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Spectra and symmetric spectra in general model categories

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Abstract

We give two general constructions for the passage from unstable to stable homotopy that apply to the known example of topological spaces, but also to new situations, such as the $\mathbb{A}^1$-homotopy theory of Morel and Voevodsky (preprint, 1998) and Voevodsky (Proceedings of the International Congress of Mathematicians, Vol. I, Berlin, Doc. Math. Extra Vol. I, 1998, pp. 579–604 (electronic)). One is based on the standard notion of spectra originated by Vogt (Boardman's Stable Homotopy Category, Lecture Notes Series, Vol. 21, Matematisk Institut Aarhus Universitet, Aarhus, 1970). Its input is a well-behaved model category $\mathscr{D}$ and an endofunctor T, generalizing the suspension. Its output is a model category $Sp^{\mathbb{N}}(\mathscr{D},T)$ on which T is a Quillen equivalence. The second construction is based on symmetric spectra (Hovey et al., J. Amer. Math. Soc. 13(1) (2000) 149–208) and applies to model categories $\mathscr{C}$ with a compatible monoidal structure. In this case, the functor T must be given by tensoring with a cofibrant object K. The output is again a model category $Sp^{\Sigma}(\mathscr{C},K)$ where tensoring with K is a Quillen equivalence, but now $Sp^{\Sigma}(\mathscr{C},K)$ is again a monoidal model category. We study general properties of these stabilizations; most importantly, we give a sufficient condition for these two stabilizations to be equivalent that applies both in the known case of topological spaces and in the case of $\mathbb{A}^1$-homotopy theory.

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0. Introduction

The object of this paper is to give two very general constructions of the passage from unstable homotopy theory to stable homotopy theory. Since homotopy theory in some form appears in many different areas of mathematics, this construction is useful

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beyond algebraic topology, where these methods originated. In particular, the two constructions we give apply not only to the usual passage from unstable homotopy theory of pointed topological spaces (or simplicial sets) to the stable homotopy theory of spectra, but also to the passage from the unstable $\mathbb{A}^1$-homotopy theory of Morel–Voevodsky [19,27] to the stable $\mathbb{A}^1$-homotopy theory. This example is obviously important, and the fact that it is an example of a widely applicable theory of stabilization may come as a surprise to readers of [14], where specific properties of sheaves are used.

Suppose, then, that we are given a (Quillen) model category $\mathscr{D}$ and a functor $T:\mathscr{D}\to\mathscr{D}$ that we would like to invert, analogous to the suspension. We will clearly need to require that T be compatible with the model structure; specifically, we require T to be a left Quillen functor. We will also need some technical hypotheses on the model category $\mathscr{D}$, which are complicated to state and to check, but which are satisfied in almost all interesting examples, including $\mathbb{A}^1$-homotopy theory. It is well known what one should do to form the category $Sp^{\mathbb{N}}(\mathscr{D},T)$ of spectra, as first written down for topological spaces in [2]. An object of $Sp^{\mathbb{N}}(\mathscr{D},T)$ is a sequence X_n of objects of $\mathscr{D}$ together with maps $TX_n\to X_{n+1}$, and a map $f:X\to Y$ is a sequence of maps $f_n:X_n\to Y_n$ compatible with the structure maps. There is an obvious model structure, called the *projective model structure*, where the weak equivalences are the maps $f:X\to Y$ such that f_n is a weak equivalence for all n. It is not difficult to show that this is a model structure and that there is a left Quillen functor $T:Sp^{\mathbb{N}}(\mathscr{D},T)\to Sp^{\mathbb{N}}(\mathscr{D},T)$ extending T on $\mathscr{D}$. But, just as in the topological case, T will not be a Quillen equivalence. So we must localize the projective model structure on $Sp^{\mathbb{N}}(\mathscr{D},T)$ to produce the *stable* model structure, with respect to which T will be a Quillen equivalence. A new feature of this paper is that we are able to construct the stable model structure with minimal hypotheses on $\mathscr{D}$, using the localization results of Hirschhorn [11] (based on work of Dror Farjoun [7]). We must pay a price for this generality, of course. That price is that stable equivalences are not stable homotopy isomorphisms, but instead are cohomology isomorphisms on all cohomology theories, just as for symmetric spectra [13]. If we put enough hypotheses on $\mathscr{D}$ and T, then stable equivalences coincide with stable homotopy isomorphisms. Using the Nisnevitch descent theorem, Jardine [14] has proved that stable equivalences coincide with stable homotopy isomorphisms in the stable $\mathbb{A}^1$-homotopy theory. His result does not follow from our general theorem, because the hypotheses we need do not hold in the Morel–Voevodsky motivic model category. However, Voevodsky (personal communication) has constructed a simpler model category equivalent to the Morel–Voevodsky one that does satisfy our hypotheses.

As is well known in algebraic topology, the category $Sp^{\mathbb{N}}(\mathscr{D},T)$ is not sufficient to understand the smash product. That is, if $\mathscr{C}$ is a symmetric monoidal model category, and T is the functor $X\mapsto X\otimes K$ for some cofibrant object K of $\mathscr{C}$, it almost never happens that $Sp^{\mathbb{N}}(\mathscr{C},T)$ is symmetric monoidal. We therefore need a different construction in this case. We define a category $Sp^{\Sigma}(\mathscr{C},K)$ just as in symmetric spectra [13]. An object of $Sp^{\Sigma}(\mathscr{C},K)$ is a sequence X_n of objects of $\mathscr{C}$ with an action of the symmetric group Σ_n on X_n. In addition, we have Σ_n-equivariant structure maps $X_n\otimes K\to X_{n+1}$, but we must further require that the iterated structure maps $X_n\otimes K^{\otimes p}\to X_{n+p}$ are

$\Sigma_n \times \Sigma_p$-equivariant, where Σ_p acts on $K^{\otimes p}$ by permuting the tensor factors. It is once again straightforward to construct the projective model structure on $Sp^{\Sigma}(\mathscr{C},K)$. The same localization methods developed for $Sp^{\mathbb{N}}(\mathscr{D},T)$ apply again here to give a stable model structure on which tensoring with K is a Quillen equivalence. Once again, stable equivalences are cohomology isomorphisms on all possible cohomology theories, but this time it is very difficult to give a better description of stable equivalences even in the case of simplicial symmetric spectra (but see [25] for the best such result I know). We point out that our construction gives a different construction of the stable model category of simplicial symmetric spectra from the one appearing in [13].

We now have competing stabilizations of $\mathscr{C}$ under the tensoring with K functor when $\mathscr{C}$ is symmetric monoidal. Naturally, we need to prove they are the same in an appropriate sense. This was done in the topological (actually, simplicial) case in [13] by constructing a functor $Sp^{\mathbb{N}}(\mathscr{C},T) \to Sp^{\Sigma}(\mathscr{C},K)$, where $K = S^1$ and T is the tensor with S^1 functor, and proving it is a Quillen equivalence. We are unable to generalize this argument. Instead, following an idea of Hopkins, we construct a zigzag of Quillen equivalences $Sp^{\mathbb{N}}(\mathscr{C},T) \to \mathscr{E} \leftarrow Sp^{\Sigma}(\mathscr{C},K)$. However, we need to require that the cyclic permutation map on $K \otimes K \otimes K$ be homotopic to the identity by an explicit homotopy for our construction to work. This hypothesis holds in the topological case with $K = S^1$ and in the $\mathbb{A}^1$-local case with K equal to either the simplicial circle or the algebraic circle $\mathbb{A}^1 - \{0\}$. This section of the paper is by far the most delicate, and it is likely that we do not have the best possible result.

We also investigate the properties of these two stabilization constructions. There are some obvious properties one would like a stabilization construction such as $Sp^{\mathbb{N}}(\mathscr{D},T)$ to have. First of all, it should be functorial in the pair $(\mathscr{D},T)$. We prove this for $Sp^{\mathbb{N}}(\mathscr{D},T)$ and an appropriate analogue of it for symmetric spectra; the most difficult point is defining what one should mean by a map from $(\mathscr{D},T)$ to $(\mathscr{D}',T')$. Furthermore, stabilization should be homotopy invariant. That is, if the map $(\mathscr{D},T) \to (\mathscr{D}',T')$ is a Quillen equivalence, the induced map of stabilizations should also be a Quillen equivalence. We also prove this for $Sp^{\mathbb{N}}(\mathscr{D},T)$ and an appropriate analogue of it for symmetric spectra; one corollary is that the Quillen equivalence class of $Sp^{\Sigma}(\mathscr{C},K)$ depends only on the homotopy type of K. Finally, the stabilization map $\mathscr{D} \to Sp^{\mathbb{N}}(\mathscr{D},T)$ should be the initial map to a model category $\mathscr{E}$ with an extension of T to a Quillen equivalence. However, this last statement seems to be asking for too much, because the category of model categories is itself something like a model category. This statement is analogous to asking for an initial map in a model category from X to a fibrant object, and such things do not usually exist. The best we can do is to say that if T is already a Quillen equivalence, then the map from $\mathscr{D} \to Sp^{\mathbb{N}}(\mathscr{D},T)$ is a Quillen equivalence. This gives a weak form of uniqueness, and is the basis for the comparison between $Sp^{\mathbb{N}}(\mathscr{D},T)$ and symmetric spectra. See also see [22,23] for uniqueness results for the usual stable homotopy category.

We point out that this paper leaves some obvious questions open. We do not have a good characterization of stable equivalences or stable fibrations in either spectra or symmetric spectra, in general, and we are unable to prove that spectra or symmetric spectra are right proper. We do have such characterizations for spectra when the original model category $\mathscr{D}$ is sufficiently well behaved, and the adjoint U of T preserves sequential colimits. These hypotheses include the cases of ordinary simplicial spectra and spectra in a new motivic model category of Voevodsky (but not the original Morel–Voevodsky motivic model category). We also prove that spectra are right proper in this situation. But we do not have a characterization of stable equivalences of symmetric spectra even with these strong assumptions. Also, we have been unable to prove that symmetric spectra satisfy the monoid axiom. Without the monoid axiom, we do not get model categories of monoids or of modules over an arbitrary monoid, though we do get a model category of modules over a cofibrant monoid. The question of whether commutative monoids form a model category is even more subtle and is not addressed in this paper. See [18] for commutative monoids in symmetric spectra of topological spaces.

There is a long history of work on stabilization, much of it not using model categories. As far as this author knows, Boardman was the first to attempt to construct a good point-set version of spectra; his work was never published (but see [28]), but it was the standard for many years. Generalizations of Boardman's construction were given by Heller in several papers, including [8,9]. Heller has continued work on these lines, most recently in [10]. The review of this paper in Mathematical Reviews by Tony Elmendorf (MR98g:55021) captures the response of many algebraic topologists to Heller's approach. I believe the central idea of Heller's approach is that the homotopy theory associated to a model category $\mathscr{D}$ is the collection of all possible homotopy categories of diagram categories ho $\mathscr{D}^I$ and all functors between them. With this definition, one can then forget one had the model category in the first place, as Heller does. Unfortunately, the resulting complexity of definition is overwhelming at present.

Of course, there has also been very successful work on stabilization by May and coauthors, the two major milestones being [16,6]. At first glance, May's approach seems wedded to the topological situation, relying as it does on homeomorphisms $X_n \to \Omega X_{n+1}$. This is the reason we have not tried to use it in this paper. However, there has been considerable recent work showing that this approach may be more flexible than one might have expected. I have mentioned [18] above, but perhaps the most ambitious attempt to generalize S-modules has been initiated by Johnson [15].

Finally, we point out that Schwede [21] has shown that the methods of Bousfield and Friedlander [2] apply to certain more general model categories. His model categories are always simplicial and proper, and he is always inverting the ordinary suspension functor. Nevertheless, the paper [21] is the first serious attempt to define a general stabilization functor of which the author is aware.

This paper is organized as follows. We begin by defining the category $Sp^{\mathbb{N}}(\mathscr{D},T)$ and the associated projective model structure in Section 1. Then there is the brief Section 2 recalling Hirschhorn's approach to localization of model categories. We construct the stable model structure modulo certain technical lemmas in Section 3. The technical lemmas we need assert that if a model category $\mathscr{D}$ is left proper cellular, then so is the projective model structure on $Sp^{\mathbb{N}}(\mathscr{D}, T)$, and therefore we can apply the localization technology of Hirschhorn. We prove these technical lemmas, and the analogous lemmas for the projective model structure on symmetric spectra, in the appendix. In Section 4, we study the simplifications that arise when the adjoint U of T preserves sequential colimits and $\mathscr{D}$ is sufficiently well behaved. We characterize stable equivalences as the appropriate generalization of stable homotopy isomorphisms in this case, and we show the stable model structure is right proper, giving a description of the stable fibrations as well. In Section 5, we prove the functoriality, homotopy invariance, and homotopy idempotence of the construction $(\mathscr{D}, T) \mapsto Sp^{\mathbb{N}}(\mathscr{D}, T)$. We investigate monoidal structure in Section 6, showing that $Sp^{\mathbb{N}}(\mathscr{C}, T)$ is almost never a symmetric monoidal model category even when $\mathscr{C}$ is so.

This demonstrates the need for a better construction, and Section 7 begins the study of symmetric spectra. Since we have developed all the necessary techniques in the first part, the proofs in this part are more concise. In Section 7 we discuss the category of symmetric spectra. In Section 8 we construct the projective and stable model structures on symmetric spectra, and in Section 9, we discuss some properties of symmetric spectra. This includes functoriality, homotopy invariance, and homotopy idempotence of the stable model structure. We conclude the paper in Section 10 by constructing the chain of Quillen equivalences between $Sp^{\mathbb{N}}(\mathscr{C}, T)$ and $Sp^{\Sigma}(\mathscr{C}, K)$, under the cyclic permutation hypothesis mentioned above. Finally, as stated previously, there is an appendix verifying that the techniques of Hirschhorn can be applied to the projective model structures on $Sp^{\mathbb{N}}(\mathscr{D}, T)$ and symmetric spectra.

Obviously, considerable familiarity with model categories will be necessary to understand this paper. The original reference is [20], but a better introductory reference is [5]. More in depth references include [4,11,12]. In particular, we rely heavily on the localization technology in [11].

1. Spectra

In this section and throughout the paper, $\mathscr{D}$ will be a model category and $T : \mathscr{D} \to \mathscr{D}$ will be a left Quillen endofunctor of $\mathscr{D}$ with right adjoint U. In this section, we define the category $Sp^{\mathbb{N}}(\mathscr{D}, T)$ of spectra and construct the projective model structure on $Sp^{\mathbb{N}}(\mathscr{D}, T)$.

The following definition is a straightforward generalization of the usual notion of spectra [2].

Definition 1.1. Suppose T is a left Quillen endofunctor of a model category $\mathscr{D}$. Define $Sp^{\mathbb{N}}(\mathscr{D}, T)$, the category of *spectra*, as follows. A *spectrum* X is a sequence $X_0, X_1, \ldots, X_n, \ldots$ of objects of $\mathscr{D}$ together with structure maps $\sigma : TX_n \to X_{n+1}$ for all n. A *map of spectra* from X to Y is a collection of maps $f_n : X_n \to Y_n$ commuting with the structure maps; this means that the diagram below

$$\begin{array}{ccc} TX_n & \xrightarrow{\sigma_X} & X_{n+1} \\ {\scriptstyle Tf_n}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ TY_n & \xrightarrow[\sigma_Y]{} & Y_{n+1} \end{array}$$

is commutative for all n.

Note that if $\mathscr{D}$ is either the model category of pointed simplicial sets or the model category of pointed topological spaces, and T is the suspension functor given by smashing with the circle S^1, then $Sp^{\mathbb{N}}(\mathscr{D},T)$ is the Bousfield–Friedlander category of spectra [2].

Definition 1.2. Given $n \geq 0$, the *evaluation functor* $\mathrm{Ev}_n : Sp^{\mathbb{N}}(\mathscr{D},T) \to \mathscr{D}$ takes X to X_n. The evaluation functor has a left adjoint $F_n : \mathscr{D} \to Sp^{\mathbb{N}}(\mathscr{D},T)$ defined by $(F_nA)_m = T^{m-n}A$ if $m \geq n$ and $(F_nA)_m = 0$ otherwise, where 0 is the initial object of $\mathscr{D}$. The structure maps are the obvious ones.

Note that F_0 is an full and faithful embedding of the category $\mathscr{D}$ into $Sp^{\mathbb{N}}(\mathscr{D},T)$.

Lemma 1.3. *The category of spectra is bicomplete.*

Proof. Given a functor G from a small category $\mathscr{I}$ into $Sp^{\mathbb{N}}(\mathscr{D},T)$, we define

$$(\operatorname{colim} G)_n = \operatorname{colim} \mathrm{Ev}_n \circ G \quad \text{and} \quad (\lim X)_n = \lim \mathrm{Ev}_n \circ G.$$

Since T is a left adjoint, it preserves colimits. The structure maps of the colimit are then the composites

$$T(\operatorname{colim} \mathrm{Ev}_n \circ G) \cong \operatorname{colim}(T \circ \mathrm{Ev}_n \circ G) \xrightarrow{\operatorname{colim}(\sigma\circ G)} \operatorname{colim} \mathrm{Ev}_{n+1} \circ G.$$

Although T does not necessarily preserve limits, there is still a natural map

$$T(\lim H) \to \lim TH$$

for any functor $H : \mathscr{I} \to \mathscr{D}$. Then the structure maps of the limit are the composites

$$T(\lim \mathrm{Ev}_n \circ G) \to \lim(T \circ \mathrm{Ev}_n \circ G) \xrightarrow{\lim(\sigma\circ G)} \lim \mathrm{Ev}_{n+1} \circ G. \quad \square$$

Remark 1.4. The evaluation functor $\mathrm{Ev}_n : Sp^{\mathbb{N}}(\mathscr{D},T) \to \mathscr{D}$ also has a right adjoint $M_n : \mathscr{D} \to Sp^{\mathbb{N}}(\mathscr{D},T)$. We define $(M_nA)_i = U^{n-i}A$ if $i \leq n$, and $(M_nA)_i = 1$ if $i > n$, where 1 denotes the terminal object of $\mathscr{D}$. The structure map $TU^{n-i}A \to U^{n-i-1}A$ is adjoint to the identity map of $U^{n-i}A$ when $i < n$. We leave it to the reader to verify that M_n is right adjoint to Ev_n.

We wish to prolong the adjunction (T, U) to an adjunction of functors between spectra. We will discuss prolonging more general adjunctions in Section 5.

Lemma 1.5. *Suppose T is a left Quillen endofunctor of a model category $\mathscr{D}$, with right adjoint U. Define a functor $T : Sp^{\mathbb{N}}(\mathscr{D}, T) \to Sp^{\mathbb{N}}(\mathscr{D}, T)$ by $(TX)_n = TX_n$, with structure map*

$$T(TX_n) \xrightarrow{T\sigma} TX_{n+1},$$

where σ is the structure map of X. Define a functor $U : Sp^{\mathbb{N}}(\mathscr{D}, T) \to Sp^{\mathbb{N}}(\mathscr{D}, T)$ by $(UX)_n = UX_n$, with structure map adjoint to

$$UX_n \xrightarrow{U\tilde{\sigma}} U(UX_{n+1}),$$

where $\tilde{\sigma}$ is adjoint to the structure map of X. Then T is left adjoint to U.

Proof. We leave it to the reader to verify the functoriality of T and U. We show they are adjoint. For convenience, let us denote the extensions of T and U to functors of spectra by $\tilde{T}$ and $\tilde{U}$. It suffices to construct unit maps $X \to \tilde{U}\tilde{T}X$ and counit maps $\tilde{T}\tilde{U}X \to X$ verifying the triangle identities, by Mac Lane [17, Theorem 4.1.2(v)]. But we can take these unit and counit maps to be the maps which are the unit and counit maps of the (T, U) adjunction in each degree. The reader should verify that these are maps of spectra. The triangle identities then follow immediately from the triangle identities of the (T, U) adjunction. □

The following remark is critically important to the understanding of our approach to spectra.

Remark 1.6. The definition we have just given of the prolongation of T to an endofunctor of $Sp^{\mathbb{N}}(\mathscr{D}, T)$ is the only possible definition under our very general hypotheses. However, this definition *does not generalize the definition of the suspension* when $\mathscr{D}$ is the category of pointed topological spaces and $TA = A \wedge S^1$. Indeed, recall from [2] that the suspension of a spectrum X in this case is defined by $(X \otimes S^1)_n = X_n \wedge S^1$, with structure map given by

$$X_n \wedge S^1 \wedge S^1 \xrightarrow{1 \wedge t} X_n \wedge S^1 \wedge S^1 \xrightarrow{\sigma \wedge 1} X_{n+1} \wedge S^1,$$

where t is the twist isomorphism. On the other hand, if we apply our definition of the prolongation of T above, we get a functor $X \mapsto X \bar{\otimes} S^1$ defined by $(X \bar{\otimes} S^1)_n = X_n \wedge S^1$ with structure map

$$X_n \wedge S^1 \wedge S^1 \xrightarrow{\sigma \wedge 1} X_{n+1} \wedge S^1.$$

This is a crucial and subtle difference whose ramifications we will study in Section 10.

We now show that $Sp^{\mathbb{N}}(\mathscr{D}, T)$ inherits a model structure from $\mathscr{D}$, called the *projective model structure*. The functor $T : Sp^{\mathbb{N}}(\mathscr{D}, T) \to Sp^{\mathbb{N}}(\mathscr{D}, T)$ will be a left Quillen functor with respect to the projective model structure, but it will not be a Quillen equivalence. Our approach to the projective model structure owes much to [2,13, Section 5.1]. At this point, we will slip into the standard model category terminology and notation, all of which can be found in [12], mostly in Section 2.1.

Definition 1.7. A map $f \in Sp^{\mathbb{N}}(\mathscr{D}, T)$ is a *level equivalence* if each map f_n is a weak equivalence in $\mathscr{D}$. Similarly, f is a *level fibration* (resp. *level cofibration*, *level trivial fibration*, *level trivial cofibration*) if each map f_n is a fibration (resp. cofibration, trivial fibration, trivial cofibration) in $\mathscr{D}$. The map f is a *projective cofibration* if f has the left lifting property with respect to every level trivial fibration.

Note that level equivalences satisfy the two out of three property, and each of the classes defined above is closed under retracts. Thus, we might be able to construct a model structure using these classes. To do so, we need the small object argument, and hence we assume that $\mathscr{D}$ is cofibrantly generated (see [12, Section 2.1] for a discussion of cofibrantly generated model categories).

Definition 1.8. Suppose $\mathscr{D}$ is a cofibrantly generated model category with generating cofibrations I and generating trivial cofibrations J. Suppose T is a left Quillen endofunctor of $\mathscr{D}$, and form the category of spectra $Sp^{\mathbb{N}}(\mathscr{D}, T)$. Define sets of maps in $Sp^{\mathbb{N}}(\mathscr{D}, T)$ by $I_T = \bigcup_n F_n I$ and $J_T = \bigcup_n F_n J$.

The sets I_T and J_T will be the generating cofibrations and trivial cofibrations for a model structure on $Sp^{\mathbb{N}}(\mathscr{D}, T)$. There is a standard method for proving this, based on the small object argument [12, Theorem 2.1.14]. The first step is to show that the domains of I_T and J_T are small, in the sense of [12, Definition 2.1.3].

Proposition 1.9. *Suppose A is small relative to the cofibrations in $\mathscr{D}$, and $n \geq 0$. Then $F_n A$ is small relative to the level cofibrations in $Sp^{\mathbb{N}}(\mathscr{D}, T)$. Similarly, if A is small relative to the trivial cofibrations in $\mathscr{D}$, then $F_n A$ is small relative to the level trivial cofibrations in $Sp^{\mathbb{N}}(\mathscr{D}, T)$.*

Proof. The main point is that Ev_n commutes with colimits. We leave the remainder of the proof to the reader. □

To apply this to the domains of I_T, we need to know that the maps of I_T-cof are level cofibrations. See [12, Definition 2.1.7] for the definition of I_T-cof, and similar notations such as I_T-inj. Recall the right adjoint M_n of Ev_n constructed in Remark 1.4.

Lemma 1.10. *A map f in $Sp^{\mathbb{N}}(\mathscr{D}, T)$ is a level cofibration if and only if it has the left lifting property with respect to $M_n p$ for all $n \geq 0$ and all trivial fibrations p in $\mathscr{D}$. Similarly, f is a level trivial cofibration if and only if it has the left lifting property with respect to $M_n p$ for all $n \geq 0$ and all fibrations $p \in \mathscr{D}$.*

Proof. By adjunction, a map f has the left lifting property with respect to $M_n p$ if and only if $\mathrm{Ev}_n f$ has the left lifting property with respect to p. Since a map is a cofibration (resp. trivial cofibration) in $\mathscr{D}$ if and only if it has the left lifting property with respect to all trivial fibrations (resp. fibrations), the lemma follows. □

Proposition 1.11. *Every map in I_T-cof is a level cofibration. Every map in J_T-cof is a level trivial cofibration.*

Proof. Since T is a left Quillen functor, every map in I_T is a level cofibration. By Lemma 1.10, this means that $M_n p \in I_T$-inj for all $n \geq 0$ and all trivial fibrations p. Since a map in I_T-cof has the left lifting property with respect to every map in I_T-inj, in particular it has the left lifting property with respect to $M_n p$. Another application of Lemma 1.10 completes the proof for I_T-cof. The proof for J_T-cof is similar. □

Corollary 1.12. *The domains of I_T are small relative to I_T-cof. The domains of J_T are small relative to J_T-cof.*

Proof. Since $\mathscr{D}$ is cofibrantly generated, the domains of I are small relative to the cofibrations in $\mathscr{D}$, and the domains of J are small relative to the trivial cofibrations in $\mathscr{D}$ (see [12, Proposition 2.1.18]). Propositions 1.9 and 1.11 complete the proof. □

We remind the reader that a model structure is *left proper* if the pushout of a weak equivalence through a cofibration is again a weak equivalence. Similarly, a model structure is *right proper* if the pullback of a weak equivalence through a fibration is again a weak equivalence. A model structure is *proper* if it is both left and right proper. See [11, Chapter 11] for more information about properness.

Theorem 1.13. *Suppose $\mathscr{D}$ is cofibrantly generated. Then the projective cofibrations, the level fibrations, and the level equivalences define a cofibrantly generated model structure on $Sp^{\mathbb{N}}(\mathscr{D}, T)$, with generating cofibrations I_T and generating trivial cofibrations J_T. We call this the projective model structure. The projective model structure is left proper (resp. right proper, proper) if $\mathscr{D}$ is left proper (resp. right proper, proper.)*

Note that if $\mathscr{D}$ is either the model category of pointed simplicial sets or pointed topological spaces, and T is the suspension functor, the projective model structure on $Sp^{\mathbb{N}}(\mathscr{D}, T)$ is the strict model structure on the Bousfield–Friedlander category of spectra [2].

Proof. The retract and two out of three axioms are immediate, as is the lifting axiom for a projective cofibration and a level trivial fibration. By adjointness, a map is a level trivial fibration if and only if it is in I_T-inj. Hence a map is a projective cofibration if and only if it is in I_T-cof. The small object argument [12, Theorem 2.1.14] applied to I_T then produces a functorial factorization into a projective cofibration followed by a level trivial fibration.

Adjointness implies that a map is a level fibration if and only if it is in J_T-inj. We have already seen in Proposition 1.11 that the maps in J_T-cof are level equivalences, and they are projective cofibrations since they have the left lifting property with respect to all level fibrations, and in particular level trivial fibrations. Hence the small object argument applied to J_T produces a functorial factorization into a projective cofibration and level equivalence followed by a level fibration.

Conversely, we claim that any projective cofibration and level equivalence f is in J_T-cof, and hence has the left lifting property with respect to level fibrations. To see this, write $f = pi$ where i is in J_T-cof and p is in J_T-inj. Then p is a level fibration. Since f and i are both level equivalences, so is p. Thus f has the left lifting property with respect to p, and so f is a retract of i by the retract argument [12, Lemma 1.1.9]. In particular $f \in J_T$-cof.

Since colimits and limits in $Sp^{\mathbb{N}}(\mathscr{D}, T)$ are taken levelwise, and since every projective cofibration is in particular a level cofibration, the statements about properness are immediate. □

We also characterize the projective cofibrations. We denote the pushout of two maps $A \to B$ and $A \to C$ by $B \amalg_A C$.

Proposition 1.14. *A map* $i : A \to B$ *of spectra is a projective cofibration if and only if the induced maps* $i_0 : A_0 \to B_0$ *and* $j_n : A_n \amalg_{TA_{n-1}} TB_{n-1} \to B_n$ *for* $n \geq 1$ *are cofibrations in* $\mathscr{D}$. *Similarly,* i *is a projective trivial cofibration if and only if* i_0 *and* j_n *for* $n \geq 1$ *are trivial cofibrations in* $\mathscr{D}$.

Proof. We only prove the cofibration case, leaving the similar trivial cofibration case to the reader. First suppose $i : A \to B$ is a projective cofibration. We have already seen in Proposition 1.11 that $A_0 \to B_0$ is a cofibration. We show that j_n is a cofibration by showing that j_n has the left lifting property with respect to any trivial fibration $p : X \to Y$ in $\mathscr{D}$. So suppose we have the commutative diagram below:

$$\begin{array}{ccc} A_n \amalg_{TA_{n-1}} TB_{n-1} & \longrightarrow & X \\ {\scriptstyle j_n}\downarrow & & \downarrow{\scriptstyle p} \\ B_n & \longrightarrow & Y. \end{array}$$

We must construct a lift in this diagram. By adjointness, it suffices to construct a lift in the induced diagram below:

$$\begin{array}{ccc} A & \longrightarrow & M_n X \\ {\scriptstyle i}\downarrow & & \downarrow \\ B & \longrightarrow & M_n Y \times_{M_{n-1}UY} M_{n-1}UX, \end{array}$$

where M_n is the right adjoint of Ev_n. Using the description of M_n given in Remark 1.4, one can check that the map $M_nX \to M_nY \times_{M_{n-1}UY} M_{n-1}UX$ is a level trivial fibration, so a lift exists.

Conversely, suppose that i_0 and j_n are cofibrations in $\mathscr{D}$ for $n > 0$. We show that i is a projective cofibration by showing that i has the left lifting property with respect to any level trivial fibration $p : X \to Y$ in $Sp^{\mathbb{N}}(\mathscr{D}, T)$. So suppose we have the commutative diagram below:

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y. \end{array}$$

We construct a lift $h_n : B_n \to X_n$, compatible with the structure maps, by induction on n. There is no difficulty defining h_0, since i_0 has the left lifting property with respect to the trivial fibration p_0. Suppose we have defined h_j for $j < n$. Then by lifting in the induced diagram below:

$$\begin{array}{ccc} A_n \amalg_{TA_{n-1}} TB_{n-1} & \xrightarrow{(f_n, \sigma \circ Th_{n-1})} & X_n \\ \big\downarrow & & \big\downarrow{\scriptstyle p_n} \\ B_n & \xrightarrow[g_n]{} & Y_n, \end{array}$$

we find the required map $h_n : B_n \to X_n$. □

Finally, we point out that the prolongation of T is still a Quillen functor.

Proposition 1.15. *Give $Sp^{\mathbb{N}}(\mathscr{D}, T)$ the projective model structure. Then the prolongation $T : Sp^{\mathbb{N}}(\mathscr{D}, T) \to Sp^{\mathbb{N}}(\mathscr{D}, T)$ of T is a Quillen functor. Furthermore, the functor $F_n : \mathscr{D} \to Sp^{\mathbb{N}}(\mathscr{D}, T)$ is a Quillen functor.*

Proof. The functor Ev_n obviously takes level fibrations to fibrations and level trivial fibrations to trivial fibrations. Hence Ev_n is a right Quillen functor, and so its left adjoint F_n is a left Quillen functor. Similarly, the prolongation of U to a functor $U : Sp^{\mathbb{N}}(\mathscr{D}, T) \to Sp^{\mathbb{N}}(\mathscr{D}, T)$ preserves level fibrations and level trivial fibrations, so its left adjoint T is a Quillen functor. □

2. Bousfield localization

We will define the stable model structure on $Sp^{\mathbb{N}}(\mathscr{D}, T)$ in Section 3 as a Bousfield localization of the projective model structure on $Sp^{\mathbb{N}}(\mathscr{D}, T)$. In this section we recall the theory of Bousfield localization of model categories from [11].

To do so, we need some preliminary remarks related to function complexes. Details can be found in [4; 11, Chapter 18; 12, Chapter 5]. First of all, given an object A in a model category, we denote by QA a functorial cofibrant replacement of A [12, p. 5]. This means that QA is cofibrant and there is a natural trivial fibration $QA \to A$. Similarly, RA denotes a functorial fibrant replacement of A, so that RA is fibrant and there is a natural trivial cofibration $A \to RA$. By repeatedly using functorial factorization, we can construct, given an object A in a model category $\mathscr{C}$, a functorial cosimplicial resolution of A. By mapping out of this cosimplicial resolution we get a simplicial set $\mathrm{Map}_\ell(A,X)$. Similarly, there is a functorial simplicial resolution of X, and by mapping into it we get a simplicial set $\mathrm{Map}_r(A,X)$. One should think of these as replacements for the simplicial structure present in a simplicial model category. These function complexes will not be homotopy invariant in general, so we define the homotopy function complex as $\mathrm{map}(A,X) = \mathrm{Map}_r(QA,RX)$. Then $\mathrm{map}(A,X)$ is canonically isomorphic in the homotopy category $\mathrm{Ho}\,\mathbf{SSet}$ of simplicial sets to $\mathrm{Map}_\ell(QA,RX)$, and defines a functor $\mathrm{Ho}\,\mathscr{C}^{\mathrm{op}} \times \mathrm{Ho}\,\mathscr{C} \to \mathrm{Ho}\,\mathbf{SSet}$. The homotopy function complex defines an enrichment of $\mathrm{Ho}\,\mathscr{C}$ over $\mathrm{Ho}\,\mathbf{SSet}$. In fact, $\mathrm{Ho}\,\mathscr{C}$ is naturally tensored and cotensored over $\mathrm{Ho}\,\mathbf{SSet}$, as well as enriched over it. In particular, if Φ is an arbitrary left Quillen functor between model categories with right adjoint Γ, we have $\mathrm{map}((L\Phi)X,Y) \cong \mathrm{map}(X,(R\Gamma)Y)$ in $\mathrm{Ho}\,\mathbf{SSet}$, where $(L\Phi)X = \Phi QX$ is the total left derived functor of Φ and $(R\Gamma)Y = \Gamma RY$ is the total right derived functor of Γ.

Definition 2.1. Suppose we have a set $\mathscr{S}$ of maps in a model category $\mathscr{C}'$.

1. A *$\mathscr{S}$-local object* of $\mathscr{C}$ is a fibrant object W such that, for every $f: A \to B$ in $\mathscr{S}$, the induced map $\mathrm{map}(B,W) \to \mathrm{map}(A,W)$ is an isomorphism in $\mathrm{Ho}\,\mathbf{SSet}$.
2. A *$\mathscr{S}$-local equivalence* is a map $g: A \to B$ in $\mathscr{C}$ such that the induced map $\mathrm{map}(B,W) \to \mathrm{map}(A,W)$ is an isomorphism in $\mathrm{Ho}\,\mathbf{SSet}$ for all $\mathscr{S}$-local objects W.

By Hirschhorn [11, Theorem 3.3.8], $\mathscr{S}$-local equivalences between $\mathscr{S}$-local objects are in fact weak equivalences. In outline, one proves this by first reducing to the case where $f: A \to B$ is a cofibration and $\mathscr{S}$-local equivalence between cofibrant $\mathscr{S}$-local objects. Then, since f is a cofibration and A is fibrant, $\mathrm{Map}_r(f,A)\colon \mathrm{Map}_r(B,A) \to \mathrm{Map}_r(A,A)$ is a fibration of simplicial sets [12, Corollary 5.4.4]. Since f is an $\mathscr{S}$-local equivalence and A is $\mathscr{S}$-local, $\mathrm{Map}_r(f,A)$ is also a weak equivalence, and so a trivial fibration of simplicial sets. In particular, $\mathrm{Map}_r(f,A)$ is surjective. Any preimage of the identity map is a homotopy inverse to f.

We will define cellular model categories, a special class of cofibrantly generated model categories, in the appendix. The main theorem of [11] is that Bousfield localizations of cellular model categories always exist. More precisely, Hirschhorn proves the following theorem.

Theorem 2.2. *Suppose $\mathscr{S}$ is a set of maps in a left proper cellular model category $\mathscr{C}$. Then there is a left proper cellular model structure on $\mathscr{C}$ where the weak equivalences* *are the $\mathscr{S}$-local equivalences and the cofibrations remain unchanged. The $\mathscr{S}$-local objects are the fibrant objects in this model structure. We denote this new model category by $L_{\mathscr{S}}\mathscr{C}$ and refer to it as the* Bousfield localization *of $\mathscr{C}$ with respect to $\mathscr{S}$. Left Quillen functors from $L_{\mathscr{S}}\mathscr{C}$ to $\mathscr{D}$ are in one to one correspondence with left Quillen functors $\Phi\colon \mathscr{C} \to \mathscr{D}$ such that $\Phi(Qf)$ is a weak equivalence for all $f \in \mathscr{S}$.*

We will also need the following fact about localizations, which is implicit in [11, Chapter 4].

Proposition 2.3. *Suppose $\mathscr{C}$ and $\mathscr{C}'$ are left proper cellular model categories, $\mathscr{S}$ is a set of maps in $\mathscr{C}$, and $\mathscr{S}'$ is a set of maps in $\mathscr{C}'$. Suppose $\Phi : \mathscr{C} \to \mathscr{C}'$ is a Quillen equivalence with right adjoint Γ, and suppose $\Phi(Qf)$ is a $\mathscr{S}'$-local equivalence for all $f \in \mathscr{S}$. Then Φ induces a Quillen* equivalence *$\Phi : L_{\mathscr{S}}\mathscr{C} \to L_{\mathscr{S}'}\mathscr{C}'$ if and only if, for every $\mathscr{S}$-local $X \in \mathscr{C}$, there is a $\mathscr{S}'$-local Y in $\mathscr{C}'$ such that X is weakly equivalent in $\mathscr{C}$ to ΓY. This condition will hold if, for all fibrant Y in $\mathscr{C}'$ such that ΓY is $\mathscr{S}$-local, Y is $\mathscr{S}'$-local.*

Proof. Suppose first that Φ does induce a Quillen equivalence on the localizations, and suppose that X is $\mathscr{S}$-local. Then QX is also $\mathscr{S}$-local, by Hirschhorn [11, Lemma 3.3.1]. Let $L_{\mathscr{S}'}$ denote a fibrant replacement functor in $L_{\mathscr{S}'}\mathscr{C}'$. Then, because Φ is a Quillen equivalence on the localizations, the map $QX \to \Gamma L_{\mathscr{S}'}\Phi QX$ is a weak equivalence in $L_{\mathscr{S}}\mathscr{C}$ (see [12, Section 1.3.3]). But both QX and $\Gamma L_{\mathscr{S}'}\Phi QX$ are $\mathscr{S}$-local, so $QX \to \Gamma L_{\mathscr{S}'}\Phi QX$ is a weak equivalence in $\mathscr{C}$. Hence X is weakly equivalent in $\mathscr{C}$ to ΓY, where Y is the $\mathscr{S}'$-local object $L_{\mathscr{S}'}\Phi QX$.

The first step in proving the converse is to note that, since Φ is a Quillen equivalence before localizing, the map $\Phi Q\Gamma X \to X$ is a weak equivalence for all fibrant X. Since the functor Q does not change upon localization, $\Phi Q\Gamma X \to X$ is a $\mathscr{S}'$-local equivalence for every $\mathscr{S}'$-local object of $\mathscr{C}'$. Thus Φ is a Quillen equivalence after localization if and only if Φ reflects local equivalences between cofibrant objects, by Hovey [12, Corollary 1.3.16].

Suppose, then, that $f\colon A \to B$ is a map between cofibrant objects such that Φf is a $\mathscr{S}'$-local equivalence. We must show that $\operatorname{map}(f,X)$ is an isomorphism in Ho **SSet** for all $\mathscr{S}$-local X. Adjointness implies that $\operatorname{map}(f,\Gamma Y)$ is an isomorphism for all $\mathscr{S}'$-local Y, and our condition then guarantees that this is enough to conclude that $\operatorname{map}(f,X)$ is an isomorphism for all $\mathscr{S}$-local X. This completes the proof of the converse.

We still need to prove the last statement of the proposition. So suppose X is $\mathscr{S}$-local. Then QX is also $\mathscr{S}$-local, again by Hirschhorn [11, Lemma 3.3.1], and, in $\mathscr{C}$, we have a weak equivalence $QX \to \Gamma R\Phi QX$. Our assumption then guarantees that $Y = R\Phi QX$ is $\mathscr{S}'$-local, and X is indeed weakly equivalent to ΓY. □

The fibrations in $L_{\mathscr{S}}\mathscr{C}$ are not completely understood [11, Section 3.6]. The $\mathscr{S}$-local fibrations between $\mathscr{S}$-local fibrant objects are just the usual fibrations. In case both

$\mathscr{C}$ and $L_{\mathscr{S}}\mathscr{C}$ are right proper, there is a characterization of the $\mathscr{S}$-local fibrations in terms of homotopy pullbacks analogous to the characterization of stable fibrations of spectra in [2]. However, $L_{\mathscr{S}}\mathscr{C}$ need not be right proper even if $\mathscr{C}$ is, as is shown by the example of Γ-spaces in [2], where it is also shown that the expected characterization of $\mathscr{S}$-local fibrations does not hold.

3. The stable model structure

Our plan now is to apply Bousfield localization to the projective model structure on $Sp^{\mathbb{N}}(\mathscr{D},T)$ to obtain a model structure with respect to which T is a Quillen equivalence. In order to do this, we will have to prove that the projective model structure makes $Sp^{\mathbb{N}}(\mathscr{D},T)$ into a cellular model category when $\mathscr{D}$ is left proper cellular. We will prove this technical result in the appendix. In this section, we will assume that $Sp^{\mathbb{N}}(\mathscr{D},T)$ is cellular, find a good set $\mathscr{S}$ of maps to form the stable model structure as the $\mathscr{S}$-localization of the projective model structure, and prove that T is a Quillen equivalence with respect to the stable model structure.

Just as in symmetric spectra [13], we want the stable equivalences to be maps which induce isomorphisms on all cohomology theories. Cohomology theories will be represented by the appropriate analogue of Ω-spectra.

Definition 3.1. A spectrum X is a *U-spectrum* if X is level fibrant and the adjoint $X_n \xrightarrow{\tilde{\sigma}} UX_{n+1}$ of the structure map of X is a weak equivalence for all $n \geq 0$.

Of course, if $\mathscr{D}$ is the category of pointed simplicial sets or pointed topological spaces, and T is the suspension functor, U-spectra are just Ω-spectra. We will find a set $\mathscr{S}$ of maps of $Sp^{\mathbb{N}}(\mathscr{D},T)$ such that the $\mathscr{S}$-local objects are the U-spectra. To do so, note that if $\operatorname{map}(A,X_n) \to \operatorname{map}(A,UX_{n+1})$ is an isomorphism in $\operatorname{Ho}\mathbf{SSet}$ for all cofibrant A in $\mathscr{D}$, then $X_n \to UX_{n+1}$ is a weak equivalence by Hirschhorn [11, Theorem 18.8.7]. Since $\mathscr{D}$ is cofibrantly generated, we should not need all cofibrant A, but only those A related to the generating cofibrations. This is true, but the proof is somewhat technical.

Proposition 3.2. *Suppose $\mathscr{C}$ is a left proper cofibrantly generated model category with generating cofibrations I, and $f: X \to Y$ is a map in $\mathscr{C}$. Then f is a weak equivalence if and only if* $\operatorname{map}(C,X) \to \operatorname{map}(C,Y)$ *is an isomorphism in* $\operatorname{Ho}\mathbf{SSet}$ *for all domains and codomains C of maps of I.*

This proof will depend on the fact that $\operatorname{Map}_r(-,RZ)$ converts colimits in $\mathscr{C}$ to limits of simplicial sets, cofibrations in $\mathscr{C}$ to fibrations of simplicial sets, trivial cofibrations in $\mathscr{C}$ to trivial fibrations of simplicial sets, and weak equivalences between cofibrant objects in $\mathscr{C}$ to weak equivalences between fibrant simplicial sets. These properties follow from [12, Corollary 5.4.4] and Ken Brown's lemma [12, Lemma 1.1.12].

Proof. The only if half follows from [11, Theorem 18.8.7]. Conversely, suppose that $\mathrm{map}(C, f)$ is an isomorphism in $\mathrm{Ho}\,\mathbf{SSet}$ for all domains and codomains of maps of I. It suffices to show that $\mathrm{map}(A, f)$ is an isomorphism for all cofibrant objects A, by Hirschhorn [11, Theorem 18.8.7]. But every cofibrant object is a retract of an I-cell complex (i.e. an object A such that the map $0 \to A$ is a transfinite composition of pushouts of maps of I), so it suffices to prove that $\mathrm{map}(A, f)$ is an isomorphism for all cell complexes A. This is equivalent to showing that $\mathrm{Map}_r(A, Rf)$ is a weak equivalence for all cell complexes A. Given a cell complex A, there is an ordinal λ and a λ-sequence

$$0 = A_0 \to A_1 \to \cdots \to A_\beta \to \cdots$$

with colimit $A_\lambda = A$, where each map $i_\beta : A_\beta \to A_{\beta+1}$ is a pushout of a map of I. We will show by transfinite induction on β that $\mathrm{Map}_r(A_\beta, Rf)$ is a weak equivalence for all $\beta \le \lambda$. Taking $\beta = \lambda$ completes the proof.

The base case of the induction is trivial, since $A_0 = 0$. For the successor ordinal case, we suppose $\mathrm{Map}_r(A_\beta, Rf)$ is a weak equivalence and prove that $\mathrm{Map}_r(A_{\beta+1}, Rf)$ is a weak equivalence. We have the pushout square below:

$$\begin{array}{ccc} C & \longrightarrow & A_\beta \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle i_\beta} \\ D & \longrightarrow & A_{\beta+1}, \end{array}$$

where g is a map of I. We must first replace this pushout square by a weakly equivalent pushout square in which all the objects are cofibrant, which we can do because $\mathscr{C}$ is left proper. Begin by factoring the composite $QC \to C \to D$ into a cofibration $\tilde{g} : QC \to \tilde{D}$ followed by a trivial fibration $\tilde{D} \to D$. In the terminology of [11], $\tilde{g}$ is a cofibrant approximation to g. By Hirschhorn [11, Proposition 11.3.2], there is a cofibrant approximation $\widetilde{i_\beta} : \widetilde{A_\beta} \to \widetilde{A_{\beta+1}}$ to i_β which is a pushout of $\tilde{g}$. That is, we have constructed the pushout square below:

$$\begin{array}{ccc} QC & \longrightarrow & \widetilde{A_\beta} \\ {\scriptstyle \tilde{g}}\downarrow & & \downarrow{\scriptstyle \widetilde{i_\beta}} \\ \tilde{D} & \longrightarrow & \widetilde{A_{\beta+1}} \end{array}$$

and a map from this pushout square to the original one that is a weak equivalence at each corner. By the properties of $\mathrm{Map}_r(-, RZ)$ mentioned in the paragraph preceding this proof, we have two pullback squares of fibrant simplicial sets as below:

$$\begin{array}{ccc} \mathrm{Map}_r(\widetilde{A_{\beta+1}}, RZ) & \longrightarrow & \mathrm{Map}_r(\tilde{D}, RZ) \\ \downarrow & & \downarrow \\ \mathrm{Map}_r(\widetilde{A_{\beta}}, RZ) & \longrightarrow & \mathrm{Map}_r(QC, RZ), \end{array}$$

where $Z=X$ and $Z=Y$, respectively. Here the vertical maps are fibrations. There is a map from the square with $Z=X$ to the square with $Z=Y$ induced by f. By hypothesis, this map is a weak equivalence on every corner except possibly the upper left. But then Dan Kan's cube lemma (see [12, Lemma 5.2.6], where the dual of the version we need is proved, or [4]) implies that the map on the upper left corner $\mathrm{Map}_r(\widetilde{A_{\beta+1}}, Rf)$ is also a weak equivalence. Since $\mathrm{Map}_r(-, RZ)$ preserves weak equivalences between cofibrant objects for any Z (see the paragraph preceding this proof), it follows that $\mathrm{Map}_r(A_{\beta+1}, Rf)$ is a weak equivalence.

We must still carry out the limit ordinal case of the induction. Suppose β is a limit ordinal and $\mathrm{Map}_r(A_\gamma, Rf)$ is a weak equivalence for all $\gamma < \beta$. We must show that $\mathrm{Map}_r(A_\beta, Rf)$ is a weak equivalence. For $Z = X$ or $Z = Y$, the simplicial sets $\mathrm{Map}_r(A_\gamma, RZ)$ define a limit-preserving functor $\beta^{\mathrm{op}} \to \mathbf{SSet}$ such that each map $\mathrm{Map}_r(A_{\gamma+1}, RZ) \to \mathrm{Map}_r(A_\gamma, RZ)$ is a fibration of fibrant simplicial sets, using the properties of $\mathrm{Map}_r(-, RZ)$ mentioned in the paragraph preceding this proof. There is a natural transformation from the functor with $Z=X$ to the functor with $Z=Y$, and by hypothesis this map is a weak equivalence at every stage. As explained in Section 5.1 of [12], there is a model structure on functors $\beta^{\mathrm{op}} \to \mathbf{SSet}$ where the weak equivalences and fibrations are taken levelwise. Both diagrams $\mathrm{Map}_r(A_\gamma, RX)$ and $\mathrm{Map}_r(A_\gamma, RY)$ are fibrant, since each simplicial set in them is fibrant. The inverse limit is a right Quillen functor [12, Corollary 5.1.6], and so preserves weak equivalences between fibrant objects by Ken Brown's lemma [12, Lemma 1.1.12]. Thus the inverse limit $\mathrm{Map}_r(A_\beta, Rf)$ is a weak equivalence, as required. This completes the transfinite induction and the proof. □

Note that the left properness assumption in Proposition 3.2 is unnecessary when the domains of the generating cofibrations are themselves cofibrant, since there is then no need to apply cofibrant approximation.

In view of Proposition 3.2, we need to choose our set $\mathscr{S}$ so as to make

$$\mathrm{map}(C, X_n) \to \mathrm{map}(C, UX_{n+1})$$

an isomorphism in $\mathrm{Ho}\,\mathbf{SSet}$ for all $\mathscr{S}$-local objects X and all domains and codomains C of the generating cofibrations I. Adjointness implies that, if X is level fibrant, $\mathrm{map}(C, X_n) \cong \mathrm{map}(F_nQC, X)$ in $\mathrm{Ho}\,\mathbf{SSet}$, since $F_nQC = (LF_n)C$, where LF_n is the total left derived functor of F_n. Also, $\mathrm{map}(C, UX_{n+1}) \cong \mathrm{map}(F_{n+1}TQC, X)$. In view of this, we make the following definition.

Definition 3.3. Suppose $\mathscr{D}$ is a left proper cellular model category with generating cofibrations I, and T is a left Quillen endofunctor of $\mathscr{D}$. Define the set $\mathscr{S}$ of maps in $Sp^{\mathbb{N}}(\mathscr{D},T)$ as $\{F_{n+1}TQC \xrightarrow{\zeta_n^{QC}} F_nQC\}$, as C runs through the set of domains and codomains of the maps of I and n runs through the non-negative integers. Here the map ζ_n^{QC} is adjoint to the identity map of TQC, and so is an isomorphism in degrees greater than n. Define the *stable model structure* on $Sp^{\mathbb{N}}(\mathscr{D},T)$ to be the localization of the projective model structure on $Sp^{\mathbb{N}}(\mathscr{D},T)$ with respect to this set $\mathscr{S}$. We refer to the $\mathscr{S}$-local weak equivalences as *stable equivalences*, and to the $\mathscr{S}$-local fibrations as *stable fibrations*.

The referee points out that Definition 3.3 is an implementation of Adams' "cells now – maps later" philosophy [1, p. 142]. Indeed, a map $F_nQC \xrightarrow{e} X$ can be thought of as a cell of the spectrum X, at least when C is a codomain of one of the generating cofibrations of I. Inverting the map $F_{n+1}TQC \to F_nQC$ is tantamount to allowing a map from X to Y to be defined on the cell e only after applying T some number of times.

Theorem 3.4. *Suppose $\mathscr{D}$ is a left proper cellular model category and T is a left Quillen endofunctor of $\mathscr{D}$. Then the stably fibrant objects in $Sp^{\mathbb{N}}(\mathscr{D},T)$ are the U-spectra. Furthermore, for all cofibrant $A \in \mathscr{D}$ and for all $n \geq 0$, the map $F_{n+1}TA \xrightarrow{\zeta_n^A} F_nA$ is a stable equivalence.*

Proof. By definition, X is $\mathscr{S}$-local if and only if X is level fibrant and

$$\operatorname{map}(F_nQC,X) \to \operatorname{map}(F_{n+1}TQC,X)$$

is an isomorphism in Ho **SSet** for all $n \geq 0$ and all domains and codomains C of maps of I. By the comments preceding Definition 3.3, this is equivalent to requiring that X be level fibrant and that the map $\operatorname{map}(C,X_n) \to \operatorname{map}(C,UX_{n+1})$ be an isomorphism for all $n \geq 0$ and all domains and codomains C of maps of I. By Proposition 3.2, this is equivalent to requiring that X be a U-spectrum.

Now, by definition, ζ_n^A is a stable equivalence if and only if $\operatorname{map}(\zeta_n^A,X)$ is a weak equivalence for all U-spectra X. But by adjointness, $\operatorname{map}(\zeta_n^A,X)$ can be identified with $\operatorname{map}(A,X_n) \to \operatorname{map}(A,UX_{n+1})$. Since $X_n \to UX_{n+1}$ is a weak equivalence between fibrant objects, $\operatorname{map}(\zeta_n^A,X)$ is an isomorphism in Ho **SSet**, by Hovey [12, Corollary 5.4.8]. □

We would now like to claim that the stable model structure on $Sp^{\mathbb{N}}(\mathscr{D},T)$ that we have just defined is a generalization of the stable model structure on spectra of topological spaces or simplicial sets defined in [2]. This cannot be a trivial observation, however, both because our approach is totally different and because of Remark 1.6.

Corollary 3.5. *If $\mathscr{D}$ is either the category of pointed simplicial sets or pointed topological spaces, and T is the suspension functor given by smashing with S^1, then the* *stable model structure on $Sp^{\mathbb{N}}(\mathscr{D},T)$ coincides with the stable model structure on the category of Bousfield–Friedlander spectra* [2].

Proof. We know already that the cofibrations are the same in the stable model structure on $Sp^{\mathbb{N}}(\mathscr{D},T)$ and the stable model structure of [2]. We will show that the weak equivalences are the same. In any model category at all, a map f is a weak equivalence if and only if $\operatorname{map}(f,X)$ is an isomorphism in Ho **SSet** for all fibrant X, by Hirschhorn [11, Theorem 18.8.7]. Construction of $\operatorname{map}(f,X)$ requires replacing f by a cofibrant approximation f' and building cosimplicial resolutions of the domain and codomain of f'. In the case at hand, we can do the cofibrant replacement and build the cosimplicial resolutions in the projective model category of spectra, since the cofibrations do not change under localization. Thus $\operatorname{map}(f,X)$ is the same in both the stable model structure on $Sp^{\mathbb{N}}(\mathscr{D},T)$ and in the stable model category of Bousfield and Friedlander. Since the stably fibrant objects are also the same, the corollary holds. □

We now begin the process of proving that the prolongation of T is a Quillen equivalence with respect to the stable model structure on $Sp^{\mathbb{N}}(\mathscr{D},T)$.

Lemma 3.6. *Suppose $\mathscr{D}$ is a left proper cellular model category and T is a left Quillen endofunctor of $\mathscr{D}$. Then the prolongation of T to a functor $T: Sp^{\mathbb{N}}(\mathscr{D},T) \to Sp^{\mathbb{N}}(\mathscr{D},T)$ is a left Quillen functor with respect to the stable model structure.*

Proof. In view of Hirschhorn's localization Theorem 2.2, we must show that $T(Qf)$ is a stable equivalence for all $f \in \mathscr{S}$. Since the domains and codomains of the maps of $\mathscr{S}$ are already cofibrant, it is equivalent to show that Tf is a stable equivalence for all $f \in \mathscr{S}$. Since $TF_n = F_nT$, we have $T(\zeta_n^A) = \zeta_n^{TA}$. In view of Theorem 3.4, this map is a stable equivalence whenever A, and hence TA, is cofibrant. Taking $A = QC$, where C is a domain or codomain of a map of I, completes the proof. □

We will now show that T is in fact a Quillen *equivalence* with respect to the stable model structure. To do so, we introduce the shift functors.

Definition 3.7. *Suppose $\mathscr{D}$ is a model category and T is a left Quillen endofunctor of $\mathscr{D}$. Define the shift functors $s_+: Sp^{\mathbb{N}}(\mathscr{D},T) \to Sp^{\mathbb{N}}(\mathscr{D},T)$ and $s_-: Sp^{\mathbb{N}}(\mathscr{D},T) \to Sp^{\mathbb{N}}(\mathscr{D},T)$ by $(s_-X)_n = X_{n+1}$, $(s_+X)_n = X_{n-1}$ for $n > 0$, and $(s_+X)_0 = 0$, with the evident structure maps. Note that s_+ is left adjoint to s_-.*

Lemma 3.8. *Suppose $\mathscr{D}$ is a left proper cellular model category and T is a left Quillen endofunctor of $\mathscr{D}$. Then*

(a) *the shift functor s_+ is a left Quillen functor with respect to the projective model structure on $Sp^{\mathbb{N}}(\mathscr{D},T)$;*

(b) *the shift functor s_+ commutes with T and s_- commutes with U;*

(c) *we have* $s_+F_n = F_{n+1}$ *and* $\mathrm{Ev}_n s_- = \mathrm{Ev}_{n+1}$.
(d) *the shift functor* s_+ *is a left Quillen functor with respect to the stable model structure on* $Sp^{\mathbb{N}}(\mathscr{D}, T)$.

Proof. For part (a), it is clear that s_- preserves level equivalences and level fibrations, so s_- is a right Quillen functor with respect to the projective model structure. Parts (b) and (c) we leave to the reader, except to note that adjointness makes the two halves of part (b) equivalent, and similarly the two halves of part (c). For part (d), note that Theorem 2.2 implies that s_+ defines a left Quillen functor with respect to the stable model structure as long as $s_+(\zeta_n^{QC})$ is a stable equivalence for all domains and codomains C of the generating cofibrations of $\mathscr{D}$. However, parts (b) and (c) imply that $s_+(\zeta_n^{QC}) = \zeta_{n+1}^{QC}$, which is certainly a stable equivalence. □

We now prove that T is a Quillen equivalence with respect to the stable model structure by comparing the T and U adjunction to the s_+ and s_- adjunction.

Theorem 3.9. *Suppose* $\mathscr{D}$ *is a left proper cellular model category and* T *is a left Quillen endofunctor of* $\mathscr{D}$. *Then the functors* $T : Sp^{\mathbb{N}}(\mathscr{D}, T) \to Sp^{\mathbb{N}}(\mathscr{D}, T)$ and $s_+ : Sp^{\mathbb{N}}(\mathscr{D}, T) \to Sp^{\mathbb{N}}(\mathscr{D}, T)$ *are Quillen equivalences with respect to the stable model structures. Furthermore*, Rs_- *is naturally isomorphic to* LT, *and* RU *is naturally isomorphic to* Ls_+.

Proof. The maps $X_n \to UX_{n+1}$ adjoint to the structure maps of a spectrum X define a natural map of spectra $X \to s_-UX$. This map is a stable equivalence (in fact, a level equivalence) when X is a stably fibrant object of $Sp^{\mathbb{N}}(\mathscr{D}, T)$. This means that the total right derived functor $R(s_-U)$ is naturally isomorphic to the identity functor on $\mathrm{Ho}\, Sp^{\mathbb{N}}(\mathscr{D}, T)$ (where we use the stable model structure). On the other hand, $R(s_-U)$ is naturally isomorphic to $Rs_- \circ RU$ and also to $RU \circ Rs_-$, since s_- and U commute with each other. Thus the natural isomorphism from the identity to $R(s_-U)$ gives rise to an natural isomorphism $1 \to Rs_- \circ RU$ and a natural isomorphism $RU \circ Rs_- \to 1$. Therefore Rs_- and RU are inverse equivalences of categories, and so both s_- and U are Quillen equivalences. Since inverse equivalences of categories can always be made into adjoint equivalences, Rs_- and RU are in fact adjoint equivalences. Since LT and Rs_- are both left adjoint to RU, LT and Rs_- are naturally isomorphic. Similarly, since Ls_+ and RU are both left adjoint to Rs_-, Ls_+ and RU are naturally isomorphic. □

We note that Theorem 3.9, when applied to the Bousfield–Friedlander model category of spectra [2], shows that the suspension functor without the twist (see Remark 1.6), $X \mapsto X \bar{\otimes} S^1$, is a Quillen equivalence. However, Theorem 3.9 does *not* show that the suspension functor with the twist, $X \mapsto X \otimes S^1$, is a Quillen equivalence. Indeed, the maps $X_n \to \Omega X_{n+1}$ only define a map of spectra if we do not put in the extra twist. We will discuss this issue further in Section 10. See also Remark 6.4.

4. The almost finitely generated case

The reader may well object at this point that we have defined the stable model structure on $Sp^{\mathbb{N}}(\mathscr{D},T)$ without ever defining stable homotopy groups. This is because stable homotopy groups do not detect stable equivalences in general. The usual simplicial and topological situation is very special. The goal of this section is to put some hypotheses on $\mathscr{D}$ and T so that the stable model structure on $Sp^{\mathbb{N}}(\mathscr{D},T)$ behaves similarly to the stable model structure on ordinary simplicial spectra. In particular, we show that, if $\mathscr{D}$ is almost finitely generated (defined below), sequential colimits in $\mathscr{D}$ preserve finite products, and U preserves sequential colimits, then the usual $\Omega^\infty\Sigma^\infty$ kind of construction gives a stable fibrant replacement functor. This implies that a map f is a stable equivalence if and only if the analogue of $\Omega^\infty\Sigma^\infty f$ is a level equivalence. This allows us to characterize $\operatorname{Ho} Sp^{\mathbb{N}}(\mathscr{D},T)(F_0A,X)$ for well-behaved A as the usual sort of colimit colim $\operatorname{Ho}\mathscr{D}(T^nA,X_n)$. It also allows us to prove that the stable model structure is right proper, under slightly more hypotheses, so we get the expected characterization of stable fibrations.

Most of the results in this section do not depend on the existence of the stable model structure on $Sp^{\mathbb{N}}(\mathscr{D},T)$, so we do not usually need to assume $\mathscr{D}$ is left proper cellular.

We now define almost finitely generated model categories, as suggested to the author by Voevodsky.

Definition 4.1. An object A of a category $\mathscr{C}$ is called *finitely presented* if the functor $\mathscr{C}(A,-)$ preserves direct limits of sequences $X_0 \to X_1 \to \cdots \to X_n \to \cdots$. A cofibrantly generated model category $\mathscr{C}$ is said to be *finitely generated* if the domains and codomains of the generating cofibrations and the generating trivial cofibrations are finitely presented. A cofibrantly generated model category is said to be *almost finitely generated* if the domains and codomains of the generating cofibrations are finitely presented, and if there is a set of trivial cofibrations J' with finitely presented domains and codomains such that a map f *whose codomain is fibrant* is a fibration if and only if f has the right lifting property with respect to J'.

This definition differs slightly from other definitions. In particular, an object A is usually said to be finitely presented [26, Section V.3] if $\mathscr{C}(A,-)$ preserves all directed (or, equivalently, filtered) colimits. We are trying to assume the minimum necessary. Finitely generated model categories were introduced in [12, Section 7.4], but in that definition we assumed only that $\mathscr{C}(A,-)$ preserves (transfinitely long) direct limits of sequences of *cofibrations*. The author would now prefer to call such model categories *compactly generated*. Thus, the model category of simplicial sets is finitely generated, but the model category on topological spaces is only compactly generated. Since we will only be working with (almost) finitely generated model categories in this section, our results will not apply to topological spaces. We will indicate where our results fail for compactly generated model categories, and a possible way to amend them in the compactly generated case.

The definition of an almost finitely generated model category was suggested by Voevodsky. The problem with finitely generated, or, indeed, compactly generated, model categories is that they are not preserved by localization. That is, if $\mathscr{C}$ is a finitely generated left proper cellular model category, and $\mathscr{S}$ is a set of maps, then the Bousfield localization [11] $L_{\mathscr{S}}\mathscr{C}$ will not be finitely generated, because we lose control of the generating trivial cofibrations in $L_{\mathscr{S}}\mathscr{C}$. However, we will show in the following proposition that Bousfield localization sometimes does preserve almost finitely generated model categories.

Recall from [12, Chapter 5] that it is possible to define $X \otimes K$ for an object X in a model category and a simplicial set K, even if the model category is not simplicial.

Proposition 4.2. *Let $\mathscr{C}$ be a left proper, cellular, almost finitely generated model category, and $\mathscr{S}$ be a set of cofibrations in $\mathscr{C}$. Suppose that, for every domain or codomain X of $\mathscr{S}$ and every finite simplicial set K, $X \otimes K$ is finitely presented. Then the Bousfield localization $L_{\mathscr{S}}\mathscr{C}$ is almost finitely generated.*

Proof. Since $\mathscr{C}$ is almost finitely generated, there is a set J' of trivial cofibrations so that a map p whose codomain is fibrant is a fibration if and only if p has the right lifting property with respect to J'. Let $\Lambda(\mathscr{S})$ denote the set of maps

$$(A \otimes \Delta[n]) \amalg_{A\otimes\Lambda^k[n]} (B \otimes \Lambda^k[n]) \to B \otimes \Delta[n],$$

where $A \to B$ is a map of $\mathscr{S}$, $n \geq 0$, $\Delta[n]$ is the standard n-simplex, and $\Lambda^k[n]$ for $0 \leq k \leq n$ is the horn obtained from $\Delta[n]$ by removing the nondegenerate n-simplex and the nondegenerate $(n-1)$-simplex not containing vertex k. As explained in [11, Proposition 4.2.4], a fibrant object X is $\mathscr{S}$-local if and only if the map $X \to 1$ has the right lifting property with respect to $\Lambda(\mathscr{S})$. Since an $\mathscr{S}$-local fibration between $\mathscr{S}$-local objects is just an ordinary fibration [11, Section 3.6], the set $\Lambda(\mathscr{S}) \cup J'$ will detect fibrations between fibrant objects in $L_{\mathscr{S}}\mathscr{C}$, and therefore $L_{\mathscr{S}}\mathscr{C}$ is almost finitely generated. □

Voevodsky has informed the author that he can make an unstable motivic model category that is almost finitely generated. For the reader's benefit, we summarize his construction. This summary will of necessity assume some familiarity with both the language of algebraic geometry and Voevodsky's central idea [27]. We begin with the category $\mathscr{E}$ of simplicial presheaves (of sets) on the category of smooth schemes over some base scheme k. There is a projective model structure on this category, where a map of simplicial presheaves $X \to Y$ is a weak equivalence (resp. fibration) if and only if the map $X(U) \to Y(U)$ is a weak equivalence (resp. fibration) of simplicial sets for all U. The projective model structure is finitely generated (using the fact that smooth schemes over k is an essentially small category). There is an embedding of smooth schemes into $\mathscr{E}$ as representable functors. We need to localize this model structure to take into account both the Nisnevich topology and the fact that the functor $X \mapsto X \times \mathbb{A}^1$ should be a Quillen equivalence. To do so, we define a set $\mathscr{S}'$ to consist of the maps

$X \times \mathbb{A}^1 \to X$ for every smooth scheme X and maps $P \to X$ for every pullback square of smooth schemes

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow p \\ A & \xrightarrow[j]{} & X, \end{array}$$

where p is etale, j is an open embedding, and $p^{-1}(X-A) \to X-A$ is an isomorphism. Here P is the mapping cylinder $(B \amalg Y) \amalg_{B \amalg B} (A \times \Delta[1])$. We then define $\mathscr{S}$ to consist of mapping cylinders on the maps of $\mathscr{S}'$. The maps of $\mathscr{S}$ are then cofibrations whose domains and codomains are finitely presented (and remain so after tensoring with any finite simplicial set), so the Bousfield localization $\mathscr{C} = L_{\mathscr{S}}\mathscr{E}$ will be almost finitely generated.

There is then some work involving properties of the Nisnevich topology to show that this model category is equivalent to the Morel–Voevodsky motivic model category of [19], and to the model category used by Jardine [14]. Given this, if we let T be the endofunctor of $\mathscr{C}$ which takes X to $X \times \mathbb{A}^1$, then $Sp^{\mathbb{N}}(\mathscr{C}, T)$ is a model for Voevodsky's stable motivic category.

The essential properties of almost finitely generated model categories that we need are contained in the following lemma.

Lemma 4.3. *Suppose $\mathscr{C}$ is an almost finitely generated model category*:

1. *If*

$$X_0 \to X_1 \to \cdots \to X_n \to \cdots$$

is a sequence of fibrant objects, *then* colim X_n *is fibrant*.

2. *Suppose we have the commutative diagram below*:

$$\begin{array}{ccccccccccc} X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots & \longrightarrow & X_n & \longrightarrow & \cdots \\ {\scriptstyle p_0}\downarrow & & {\scriptstyle p_1}\downarrow & & {\scriptstyle p_2}\downarrow & & & & {\scriptstyle p_n}\downarrow & & \\ Y_0 & \longrightarrow & Y_1 & \longrightarrow & Y_2 & \longrightarrow & \cdots & \longrightarrow & Y_n & \longrightarrow & \cdots \end{array}$$

If each p_n is a trivial fibration, *so is* colim p_n. *If each p_n is a fibration between fibrant objects*, *so is* colim p_n.

Proof. Let J' denote a set of trivial cofibrations in $\mathscr{C}$ with finitely presented domains and codomains that detect fibrations with fibrant codomain. For part (a), it suffices to show that colim $X_n \to 1$ has the right lifting property with respect to J'. But this is clear, since any map from a domain of J' to colim X_n factors through some X_k, and X_k is fibrant. The second part is proved similarly. □

We now consider the stable model structure on $Sp^{\mathbb{N}}(\mathscr{D}, T)$ when $\mathscr{D}$ is an almost finitely generated model category and T is a left Quillen endofunctor of $\mathscr{D}$. In analogy with ordinary Bousfield–Friedlander spectra, there is an obvious candidate for a stable fibrant replacement of a spectrum X.

Definition 4.4. Suppose T is a left Quillen endofunctor of a model category $\mathscr{D}$ with right adjoint U. Define $\Theta : Sp^{\mathbb{N}}(\mathscr{D}, T) \to Sp^{\mathbb{N}}(\mathscr{D}, T)$ to be the functor $s_- U$, where s_- is the shift functor. Then we have a natural map $\iota_X : X \to \Theta X$, and we define

$$\Theta^\infty X = \operatorname{colim}(X \xrightarrow{\iota_X} \Theta X \xrightarrow{\Theta \iota_X} \Theta^2 X \xrightarrow{\Theta^2 \iota_X} \cdots \xrightarrow{\Theta^{n-1} \iota_X} \Theta^n X \xrightarrow{\Theta^n \iota_X} \ldots).$$

Let $j_X : X \to \Theta^\infty X$ denote the obvious natural transformation.

The following lemma, though elementary, is crucial.

Lemma 4.5. *The maps $\iota_{\Theta X}, \Theta \iota_X : \Theta X \to \Theta^2 X$ coincide.*

Proof. The map $\iota_X : X_n \to UX_{n+1}$ is the adjoint $\tilde{\sigma}$ of the structure map of X. Hence $\Theta \iota_X$ is just $U\tilde{\sigma}$ in each degree. Since the adjoint of the structure map of UX is just $U\tilde{\sigma}$ (see Lemma 1.5), $\iota_{\Theta X} = \Theta \iota_X$. □

We stress that Lemma 4.5 fails for symmetric spectra, and it is the major reason we must work with finitely generated model categories rather than compactly generated model categories. Indeed, in the compactly generated case, Θ^∞ is not a good functor, since maps out of one of the domains of the generating cofibrations will not preserve the colimit that defines $\Theta^\infty X$. The obvious thing to try is to replace the functor Θ by a functor W, obtained by factoring $X \to \Theta X$ into a projective cofibration $X \to WX$ followed by a level trivial fibration $WX \to \Theta X$. The difficulty with this plan is that we do not see how to prove Lemma 4.5 for W. An alternative plan would be to use the mapping cylinder $X \to W'X$ on $X \to \Theta X$; this might make Lemma 4.5 easier to prove, but the map $X \to W'X$ will not be a cofibration. The map $X \to W'X$ may, however, be good enough for the required smallness properties to hold. It is a closed inclusion if $\mathscr{D}$ is topological spaces, for example. The author knows of no good general theorem in the compactly generated case.

This lemma leads immediately to the following proposition.

Proposition 4.6. *Suppose T is a left Quillen endofunctor of a model category $\mathscr{D}$, and suppose that its right adjoint U preserves sequential colimits. Then the map $\iota_{\Theta^\infty X} : \Theta^\infty X \to \Theta(\Theta^\infty X)$ is an isomorphism. In particular, if X is level fibrant, $\mathscr{D}$ is almost finitely generated, and U preserves sequential colimits, then $\Theta^\infty X$ is a U-spectrum.*

Proof. The map $\iota_{\Theta^\infty X}$ is the colimit of the vertical maps in the diagram below:

$$\begin{array}{ccccccccccc}
X & \xrightarrow{\iota_X} & \Theta X & \xrightarrow{\Theta\iota_X} & \Theta^2 X & \xrightarrow{\Theta^2\iota_X} & \cdots & \xrightarrow{\Theta^{n-1}\iota_X} & \Theta^n X & \xrightarrow{\Theta^n\iota_X} & \cdots \\
\downarrow{\scriptstyle \iota_X} & & \downarrow{\scriptstyle \iota_{\Theta X}} & & \downarrow{\scriptstyle \iota_{\Theta^2 X}} & & & & \downarrow{\scriptstyle \iota_{\Theta^n X}} & & \\
\Theta X & \xrightarrow[\Theta\iota_X]{} & \Theta^2 X & \xrightarrow[\Theta^2\iota_X]{} & \Theta^3 X & \xrightarrow[\Theta^3\iota_X]{} & \cdots & \xrightarrow[\Theta^n\iota_X]{} & \Theta^{n+1} X & \xrightarrow[\Theta^{n+1}\iota_X]{} & \cdots.
\end{array}$$

Since the vertical and horizontal maps coincide, the result follows. For the second statement, we note that if X is level fibrant, then each $\Theta^n X$ is level fibrant since Θ is a right Quillen functor (with respect to the projective model structure). Since sequential colimits in $\mathscr{D}$ preserve fibrant objects by Lemma 4.3, $\Theta^\infty X$ is level fibrant, and hence a U-spectrum. □

Proposition 4.7. *Suppose T is a left Quillen endofunctor of a model category $\mathscr{D}$ with right adjoint U. If $\mathscr{D}$ is almost finitely generated, and X is a U-spectrum, then the map $j_X : X \to \Theta^\infty X$ is a level equivalence.*

Proof. By assumption, the map $\iota_X : X \to \Theta X$ is a level equivalence between level fibrant objects. Since Θ is a right Quillen functor, $\Theta^n \iota_X$ is a level equivalence as well. Then the method of [12, Corollary 7.4.2] completes the proof. Recall that this method is to use factorization to construct a sequence of projective trivial cofibrations $Y_n \to Y_{n+1}$ with $Y_0 = X$ and a level trivial fibration of sequences $Y_n \to \Theta^n X$. Then the map $X \to \operatorname{colim} Y_n$ will be a projective trivial cofibration. Since sequential colimits in $\mathscr{D}$ preserve trivial fibrations by Lemma 4.3, the map $\operatorname{colim} Y_n \to \Theta^\infty X$ will still be a level trivial fibration. □

Proposition 4.7 gives us a slightly better method of detecting stable equivalences.

Corollary 4.8. *Suppose T is a left Quillen endofunctor of a model category $\mathscr{D}$ with right adjoint U. Suppose $\mathscr{D}$ is almost finitely generated and U preserves sequential colimits. Then a map $f : A \to B$ is a stable equivalence in $Sp^{\mathbb{N}}(\mathscr{D}, T)$ if and only if* $\operatorname{map}(f, X)$ *is an isomorphism in* Ho **SSet** *for all level fibrant spectra X such that $\iota_X : X \to \Theta X$ is an isomorphism.*

Proof. By definition, f is a stable equivalence if and only if $\operatorname{map}(f, Y)$ is an isomorphism for all U-spectra Y. But we have a level equivalence $Y \to \Theta^\infty Y$ by Proposition 4.7, and so it suffices to know that $\operatorname{map}(f, \Theta^\infty Y)$ is an isomorphism for all U-spectra Y. But, by Proposition 4.6, $\iota_{\Theta^\infty Y}$ is an isomorphism. □

This corollary, in turn, allows us to prove that Θ^∞ detects stable equivalences. The following theorem is similar to [13, Theorem 3.1.11].

Theorem 4.9. *Suppose T is a left Quillen endofunctor of a model category $\mathscr{D}$ with right adjoint U. Suppose that $\mathscr{D}$ is almost finitely generated and sequential colimits in $\mathscr{D}$ preserve finite products. Suppose also that U preserves sequential colimits. If $f: A \to B$ is a map in $Sp^{\mathbb{N}}(\mathscr{D}, T)$ such that $\Theta^{\infty} f$ is a level equivalence, then f is a stable equivalence.*

Before we can prove this theorem, however, we need to study how Θ^{∞} interacts with the enrichment $\mathrm{map}(X,Y)$. This requires some model category theory based on [12, Chapter 5].

Lemma 4.10. *Suppose $H: \mathscr{D} \to \mathscr{E}$ is a functor between model categories that preserves fibrant objects, weak equivalences between fibrant objects, fibrations between fibrant objects, and finite products. Let C be a cofibrant object of $\mathscr{D}$ and let X be a fibrant object of $\mathscr{D}$. Then there is a natural map $\mathrm{map}(C,X) \xrightarrow{\rho} \mathrm{map}(HC,HX)$ in* $\mathrm{Ho}\,\mathbf{SSet}$.

Proof. This proof will assume familiarity with [12, Chapter 5]. In particular, we need the notions of a simplicial frame X_* on a fibrant object X in a model category $\mathscr{C}$ from [12, Section 5.2] and the associated functor $\mathbf{SSet} \to \mathscr{C}$ that takes K to $(X_*)^K$. Here $(X_*)_n = (X_*)^{\Delta[n]}$, and the recipe for building $(X_*)^K$ from the $(X_*)^{\Delta[n]}$'s is derived from the recipe for building K rom the $\Delta[n]$'s (see [12, Proposition 3.1.5]). Recall that a simplicial frame X_* on X is a simplicial object in $\mathscr{D}$ with $X_0 \cong X$ and a factorization $\ell_\bullet X \to X_* \to r_\bullet X$ into a weak equivalence followed by a fibration in the category of simplicial objects in $\mathscr{D}$. Here $\ell_\bullet X$ is the constant simplicial object on X, $(r_\bullet X)_{n+1}$ is the $(n+1)$-fold product of X, the map $\ell_\bullet X \to r_\bullet X$ is the diagonal map. The hypotheses on H guarantee that, if X_* is a simplicial frame on the fibrant object X, then $H(X_*)$ is a simplicial frame on HX. This is the key fact that this lemma relies on.

Now, given a choice of functorial factorization on $\mathscr{C}$, there is a canonical simplicial frame $X_\circ$ associated to X, and the associated functor defines the cotensor action of $\mathrm{Ho}\,\mathbf{SSet}$ on $\mathrm{Ho}\,\mathscr{D}$ by taking (K,X) to $X^K = (X_\circ)^K$. For any other simplicial frame X_* there is a weak equivalence $X_* \to X_\circ$ inducing an isomorphism $(X_*)^K \to X^K$ in $\mathrm{Ho}\,\mathscr{C}$ that is natural in K and only depends on the isomorphism $(X_*)_0 \to X$ (see [12, Lemma 5.5.2]). In particular, there is a weak equivalence of simplicial frames $H(X_\circ) \to (HX)_\circ$ inducing an isomorphism $H(X^K) \to (HX)^K$ in $\mathrm{Ho}\,\mathscr{E}$ that is natural in both X and K.

Finally, we have an adjointness isomorphism

$$\mathrm{Ho}\,\mathbf{SSet}(K, \mathrm{map}(C,X)) \cong \mathrm{Ho}\,\mathscr{D}(C, X^K).$$

Thus, the identity map of $\mathrm{map}(C,X)$ gives us a map $C \to X^{\mathrm{map}(C,X)}$ in $\mathrm{Ho}\,\mathscr{D}$. Since C is cofibrant and X is fibrant, this map is represented by a map $C \to X^{\mathrm{map}(C,X)}$ in $\mathscr{D}$. By applying H, we get a map

$$HC \to H(X^{\mathrm{map}(C,X)}) \cong (HX)^{\mathrm{map}(C,X)}.$$

Then, applying adjointness again, we get the desired natural map $\mathrm{map}(C,X) \to \mathrm{map}(HC,HX)$. $\square$

With this lemma in hand, we can prove Theorem 4.9.

Proof of Theorem 4.9. We are given a map f such that $\Theta^{\infty} f$ is a level equivalence. Since $\mathscr{D}$ is almost finitely generated and U preserves sequential colimits, Θ^{∞} preserves level trivial fibrations. Therefore, $\Theta^{\infty}(Qf)$ is also a level equivalence. Hence we may as well assume that f is a map of cofibrant spectra. Now, suppose X is a U-spectrum such that the map $\iota_X : X \to \Theta X$ is an isomorphism. By Corollary 4.8, it suffices to show that $\mathrm{map}(f,X)$ is an isomorphism in $\mathrm{Ho}\,\mathbf{SSet}$. Since $\Theta^{\infty} f$ is a level equivalence, $\mathrm{map}(\Theta^{\infty} f, \Theta^{\infty} X)$ is an isomorphism. It therefore suffices to show that $\mathrm{map}(f,X)$ is a retract of $\mathrm{map}(\Theta^{\infty} f, \Theta^{\infty} X)$. We will prove this by showing that $\mathrm{map}(C,X)$ is naturally a retract of $\mathrm{map}(\Theta^{\infty} C, \Theta^{\infty} X)$ for any cofibrant spectrum C. Our hypotheses guarantee that Θ^{∞} preserves level fibrant objects, level fibrations between level fibrant objects, and all level trivial fibrations, since $\mathscr{D}$ is almost finitely generated. Furthermore, Θ^{∞} preserves finite products, since sequential colimits commute with finite products. Thus, Lemma 4.10 gives us a natural map $\mathrm{map}(C,X) \to \mathrm{map}(\Theta^{\infty} C, \Theta^{\infty} X)$. There is also a natural map $\alpha_C : \mathrm{map}(\Theta^{\infty} C, \Theta^{\infty} X) \to \mathrm{map}(C,X)$ defined as the composite

$$\mathrm{map}(\Theta^{\infty} C, \Theta^{\infty} X) \xrightarrow{\mathrm{map}(j_C, \Theta^{\infty} X)} \mathrm{map}(C, \Theta^{\infty} X) \xrightarrow{\mathrm{map}(C, j_X^{-1})} \mathrm{map}(C,X),$$

where we have used the fact that i_X is an isomorphism to conclude that j_X is also an isomorphism. Naturality means that, given a map $g : C \to D$, we have the commutative diagram below:

$$\begin{array}{ccc} \mathrm{map}(\Theta^{\infty} D, \Theta^{\infty} X) & \xrightarrow{\mathrm{map}(\Theta^{\infty} g, \Theta^{\infty} X)} & \mathrm{map}(\Theta^{\infty} C, \Theta^{\infty} X) \\ \downarrow{\scriptstyle \alpha_D} & & \downarrow{\scriptstyle \alpha_C} \\ \mathrm{map}(D,X) & \xrightarrow[\mathrm{map}(g,X)]{} & \mathrm{map}(C,X). \end{array}$$

We claim that the composite $\mathrm{map}(C,X) \to \mathrm{map}(\Theta^{\infty} C, \Theta^{\infty} X) \to \mathrm{map}(C,X)$ is the identity, so that $\mathrm{map}(C,X)$ is naturally a retract of $\mathrm{map}(\Theta^{\infty} C, \Theta^{\infty} X)$. This argument, which will depend heavily on the method of Lemma 4.10, will complete the proof. Let $\gamma : C \to X^{\mathrm{map}(C,X)}$ denote the adjoint of the identity of $\mathrm{map}(C,X)$. Then the composite $\mathrm{map}(C,X) \to \mathrm{map}(\Theta^{\infty} C, \Theta^{\infty} X) \to \mathrm{map}(C,X)$ is adjoint to the counter-clockwise composite in the following commutative diagram:

$$\begin{array}{ccccccc} C & \xrightarrow[\gamma]{} & X^{\mathrm{map}(C,X)} & = & X^{\mathrm{map}(C,X)} & & \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \delta} & & \\ \Theta^{\infty} C & \xrightarrow[Q\gamma]{} & \Theta^{\infty}(X^{\mathrm{map}(C,X)}) & \xrightarrow[\cong]{} & (\Theta^{\infty} X)^{\mathrm{map}(C,X)} & \xrightarrow[(j_X^{\mathrm{map}(C,X)})^{-1}]{\cong} & X^{\mathrm{map}(C,X)}. \end{array}$$

The left square of this diagram is commutative because j is natural. There is some choice of δ that makes the right square commutative, but we claim that $\delta = j_X^{\mathrm{map}(C,X)}$ will work. This completes the proof, and follows from [12, Lemma 5.5.2], because the natural isomorphism $\Theta^\infty(X^K) \to (\Theta^\infty X)^K$ is induced by a map of simplicial frames which is the identity in degree 0. □

Corollary 4.11. *Let T be a left Quillen endofunctor of a model category $\mathscr{D}$ with right adjoint U. Suppose that $\mathscr{D}$ is almost finitely generated, that sequential colimits in $\mathscr{D}$ preserve finite products, and that U preserves sequential colimits. Then $j_A : A \to \Theta^\infty A$ is a stable equivalence for all $A \in Sp^{\mathbb{N}}(\mathscr{D}, T)$.*

Proof. One can easily check that $\Theta^\infty j_A$ is an isomorphism, using Proposition 4.6. □

Finally, we get the desired characterization of stable equivalences.

Theorem 4.12. *Let T be a left Quillen endofunctor of a model category $\mathscr{D}$ with right adjoint U. Suppose that $\mathscr{D}$ is almost finitely generated, that sequential colimits in $\mathscr{D}$ preserve finite products, and that U preserves sequential colimits. Let L' denote a fibrant replacement functor in the projective model structure on $Sp^{\mathbb{N}}(\mathscr{D}, T)$. Then, for all $A \in Sp^{\mathbb{N}}(\mathscr{D}, T)$, the map $A \to \Theta^\infty L'A$ is a stable equivalence into a U-spectrum. Also, a map $f : A \to B$ is a stable equivalence if and only if $\Theta^\infty L' f$ is a level equivalence.*

Proof. The first statement follows immediately from Proposition 4.6 and Corollary 4.11. By the first statement, if f is a stable equivalence, so is $\Theta^\infty L' f$. Since $\Theta^\infty L' f$ is a map between U-spectra, it is a stable equivalence if and only if it is a level equivalence. The converse follows from Theorem 4.9. □

Since we did not need the existence of the stable model structure to prove Theorem 4.12, one can imagine attempting to construct it from the functor $\Theta^\infty L'$. This is, of course, the original approach of Bousfield–Friedlander [2], and this approach has been generalized by Schwede [21]. Also, if there is some functor F, like homotopy groups, that detects level equivalences in $\mathscr{D}$, then Theorem 4.12 implies that $F \circ \Theta^\infty \circ L'$ detects stable equivalences in $Sp^{\mathbb{N}}(\mathscr{D}, T)$. The functor $F \circ \Theta^\infty \circ L'$ is a generalization of the usual stable homotopy groups. One can see these generalizations in the following corollary as well.

Corollary 4.13. *Let $\mathscr{D}$ be a left proper, cellular, almost finitely generated model category where sequential colimits preserve finite products. Suppose $T : \mathscr{D} \to \mathscr{D}$ is a left Quillen functor whose right adjoint U commutes with sequential colimits. Finally, suppose A is a finitely presented cofibrant object of $\mathscr{D}$ that has a finitely presented cylinder object $A \times I$. Then*

$$\mathrm{Ho}\, Sp^{\mathbb{N}}(\mathscr{D}, T)(F_k A, Y) = \mathrm{colim}_m \, \mathrm{Ho}\, \mathscr{D}(A, U^m Y_{k+m})$$

for all level fibrant $Y \in Sp^{\mathbb{N}}(\mathscr{D}, T)$.

Here we are using the stable model structure to form $\operatorname{Ho} Sp^{\mathbb{N}}(\mathscr{D},T)$, of course.

Proof. We have $\operatorname{Ho} Sp^{\mathbb{N}}(\mathscr{D},T)(F_kA,Y)=Sp^{\mathbb{N}}(\mathscr{D},T)(F_kA,\Theta^{\infty}Y)/\sim$, by Theorem 4.12, where $\sim$ denotes the left homotopy relation. We can use the cylinder object $F_k(A\times I)$ as the source for our left homotopies. Then adjointness implies that $Sp^{\mathbb{N}}(\mathscr{D},T)(F_kA,\Theta^{\infty}Y)/\sim\, =\mathscr{D}(A,\operatorname{Ev}_k\Theta^{\infty}Y)/\sim$. Since A and $A\times I$ are finitely presented, we get the required result. □

By assuming slightly more about $\mathscr{D}$, we can also characterize the stable fibrations.

Corollary 4.14. *Let $\mathscr{D}$ be a proper, cellular, almost finitely generated model category such that sequential colimits preserve pullbacks. Suppose $T:\mathscr{D}\to\mathscr{D}$ is a left Quillen functor whose right adjoint U commutes with sequential colimits. Then the stable model structure on $Sp^{\mathbb{N}}(\mathscr{D},T)$ is proper. In particular, a map $p:X\to Y$ is a stable fibration if and only if p is a level fibration and the diagram*

$$\begin{array}{ccc} X & \longrightarrow & \Theta^{\infty}L'X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle L_p} \\ Y & \longrightarrow & \Theta^{\infty}L'Y \end{array}$$

is a homotopy pullback square in the projective model structure, where L' is a fibrant replacement functor in the projective model structure.

Proof. We will actually show that, if $p:X\to Y$ is a level fibration and $f:B\to Y$ is a stable equivalence, the pullback $B\times_Y X\to X$ is a stable equivalence. This means the the stable model structure on $Sp^{\mathbb{N}}(\mathscr{D},T)$ is right proper, and then the characterization of stable fibrations follows from [11, Proposition 3.6.8].

The first step is to use the right properness of the projective model structure on $Sp^{\mathbb{N}}(\mathscr{D},T)$ to reduce to the case where B and Y are level fibrant. Indeed, let $Y'=L'Y$, $B'=L'B$, and $f'=L'f$. Then factor the composite $X\to Y\to Y'$ into a projective trivial cofibration $X\to X'$ followed by a level fibration $p':X'\to Y'$. Then we have the commutative diagram below:

$$\begin{array}{ccccc} B & \xrightarrow{f} & Y & \xleftarrow{p} & X \\ \downarrow & & \downarrow & & \downarrow \\ B' & \xrightarrow[f']{} & Y' & \xleftarrow[p'']{} & X', \end{array}$$

where the vertical maps are level equivalences. Proposition 11.2.4 and Corollary 11.2.8 of [11], which depend on the projective model structure being right proper, imply that the induced map $B\times_Y X\to B'\times_{Y'}X'$ is a level equivalence. Hence $B\times_Y X\to X$ is a stable equivalence if and only if $B'\times_{Y'}X'\to X'$ is a stable equivalence, and so we can assume B and X are level fibrant.

Now let S denote the pullback square below:

$$\begin{array}{ccc} B\times_Y X & \longrightarrow & X \\ \Big\downarrow & & \Big\downarrow{\scriptstyle p} \\ B & \underset{f}{\longrightarrow} & Y. \end{array}$$

Then $\Theta^n S$ is a pullback square for all n, and there are maps $\Theta^n S \xrightarrow{\Theta^n i_S} \Theta^{n+1} S$. Since pullbacks commute with sequential colimits, $\Theta^\infty S$ is a pullback square. Furthermore, $\Theta^\infty p$ is a level fibration, since sequential colimits in $\mathscr{D}$ preserve fibrations between level fibrant objects. Since f is a stable equivalence between level fibrant spectra, $\Theta^\infty f$ is a level equivalence by Theorem 4.12. So, since the projective model structure is right proper, the map $\Theta^\infty(B\times_Y X \to X)$ is a level equivalence, and thus $B\times_Y X\to X$ is a stable equivalence. □

5. Functoriality of the stable model structure

In this section, we consider the stable model structure on $Sp^{\mathbb{N}}(\mathscr{D},T)$ as a functor of the pair $(\mathscr{D},T)$. The most important result in this section is that $Sp^{\mathbb{N}}(\mathscr{D},T)$ is Quillen equivalent to $\mathscr{D}$ if T is already a Quillen equivalence on $\mathscr{D}$. This means that the functor $(\mathscr{D},T)\mapsto(Sp^{\mathbb{N}}(\mathscr{D},T),T)$ is idempotent up to Quillen equivalence. This is as close as we can get to our belief that, up to Quillen equivalence, $Sp^{\mathbb{N}}(\mathscr{D},T)$ should be the initial stabilization of $\mathscr{D}$ with respect to T.

We also show that $Sp^{\mathbb{N}}(\mathscr{D},T)$ is functorial in the pair $(\mathscr{D},T)$, with a suitable definition of maps of pairs. Under mild hypotheses, we show that $Sp^{\mathbb{N}}(\mathscr{D},T)$ preserves Quillen equivalences in the pair $(\mathscr{D},T)$. Applying this to the Bousfield–Friedlander category of spectra of simplicial sets, we find that the choice of simplicial model of the circle has no effect on the Quillen equivalence class of the stable model category of spectra.

Theorem 5.1. *Suppose $\mathscr{D}$ is a left proper cellular model category, and suppose T is a left Quillen endofunctor of $\mathscr{D}$ that is a Quillen equivalence. Then $F_0:\mathscr{D}\to Sp^{\mathbb{N}}(\mathscr{D},T)$ is a Quillen equivalence, where $Sp^{\mathbb{N}}(\mathscr{D},T)$ has the stable model structure.*

Proof. By Hovey [12, Corollary 1.3.16], it suffices to check two conditions. We first show that $\mathrm{Ev}_0: Sp^{\mathbb{N}}(\mathscr{D},T)\to\mathscr{D}$ reflects weak equivalences between stably fibrant objects. We then show that the map $A\to \mathrm{Ev}_0 L_{\mathscr{S}} F_0 A$ is a weak equivalence for all cofibrant $A\in\mathscr{D}$, where $L_{\mathscr{S}}$ denotes a stable fibrant replacement functor in $Sp^{\mathbb{N}}(\mathscr{D},T)$.

Suppose X and Y are U-spectra, and $f:X\to Y$ is a map such that $\mathrm{Ev}_0 f = f_0$ is a weak equivalence. We claim that f is a level equivalence, so that Ev_0 reflects weak equivalences between stably fibrant objects. Since f is a map of spectra, we have $\sigma_Y\circ Tf_{n-1} = f_n\circ\sigma_X$ for $n\geq 1$. Using adjointness, we find that $Uf_n\circ\tilde{\sigma}_X = \tilde{\sigma}_Y\circ f_{n-1}$

for $n \geq 1$. Since X and Y are U-spectra, we find that f_{n-1} is a weak equivalence if and only if Uf_n is a weak equivalence. On the other hand, since T is a Quillen equivalence on $\mathscr{D}$, U reflects weak equivalences between fibrant objects of $\mathscr{D}$, by Hovey [12, Corollary 1.3.16]. Therefore Uf_n is a weak equivalence if and only if f_n is a weak equivalence. Altogether, f_{n-1} is a weak equivalence if and only if f_n is a weak equivalence. Since f_0 is a weak equivalence by hypothesis, f_n is a weak equivalence for all n, and so f is a level equivalence, as required.

We now show that $A \to \mathrm{Ev}_0 L_{\mathscr{S}} F_0 A$ is a weak equivalence for all cofibrant $A \in \mathscr{D}$. Let R' denote a fibrant replacement functor in the projective model structure on $Sp^{\mathbb{N}}(\mathscr{D}, T)$. We claim that $R'F_0A$ is already a U-spectrum. Suppose for the moment that this is true. Then we have the commutative diagram below:

$$\begin{array}{ccc} F_0A & \xrightarrow{i} & R'F_0A \\ {\scriptstyle j}\downarrow & & \downarrow \\ L_{\mathscr{S}}F_0A & \longrightarrow & 0. \end{array}$$

Since $R'F_0A$ is a U-spectrum, the right-hand vertical map is a stable fibration. The left-hand vertical map is a stable trivial cofibration, so there is a lift $h : L_{\mathscr{S}}F_0A \xrightarrow{h} R'F_0A$ with $hi = j$. Since i is a level equivalence and j is a stable equivalence, h is a stable equivalence. But both $L_{\mathscr{S}}F_0A$ and $R'F_0A$ are U-spectra, so h is a level equivalence (see the discussion following Definition 2.1). This means that j is also a level equivalence. Hence the map

$$A = \mathrm{Ev}_0\ F_0\ A \to \mathrm{Ev}_0\ L_{\mathscr{S}}\ F_0\ A$$

is a weak equivalence, as required.

It remains to prove that $R'F_0A$ is a U-spectrum when A is cofibrant. Let R denote a fibrant replacement functor in $\mathscr{D}$. Since T is a Quillen equivalence, and T^nA is cofibrant, the map

$$(F_0A)_n = T^nA \to URT^{n+1}A = UR(F_0A)_{n+1}$$

is a weak equivalence, by Hovey [12, Corollary 1.3.16]. In the commutative diagram below

$$\begin{array}{ccc} (F_0A)_{n+1} & \longrightarrow & (R'F_0A)_{n+1} \\ \downarrow & & \downarrow \\ R(F_0A)_{n+1} & \longrightarrow & 0, \end{array}$$

the right-hand vertical map is a fibration, and the left-hand vertical map is a trivial cofibration. Thus, there is a lift $R(F_0A)_{n+1} \to (R'F_0A)_{n+1}$, which must be a weak equivalence by the two-out-of-three property. Applying U, which preserves weak equivalences between fibrant objects, we find that

$$(F_0A)_n \to UR(F_0A)_{n+1} \to U(R'F_0A)_{n+1}$$

is a weak equivalence. This means that $R'F_0A$ is a U-spectrum, as required. □

In particular, this theorem means that the passage $(\mathscr{D}, T) \mapsto (Sp^{\mathbb{N}}(\mathscr{D}, T), T)$ is idempotent, up to Quillen equivalence. This suggests that we are doing some kind of fibrant replacement of $(\mathscr{D}, T)$ in an appropriate model category of model categories, but the author knows no way of making this precise.

We now examine the functoriality of the stable model structure on $Sp^{\mathbb{N}}(\mathscr{D}, T)$. We first consider what information we need to extend a functor on a category to a functor on the category of spectra.

Lemma 5.2. *Suppose $\mathscr{D}$ and $\mathscr{D}'$ are model categories equipped with left Quillen endofunctors $T:\mathscr{D} \to \mathscr{D}$ and $T':\mathscr{D}' \to \mathscr{D}'$. Let U denote a right adjoint of T and let U' denote a right adjoint of T'. Suppose $\Gamma:\mathscr{D}' \to \mathscr{D}$ is a functor and $\rho:\Gamma U' \to U\Gamma$ is a natural transformation. Then there is an induced functor $Sp^{\mathbb{N}}(\Gamma, \rho):Sp^{\mathbb{N}}(\mathscr{D}', T') \to Sp^{\mathbb{N}}(\mathscr{D}, T)$, sometimes denoted simply $Sp^{\mathbb{N}}(\Gamma)$ when the choice of ρ is clear, called the* prolongation *of Γ.*

Of course, the model structures are irrelevant to this lemma.

Proof. Given $X \in Sp^{\mathbb{N}}(\mathscr{D}', T')$, we define $(Sp^{\mathbb{N}}(\Gamma, \rho)(X))_n = \Gamma X_n$. The structure map of $Sp^{\mathbb{N}}(\Gamma, \rho)(X)$ is adjoint to the composite

$$\Gamma X_n \xrightarrow{\Gamma\tilde{\sigma}} \Gamma U' X_{n+1} \xrightarrow{\rho} U\Gamma X_{n+1},$$

where $\tilde{\sigma}$ is adjoint to the structure map of X. Given a map $f:X \to Y$, we define $Sp^{\mathbb{N}}(\Gamma, \rho)(f)$ by $(Sp^{\mathbb{N}}(\Gamma, \rho)(f))_n = \Gamma f_n$. □

Note that the natural transformation $\rho:\Gamma U' \to U\Gamma$ is equivalent to a natural transformation $\bar{\rho}:T\Gamma \to \Gamma T'$. Indeed, given ρ, we define $\bar{\rho}$ to be the composite

$$T\Gamma X \xrightarrow{T\Gamma\eta_X} T\Gamma U'T'X \xrightarrow{T\rho_{T'X}} TU\Gamma T'X \xrightarrow{\varepsilon_{\Gamma T'X}} \Gamma T'X,$$

where η denotes the unit and ε the counit of the appropriate adjunctions. Conversely, given $\bar{\rho}$, we can recover ρ as the composite

$$\Gamma U'X \xrightarrow{\eta_{\Gamma U'X}} UT\Gamma U'X \xrightarrow{U\bar{\rho}_{U'X}} U\Gamma T'U'X \xrightarrow{U\Gamma\varepsilon_X} U\Gamma X.$$

We can describe the prolongation $Sp^{\mathbb{N}}(\Gamma, \rho)$ in terms of this associated natural transformation $\bar{\rho}$. Indeed, the structure map of $Sp^{\mathbb{N}}(\Gamma, \rho)(X)$ is the composite

$$T\Gamma X_n \xrightarrow{\bar{\rho}} \Gamma T'X_n \xrightarrow{\Gamma\sigma} \Gamma X_{n+1},$$

where σ is the structure map of X.

In practice, the functor $\Gamma:\mathscr{D}' \to \mathscr{D}$ that we wish to prolong will usually have a left adjoint $\Phi:\mathscr{D} \to \mathscr{D}'$. We would like the prolongation $Sp^{\mathbb{N}}(\Gamma)$ to also have a left adjoint. The following lemma deals with this issue.

Lemma 5.3. *Suppose $\mathscr{D}$ and $\mathscr{D}'$ are model categories equipped with left Quillen endofunctors $T:\mathscr{D} \to \mathscr{D}$ and $T':\mathscr{D}' \to \mathscr{D}'$. Let U denote a right adjoint of T and let U' denote a right adjoint of T'. Suppose that $\Gamma:\mathscr{D}' \to \mathscr{D}$ is a functor with left adjoint Φ, and $\rho:\Gamma U' \to U\Gamma$ is a natural transformation. Then the prolongation $Sp^{\mathbb{N}}(\Gamma,\rho):Sp^{\mathbb{N}}(\mathscr{D}',T') \to Sp^{\mathbb{N}}(\mathscr{D},T)$ has a left adjoint $\tilde{\Phi}$ satisfying $\tilde{\Phi}F_n \cong F_n\Phi$. If ρ is a natural isomorphism, then $\tilde{\Phi}$ is a prolongation of Φ.*

The reason we do not denote $\tilde{\Phi}$ by $Sp^{\mathbb{N}}(\Phi)$ is that $\tilde{\Phi}$ is not generally a prolongation of Φ.

Proof. Since $\mathrm{Ev}_n\, Sp^{\mathbb{N}}(\Gamma,\rho) = \Gamma\mathrm{Ev}_n$, if $\tilde{\Phi}$ exists we must have $\tilde{\Phi}F_n \cong F_n\Phi$. To construct $\tilde{\Phi}$, first note that the natural transformation ρ has a dual, or conjugate, natural transformation $\tau = D\rho:\Phi T \to T'\Phi$, discussed in [12, p. 24] and in [17, Section IV.7]. By iteration, we get induced natural transformations $\tau^q:\Phi T^q \to (T')^q\Phi$ for all integers $q \geq 1$. We now define $(\tilde{\Phi}X)_n$ to be the coequalizer of two maps

$$\alpha_n,\beta_n\colon \coprod_{p+q+r=n} (T')^p\Phi T^q X_r \rightrightarrows \coprod_{p+q=n} (T')^p\Phi X_q.$$

On the summand $(T')^p\Phi T^q X_r$, the top map α_n is $(T')^p\Phi$ applied to the iterated structure map $T^qX_r \to X_{q+r}$ of X. On the same summand, the bottom map β_n is $(T')^p\tau^q:(T')^p\Phi T^qX_r \to (T')^{p+q}\Phi X_r$. Note that $T'\alpha_n$ is the retract of α_{n+1} consisting of those terms $(T')^p\Phi T^qX_r$ with $p+q+r=n+1$ and $p>0$. A similar statement is true for $T'\beta_n$. Since T' preserves coequalizers, there is an induced map $T'(\tilde{\Phi}X)_n \to (\tilde{\Phi}X)_{n+1}$, which is the structure map of $\tilde{\Phi}X$. We leave to the reader the definition of $\tilde{\Phi}f$ for a map of spectra $f:X \to Y$.

The argument that $\tilde{\Phi}$ is left adjoint to $Sp^{\mathbb{N}}(\Gamma,\rho)$ is intricate, but straightforward. For example, a map $\tilde{\Phi}X \to Y$ induces a map $\Phi X_n \to Y_n$ by restriction to the $p=0$ summand in $\amalg_{p+q=n}(T')^p\Phi X_q$. The adjoint of this is a map $X_n \to \Gamma Y_n$, which induces a map of spectra $X \to Sp^{\mathbb{N}}(\Gamma,\rho)(Y)$. To see that this is indeed a map of spectra, consider the ΦTX_n summand in $\amalg_{p+q+r=n+1}(T')^p\Phi T^qX_r$. Conversely, a map $X \xrightarrow{f} Sp^{\mathbb{N}}(\Gamma,\rho)Y$ consists of maps $f_n:X_n \to \Gamma Y_n$. The adjoints $\widetilde{f_n}:\Phi X_n \to Y_n$ define a map $\amalg_{p+q=n}(T')^p\Phi X_q \xrightarrow{g_n} Y_n$ via the composite

$$(T')^p\Phi X_q \xrightarrow{(T')^p\tilde{f}_q} (T')^pY_q \to Y_n,$$

where the second map is a composite of structure maps of Y. The fact that the original map f is a map of spectra implies that these maps g_n descend to define a map $(\tilde{\Phi}X)_n \to$

Y_n, which is automatically a map of spectra. We leave the rest of the argument to the reader.

It remains to show that if ρ is a natural isomorphism, then $\tilde{\Phi}$ is a prolongation of Φ. If ρ is a natural isomorphism, then $\tau = D\rho : \Phi T \to T'\Phi$ is also a natural isomorphism. Let $v : T'\Phi \to \Phi T$ denote the inverse of τ. Then, by Lemma 5.2, there is a prolongation $Sp^{\mathbb{N}}(\Phi, v)$ of Φ. We claim that $Sp^{\mathbb{N}}(\Phi, v)$ is left adjoint to $Sp^{\mathbb{N}}(\Gamma, \rho)$. Indeed, there are natural candidates for the unit and counit of this purported adjunction; namely, the maps which are levelwise the unit and counit of the (Φ, Γ) adjunction. We leave it to the reader to check that these maps are maps of spectra and are natural. To ensure that they are the unit and counit of an adjunction, we need to verify the triangle identities [17, Theorem 4.1.2(v)], but these follow immediately from the triangle identities of the (Φ, Γ) adjunction. □

In view of the preceding lemma, we make the following definition.

Definition 5.4. Suppose $\mathscr{D}$ and $\mathscr{D}'$ are left proper cellular model categories, T is a left Quillen endofunctor of $\mathscr{D}$, and T' is a left Quillen endofunctor of $\mathscr{D}'$. A *map of pairs* $(\Phi, \tau) : (\mathscr{D}, T) \to (\mathscr{D}', T')$ is a functor $\Phi : \mathscr{D} \to \mathscr{D}'$ with a right adjoint Γ, and a natural transformation $\tau : \Phi T \to T'\Phi$. We say that a map of pairs (Φ, τ) is a *Quillen map of pairs* if Φ is a left Quillen functor and τ_A is a weak equivalence for all cofibrant $A \in \mathscr{D}$.

Note that the natural transformation τ in this definition is the natural transformation $D\rho$ in Lemma 5.3. Our goal is for a Quillen map of pairs to induce a corresponding map of pairs on the stable model category of spectra. For this to be true, we need some condition on τ, and requiring τ_A to be a weak equivalence for all cofibrant A seems to be the least we can assume.

Note that there is an obvious associative and unital composition of maps of pairs.

Proposition 5.5. *Suppose $(\Phi, \tau) : (\mathscr{D}, T) \to (\mathscr{D}', T')$ is a Quillen map of pairs. Then there is an induced Quillen map of pairs*

$$(\tilde{\Phi}, \tilde{\tau}) : (Sp^{\mathbb{N}}(\mathscr{D}, T), T) \to (Sp^{\mathbb{N}}(\mathscr{D}', T'), T'),$$

where $Sp^{\mathbb{N}}(\mathscr{D}, T)$ and $Sp^{\mathbb{N}}(\mathscr{D}', T')$ are given the stable model structures, such that $\tilde{\Phi} \circ F_n \cong F_n\Phi$. This induced map of pairs is compatible with composition and identities.

Proof. We constructed $\tilde{\Phi}$ and its right adjoint $Sp^{\mathbb{N}}(\Gamma) = Sp^{\mathbb{N}}(\Gamma, \rho)$ in Lemma 5.3. Here Γ denotes a right adjoint of Φ, and ρ is the dual natural transformation $D\tau : \Gamma U' \to U\Gamma$, where U is right adjoint to T and U' is right adjoint to T'. Since $Sp^{\mathbb{N}}(\Gamma)(X)_n = \Gamma X_n$, and Γ is a right Quillen functor, $Sp^{\mathbb{N}}(\Gamma)$ preserves level fibrations and level trivial fibrations. Hence $\tilde{\Phi}$ is a left Quillen functor with respect to the projective model structures. To show that $\tilde{\Phi}$ is a left Quillen functor with respect to the stable model structures, it suffices to show that $\tilde{\Phi}\zeta_n^A$ is a stable equivalence for all cofibrant A, by Theorem 2.2. Since $\tilde{\Phi}F_n \cong F_n\Phi$ by Lemma 5.3, $\tilde{\Phi}F_{n+1}TA \cong F_{n+1}\Phi TA$ and $\tilde{\Phi}F_nA \cong F_n\Phi A$. Therefore, the map $\tilde{\Phi}\zeta_n^A$ differs from the map $\zeta_n^{\Phi A}$ by the map $F_{n+1}\tau_A$. Since $\zeta_n^{\Phi A}$ is a stable equivalence, and τ_A is a weak equivalence by hypothesis, $\tilde{\Phi}\zeta_n^A$ is a stable equivalence. Therefore, $\tilde{\Phi}$ is a left Quillen functor with respect to the stable model structures.

We define $\tilde{\tau}$ by defining its dual natural transformation

$$D\tilde{\tau} = Sp^{\mathbb{N}}(\rho): Sp^{\mathbb{N}}(\Gamma)U' \to USp^{\mathbb{N}}(\Gamma).$$

Indeed, we just define $Sp^{\mathbb{N}}(\rho)$ to be ρ in each degree. We leave it to the reader to verify that this defines a natural map of spectra $Sp^{\mathbb{N}}(\rho)$. Since τ is a weak equivalence on all cofibrant objects of $\mathscr{D}$, $\rho = D\tau$ is a weak equivalence on all fibrant objects of $\mathscr{D}'$. To see this, note that τ induces a natural isomorphism in the homotopy category. Adjointness implies that ρ also induces a natural isomorphism in the homotopy category, and it follows that ρ is a weak equivalence on all fibrant objects of $\mathscr{D}$. Thus $Sp^{\mathbb{N}}(\rho)$ will be a level equivalence on all level fibrant objects of $Sp^{\mathbb{N}}(\mathscr{D}', T')$, so, by reversing the homotopy category argument just given, $\tilde{\tau}$ is a level equivalence on all cofibrant objects of $Sp^{\mathbb{N}}(\mathscr{D}, T)$. We leave it to the reader to check compatibility of $(\tilde{\Phi}, \tilde{\tau})$ with compositions and identities. □

Proposition 5.5 and Theorem 5.1 give us a weak universal property of $Sp^{\mathbb{N}}(\mathscr{D}, T)$.

Corollary 5.6. *Suppose $(\Phi, \tau): (\mathscr{D}, T) \to (\mathscr{D}', T')$ is a Quillen map of pairs such that T' is a Quillen equivalence. Give $Sp^{\mathbb{N}}(\mathscr{D}, T)$ its stable model structure. Then there is a functor $\Psi: \operatorname{Ho} Sp^{\mathbb{N}}(\mathscr{D}, T) \to \operatorname{Ho} \mathscr{D}'$ such that $\Psi \circ LF_0 = L\Phi : \operatorname{Ho} \mathscr{D} \to \operatorname{Ho} \mathscr{D}'$.*

This corollary is trying to say that $(Sp^{\mathbb{N}}(\mathscr{D}, T), T)$ is homotopy initial among Quillen maps of pairs $(\mathscr{D}, T) \to (\mathscr{D}', T')$ where T' is a Quillen equivalence. Though the statement of the corollary is the best statement of this concept we have been able to find, we suspect there is a better one.

Proof. By Proposition 5.5 there is a Quillen map of pairs

$$(\tilde{\Phi}, \tilde{\tau}): (Sp^{\mathbb{N}}(\mathscr{D}, T), T) \to (Sp^{\mathbb{N}}(\mathscr{D}', T'), T')$$

induced by (Φ, τ). By Theorem 5.1, $F_0: \mathscr{D}' \to Sp^{\mathbb{N}}(\mathscr{D}', T')$ is a Quillen equivalence. Define Ψ to be the composite $R\mathrm{Ev}_0 \circ L\tilde{\Phi}$. □

Proposition 5.5 shows that the correspondence $(\mathscr{D}, T) \mapsto (Sp^{\mathbb{N}}(\mathscr{D}, T), T)$ is functorial. We would like to know that it is homotopy invariant. In particular, we would like to know that if (Φ, τ) is a Quillen equivalence of pairs, then the induced Quillen map of pairs $(\tilde{\Phi}, \tilde{\tau})$ on spectra is a Quillen equivalence with respect to the stable model structures. Our proof of this seems to require some hypotheses.

Theorem 5.7. *Suppose $(\Phi, \tau): (\mathscr{D}, T) \to (\mathscr{D}', T')$ is a Quillen map of pairs such that Φ is a Quillen equivalence. Suppose as well that either the domains of the generating cofibrations for $\mathscr{D}$ can be taken to be cofibrant, or that τ_A is a weak equivalence for all A. Then, in the induced Quillen map of pairs*

$$(\tilde{\Phi}, \tilde{\tau}): (Sp^{\mathbb{N}}(\mathscr{D}, T), T) \to (Sp^{\mathbb{N}}(\mathscr{D}', T'), T'),$$

the Quillen functor $\tilde{\Phi}$ is a Quillen equivalence with respect to the stable model structures.

Proof. We will first show that $\tilde{\Phi}$ is a Quillen equivalence with respect to the projective model structures. Use the same notation as in the proof of Proposition 5.5, so that Γ denotes the right adjoint of Φ. Then Γ reflects weak equivalences between fibrant objects, by Hovey [12, Corollary 1.3.16]. Thus $Sp^{\mathbb{N}}(\Gamma)$ reflects level equivalences between level fibrant objects. By Hovey [12, Corollary 1.3.16], to show that $\tilde{\Phi}$ is a Quillen equivalence with respect to the projective model structures, it suffices to show that $X \to Sp^{\mathbb{N}}(\Gamma)R'\tilde{\Phi}X$ is a level equivalence for all cofibrant $X \in Sp^{\mathbb{N}}(\mathscr{D}, T)$, where R' is a fibrant replacement functor in the projective model structure on $Sp^{\mathbb{N}}(\mathscr{D}', T')$. Let R denote a fibrant replacement functor in $\mathscr{D}'$. By a lifting argument, the weak equivalence $(\tilde{\Phi}X)_n : (R'\tilde{\Phi}X)_n$ factors through the trivial cofibration $(\tilde{\Phi}X)_n \to R(\tilde{\Phi}X)_n$. We therefore get a weak equivalence $R(\tilde{\Phi}X)_n \to (R'\tilde{\Phi}X)_n$, and so a weak equivalence $\Gamma R(\tilde{\Phi}X)_n \to (Sp^{\mathbb{N}}(\Gamma)R'\tilde{\Phi}X)_n$. Therefore, it suffices to show that $X_n \to \Gamma R(\tilde{\Phi}X)_n$ is a weak equivalence for all n and all cofibrant X. Since X_n is cofibrant and Φ is a Quillen equivalence, it suffices to show that $\Phi X_n \to (\tilde{\Phi}X)_n$ is a weak equivalence for all n and all cofibrant X.

Since every cofibrant X is a retract of a transfinite composition of pushouts of maps of I_T, we can in fact assume that X is the colimit of a λ-sequence

$$0 = X^0 \to X^1 \to X^2 \to \cdots \to X^{\beta} \to \cdots \to X^{\lambda} = X$$

for some ordinal λ, where each map $X^{\beta} \to X^{\beta+1}$ is a pushout of a map of I_T. We will prove by transfinite induction on β that $\Phi X_n^{\beta} \to (\tilde{\Phi}X^{\beta})_n$ is a weak equivalence for all n and all $\beta \leq \lambda$. The base case of the induction is trivial. The limit ordinal step of the induction follows from [11, Proposition 18.10.1], since each of the maps $\Phi X_n^{\beta} \to \Phi X_n^{\beta+1}$ and each of the maps $(\tilde{\Phi}X^{\beta})_n \to (\tilde{\Phi}X^{\beta+1})_n$ is a cofibration of cofibrant objects.

For the succesor ordinal step of the induction, suppose $X^{\beta} \to X^{\beta+1}$ is a pushout of the map $F_m C \xrightarrow{F_m f} F_m D$ of I_T. Then we have a pushout diagram

$$\begin{array}{ccc}
\Phi(F_m C)_n & \longrightarrow & \Phi(F_m D)_n \\
\downarrow & & \downarrow \\
\Phi X_n^{\beta} & \longrightarrow & \Phi X_n^{\beta+1}
\end{array}$$

and another pushout diagram

$$\begin{array}{ccc} (\tilde{\Phi}F_mC)_n & \longrightarrow & (\tilde{\Phi}F_mD)_n \\ \downarrow & & \downarrow \\ (\tilde{\Phi}X^{\beta})_n & \longrightarrow & (\tilde{\Phi}X^{\beta+1})_n \end{array}$$

in $\mathscr{D}'$. Note that $\Phi(F_mC)_n = \Phi T^{n-m}C$, where we interpret $T^{n-m}C$ to be the initial object if $n < m$. Similarly, $(\tilde{\Phi}F_mC)_n \cong (F_m\Phi C)_n = (T')^{n-m}\Phi C$. Thus the natural transformation τ induces a map from the first of these pushout squares to the second. If C (and hence also D) is cofibrant, then this map of pushout squares is a weak equivalence at both the upper left and upper right corners. Or, if τ_A is a weak equivalence for all A, then again this map is a weak equivalence at both the upper left and upper right corners. It is also a weak equivalence at the lower left corner, by the induction hypothesis. Since both of the top horizontal maps are cofibrations in $\mathscr{D}'$, Dan Kan's cube lemma [12, Lemma 5.2.6] implies that the map is a weak equivalence on the lower right corner. This completes the induction.

We have now proved that $\tilde{\Phi}$ is a Quillen equivalence with respect to the projective model structures. We have already seen that $\tilde{\Phi}$ is a Quillen functor with respect to the stable model structures in Proposition 5.5. In view of Proposition 2.3, to show that $\tilde{\Phi}$ is a Quillen equivalence with respect to the stable model structures, it suffices to show that if Y is level fibrant in $Sp^{\mathbb{N}}(\mathscr{D}', T')$ and $Sp^{\mathbb{N}}(\Gamma)Y$ is a U-spectrum, then Y is a U'-spectrum. Since $Sp^{\mathbb{N}}(\Gamma)Y$ is a U-spectrum, the composite

$$\Gamma Y_n \xrightarrow{\Gamma\tilde{\sigma}} \Gamma U' Y_{n+1} \xrightarrow{D\tau} U\Gamma Y_{n+1}$$

is a weak equivalence for all n, where $\tilde{\sigma}$ is adjoint to the structure map of Y and $D\tau$ is the natural transformation dual to τ. Since τ_A is a weak equivalence for all cofibrant A, $(D\tau)_X$ is a weak equivalence for all fibrant X. This was explained in the last paragraph of the proof of Proposition 5.5. Therefore,

$$\Gamma\tilde{\sigma} : \Gamma Y_n \to \Gamma U' Y_{n+1}$$

is a weak equivalence. But Γ reflects weak equivalences between fibrant objects, by Hovey [12, Corollary 1.3.16]. Hence $\tilde{\sigma}$ is a weak equivalence for all n, and so Y is a U'-spectrum. □

As an example of Theorem 5.7, suppose we take a pointed simplicial set K weakly equivalent to S^1. Then there is a weak equivalence $K \to RS^1$, where R is a fibrant replacement functor on simplicial sets. This induces a natural transformation of left Quillen functors $\tau : - \wedge K \to - \wedge RS^1$. In Theorem 5.7, take $\mathscr{D} = \mathscr{D}'$ equal to the model category of pointed simplicial sets, take Φ to be the identity, and take τ to be this natural transformation. Then we get a Quillen equivalence between the stable model categories of spectra obtained by inverting K and inverting RS^1. Therefore, the choice of simplicial circle does not matter, up to Quillen equivalence, for Bousfield–Friedlander spectra.

6. Monoidal structure

In this section, we show that our stabilization construction preserves some monoidal structure. For example, if $\mathscr{D}$ is a simplicial model category, and T is a simplicial functor, then the category $Sp^{\mathbb{N}}(\mathscr{D},T)$ of spectra is again a simplicial model category, in both the projective and stable model structures, and the extension of T is again a simplicial functor. However, if $\mathscr{C}$ is a symmetric monoidal model category (see [12, Chapter 4]), and $T:\mathscr{C}\to\mathscr{C}$ is a monoidal functor, the category $Sp^{\mathbb{N}}(\mathscr{C},T)$ of spectra will almost never be a monoidal category. This is the reason we need the symmetric spectra introduced in the next section.

Throughout this section, $\mathscr{C}$ will be a symmetric monoidal model category, and $\mathscr{D}$ will be a $\mathscr{C}$-model category. This means that $\mathscr{C}$ is both a closed symmetric monoidal category and a model category, and the model structure is compatible with the symmetric monoidal structure in a precise sense that we will recall below. It also means that $\mathscr{D}$ is a right $\mathscr{C}$-module with a compatible model structure. That is, $\mathscr{D}$ is tensored, cotensored, and enriched over $\mathscr{C}$. See [12, Chapter 4] for complete definitions.

We remind the reader of the precise definition of the compatibility between the monoidal structure and the model structure.

Definition 6.1. Suppose $\mathscr{C}$ is a monoidal category. Given maps $f:A\to B$ and $g:C\to D$ in $\mathscr{C}$, we define the *pushout product* $f\,\square\,g$ of f and g to be the map

$$f\,\square\,g:(A\otimes D)\amalg_{A\otimes C}(B\otimes C)\to B\otimes D$$

induced by the commutative square below:

$$\begin{array}{ccc} A\otimes C & \xrightarrow{f\otimes 1} & B\otimes C \\ {\scriptstyle 1\otimes g}\downarrow & & \downarrow{\scriptstyle 1\otimes g} \\ A\otimes D & \xrightarrow[f\otimes 1]{} & B\otimes D. \end{array}$$

Note that $f\,\square\,g$ also makes perfect sense in case $f\in\mathscr{D}$ and $g\in\mathscr{C}$, since $\mathscr{D}$ is a right $\mathscr{C}$-module.

Definition 6.2. A *symmetric monoidal model category* is a symmetric monoidal category $\mathscr{C}$ equipped with a model structure satisfying the following two conditions:

1. If f and g are cofibrations in $\mathscr{C}$, then $f\,\square\,g$ is a cofibration. If, in addition, one of f or g is a trivial cofibration, so is $f\,\square\,g$.

2. Let $QS \xrightarrow{r} S$ be a cofibrant replacement for the unit S of the monoidal structure, so that QS is cofibrant and r is a trivial fibration. Then, for all cofibrant X, the induced map

$$X \otimes QS \xrightarrow{1 \otimes r} X \otimes S \cong X$$

is a weak equivalence.

Similarly, if $\mathscr{D}$ is a model category that is enriched, tensored, and cotensored over $\mathscr{C}$, then $\mathscr{D}$ is a *$\mathscr{C}$-model category* when the following two conditions are satisfied:

1. If f is a cofibration in $\mathscr{D}$ and g is a cofibration in $\mathscr{C}$, then $f \mathbin{\square} g$ is a cofibration in $\mathscr{D}$. If, in addition, one of f or g is a trivial cofibration, so is $f \mathbin{\square} g$.
2. For all cofibrant X in $\mathscr{D}$, the induced map

$$X \otimes QS \xrightarrow{1 \otimes r} X \otimes S \cong X$$

is a weak equivalence.

Note that a left Quillen functor between $\mathscr{C}$-model categories is called a *$\mathscr{C}$-Quillen functor* if it preserves the action of $\mathscr{C}$ up to natural isomorphism. See [12, Definitions 4.1.7, 4.2.18]. A $\mathscr{C}$-Quillen functor that is a Quillen equivalence is called a *$\mathscr{C}$-Quillen equivalence*.

We then have the following theorem.

Theorem 6.3. *Let $\mathscr{C}$ be a cofibrantly generated symmetric monoidal model category, and suppose the domains of the generating cofibrations of $\mathscr{C}$ can be taken to be cofibrant. Suppose $\mathscr{D}$ is a left proper cellular $\mathscr{C}$-model category, and that T is a left $\mathscr{C}$-Quillen endofunctor of $\mathscr{D}$. Then $Sp^{\mathbb{N}}(\mathscr{D}, T)$, with the stable model structure, is again a $\mathscr{C}$-model category, and the extension of T is a $\mathscr{C}$-Quillen self-equivalence of $Sp^{\mathbb{N}}(\mathscr{D}, T)$. The functors $F_n : \mathscr{D} \to Sp^{\mathbb{N}}(\mathscr{D}, T)$ are $\mathscr{C}$-Quillen functors.*

Proof. We define the action of $\mathscr{C}$ on $Sp^{\mathbb{N}}(\mathscr{D}, T)$ levelwise. That is, given $X \in Sp^{\mathbb{N}}(\mathscr{D}, T)$ and $K \in \mathscr{C}$, we define $(X \otimes K)_n = X_n \otimes K$. The structure map is given by

$$T(X_n \wedge K) \cong TX_n \wedge K \xrightarrow{\sigma \wedge 1} X_{n+1} \wedge K,$$

where the first isomorphism comes from the fact that T preserves the $\mathscr{C}$-action, and σ is the structure map of X. One can easily verify that this makes $Sp^{\mathbb{N}}(\mathscr{D}, T)$ tensored over $\mathscr{C}$. Similarly, if we denote the cotensor of $Z \in \mathscr{D}$ and $K \in \mathscr{C}$ by Z^K, we can define the cotensor X^K of $X \in Sp^{\mathbb{N}}(\mathscr{D}, T)$ and $K \in \mathscr{C}$ by $(X^K)_n = X_n^K$. The structure map $T(X_n^K) \to X_{n+1}^K$ of X^K is adjoint to the composite

$$T(X_n^K) \wedge K \cong T(X_n^K \wedge K) \xrightarrow{T(\mathrm{ev})} TX_n \xrightarrow{\sigma} X_{n+1},$$

where $\mathrm{ev} : X_n^K \wedge K$ is the evaluation map, adjoint to the identity of X_n^K. This makes $Sp^{\mathbb{N}}(\mathscr{D}, T)$ cotensored over $\mathscr{C}$. Finally, if we denote the enrichment of Z and W in $\mathscr{D}$

by $\mathrm{Map}(Z,W) \in \mathscr{C}$, we define the enrichment $\mathrm{Map}(X,Y) \in \mathscr{C}$ of X and Y in $Sp^{\mathbb{N}}(\mathscr{D},T)$ to be the equalizer of two maps

$$\alpha, \beta : \prod_n \mathrm{Map}(X_n, Y_n) \rightrightarrows \prod_n \mathrm{Map}(X_n, UY_{n+1}).$$

Here α is the product of the maps $\mathrm{Map}(X_n, Y_n) \xrightarrow{\mathrm{Map} X_n, \tilde{\sigma}} \mathrm{Map}(X_n, UY_{n+1})$ where $\tilde{\sigma}$ denotes the adjoint of the structure map, and β is the product of the maps

$$\mathrm{Map}(X_{n+1}, Y_{n+1}) \xrightarrow{\mathrm{Map}(\tilde{\sigma}, Y_{n+1})} \mathrm{Map}(TX_n, Y_{n+1}) \cong \mathrm{Map}(X_n, UY_{n+1}),$$

where the isomorphism exists since T preserves the action of $\mathscr{C}$. This functor $\mathrm{Map}(X,Y)$ makes $Sp^{\mathbb{N}}(\mathscr{D},T)$ enriched over $\mathscr{C}$.

We must now check that these structures are compatible with the model structure. We begin with the projective model structure on $Sp^{\mathbb{N}}(\mathscr{D},T)$. One can easily check that if h is a map in $\mathscr{D}$ and g is a map in $\mathscr{C}$, then $F_n h \square g = F_n(h \square g)$. Thus, if $f = F_n h$ is one of the generating cofibrations of the projective model structure on $Sp^{\mathbb{N}}(\mathscr{D},T)$, and g is a cofibration in $\mathscr{C}$, then $f \square g$ is a cofibration in $Sp^{\mathbb{N}}(\mathscr{D},T)$. It follows that $f \square g$ is a cofibration for f an arbitrary cofibration of $Sp^{\mathbb{N}}(\mathscr{D},T)$ (see [12, Corollary 4.2.5]; [13, Corollary 5.3.5], or [24, Lemma 2.3]). A similar argument shows that $f \square g$ is a projective trivial cofibration in $Sp^{\mathbb{N}}(\mathscr{D},T)$ if, in addition, either f is a projective trivial cofibration in $Sp^{\mathbb{N}}(\mathscr{D},T)$ or g is a trivial cofibration in $\mathscr{C}$. Finally, if $QS \to S$ is a cofibrant approximation to the unit S in $\mathscr{C}$, and X is cofibrant in $Sp^{\mathbb{N}}(\mathscr{D},T)$, then each X_n is cofibrant in $\mathscr{D}$, so the map $X \otimes QS \to X$ is a level equivalence as required. Thus $Sp^{\mathbb{N}}(\mathscr{D},T)$ with its projective model structure is a $\mathscr{C}$-model category.

Since the cofibrations in the projective and stable model structures on $Sp^{\mathbb{N}}(\mathscr{D},T)$ coincide, to show that $Sp^{\mathbb{N}}(\mathscr{D},T)$ with its stable model structure is also a $\mathscr{C}$-model category, we only need to show that, if f is a stable trivial cofibration in $Sp^{\mathbb{N}}(\mathscr{D},T)$ and g is a cofibration in $\mathscr{C}$, then $f \square g$ is a stable equivalence. It suffices to check this for $g: K \to L$ one of the generating cofibrations of $\mathscr{C}$, again using [12, Corollary 4.2.5]. In this case, by hypothesis, K and L are cofibrant in $\mathscr{C}$. Thus, the functor $-\otimes K$ is a Quillen functor with respect to the projective model structure on $Sp^{\mathbb{N}}(\mathscr{D},T)$, and similarly for L. We will show that $-\otimes K$ is a Quillen functor with respect to the stable model structure as well. To see this, note that if $\zeta_n^{QC}: F_{n+1}TQC \to F_n QC$ is an element of the set $\mathscr{S}$, then $\zeta_n^{QC} \otimes K \cong \zeta_n^{QC\otimes K}$, since T preserves the $\mathscr{C}$-action. In view of Theorem 3.4, the map $\zeta_n^{QC\otimes K}$ is a stable equivalence. Theorem 2.2 then implies that $-\otimes K$ is a Quillen functor with respect to the stable model structure on $Sp^{\mathbb{N}}(\mathscr{D},T)$, and similarly for $-\otimes L$. Thus, if f is a stable trivial cofibration, so are $f \otimes K$ and $f \otimes L$. It follows from the two out of three property that $f \square g$ is a stable equivalence, as required. □

Remark 6.4. Suppose that the functor T is actually given by $TX = X \otimes K$ for some cofibrant object K of $\mathscr{C}$. We then have *two different ways* of tensoring with K on $Sp^{\mathbb{N}}(\mathscr{D},T)$. We have the functor $X \mapsto X \otimes K$ that we have just constructed as part of the $\mathscr{C}$-action on $Sp^{\mathbb{N}}(\mathscr{D},T)$. We also have the functor $X \mapsto X \bar{\otimes} K$ which is the extension of T to a Quillen equivalence on the stable model structure on $Sp^{\mathbb{N}}(\mathscr{D},T)$. As explained in Remark 1.6, $X\bar{\otimes}K$ does not involve the twist map t of the symmetric monoidal structure on $\mathscr{C}$. However, the functor $X\otimes K$ *does* use the twist map as part of its structure map; indeed, in order to construct the isomorphism $T(X\otimes K)\cong TX\otimes K$ we need to permute the two different copies of K. Therefore, we *do not know* that $X\mapsto X\otimes K$ is a Quillen equivalence, even though $X\mapsto X\bar{\otimes}K$ is. We will have to deal with this point more thoroughly in Section 10, when we compare $Sp^{\mathbb{N}}(\mathscr{D},T)$ with symmetric spectra.

Theorem 6.3 gives us a functorial stabilization. We first simplify the notation. Suppose K is a cofibrant object of a symmetric monoidal model category $\mathscr{C}$. Then $T=-\otimes K$ is a left Quillen functor on any $\mathscr{C}$-model category $\mathscr{D}$. In this case, we denote $Sp^{\mathbb{N}}(\mathscr{D},T)$ by $Sp^{\mathbb{N}}(\mathscr{D},K)$.

Corollary 6.5. *Suppose K is a cofibrant object of a cofibrantly generated symmetric monoidal model category $\mathscr{C}$ where the domains of the generating cofibrations can be taken to be cofibrant. Then the correspondence $\mathscr{D}\mapsto Sp^{\mathbb{N}}(\mathscr{D},K)$, where $Sp^{\mathbb{N}}(\mathscr{D},K)$ is given the stable model structure, defines an endofunctor of the category of left proper cellular $\mathscr{D}$-model categories.*

Note that the "category" of left proper cellular $\mathscr{C}$-model categories is not really a category, because the Hom-sets need not be sets. It is really a 2-category, and the correspondence $\mathscr{D}\mapsto Sp^{\mathbb{N}}(\mathscr{D},K)$ is actually a 2-functor. See [12] for a description of this point of view on model categories.

Proof. Given a left proper cellular $\mathscr{C}$-model category $\mathscr{D}$, we have seen in Theorem 6.3 that $Sp^{\mathbb{N}}(\mathscr{D},K)$ is a $\mathscr{C}$-model category. Given a left $\mathscr{C}$-Quillen functor $\Phi:\mathscr{D}\to\mathscr{D}'$ between two left proper cellular $\mathscr{C}$-model categories $\mathscr{D}$ and $\mathscr{D}'$ with right adjoint Γ, there is a natural isomorphism $\tau:\Phi(-\otimes K)\to\Phi(-)\otimes K$. Taking $\rho=D\tau$ in Lemma 5.3, we get an induced functor $\tilde{\Phi}:Sp^{\mathbb{N}}(\mathscr{D},K)\to Sp^{\mathbb{N}}(\mathscr{D}',K)$ with right adjoint $Sp^{\mathbb{N}}(\Gamma,\rho)$. Since τ is an isomorphism, $\tilde{\Phi}$ is a prolongation of Φ, and so one can easily check that $\tilde{\Phi}$ preserves the action of $\mathscr{C}$. Proposition 5.5 guarantees that $\tilde{\Phi}$ is a Quillen functor with respect to the stable model structures. □

We now point out that, if $\mathscr{C}$ is a symmetric monoidal model category, and T is a $\mathscr{C}$-Quillen endofunctor of $\mathscr{C}$, then the category $Sp^{\mathbb{N}}(\mathscr{C},T)$ is almost never itself monoidal, though, as we have seen, it has an action of $\mathscr{C}$. To see this, first note that T is naturally isomorphic to $-\otimes K$ for $K=TS$. Now consider the category $\mathscr{C}^{\mathbb{N}}$ of sequences from $\mathscr{C}$. An object of $\mathscr{C}^{\mathbb{N}}$ is a sequence X_n of objects of $\mathscr{C}$, and a map $f:X\to Y$ is a sequence of maps $f_n:X_n\to Y_n$. Then $\mathscr{C}^{\mathbb{N}}$ is a symmetric monoidal category, where we define $(X\otimes Y)_n=\amalg_{p+q=n}X_p\otimes Y_q$. Furthermore, if $\mathscr{D}$ is a $\mathscr{C}$-model category, then $\mathscr{D}^{\mathbb{N}}$ is a right $\mathscr{C}^{\mathbb{N}}$-module, using the same definition of the tensor product. The sequence $F_0S=(S,K,K\otimes K,\dots,K^{\otimes n},\dots)$ is a monoid in $\mathscr{C}^{\mathbb{N}}$.

Lemma 6.6. *Suppose $\mathscr{C}$ is a symmetric monoidal model category and T is a left $\mathscr{C}$-Quillen functor with $K = TS$. Then $Sp^{\mathbb{N}}(\mathscr{C}, T)$ is the category of right modules in $\mathscr{C}^{\mathbb{N}}$ over the monoid F_0S. Furthermore, if $\mathscr{D}$ is a $\mathscr{C}$-model category, $Sp^{\mathbb{N}}(\mathscr{D}, K)$ is the category of right modules in $\mathscr{D}^{\mathbb{N}}$ over F_0S.*

We leave the proof of this lemma to the reader, as it is a matter of unwinding definitions. The important corollary of this lemma is that the monoid F_0S is almost never commutative, and therefore $Sp^{\mathbb{N}}(\mathscr{C}, K)$ can not be a symmetric monoidal category with unit F_0S. Indeed, F_0S is commutative if and only if the commutativity isomorphism of $\mathscr{C}$ applied to $K \otimes K$ is the identity. This happens only very rarely.

7. Symmetric spectra

We have just seen that the stabilization functor $Sp^{\mathbb{N}}(\mathscr{C}, T)$ is not good enough in case $\mathscr{C}$ is a symmetric monoidal model category and T is a $\mathscr{C}$-Quillen functor, because $Sp^{\mathbb{N}}(\mathscr{C}, T)$ is not usually itself a symmetric monoidal model category. In this section, we begin the construction of a better stabilization functor for this case. We will concentrate on the category theory in this section, leaving the model structures for the next section. The terms used for the algebra of symmetric monoidal categories and modules over them are defined in [12, Section 4.1].

Through most of this section, then, $\mathscr{C}$ will be a bicomplete closed symmetric monoidal category with unit S, and K will be an object of $\mathscr{C}$. The category $\mathscr{D}$ will be a bicomplete right $\mathscr{C}$-module category; this means that $\mathscr{D}$ is enriched, tensored, and cotensored over $\mathscr{C}$. Note that any $\mathscr{C}$-functor T on $\mathscr{C}$ itself is of the form $T(L) = L \otimes K$ for $K = TS$, so we will only consider such functors. Because of this, we will drop the letter T from our notations and replace it with K.

This section is based on the symmetric spectra and sequences of [13]. The main idea of symmetric spectra is that the associativity and commutativity isomorphisms of $\mathscr{C}$ make $K^{\otimes n}$ into a Σ_n-object of $\mathscr{C}$, where Σ_n is the symmetric group on n letters. We must keep track of this action if we expect to get a symmetric monoidal category of K-spectra.

The following definition is [13, Definition 2.1.1].

Definition 7.1. Let $\Sigma = \amalg_{n \geq 0} \Sigma_n$ be the category whose objects are the sets $\bar{n} = \{1, 2, \dots, n\}$ for $n \geq 0$, where $\bar{0} = \emptyset$. The morphisms of Σ are the isomorphisms of $\bar{n}$. Given a category $\mathscr{E}$, a *symmetric sequence* in $\mathscr{E}$ is a functor $\Sigma \to \mathscr{E}$. The category of symmetric sequences is the functor category $\mathscr{E}^{\Sigma}$.

A symmetric sequence X in a category $\mathscr{E}$ is a sequence $X_0, X_1, \dots, X_n, \dots$ of objects of $\mathscr{E}$ with an action of Σ_n on X_n. It is sometimes more useful to consider a symmetric sequence as a functor from the category of finite sets and isomorphisms to $\mathscr{E}$. Since the category Σ is a skeleton of the category of finite sets and isomorphisms, there is no difficulty in such a change of viewpoint.

As a functor category, the category of symmetric sequences in $\mathscr{E}$ is bicomplete if $\mathscr{E}$ is so; limits and colimits are taken levelwise. Furthermore, since our category $\mathscr{C}$ is closed symmetric monoidal, so is $\mathscr{C}^{\Sigma}$, as explained in [13, Section 2.1]. This result is a special case of the much more general work of Day [3]. Recall that the monoidal structure is given by

$$(X \otimes Y)(C) = \coprod_{A\cup B=C, A\cap B=\emptyset} X(A) \otimes Y(B),$$

where we think of X, Y, and $X \otimes Y$ as functors from finite sets to $\mathscr{C}$. Equivalently, though less canonically, we have

$$(X \otimes Y)_n = \coprod_{p+q=n} \Sigma_n \times_{\Sigma_p\times\Sigma_q} (X_p \otimes Y_q).$$

This notation may need some explanation. Given a set Γ and an object A of a cocomplete category $\mathscr{E}$, $\Gamma \times A$ is the coproduct of $|\Gamma|$ copies of A. If Γ is a group, then $\Gamma \times A$ has an obvious left Γ-action; $\Gamma \times A$ is the free Γ-object on A. Note that a Γ-action on A is then equivalent to a map $\Gamma \times A \to A$ satisfying the usual unit and associativity conditions. Also, if Γ admits a right action by a group Γ', and A is a left Γ'-object, then we can form $\Gamma \times_{\Gamma'} A$ as the colimit of the Γ'-action on $\Gamma \times A$, where $\alpha \in \Gamma'$ takes the copy of A corresponding to $\beta \in \Gamma$ to the copy of A corresponding to $\beta\alpha^{-1}$ by the action of α.

The unit of the monoidal structure on $\mathscr{C}^{\Sigma}$ is the symmetric sequence $(S, 0, 0, \ldots)$, where 0 is the initial object of $\mathscr{C}$. To define the closed structure on $\mathscr{C}^{\Sigma}$, we first define $\mathrm{Hom}_{\Sigma_n}(L, L')$ for Σ_n-objects L and L' in $\mathscr{C}$ in the usual way, as an equalizer of the two maps $\mathrm{Hom}(L, L') \to \mathrm{Hom}(\Sigma_n \times L, L')$ defined using the structure maps of L and L'. The closed structure is then given by

$$\mathrm{Hom}(X, Y)_k = \prod_n \mathrm{Hom}_{\Sigma_n}(X_n, Y_{n+k}).$$

Since our category $\mathscr{D}$ is enriched, tensored, and cotensored over $\mathscr{C}$, $\mathscr{D}^{\Sigma}$ is enriched, tensored, and cotensored over $\mathscr{C}^{\Sigma}$, making $\mathscr{D}^{\Sigma}$ a right $\mathscr{C}^{\Sigma}$-module category. Indeed, the same definition as above works to define the tensor structure. The cotensor structure is defined as follows. First we define $\mathrm{Hom}_{\Sigma_n}(L, A)$ for Σ_n-objects L of $\mathscr{C}$ and A of $\mathscr{D}$ as an appropriate equalizer. Then, for $X \in \mathscr{D}^{\Sigma}$ and $Z \in \mathscr{C}^{\Sigma}$, we define $X^Z_k = \prod_n \mathrm{Hom}_{\Sigma_n}(Z_n, X_{n+k})$. The enrichment $\mathrm{Map}(X, Y) \in \mathscr{C}^{\Sigma}$ for X and Y in $\mathscr{D}^{\Sigma}$ is defined similarly. In the same way, if $\mathscr{D}$ is an enriched monoidal category over $\mathscr{C}$ (a $\mathscr{C}$-algebra, in the terminology of [12, Section 4.1]), then $\mathscr{D}^{\Sigma}$ is an enriched monoidal category over $\mathscr{C}^{\Sigma}$.

Consider the free commutative monoid $\mathrm{Sym}(K)$ on the object $(0, K, 0, \ldots, 0, \ldots)$ of $\mathscr{C}^{\Sigma}$. One can easily check that $\mathrm{Sym}(K)$ is the symmetric sequence $(S, K, K \otimes K, \ldots, K^{\otimes n}, \ldots)$ where Σ_n acts on $K^{\otimes n}$ by permutation, using the commutativity and associativity isomorphisms.

Definition 7.2. Suppose $\mathscr{C}$ is a symmetric monoidal model category, $\mathscr{D}$ is a $\mathscr{C}$-model category, and K is an object of $\mathscr{C}$. The category of *symmetric spectra* $Sp^{\Sigma}(\mathscr{D},K)$ is the category of modules in $\mathscr{D}^{\Sigma}$ over the commutative monoid $\mathrm{Sym}(K)$ in $\mathscr{C}^{\Sigma}$. That is, a symmetric spectrum X is a sequence of Σ_n-objects $X_n \in \mathscr{C}$ and Σ_n-equivariant maps $X_n \otimes K \to X_{n+1}$, such that the composite

$$X_n \otimes K^{\otimes p} \to X_{n+1} \otimes K^{\otimes p-1} \to \cdots \to X_{n+p}$$

is $\Sigma_n \times \Sigma_p$-equivariant for all $n, p \geq 0$. A map of symmetric spectra is a collection of Σ_n-equivariant maps $X_n \to Y_n$ compatible with the structure maps of X and Y.

Because $\mathrm{Sym}(K)$ is a commutative monoid, the category $Sp^{\Sigma}(\mathscr{C},K)$ is a bicomplete closed symmetric monoidal category, with $\mathrm{Sym}(K)$ itself as the unit (see Lemmas 2.2.2 and 2.2.8 of [13]). We denote the monoidal structure by $X \wedge Y = X \otimes_{\mathrm{Sym}(K)} Y$, and the closed structure by $\mathrm{Hom}_{\mathrm{Sym}(K)}(X,Y)$. Similarly, $Sp^{\Sigma}(\mathscr{D},K)$ is bicomplete, enriched, tensored, and cotensored over $Sp^{\Sigma}(\mathscr{C},K)$ with the tensor structure denoted $X \wedge Y$ again, and, if $\mathscr{D}$ is a $\mathscr{C}$-monoidal model category, then $Sp^{\Sigma}(\mathscr{D},K)$ will be a monoidal category enriched over $Sp^{\Sigma}(\mathscr{C},K)$.

Of course, if we take $\mathscr{C} = \mathbf{SSet}_*$ and $K = S^1$, we recover the definition of symmetric spectra given in [13], except that we are using right $\mathrm{Sym}(K)$-modules instead of left $\mathrm{Sym}(K)$-modules.

Definition 7.3. Given $n \geq 0$, the *evaluation functor* Ev_n: $Sp^{\Sigma}(\mathscr{D},K) \to \mathscr{D}$ takes X to X_n. The evaluation functor has a left adjoint F_n: $\mathscr{D} \to Sp^{\Sigma}(\mathscr{D},K)$, defined by $F_nA = \tilde{F}_nA \otimes \mathrm{Sym}(K)$, where $\tilde{F}_nA$ is the symmetric sequence $(0,\ldots,0,\Sigma_n \times A, 0, \ldots)$.

Note that $F_0A = (A, A\otimes K, \ldots, A\otimes K^{\otimes n}, \ldots)$, and in particular $F_0S = \mathrm{Sym}(K)$. In general, we have $(F_nA)_m = \Sigma_m \times_{\Sigma_{m-n}} (A \otimes K^{\otimes(m-n)})$ for $m \geq n$. Also, if $A \in \mathscr{D}$ and $L \in \mathscr{C}$, there is a natural isomorphism $F_nA \wedge F_mL \cong F_{n+m}(A \otimes L)$, just as in [13, Proposition 2.2.6]. In particular, F_0: $\mathscr{C} \to Sp^{\Sigma}(\mathscr{C},K)$ is a (symmetric) monoidal functor, and so $Sp^{\Sigma}(\mathscr{D},K)$ is naturally enriched, tensored, and cotensored over $\mathscr{C}$. In fact, this structure is very simple. Indeed, if $X \in Sp^{\Sigma}(\mathscr{D},K)$ and $L \in \mathscr{C}$, $X \otimes L = X \otimes_{\mathrm{Sym}(K)} F_0L$ is just the symmetric sequence whose nth term is $X_n \otimes L$. The structure map is the composite

$$X_n \otimes L \otimes K \xrightarrow{1\otimes t} X_n \otimes K \otimes L \to X_{n+1} \otimes L.$$

Note the presence of the twist map t; this is required even when $L = K$ to get a symmetric spectrum, unlike the case of ordinary spectra. Similarly, $X^L = \mathrm{Hom}_{\mathrm{Sym}(K)}(F_0L, X)$ is the symmetric sequence whose nth term is X_n^L, with the twist map again appearing as part of the structure map.

Remark 7.4. The evaluation functor Ev_n has a right adjoint

$$M_n: \mathscr{D} \to Sp^{\Sigma}(\mathscr{D},K),$$

just as in the spectrum case (see Remark 1.4). Indeed, we define

$$M_nA = \mathrm{Hom}(\mathrm{Sym}(K), \tilde{M}_nA),$$

where $\tilde{M}_n A$ is the symmetric sequence that is the terminal object in dimensions other than n, and is the cofree Σ_n-object $\mathscr{C}(\Sigma_n, A)$ in dimension n. As an object of $\mathscr{C}$, $\mathscr{C}(\Sigma_n, A)$ is just the $n!$-fold product of A. Given $\rho \in \Sigma_n$ and $f \in \mathscr{C}(\Sigma_n, A)$, the Σ_n-action is defined by $(\rho f)(\rho') = f(\rho'\rho)$. Just as in Remark 1.4, $M_n A$ is the terminal object in dimensions greater than n.

8. Model structures on symmetric spectra

Throughout this section, $\mathscr{C}$ will denote a left proper cellular symmetric monoidal model category, $\mathscr{D}$ will denote a left proper cellular $\mathscr{C}$-model category, and K will denote a cofibrant object of $\mathscr{C}$. In this section, we discuss the projective and stable model structures on the category $Sp^{\Sigma}(\mathscr{D}, K)$ of symmetric spectra. The results in this section are very similar to the corresponding results in Section 3, so we will leave most of the proofs to the reader.

Definition 8.1. A map $f \in Sp^{\Sigma}(\mathscr{D}, K)$ is a *level equivalence* if each map f_n is a weak equivalence in $\mathscr{D}$. Similarly, f is a *level fibration* (resp. *level cofibration*, *level trivial fibration*, *level trivial cofibration*) if each map f_n is a fibration (resp. cofibration, trivial fibration, trivial cofibration) in $\mathscr{D}$. The map f is a *projective cofibration* if f has the left lifting property with respect to every level trivial fibration.

Then, just as in Definition 1.8, if we denote the generating cofibrations of $\mathscr{D}$ by I and the generating trivial cofibrations by J, we define $I_K = \bigcup_n F_n I$ and $J_K = \bigcup_n F_n J$.

We have analogues of 1.9–1.12 with almost the same proofs. The only real difference is that it is less obvious that the maps of I_K are level cofibrations, and that the maps of J_K are level trivial cofibrations. If $g: A \to B$ is a map in $\mathscr{D}$, and $m \geq n$, then $(F_n g)_m$ is the map

$$\Sigma_m \times_{\Sigma_{m-n}} (g \otimes K^{\otimes m-n}) \cong g \otimes (\Sigma_m \times_{\Sigma_{m-n}} K^{\otimes m-n}).$$

As a map in $\mathscr{D}$, this is the coproduct of $m!/(m-n)!$ copies of $g \otimes K^{\otimes m-n}$. Since K is cofibrant, if g is a cofibration (trivial cofibration), then $F_n g$ is a level cofibration (level trivial cofibration). This uses the fact that $\mathscr{D}$ is a $\mathscr{C}$-model category, and also that $(F_n g)_m = 0$ for $m < n$.

We then construct the projective model structure just as in the proof of Theorem 1.13.

Theorem 8.2. *The projective cofibrations, the level fibrations, and the level equivalences define a left proper cellular model structure on* $Sp^{\Sigma}(\mathscr{D}, K)$.

The set I_K is the set of generating cofibrations of the projective model structure, and J_K is the set of generating trivial cofibrations. The cellularity of the projective model structure is proved in the appendix.

Note that Ev_n takes level (trivial) fibrations to (trivial) fibrations, so Ev_n is a right Quillen functor and F_n is a left Quillen functor, with respect to the projective model structure.

Theorem 8.3. *The category $Sp^{\Sigma}(\mathscr{C},K)$, with the projective model structure, is a symmetric monoidal model category. The category $Sp^{\Sigma}(\mathscr{D},K)$, with its projective model structure, is a $Sp^{\Sigma}(\mathscr{C},K)$-model category.*

See the discussion following Definition 6.1 for the definition of a symmetric monoidal model category, or see [12, Chapter 4] for more detail.

Proof. We first show that the pushout product $f \square g$ is a cofibration when f is a cofibration in $Sp^{\Sigma}(\mathscr{D},K)$ and g is a cofibration in $Sp^{\Sigma}(\mathscr{C},K)$, and that $f \square g$ is a trivial cofibration when, in addition, one of f or g is a level equivalence. As explained in [12, Corollary 4.2.5], we may as well assume that f and g belong to the sets of generating cofibrations or generating trivial cofibrations. In either case, we have $f = F_m f'$ and $g = F_n g'$. But then $f \square g = F_{m+n}(f' \square g')$. Since F_{m+n} is a left Quillen functor, the result follows.

It remains to show that, if X is cofibrant in $Sp^{\Sigma}(\mathscr{D},K)$ and $Q(\mathrm{Sym}(K))$ is a cofibrant replacement for the unit $\mathrm{Sym}(K)$ of $Sp^{\Sigma}(\mathscr{C},K)$, then the map $X \otimes Q(\mathrm{Sym}(K)) \to X \otimes \mathrm{Sym}(K) \cong X$ is a level equivalence. Let QS denote a cofibrant replacement for the unit S in $\mathscr{C}$. Then we claim that F_0QS is a cofibrant replacement for $F_0S = \mathrm{Sym}(K)$ in $Sp^{\Sigma}(\mathscr{C},K)$. Indeed, F_0QS is cofibrant, and $\mathrm{Ev}_n F_0 QS$ is just $QS \otimes K^{\otimes n}$. Since K is cofibrant and $\mathscr{C}$ is a monoidal model category, the map $F_0QS \to F_0S$ is a level equivalence. Now, since X is cofibrant in $Sp^{\Sigma}(\mathscr{D},K)$, each X_n is cofibrant. Hence the map $X_n \otimes QS \to X_n$ is a weak equivalence for all n, and so the map $X \wedge F_0QS \to X$ is a level equivalence, as required. □

We point out that one can show that the projective model structure on $Sp^{\Sigma}(\mathscr{C},K)$ satisfies the monoid axiom of [24], assuming that $\mathscr{C}$ itself does so. This means there is a projective model structure on the category of monoids in $Sp^{\Sigma}(\mathscr{C},K)$ and on the category of modules over any monoid. We do not include the proofs of these statements since we have been unable to prove the analogous statements for the stable model structure.

The projective cofibrations of symmetric spectra are more complicated than they are in the case of ordinary spectra. The idea is the same as that for ordinary spectra. Recall that an ordinary spectrum A is cofibrant if A_0 is cofibrant and each map $A_{n-1} \otimes K \to A_n$ is a cofibration (see Proposition 1.14). This then implies that each map $A_k \otimes K^{\otimes(n-k)} \to A_n$ for $k < n$ is a cofibration as well. However, in the case of symmetric spectra, we need to guarantee that $A_k \otimes K^{\otimes(n-k)} \to A_n$ is a cofibration in the category of $\Sigma_k \times \Sigma_{n-k}$ objects. We therefore introduce a more complicated object, called the *latching space* L_nA, which is an amalgam of all the objects $A_k \otimes K^{\otimes(n-k)}$, induced up from $\Sigma_k \times \Sigma_{n-k}$-objects to Σ_n-objects.

Definition 8.4. Define the symmetric spectrum $\overline{\mathrm{Sym}(K)}$ in $Sp^{\Sigma}(\mathscr{C},K)$ to be 0 in degree 0 and $K^{\otimes n}$ in degree n, for $n>0$, with the obvious structure maps. Define the nth *latching space* L_nA of $A \in Sp^{\Sigma}(\mathscr{D},K)$ by $L_nA = \mathrm{Ev}_n(A \wedge \overline{\mathrm{Sym}(K)})$.

Note that L_nA is precisely the colimit of the objects $\Sigma_n \times_{\Sigma_k \times \Sigma_{n-k}} (A_k \otimes K^{\otimes(n-k)})$ for $k<n$, where the colimit is taken using the $\mathrm{Sym}(K)$-module structure. The obvious map of spectra $\overline{\mathrm{Sym}(K)} \to \mathrm{Sym}(K)$ induces a Σ_n-equivariant natural transformation $i\colon L_nA \to A_n$. When restricted to $\Sigma_n \times_{\Sigma_k\times\Sigma_{n-k}} (A_k \otimes K^{\otimes n-k})$, this map is just the structure map of A.

Note that the latching space is a Σ_n-object of $\mathscr{D}$. There is a model structure on Σ_n-objects of $\mathscr{D}$ where the fibrations and weak equivalences are the underlying ones. This model structure is cofibrantly generated: if I is the set of generating cofibrations of $\mathscr{D}$, then the set of generating cofibrations of $\mathscr{D}^{\Sigma_n}$ is the set $\Sigma_n \times I$. Recall that, for an object N, $\Sigma_n \times N$ is the coproduct of $n!$ copies of N, given the obvious Σ_n-structure. The cofibrations in $\mathscr{D}^{\Sigma_n}$ are cofibrations f in $\mathscr{D}$ where Σ_n acts freely away from the image of f.

Proposition 8.5. *A map $f\colon A \to B$ in $Sp^{\Sigma}(\mathscr{D},K)$ is a projective cofibration if and only if the induced map $\mathrm{Ev}_n(f \,\Box\, i)\colon A_n \amalg_{L_nA} L_nB \to B_n$ is a cofibration in $\mathscr{D}^{\Sigma_n}$ for all n. Similarly, f is a projective trivial cofibration if and only if $\mathrm{Ev}_n(f \,\Box\, i)$ is a trivial cofibration in $\mathscr{D}^{\Sigma_n}$ for all n.*

Note the similarity of Proposition 8.5 to Proposition 1.14.

Proof. We only prove the cofibration case, as the trivial cofibration case is analogous. Suppose first that each map $\mathrm{Ev}_n(f \,\Box\, i)$ is a cofibration in $\mathscr{D}^{\Sigma_n}$. We show that f is a projective cofibration by showing f has the left lifting property with respect to every level trivial fibration $p\colon X \to Y$. Suppose we have a commutative square as below:

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y. \end{array}$$

We construct a lift $B_n \to X_n$ in this diagram by induction on n. When $n=0$, this is easy since f_0 is a cofibration and p_0 is a trivial fibration in $\mathscr{D}$. Suppose we have constructed compatible partial lifts $B_k \to X_k$ for $k<n$. These partial lifts assemble into a Σ_n-equivariant map $L_nB \to X_n$. Combining this with the given map $A_n \to X_n$, we get the commutative diagram of Σ_n-equivariant maps below:

$$\begin{array}{ccc} A_n \amalg_{L_nA} L_nB & \longrightarrow & X_n \\ {\scriptstyle \mathrm{Ev}_n(f \,\Box\, i)}\downarrow & & \downarrow{\scriptstyle p_n} \\ B_n & \longrightarrow & Y_n. \end{array}$$

Since $\mathrm{Ev}_n(f \mathbin{\square} i)$ is a cofibration in $\mathscr{D}^{\Sigma_n}$ and p_n is a trivial fibration, there is a lift $B_n \to X_n$ in this diagram. This completes the induction step. The resulting map $B \to X$ is a map of spectra, since the structure maps of B are encoded into the map $L_n B \to B_n$, and gives us the desired lift.

To prove the converse, we need to show that if f is a cofibration of symmetric spectra, then $\mathrm{Ev}_n(f \mathbin{\square} i)$ is a Σ_n-cofibration. Since f is a retract of a transfinite composition of pushouts of maps of I_K, and $\mathrm{Ev}_n(- \mathbin{\square} i)$ preserves retracts, transfinite compositions, and pushouts, we can assume f is a map in I_K. Then we can write $f = F_m g$ for some integer m and some map $g\colon C \to D$ in $\mathscr{D}$. In this case, one can check that the map $L_n F_m C \to F_m C$ is an isomorphism except when $n = m$, in which case it is the map from the initial object to $\Sigma_m \times C$. It follows that $\mathrm{Ev}_n(f \mathbin{\square} i)$ is an isomorphism when $n \neq m$, and $\mathrm{Ev}_m(f \mathbin{\square} i) = \Sigma_m \times g$. □

We must now localize the projective model structure to obtain the stable model structure.

Definition 8.6. A symmetric spectrum $X \in Sp^{\Sigma}(\mathscr{D}, K)$ is an *Ω-spectrum* if X is level fibrant and the adjoint $X_n \to X_{n+1}^K$ of the structure map of X is a weak equivalence for all n.

Just as with Bousfield–Friedlander spectra, we would like the Ω-spectra to be the fibrant objects in the stable model structure. We therefore invert analogous maps.

Definition 8.7. Define the set of maps $\mathscr{S}$ in $Sp^{\Sigma}(\mathscr{D}, K)$ to be

$$\{F_{n+1}(QC \otimes K) \xrightarrow{\zeta_n^{QC}} F_n QC\}$$

as C runs through the domains and codomains of the generating cofibrations of $\mathscr{D}$, and $n \geq 0$. The map ζ_n^{QC} is adjoint to the map

$$QC \otimes K \to \mathrm{Ev}_{n+1} F_n QC = \Sigma_{n+1} \times (QC \otimes K)$$

corresponding to the identity of Σ_{n+1}. We then define the *stable model structure* on $Sp^{\Sigma}(\mathscr{D}, K)$ to be the Bousfield localization with respect to $\mathscr{S}$ of the projective model structure on $Sp^{\Sigma}(\mathscr{D}, K)$. The $\mathscr{S}$-local weak equivalences are called the *stable equivalences*, and the $\mathscr{S}$-local fibrations are called the *stable fibrations*.

The following theorem is then analogous to Theorem 3.4, and has the same proof.

Theorem 8.8. *The stably fibrant symmetric spectra are the Ω-spectra. Furthermore, for all cofibrant $A \in \mathscr{D}$ and for all $n \geq 0$, the map $F_{n+1}(A \otimes K) \xrightarrow{\zeta_n^A} F_n A$ is a stable equivalence.*

Just as in Corollary 3.5, this theorem implies that, when $\mathscr{D} = \mathbf{SSet}_*$ or $\mathbf{Top}_*$, $Sp^{\Sigma}(\mathscr{C}, K)$ is the same as the stable model category on (simplicial or topological) symmetric spectra discussed in [13].

The analog of Lemma 3.6 also holds, with the same proof, so that tensoring with K is a Quillen endofunctor of $Sp^{\Sigma}(\mathscr{D},K)$. Of course, we want this functor to be a Quillen equivalence. As in Definition 3.7, we prove this by introducing the shift functors.

Definition 8.9. Define the *right shift functor* s_-: $Sp^{\Sigma}(\mathscr{D},K) \to Sp^{\Sigma}(\mathscr{D},K)$ by $s_-X = \mathrm{Hom}_{\mathrm{Sym}(K)}(F_1S,X)$. Thus $(s_-X)_n = X_{n+1}$, where the Σ_n-action on X_{n+1} is induced by the usual inclusion $\Sigma_n \to \Sigma_{n+1}$. The structure maps of s_-X are the same as the structure maps of X. Define the *left shift functor* s_+: $Sp^{\Sigma}(\mathscr{D},K) \to Sp^{\Sigma}(\mathscr{D},K)$ by $s_+X = X \otimes_{\mathrm{Sym}(K)} F_1S$, so that s_+ is left adjoint to s_-. We have $(s_+X)_n = \Sigma_n \times_{\Sigma_{n-1}} X_{n-1}$ for $n > 0$, and $(s_+X)_0$ is the initial object of $\mathscr{D}$. The structure maps of s_+X are induced from the structure maps of X.

Note that adjointness gives natural isomorphisms

$$\begin{aligned}(s_-X)^K &\cong \mathrm{Hom}_{\mathrm{Sym}(K)}(F_0K, \mathrm{Hom}_{\mathrm{Sym}(K)}(F_1S,X))\\ &\cong \mathrm{Hom}_{\mathrm{Sym}(K)}(F_0K \wedge F_1S,X) \cong \mathrm{Hom}_{\mathrm{Sym}(K)}(F_1K,X).\end{aligned}$$

A similar chain of isomorphisms shows that $s_-(X^K)$ is also naturally isomorphic to $\mathrm{Hom}_{\mathrm{Sym}(K)}(F_1K,X)$.

There is a map $F_1K \to F_0S$ which is the identity in degree 1. By adjointness, this map induces a map

$$X = \mathrm{Hom}_{\mathrm{Sym}(K)}(F_0S,X) \to \mathrm{Hom}_{\mathrm{Sym}(K)}(F_1K,X) = (s_-X)^K.$$

By applying Ev_n, we get a map $X_n \to X_{n+1}^K$. This map is adjoint to the structure map of X, as is explained for simplicial symmetric spectra in [13, Remark 2.2.12]. Therefore, X is an Ω-spectrum if and only if this map $X \to (s_-X)^K$ is a level equivalence and X is level fibrant. Hence the same method used to prove Theorem 3.9 also proves the following theorem.

Theorem 8.10. *The functors $X \mapsto X \otimes K$ and s_+ are Quillen equivalences with respect to the stable model structure on $Sp^{\Sigma}(\mathscr{D},K)$. Furthermore, Rs_- is naturally isomorphic to $L(- \otimes K)$ and $R((-)^K)$ is naturally isomorphic to Ls_+.*

We have now shown that $Sp^{\Sigma}(\mathscr{D},K)$ is a K-stabilization of $\mathscr{D}$. However, for symmetric spectra to be better than ordinary spectra, we must show that $Sp^{\Sigma}(\mathscr{C},K)$ is a symmetric monoidal model category.

Theorem 8.11. *Suppose that the domains of the generating cofibrations of both $\mathscr{C}$ and $\mathscr{D}$ are cofibrant. Then the stable model structure makes $Sp^{\Sigma}(\mathscr{C},K)$ into a symmetric monoidal model category, and the stable model structure makes $Sp^{\Sigma}(\mathscr{D},K)$ into a $Sp^{\Sigma}(\mathscr{C},K)$-model category.*

Proof. We prove this theorem in the same way as Theorem 6.3. Since the cofibrations in the stable model structure are the same as the cofibrations in the projective model structure, the only thing to check is that $f \,\square\, g$ is a stable equivalence when f and

g are cofibrations and one of them is a stable equivalence. We may as well assume that $f: F_nA \to F_nB$ is a generating cofibration in $Sp^{\Sigma}(\mathscr{D},K)$ and g is a stable trivial cofibration in $Sp^{\Sigma}(\mathscr{C},K)$, by Hovey [12, Corollary 4.2.5]; the argument for f a stable trivial cofibration and g a generating cofibration in $Sp^{\Sigma}(\mathscr{C},K)$ is the same. Then, by hypothesis, A and B are cofibrant in $\mathscr{D}$. We claim that $F_nA \wedge (-): Sp^{\Sigma}(\mathscr{C},K) \to Sp^{\Sigma}(\mathscr{D},K)$ is a Quillen functor with respect to the stable model structures, and similarly for $F_nB \wedge (-)$. Indeed, in view of Theorem 2.2, it suffices to show that $F_nA \wedge \zeta_m^{QC}$ is a stable equivalence for all $m \geq 0$ and all domains or codomains C of the generating cofibrations of $\mathscr{C}$. But one can easily check that $F_nA \wedge \zeta_m^{QC} = \zeta_{n+m}^{A\otimes QC}$. Then Theorem 8.8 implies that this map is a stable equivalence, as required.

Thus, both functors $F_nA \wedge (-)$ and $F_nB \wedge (-)$ are Quillen functors with respect to the stable model structures. Consider the commutative diagram below:

$$\begin{array}{ccccc}
F_nA \wedge X & \xrightarrow{f\wedge X} & F_nB \wedge X & = & F_nB \wedge X \\
\Big\downarrow{\scriptstyle F_nA\wedge g} & & \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle F_nB\wedge g} \\
F_nA \wedge Y & \longrightarrow & P & \xrightarrow{f \,\square\, g} & F_nB \wedge Y,
\end{array}$$

where the left-hand square is a pushout square. Since g is a stable trivial cofibration, so are $F_nA \wedge g$ and $F_nB \wedge g$. Since the left-hand square is a pushout square α is also a stable trivial cofibration. By the two out of three property of stable equivalences, $f \,\square\, g$ is a stable equivalence, as required. □

The functor $F_0: \mathscr{C} \to Sp^{\Sigma}(\mathscr{C},K)$ is a symmetric monoidal Quillen functor, so, under the hypotheses of Theorem 8.11, $Sp^{\Sigma}(\mathscr{D},K)$ is a $\mathscr{C}$-model category as well. In fact, we only need the domains of the generating cofibrations of $\mathscr{C}$ to be cofibrant to conclude that $Sp^{\Sigma}(\mathscr{D},K)$ is a $\mathscr{C}$-model category, using the argument of Theorem 8.11.

As we mentioned above, we do not know if the stable model structure satisfies the monoid axiom. Given a particular monoid R, one could attempt to localize the projective model structure on R-modules to obtain a stable model structure. However, for this to work one would need to know that the projective model structure on R-modules is cellular, and the author does not see how to prove this. This plan will certainly fail for the category of monoids, since the projective model structure on monoids will not be left proper in general.

We also point out that it may be possible to prove some of the results of Section 4 for symmetric spectra. Not all of those results can hold, since stable homotopy isomorphisms do not coincide with stable equivalences even for symmetric spectra of simplicial sets. Nevertheless, in that case, every stable homotopy isomorphism is a stable equivalence [13, Theorem 3.1.11]. Shipley [25] has constructed a fibrant replacement functor for simplicial symmetric spectra as well. We do not know if these results hold for symmetric spectra over a general well-behaved almost finitely generated model category.

9. Properties of symmetric spectra

In this section, we point out that the arguments of Section 5 also apply to symmetric spectra. In particular, if $(-)\otimes K$ is already a Quillen equivalence on $\mathscr{D}$, then $F_0: \mathscr{D} \to Sp^{\Sigma}(\mathscr{D},K)$ is a Quillen equivalence. In this case, with some additional mild hypotheses, the homotopy category of $\mathscr{D}$ is enriched, tensored, and cotensored over $\operatorname{Ho} Sp^{\Sigma}(\mathscr{C},K)$. We also show symmetric spectra are functorial in an appropriate sense. In particular, we show that the Quillen equivalence class of $Sp^{\Sigma}(\mathscr{D},K)$ is an invariant of the homotopy type of K.

Throughout this section, $\mathscr{C}$ will denote a left proper cellular symmetric monoidal model category, $\mathscr{D}$ will denote a left proper cellular $\mathscr{C}$-model category, and K will denote a cofibrant object of $\mathscr{C}$.

The proof of the following theorem is the same as the proof of Theorem 5.1.

Theorem 9.1. *Suppose $(-)\otimes K$ is a Quillen equivalence of $\mathscr{D}$. Then $F_0 : \mathscr{D} \to Sp^{\Sigma}(\mathscr{D},K)$ is a Quillen equivalence.*

Corollary 9.2. *Suppose that the domains of the generating cofibrations of both $\mathscr{C}$ and $\mathscr{D}$ are cofibrant, and suppose that $(-)\otimes K$ is a Quillen equivalence of $\mathscr{D}$. Then $\operatorname{Ho}\mathscr{D}$ is enriched, tensored, and cotensored over $\operatorname{Ho} Sp^{\Sigma}(\mathscr{C},K)$.*

Proof. Note that $\operatorname{Ho} Sp^{\Sigma}(\mathscr{D},K)$ is certainly enriched, tensored, and cotensored over $\operatorname{Ho} Sp^{\Sigma}(\mathscr{C},K)$. Now use the equivalence of categories $LF_0 : \operatorname{Ho}\mathscr{D} \to \operatorname{Ho} Sp^{\Sigma}(\mathscr{D},K)$ to transport this structure back to $\operatorname{Ho}\mathscr{D}$. □

Recall that the homotopy category of any model category is naturally enriched, tensored, and cotensored over Ho **SSet** [12, Chapter 5]. Corollary 9.2 is the first step to the assertion that the homotopy category of any stable (with respect to the suspension) model category is naturally enriched, tensored, and cotensored over the homotopy category of (simplicial) symmetric spectra. See [23] for further results along these lines.

Symmetric spectra are functorial in a natural way. The following theorem is analogous to Proposition 5.5. We use the notation $Sp^{\Sigma}(\Phi)$ for the map of symmetric spectra induced by a functor Φ because $Sp^{\Sigma}(\Phi)$ is always a prolongation of Φ, unlike the situation of Proposition 5.5.

Theorem 9.3. *Suppose $\mathscr{C}$ is a left proper cellular symmetric monoidal model category, and $\mathscr{D}$ and $\mathscr{D}'$ are left proper cellular $\mathscr{C}$-model categories. Suppose also that the domains of the generating cofibrations of $\mathscr{C}$, $\mathscr{D}$, and $\mathscr{D}'$ can be taken to be cofibrant. Then any $\mathscr{C}$-Quillen functor $\Phi : \mathscr{D} \to \mathscr{D}'$ extends naturally to a $Sp^{\Sigma}(\mathscr{C},K)$-Quillen functor*

$$Sp^{\Sigma}(\Phi) : Sp^{\Sigma}(\mathscr{D},K) \to Sp^{\Sigma}(\mathscr{D}',K).$$

Furthermore, if Φ is a Quillen equivalence, so is $Sp^{\Sigma}(\Phi)$.

Proof. The functor Φ induces a $\mathscr{C}^{\Sigma}$-functor $\mathscr{D}^{\Sigma} \to (\mathscr{D}')^{\Sigma}$, which takes the symmetric sequence (X_n) to the symmetric sequence (ΦX_n). It follows that Φ induces a $Sp^{\Sigma}(\mathscr{C},K)$-functor $Sp^{\Sigma}(\Phi): Sp^{\Sigma}(\mathscr{D},K) \to Sp^{\Sigma}(\mathscr{C}',K)$, which takes the symmetric spectrum (X_n) to the symmetric spectrum (ΦX_n), with structure maps

$$\Phi X_n \otimes K \cong \Phi(X_n \otimes K) \xrightarrow{\Phi\sigma} \Phi X_{n+1},$$

where σ denotes the structure map of X. Let Γ denote the right adjoint of Φ. Since Φ preserves the tensor action of $\mathscr{C}$, Γ preserves the cotensor action of $\mathscr{C}$. Then the right adjoint of $Sp^{\Sigma}(\Phi)$ is $Sp^{\Sigma}(\Gamma)$, which takes the symmetric spectrum (Y_n) to the symmetric spectrum (ΓY_n), with structure maps adjoint to the composite

$$\Gamma Y_n \xrightarrow{\Gamma\tilde{\sigma}} \Gamma Y_{n+1}^K \cong (\Gamma Y_{n+1})^K,$$

where $\tilde{\sigma}$ denotes the adjoint to the structure map of Y. Since $\mathrm{Ev}_n Sp^{\Sigma}(\Gamma) = \Gamma \mathrm{Ev}_n$, it follows that $Sp^{\Sigma}(\Phi)F_n = F_n\Phi$.

It is clear that $Sp^{\Sigma}(\Gamma)$ preserves level fibrations and level equivalences, so $Sp^{\Sigma}(\Phi)$ is a Quillen functor with respect to the projective model structures. In view of Theorem 2.2, to see that $Sp^{\Sigma}(\Phi)$ defines a Quillen functor with respect to the stable model structures, it suffices to show that $Sp^{\Sigma}(\Phi)(\zeta_n^{QC})$ is a stable equivalence for all domains and codomains of C of the generating cofibrations of $\mathscr{C}$. But one can readily verify that $Sp^{\Sigma}(\Phi)(\zeta_n^{QC}) = \zeta_n^{\Phi QC}$, which is a stable equivalence as required. Thus $Sp^{\Sigma}(\Phi)$ is a Quillen functor with respect to the stable model structures.

If Φ is a Quillen equivalence, one can check that $Sp^{\Sigma}(\Phi)$ is a Quillen equivalence with respect to the projective model structure. Indeed, in that case, Γ reflects weak equivalences between fibrant objects [12, Corollary 1.3.16]), so $Sp^{\Sigma}(\Gamma)$ reflects level equivalences between level fibrant objects. Let R' denote a fibrant replacement functor in the projective model structure and let R denote a fibrant replacement functor in $\mathscr{D}'$. Then, by using lifting as in the proof of Theorem 5.7, we find that the map $X \to Sp^{\Sigma}(\Gamma)R'Sp^{\Sigma}(\Phi)X$ is a level equivalence if and only if $X_n \to \Gamma R(Sp^{\Sigma}(\Phi)X)_n = \Gamma R\Phi X_n$ is a weak equivalence for all n. Note that the additional complexity of R' coming from the symmetric group actions is irrelevant to this reduction argument. The latter map is a weak equivalence since Φ is a Quillen equivalence.

To see that $Sp^{\Sigma}(\Phi)$ is still a Quillen equivalence with respect to the stable model structures, it suffices to show that $Sp^{\Sigma}(\Gamma)$ reflects stably fibrant objects, in view of Proposition 2.3. Suppose X is level fibrant and $Sp^{\Sigma}(\Gamma)(X)$ is an Ω-spectrum. Then $\Gamma X_n \xrightarrow{\Gamma\tilde{\sigma}} \Gamma(X_{n+1}^K)$ is a weak equivalence for all n. Since Γ reflects weak equivalences between fibrant objects by Hovey [12, Corollary 1.3.16], this means that X is an Ω-spectrum, as required. □

Symmetric spectra are also functorial, in a limited sense, in the cofibrant object K.

Theorem 9.4. *Suppose $\mathscr{C}$ is a left proper cellular symmetric monoidal model category, and $\mathscr{D}$ is a left proper cellular $\mathscr{C}$-model category. Suppose the domains of the generating cofibrations of $\mathscr{C}$ and $\mathscr{D}$ can be taken to be cofibrant. Finally, suppose $f: K \to K'$*

is a weak equivalence of cofibrant objects of $\mathscr{C}$. Then f induces a natural Quillen equivalence $(-) \otimes_{\mathrm{Sym}(K)} \mathrm{Sym}(K') : Sp^{\Sigma}(\mathscr{D},K) \to Sp^{\Sigma}(\mathscr{D},K')$.

Proof. The map f induces a map of commutative monoids $\mathrm{Sym}(K) \to \mathrm{Sym}(K')$. This induces the usual induction map

$$(-) \otimes_{\mathrm{Sym}(K)} \mathrm{Sym}(K') : Sp^{\Sigma}(\mathscr{D},K) \to Sp^{\Sigma}(\mathscr{D},K'),$$

and its right adjoint, the restriction map that takes a $\mathrm{Sym}(K')$-module X to X itself, thought of as a $\mathrm{Sym}(K)$-module via the map $\mathrm{Sym}(K) \to \mathrm{Sym}(K')$. Restriction obviously preserves level fibrations and level equivalences, so is a Quillen functor with respect to the projective model structure. Also, restriction preserves the evaluation functors Ev_n, so, by adjointness, $F_nA \otimes_{\mathrm{Sym}(K)} \mathrm{Sym}(K') = F_nA$. Here we must interpret F_nA as an object of $Sp^{\Sigma}(\mathscr{D},K)$ on the left side of this equation and as an object of $Sp^{\Sigma}(\mathscr{D},K')$ on the right side. It follows that, if C is a domain or codomain of a generating cofibration of $\mathscr{D}$, $\zeta_n^{QC} \otimes_{\mathrm{Sym}(K)} \mathrm{Sym}(K')$ is the map

$$F_{n+1}(QC \otimes K) \to F_nQC$$

in $Sp^{\Sigma}(\mathscr{C},K')$. The weak equivalence $QC \otimes K \to QC \otimes K'$ induces a level equivalence $F_{n+1}(QC \otimes K) \to F_{n+1}(QC \otimes K')$. Since the map $F_{n+1}(QC \otimes K') \to F_nQC$ is a stable equivalence, so is $\zeta_n^{QC} \otimes_{\mathrm{Sym}(K)} \mathrm{Sym}(K')$. Thus, by Theorem 2.2, induction is a Quillen functor with respect to the stable model structures.

We now prove that induction is a Quillen equivalence between the projective model structures. The proof of this is similar to the proof of Theorem 5.7. That is, restriction certainly reflects level equivalences between level fibrant objects. By a lifting argument as in the proof of Theorem 5.7, it therefore suffices to show that the map $X \to X \otimes_{\mathrm{Sym}(K)} \mathrm{Sym}(K')$ is a level equivalence for all cofibrant X. The transfinite induction argument of Theorem 5.7 will prove this without difficulty.

To show that induction is a Quillen equivalence between the stable model structures, we need only check that restriction reflects stably fibrant objects. This follows from the fact that the map $Z^{K'} \to Z^K$ is a weak equivalence for all fibrant Z. □

In particular, it does not matter, up to Quillen equivalence, what model of the simplicial circle one takes in forming the symmetric spectra of [13].

10. Comparison of spectra and symmetric spectra

In this section, $\mathscr{C}$ will be a left proper cellular symmetric monoidal model category, K will be a cofibrant object of $\mathscr{C}$, and $\mathscr{D}$ will be a left proper cellular $\mathscr{C}$-model category. Then we have two different stabilizations of $\mathscr{D}$, namely the stable model structures on $Sp^{\mathbb{N}}(\mathscr{D},K)$ and $Sp^{\Sigma}(\mathscr{D},K)$, where $Sp^{\mathbb{N}}(\mathscr{D},K)$ is the category of T-spectra $Sp^{\mathbb{N}}(\mathscr{D},T)$ when T is the functor $TX = X \otimes K$. The object of this section is to compare them. The quick summary of our result is that $Sp^{\mathbb{N}}(\mathscr{D},K)$ and $Sp^{\Sigma}(\mathscr{D},K)$ are related by a chain of Quillen equivalences whenever the cyclic permutation self-map of $K \otimes K \otimes K$ is homotopic to the identity. The precise statement requires a few more hypotheses; see the statements of Theorems 10.1 and 10.3.

This is not the ideal theorem; one might hope for a direct Quillen equivalence rather than a zigzag of Quillen equivalences, and one might hope for weaker hypotheses, or even no hypotheses. However, some hypotheses are necessary, as pointed out to the author by Jeff Smith. Indeed, the category $\operatorname{Ho} Sp^{\Sigma}(\mathscr{C},K)$ is symmetric monoidal, and therefore $\operatorname{Ho} Sp^{\Sigma}(\mathscr{C},K)(F_0S,F_0S)$, the set of self-maps of the unit object of $\operatorname{Ho} Sp^{\Sigma}(\mathscr{C},K)$, is a commutative monoid. If we have a chain of Quillen equivalences between the stable model structures on $Sp^{\mathbb{N}}(\mathscr{C},K)$ and $Sp^{\Sigma}(\mathscr{C},K)$ that preserves the functors F_0, then $\operatorname{Ho} Sp^{\mathbb{N}}(\mathscr{C},K)(F_0S,F_0S)$ would also have to be a commutative monoid. With sufficiently many hypotheses on $\mathscr{C}$ and K, for example if $\mathscr{C}$ is the category of simplicial sets and K is a finite simplicial set, we have seen in Section 4 that this mapping set is the colimit $\operatorname{colim} \operatorname{Ho} \mathscr{C}(K^{\otimes n},K^{\otimes n})$. There are certainly examples where this monoid is not commutative; for example K could be the mod p Moore space, and then homology calculations show this colimit is not commutative. In fact, this monoid will be commutative if and only if the cyclic permutation map of $K\otimes K\otimes K$ becomes the identity in $\operatorname{Ho}\mathscr{C}$ after tensoring with sufficiently many copies of K. Hence we need some hypothesis on the cyclic permutation map.

The central idea of this section is that commuting stabilization functors must be equivalent, an idea suggested to the author by Mike Hopkins in a different context. The following theorem is a consequence of this idea. To make sense of it, recall that there are two different ways to tensor with K on $Sp^{\mathbb{N}}(\mathscr{D},K)$; the functor $X \mapsto X \bar{\otimes} K$ that is a Quillen equivalence but does not involve the twist map, and the functor $X \mapsto X \otimes K$ that may not be a Quillen equivalence but does involve the twist map.

Theorem 10.1. *Suppose $\mathscr{C}$ is a left proper cellular symmetric monoidal model category, and that the domains of the generating cofibrations of $\mathscr{C}$ can be taken to be cofibrant. Suppose $\mathscr{D}$ is a left proper cellular $\mathscr{C}$-model category such that the functor $X \mapsto X\otimes K$ is a Quillen equivalence of the stable model structure on $Sp^{\mathbb{N}}(\mathscr{D},K)$. Then there is a $\mathscr{C}$-model category $\mathscr{E}$ together with $\mathscr{C}$-Quillen equivalences $Sp^{\Sigma}(\mathscr{D},K) \to \mathscr{E} \leftarrow Sp^{\mathbb{N}}(\mathscr{D},K)$, where $Sp^{\Sigma}(\mathscr{D},K)$ and $Sp^{\mathbb{N}}(\mathscr{D},K)$ are given the stable model structures. Furthermore, we have a natural isomorphism $[\operatorname{Ho} Sp^{\Sigma}(\mathscr{D},K)](F_0A,F_0B) \cong [\operatorname{Ho} Sp^{\mathbb{N}}(\mathscr{D},K)](F_0A,F_0B)$ for $A,B \in \mathscr{D}$.*

Proof. We take $\mathscr{E} = Sp^{\mathbb{N}}(Sp^{\Sigma}(\mathscr{D},K),K)$ with its stable model structure. This makes sense since $Sp^{\Sigma}(\mathscr{D},K)$ is a $\mathscr{C}$-model category, by the comments following Theorem 8.11. By Theorem 5.1, $F_0 : Sp^{\Sigma}(\mathscr{D},K) \to \mathscr{E}$ is a $\mathscr{C}$-Quillen equivalence. On the other hand, since $Sp^{\mathbb{N}}(\mathscr{D},K)$ is a $\mathscr{C}$-model category by Theorem 6.3, we can also consider $\mathscr{E}' = Sp^{\Sigma}(Sp^{\mathbb{N}}(\mathscr{D},K),K)$ with its stable model structure. The action of K on $Sp^{\mathbb{N}}(\mathscr{D},K)$ is then $X \mapsto X\otimes K$, as pointed out in Remark 6.4. By hypothesis, this functor is already a Quillen equivalence, so Theorem 9.1 implies that $F_0 : Sp^{\mathbb{N}}(\mathscr{D},K) \to \mathscr{E}'$ is a $\mathscr{C}$-Quillen equivalence.

We claim that $\mathscr{E}$ is isomorphic to $\mathscr{E}'$ as a model category. This is the precise sense in which our two stabilization functors commute with each other, and obviously will complete the proof. An object of $\mathscr{E}$ is a set $\{Y_{m,n}\}$ of objects of $\mathscr{D}$, where $m,n \geq 0$, together with certain maps. There is an action of Σ_n on $Y_{m,n}$, and there are Σ_n-equivariant maps $Y_{m,n} \otimes K \xrightarrow{v} Y_{m+1,n}$ and $Y_{m,n} \otimes K \xrightarrow{\rho} Y_{m,n+1}$. In addition, the composite $Y_{m,n} \otimes K^{\otimes p} \to Y_{m,n+p}$ is $\Sigma_n \times \Sigma_p$-equivariant, and there is a compatibility between v and ρ, expressed in the commutativity of the following diagram:

$$\begin{array}{ccccc}
Y_{m,n} \otimes K \otimes K & \xrightarrow{1\otimes T} & Y_{m,n} \otimes K \otimes K & \xrightarrow{\rho\otimes 1} & Y_{m,n+1} \otimes K \\
\downarrow{\scriptstyle v\otimes 1} & & & & \downarrow{\scriptstyle v} \\
Y_{m+1,n} \otimes K & = & Y_{m+1,n} \otimes K & \xrightarrow[\rho]{} & Y_{m+1,n+1}.
\end{array}$$

An object of the category $\mathscr{E}'$ is a set $\{Y'_{m,n}\}$ of objects of $\mathscr{D}$ for $m,n \geq 0$ together with certain maps. In this case, we have an action of Σ_m on $Y'_{m,n}$ and Σ_m-equivariant maps $\rho' : Y'_{m,n} \otimes K \to Y'_{m+1,n}$ and $v' : Y'_{m,n} \otimes K \to Y'_{m,n+1}$. The composite $Y'_{m,n} \otimes K^{\otimes p} \to Y'_{m+p,n}$ is $\Sigma_m \times \Sigma_p$-equivariant, and there is a similar compatibility relationship between ρ' and v'. There is therefore an isomorphism of categories from $\mathscr{E}$ to $\mathscr{E}'$ that takes Y to Y', where $Y'_{m,n} = Y_{n,m}$, $\rho' = v$, and $v' = \rho$.

There is a projective model structure on both $\mathscr{E}$ and $\mathscr{E}'$, where a map f is a weak equivalence (or fibration) if and only if $f_{m,n}$ is a weak equivalence (or fibration) in $\mathscr{D}$ for all m,n. The isomorphism between $\mathscr{E}$ and $\mathscr{E}'$ obviously preserves this projective model structure. The stable model structure on both $\mathscr{E}$ and $\mathscr{E}'$ is the Bousfield localization of the projective model structure with respect to the maps $F_{m,n+1}(QC \otimes K) \to F_{m,n}QC$ and $F_{m+1,n}(QC \otimes K) \to F_{m,n}QC$, where $F_{m,n}$ is left adjoint to the evaluation functor $\mathrm{Ev}_{m,n}$, and C runs through the domains and codomains of the generating cofibrations of $\mathscr{D}$. The isomorphism between $\mathscr{E}$ and $\mathscr{E}'$ preserves this set of maps, so must preserve the entire stable model structure.

The composites

$$\mathscr{D} \xrightarrow{F_0} Sp^{\Sigma}(\mathscr{D},K) \xrightarrow{F_0} \mathscr{E}$$

and

$$\mathscr{D} \xrightarrow{F_0} Sp^{\mathbb{N}}(\mathscr{D},K) \xrightarrow{F_0} \mathscr{E}' \cong \mathscr{E}$$

are both naturally isomorphic to $F_{0,0}$, since both are left adjoint to $\mathrm{Ev}_{0,0}$. Therefore, we have natural isomorphisms

$$\begin{aligned}
[\mathrm{Ho}\, Sp^{\Sigma}(\mathscr{D},K)](F_0A, F_0B) &\cong [\mathrm{Ho}\,\mathscr{E}](F_{0,0}A, F_{0,0}B) \\
&\cong [\mathrm{Ho}\,\mathscr{E}'](F_{0,0}A, F_{0,0}B) \\
&\cong [\mathrm{Ho}\, Sp^{\mathbb{N}}(\mathscr{D},K)](F_0A, F_0B)
\end{aligned}$$

which complete the proof. □

In particular, we have calculated $[\operatorname{Ho} Sp^{\mathbb{N}}(\mathscr{D},K)](F_0A,F_0B)$ in Corollary 4.13; when both the hypotheses of that corollary and the hypotheses of Theorem 10.1 hold, we get the expected result

$$[\operatorname{Ho} Sp^{\Sigma}(\mathscr{D},K)](F_0A,F_0B) = \operatorname{colim} \operatorname{Ho} \mathscr{D}(A\otimes K^{\otimes n}, B\otimes K^{\otimes n})$$

for cofibrant A and B.

Theorem 10.1 indicates that we should try to prove that $(-)\otimes K$ is a Quillen equivalence of $Sp^{\mathbb{N}}(\mathscr{D},K)$. The only way the author can see to do this is by comparing $(-)\otimes K$ to $(-)\bar{\otimes}K$, which we know is a Quillen equivalence. The basic idea is to compare $X\otimes K\otimes K$ to $X\bar{\otimes}K\bar{\otimes}K$. Both of these spectra have the same spaces, and their structure maps differ precisely by the cyclic permutation self-map of $K\otimes K\otimes K$. So if we knew that this map were the identity, they would be the same spectrum. The hope is then that, if we knew that the cyclic permutation map were only homotopic to the identity, these two spectra would still be equivalent in $\operatorname{Ho} Sp^{\mathbb{N}}(\mathscr{D},K)$. One can, in fact, construct a map of spectra $X\bar{\otimes}K\bar{\otimes}K \to R'(X\otimes K\otimes K)$, where R' is a level fibrant replacement functor and X is cofibrant, by inductively modifying the identity map. Unfortunately, the author does not know how to do this modification in a natural way, so is unable to prove that the derived functors $L(X\otimes K\otimes K)$ and $L(X\bar{\otimes}K\bar{\otimes}K)$ are equivalent using this method.

Instead, we will follow a suggestion of Dan Dugger. We will construct a new functor F on cofibrant objects X of $Sp^{\mathbb{N}}(\mathscr{D},K)$ and natural level equivalences $FX\to X\bar{\otimes}K\bar{\otimes}K$ and $FX\to X\otimes K\otimes K$. It will follow immediately that $L(X\bar{\otimes}K\bar{\otimes}K)$ is naturally equivalent to $L(X\otimes K\otimes K)$, and therefore that $X\mapsto X\otimes K$ is a Quillen equivalence on $Sp^{\mathbb{N}}(\mathscr{D},K)$. Unfortunately, to construct F, we will need to make some unpleasant assumptions that ought to be unnecessary.

Definition 10.2. Given a symmetric monoidal model category $\mathscr{C}$ whose unit S is cofibrant, a *unit interval* in $\mathscr{C}$ is a cylinder object I for S such that there exists a map $H_I: I\otimes I\to I$ satisfying $H_I\circ(1\otimes i_0)=H_I\circ(i_0\otimes 1)=i_0\pi$ and $H_I\circ(1\otimes i_1)$ is the identity. Here $i_0,i_1:S\to I$ and $\pi:I\to S$ are the structure maps of I. Given a cofibrant object K of $\mathscr{C}$, we say that K is *symmetric* if there is a unit interval I and a homotopy

$$H: K\otimes K\otimes K\otimes I\to K\otimes K\otimes K$$

from the cyclic permutation to the identity.

Note that $[0,1]$ is a unit interval in the usual model structure on compactly generated topological spaces, and $\Delta[1]$ is a unit interval in the category of simplicial sets. Indeed, the required map $H_1:\Delta[1]\times\Delta[1]$ takes both of the nondegenerate 2-simplices 011×001 and 001×011 to 001. Similarly, the standard unit interval chain complex of abelian groups is a unit interval in the projective model structure on chain complexes. Also, any symmetric monoidal left Quillen functor preserves unit intervals. It follows, for example, that the unstable A^1-model category of Morel-Voevodksy has a unit interval.

Our goal, then, is to prove the following theorem.

Theorem 10.3. *Let $\mathscr{C}$ be a symmetric monoidal model category with cofibrant unit S, and let $\mathscr{D}$ be a left proper cellular $\mathscr{C}$-model category. Suppose that K is a cofibrant object of $\mathscr{C}$, and that either K is itself symmetric or the domains of the generating cofibrations of $\mathscr{D}$ are cofibrant and K is weakly equivalent to a symmetric object of $\mathscr{C}$. Then the functor $X \mapsto X \otimes K$ is a Quillen equivalence of $Sp^{\mathbb{N}}(\mathscr{D},K)$.*

This theorem is not the best one ought to be able to do. For example, by considering the analogous functors with more than three tensor factors of K, it should be possible to show that the same theorem holds if there is a left homotopy between some even permutation of $K^{\otimes n}$ and the identity. Also, it seems clear that one should only need the cyclic permutation, or more generally some even permutation, to be equal to the identity in $\operatorname{Ho}\mathscr{C}$. That is, we should not need a specific left homotopy between an even permutation and the identity. But the author does not know how to remove this hypothesis.

In any case, we have the following corollary.

Corollary 10.4. *Let $\mathscr{C}$ be a left proper cellular symmetric monoidal model category whose unit S is cofibrant, and whose generating cofibrations can be taken to have cofibrant domains. Let $\mathscr{D}$ be a left proper $\mathscr{C}$-model category. Suppose K is a cofibrant object of $\mathscr{C}$, and either that K is itself symmetric, or else that the domains of the generating cofibrations of $\mathscr{D}$ are cofibrant and K is weakly equivalent to a symmetric object of $\mathscr{C}$. Then there is a $\mathscr{C}$-model category $\mathscr{E}$ and $\mathscr{C}$-Quillen equivalences*

$$Sp^{\Sigma}(\mathscr{D},K) \to \mathscr{E} \leftarrow Sp^{\mathbb{N}}(\mathscr{D},K).$$

We will prove Theorem 10.3 in a series of lemmas.

Lemma 10.5. *Let $\mathscr{C}$ be a symmetric monoidal model category whose unit S is cofibrant. Suppose we have the (not necessarily commutative) square below*

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} \\ B & \xrightarrow[g]{} & Y \end{array}$$

in a $\mathscr{C}$-model category $\mathscr{D}$, where A and B are cofibrant. Let $H: A \otimes I \to Y$ be a left homotopy from gr to sf, for some unit interval I. Then there is an object B' of $\mathscr{D}$, a weak equivalence $B' \xrightarrow{q} B$, a commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle r'}\downarrow & & \downarrow{\scriptstyle s} \\ B' & \xrightarrow[g']{} & Y \end{array}$$

such that $qr' = r$, *and a left homotopy* $H' : B' \otimes I \to Y$ *between* gq *and* g'. *Furthermore, this construction is natural in an appropriate sense.*

Naturality means that, if we have a map of such homotopy commutative squares that preserves the homotopies, then we get a map of the resulting commutative squares that preserves the maps q and H'. The precise statement is complicated, and we leave it to the reader.

Proof. We let B' be the mapping cylinder of r. That is, we take B' to be the pushout in the diagram below:

$$\begin{array}{ccc} A & \xrightarrow{i_0} & A \otimes I \\ {\scriptstyle r}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ B & \xrightarrow{j} & B'. \end{array}$$

The map r' is then the composite hi_1, and the map g' is the map that is g on B and H on $A \otimes I$. It follows that $g'r' = Hi_1 = sf$, as required. The map $q : B' \to B$ is defined to be the identity on B and the composite $A \otimes I \xrightarrow{\pi} A \xrightarrow{r} B$ on $A \otimes I$. Since j is a trivial cofibration (as a pushout of i_0), it follows that q is a weak equivalence, and it is clear that $qr' = r$. We must now construct the homotopy H'. First note that B' is cofibrant, since B is so and j is a trivial cofibration, and so $B' \otimes I$ is a cylinder object for B'. In fact, $B' \otimes I$ is the pushout of $A \otimes I \otimes I$ and $B \otimes I$ over $A \otimes I$. Define H' to be the constant homotopy $B \otimes I \xrightarrow{\pi} B \xrightarrow{g} Y$ on $B \otimes I$ and the homotopy

$$A \otimes I \otimes I \xrightarrow{1 \otimes H_I} A \otimes I \xrightarrow{H} Y$$

on $A \otimes I \otimes I$. The fact that $H_I \circ (i_0 \otimes 1) = i_0 \pi$ guarantees that H' is well defined, and the other conditions on H_I guarantee that H' is a left homotopy from gq to g'. We leave the naturality of this construction to the reader. □

We also need the following lemma about the behavior of unit intervals.

Lemma 10.6. *Suppose* $\mathscr{C}$ *is a symmetric monoidal model category whose unit* S *is cofibrant. Let* I *and* I' *be unit intervals with structure maps* $i_0, i_1 : S \to I$ *and* $i'_0, i'_1 : S \to I'$, *and define* J *by the pushout diagram below*:

$$\begin{array}{ccc} S & \xrightarrow{i'_o} & I' \\ {\scriptstyle i_1}\big\downarrow & & \big\downarrow{\scriptstyle \beta} \\ I & \xrightarrow{\alpha} & J. \end{array}$$

Then J *is a unit interval.*

Proof. The reader is well advised to draw a picture in the topological or simplicial case, from which the proof should be clear. We think of J as the interval whose left half is I and whose right half is I'. The fact that J is a cylinder object for S, with structure maps $j_0 = \alpha i_0$ and $j_1 = \beta i_1'$, is proved in [11, Lemma 7.3.11]. Because the tensor product preserves pushouts, we can think of $J \otimes J$ as a square consisting of a copy of $I \otimes I$ in the lower left, a copy of $I \otimes I'$ in the upper left, a copy of $I' \otimes I$ is the lower right, and a copy of $I' \otimes I'$ in the upper right. We define the necessary map $G : J \otimes J \to J$ by defining G on each subsquare. On the lower left, we use the composite $I \otimes I \xrightarrow{H} I \xrightarrow{\alpha} J$, where H is the homotopy making I into a unit interval. Similarly, on the upper right square, we use the composite $\beta H'$. On the upper left square we use the constant homotopy $\alpha(1 \otimes \pi_{I'})$. On the lower right square we use the composite

$$I' \otimes I \xrightarrow{\pi_{I'} \otimes 1} I \xrightarrow{i_1 \otimes 1} I \otimes I \xrightarrow{H} I \xrightarrow{\alpha} J.$$

We leave it to the reader to check that this makes J into a unit interval. □

The importance of these two lemmas for spectra is indicated in the following consequence.

Lemma 10.7. *Suppose $\mathscr{C}$ is a left proper cellular symmetric monoidal model category with cofibrant unit S, unit interval I, and cofibrant object K. Let $\mathscr{D}$ be a left proper cellular $\mathscr{C}$-model category. Suppose $A, B \in Sp^{\mathbb{N}}(\mathscr{D}, K)$, where A is cofibrant, and we have maps $f_n : A_n \to B_n$ for all n and homotopies $H_n : A_n \otimes K \otimes I \to B_{n+1}$ from $f_{n+1}\sigma_A$ to $\sigma_B(f_n \otimes 1)$, where $\sigma_{(-)}$ is the structure map of the spectrum $(-)$. Then there is a spectrum C, a level equivalence $C \xrightarrow{h} A$, and a map of spectra $C \xrightarrow{g} B$ such that g_n is homotopic to $f_n h_n$. Furthermore, this construction is natural in an appropriate sense.*

Once again, the naturality involves the homotopies H_n as well as the maps f_n. We leave the precise statement to the reader.

Proof. We define C_n, h_n, g_n and a homotopy $H_n' : C_n \otimes I_n \to B_n$ from $f_n h_n$ to g_n, where I_n is a unit interval, inductively on n, using Lemma 10.5. To get started, we take $C_0 = A_0$, h_0 to be the identity, g_0 to be f_0, and H_0' to be the constant homotopy (with $I_0 = I$). For the inductive step, we apply Lemma 10.5 to the diagram

$$\begin{array}{ccc} C_n \otimes K & \xrightarrow{g_n \otimes 1} & B_n \otimes K \\ {\scriptstyle \sigma_A(h_n \otimes 1)} \downarrow & & \downarrow {\scriptstyle \sigma_B} \\ A_{n+1} & \xrightarrow[f_{n+1}]{} & B_{n+1} \end{array}$$

and the homotopy obtained as follows. We have a homotopy

$$C_n \otimes K \otimes I_n \xrightarrow{1 \otimes T} C_n \otimes I_n \otimes K \xrightarrow{H_n' \otimes 1} B_n \otimes K \xrightarrow{\sigma_B} B_{n+1}$$

from $\sigma_B(f_n \otimes 1)(h_n \otimes 1)$ to $\sigma_B(g_n \otimes 1)$. On the other hand, we also have the homotopy $H_n(h_n \otimes 1)$ from $f_{n+1}\sigma_A(h_n \otimes 1)$ to $\sigma_B(f_n \otimes 1)(h_n \otimes 1)$. We can amalgamate these to get a homotopy $G_n : C_n \otimes K \otimes I_{n+1} \to B_{n+1}$ from $f_{n+1}\sigma_A(h_n \otimes 1)$ to $\sigma_B(g_n \otimes 1)$, and I_{n+1} is still a unit interval by Lemma 10.6. Hence Lemma 10.5 gives us an object C_{n+1}, a map $\sigma_C : C_n \otimes K \to C_{n+1}$, and a map $g_{n+1} : C_{n+1} \to B_{n+1}$ such that $g_{n+1}\sigma_C = \sigma_B(g_n \otimes 1)$. It also gives us a map $h_{n+1} : C_{n+1} \to A_{n+1}$ such that $h_{n+1}\sigma_C = \sigma_A(h_n \otimes 1)$ and a homotopy $H'_{n+1} : C_{n+1} \otimes I_{n+1} \to B_{n+1}$ from $f_{n+1}h_{n+1}$ to g_{n+1}. This completes the induction step and the proof (we leave naturality to the reader). □

With this lemma in hand we can now give the proof of Theorem 10.3.

Proof of Theorem 10.3. We first reduce to the case where K is itself symmetric. So suppose the generating cofibrations of $\mathscr{D}$ have cofibrant domains, and suppose K' is symmetric and weakly equivalent to K; this means there are weak equivalences $K \to RK \to RK' \leftarrow K'$, where R denotes a fibrant replacement functor. This implies that the total left derived functors $X \mapsto X \otimes^L K$ and $X \mapsto X \otimes^L K'$ are naturally isomorphic on the homotopy category of any $\mathscr{C}$-model category. In particular, it suffices to show that $X \mapsto X \otimes K'$ is a Quillen equivalence on $Sp^{\mathbb{N}}(\mathscr{D}, K)$. On the other hand, by Theorem 5.7, there are $\mathscr{C}$-Quillen equivalences

$$Sp^{\mathbb{N}}(\mathscr{D}, K) \to Sp^{\mathbb{N}}(\mathscr{D}, RK) \to Sp^{\mathbb{N}}(\mathscr{D}, RK') \leftarrow Sp^{\mathbb{N}}(\mathscr{D}, K').$$

It therefore suffices to show that $X \mapsto X \otimes K'$ is a Quillen equivalence of $Sp^{\mathbb{N}}(\mathscr{D}, K')$; that is, we can assume that K itself is symmetric.

Let H denote the given homotopy from the cyclic permutation to the identity of $K \otimes K \otimes K$. Let X be a cofibrant spectrum, let $\tilde{\sigma}$ denote the structure map of $X \bar{\otimes} K \bar{\otimes} K$, and let σ denote the structure map of $X \otimes K \otimes K$. These two structure maps differ by the cyclic permutation, and therefore we are in the situation of Lemma 10.7, with $A = X \bar{\otimes} K \bar{\otimes} K$, $B = X \otimes K \otimes K$, f_n equal to the identity map, and $H_n = (\sigma_X \otimes 1 \otimes 1)(1 \otimes H)$. It follows that we get a functor F defined on cofibrant objects of $Sp^{\mathbb{N}}(\mathscr{D}, K)$ and natural level equivalences $FX \xrightarrow{h} X \bar{\otimes} K \bar{\otimes} K$ and $FX \xrightarrow{g} X \otimes K \otimes K$, where the latter map is a level equivalence since g_n is homotopic to h_n. Thus the total left derived functors of $(-) \bar{\otimes} K \bar{\otimes} K$ and $(-) \otimes K \otimes K$ are naturally isomorphic. Since we know already that $(-) \bar{\otimes} K \bar{\otimes} K$ is a Quillen equivalence, so is $(-) \otimes K \otimes K$, and hence so is $(-) \otimes K$. □

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Appendix A. Cellular model categories

In this section we define cellular model categories and show that the projective model structures on $Sp^{\mathbb{N}}(\mathscr{D},T)$ and $Sp^{\Sigma}(\mathscr{D},K)$ are cellular when the model structure on $\mathscr{D}$ is cellular. This is necessary to ensure that the Bousfield localizations used in the paper do in fact exist. The definitions in this section are taken from [11]. Throughout this section, then, T will be a left Quillen endofunctor of a model category $\mathscr{D}$; when we refer to $Sp^{\Sigma}(\mathscr{D},K)$, we will be thinking of $\mathscr{D}$ as a $\mathscr{C}$-model category, where $\mathscr{C}$ is some symmetric monoidal model category, and of K as a cofibrant object of $\mathscr{C}$.

A cellular model category is a special kind of cofibrantly generated model category. Three additional hypotheses are needed.

Definition A.1. A model category $\mathscr{E}$ is *cellular* if there is a set of cofibrations I and a set of trivial cofibrations J making $\mathscr{E}$ into a cofibrantly generated model category and also satisfying the following conditions:

1. The domains and codomains of I are compact relative to I.
2. The domains of J are small relative to the cofibrations.
3. Cofibrations are effective monomorphisms.

The first hypothesis above requires considerable explanation, which we will provide below. We first point out that the second hypothesis will hold in the projective model structure on $Sp^{\mathbb{N}}(\mathscr{D},T)$ or $Sp^{\Sigma}(\mathscr{D},K)$ when it holds in $\mathscr{D}$.

Lemma A.2. *Suppose $\mathscr{D}$ is a cofibrantly generated model category with generating cofibrations I and generating trivial cofibrations J, and T is a left Quillen endofunctor of $\mathscr{D}$. Suppose the domains of J are small relative to the cofibrations in $\mathscr{D}$. Then the domains of the generating trivial cofibrations J_T of the projective model structure on $Sp^{\mathbb{N}}(\mathscr{D},T)$ are small relative to the cofibrations in $Sp^{\mathbb{N}}(\mathscr{D},T)$. Similarly, if $\mathscr{D}$ is a $\mathscr{C}$-model category, K is a cofibrant object of $\mathscr{C}$, and $T=(-)\otimes K$, then the domains of the generating trivial cofibrations J_K of the projective model structure on $Sp^{\Sigma}(\mathscr{D},K)$ are small relative to the cofibrations in $Sp^{\Sigma}(\mathscr{D},K)$.*

Proof. For $Sp^{\mathbb{N}}(\mathscr{D},T)$, this follows immediately from the definition of J_T, Propositions 1.9 and 1.11. The proof for $Sp^{\Sigma}(\mathscr{D},K)$ is similar. □

We now discuss the third hypothesis.

Definition A.3. Suppose $\mathscr{E}$ is a category. A map $f:X \to Y$ in $\mathscr{E}$ is an *effective monomorphism* if f is the equalizer of the two obvious maps $Y \rightrightarrows Y \amalg_X Y$.

Proposition A.4. *Suppose $\mathscr{D}$ is a cofibrantly generated model category and T is a left Quillen endofunctor of $\mathscr{D}$. Suppose that cofibrations are effective monomorphisms in $\mathscr{D}$. Then level cofibrations, and in particular projective cofibrations, are effective monomorphisms in $Sp^{\mathbb{N}}(\mathscr{D},T)$. Similarly, when $\mathscr{D}$ is a $\mathscr{C}$-model category and $T=(-)\otimes K$ for some cofibrant $K\in\mathscr{C}$, level cofibrations are effective monomorphisms in $Sp^{\Sigma}(\mathscr{D},K)$.*

Proof. This is immediate, since colimits and limits in $Sp^{\mathbb{N}}(\mathscr{D},T)$ and $Sp^{\Sigma}(\mathscr{D},K)$ are taken levelwise. □

We must now define compactness. This will involve some preliminary definitions. These definitions are based on the idea of CW-complexes, but are of necessity somewhat technical in the general case.

Definition A.5. Suppose I is a set of maps in a cocomplete category $\mathscr{E}$. A *relative I-cell complex* is a map which can be written as the transfinite composition of pushouts of coproducts of maps of I. That is, given a relative I-cell complex f, there is an ordinal λ and a colimit-preserving functor $X:\lambda\to\mathscr{E}$ and a collection $\{(T^{\beta},e^{\beta},h^{\beta})_{\beta<\lambda}\}$ satisfying the following properties:

1. f is isomorphic to the transfinite composition of X.
2. Each T^{β} is a set.
3. Each e^{β} is a function $e^{\beta}:T^{\beta}\to I$.
4. Given $\beta<\lambda$ and $i\in T^{\beta}$, if $e_i^{\beta}:C_i\to D_i$ is the image of i under e^{β}, then h_i^{β} is a map $h_i^{\beta}:C_i\to X_{\beta}$.
5. Each $X_{\beta+1}$ is the pushout in the diagram

$$\begin{array}{ccc} \amalg_{T^{\beta}} C_i & \xrightarrow{\amalg e_i^{\beta}} & \amalg_{T^{\beta}} D_i \\ {\scriptstyle \amalg h_i^{\beta}}\downarrow & & \downarrow \\ X_{\beta} & \longrightarrow & X_{\beta+1}. \end{array}$$

The ordinal λ together with the colimit-preserving functor X and the collection $\{(T^{\beta},e^{\beta},h^{\beta})_{\beta<\lambda}\}$ is called a *presentation* of f. The set $\amalg_{\beta}T_{\beta}$ is the *set of cells* of f, and given a cell e, its *presentation ordinal* is the ordinal β such that $e\in T^{\beta}$. The *presentation ordinal* of f is λ.

We also need to define subcomplexes of relative I-cell complexes.

Definition A.6. Suppose $\mathscr{E}$ is a cocomplete category and I is a set of maps in $\mathscr{E}$. Given a presentation λ, $X:\lambda\to\mathscr{C}$, and $\{(T^{\beta},e^{\beta},h^{\beta})_{\beta<\lambda}\}$ of a map f as a relative

I-cell complex, a *subcomplex* of f (or really of the presentation of f), is a collection $\{(\tilde{T}^{\beta}, \tilde{e}^{\beta}, \tilde{h}^{\beta})_{\beta<\lambda}\}$ such that the following properties hold:

1. Every $\tilde{T}^{\beta}$ is a subset of T^{β}, and $\tilde{e}^{\beta}$ is the restriction of e^{β} to $\tilde{T}^{\beta}$.
2. There is a colimit-preserving functor $\tilde{X}: \lambda \to \mathscr{E}$ such that $\tilde{X}_0 = X_0$ and a natural transformation $\tilde{X} \to X$ such that, for every $\beta < \lambda$ and $i \in \tilde{T}^{\beta}$, the map $\tilde{h}_i^{\beta}: C_i \to \tilde{X}_{\beta}$ is a factorization of $h_i^{\beta}: C_i \to X_{\beta}$ through the map $\tilde{X}_{\beta} \to X_{\beta}$.
3. For all $\beta < \lambda$, $\widetilde{X_{\beta+1}}$ is the pushout in the diagram

$$\begin{array}{ccc} \amalg_{\tilde{T}^{\beta}} C_i & \xrightarrow{\amalg \tilde{e}_i^{\beta}} & \amalg_{\tilde{T}^{\beta}} D_i \\ {\scriptstyle \amalg \tilde{h}_i^{\beta}} \downarrow & & \downarrow \\ \tilde{X}_{\beta} & \longrightarrow & \tilde{X}_{\beta+1}. \end{array}$$

Given a subcomplex of f, the *size* of that subcomplex is the cardinality of its set of cells $\amalg_{\beta<\lambda} \tilde{T}^{\beta}$. Usually, I will be a set of cofibrations in a model category where the cofibrations are (essential) monomorphisms. This condition guarantees that a subcomplex of a presentation is uniquely determined by its collection of cells $\{\tilde{T}^{\beta}\}_{\beta<\lambda}$, since $\tilde{e}^{\beta}$ is always determined by e^{β} and there is a unique choice for the factorization $\tilde{h}^{\beta}$ [11, Proposition 12.5.10]. Of course, not every set of cells gives rise to a subcomplex.

We can now define compactness.

Definition A.7. Suppose $\mathscr{E}$ is a cocomplete category and I is a set of maps in $\mathscr{E}$.

1. Given a cardinal κ, an object X is *κ-compact relative to I* if, for every relative I-cell complex $f: Y \to Z$ and for every presentation of f, every map $X \to Z$ factors through a subcomplex of that presentation with size at most κ.
2. An object X is *compact relative to I* if X is κ-compact relative to I for some cardinal κ.

The following proposition is adapted from an argument of Phil Hirschhorn's.

Proposition A.8. *Suppose $\mathscr{D}$ is a cellular model category with generating cofibrations I. Let A be a domain or codomain of I. If T is a left Quillen endofunctor of $\mathscr{D}$, then F_nA is compact relative to I_T in $Sp^{\mathbb{N}}(\mathscr{D}, T)$. Similarly, if $\mathscr{D}$ is a $\mathscr{C}$-model category and K is a cofibrant object of $\mathscr{C}$, then F_nA is compact relative to I_K in $Sp^{\Sigma}(\mathscr{D}, K)$.*

Proof. We will prove the proposition only for $Sp^{\mathbb{N}}(\mathscr{D}, T)$, as the $Sp^{\Sigma}(\mathscr{D}, K)$ case is similar. Throughout this proof we will use Proposition A.4, which guarantees that subcomplexes in $Sp^{\mathbb{N}}(\mathscr{D}, T)$ are determined by their cells. Choose an infinite cardinal

γ such that the domains and codomains of I are all γ-compact relative to I. When dealing with relative I-cell complexes, we can assume that we have a presentation as a transfinite composition of pushouts of maps of I, rather than as a transfinite composition of pushouts of coproducts of maps of I, using [12, Lemma 2.1.13] or [11, Section 12.2]. A similar comment holds for relative I_T-cell complexes. We will proceed by transfinite induction on β, where the induction hypothesis is that for every presented relative I_T-cell complex $f : X \to Y$ whose presentation ordinal is $\leq \beta$, and for every map $F_n A \xrightarrow{f} Y$ where n is an integer and A is a domain or codomain of I, f factors through a subcomplex with at most γ I_T-cells. Getting the induction started is easy. For the induction step, suppose the induction hypothesis holds for all ordinals $\alpha < \beta$, and suppose we have a presentation

$$X = X^0 \to X^1 \to \cdots \to X^\alpha \to \cdots \to X^\beta = Y$$

of $f : X \to Y$ as a transfinite composition of pushouts of maps of I_T. Then the boundary of each I_T-cell of this presentation is represented by a map $F_m C \to X^\alpha$, for some $\alpha < \beta$, some $m \geq 0$, and some domain C of a map of I. This map factors through a subcomplex with at most γ I_T-cells, by induction. It follows that the I_T-cell itself is contained in a subcomplex of at most γ I_T-cells, since we can just attach the interior of the I_T-cell to the given subcomplex.

Now suppose we have an arbitrary map $F_n A \to Y$, where A is a domain or codomain of a map of I. Such a map is determined by a map $A \to Y_n$ in $\mathscr{D}$. The map $f_n : X_n \to Y_n$ is the transfinite composition of the cofibrations $X_n^\alpha \to X_n^{\alpha+1}$. For each α, there is an m and a map h of I such that $X_n^\alpha \to X_n^{\alpha+1}$ is the pushout of $G^m h$, where we interpret $G^m h$ as the identity map if m is negative. The difficulty is that the cofibration $X_n^\alpha \to X_n^{\alpha+1}$ may not itself be a relative I-cell complex, though it must be a retract of one by Hovey [12, Proposition 2.1.18]. A stronger version of that proposition, [11, Lemma 12.4.21], allows us to write the colimit-preserving functor $\beta : \mathscr{D}$ that takes α to X_n^α as a retract of a colimit-preserving functor

$$X_n = Z^0 \to Z^1 \to \cdots \to Z^\alpha \to \cdots Z^\beta = Z,$$

where each map $Z^\alpha \to Z^{\alpha+1}$ is a relative I-cell complex. We denote the retraction by $r : Z \to Y_n$, noting that the restriction of r to Z^α factors (uniquely) through X_n^α. We can think of the entire map $X_n \to Z$ as a relative I-cell complex, each cell e of which appears in the relative I-cell complex $Z^{t(e)} \to Z^{t(e)+1}$ for some unique ordinal $t(e)$, and so has associated to it the I_G-cell $c(e)$ of f used to form $X^{t(e)} \to X^{t(e)+1}$. The composite $A \to Y_n \to Z$ then factors through a subcomplex V with at most γ I-cells. The proof will be completed if we can find a subcomplex W of Y with at most γ I_T-cells such that the restriction of r to V factors through W_n.

Take W to be a subcomplex of Y containing the I_T-cells $c(e)$ as e runs through the cells of V. Then W_n contains $r(e)$ for every cell of V, so W_n contains rV, as required. Furthermore, since each I_T-cell $c(e)$ lies in a subcomplex with no more that γ I_T-cells, and V has no more than γ cells, there is a choice for W which has no more than $\gamma^2=\gamma$ cells. This completes the induction step and the proof. □

Altogether then, we have the following theorem.

Theorem A.9. *Suppose $\mathscr{D}$ is a left proper cellular model category, and T is a left Quillen endofunctor on $\mathscr{D}$. Then the category $Sp^{\mathbb{N}}(\mathscr{D},T)$ of T-spectra, with the projective model structure, is a left proper cellular model category. Similarly, if $\mathscr{D}$ is a left proper cellular $\mathscr{C}$-model category, and K is a cofibrant object of $\mathscr{C}$, then the category $Sp^{\Sigma}(\mathscr{D},K)$ of symmetric spectra, with the projective model structure, is a left proper cellular model category.*

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