta, N_{\hat{j}_e}c_ky) \\
&= \mu_L^K(R_\eta, N_{\hat{\eta}}\mathrm{norm}_{{}^kH\cap L}^L(c_ky)) \\
&= \mathrm{norm}_{{}^kH\cap L}^L(c_ky)\,.
\end{aligned}
$$

For the first equality, we are using $\hat{j}_k\colon H\cap L^k/H\cap L^k \to i_{H\cap L^k}(K/L)$ to denote the following composition:

$$\frac{H\cap L^k}{H\cap L^k} \xrightarrow{\hat{\eta}} i_{H\cap L^k}\left(\frac{H}{H\cap L^k}\right) \xrightarrow{i_{H\cap L^k}(j_k)} i_{H\cap L^k}(K/L)\,.$$

Since $\mu_{H\cap L^k}^K = \mu_H^K \circ N_H^K(\mu_{H\cap L^k}^H)$, we see that the first equality follows from the identity $\mu_{H\cap L^k}^H(R_\eta, N_{\hat{j}_k}y) = N_{j_k}y$. This can be observed from the triangular axiom and the decomposition $\hat{j}_k = \hat{\eta}\circ i_{H\cap L^k}(j_k)$.

The second equality uses $\hat{j}_e\colon {}^kH\cap L/\,{}^kH\cap L \to i_{{}^kH\cap L}(K/L)$ to similarly denote the map $i_{{}^kH\cap L}(\hat{\eta})$. The second equality follows from the compatibility of our action maps with conjugations, and the equation $N_{\hat{j}_e}(c_ky) = N_{\hat{j}_e}R_{\hat{k}}y = R_{\hat{k}}N_{\hat{j}_k}y = w_k(N_{\hat{j}_k}y)$, which is demonstrated by the following pullback square:

$$
\begin{tikzcd}
K \underset{H\cap L^k}{\times} \left(\dfrac{H\cap L^k}{H\cap L^k}\right) \arrow[r, "K \underset{H\cap L^k}{\times} \hat{j}_k"'] & K \underset{H\cap L^k}{\times} \left(\dfrac{K}{L}\right) \\
K \underset{{}^kH\cap L}{\times} \left(\dfrac{{}^kH\cap L}{{}^kH\cap L}\right) \arrow[u, "\hat{k}", "\cong"'] \arrow[r, "K \underset{{}^kH\cap L}{\times} \hat{j}_e"'] & K \underset{{}^kH\cap L}{\times} \left(\dfrac{K}{L}\right) \arrow[u, "\hat{k}", "\cong"']
\end{tikzcd}
.
$$

The third equality follows from the factorization $\mu^K_{k_{H\cap L}} = \mu^K_L \circ N^K_L(\mu^L_{k_{H\cap L}})$, where the triangular axiom gives $\mu^L_{k_{H\cap L}}(R_\eta, N_{\hat{j}_e} c_k y) = N_{\hat{\eta}} \mathrm{norm}^L_{k_{H\cap L}}(c_k y)$. Here we use the fact that $\mu^L_{k_{H\cap L}}$ is a map of unital functors. We also need to apply Lemma 2.7.6, again noting $j_{\hat{e}} = i_{k_{H\cap L}}(\hat{\eta})$.

The fourth equality is another application of the triangular axiom. $\square$

CHAPTER 3

EILENBERG-MACLANE SPECTRA AS RELATIVE E_∞ CELL COMPLEXES

3.1 Introduction

In this chapter, which is part of a joint project with Andrew Baker, we consider what happens when we take the sphere spectrum, and kill elements of homotopy in an E_∞ fashion. This process starts by killing the element $2 \in \pi_0 S$ and is repeated in order to kill the higher homotopy groups. In the colimit, this provides a model of the Eilenberg-MacLane spectrum $H\mathbb{F}_2$ as a relative cellular complex in the E_∞ sense over the sphere spectrum S.

In Section 3.2, we provide an inductive description of this construction as well as a description of the cells in the complex as admissible Steenrod monomials whose last term is 4 or greater. The key step in this construction is establishing algebraic control over the (additive) splitting of each stage into wedges of Eilenberg-MacLane spectra, which is guaranteed due to work of Steinberger [8].

In Section 3.3, we outline how this description is relevant to yielding a calculation of topological André-Quillen homology of $H\mathbb{F}_2$. Modulo some assumptions, the description agrees with the calculations present in folklore and sketched in the literature.

In Section 3.4 and 3.5, we establish algebraic control over the homology of our construction. This involves providing explicit formulae for spherical classes and computing the action of the Dyer-Lashof algebra at each stage.

In Section 3.6, we construct primitive elements in the homology of the analogous first stage of a cellular complex for the Eilenberg-MacLane spectrum $H\mathbb{Z}_{(2)}$. It is hoped that similar methods lead to a corresponding computation of $\mathrm{TAQ}_*(H\mathbb{Z}/S)$ with $\mathbb{F}_2$ coefficients.

3.2 A description of the cell structure

We henceforth only consider homology with $\mathbb{F}_2$ coefficients, and work in the category of spectra localized at 2.

We let $S//2$ denote the commutative S-algebra constructed by killing $2 \in \pi_0(S)$ in the E_∞ sense. This means taking the map of commutative S-algebras $\mathbb{P}2\colon \mathbb{P}S^0 \to S$ defined on the free commutative S-algebra $\mathbb{P}S^0$, and then forming the following pushout of commutative S-algebras, where $\tilde{S}$ is an appropriate cofibrant replacement for S:

$$S//2 := \tilde{S} \wedge_{\mathbb{P}S^0} \mathbb{P}D^1 \,.$$

Here $S//2$ is the first stage of our cellular complex.

We recall some facts about $H_*(S//2)$ from [3]. There is an element $x_1 \in H_1(S//2)$ such that $H_*(S//2)$ is a polynomial algebra on generators $Q^I x_1$, where Q^I is an admissible Dyer-Lashof monomial of excess greater than 1.

The work of Steinberger [8] tells us that $S//2$ splits as a wedge of Eilenberg-Maclane spectra. We now define elements $X_i \in H_*(S//2)$ that correspond to the polynomial generators $\zeta_i \in \mathcal{A}_* = H_*(H\mathbb{F}_2)$. We inductively define $X_1 = x_1$ and $X_{i+1} = Q^{2^i} X_i$. The ideal generated by the X_i is invariant under the coaction of the dual Steenrod algebra, as shown by Proposition 9.2 of [3].

This allows us to view $H_*(S//2)$ as a polynomial algebra over the ring $\mathcal{A}_*$ on elements of the form $Q^I x_1$, where Q^I is admissible, excess greater than 1, and also has its last index at least 3. This expression algebraicizes the splitting of $S//2$ into Eilenberg-Maclane spectra. Furthermore, as explained in Section 9 of [3], this is the universal example of such a splitting.

We wish to to attach E_∞ cells to annihilate every summand except for the one in degree zero. We note that if we kill a homology element z by such an E_∞ cone, then we must also kill every possible element of the form $Q^I z$. Thus, we focus on the "bottom cells", namely those summands corresponding to the $Q^r x_1$ elements in homology for $r \geqslant 3$. The $Q^r x_1$ are not spherical, so we replace them with polynomial generators u_{r+1} that are spherical by adding decomposable terms and also elements X_i. Such a decomposition must exist due to the aforementioned splitting, and using the dual Steenrod coaction we compute explicit formulae for the u_{r+1} in Section 3.4. For instance, $u_4 = Q^3 x_1 + x_1^4$ is spherical.

To construct the E_∞ cone, we combine our maps $u_{r+1}\colon S^{r+1} \to S//2$ to give us a map $\mathbb{P}\left(\bigvee_r S^{r+1}\right) \to S//2$, allowing us to define the spectrum Y_2 as the following pushout of commutative S-algebras: (We set Y_1 as notation for $S//2$.)

$$Y_2 := Y_1 \wedge_{\mathbb{P}(\bigvee_r S^{r+1})} \mathbb{P}\left(\bigvee_r D^{r+2}\right).$$

The mod 2 homology of $\mathbb{P}\left(\bigvee_r S^{r+1}\right)$ is a polynomial algebra generated by elements of the form $Q^I s_{r+1}$, where s_{r+1} corresponds to the generator in homology of S^{r+1} and I is a Dyer-Lashof monomial of excess greater than $r+1$. Thus the image in homology of $\mathbb{P}(\vee_r S^{r+1}) \to Y_1$ is the subalgebra generated by the generators of form $Q^I u_{r+1}$.

The image in homology of the map $H_*(Y_1) \to H_*(Y_2)$ is then a copy of $\mathcal{A}_*$, but the target has additional homology due to the Adem relations. For instance, since $Q^7 Q^3 = 0$, we see that $Q^7 u_4 = 0$, which indicates the existence of a new degree 12 generator in $H_*(Y_2)$. We have that Y_2 splits as a wedge of $H\mathbb{F}_2$'s, so its homology contains a copy of $\mathbb{F}_2[X_i]$, the homology of the degree zero $H\mathbb{F}_2$ summand. We have made progress in getting closer to $H\mathbb{F}_2$, since the next summand in Y_1 is in degree four (corresponding to u_4), but here the next summand is degree 12.

We now repeat our construction. We produce an analogous spectrum Y_{j+1} from Y_j by coning off the relevant elements u_R in the E_∞ sense. Here the u_R represent a (minimal) set of spherical generators for $H_*(Y_j)/J$ as an algebra over the Dyer-Lashof algebra, where J is the ideal generated by the image of the X_i.

More explicitly, we form the following pushout of commutative S-algebras: (We henceforth write $|R|$ for the degree $r_1 + \ldots + r_j$ of u_R.)

$$Y_{j+1} = Y_j \wedge_{\mathbb{P}(\vee_R S^{|R|})} \mathbb{P}(\vee_R D^{|R|+1}) \,.$$

The following result gives us a homological description of the j-th stage of our construction, in terms of the set of cells to be attached to construct the $(j+1)$-th stage.

Theorem 3.2.1. *If we construct Y_j as above, then H_*Y_j is a polynomial algebra over $\mathcal{A}_*$ with generators of the form $Q^I u_R$. The $u_R \in H_{|R|}Y_j$ are spherical classes indexed by a length-j sequence $R = (r_1, \ldots, r_j)$ satisfying $r_k \geqslant 2r_{k+1}$ for $k < j$ and $r_j \geqslant 4$. Here I ranges over all indices $(i_1, \ldots, i_l)$ of admissible Dyer-Lashof monomials satisfying both $ex(I) > |R|$ and $i_l < 2r_1 - 1$.*

Note that the indices R are given precisely by the indices for admissible Steenrod monomials with last term at least 4. We see that the colimit of the Y_j is a model for $H\mathbb{F}_2$, as desired.

Proof. We inductively derive the description of H_*Y_{j+1} from the description of H_*Y_j.

We start by noting that we have a Künneth spectral sequence of the following form:

$$E^2_{*,*} = \operatorname{Tor}_*^{H_*\mathbb{P}(\vee_R S^{|R|})}(H_*Y_j, \mathbb{F}_2) \Rightarrow H_*Y_{j+1} \,.$$

The spectral sequence is set up and referred to as the bar construction spectral sequence in [9], and is proven to be multiplicative in [5].

The algebra $H_*\mathbb{P}(\vee_R S^{|R|})$ is a polynomial algebra generated by elements of the form $Q^I s_R$. Here s_R is the generator in homology of the corresponding sphere $S^{|R|}$ and Q^I ranges across all admissible Dyer-Lashof monomials of excess greater than $|R|$. In our construction of Y_{j+1}, the element s_R is mapped to the spherical class u_R, and therefore $Q^I s_R$ is mapped to $Q^I u_R$.

Now there are two cases. If the last element of I satisfies $i_l < 2r_1 - 1$, then $Q^I u_R$ is one of our given polynomial generators for H_*Y_i as an algebra over $\mathcal{A}_*$. In the case where $i_l \geqslant 2r_1 - 1$, we see that there is some expression for $Q^I u_R$ in terms of our given generating set. We define polynomials $F_{I,R}$ as follows to encode this action:

$$Q^I u_R = F_{I,R}(Q^J u_T, X_i)\,.$$

Here the $Q^J u_T$ range over indexing sets $J = (j_1, \ldots, j_m), T = (t_1, \ldots, t_j)$. These must satisfy the same admissibility constraints $j_k \leqslant 2j_{k+1}$ and $t_k \geqslant 2t_{k+1}$ that constrain the indices I, R, but must additionally satisfy $j_m < 2t_1 - 1$.

In subsequent sections we develop techniques to explicitly compute the polynomials $F_{I,R}$.

Our description of the Dyer-Lashof action on H_*Y_j allows us to compute the $E^2_{*,*}$ term of our spectral sequence by means of a Koszul resolution of $\mathbb{F}_2$ over $\mathbb{F}_2[Q^I s_R]$. We see that $E^2_{*,*}$ is an exterior algebra over $\mathcal{A}_*$ with generators lying in $\mathrm{Tor}_{1,*}$, which can be represented by the following elements in the bar construction:

$$[Q^I s_R + F_{I,R}(Q^J s_T, X_i)] \in E^2_{1,|I|+|R|}\,.$$

Here I, R run over all pairs of admissible indexing sequences such that I is nonempty and the last term of I exceeds $2r_1 - 1$. Here we must symbolically replace u_R by s_R to get the relevant polynomial $F_{I,R} \in \mathbb{F}_2[Q^I s_R, X_i] \cong \mathbb{F}_2[Q^I s_R] \otimes \mathcal{A}_*$. To follow the standard notation for the bar construction the X_i should also be separated from the $Q^I s_R$ and pulled outside of the brackets. As an example, we can interpret $[Q^9 s_4 + Q^7 s_6 + X_1^4 Q^5 s_4]$ as $[Q^9 s_4 + Q^7 s_6] + X_1^4[Q^5 s_4]$. (This ends up being an expansion of $F_{(9),(4)}$.)

Since this is a homology spectral sequence of algebras generated by elements on the $E^2_{1,*}$-line, the spectral sequence collapses.

We must resolve any multiplicative extensions and in particular check that no element is nilpotent. This is a straightforward argument using the compatibility of the Dyer-Lashof actions with the spectral sequence. A classical account of such compatibility is given in [17]; in our case we shall appeal to the technology of [9]. Compare with the proof of Theorem 2.7 of [2]. Explicitly, we start with an arbitrary generator $Q^I s_R$ and compute:

$$\begin{aligned}
&[Q^I s_R + F_{I,R}(Q^J s_T, X_i)]^2 \\
&= Q^{|I|+|R|+1}[Q^I s_R + F_{I,R}(Q^J s_T, X_i)] \\
&= [Q^{|I|+|R|+1}Q^I s_R + Q^{|I|+|R|+1}F_{I,R}(Q^J s_T, X_i)] \\
&= [Q^{(|I|+|R|+1,I)} s_R + F_{(|I|+|R|+1,i_1,\ldots,i_l),R}(Q^J s_T, X_i)]\,.
\end{aligned}$$

The second equality follows from our definition of the $F_{I,R}$. We note that since the Dyer-Lashof monomial in this case has excess exactly 1, the only relevant Dyer-Lashof actions on the $X_i \in E^2_{0,*}$ are $Q^{2^i}X_i = X_{i+1}$ and $Q^{2^i-1}X_i = X_i^2$. We note that our condition $\mathrm{ex}(I) > |R|$ compels $(|I| + |R| + 1, i_1, \ldots, i_l)$ to also be an admissible sequence with excess greater than $|R|$.

This argument tells us that our polynomial generators are given by representatives

$$[Q^I s_R + F_{I,R}(Q^J s_T, X_i)]$$

such that the sequences I have excess at least $|R|+2$. For any choice of r and $R = (r_1, \ldots, r_j)$ with $r \geqslant 2r_1 - 1$, we define a sequence $R' = (r + 1, r_1, \ldots r_j)$, and use the symbol $y_{R'}$ to denote an element represented by $[Q^r s_R + F_{(r),R}(Q^J s_T, X_i)]$. We can then describe H_*Y_{j+1} as a polynomial algebra over $\mathcal{A}_*$ with generators $Q^I y_{R'}$ with the specified conditions on I and R'.

The final step is to replace the y_R with spherical classes u_R. We observe that Y_{j+1} also splits into Eilenberg-MacLane spectra due to Steinberger's result [8], and thus y_R is spherical modulo decomposables and the images of the X_i. We therefore replace each y_R

with a corresponding spherical u_R, and replace any $Q^I y_R$ with the corresponding $Q^I u_R$. $\square$

3.3 Using the cell structure to compute $\mathrm{TAQ}_*(H\mathbb{F}_2/S, H\mathbb{F}_2)$

Here we use the notion of Topological André-Quillen homology developed by Basterra in [6]. This generalizes a theory defined on the ordinary category of commutative rings. In our case the coefficient module is always $H\mathbb{F}_2$, and this means that the corresponding homology and cohomology theories are $\mathbb{F}_2$-linear duals to one another.

By a cell complex, we assume that we start with some algebra Y_0 (which in our case is the sphere S), and that we inductively obtain a complex Y_{j+1} from Y_j by attaching cells in the E_∞ sense.

To clarify what this means, we think of a single cell as $\mathbb{P}S^n$, where $\mathbb{P}$ is the free functor from S-modules to commutative S-algebras. The adjunction gives a map of algebras $\mathbb{P}S^n \to Y_j$ for each element of $\pi_n Y_j$. To attach multiple cells at once, set W to be a wedge of spheres, and form the following pushout in the category of commutative S-algebras:

$$\begin{tikzcd}
\mathbb{P}W \arrow[r] \arrow[d] & \mathbb{P}CW \arrow[d] \\
Y_j \arrow[r] & Y_{j+1}
\end{tikzcd}.$$

We require the cofibrancy conditions ensuring $Y_{j+1} \cong Y_j \wedge_{\mathbb{P}W} \mathbb{P}CW$. Here the cone CW is contractible which implies that $\mathbb{P}CW \cong S$.

We consider what this does in TAQ-homology, where we suppress the coefficient module $H\mathbb{F}_2$ from the notation. The long exact sequence in TAQ_* for the two maps $Y_0 \to Y_j \to Y_{j+1}$

is then of the following form:

$$\begin{aligned} \cdots \to \mathrm{TAQ}_*(Y_j/Y_0) \to \mathrm{TAQ}_*(Y_{j+1}/Y_0) \to \mathrm{TAQ}_*(Y_{j+1}/Y_j) \\ \to \mathrm{TAQ}_{*-1}(Y_j/Y_0) \to \cdots . \end{aligned}$$

Here we have that the term $\mathrm{TAQ}_*(Y_{j+1}/Y_j)$ is isomorphic to $\mathrm{TAQ}_*(S/\mathbb{P}W)$ by Proposition 4.6 of [6]. (This is the analogue to flat base change in this setting.)

Proposition 1.8 of [4] computes $\mathrm{TAQ}_*(S/\mathbb{P}W)$ to be a single generator in degree $n+1$ for each sphere S^n in the wedge W. This calculation is analogous to the ordinary homology of cells in the traditional context.

We then see that we can inductively compute $\mathrm{TAQ}_*(Y_j/Y_0)$ as long as we have a description of the cells at each stage, as well as an understanding of the boundary map. This is algebraically identical to the computation of cellular homology given the degrees of the attaching maps of a CW complex. In particular, we have the following.

Lemma 3.3.1. *If Y is the colimit of Y_j as above, and each boundary map in the above long exact sequence is trivial, then the set of generators of $\mathrm{TAQ}_*(Y/Y_0)$ is given by the set of cells in the construction, with a shift of degree by 1.*

Conjecture 3.3.2. *For our construction the boundary map is trivial.*

Combining the description of cells to follow in Theorem 3.2.1 with the above conjecture tells us that $\mathrm{TAQ}_*(H\mathbb{F}_2/S)$ has generators corresponding to (shifted) admissible Steenrod monomials with last term at least 4. This recovers the calculation sketched in [16].

Checking the conjecture requires showing that the induced map

$$\mathrm{TAQ}_*(\mathbb{P}S^n/S) \to \mathrm{TAQ}_*(Y_j/S)$$

is trivial for every cell. This can be reduced to showing that the corresponding element in

$H_*(Y_j)$ is in the kernel of the TAQ_*-Hurewicz map. For $j = 1$ this follows from the fact that nontrivial Dyer-Lashof operations always take values in this kernel. (This is Theorem 4.4 of [1].) Similarly, all decomposable elements lie in this kernel.

3.4 Primitives in $H_*(S//2)$

The first step in establishing explicit algebraic control over the constructions in the previous section is to understand how to replace generators of the form $Q^I x_1 \in H_*(S//2)$ with generators of the form $Q^I u_{r+1}$, where the u_{r+1} are spherical. Since $S//2$ is a wedge of Eilenberg-Maclane spectra, the spherical elements coincide with the primitive elements under the coaction of the dual Steenrod algebra.

We use the previously defined elements $X_i \in H_{2^i-1}(S//2)$, as well as the usual polynomial generators $\xi_i \in H_*H\mathbb{F}_2$ for the Milnor basis of the dual Steenrod algebra. We also use the conjugates $\zeta_i = \chi(\xi_i)$.

These computations rely on the power series expression for the coaction as explained in [3]. To develop this notationally, we start by defining the power series $X(t) \in H_*(S//2)[[t]]$ and $\xi(t), \zeta(t) \in H_*(H\mathbb{F}_2 \wedge S//2)[[t]] \cong \mathcal{A}_* \otimes H_*(S//2)[[t]]$:

$$
\begin{aligned}
X(t) &= \sum_i X_i t^{2^i} = t + X_1 t^2 + X_2 t^4 + X_3 t^8 + \dots \\
\zeta(t) &= \sum_i (\zeta_i \otimes 1) t^{2^i} = t + (\zeta_1 \otimes 1) t^2 + (\zeta_2 \otimes 1) t^4 + \dots \\
\xi(t) &= \sum_i (\xi_i \otimes 1) t^{2^i} = t + (\xi_1 \otimes 1) t^2 + (\xi_2 \otimes 1) t^4 + \dots .
\end{aligned}
$$

We also establish the following notational convention for a power series with coefficients encoding the Dyer-Lashof action on any element y:

$$
Qy(t) = \sum_i (Q^i y) t^i = y^2 t^{|y|} + Q^{|y|+1} y t^{|y|+1} + Q^{|y|+2} y t^{|y|+2} + \dots .
$$

This notation allows us to concisely formulate the interaction of (left) coaction of the dual Steenrod algebra with the Dyer-Lashof action: (This is Theorem 4.1 of [3]).

$$\psi(Qy(t)) = \big[(\chi \otimes 1)Q((\chi \otimes 1)\psi y)\big](\xi(t)) \, .$$

Thus, for our given element $x_1 \in H_*S//2$, the coaction $\psi(x_1) = 1 \otimes x_1 + \xi_1 \otimes 1$ determines the coaction on an arbitrary $Q^r x_1$:

$$\psi(Qx_1(t)) = (1 \otimes Qx_1)(\xi(t)) + (\chi(Q\zeta_1) \otimes 1)(\xi(t)) \, .$$

We recall the Dyer-Lashof action on ζ_1: (see III.2 of [8])

$$\frac{1}{\xi(t)} = \frac{1}{t} + \xi_1 \otimes 1 + Q\zeta_1(t) \otimes 1 \, .$$

This allows us to rewrite our coaction formula: (Recall that $\xi(t)$ is the composition inverse to $\zeta(t)$.)

$$\begin{aligned} \chi(Q\zeta_1 \otimes 1)(t) &= \frac{1}{t} + \frac{1}{\zeta(t)} + \xi_1 \otimes 1 \\ \psi(Qx_1(t)) &= (1 \otimes Qx_1)(\xi(t)) + \left[\chi(Q\zeta_1 \otimes 1)\right](\xi(t)) \\ &= (1 \otimes Qx_1)(\xi(t)) + \frac{1}{\xi(t)} + \frac{1}{t} + \xi_1 \otimes 1 \, . \end{aligned}$$

We can now define a series $u(t)$ by the following formula:

$$u(t) = Qx_1(X(t)) + \frac{1}{X(t)} + \frac{1}{t} + X_1 \, .$$

We note that every coefficient is of the form

$$Q^r x_1 + \left(\sum_{k=1}^{r-1} f_{k,r}(X_i) Q^k x_1 \right) + g_r(X_i)\,,$$

where $f_{k,r}$ and g_r are degrees $r-k$ and $r+1$, respectively. This shows us that this coefficient differs from $Q^r x_1$ only by decomposables and the elements X_i.

We now check directly that the first nonzero term of $u(t)$ is $u_4 t^3$ with $u_4 = Q^3 x_1 + X_1^4$. This follows from a computation of the lowest terms of the following power series:

$$\begin{aligned} Qx_1(X(t)) &= (Q^1 x_1)t + (Q^2 x_1 + X_1 Q^1 x_1)t^2 + (Q^3 x_1)t^3 + \dots \\ &= X_1^2 t + (X_1^3 + X_2)t^2 + (Q^3 x_1)t^3 + \dots \\ \frac{1}{X(t)} &= \frac{1}{t} + X_1 + X_1^2 t + (X_1^3 + X_2)t^2 + X_1^4 t^3 + \dots\,. \end{aligned}$$

If we want to express the elements $Q^r x_1$ in terms of the elements u_{r+1}, (instead of our more usual goal of expressing the latter in terms of the former) we can use the following formula, where $\overline{X}(t)$ denotes the composition inverse to $X(t)$:

$$Qx_1(t) = u\left(\overline{X}(t)\right) + \frac{1}{t} + \frac{1}{\overline{X}(t)} + X_1\,.$$

Through power series inversion, this is equivalent to our formula defining $u(t)$.

Proposition 3.4.1. *Each coefficent of $u(t)$ is primitive.*

Proof. We must apply the coaction on the X_i, which can be written in the form $\psi(X(t)) = \zeta(1 \otimes X(t))$. (This is verified in Proposition 9.2 of [3], although the reader should be warned that our X_i is written X_{i-1} there.) Note that since ψ is a multiplicative homomorphism, we have the relation $\psi(F \circ G(t)) = \psi F(\psi G(t))$ for any power series F, G.

We have

$$
\begin{aligned}
\psi(u(t)) &= (\psi Q x_1)(\psi X(t)) + \frac{1}{\psi X(t)} + \frac{1}{t} + \psi(X_1) \\
&= \left(Q x_1(\xi(\psi X(t))) + \frac{1}{\xi(\psi X(t))} + \frac{1}{\psi X(t)} + \xi_1 \otimes 1 \right) \\
&\quad + \frac{1}{\psi X(t)} + \frac{1}{t} + 1 \otimes X_1 + \xi_1 \otimes 1 \\
&= (1 \otimes Q x_1)(\xi(\zeta(1 \otimes X(t)))) + \frac{1}{\xi(\zeta(1 \otimes X(t)))} + \frac{1}{t} + 1 \otimes X_1 \\
&= (1 \otimes Q x_1)(1 \otimes X(t)) + \frac{1}{1 \otimes X(t)} + \frac{1}{t} + 1 \otimes X_1 \\
&= (1 \otimes u)(t)\,.
\end{aligned}
$$

□

Note that so far we have only replaced the elements $Q^r x_1$ with primitives. If we wish to replace arbitrary $Q^I x_1$ with primitives, we can iteratively apply the following observation of Andrew Baker. Such replacement allows one to completely algebraicize the splitting of $S//2$ into Eilenberg-MacLane spectra.

Lemma 3.4.2. *If z is primitive, then so too are the coefficients of $Qz(X(t))$.*

Proof. We evaluate

$$
\begin{aligned}
\psi(Qz(X(t))) &= (Q(1 \otimes z))(\xi(\psi X(t))) \\
&= (1 \otimes Qz)(\xi(\zeta(1 \otimes X(t)))) \\
&= 1 \otimes Qz(X(t))\,.
\end{aligned}
$$

□

Here the resulting primitives are of the form $Q^r z + \sum_{k=|z|}^{r-1} f_{k,r}(X_i) Q^k z$. In particular, we note that there are no terms without some some factor of the form $Q^k z$.

3.5 Computation of the polynomials $F_{I,R}$

Explicit algebraic control of H_*Y_j requires computing the polynomials $F_{I,R}$. We note that in the induction we replaced generators y_R with spherical generators u_R, and must be able to account for this replacement in order to be able to compute the Dyer-Lashof action on the H_*Y_j.

For an example, we see that in the case $j = 1$ we have the following spherical classes:

$$\begin{aligned}
u_4 &= Q^3 x_1 + X_1^4 \\
u_5 &= Q^4 x_1 + X_1 u_4 + X_1^2 X_2 \\
u_6 &= Q^5 x_1 + X_1^2 u_4 + X_2^2 \\
u_7 &= Q^6 x_1 + X_1 u_6 + X_2 u_4 + X_3 \,.
\end{aligned}$$

These formulae and the Adem relations allow us to compute the following Dyer-Lashof actions:

$$\begin{aligned}
Q^7 u_4 &= 0 \\
Q^8 u_4 &= Q^8 Q^3 x_1 + Q^8(X_1^4) = Q^7 Q^4 x_1 + (Q^2 X_1)^4 \\
&= Q^7(u_5 + X_1 u_4 + X_1^2 X_2) + X_2^4 \\
&= Q^7 u_5 + X_1^2 Q^6 u_5 + X_2 Q^5 u_4 + (Q^3 x_1) u_4^2 + X_1^4 Q^5 Q^2 x_1 \\
&= Q^7 u_5 + X_1^2 Q^6 u_5 + X_2 Q^5 u_4 + X_1^4 u_4^2 + u_4^3 \\
Q^9 u_4 &= Q^9 Q^3 x_1 = Q^7 Q^5 x_1 = Q^7(u_6 + X_1^2 u_4 + X_2^2) \\
&= Q^7 u_6 + X_1^4 Q^5 u_4
\end{aligned}$$

$$
\begin{aligned}
Q^{10}u_4 &= Q^{10}Q^3x_1 = Q^8Q^5x_1 + Q^7Q^6x_1 \\
&= Q^8(u_6 + X_1^2u_4 + X_2^2) + Q^7(u_7 + X_1u_6 + X_2u_4 + X_3) \\
&= Q^8u_6 + u_7^2 + X_1^4Q^6u_4 + X_1^2u_6^2 \\
Q^{11}u_4 &= Q^{11}Q^3x_1 = 0\,.
\end{aligned}
$$

Such expansions yield a method for computing $F_{I,R}$ for all R of length 1. In order to compute the $F_{I,R}$ for $j > 1$, we need to do two things. First, we must compute the Dyer-Lashof action on H_*Y_j, and for that we should compute the action on the X_i. We have already noted that for all j we have $Q^{2^i}X_i = X_{i+1}$ and $Q^{2^i-1}X_i = X_i^2$. For the rest of the action, we break into the case $j = 1$ and the case $j > 1$.

For $j = 1$, the action of higher Dyer-Lashof terms is directly computable as follows:

Lemma 3.5.1. *In $H_*(S//2)$, we have the following Dyer-Lashof action on X_2:*

$$
Q^rX_2 = \begin{cases} Q^{k+2}Q^kx_1 & \textit{if } r = 2k \\ (Q^{2k}x_1)^2 & \textit{if } r = 4k-1 \\ 0 & \textit{otherwise}\,. \end{cases}
$$

For $i > 2$, we have the following formulae instead:

$$
Q^rX_i = \begin{cases} Q^{2^{i-2}(k+2)}Q^{2^{i-3}(k+2)}\cdots Q^{2(k+2)}Q^{k+2}Q^kx_1 & \textit{if } r = 2^{i-1}k \\ (Q^{2^{i-3}(k+2)}Q^{2^{i-4}(k+2)}\cdots Q^{2(k+2)}Q^{k+2}Q^kx_1)^2 & \textit{if } r = 2^{i-1}k-1 \\ 0 & \textit{otherwise}\,. \end{cases}
$$

Proof. The formulae follow by inductively applying the Adem relations. □

For $j > 1$, we claim that the Dyer-Lashof action on the X_i is analogous to Steinberger's description of $Q^k\zeta_i$ in III.2.2 of [8]. For this description, we use $\overline{X}_i$ to denote the coefficients of t^{2^i} in the series $\overline{X}(t)$ defined as the composition inverse to $X(t)$, analogous to how the

ξ_i are defined in terms of the ζ_i. To express our answer, we use the notation N_k to denote the k-th Newton polynomial. We then form an expression $N_k(\overline{X})$ by substituting $\overline{X}_i$ for the (2^i-1)-th elementary symmetric polynomial, and 0 for all other elementary symmetric polynomials. The Newton recurrence relation tells us that $N_{k+1}(\overline{X})$ is the coefficient for t^k in the series $1/\overline{X}(t)$.

Proposition 3.5.2. *We have the following Dyer-Lashof action for $j \geqslant 2$:*

$$Q^r X_i = \begin{cases} Q^{r-2^i+1}X_1 = N_{r-2^i+1}(\overline{X}) & \text{if } r \geqslant 2^i-1 \text{ and } r = 0,-1 \pmod{2^i} \\ 0 & \text{otherwise}. \end{cases}$$

We specifically note that when $k = 2^i-1$ we have $N_k(\overline{X}) = X_i$.

Proof. An examination of the definition of Y_j for $j > 2$ makes it clear that we can restrict to $j = 2$ without loss of generality.

We first consider the case $i = 1$.

We recall that in the previous section we have the following equation:

$$Qx_1(t) = u\left(\overline{X}(t)\right) + \frac{1}{t} + \frac{1}{\overline{X}(t)} + X_1\,.$$

In order to pass to Y_2, we killed every coefficient of $u(t)$, so the coefficient of t^r in $1/\overline{X}(t)$ is precisely Q^rX_1 in the target. We have already observed that this is $N_{k+1}(\overline{X})$.

To evaluate Q^rX_i for $i > 1$, we use the previous lemma to convert

$$Q^rX_i = Q^rQ^{2^{i-1}} \cdots Q^4Q^2X_1$$

into an admissible format. We must then apply Dyer-Lashof monomials to terms of the form

$N_{k+1}(\overline{X})$. The proposition follows by iteratively applying the following identities:

$$(N_s(\overline{X}))^2 = N_{2s}(\overline{X})$$
$$Q^{s+1}N_s(\overline{X}) = \begin{cases} N_{2s+1}(\overline{X}) & \text{if } s \text{ odd} \\ 0 & \text{if } s \text{ even}\,. \end{cases}$$

The first identity follows from examination of the series $1/\overline{X}(t)$, and by the Cartan formula implies the even case of the second. For the odd case, we write out our polynomial in X_i's and observe that for degree considerations the Cartan formula reduces to terms of the form $Q^{2^i}X_i$ and $Q^{2^i-1}X_i$ only. The former is always X_{i+1} and the latter is always X_i^2. At this point we could give a direct combinatorial proof (the formula $Q^{2^i}\overline{X_i} = \overline{X}_{i+1} + X_1\overline{X}_i^2$ can be proved as in Lemma 4.4 of [3]), but it suffices to note that the algebra is identical to the case of the dual Steenrod algebra, where the following more general identity is known:

$$Q^r N_k(\xi) = \binom{r-1}{k-1} N_{r+k}(\xi)\,.$$

□

We note that Proposition 3.5.2 now formally implies that the following analogous formula holds in the case $j \geqslant 2$:

$$Q^r N_k(\overline{X}) = \binom{r-1}{k-1} N_{r+k}(\overline{X})\,.$$

To complete our understanding of the Dyer-Lashof action, we must examine the action on generators of form y_R. If $i \leqslant 2r$, then $Q^i y_{(r+1,R)}$ lies in our set of polynomial generators, so we need only work in the case $i > 2r$. In this case the action resembles the Adem relations.

Proposition 3.5.3. *Assume $i > 2r$, and let $(r+1, R)$ be a length j sequence admissible in* *the sense of Steenrod monomials. Then*

$$Q^i y_{(r+1,R)} = \sum_k \binom{k-r-1}{2k-i} Q^{i+r-k} y_{(k+1,R)} \,.$$

Here the k are those satisfying $i+r-k \leqslant 2k$.

Proof. The Adem relations give us the following relation in $H_*\mathbb{P}S^{|R|}$:

$$Q^i Q^r s_R = \sum_k \binom{k-r-1}{2k-i} Q^{i+r-k} Q^k s_R \,.$$

The two sides must map to the same element of H_*Y_{j-1}, so this yields the following relations:

$$F_{(i,r),R}(Q^J u_T, X_i) = \sum_k \binom{k-r-1}{2k-i} F_{(i+r-k,k),R}(Q^J u_T, X_i) \,.$$

We apply the compatibility of the Dyer-Lashof operations with our Künneth spectral sequence converging to H_*Y_j from the proof of Theorem 3.2.1:

$$\begin{aligned}
&Q^i[Q^r s_R + F_{(r),R}(Q^J s_T, X_i)] \\
&\quad = [Q^i Q^r s_R + Q^i F_{(r),R}(Q^J s_T, X_i)] \\
&\quad = [Q^i Q^r s_R + F_{(i,r),R}(Q^J s_T, X_i)] \\
&\quad = \sum_k \binom{k-r-1}{2k-i} [Q^{i+r-k} Q^k s_R + F_{(i+r-k,k),R}(Q^J s_T, X_i)] \\
&\quad = \sum_k \binom{k-r-1}{2k-i} Q^{i+r-k} [Q^k s_R + F_{(k),R}(Q^J s_T, X_i)] \,.
\end{aligned}$$

We claim that we are done by how we defined the representatives for $y_{(r+1,R)}$ and $y_{(k+1,R)}$. In the case where there is a k such that $i+r-k = k+|R|+1$, we have already seen in the proof of Theorem 3.2.1 how the element

$$Q^{k+|R|+1}[Q^k s_R + F_{(k),R}(Q^J s_T, X_i)]$$

represents the corresponding square element $y^2_{(k+1,R)}$. In the case $i + r - k = k + |R|$, we see that the element

$$Q^{k+|R|}[Q^k s_R + F_{(k),R}(Q^J s_T, X_i)]$$

vanishes for dimensional reasons since the dimension of the bracketed representative is $k + |R| + 1$.

□

The last thing we need for explicit algbebraic control is to establish an analogue of the previous section for determining how to replace the elements y_R with primitive elements u_R. The formula for this is more straightforward in some sense than the formula for $j = 1$, because the $j = 1$ case is the only one where the Steenrod action takes the X_i's to expressions using the other generators.

Proposition 3.5.4. *Let R be a sequence of length $j - 1$ and let the power series $y_{(t,R)}$ be given by the following formula:*

$$y_{(t,R)} = \sum_{k+1=2|R|}^{\infty} y_{(k+1,R)} t^k .$$

Then the coefficients of the series $u_{(t,R)} = y_{(X(t),R)}$ are primitives in $H_* Y_j$.

Proof. We begin by defining an analogous power series $s_{(t,R)} \in H_* \mathbb{P} S^{|R|}[[t]] \cong \mathbb{F}_2[Q^I s_R][[t]]$:

$$s_{(t,R)} = \sum_{k=|R|}^{\infty} Q^k s_R t^k .$$

Lemma 3.4.2 guarantees that the cofficients of $s_{(X(t),R)}$ are primitives in the $\mathcal{A}_*$-comodule algebra $\mathbb{F}_2[Q^I s_R, X_i]$.

We recall that in the proof of Theorem 3.2.1 we examined the Künneth spectral sequence converging to $H_* Y_j$. We use the notation $z_{(X(t),R)}$ to denote the power series whose coefficients are elements in $\mathbb{F}_2[Q^I s_R, X_i]$ whose images in the subquotient $E^2_{1,*}$ are representatives for the coefficients of $y_{(X(t),R)}$. The homological dimension in the spectral sequence accounts for the shift from the k in the definition of $s_{(t,R)}$ to the $k+1$ in the definition of $y_{(X(t),R)}$.

We see directly by the definition of the $y_{(k+1,R)}$ that the coefficients of $z_{(X(t),R)}$ share leading terms with the coefficients of $[s_{(X(t),R)}]$ in the sense that in suitable degrees the difference is a polynomial in the X_i and the $[F_{I,R}(Q^J s_T, X_i)]$, and is therefore in the subring generated by the X_i and those $[Q^I s_R]$ with $i_l < 2r_1 - 1$.

This forces the coaction on the coefficients of $z_{(X(t),R)}$ and of $[s_{(X(t),R)}]$ to be the same modulo elements in the $\mathcal{A}_*$-subalgebra T generated by the X_i and those $[Q^I s_R]$ with $i_l < 2r_1 - 1$ in $\mathcal{A}_* \otimes \mathbb{F}_2[Q^I s_R, X_i]$. In particular, we have that $\psi z_{(X(t),R)} - 1 \otimes z_{(X(t),R)}$ lies in $\mathcal{A}_* \otimes T$. But this means that it is zero in $\mathcal{A}_* \otimes E^2_{1,*}$, since by inspection the representatives for additive generators of $E^2_{1,*}$ each have a distinct unique indecomposable leading term lying outside of T. $\square$

As an example computation, we consider $F_{(16),(8,4)}$. We first use the Adem relation $Q^{16}Q^7 = Q^{15}Q^8$ to establish $Q^{16}y_{(8,4)} = Q^{15}y_{(9,4)}$ by Proposition 3.5.3. Next, we see that our primitive replacement yields $u_{(8,4)} = y_{(8,4)}$ and $u_{(9,4)} = y_{(9,4)} + X_1 y_{(8,4)}$ by Proposition 3.5.4. We now evaluate:

$$\begin{aligned} Q^{16}u_{(8,4)} &= Q^{16}y_{(8,4)} \\ &= Q^{15}y_{(9,4)} \\ &= Q^{15}(u_{(9,4)} + X_1 u_{(8,4)}) \\ &= Q^{15}u_{(9,4)} + X_1^2 Q^{14}u_{(8,4)} + X_2 Q^{13}u_{(8,4)} + X_1^4 u_{(8,4)}^2 . \end{aligned}$$

3.6 Primitives in $S//\eta, u$

Here we use $S//\eta, u$ to denote the commutative S-algebra constructed by first coning off $\eta \in \pi_1(S)$ in the E_∞ sense, and then coning off the resulting class $u \in \pi_2(S//\eta)$ in the E_∞ sense.

We have generators x_2, x_3 such that the homology ring $H_*(S//\eta, u)$ is a polynomial algebra with generators $Q^I x_2$ and $Q^I x_3$ for I admissible Dyer-Lashof monomials of excess greater than 2 or greater than 3, respectively. In this section we consider replacing $Q^r x_2$ and $Q^r x_3$ with primitive elements in the cases $r > 2$ and $r > 4$, respectively. Conceptually, this corresponds to the first step in constructing an E_∞ cell complex for $H\mathbb{Z}$.

We have the following (left) coaction on the generators x_2, x_3:

$$\psi(x_2) = 1 \otimes x_2 + \zeta_1^2 \otimes 1$$
$$\psi(x_3) = 1 \otimes x_3 + \zeta_1 \otimes x_2 + \zeta_2 \otimes 1\,.$$

This is enough information to get a 2-local Steinberger-type splitting into $H\mathbb{Z}_{(2)}$'s and $H\mathbb{Z}/2^s$'s. To apply III.4.2 of [8] we observe that the above coaction implies $Sq_*^3 x_3 = 1$, so in cohomology $Sq^3(1)$ is the nonzero dual to x_3. We see that the only $H\mathbb{Z}_{(2)}$ term occurs in degree zero by considering rationalization.

Our intuition is that x_2 corresponds to ζ_1^2 and x_3 corresponds to ζ_2 in the homology of the the degree zero summand $H_*H\mathbb{Z}_{(2)} \cong \mathbb{F}_2[\zeta_1^2, \zeta_2, \ldots]$. This leads us to define $X_1^2 = x_2$ and $X_2 = x_3$. Note that any expression involving an odd power of X_1 does not lie in the homology of $S//\eta, u$, but rather lies in the formal algebraic extension $\mathbb{F}_2[\sqrt{x_2}, Q^I x_2, Q^I x_3] \cong H_*(S//\eta, u)[\sqrt{x_2}]$. Here we can give the coaction on $\sqrt{x_2} = X_1$ by the usual formula $\psi(X_1) = 1 \otimes X_1 + \zeta_1 \otimes 1$, so that our extension has the structure of a comodule algebra over the dual Steenrod algebra.

To find elements corresponding to the higher ζ_i, we use the coaction as our guide. We desire the following formula:

$$\psi(X_3) = 1 \otimes X_3 + \zeta_1 \otimes X_2^2 + \zeta_2 \otimes X_1^4 .$$

This is satisfied by $X_3 = Q^4x_3 + x_2Q^3x_2$. The extra term is necessary because an error term arises in $\psi(Q^4x_3)$ due to the fact that $Q^3X_1^2 \neq 0$, recalling that x_2 is not actually a square.

We can now define $X_{i+1} = Q^{2^{i+1}}X_i$ for $i \geqslant 3$, and the same proof as in the case $S//2$ now runs into no difficulties in verifying our coaction formula

$$\psi(X(t)) = \zeta(1 \otimes X(t)) ,$$

since the exponent of X_1 is at least 4.

The Nishida relations tell us that $\beta_*Q^{2k} = Q^{2k-1}$, which tells us to group generators of length 1 in pairs of the form $\{Q^{2k}x_2, Q^{2k-1}x_2\}$ and $\{Q^{2k}x_3, Q^{2k-1}x_3\}$. In our algebraicized splitting into Eilenberg-Maclane spectra, the bottom terms $Q^{2k-1}x_i$ correspond to actual $H\mathbb{F}_2$ summands and we must first cone off those generators before spherical classes corresponding to the $Q^{2k}x_i$ can exist.

To figure out the formulae for primitives, we first compute the coaction on the $Q^r x_2$:

$$\begin{aligned}\psi(Qx_2(t)) &= (1 \otimes Qx_2)(\xi(t)) + (\chi Q\zeta_1^2 \otimes 1)(\xi(t)) \\ &= (1 \otimes Qx_2)(\xi(t)) + \frac{1}{\xi(t)^2} + \frac{1}{t^2} + \xi_1^2 \otimes 1 .\end{aligned}$$

Again, we derive formulae for the primitives by considering an analogous series with X_i replacing the ξ_i whenever possible.

Proposition 3.6.1. *Define a series by the following:*

$$A(t) = Qx_2(X(t)) + \frac{1}{X(t)^2} + \frac{1}{t^2} + X_1^2 .$$

Then each coefficient of $A(t)$ is a primitive in the comodule $H_*(S//\eta, u)[\sqrt{x_2}]$.

Proof. This is analogous to Proposition 3.4.1:

$$\begin{aligned}\psi(A(t)) &= \left((1\otimes Qx_2)\xi(\psi X(t)) + \frac{1}{\xi(\psi X(t))^2} + \frac{1}{\psi X(t)^2} + \xi_1^2\otimes 1\right)\\ &\quad + \frac{1}{\psi X(t)^2} + \frac{1}{t^2} + 1\otimes X_1^2 + \xi_1^2\otimes 1\\ &= (1\otimes Qx_2)(1\otimes X(t)) + \frac{1}{1\otimes X(t)^2} + \frac{1}{t^2} + 1\otimes X_1^2\\ &= 1\otimes A(t)\,.\end{aligned}$$

□

We only care about actual elements with no radicals, so we must examine which primitives only contain even powers of X_1. We have the following statement about when elements are well-defined.

Lemma 3.6.2. *The odd coefficients of $A(t)$ are well-defined in that their terms only contain even powers of X_1. The even coefficients become well-defined in this sense if we replace all expressions $Q^{2k-1}x_2$ by zero for $k > 1$.*

The replacement in the second statement tells us what happens to $Q^{2k-1}x_2$ after we do our first round of E_∞ coning. Only the elements corresponding to odd operations can yield actual primitives since the others are not in the image of the Hurewicz map, as witnessed the nontrivial action $\beta_* Q^{2k}x_i = Q^{2k-1}x_i$. After those are killed, we are able to cone off the remaining length 1 generators.

Proof. The second statement is obvious, since once the odd Dyer-Lashof elements are removed all terms are squares. For the first statement, we must check that for each r, k the coefficient of t^{2k+1} in the series $X(t)^r$ must have an odd power of X_1. If r is even this is obvious. For odd numbers we can decompose $X(t)^{r+2}$ as $X(t)^2X(t)^r$ and proceed by induction, with the case $r = 1$ obvious.

$\square$

For our second class of polynomial generators, we must compute the coaction on elements of the form $Q^r x_3$. Here the new ingredient is to identify $Q\zeta_2(t)$. For this we use the computation of Steinberger [8]:

$$Q^r\zeta_i = \begin{cases} Q^{r-2^i+1}\zeta_1 & \text{if } r \geqslant 2^i - 1 \text{ and } r = 0, -1 \pmod{2^i} \\ 0 & \text{otherwise}. \end{cases}$$

The nonzero terms of $Q^r\zeta_2$ are equal to $Q^{r+2}\zeta_1$ for $r = 3, 4 \pmod 4$ and $r \geqslant 3$. To isolate terms we apply the identity

$$Q^{2k+1}\zeta_1 = Q^{2k}\zeta_1^2 = (Q^k\zeta_1)^2\,.$$

We have the following series expansions:

$$\begin{aligned}
\frac{1}{t^2\xi(t)} &= \frac{1}{t^3} + \frac{\xi_1}{t^2} + \frac{\xi_1^2}{t} + \zeta_2 + \xi_1^4 t + \ldots + Q^{r+2}\zeta_1 t^r + \ldots \\
\frac{1}{t\xi(t)^2} &= \frac{1}{t^3} + \frac{\zeta_1^2}{t} + \zeta_1^4 t + \zeta_2^2 t^3 + \ldots + Q^{2r+3}\zeta_1 t^{2r+1} + \ldots \\
\frac{t}{\xi(t)^4} &= \frac{1}{t^3} + \zeta_1^4 t + \zeta_1^8 t^5 + \zeta_2^4 t^9 + \ldots + Q^{4r+3}\zeta_1 t^{4r+1} + \ldots .
\end{aligned}$$

The second two of these tell us that the odd-dimensional terms of $Q\zeta_2(t)$ are given by the sum

$$\frac{\zeta_1^2}{t} + \frac{1}{t\xi(t)^2} + \frac{t}{\xi(t)^4}\,.$$

To get the even-dimensional terms, we use the following manipulations. Here we define $\overline{\xi}(t)$ to be the power series $\xi_1 t + \xi_2 t^2 + \xi_3 t^4 + \ldots$, so that $t + \xi(t) = \overline{\xi}(t^2)$. We must use the formal derivative to get the third expression below:

$$\begin{aligned}
\frac{t+\xi(t)}{\xi(t)^2} &= \zeta_1 + \zeta_2 t^2 + \zeta_1^2\zeta_2 t^4 + \zeta_3 t^6 + \ldots + Q^{2r}\zeta_1 t^{2r} + \ldots \\
\frac{\overline{\xi}(t)}{\xi(t)} &= \zeta_1 + \zeta_2 t + \zeta_1^2\zeta_2 t^2 + \zeta_3 t^3 + \ldots + Q^{2r}\zeta_1 t^{r} + \ldots \\
\frac{\overline{\xi}(t)+\xi_1\xi(t)}{\xi(t)^2} &= \zeta_2 + \zeta_3 t^2 + \zeta_1^4\zeta_3 t^4 + \zeta_4 t^6 + \ldots + Q^{4r+2}\zeta_1 t^{2r} + \ldots \\
\frac{t+\xi(t)+\xi_1\xi(t)^2}{\xi(t)^4} &= \zeta_2 + \zeta_3 t^4 + \zeta_1^4\zeta_3 t^8 + \zeta_4 t^{12} + \ldots + Q^{4r+2}\zeta_1 t^{4r} + \ldots .
\end{aligned}$$

We combine even and odd terms to get the following power series expression:

$$\begin{aligned}
Q\zeta_2(t) &= \left(\frac{\zeta_1^2}{t} + \frac{1}{t\xi(t)^2} + \frac{t}{\xi(t)^4}\right) + \left(\zeta_2 + \frac{t+\xi(t)+\xi_1\xi(t)^2}{\xi(t)^4}\right) \\
&= \zeta_2 + \frac{\zeta_1^2}{t} + \frac{\xi_1}{\xi(t)^2} + \frac{1}{\xi(t)^3} + \frac{1}{t\xi(t)^2} \\
\chi Q\zeta_2(\xi(t)) &= \xi_2 + \frac{\xi_1^2}{\xi(t)} + \frac{\xi_1}{t^2} + \frac{1}{t^3} + \frac{1}{t^2\xi(t)} .
\end{aligned}$$

We now compute the coaction on elements of form $Q^r x_3$ as follows:

$$\begin{aligned}
\psi(Qx_3(t)) &= (1\otimes Qx_3)(\xi(t)) + (1\otimes Qx_2)(\xi(t))(\chi Q\zeta_1\otimes 1)(\xi(t)) \\
&\quad + (\chi Q(\zeta_1^3+\zeta_2)\otimes 1)(\xi(t)) \\
&= (1\otimes Qx_3)(\xi(t)) + (1\otimes Qx_2)(\xi(t))\cdot\left(\frac{1}{\xi(t)} + \frac{1}{t} + \xi_1\otimes 1\right) \\
&\quad + \left(\frac{1}{\xi(t)} + \frac{1}{t} + \xi_1\otimes 1\right)^3 + \xi_2\otimes 1 + \frac{\xi_1^2\otimes 1}{\xi(t)} + \frac{\xi_1\otimes 1}{t^2} + \frac{1}{t^3} + \frac{1}{t^2\xi(t)} .
\end{aligned}$$

We are now ready to provide the relevant primitives corresponding to $Q^r x_3$.

Proposition 3.6.3. *Define a series by the following:*

$$B(t) = Qx_3(X(t)) + A(t)\cdot\left(\frac{1}{t} + X_1 + \frac{1}{X(t)}\right) + X_2 + \frac{X_1^2}{X(t)} + \frac{X_1}{t^2} + \frac{1}{t^3} + \frac{1}{t^2X(t)} .$$

Then each coefficient of $B(t)$ is a primitive element of the comodule $H_*(S//\eta, u)[\sqrt{x_2}]$.

Proof. We first start to compute the coaction on $B(t)$, using the fact that $A(t)$ has primitive coefficients:

$$\begin{aligned}\psi(B(t)) = \psi(Qx_3(X(t))) + (1\otimes A(t))\cdot\left(\frac{1}{t} + 1\otimes X_1 + \xi_1\otimes 1 + \frac{1}{\zeta(1\otimes X(t))}\right)\\ + 1\otimes X_2 + \xi_1\otimes X_1^2 + \zeta_2\otimes 1 + \frac{\xi_1^2\otimes 1 + 1\otimes X_1^2}{\zeta(1\otimes X(t))}\\ + \frac{\xi_1\otimes 1 + 1\otimes X_1}{t^2} + \frac{1}{t^3} + \frac{1}{t^2\zeta(1\otimes X(t))}.\end{aligned}$$

To expand the first term we need to rely on our coaction formula:

$$\begin{aligned}\psi(Qx_3(X(t))) = (1\otimes Qx_3(X(t)))\\ + \left(1\otimes Qx_2(X(t)) + \frac{1}{1\otimes X(t)^2} + \frac{1}{\zeta(1\otimes X(t))^2} + \xi_1^2\otimes 1\right)\\ \cdot\left(\frac{1}{1\otimes X(t)} + \frac{1}{\zeta(1\otimes X(t))} + \xi_1\otimes 1\right) + \xi_2\otimes 1 + \frac{\xi_1^2\otimes 1}{1\otimes X(t)}\\ + \frac{\xi_1\otimes 1}{\zeta(1\otimes X(t))^2} + \frac{1}{\zeta(1\otimes X(t))^3} + \frac{1}{(1\otimes X(t))\zeta(1\otimes X(t))^2}.\end{aligned}$$

Now the desired statement $\psi(B(t)) - 1\otimes B(t) = 0$ is straightforward to verify.

∎

Just like the coefficients of $A(t)$, we need to check which coefficients have only even powers of X_1 in order to get honest elements in homology. Again, this turns out to be the odd coefficients of our series.

Lemma 3.6.4. *The odd coefficients of $B(t)$ are well-defined in the sense that these terms only contain odd powers of X_1.*

Proof. In the proof of Lemma 3.6.2 we noted that the coefficients of t^{2k+1} in $X(t)^r$ are well-defined for positive r, and this works for negative odd k as well by the decomposition

$X(t)^{r-2} = X(t)^{-2}X(t)^r$. The terms involving a $Q^r x_2$ are now the only ones remaining with a risk of having an odd power of X_1 in the odd coefficient. The coefficent of $Q^r x_2$ is seen to be $X(t)^r(t^{-1} + X_1 + X(t)^{-1})$, which can be rewritten $X(t)^r + t^{-2}X(t)^r(t + X_1 t^2)$. Here the term $(t + X_1 t^2)$ is equal to $X(t)$ modulo terms not involving X_1, so we have reduced everything to our analysis of $X(t)^r$. □

We must similarly replace $Q^{2k+1}x_3$ with something new in order for the even coefficients to be well-defined, to reflect the fact that after we first cone off odd coefficient primitives, the Dyer-Lashof action on x_3 changes. In the earlier case the modification is simple: $Q^{2k+1}x_2 = Q^{2k+1}X_1^2 = 0$. In this case there is a nontrivial action, which we must compute.

Notationally, let $F'(t)$ be the formal derivative of a power series $F(t)$ and $F^{odd}(t)$ and $F^{even}(t)$ be the terms with only odd powers of t and only even powers of t, respectively. We are working mod 2 so we have $F^{odd}(t) = tF'(t)$. We want to see what happens when we cone off the odd coefficients of $B(t)$, so we must isolate these coefficients:

$$\begin{aligned}
B'(t) &= Qx_3(X(t)) + A'(t)\cdot\left(\frac{1}{t} + X_1 + \frac{1}{X(t)}\right) + A(t)\cdot\left(\frac{1}{t^2} + \frac{1}{X(t)^2}\right) \\
&\quad + \frac{X_1^2}{X(t)^2} + \frac{1}{t^4} + \frac{1}{t^2 X(t)^2} \\
B^{odd}(t) &= \frac{t}{X(t)}Qx_3^{odd}(X(t)) + \frac{t}{X(t)}Qx_2^{odd}(X(t))\cdot\left(\frac{1}{t} + X_1 + \frac{1}{X(t)}\right) \\
&\quad + A(t)\cdot\left(\frac{1}{t} + \frac{t}{X(t)^2}\right) + \frac{tX_1^2}{X(t)^2} + \frac{1}{t^3} + \frac{1}{tX(t)^2}\,.
\end{aligned}$$

Now after coning off the odd coefficients of $B(t)$, we can use the above to see the resulting Dyer-Lashof action on x_3. This action is encoded by the series $Qx_3^{odd}(t)$, but it is $Qx_3^{odd}(X(t))$

that we need to replace in our definition of $B(t)$:

$$\begin{aligned} Qx_3^{odd}(X(t)) &= \frac{X(t)}{t}B^{odd}(t) + Qx_2^{odd}(X(t)) \cdot \left(\frac{1}{t} + X_1 + \frac{1}{X(t)}\right) \\ &\quad + A(t) \cdot \left(\frac{X(t)}{t^2} + \frac{1}{X(t)}\right) + \frac{X_1^2}{X(t)} + \frac{X(t)}{t^4} + \frac{1}{t^2 X(t)} \\ Qx_3^{odd}(t) &= \frac{t}{\overline{X}(t)}B^{odd}(\overline{X}(t)) + Qx_2^{odd}(t) \cdot \left(\frac{1}{t} + X_1 + \frac{1}{\overline{X}(t)}\right) \\ &\quad + A(\overline{X}(t)) \cdot \left(\frac{t}{\overline{X}(t)^2} + \frac{1}{t}\right) + \frac{X_1^2}{t} + \frac{t}{\overline{X}(t)^4} + \frac{1}{t\overline{X}(t)^2}. \end{aligned}$$

We note that the last three terms $\frac{X_1^2}{t} + \frac{t}{\overline{X}(t)^4} + \frac{1}{t\overline{X}(t)^2}$ encode the odd Dyer-Lashof action on $X_2 = x_3$ after we cone off both even and odd coefficients for the $A(t)$ and the $B(t)$. This series has coefficients analogous to the expression for $Q^{4k-1}\zeta_2 = Q^{4k+1}\zeta_1$ in terms of the ζ_i's or ξ_i's.

Lemma 3.6.5. *If we assume $B^{odd}(t) = 0$ and examine the series $\hat{B}(t)$ derived from $B(t)$ by replacing $Qx_3^{odd}(X(t))$ with the corresponding terms above, then the resulting series has well-defined coefficients in the sense that all powers of X_1 are even.*

Proof. If we do the given replacement, our series can be reorganized as follows:

$$\begin{aligned} \hat{B}(t) &= Qx_3^{even}(X(t)) + (A(t) + Qx_2^{odd}(X(t))) \cdot \left(\frac{1}{t} + X_1 + \frac{1}{X(t)}\right) \\ &\quad + A(t) \cdot \left(\frac{X(t)}{t^2} + \frac{1}{X(t)}\right) + X_2 + \frac{X_1}{t^2} + \frac{X(t)}{t^4} + \frac{1}{t^3} \\ &= Qx_3^{even}(X(t)) + \left(Qx_2^{even}(X(t)) + \frac{1}{t^2} + X_1^2 + \frac{1}{X(t)^2}\right) \cdot \left(\frac{t + X_1 t^2 + X(t)}{t^2}\right) \\ &\quad + Qx_2^{odd}(X(t)) \cdot \left(\frac{X(t)}{t^2} + \frac{1}{X(t)}\right) + X_2 + \frac{t + X_1 t^2 + X(t)}{t^4}. \end{aligned}$$

In this form the result is clear.

$\square$

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