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P.J. Hilton U. Stammbach

A Course in Homological Algebra

Second Edition

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To Margaret and Irene

Preface to the Second Edition

We have inserted, in this edition, an extra chapter (Chapter X) entitled "Some Applications and Recent Developments." The first section of this chapter describes how homological algebra arose by abstraction from algebraic topology and how it has contributed to the knowledge of topology. The other four sections describe applications of the methods and results of homological algebra to other parts of algebra. Most of the material presented in these four sections was not available when this text was first published. Naturally, the treatments in these five sections are somewhat cursory, the intention being to give the flavor of the homological methods rather than the details of the arguments and results.

We would like to express our appreciation of help received in writing Chapter X; in particular, to Ross Geoghegan and Peter Kropholler (Section 3), and to Jacques Thévenaz (Sections 4 and 5).

The only other changes consist of the correction of small errors and, of course, the enlargement of the Index.

Binghamton, New York, USA — Peter Hilton
Zürich, Switzerland — Urs Stammbach

Contents

Introduction*

This book arose out of a course of lectures given at the Swiss Federal Institute of Technology (ETH), Zürich, in 1966–67. The course was first set down as a set of lecture notes, and, in 1968, Professor Eckmann persuaded the authors to build a graduate text out of the notes, taking account, where appropriate, of recent developments in the subject.

The level and duration of the original course corresponded essentially to that of a year-long, first-year graduate course at an American university. The background assumed of the student consisted of little more than the algebraic theories of finitely-generated abelian groups and of vector spaces over a field. In particular, he was not supposed to have had any formal instruction in categorical notions beyond simply some understanding of the basic terms employed (category, functor, natural transformation). On the other hand, the student was expected to have some sophistication and some preparation for rather abstract ideas. Further, no knowledge of algebraic topology was assumed, so that such notions as chain-complex, chain-map, chain-homotopy, homology were not already available and had to be introduced as purely algebraic constructs. Although references to relevant ideas in algebraic topology do feature in this text, as they did in the course, they are in the nature of (two-way) motivational enrichment, and the student is not left to depend on any understanding of topology to provide a justification for presenting a given topic.

The level and knowledge assumed of the student explains the order of events in the opening chapters. Thus, Chapter I is devoted to the theory of modules over a unitary ring Λ. In this chapter, we do little more than introduce the category of modules and the basic functors on modules and the notions of projective and injective modules, together with their most easily accessible properties. However, on completion of Chapter I, the student is ready with a set of examples to illumine his understanding of the abstract notions of category theory which are presented in Chapter II.

* Sections of this Introduction in small type are intended to give amplified motivation and background for the more experienced algebraist. They may be ignored, at least on first reading, by the beginning graduate student.

In this chapter we are largely influenced in our choice of material by the demands of the rest of the book. However, we take the view that this is an opportunity for the student to grasp basic categorical notions which permeate so much of mathematics today, including, of course, algebraic topology, so that we do not allow ourselves to be rigidly restricted by our immediate objectives. A reader totally unfamiliar with category theory may find it easiest to restrict his first reading of Chapter II to Sections 1 to 6; large parts of the book are understandable with the material presented in these sections. Another reader, who had already met many examples of categorical formulations and concepts might, in fact, prefer to look at Chapter II before reading Chapter I. Of course the reader thoroughly familiar with category theory could, in principal, omit Chapter II, except perhaps to familiarize himself with the notations employed.

In Chapter III we begin the proper study of homological algebra by looking in particular at the group $\operatorname{Ext}_\Lambda(A, B)$, where A and B are Λ-modules. It is shown how this group can be calculated by means of a projective presentation of A, or an injective presentation of B; and how it may also be identified with the group of equivalence classes of extensions of the quotient module A by the submodule B. These facets of the Ext functor are prototypes for the more general theorems to be presented later in the book. Exact sequences are obtained connecting Ext and Hom, again preparing the way for the more general results of Chapter IV. In the final sections of Chapter III, attention is turned from the Ext functor to the Tor functor, $\operatorname{Tor}^\Lambda(A, B)$, which is related to the tensor product of a right Λ-module A and a left Λ-module B rather in the same way as Ext is related to Hom.

With the special cases of Chapter III mastered, the reader should be ready at the outset of Chapter IV for the general idea of a derived functor of an additive functor which we regard as the main motif of homological algebra. Thus, one may say that the material prior to Chapter IV constitutes a build-up, in terms of mathematical knowledge and the study of special cases, for the central ideas of homological algebra which are presented in Chapter IV. We introduce, quite explicitly, left and right derived functors of both covariant and contravariant additive functors, and we draw attention to the special cases of right-exact and left-exact functors. We obtain the basic exact sequences and prove the *balance* of $\operatorname{Ext}^n_\Lambda(A, B)$, $\operatorname{Tor}^\Lambda_n(A, B)$ as bifunctors. It would be reasonable to regard the first four chapters as constituting the first part of the book, as they did, in fact, of the course.

Chapter V is concerned with a very special situation of great importance in algebraic topology where we are concerned with tensor products of free abelian chain-complexes. There it is known that there is a formula expressing the homology groups of the tensor product of the free abelian chain-complexes $\boldsymbol{C}$ and $\boldsymbol{D}$ in terms of the homology groups of $\boldsymbol{C}$ and $\boldsymbol{D}$. We generalize this Künneth formula and we also give a corresponding formula in which the tensor product is replaced by Hom. This corresponding formula is not of such immediate application to topology (where the Künneth formula for the tensor product yields a significant result in the homology of topological products), but it is valuable in homological algebra and leads to certain important identities relating Hom, Ext, tensor and Tor.

Chapters VI and VII may, in a sense, be regarded as individual monographs. In Chapter VI we discuss the homology theory of abstract groups. This is the most classical topic in homological algebra and really provided the original impetus for the entire development of the subject. It has seemed to us important to go in some detail into this theory in order to provide strong motivation for the abstract ideas introduced. Thus, we have been concerned in particular to show how homological ideas may yield proofs of results in group theory which do not require any homology theory for their formulation – and indeed, which were enunciated and proved in some cases before or without the use of homological ideas. Such an example is Maschke's theorem which we state and prove in Section 16.

The relation of the homology theory of groups to algebraic topology is explained in the introductory remarks in Chapter VI itself. It would perhaps be appropriate here to give some indication of the scope and application of the homology theory of groups in group theory. Eilenberg and MacLane [15] showed that the second cohomology group, $H^2(G, A)$, of the group G with coefficients in the G-module A, may be used to formalize the extension theory of groups due to Schreier, Baer, and Fitting. They also gave an interpretation of $H^3(G,A)$ in terms of group extensions with non-abelian kernel, in which A plays the role of the center of the kernel. For a contemporary account of these theories, see Gruenberg [20]. In subsequent developments, the theory has been applied extensively to finite groups and to class field theory by Hochschild, Tate, Artin, etc.; see Weiss [49]. A separate branch of cohomology, the so-called Galois cohomology, has grown out of this connection and has been extensively studied by many algebraists (see Serre [41]).

The natural ring structure in the cohomology of groups, which is clearly in evidence in the relation of the cohomology of a group to that of a space, has also been studied, though not so extensively. However, we should mention here the deep result of L. Evens [17] that the cohomology ring of a finite group is finitely generated.

It would also be appropriate to mention the connection which has been established between the homology theory of groups and algebraic K-theory, a very active area of mathematical research today, which seems to offer hope of providing us with an effective set of invariants of unitary rings. Given a unitary ring Λ we may form the general linear group, $GL_n(\Lambda)$, of invertible $(n \times n)$ matrices over Λ, and then the group $GL(\Lambda)$ is defined to be the union of the groups $GL_n(\Lambda)$ under the natural inclusions. If $E(\Lambda)$ is the commutator subgroup of $GL(\Lambda)$, then a definition given by Milnor for $K_2(\Lambda)$, in terms of the Steinberg group, amounts to saying that $K_2(\Lambda) = H_2(E(\Lambda))$. Moreover, the group $E(\Lambda)$ is perfect, that is to say, $H_1(E(\Lambda)) = 0$, so that the study of the K-groups of Λ leads to the study of the second homology group of perfect groups. The second homology group of the group G actually has an extremely long history, being effectively the *Schur multiplicator* of G, as introduced by Schur [40] in 1904.

Finally, to indicate the extent of activity in this area of algebra, without in any way trying to be comprehensive, we should refer to the proof by Stallings [45] and Swan [48], that a group G is free if and only if $H^n(G, A) = 0$ for all G-modules A and all $n \geqq 2$. That the cohomology vanishes in dimensions $\geqq 2$ when G is free is quite trivial (and is, of course, proved in this book); the opposite implication, however, is deep and difficult to establish. The result has particularly interesting consequences for torsion-free groups.

In Chapter VII we discuss the cohomology theory of Lie algebras. Here the spirit and treatment are very much the same as in Chapter VI, but we do not treat Lie algebras so extensively, principally because so much of the development is formally analogous to that for the cohomology of groups. As explained in the introductory remarks to the chapter, the cohomology theory of Lie algebras, like the homology theory of groups, arose originally from considerations of algebraic topology, namely, the cohomology of the underlying spaces of Lie groups. However, the theory of Lie algebra cohomology has developed independently of its topological origins.

This development has been largely due to the work of Koszul [31]. The cohomological proofs of two main theorems of Lie algebra theory which we give in Sections 5 and 6 of Chapter VII are basically due to Chevalley-Eilenberg [8]. Hochschild [24] showed that, as for groups, the three-dimensional cohomology group $H^3(\mathfrak{g}, A)$ of the Lie algebra $\mathfrak{g}$ with coefficients in the $\mathfrak{g}$-module A classifies obstructions to extensions with non-abelian kernel.

Cartan and Eilenberg [7] realized that group cohomology and Lie algebra cohomology (as well as the cohomology of associative algebras over a field) may all be obtained by a general procedure, namely, as derived functors in a suitable module-category. It is, of course, this procedure which is adopted in this book, so that we have presented the theory of derived functors in Chapter IV as the core of homological algebra, and Chapters VI and VII are then treated as important special cases.

Chapters VIII and IX constitute the third part of the book. Chapter VIII consists of an extensive treatment of the theory of spectral sequences. Here, as in Chapter II, we have gone beyond the strict requirements of the applications which we make in the text. Since the theory of spectral sequences is so ubiquitous in homological algebra and its applications, it appeared to us to be sensible to give the reader a thorough grounding in the topic. However, we indicate in the introductory remarks to Chapter VIII, and in the course of the text itself, those parts of the chapter which may be omitted by the reader who simply wishes to be able to understand those applications which are explicitly presented. Our own treatment gives prominence to the idea of an exact couple and emphasizes the notion of the spectral sequence functor on the category of exact couples. This is by no means the unique way of presenting spectral sequences and the reader should, in particular, consult the book of Cartan-Eilenberg [7] to see an alternative approach. However, we do believe that the approach adopted is a reasonable one and a natural one. In fact, we have presented an elaboration of the notion of an exact couple, namely, that of a Rees system, since within the Rees system is contained all the information necessary to deduce the crucial convergence properties of the spectral sequence. Our treatment owes much to the study by Eckmann-Hilton [10] of exact couples in an abelian category. We take from them the point of view that the grading on the objects should only be introduced at such time as it is crucial for the study of convergence; that is to say, the purely algebraic constructions are carried out without any reference to grading. This, we believe, simplifies the presentation and facilitates the understanding.

We should point out that we depart in Chapter VIII from the standard conventions with regard to spectral sequences in one important and one less important respect. We index the original exact couple by the symbol 0 so that the first derived couple is indexed by the symbol 1 and, in general, the nth derived couple by the symbol n. This has the effect that what is called by most authorities the E_2-term appears with us as the E_1-term. We do not believe that this difference of convention, once it has been drawn to the attention of the reader, should cause any difficulties. On the other hand, we claim that the convention we adopt has many advantages. Principal among them, perhaps, is the fact that in the exact couple

$$\begin{tikzcd} D \arrow[rr, "\alpha"] & & D \arrow[dl, "\beta"] \\ & E \arrow[ul, "\gamma"] & \end{tikzcd}$$

the nth differential in the associated spectral sequence d_n is, by our convention, induced by $\beta\alpha^{-n}\gamma$. With the more habitual convention d_n would be induced by $\beta\alpha^{-n+1}\gamma$. It is our experience that where a difference of unity enters gratuitously into a formula like this, there is a great danger that the sign is misremembered – or that the difference is simply forgotten. A minor departure from the more usual convention is that the second index, or q index, in the spectral sequence term, $E_r^{p,q}$, signifies the total degree and not the complementary degree. As a result, we have the situation that if C is a filtered chain-complex, then $H_q(C)$ is filtered by subgroups whose associated graded group is $\{E_\infty^{p,q}\}$. Our convention is the one usually adopted for the generalized Atiyah-Hirzebruch spectral sequence, but it is not the one introduced by Serre in his seminal paper on the homology of fiber spaces, which has influenced the adoption of the alternative convention to which we referred above. However, since the translation from one convention to another is, in this case, absolutely trivial (with our convention, the term $E_r^{p,q}$ has complementary degree $q-p$), we do not think it necessary to lay further stress on this distinction.

Chapter IX is somewhat different from the other chapters in that it represents a further development of many of the ideas of the rest of the text, in particular, those of Chapters IV and VIII. This chapter did not appear in its present form in the course, which concluded with applications of spectral sequences available through the material already familiar to the students. In the text we have permitted ourselves further theoretical developments and generalizations. In particular, we present the theory of satellites, some relative homological algebra, and the theory of the homology of small categories. Since this chapter does constitute further development of the subject, one might regard its contents as more arbitrary than those of the other chapters and, in the same way, the chapter itself is far more open-ended than its predecessors. In particular, ideas are presented in the expectation that the student will be encouraged to make a further study of them beyond the scope of this book.

Each chapter is furnished with some introductory remarks describing the content of the chapter and providing some motivation and background. These introductory remarks are particularly extensive in the case of Chapters VI and VII in view of their special nature. The chapters are divided into sections and each section closes with a set of exercises.* These exercises are of many different kinds; some are purely computational, some are of a theoretical nature, and some ask the student to fill in gaps in the text where we have been content to omit proofs. Sometimes we suggest exercises which take the reader beyond the scope of the text. In some cases, exercises appearing at the end of a given section may reappear as text material in a later section or later chapter; in fact, the results stated in an exercise may even be quoted subsequently with appropriate reference, but this procedure is adopted only if their demonstration is incontestably elementary.

Although this text is primarily intended to accompany a course at the graduate level, we have also had in mind the obligation to write a book which can be used as a work of reference. Thus, we have endeavored, by giving very precise references, by making self-contained statements, and in other ways, to ensure that the reader interested in a particular aspect of the theory covered by the text may dip into the book at any point and find the material intelligible – always assuming, of course, that he is prepared to follow up the references given. This applies in particular to Chapters VI and VII, but the same principles have been adopted in designing the presentation in all the chapters.

The enumeration of items in the text follows the following conventions. The chapters are enumerated with Roman numerals and the

* Of course, Chapter X is different.

sections with Arabic numerals. Within a given chapter, we have two series of enumerations, one for theorems, lemmas, propositions, and corollaries, the other for displayed formulas. The system of enumeration in each of these series consists of a pair of numbers, the first referring to the section and the second to the particular item. Thus, in Section 5 of Chapter VI, we have Theorem 5.1 in which a formula is displayed which is labeled (5.2). On the subsequent page there appears Corollary 5.2 which is a corollary to Theorem 5.1. When we wish to refer to a theorem, etc., or a displayed formula, we simply use the same system of enumeration, provided the item to be cited occurs in the same chapter. If it occurs in a different chapter, we will then precede the pair of numbers specifying the item with the Roman numeral specifying the chapter. The exercises are enumerated according to the same principle. Thus, Exercise 1.2 of Chapter VIII refers to the second exercise at the end of the first section of Chapter VIII. A reference to Exercise 1.2, occurring in Chapter VIII, means Exercise 1.2 of that chapter. If we wish to refer to that exercise in the course of a different chapter, we would refer to Exercise VIII.1.2.

This text arose from a course and is designed, itself, to constitute a graduate course, at the first-year level at an American university. Thus, there is no attempt at complete coverage of all areas of homological algebra. This should explain the omission of such important topics as Hopf algebras, derived categories, triple cohomology, Galois cohomology, and others, from the content of the text. Since, in planning a course, it is necessary to be selective in choosing applications of the basic ideas of homological algebra, we simply claim that we have made one possible selection in the second and third parts of the text. We hope that the reader interested in applications of homological algebra not given in the text will be able to consult the appropriate authorities.

We have not provided a bibliography beyond a list of references to works cited in the text. The comprehensive listing by Steenrod of articles and books in homological algebra* should, we believe, serve as a more than adequate bibliography. Of course it is to be expected that the instructor in a course in homological algebra will, himself, draw the students' attention to further developments of the subject and will thus himself choose what further reading he wishes to advise. As a single exception to our intention not to provide an explicit bibliography, we should mention the work by Saunders MacLane, *Homology*, published by Springer-Verlag, which we would like to view as a companion volume to the present text.

Some remarks are in order about notational conventions. First, we use the left-handed convention, whereby the composite of the morphism φ

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* *Reviews of Papers in Algebraic and Differential Topology, Topological Groups and Homological Algebra*, Part II (American Mathematical Society).

followed by the morphism ψ is written as $\psi\varphi$ or, where the morphism symbols may themselves be complicated, $\psi \circ \varphi$. We allow ourselves to simplify notation once the strict notation has been introduced and established. Thus, for example, $f(x)$ may appear later simply as fx and $F(A)$ may appear later as FA. We also adapt notation to local needs in the sense that we may very well modify a notation already introduced in order to make it more appropriate to a particular context. Thus, for instance, although our general rule is that the dimension symbol in cohomology appears as a superscript (while in homology it appears as a subscript), we may sometimes find it convenient to write the dimension index as a subscript in cohomology; for example, in discussing certain right-derived functors. We use the symbol $\square$ to indicate the end of a proof even if the proof is incomplete; as a special case we may very well place the symbol at the end of the statement of a theorem (or proposition, lemma, corollary) to indicate that no proof is being offered or that the remarks preceding the statement constitute a sufficient demonstration. In diagrams, the firm arrows represent the data of the diagram, and dotted arrows represent new morphisms whose existence is attested by arguments given in the text. We generally use MacLane's notation $\rightarrowtail$, $\twoheadrightarrow$ to represent monomorphisms and epimorphisms respectively. We distinguish between the symbols $\cong$ and $\xrightarrow{\sim}$. In the first case we would write $X \cong Y$ simply to indicate that X and Y are isomorphic objects in the given category, whereas the symbol $\varphi: X \xrightarrow{\sim} Y$ indicates that the morphism φ is itself an isomorphism.

It is a pleasure to make many acknowledgments. First, we would like to express our appreciation to our good friend Beno Eckmann for inviting one of us (P.H.) to Zürich in 1966–67 as Visiting Professor at the ETH, and further inviting him to deliver the course of lectures which constitutes the origin of this text. Our indebtedness to Beno Eckmann goes much further than this and we would be happy to regard him as having provided us with both the intellectual stimulus and the encouragement necessary to bring this book into being. In particular, we would also like to mention that it was through his advocacy that Springer-Verlag was led to commission this text from us. We would also like to thank Professor Paul Halmos for accepting this book into the series Graduate Texts in Mathematics. Our grateful thanks go to Frau Marina von Wildemann for her many invaluable services throughout the evolution of the manuscript from original lecture notes to final typescript. Our thanks are also due to Frau Eva Minzloff, Frau Hildegard Mourad, Mrs. Lorraine Pritchett, and Mrs. Marlys Williams for typing the manuscript and helping in so many ways in the preparation of the final text. Their combination of cheerful good will and quiet efficiency has left us forever in their debt. We are also grateful to Mr. Rudolf Beyl for his careful reading of the text and exercises of Chapters VI and VII.

We would also like to thank our friend Klaus Peters of Springer-Verlag for his encouragement to us and his ready accessibility for the discussion of all technical problems associated with the final production of the book. We have been very fortunate indeed to enjoy such pleasant informal relations with Dr. Peters and other members of the staff of Springer-Verlag, as a result of which the process of transforming this book from a rather rough set of lecture notes to a final publishable document has proved unexpectedly pleasant.

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April, 1971

I. Modules

The algebraic categories with which we shall be principally concerned in this book are categories of modules over a fixed (unitary) ring Λ and module-homomorphisms. Thus we devote this chapter to a preliminary discussion of Λ-modules.

The notion of *Λ-module* may be regarded as providing a common generalization of the notions of *vector space* and *abelian group*. Thus if Λ is a field K then a K-module is simply a vector space over K and a K-module homomorphism is a linear transformation; while if $\Lambda = \mathbb{Z}$ then a $\mathbb{Z}$-module is simply an abelian group and a $\mathbb{Z}$-module homomorphism is a homomorphism of abelian groups. However, the facets of module theory which are of interest in homological algebra tend to be trivial in vector space theory; whereas the case $\Lambda = \mathbb{Z}$ will often yield interesting specializations of our results, or motivations for our constructions.

Thus, for example, in the theory of vector spaces, there is no interest in the following question: given vector spaces A, B over the field K, find all vector spaces E over K having B as subspace with A as associated quotient space. For any such E is isomorphic to $A \oplus B$. However, the question is interesting if A, B, E are now abelian groups; and it turns out to be a very basic question in homological algebra (see Chapter III).

Again it is trivial that, given a diagram of linear transformations of K-vector spaces

$$\begin{array}{ccc} & & P \\ & & \downarrow{\scriptstyle \gamma} \\ B & \xrightarrow{\varepsilon} & C \end{array} \tag{0.1}$$

where ε is surjective, there is a linear transformation $\beta: P \to B$ with $\varepsilon\beta = \gamma$. However, it is a very special feature of an abelian group P that, for all diagrams of the form (0.1) of abelian groups and homomorphisms, with ε surjective, such a homomorphism β exists. Indeed, for abelian groups, this characterizes the *free* abelian groups (thus one might say that *all* vector spaces are free). Actually, in this case, the example $\Lambda = \mathbb{Z}$ is somewhat misleading. For if we define a Λ-module P to be projective if, given any diagram (0.1) with ε surjective, we may find β with $\varepsilon\beta = \gamma$, then it is always the case that free Λ-modules are projective but, for some rings Λ, there are projective Λ-modules which are not free. The relation between those two concepts is elucidated in Sections 4 and 5, where we see that the concepts coincide if Λ is a principal ideal domain (p.i.d.) – this explains the phenomenon in the case of abelian groups.

In fact, the matters of concern in homological algebra tend very much to become simplified – but not trivial – if Λ is a p.i.d., so that this special case recurs frequently in the text. It is thus an important special case, but nevertheless atypical in certain respects. In fact, there is a precise numerical index (the so-called *global dimension* of Λ) whereby the case Λ a field appears as case 0 and Λ a p.i.d. as case 1.

The categorical notion of *duality* (see Chapter II) may be applied to the study of Λ-modules and leads to the concept of an *injective* module, dual to that of a projective module. In this case, the theory for $\Lambda = \mathbb{Z}$, or, indeed, for Λ any p.i.d., is surely not as familiar as that of free modules; nevertheless, it is again the case that the theory is, for modules over a p.i.d., much simpler than for general rings Λ – and it is again trivial for vector spaces!

We should repeat (from the main Introduction) our rationale for placing this preparatory chapter on modules before the chapter introducing the basic categorical concepts which will be used throughout the rest of the book. Our justification is that we wish, in Chapter II, to have some mathematics available from which we may make meaningful abstractions. This chapter provides that mathematics; had we reversed the order of these chapters, the reader would have been faced with a battery of "abstract" ideas lacking in motivation. Although it is, of course, true that motivation, or at least exemplification, could in many cases be provided by concepts drawn from other parts of mathematics familiar to the reader, we prefer that the motivation come from concrete instances of the abstract ideas germane to homological algebra.

1. Modules

We start with some introductory remarks on the notion of a ring. In this book a ring Λ will always have a unity element $1_\Lambda \neq 0$. A homomorphism of rings $\omega: \Lambda \to \Gamma$ will always carry the unity element of the first ring Λ into the unity element of the second ring Γ. Recall that the endomorphisms of an abelian group A form a ring $\mathrm{End}(A, A)$.

Definition. A *left module* over the ring Λ or a *left Λ-module* is an abelian group A together with a ring homomorphism $\omega: \Lambda \to \mathrm{End}(A, A)$.

We write λa for $(\omega(\lambda))(a)$, $a \in A$, $\lambda \in \Lambda$. We may then talk of Λ *operating* (on the left) on A, in the sense that we associate with the pair (λ, a) the element λa. Clearly the following rules are satisfied for all $a, a_1, a_2 \in A$, $\lambda, \lambda_1, \lambda_2 \in \Lambda$:

M 1: $(\lambda_1 + \lambda_2)a = \lambda_1 a + \lambda_2 a$
M 2: $(\lambda_1 \lambda_2)a = \lambda_1(\lambda_2 a)$
M 3: $1_\Lambda a = a$
M 4: $\lambda(a_1 + a_2) = \lambda a_1 + \lambda a_2$.

On the other hand, if an operation of Λ on the abelian group A satisfies **M 1**, ..., **M 4**, then it obviously defines a ring homomorphism

$$\omega : \Lambda \to \mathrm{End}(A, A), \qquad \text{by the rule} \qquad (\omega(\lambda))(a) = \lambda a .$$

Denote by Λ^{opp} the opposite ring of Λ. The elements $\lambda^{\mathrm{opp}} \in \Lambda^{\mathrm{opp}}$ are in one-to-one correspondence with the elements $\lambda \in \Lambda$. As abelian groups Λ and Λ^{opp} are isomorphic under this correspondence. The product in Λ^{opp} is given by $\lambda_1^{\mathrm{opp}} \lambda_2^{\mathrm{opp}} = (\lambda_2 \lambda_1)^{\mathrm{opp}}$. We naturally identify the underlying sets of Λ and Λ^{opp}.

A *right module* over Λ or *right Λ-module* is simply a left Λ^{opp}-module, that is, an abelian group A together with a ring map $\omega' : \Lambda^{\mathrm{opp}} \to \mathrm{End}(A, A)$. We leave it to the reader to state the axioms **M 1′**, **M 2′**, **M 3′**, **M 4′** for a right module over Λ. Clearly, if Λ is commutative, the notions of a left and a right module over Λ coincide. *For convenience, we shall use the term "module" always to mean "left module".*

Let us give a few examples:

(a) The left-multiplication in Λ defines an operation of Λ on the underlying abelian group of Λ, satisfying **M 1**, ..., **M 4**. Thus Λ is a left module over Λ. Similarly, using right multiplication, Λ is a right module over Λ. Analogously, any left-ideal of Λ becomes a left module over Λ, any right-ideal of Λ becomes a right module over Λ.

(b) Let $\Lambda = \mathbb{Z}$, the ring of integers. Every abelian group A possesses the structure of a $\mathbb{Z}$-module; for $a \in A$, $n \in \mathbb{Z}$ define $na = 0$, if $n = 0$, $na = a + \cdots + a$ (n times), if $n > 0$, and $na = -(-na)$, if $n < 0$.

(c) Let $\Lambda = K$, a field. A K-module is a vector space over K.

(d) Let V be a vector space over the field K, and T a linear transformation from V into V. Let $\Lambda = K[T]$, the polynomial ring in T over K. Then V becomes a $K[T]$-module, with the obvious operation of $K[T]$ on V.

(e) Let G be a group and let K be a field. Consider the K-vector-space of all linear combinations $\sum_{x \in G} k_x x$, $k_x \in K$. One checks quite easily that the definition

$$\Big(\sum_{x \in G} k_x x\Big)\Big(\sum_{y \in G} k'_y y\Big) = \sum_{x, y \in G} (k_x k'_y)\, x y ,$$

where xy denotes the product in G, makes this vector space into a K-algebra KG, called the *group algebra* of G over K. Let V be a vector space over K. A *K-representation* of G in V is a group homomorphism $\sigma: G \to \operatorname{Aut}_K(V, V)$. The map σ gives rise to a ring homomorphism $\sigma': KG \to \operatorname{End}_K(V, V)$ by setting

$$\sigma'\left(\sum_{x\in G} k_x x\right) = \sum_{x\in G} k_x \sigma(x).$$

Since every K-linear endomorphism of V is also a homomorphism of the underlying abelian group, we obtain from σ' a ring homomorphism $\varrho: KG \to \operatorname{End}_{\mathbb{Z}}(V, V)$, making V into a KG-module. Conversely, let V be a KG-module. Clearly V has a K-vector-space structure, and the structure map $\varrho: KG \to \operatorname{End}_{\mathbb{Z}}(V, V)$ factors through $\operatorname{End}_K(V, V)$. Its restriction to the elements of G defines a K-representation of G. We see that the K-representations of G are in one-to-one correspondence with the KG-modules. (We leave to the reader to check the assertions in this example.)

Definition. Let A, B two Λ-modules. A *homomorphism* (or *map*) $\varphi: A \to B$ of Λ-modules is a homomorphism of abelian groups such that $\varphi(\lambda a) = \lambda(\varphi a)$ for all $a \in A$, $\lambda \in \Lambda$.

Clearly the identity map of A is a homomorphism of Λ-modules; we denote it by $1_A: A \to A$.

If φ is surjective, we use the symbol $\varphi: A \twoheadrightarrow B$. If φ is injective, we use the symbol $\varphi: A \rightarrowtail B$. We call $\varphi: A \to B$ *isomorphic* or an *isomorphism*, and write $\varphi: A \xrightarrow{\sim} B$, if there exists a homomorphism $\psi: B \to A$ such that $\psi\varphi = 1_A$ and $\varphi\psi = 1_B$. Plainly, if it exists, ψ is uniquely determined; it is denoted by φ^{-1} and called the *inverse* of φ. If $\varphi: A \to B$ is isomorphic, it is clearly injective and surjective. Conversely, if the module homomorphism $\varphi: A \to B$ is both injective and surjective, it is isomorphic. We shall call A and B *isomorphic*, $A \cong B$, if there exists an isomorphism $\varphi: A \xrightarrow{\sim} B$.

If A' is a subgroup of A with $\lambda a' \in A'$ for all $\lambda \in \Lambda$ and all $a' \in A'$, then A' together with the induced operation of Λ is called a *submodule* of A. Let A' be a submodule of A. Then the quotient group A/A' may be given the structure of a Λ-module by defining $\lambda(a + A') = (\lambda a + A')$ for all $\lambda \in \Lambda$, $a \in A$. Clearly, we have an injective homomorphism $\mu: A' \rightarrowtail A$ and a surjective homomorphism $\pi: A \twoheadrightarrow A/A'$.

For an arbitrary homomorphism $\varphi: A \to B$, we shall use the notation $\ker\varphi = \{a \in A \mid \varphi a = 0\}$ for the *kernel* of φ and

$$\operatorname{im}\varphi = \varphi A = \{b \in B \mid b = \varphi a \quad \text{for some } a \in A\}$$

for the *image* of φ. Obviously $\ker\varphi$ is a submodule of A and $\operatorname{im}\varphi$ is a submodule of B. One easily checks that the canonical isomorphism of abelian groups $A/\ker\varphi \xrightarrow{\sim} \operatorname{im}\varphi$ is actually an isomorphism of Λ-modules. We also introduce the notation $\operatorname{coker}\varphi = B/\operatorname{im}\varphi$ for the *cokernel* of φ. Just as $\ker\varphi$ measures how far φ differs from being injective, so $\operatorname{coker}\varphi$ measures how far φ differs from being surjective. If $\mu: A' \rightarrowtail A$ is injective, we can identify A' with the submodule $\mu A'$ of A. Similarly, if $\varepsilon: A \twoheadrightarrow A''$ is surjective, we can identify A'' with $A/\ker\varepsilon$.

Definition. Let $\varphi: A \to B$ and $\psi: B \to C$ be homomorphisms of Λ-modules. The sequence $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ is called *exact* (at B) if $\ker\psi = \operatorname{im}\varphi$. If a sequence $A_0 \to A_1 \to \cdots \to A_n \to A_{n+1}$ is exact at $A_1, \ldots, A_n$, then the sequence is simply called *exact.*

As examples we mention

(a) $0 \to A \xrightarrow{\varphi} B$ is exact (at A) if and only if φ is injective.

(b) $A \xrightarrow{\varphi} B \to 0$ is exact (at B) if and only if φ is surjective.

(c) The sequence $0 \to A' \xrightarrow{\mu} A \xrightarrow{\varepsilon} A'' \to 0$ is exact (at A', A, A'') if and only if μ induces an isomorphism $A' \xrightarrow{\sim} \mu A'$ and ε induces an isomorphism $A/\ker\varepsilon = A/\mu A' \xrightarrow{\sim} A''$. Essentially A' is then a submodule of A and A'' the corresponding quotient module. Such an exact sequence is called *short exact*, and often written $A' \rightarrowtail A \twoheadrightarrow A''$.

The proofs of these assertions are left to the reader. Let A, B, C, D be Λ-modules and let α, β, γ, δ be Λ-module homomorphisms. We say that the *diagram*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\gamma"] & B \arrow[d, "\beta"] \\ C \arrow[r, "\delta"] & D \end{tikzcd}$$

is *commutative* if $\beta\alpha = \delta\gamma: A \to D$. This notion generalizes in an obvious way to more complicated diagrams. Among the many propositions and lemmas about diagrams we shall need the following:

Lemma 1.1. *Let $A' \rightarrowtail A \twoheadrightarrow A''$ and $B' \rightarrowtail B \twoheadrightarrow B''$ be two short exact sequences. Suppose that in the commutative diagram*

$$\begin{tikzcd} A' \arrow[r, tail, "\mu"] \arrow[d, "\alpha'"] & A \arrow[r, two heads, "\varepsilon"] \arrow[d, "\alpha"] & A'' \arrow[d, "\alpha''"] \\ B' \arrow[r, tail, "\mu'"] & B \arrow[r, two heads, "\varepsilon'"] & B'' \end{tikzcd} \tag{1.2}$$

any two of the three homomorphisms α', α, α'' are isomorphisms. Then the third is an isomorphism, too.

Proof. We only prove one of the possible three cases, leaving the other two as exercises. Suppose α', α'' are isomorphisms; we have to show that α is an isomorphism.

First we show that $\ker\alpha = 0$. Let $a \in \ker\alpha$, then $0 = \varepsilon'\alpha a = \alpha''\varepsilon a$. Since α'' is an isomorphism, it follows that $\varepsilon a = 0$. Hence there exists $a' \in A'$ with $\mu a' = a$ by the exactness of the upper sequence. Then $0 = \alpha\mu a' = \mu'\alpha' a'$. Since $\mu'\alpha'$ is injective, it follows that $a' = 0$. Hence $a = \mu a' = 0$.

Secondly, we show that α is surjective. Let $b \in B$; we have to show that $b = \alpha a$ for some $a \in A$. Since α'' is an isomorphism, there exists $a'' \in A''$ with $\alpha'' a'' = \varepsilon' b$. Since ε is surjective, there exists $\bar{a} \in A$ such that $\varepsilon \bar{a} = a''$. We obtain $\varepsilon'(b - \alpha\bar{a}) = \varepsilon' b - \varepsilon'\alpha\bar{a} = \varepsilon' b - \alpha''\varepsilon\bar{a} = 0$. Hence by the exactness of the lower sequence there exists $b' \in B'$ with $\mu' b' = b - \alpha\bar{a}$. Since α' is isomorphic there exists $a' \in A'$ such that $\alpha' a' = b'$. Now

$$\alpha(\mu a' + \bar{a}) = \alpha\mu a' + \alpha\bar{a} = \mu'\alpha' a' + \alpha\bar{a} = \mu' b' + \alpha\bar{a} = b\,.$$

So setting $a = \mu a' + \bar{a}$, we have $\alpha a = b$. $\square$

Notice that Lemma 1.1 does *not* imply that, given exact sequences $A' \rightarrowtail A \twoheadrightarrow A''$, $B' \rightarrowtail B \twoheadrightarrow B''$, with $A' \cong B'$, $A'' \cong B''$, then $A \cong B$. It is crucial to the proof of Lemma 1.1 that there is a map $A \to B$ compatible with the isomorphisms $A' \cong B'$, $A'' \cong B''$, in the sense that (1.2) commutes.

Exercises:

1.1. Complete the proof of Lemma 1.1. Show moreover that, in (1.2), α is surjective (injective) if α', α'' are surjective (injective).

1.2. *(Five Lemma)* Show that, given a commutative diagram

$$\begin{tikzcd} {} \arrow[r] & A_1 \arrow[r] \arrow[d, "\varphi_1"] & A_2 \arrow[r] \arrow[d, "\varphi_2"] & A_3 \arrow[r] \arrow[d, "\varphi_3"] & A_4 \arrow[r] \arrow[d, "\varphi_4"] & A_5 \arrow[r] \arrow[d, "\varphi_5"] & {} \\ {} \arrow[r] & B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5 \arrow[r] & {} \end{tikzcd}$$

with exact rows, in which $\varphi_1, \varphi_2, \varphi_4, \varphi_5$ are isomorphisms, then φ_3 is also an isomorphism. Can we weaken the hypotheses in a reasonable way?

1.3. Give examples of short exact sequences of abelian groups

$$0 \to A' \to A \to A'' \to 0\,, \qquad 0 \to B' \to B \to B'' \to 0$$

such that

(i) $A' \cong B'$, $A \cong B$, $A'' \ncong B''$;
(ii) $A' \cong B'$, $A \ncong B$, $A'' \cong B''$;
(iii) $A' \ncong B'$, $A \cong B$, $A'' \cong B''$.

1.4. Show that the abelian group A admits the structure of a $\mathbb{Z}_m$-module if and only if $mA = 0$.

1.5. Define the group algebra KG for K an arbitrary commutative ring. What are the KG-modules?

1.6. Let V be a non-trivial (left) KG-module. Show how to give V the structure of a non-trivial *right* KG-module. (Use the group inverse.)

1.7. Let $0 \to A' \xrightarrow{\mu} A \xrightarrow{\varepsilon} A'' \to 0$ be a short exact sequence of abelian groups. We say that the sequence is *pure* if, whenever $\mu(a') = ma$, $a' \in A'$, $a \in A$, m a positive integer, there exists $b' \in A'$ with $a' = mb'$. Show that the following statements are equivalent:

(i) the sequence is pure;

(ii) the induced sequence (*reduction* mod m) $0 \to A'_m \xrightarrow{\mu_m} A_m \xrightarrow{\varepsilon_m} A''_m \to 0$ is exact for all m; ($A_m = A/mA$, etc.)

(iii) given $a'' \in A''$ with $ma'' = 0$, there exists $a \in A$ with $\varepsilon(a) = a''$, $ma = 0$ (for all m).

2. The Group of Homomorphisms

Let $\mathrm{Hom}_\Lambda(A, B)$ denote the set of all Λ-module homomorphisms from A to B. Clearly, this set has the structure of an abelian group; if $\varphi : A \to B$ and $\psi : A \to B$ are Λ-module homomorphisms, then $\varphi + \psi : A \to B$ is defined as $(\varphi + \psi)a = \varphi a + \psi a$ for all $a \in A$. The reader should check that $\varphi + \psi$ is a Λ-module homomorphism. Note, however, that $\mathrm{Hom}_\Lambda(A, B)$ is *not*, in general, a Λ-module in any obvious way (see Exercise 2.3).

Let $\beta : B_1 \to B_2$ be a homomorphism of Λ-modules. We can assign to a homomorphism $\varphi : A \to B_1$, the homomorphism $\beta\varphi : A \to B_2$, thus defining a map $\beta_* = \mathrm{Hom}_\Lambda(A, \beta) : \mathrm{Hom}_\Lambda(A, B_1) \to \mathrm{Hom}_\Lambda(A, B_2)$. It is left to the reader to verify that β_* is actually a homomorphism of abelian groups. Evidently the following two rules hold:

(i) If $\beta : B_1 \to B_2$ and $\beta' : B_2 \to B_3$, then

$$(\beta'\beta)_* = \beta'_*\beta_* : \mathrm{Hom}_\Lambda(A, B_1) \to \mathrm{Hom}_\Lambda(A, B_3)\,.$$

(ii) If $\beta : B_1 \to B_1$ is the identity, then $\beta_* : \mathrm{Hom}_\Lambda(A, B_1) \to \mathrm{Hom}_\Lambda(A, B_1)$ is the identity, also.

In short, the symbol $\mathrm{Hom}_\Lambda(A, -)$ assigns to every Λ-module B an abelian group $\mathrm{Hom}_\Lambda(A, B)$, and to every homomorphism of Λ-modules $\beta : B_1 \to B_2$ a homomorphism of abelian groups

$$\beta_* = \mathrm{Hom}_\Lambda(A, \beta) : \mathrm{Hom}_\Lambda(A, B_1) \to \mathrm{Hom}_\Lambda(A, B_2)$$

such that the above two rules hold. In Chapter II, we shall see that this means that $\mathrm{Hom}_\Lambda(A, -)$ is a (covariant) functor from the category of Λ-modules to the category of abelian groups.

On the other hand, if $\alpha : A_2 \to A_1$ is a Λ-module homomorphism, then we assign to every homomorphism $\varphi : A_1 \to B$ the homomorphism $\varphi\alpha : A_2 \to B$, thus defining a map

$$\alpha^* = \mathrm{Hom}_\Lambda(\alpha, B) : \mathrm{Hom}_\Lambda(A_1, B) \to \mathrm{Hom}_\Lambda(A_2, B)\,.$$

Again we leave it to the reader to verify that α^* is actually a homomorphism of abelian groups. Evidently, we have:

(i)′ If $\alpha : A_2 \to A_1$ and $\alpha' : A_3 \to A_2$, then $(\alpha\alpha')^* = \alpha'^*\alpha^*$ (inverse order!).

(ii)′ If $\alpha : A_1 \to A_1$ is the identity, then α^* is the identity.

$\mathrm{Hom}_\Lambda(-, B)$ is an instance of a contravariant functor (from Λ-modules to abelian groups).

Theorem 2.1. *Let $B' \overset{\mu}{\rightarrowtail} B \overset{\varepsilon}{\rightarrow} B''$ be an exact sequence of Λ-modules. For every Λ-module A the induced sequence*

$$0 \rightarrow \mathrm{Hom}_\Lambda(A, B') \xrightarrow{\mu_*} \mathrm{Hom}_\Lambda(A, B) \xrightarrow{\varepsilon_*} \mathrm{Hom}_\Lambda(A, B'')$$

is exact.

Proof. First we show that μ_* is injective.

Assume that $\mu\varphi$ in the diagram

$$\begin{tikzcd} A \arrow[d, "\varphi"] & & \\ B' \arrow[r, tail, "\mu"] & B \arrow[r, "\varepsilon"] & B'' \end{tikzcd}$$

is the zero map. Since $\mu : B' \rightarrowtail B$ is injective this implies that $\varphi : A \rightarrow B'$ is the zero map, so μ_* is injective.

Next we show that $\ker \varepsilon_* \supset \mathrm{im}\,\mu_*$. Consider the above diagram. A map in $\mathrm{im}\,\mu_*$ is of the form $\mu\varphi$. Plainly $\varepsilon\mu\varphi$ is the zero map, since $\varepsilon\mu$ already is. Finally we show that $\mathrm{im}\,\mu_* \supset \ker \varepsilon_*$. Consider the diagram

$$\begin{tikzcd} & A \arrow[d, "\psi"] & \\ B' \arrow[r, tail, "\mu"] & B \arrow[r, "\varepsilon"] & B'' . \end{tikzcd}$$

We have to show that if $\varepsilon\psi$ is the zero map, then ψ is of the form $\mu\varphi$ for some $\varphi : A \rightarrow B'$. But, if $\varepsilon\psi = 0$ the image of ψ is contained in $\ker \varepsilon = \mathrm{im}\,\mu$. Since μ is injective, ψ gives rise to a (unique) map $\varphi : A \rightarrow B'$ such that $\mu\varphi = \psi$. □

We remark that even in case ε is surjective the induced map ε_* is *not* surjective in general (see Exercise 2.1).

Theorem 2.2. *Let $A' \overset{\mu}{\rightarrow} A \overset{\varepsilon}{\twoheadrightarrow} A''$ be an exact sequence of Λ-modules. For every Λ-module B the induced sequence*

$$0 \rightarrow \mathrm{Hom}_\Lambda(A'', B) \xrightarrow{\varepsilon^*} \mathrm{Hom}_\Lambda(A, B) \xrightarrow{\mu^*} \mathrm{Hom}_\Lambda(A', B)$$

is exact.

The proof is left to the reader. □

Notice that even in case μ is injective μ^* is *not* surjective in general (see Exercise 2.2).

We finally remark that Theorem 2.1 provides a universal characterization of $\ker \varepsilon$ (in the sense of Sections II.5 and II.6): To every homomorphism $\varphi : A \rightarrow B$ with $\varepsilon_*(\varphi) = \varepsilon\varphi : A \rightarrow B''$ the zero map there exists a unique homomorphism $\varphi' : A \rightarrow B'$ with $\mu_*(\varphi') = \mu\varphi' = \varphi$. Similarly Theorem 2.2 provides a universal characterization of $\mathrm{coker}\,\mu$.

Exercises:

2.1. Show that in the setting of Theorem 2.1 $\varepsilon_* = \mathrm{Hom}(A, \varepsilon)$ is not, in general, surjective even if ε is. (Take $\Lambda = \mathbb{Z}$, $A = \mathbb{Z}_n$, the integers mod n, and the short exact sequence $\mathbb{Z} \overset{\mu}{\rightarrowtail} \mathbb{Z} \twoheadrightarrow \mathbb{Z}_n$ where μ is multiplication by n.)

2.2. Prove Theorem 2.2. Show that $\mu^* = \mathrm{Hom}_\Lambda(\mu, B)$ is not, in general, surjective even if μ is injective. (Take $\Lambda = \mathbb{Z}$, $B = \mathbb{Z}_n$, the integers mod n, and the short exact sequence $\mathbb{Z} \overset{\mu}{\rightarrowtail} \mathbb{Z} \twoheadrightarrow \mathbb{Z}_n$, where μ is multiplication by n.)

2.3. Suppose Λ commutative, and A and B two Λ-modules. Define for a Λ-module homomorphism $\varphi: A \to B$, $(\lambda\varphi)(a) = \varphi(\lambda a)$, $a \in A$. Show that this definition makes $\mathrm{Hom}_\Lambda(A, B)$ into a Λ-module. Also show that this definition does not work in case Λ is *not* commutative.

2.4. Let A be a Λ-module and B be an abelian group. Show how to give $\mathrm{Hom}_\mathbb{Z}(A, B)$ the structure of a *right* Λ-module.

2.5. Interpret and prove the assertions $0_* = 0$, $0^* = 0$.

2.6. Compute $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_n)$, $\mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}_n)$, $\mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z})$, $\mathrm{Hom}(\mathbb{Q}, \mathbb{Z})$, $\mathrm{Hom}(\mathbb{Q}, \mathbb{Q})$. [Here "Hom" means "$\mathrm{Hom}_\mathbb{Z}$" and $\mathbb{Q}$ is the group of rationals.]

2.7. Show (see Exercise 1.7) that the sequence $0 \to A' \to A \to A'' \to 0$ is pure if and only if $\mathrm{Hom}(\mathbb{Z}_m, -)$ preserves exactness, for all $m > 0$.

2.8. If A is a left Λ-module and a right Γ-module such that the Λ-action commutes with the Γ-action, then A is called a left Λ-right *Γ-bimodule*. Show that if A is a left Λ-right Σ-bimodule and B is a left Λ-right Γ-bimodule then $\mathrm{Hom}_\Lambda(A, B)$ is naturally a left Σ-right Γ-bimodule.

3. Sums and Products

Let A and B be Λ-modules. We construct the *direct sum* $A \oplus B$ of A and B as the set of pairs (a, b) with $a \in A$ and $b \in B$ together with componentwise addition $(a, b) + (a', b') = (a + a', b + b')$ and componentwise Λ-operation $\lambda(a, b) = (\lambda a, \lambda b)$. Clearly, we have injective homomorphisms of Λ-modules $\iota_A: A \to A \oplus B$ defined by $\iota_A(a) = (a, 0)$ and $\iota_B: B \to A \oplus B$ defined by $\iota_B(b) = (0, b)$.

Proposition 3.1. *Let M be a Λ-module, $\psi_A: A \to M$ and $\psi_B: B \to M$ Λ-module homomorphisms. Then there exists a unique map*

$$\psi = \langle \psi_A, \psi_B \rangle : A \oplus B \to M$$

such that $\psi\iota_A = \psi_A$ and $\psi\iota_B = \psi_B$.

We can express Proposition 3.1 in the following way: For any Λ-module M and any maps ψ_A, ψ_B the diagram

$$\begin{tikzcd}
A \arrow[dr, "\iota_A"'] \arrow[drr, "\psi_A"] & & \\
& A \oplus B & M \\
B \arrow[ur, "\iota_B"] \arrow[urr, "\psi_B"'] & &
\end{tikzcd}$$

can be completed by a unique homomorphism $\psi: A\oplus B \to M$ such that the two triangles are commutative.

In situations like this where the existence of a map is claimed which makes a diagram commutative, we shall use a *dotted arrow* to denote this map. Thus the above assertion will be summarized by the diagram

$$\begin{tikzcd}
A \arrow[dr, "\iota_A"'] \arrow[drr, "\psi_A"] & & \\
& A\oplus B \arrow[r, dotted, "\psi"] & M \\
B \arrow[ur, "\iota_B"] \arrow[urr, "\psi_B"'] & &
\end{tikzcd}$$

and the remark that ψ is uniquely determined.

Proof. Define $\psi(a,b)=\psi_A(a)+\psi_B(b)$. This obviously is the only homomorphism $\psi: A\oplus B\to M$ satisfying $\psi\iota_A=\psi_A$ and $\psi\iota_B=\psi_B$. □

We can easily expand this construction to more than two modules: Let $\{A_j\}$, $j\in J$ be a family of Λ-modules indexed by J. We define the *direct sum* $\bigoplus_{j\in J} A_j$ of the modules A_j as follows: An element of $\bigoplus_{j\in J} A_j$ is a family $(a_j)_{j\in J}$ with $a_j\in A_j$ and $a_j\neq 0$ *for only a finite number of subscripts*. The addition is defined by $(a_j)_{j\in J}+(b_j)_{j\in J}=(a_j+b_j)_{j\in J}$ and the Λ-operation by $\lambda(a_j)_{j\in J}=(\lambda a_j)_{j\in J}$. For each $k\in J$ we can define *injections* $\iota_k: A_k\to \bigoplus_{j\in J} A_j$ by $\iota_k(a_k)=(b_j)_{j\in J}$ with $b_j=0$ for $j\neq k$ and $b_k=a_k$, $a_k\in A_k$.

Proposition 3.2. *Let M be a Λ-module and let $\{\psi_j: A_j\to M\}$, $j\in J$, be a family of Λ-module homomorphisms. Then there exists a unique homomorphism $\psi=\langle\psi_j\rangle: \bigoplus_{j\in J} A_j\to M$, such that $\psi\iota_j=\psi_j$ for all $j\in J$.*

Proof. We define $\psi((a_j)_{j\in J})=\sum_{j\in J}\psi_j(a_j)$. This is possible because $a_j=0$ except for a *finite* number of indices. The map ψ so defined is obviously the only homomorphism $\psi: \bigoplus_{j\in J} A_j\to M$ such that $\psi\iota_j=\psi_j$ for all $j\in J$. □

We remark the important fact that the property stated in Proposition 3.2 characterizes the direct sum together with the injections up to a unique isomorphism. To see this, let the Λ-module S together with injections $\iota_j': A_j\to S$ also have the property $\mathscr{P}$ claimed for $\left(\bigoplus_{j\in J} A_j;\ \iota_j\right)$ in Proposition 3.2. Write (temporarily) T for $\bigoplus_{j\in J} A_j$. First choose $M=T$ and $\psi_j=\iota_j$, $j\in J$. Since $(S;\iota_j')$ has property $\mathscr{P}$, there exists a unique homomorphism ψ: $S\to T$ such that the diagram

$$\begin{tikzcd}
A_j \arrow[dr, "\iota_j'"'] \arrow[drr, "\iota_j"] & & \\
& S \arrow[r, dotted, "\psi"'] & T
\end{tikzcd}$$

is commutative for every $j \in J$. Choosing $M = S$ and $\psi'_j = \iota'_j$ and invoking property $\mathscr{P}$ for $(T;\ \iota_j)$ we obtain a map $\psi' : T \to S$ such that the diagram

$$\begin{tikzcd} A_j \arrow[dr, "\iota_j"'] \arrow[drr, "\iota'_j"] & & \\ & T \arrow[r, dotted, "\psi'"'] & S \end{tikzcd}$$

is commutative for every $j \in J$. In order to show that $\psi\psi'$ is the identity, we remark that the diagram

$$\begin{tikzcd} A_j \arrow[dr, "\iota_j"'] \arrow[drr, "\iota_j"] & & \\ & T \arrow[r, dotted, shift left, "1_T"] \arrow[r, dotted, shift right, "\psi\psi'"'] & T \end{tikzcd}$$

is commutative for both $\psi\psi'$ and the identity. By the uniqueness part of property $\mathscr{P}$ we conclude that $\psi\psi' = 1_T$. Similarly we prove that $\psi'\psi = 1_S$. Thus both ψ and ψ' are isomorphisms.

A property like the one stated in Proposition 3.2 for the direct sum of modules is called *universal*. We shall treat these universal properties in detail in Chapter II. Here we are content to remark that the construction of the direct sum yields an *existence* proof for a module having property $\mathscr{P}$.

Next we define the *direct product* $\prod\limits_{j \in J} A_j$ of a family of modules $\{A_j\}, j \in J$. An element of $\prod\limits_{j \in J} A_j$ is a family $(a_j)_{j \in J}$ of elements $a_j \in A_j$. No restrictions are placed on the elements a_j; in particular, the elements a_j may be non-zero for an infinite number of subscripts. The addition is defined by $(a_j)_{j \in J} + (b_j)_{j \in J} = (a_j + b_j)_{j \in J}$ and the Λ-operation by $\lambda(a_j)_{j \in J} = (\lambda a_j)_{j \in J}$. For each $k \in J$ we can define *projections* $\pi_k : \prod\limits_{j \in J} A_j \to A_k$ by $\pi_k(a_j)_{j \in J} = a_k$.

For a finite family of modules $A_j, j = 1, \ldots, n$, it is readily seen that the *modules* $\prod\limits_{j=1}^{n} A_j$ and $\bigoplus\limits_{j=1}^{n} A_j$ are identical; however in considering the direct sum we put emphasis on the injections ι_j and in considering the direct product we put emphasis on the projections π_j.

Proposition 3.3. *Let M be a Λ-module and let $\{\varphi_j : M \to A_j\}$, $j \in J$, be a family of Λ-module homomorphisms. Then there exists a unique homomorphism $\varphi = \{\varphi_j\} : M \to \prod\limits_{j \in J} A_j$ such that for every $j \in J$ the diagram*

$$\begin{tikzcd} & A_j \\ M \arrow[ur, "\varphi_j"] \arrow[r, dotted, "\varphi"'] & \prod\limits_{j \in J} A_j \arrow[u, "\pi_j"'] \end{tikzcd}$$

is commutative, i.e. $\pi_j \varphi = \varphi_j$. $\square$

The proof is left to the reader; also the reader will see that the universal property of the direct product $\prod_{j \in J} A_j$ and the projections π_j characterizes it up to a unique isomorphism. Finally we prove

Proposition 3.4. *Let B be a Λ-module and $\{A_j\}, j \in J$ be a family of Λ-modules. Then there is an isomorphism*

$$\eta : \mathrm{Hom}_\Lambda\left(\bigoplus_{j \in J} A_j, B\right) \xrightarrow{\sim} \prod_{j \in J} \mathrm{Hom}_\Lambda(A_j, B)\,.$$

Proof. The proof reveals that this theorem is merely a restatement of the universal property of the direct sum. For $\psi : \bigoplus_{j \in J} A_j \to B$, define $\eta(\psi) = (\psi \iota_j : A_j \to B)_{j \in J}$. Conversely a family $\{\psi_j : A_j \to B\}$, $j \in J$, gives rise to a unique map $\psi : \bigoplus_{j \in J} A_j \to B$. The projections $\pi_j : \prod_{j \in J} \mathrm{Hom}_\Lambda(A_j, B) \to \mathrm{Hom}_\Lambda(A_j, B)$ are given by $\pi_j \eta = \mathrm{Hom}_\Lambda(\iota_j, B)$. □

Analogously one proves:

Proposition 3.5. *Let A be a Λ-module and $\{B_j\}$, $j \in J$ be a family of Λ-modules. Then there is an isomorphism*

$$\zeta : \mathrm{Hom}_\Lambda\left(A, \prod_{j \in J} B_j\right) \xrightarrow{\sim} \prod_{j \in J} \mathrm{Hom}_\Lambda(A, B_j)\,.$$

The proof is left to the reader. □

Exercises:

3.1. Show that there is a canonical map $\sigma : \bigoplus_j A_j \to \prod_j A_j$.

3.2. Show how a map from $\bigoplus_{i=1}^m A_i$ to $\bigoplus_{j=1}^n B_j$ may be represented by a matrix

$$\Phi = (\varphi_{ij})\,,$$

where $\varphi_{ij} : A_i \to B_j$. Show that, if we write the composite of $\varphi : A \to B$ and $\psi : B \to C$ as $\varphi\psi$ *(not $\psi\varphi$)*, then the composite of

$$\Phi = (\varphi_{ij}) : \bigoplus_{i=1}^m A_i \to \bigoplus_{j=1}^n B_j$$

and

$$\Psi = (\psi_{jk}) : \bigoplus_{j=1}^n B_j \to \bigoplus_{k=1}^q C_k$$

is the *matrix product* $\Phi\Psi$.

3.3. Show that if, in (1.2), α' is an isomorphism, then the sequence

$$0 \to A \xrightarrow{\{\varepsilon, \alpha\}} A'' \oplus B \xrightarrow{\langle \alpha'', -\varepsilon' \rangle} B'' \to 0$$

is exact. State and prove the converse.

3.4. Carry out a similar exercise to the one above, assuming α'' is an isomorphism.

3.5. Use the *universal property* of the direct sum to show that

$$(A_1 \oplus A_2) \oplus A_3 \cong A_1 \oplus (A_2 \oplus A_3).$$

3.6. Show that $\mathbb{Z}_m \oplus \mathbb{Z}_n = \mathbb{Z}_{mn}$ if and only if m and n are mutually prime.

3.7. Show that the following statements about the exact sequence

$$0 \to A' \xrightarrow{\alpha'} A \xrightarrow{\alpha''} A'' \to 0$$

of Λ-modules are equivalent:

(i) there exists $\mu: A'' \to A$ with $\alpha''\mu = 1$ on A'';

(ii) there exists $\varepsilon: A \to A'$ with $\varepsilon\alpha' = 1$ on A';

(iii) $0 \to \mathrm{Hom}_\Lambda(B, A') \xrightarrow{\alpha'_*} \mathrm{Hom}_\Lambda(B, A) \xrightarrow{\alpha''_*} \mathrm{Hom}_\Lambda(B, A'') \to 0$ is exact for all B;

(iv) $0 \to \mathrm{Hom}_\Lambda(A'', C) \xrightarrow{\alpha''^*} \mathrm{Hom}_\Lambda(A, C) \xrightarrow{\alpha'^*} \mathrm{Hom}_\Lambda(A', C) \to 0$ is exact for all C;

(v) there exists $\mu: A'' \to A$ such that $\langle \alpha', \mu \rangle : A' \oplus A'' \xrightarrow{\sim} A$.

3.8. Show that if $0 \to A' \xrightarrow{\alpha'} A \xrightarrow{\alpha''} A'' \to 0$ is pure and if A'' is a direct sum of cyclic groups then statement (i) above holds (see Exercise 2.7).

4. Free and Projective Modules

Let A be a Λ-module and let S be a subset of A. We consider the set A_0 of all elements $a \in A$ of the form $a = \sum_{s \in S} \lambda_s s$ where $\lambda_s \in \Lambda$ and $\lambda_s \neq 0$ for only a finite number of elements $s \in S$. It is trivially seen that A_0 is a submodule of A; hence it is the smallest submodule of A containing S.

If for the set S the submodule A_0 is the whole of A, we shall say that S is a set of *generators* of A. If A admits a finite set of generators it is said to be *finitely generated.* A set S of generators of A is called a *basis* of A if every element $a \in A$ may be expressed *uniquely* in the form $a = \sum_{s \in S} \lambda_s s$ with $\lambda_s \in \Lambda$ and $\lambda_s \neq 0$ for only a finite number of elements $s \in S$. It is readily seen that a set S of generators is a basis if and only if it is *linearly independent*, that is, if $\sum_{s \in S} \lambda_s s = 0$ implies $\lambda_s = 0$ for all $s \in S$. The reader should note that not every module possesses a basis.

Definition. If S is a basis of the Λ-module P, then P is called *free on the set* S. We shall call P *free* if it is free on some subset.

Proposition 4.1. *Suppose the Λ-module P is free on the set S. Then $P \cong \bigoplus_{s \in S} \Lambda_s$ where $\Lambda_s = \Lambda$ as a left module for $s \in S$. Conversely, $\bigoplus_{s \in S} \Lambda_s$ is free on the set $\{1_{\Lambda_s}, s \in S\}$.*

Proof. We define $\varphi: P \to \bigoplus_{s \in S} \Lambda_s$ as follows: Every element $a \in P$ is expressed uniquely in the form $a = \sum_{s \in S} \lambda_s s$; set $\varphi(a) = (\lambda_s)_{s \in S}$. Conversely, for $s \in S$ define $\psi_s : \Lambda_s \to P$ by $\psi_s(\lambda_s) = \lambda_s s$. By the universal property of the direct sum the family $\{\psi_s\}$, $s \in S$, gives rise to a map $\psi = \langle \psi_s \rangle : \bigoplus_{s \in S} \Lambda_s \to P$. It is readily seen that φ and ψ are inverse to each other. The remaining assertion immediately follows from the construction of the direct sum. □

The next proposition yields a universal characterization of the free module on the set S.

Proposition 4.2. *Let P be free on the set S. To every Λ-module M and to every function f from S into the set underlying M, there is a unique Λ-module homomorphism $\varphi : P \to M$ extending f.*

Proof. Let $f(s) = m_s$. Set $\varphi(a) = \varphi\left(\sum_{s \in S} \lambda_s s\right) = \sum_{s \in S} \lambda_s m_s$. This obviously is the only homomorphism having the required property. □

Proposition 4.3. *Every Λ-module A is a quotient of a free module P.*

Proof. Let S be a set of generators of A. Let $P = \bigoplus_{s \in S} \Lambda_s$ with $\Lambda_s = \Lambda$ and define $\varphi : P \to A$ to be the extension of the function f given by $f(1_{\Lambda_s}) = s$. Trivially φ is surjective. □

Proposition 4.4. *Let P be a free Λ-module. To every surjective homomorphism $\varepsilon : B \twoheadrightarrow C$ of Λ-modules and to every homomorphism $\gamma : P \to C$ there exists a homomorphism $\beta : P \to B$ such that $\varepsilon\beta = \gamma$.*

Proof. Let P be free on S. Since ε is surjective we can find elements $b_s \in B$, $s \in S$ with $\varepsilon(b_s) = \gamma(s)$, $s \in S$. Define β as the extension of the function $f : S \to B$ given by $f(s) = b_s$, $s \in S$. By the uniqueness part of Proposition 4.2 we conclude that $\varepsilon\beta = \gamma$. □

To emphasize the importance of the property proved in Proposition 4.4 we make the following remark: Let $A \overset{\mu}{\rightarrowtail} B \overset{\varepsilon}{\twoheadrightarrow} C$ be a short exact sequence of Λ-modules. If P is a free Λ-module Proposition 4.4 asserts that every homomorphism $\gamma : P \to C$ is induced by a homomorphism $\beta : P \to B$. Hence using Theorem 2.1 we can conclude that the induced sequence

$$0 \to \mathrm{Hom}_\Lambda(P, A) \xrightarrow{\mu_*} \mathrm{Hom}_\Lambda(P, B) \xrightarrow{\varepsilon_*} \mathrm{Hom}_\Lambda(P, C) \to 0 \tag{4.1}$$

is exact, i.e. that ε_* is surjective. Conversely, it is readily seen that exactness of (4.1) for all short exact sequences $A \rightarrowtail B \twoheadrightarrow C$ implies for the module P the property asserted in Proposition 4.4 for P a free module. Therefore there is considerable interest in the class of modules having this property. These are by definition the projective modules:

Definition. A Λ-module P is *projective* if to every surjective homomorphism $\varepsilon : B \twoheadrightarrow C$ of Λ-modules and to every homomorphism $\gamma : P \to C$ there exists a homomorphism $\beta : P \to B$ with $\varepsilon\beta = \gamma$. Equivalently, to any homomorphisms ε, γ with ε surjective in the diagram below there exists

β such that the triangle

$$\begin{tikzcd} & P \arrow[dl, dotted, "\beta"'] \arrow[d, "\gamma"] \\ B \arrow[r, two heads, "\varepsilon"] & C \end{tikzcd}$$

is commutative.

As mentioned above, every free module is projective. We shall give some more examples of projective modules at the end of this section.

Proposition 4.5. *A direct sum* $\bigoplus_{i\in I} P_i$ *is projective if and only if each* P_i *is.*

Proof. We prove the proposition only for $A = P \oplus Q$. The proof in the general case is analogous. First assume P and Q projective. Let $\varepsilon: B \twoheadrightarrow C$ be surjective and $\gamma: P \oplus Q \to C$ a homomorphism. Define $\gamma_P = \gamma\iota_P: P \to C$ and $\gamma_Q = \gamma\iota_Q: Q \to C$. Since P, Q are projective there exist β_P, β_Q such that $\varepsilon\beta_P = \gamma_P$, $\varepsilon\beta_Q = \gamma_Q$. By the universal property of the direct sum there exists $\beta: P \oplus Q \to B$ such that $\beta\iota_P = \beta_P$ and $\beta\iota_Q = \beta_Q$. It follows that $(\varepsilon\beta)\iota_P = \varepsilon\beta_P = \gamma_P = \gamma\iota_P$ and $(\varepsilon\beta)\iota_Q = \varepsilon\beta_Q = \gamma_Q = \gamma\iota_Q$. By the uniqueness part of the universal property we conclude that $\varepsilon\beta = \gamma$. Of course, this could be proved using the explicit *construction* of $P \oplus Q$, but we prefer to emphasize the universal property of the direct sum.

Next assume that $P \oplus Q$ is projective. Let $\varepsilon: B \twoheadrightarrow C$ be a surjection and $\gamma_P: P \to C$ a homomorphism. Choose $\gamma_Q: Q \to C$ to be the zero map. We obtain $\gamma: P \oplus Q \to C$ such that $\gamma\iota_P = \gamma_P$ and $\gamma\iota_Q = \gamma_Q = 0$. Since $P \oplus Q$ is projective there exists $\beta: P \oplus Q \to B$ such that $\varepsilon\beta = \gamma$. Finally we obtain $\varepsilon(\beta\iota_P) = \gamma\iota_P = \gamma_P$. Hence $\beta\iota_P: P \to B$ is the desired homomorphism. Thus P is projective; similarly Q is projective. $\square$

In Theorem 4.7 below we shall give a number of different characterizations of projective modules. As a preparation we define:

Definition. A short exact sequence $A \overset{\mu}{\rightarrowtail} B \overset{\varepsilon}{\twoheadrightarrow} C$ of Λ-modules *splits* if there exists a right inverse to ε, i.e. a homomorphism $\sigma: C \to B$ such that $\varepsilon\sigma = 1_C$. The map σ is then called a *splitting.*

We remark that the sequence $A \overset{\iota_A}{\rightarrowtail} A \oplus C \overset{\pi_C}{\twoheadrightarrow} C$ is exact, and splits by the homomorphism ι_C. The following lemma shows that all split short exact sequences of modules are of this form (see Exercise 3.7).

Lemma 4.6. *Suppose that* $\sigma: C \to B$ *is a splitting for the short exact sequence* $A \overset{\mu}{\rightarrowtail} B \overset{\varepsilon}{\twoheadrightarrow} C$. *Then* B *is isomorphic to the direct sum* $A \oplus C$. *Under this isomorphism,* μ *corresponds to* ι_A *and* σ *to* ι_C.

In this case we shall say that C (like A) is a *direct summand* in B.

Proof. By the universal property of the direct sum we define a map ψ as follows

$$\begin{tikzcd} A \arrow[dr, "\iota_A"'] \arrow[drr, "\mu"] & & \\ & A \oplus C \arrow[r, dotted, "\psi"] & B \\ C \arrow[ur, "\iota_C"] \arrow[urr, "\sigma"'] & & \end{tikzcd}$$

Then the diagram

$$\begin{tikzcd} A \arrow[r, tail, "\iota_A"] \arrow[d, equal] & A\oplus C \arrow[r, two heads, "\pi_C"] \arrow[d, "\psi"] & C \arrow[d, equal] \\ A \arrow[r, tail, "\mu"] & B \arrow[r, two heads, "\varepsilon"] & C \end{tikzcd}$$

is commutative; the left-hand square trivially is; the right-hand square is by $\varepsilon\psi(a, c)=\varepsilon(\mu a+\sigma c)=0+\varepsilon\sigma c=c$, and $\pi_C(a, c)=c$, $a\in A$, $c\in C$. By Lemma 1.1 ψ is an isomorphism. ▯

Theorem 4.7. *For a Λ-module P the following statements are equivalent:*

(1) *P is projective;*

(2) *for every short exact sequence $A \rightarrowtail^{\mu} B \overset{\varepsilon}{\twoheadrightarrow} C$ of Λ-modules the induced sequence*

$$0\to \mathrm{Hom}_\Lambda(P, A)\xrightarrow{\mu_*}\mathrm{Hom}_\Lambda(P, B)\xrightarrow{\varepsilon_*}\mathrm{Hom}_\Lambda(P, C)\to 0$$

is exact;

(3) *if $\varepsilon: B\twoheadrightarrow P$ is surjective, then there exists a homomorphism $\beta: P\to B$ such that $\varepsilon\beta=1_P$;*

(4) *P is a direct summand in every module of which it is a quotient;*

(5) *P is a direct summand in a free module.*

Proof. (1)⇒(2). By Theorem 2.1 we only have to show exactness at $\mathrm{Hom}_\Lambda(P, C)$, i.e. that ε_* is surjective. But since $\varepsilon: B\to C$ is surjective this is asserted by the fact that P is projective.

(2)⇒(3). Choose as exact sequence $\ker\varepsilon \rightarrowtail B \overset{\varepsilon}{\twoheadrightarrow} P$. The induced sequence

$$0\to \mathrm{Hom}_\Lambda(P, \ker\varepsilon)\to \mathrm{Hom}_\Lambda(P, B)\xrightarrow{\varepsilon_*}\mathrm{Hom}_\Lambda(P, P)\to 0$$

is exact. Therefore there exists $\beta: P\to B$ such that $\varepsilon\beta=1_P$.

(3)⇒(4). Let $P\cong B/A$, then we have an exact sequence $A\rightarrowtail B\overset{\varepsilon}{\twoheadrightarrow} P$. By (3) there exists $\beta: P\to B$ such that $\varepsilon\beta=1_P$. By Lemma 4.6 we conclude that P is a direct summand in B.

(4)⇒(5). By Proposition 4.3 P is a quotient of a free module P'. By (4) P is a direct summand in P'.

(5)⇒(1). By (5) $P'\cong P\oplus Q$, where P' is a free module. Since free modules are projective, it follows from Proposition 4.5 that P is projective. ▯

Next we list some examples:

(a) If $\Lambda=K$, a field, then every K-module is free, hence projective.

(b) By Exercise 2.1 and (2) of Theorem 4.7, $\mathbb{Z}_n$ is not projective as a module over the integers. Hence a finitely generated abelian group is projective if and only if it is free.

(c) Let $\Lambda=\mathbb{Z}_6$, the ring of integers modulo 6. Since $\mathbb{Z}_6=\mathbb{Z}_3\oplus\mathbb{Z}_2$ as a $\mathbb{Z}_6$-module, Proposition 4.5 shows that $\mathbb{Z}_2$ as well as $\mathbb{Z}_3$ are projective $\mathbb{Z}_6$-modules. However, they are plainly not free $\mathbb{Z}_6$-modules.

Exercises:

4.1. Let V be a vector space of countable dimension over the field K. Let $\Lambda = \mathrm{Hom}_K(V, V)$. Show that, as K-vector spaces V, is isomorphic to $V \oplus V$. We therefore obtain

$$\Lambda = \mathrm{Hom}_K(V, V) \cong \mathrm{Hom}_K(V \oplus V, V) \cong \mathrm{Hom}_K(V, V) \oplus \mathrm{Hom}_K(V, V) = \Lambda \oplus \Lambda .$$

Conclude that, in general, the free module on a set of n elements may be isomorphic to the free module on a set of m elements, with $n \neq m$.

4.2. Given two projective Λ-modules P, Q, show that there exists a *free* Λ-module R such that $P \oplus R \cong Q \oplus R$ is free. (Hint: Let $P \oplus P'$ and $Q \oplus Q'$ be free. Define $R = P' \oplus (Q \oplus Q') \oplus (P \oplus P') \oplus \cdots \cong Q' \oplus (P \oplus P') \oplus (Q \oplus Q') \oplus \cdots$.)

4.3. Show that $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.

4.4. Need a direct product of projective modules be projective?

4.5. Show that if $0 \to N \to P \to A \to 0$, $0 \to M \to Q \to A \to 0$ are exact with P, Q projective, then $P \oplus M \cong Q \oplus N$. (Hint: Use Exercise 3.4.)

4.6. We say that A has a *finite presentation* if there is a short exact sequence $0 \to N \to P \to A \to 0$ with P finitely-generated projective and N finitely-generated. Show that

(i) if A has a finite presentation, then, for every exact sequence

$$0 \to R \to S \to A \to 0$$

with S finitely-generated, R is also finitely-generated;

(ii) if A has a finite presentation, it has a finite presentation with P free;

(iii) if A has a finite presentation every presentation $0 \to N \to P \to A \to 0$ with P projective, N finitely-generated is finite, and every presentation $0 \to N \to P \to A \to 0$ with P finitely-generated projective is finite;

(iv) if A has a presentation $0 \to N_1 \to P_1 \to A \to 0$ with P_1 finitely-generated projective, and a presentation $0 \to N_2 \to P_2 \to A \to 0$ with P_2 projective, N_2 finitely-generated, then A has a finite presentation (indeed, both the given presentations are finite).

4.7. Let $\Lambda = K(x_1, \ldots, x_n, \ldots)$ be the polynomial ring in countably many indeterminates $x_1, \ldots, x_n, \ldots$ over the field K. Show that the ideal I generated by $x_1, \ldots, x_n, \ldots$ is not finitely generated. Hence we may have a presentation $0 \to N \to P \to A \to 0$ with P finitely generated projective and N *not* finitely-generated.

5. Projective Modules over a Principal Ideal Domain

Here we shall prove a rather difficult theorem about principal ideal domains. We remark that a very simple proof is available if one is content to consider only finitely generated Λ-modules; then the theorem forms a part of the fundamental classical theorem on the structure of finitely generated modules over principal ideal domains.

Recall that a principal ideal domain Λ is a commutative ring without divisors of zero in which every ideal is principal, i.e. generated by one element. It follows that as a module every ideal in Λ is isomorphic to Λ itself.

Theorem 5.1. *Over a principal ideal domain Λ every submodule of a free Λ-module is free.*

Since projective modules are direct summands in free modules, this implies

Corollary 5.2. *Over a principal ideal domain, every projective module is free.*

Corollary 5.3. *Over a principal ideal domain, every submodule of a projective module is projective.*

Proof of Theorem 5.1. Let $P = \bigoplus_{j \in J} \Lambda_j$, where $\Lambda_j = \Lambda$, be a free module and let R be a submodule of P. We shall show that R has a basis. Assume J well-ordered and define for every $j \in J$ modules

$$\bar{P}_{(j)} = \bigoplus_{i<j} \Lambda_i, \qquad P_{(j)} = \bigoplus_{i \leqq j} \Lambda_i .$$

Then every element $a \in P_{(j)} \cap R$ may be written uniquely in the form (b, λ) where $b \in \bar{P}_{(j)}$ and $\lambda \in \Lambda_j$. We define a homomorphism $f_j : P_{(j)} \cap R \to \Lambda$ by $f_j(a) = \lambda$. Since the kernel of f_j is $\bar{P}_{(j)} \cap R$ we obtain an exact sequence

$$\bar{P}_{(j)} \cap R \rightarrowtail P_{(j)} \cap R \twoheadrightarrow \operatorname{im} f_j .$$

Clearly $\operatorname{im} f_j$ is an ideal in Λ. Since Λ is a principal ideal domain, this ideal is generated by one element, say λ_j. For $\lambda_j \neq 0$ we choose $c_j \in P_{(j)} \cap R$, such that $f_j(c_j) = \lambda_j$. Let $J' \subseteqq J$ consist of those j such that $\lambda_j \neq 0$. We claim that the family $\{c_j\}, j \in J'$, is a basis of R.

First we show that $\{c_j\}, j \in J'$, is linearly independent. Let $\sum_{k=1}^{n} \mu_k c_{j_k} = 0$ and let $j_1 < j_2 < \cdots < j_n$. Then applying the homomorphism f_{j_n}, we get $\mu_n f_{j_n}(c_{j_n}) = \mu_n \lambda_{j_n} = 0$. Since $\lambda_{j_n} \neq 0$ this implies $\mu_n = 0$. The assertion then follows by induction on n.

Finally, we show that $\{c_j\}, j \in J'$, generates R. Assume the contrary. Then there is a least $i \in J$ such that there exists $a \in P_{(i)} \cap R$ which cannot be written as a linear combination of $\{c_j\}, j \in J'$. If $i \notin J'$, then $a \in \bar{P}_{(i)} \cap R$; but then there exists $k < i$ such that $a \in P_{(k)} \cap R$, contradicting the minimality of i. Thus $i \in J'$.

Consider $f_i(a) = \mu \lambda_i$ and form $b = a - \mu c_i$. Clearly

$$f_i(b) = f_i(a) - f_i(\mu c_i) = 0 .$$

Hence $b \in \bar{P}_{(i)} \cap R$, and b cannot be written as a linear combination of $\{c_j\}, j \in J'$. But there exists $k < i$ with $b \in P_{(k)} \cap R$, thus contradicting the minimality of i. Hence $\{c_j\}, j \in J'$, is a basis of R. □

Exercises:

5.1. Prove the following proposition, due to Kaplansky: Let Λ be a ring in which every left ideal is projective. Then every submodule of a free Λ-module is isomorphic to a direct sum of modules each of which is isomorphic to a left ideal in Λ. Hence every submodule of a projective module is projective. (Hint: Proceed as in the proof of Theorem 5.1.)

5.2. Prove that a submodule of a finitely-generated module over a principal ideal domain is finitely-generated. State the fundamental theorem for finitely-generated modules over principal ideal domains.

5.3. Let A, B, C be finitely generated modules over the principal ideal domain Λ. Show that if $A \oplus C \cong B \oplus C$, then $A \cong B$. Give counterexamples if one drops (a) the condition that the modules be finitely generated, (b) the condition that Λ is a principal ideal domain.

5.4. Show that submodules of projective modules need not be projective. ($\Lambda = \mathbb{Z}_{p^2}$, where p is a prime. $\mathbb{Z}_p \rightarrowtail \mathbb{Z}_{p^2} \twoheadrightarrow \mathbb{Z}_p$ is short exact but does not split!)

5.5. Develop a theory of linear transformations $T: V \to V$ of finite-dimensional vector spaces over a field K by utilizing the fundamental theorem in the integral domain $K[T]$.

6. Dualization, Injective Modules

We introduce here the process of dualization only as a heuristic procedure. However, we shall see in Chapter II that it is a special case of a more general and canonical procedure. Suppose given a statement involving only modules and homomorphisms of modules; for example, the characterization of the direct sum of modules by its universal property given in Proposition 3.2:

"The system consisting of the direct sum S of modules $\{A_j\}$, $j \in J$, together with the homomorphisms $\iota_j: A_j \to S$, is characterized by the following property. To any module M and homomorphisms $\{\psi_j: A_j \to M\}$, $j \in J$, there is a unique homomorphism $\psi: S \to M$ such that for every $j \in J$ the diagram

$$\begin{tikzcd} A_j \arrow[d, "\iota_j"'] \arrow[dr, "\psi_j"] & \\ S \arrow[r, dotted, "\psi"'] & M \end{tikzcd}$$

is commutative."

The *dual* of such a statement is obtained by "reversing the arrows"; more precisely, whenever in the original statement a homomorphism occurs we replace it by a homomorphism in the opposite direction. In our example the dual statement reads therefore as follows:

"Given a module T and homomorphisms $\{\pi_j: T \to A_j\}$, $j \in J$. To any module M and homomorphisms $\{\varphi_j: M \to A_j\}$, $j \in J$, there exists a unique homomorphism $\varphi: M \to T$ such that for every $j \in J$ the diagram

$$\begin{tikzcd} A_j & \\ T \arrow[u, "\pi_j"] & M \arrow[ul, "\varphi_j"'] \arrow[l, dotted, "\varphi"] \end{tikzcd}$$

is commutative."

It is readily seen that this is the universal property characterizing the direct product of modules $\{A_j\}$, $j \in J$, the π_j being the canonical projections (Proposition 3.3). We therefore say that the notion of the direct product is *dual* to the notion of the direct sum.

Clearly to dualize a given statement we have to express it entirely in terms of modules and homomorphisms (not elements etc.). This can be done for a great many – though not all – of the basic notions introduced in Sections 1, ..., 5. In the remainder of this section we shall deal with a very important special case in greater detail: We define the class of injective modules by a property dual to the defining property of projective modules. Since in our original definition of projective modules the term "surjective" occurs, we first have to find a characterization of surjective homomorphisms in terms of modules and homomorphisms only. This is achieved by the following definition and Proposition 6.1.

Definition. A module homomorphism $\varepsilon: B \to C$ is *epimorphic* or an *epimorphism* if $\alpha_1 \varepsilon = \alpha_2 \varepsilon$ implies $\alpha_1 = \alpha_2$ for any two homomorphisms $\alpha_i: C \to M$, $i = 1, 2$.

Proposition 6.1. *$\varepsilon: B \to C$ is epimorphic if and only if it is surjective.*

Proof. Let $B \xrightarrow{\varepsilon} C \overset{\alpha_1}{\underset{\alpha_2}{\rightrightarrows}} M$. If ε is surjective then clearly $\alpha_1 \varepsilon b = \alpha_2 \varepsilon b$ for all $b \in B$, implies $\alpha_1 c = \alpha_2 c$ for all $c \in C$. Conversely, suppose ε epimorphic and consider $B \xrightarrow{\varepsilon} C \overset{\pi}{\underset{0}{\rightrightarrows}} C/\varepsilon B$, where π is the canonical projection and 0 is the zero map. Since $0\varepsilon = 0 = \pi\varepsilon$, we obtain $0 = \pi$ and therefore $C/\varepsilon B = 0$ or $C = \varepsilon B$. □

Dualizing the above definition in the obvious way we have

Definition. The module homomorphism $\mu: A \to B$ is *monomorphic* or a *monomorphism* if $\mu\alpha_1 = \mu\alpha_2$ implies $\alpha_1 = \alpha_2$ for any two homomorphisms $\alpha_i: M \to A$, $i = 1, 2$.

Of course one expects that "monomorphic" means the same thing as "injective". For modules this is indeed the case; thus we have

Proposition 6.2. *$\mu: A \to B$ is monomorphic if and only if it is injective.*

Proof. If μ is injective, then $\mu\alpha_1 x = \mu\alpha_2 x$ for all $x \in M$ implies $\alpha_1 x = \alpha_2 x$ for all $x \in M$. Conversely, suppose μ monomorphic and $a_1, a_2 \in A$ such that $\mu a_1 = \mu a_2$. Choose $M = \Lambda$ and $\alpha_i: \Lambda \to A$ such that $\alpha_i(1) = a_i$, $i = 1, 2$. Then clearly $\mu\alpha_1 = \mu\alpha_2$; hence $\alpha_1 = \alpha_2$ and $a_1 = a_2$. □

It should be remarked here that from the categorical point of view (Chapter II) definitions should whenever possible be worded in terms of maps only. The basic notions therefore are "epimorphism" and "monomorphism", both of which are defined entirely in terms of maps. It is a fortunate coincidence that, for *modules*, "monomorphic" and "injective" on the one hand and "epimorphic" and "surjective" on the other hand mean the same thing. We shall see in Chapter II that in other categories monomorphisms do not have to be injective and epimorphisms do not have to be surjective. Notice that, to test whether a homomorphism is injective (surjective) one simply has to look at the homomorphism itself, whereas to test whether a homomorphism is monomorphic (epimorphic) one has, in principle, to consult all Λ-module homomorphisms.

We are now prepared to dualize the notion of a projective module.

Definition. A Λ-module I is called *injective* if for every homomorphism $\alpha: A \to I$ and every monomorphism $\mu: A \rightarrowtail B$ there exists a homomorphism $\beta: B \to I$ such that $\beta\mu = \alpha$, i.e. such that the diagram

$$\begin{tikzcd} A \arrow[r, tail, "\mu"] \arrow[d, "\alpha"'] & B \arrow[dl, dotted, "\beta"] \\ I & \end{tikzcd}$$

is commutative. Since μ may be regarded as an embedding, it is natural simply to say that I is injective if homomorphisms into I may be extended (from a given domain A to a larger domain B).

Clearly, one will expect that propositions about projective modules will dualize to propositions about injective modules. The reader must be warned, however, that even if the statement of a proposition is dualizable, the proof may *not* be. Thus it may happen that the dual of a true proposition turns out to be false. One must therefore give a proof of the dual proposition. One of the main objectives of Section 8 will, in fact, be to formulate and prove the dual of Theorem 4.7 (see Theorem 8.4). However, we shall need some preparation; first we state the dual of Proposition 4.5.

Proposition 6.3. *A direct product of modules* $\prod_{j \in J} I_j$ *is injective if and only if each* I_j *is injective.* $\square$

The reader may check that in this particular instance the proof of Proposition 4.5 is dualizable. We therefore leave the details to the reader.

Exercises:

6.1. (a) Show that the zero module 0 is characterized by the property: To any module M there exists precisely one homomorphism $\varphi: 0 \to M$.
(b) Show that the dual property also characterizes the zero module.

6.2. Give a universal characterization of kernel and cokernel, and show that kernel and cokernel are dual notions.

6.3. Dualize the assertions of Lemma 1.1, the Five Lemma (Exercise 1.2) and those of Exercises 3.4 and 3.5.

6.4. Let $\varphi: A \to B$. Characterize $\operatorname{im}\varphi$, $\varphi^{-1}B_0$ for $B_0 \subseteq B$, without using elements. What are their duals? Hence (or otherwise) characterize exactness.

6.5. What is the dual of the canonical homomorphism $\sigma: \bigoplus_{i\in J} A_i \to \prod_{i\in J} A_i$? What is the dual of the assertion that σ is an injection? Is the dual true?

7. Injective Modules over a Principal Ideal Domain

Recall that by Corollary 5.2 every projective module over a principal ideal domain is free. It is reasonable to expect that the injective modules over a principal ideal domain also have a simple structure. We first define:

Definition. Let Λ be an integral domain. A Λ-module D is *divisible* if for every $d \in D$ and every $0 \neq \lambda \in \Lambda$ there exists $c \in D$ such that $\lambda c = d$. Note that we do not require the uniqueness of c.

We list a few examples:

(a) As $\mathbb{Z}$-module the additive group of the rationals $\mathbb{Q}$ is divisible. In this example c is uniquely determined.

(b) As $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is divisible. Here c is *not* uniquely determined.

(c) The additive group of the reals $\mathbb{R}$, as well as $\mathbb{R}/\mathbb{Z}$, are divisible.

(d) A non-trivial finitely generated abelian group A is never divisible. Indeed, A is a direct sum of cyclic groups, which clearly are not divisible.

Theorem 7.1. *Let Λ be a principal ideal domain. A Λ-module is injective if and only if it is divisible.*

Proof. First suppose D is injective. Let $d \in D$ and $0 \neq \lambda \in \Lambda$. We have to show that there exists $c \in D$ such that $\lambda c = d$. Define $\alpha: \Lambda \to D$ by $\alpha(1) = d$ and $\mu: \Lambda \to \Lambda$ by $\mu(1) = \lambda$. Since Λ is an integral domain, $\mu(\xi) = \xi\lambda = 0$ if and only if $\xi = 0$. Hence μ is monomorphic. Since D is injective, there exists $\beta: \Lambda \to D$ such that $\beta\mu = \alpha$. We obtain

$$d = \alpha(1) = \beta\mu(1) = \beta(\lambda) = \lambda\beta(1).$$

Hence by setting $c = \beta(1)$ we obtain $d = \lambda c$. (Notice that so far no use is made of the fact that Λ is a principal ideal domain.)

Now suppose D is divisible. Consider the following diagram

$$\begin{tikzcd} A \arrow[r, tail, "\mu"] \arrow[d, "\alpha"'] & B \\ D & \end{tikzcd}$$

We have to show the existence of $\beta: B \to D$ such that $\beta\mu = \alpha$. To simplify the notation we consider μ as an embedding of a submodule A into B. We look at pairs (A_j, α_j) with $A \subseteqq A_j \subseteqq B$, $\alpha_j: A_j \to D$ such that $\alpha_j|_A = \alpha$. Let Φ be the set of all such pairs. Clearly Φ is nonempty, since (A, α) is in Φ. The relation $(A_j, \alpha_j) \leqq (A_k, \alpha_k)$ if $A_j \subseteqq A_k$ and $\alpha_k|_{A_j} = \alpha_j$ defines an ordering in Φ. With this ordering Φ is inductive. Indeed, every chain (A_j, α_j), $j \in J$ has an upper bound, namely $(\bigcup A_j, \bigcup \alpha_j)$ where $\bigcup A_j$ is simply the union, and $\bigcup \alpha_j$ is defined as follows: If $a \in \bigcup A_j$, then $a \in A_k$ for some $k \in J$. We define $\bigcup \alpha_j(a) = \alpha_k(a)$. Plainly $\bigcup \alpha_j$ is well-defined and is a homomorphism, and

$$(A_j, \alpha_j) \leqq \left(\bigcup A_j, \bigcup \alpha_j\right).$$

By Zorn's Lemma there exists a maximal element $(\bar{A}, \bar{\alpha})$ in Φ. We shall show that $\bar{A} = B$, thus proving the theorem. Suppose $\bar{A} \neq B$; then there exists $b \in B$ with $b \notin \bar{A}$. The set of $\lambda \in \Lambda$ such that $\lambda b \in \bar{A}$ is readily seen to be an ideal of Λ. Since Λ is a principal ideal domain, this ideal is generated by one element, say λ_0. If $\lambda_0 \neq 0$, then we use the fact that D is divisible to find $c \in D$ such that $\bar{\alpha}(\lambda_0 b) = \lambda_0 c$. If $\lambda_0 = 0$, we choose an arbitrary c. The homomorphism $\bar{\alpha}$ may now be extended to the module $\tilde{A}$ generated by $\bar{A}$ and b, by setting $\tilde{\alpha}(\bar{a} + \lambda b) = \bar{\alpha}(\bar{a}) + \lambda c$. We have to check that this definition is consistent. If $\lambda b \in \bar{A}$, we have $\tilde{\alpha}(\lambda b) = \lambda c$. But $\lambda = \xi\lambda_0$ for some $\xi \in \Lambda$ and therefore $\lambda b = \xi\lambda_0 b$. Hence

$$\bar{\alpha}(\lambda b) = \bar{\alpha}(\xi\lambda_0 b) = \xi\bar{\alpha}(\lambda_0 b) = \xi\lambda_0 c = \lambda c.$$

Since $(\bar{A}, \bar{\alpha}) < (\tilde{A}, \tilde{\alpha})$, this contradicts the maximality of $(\bar{A}, \bar{\alpha})$, so that $\bar{A} = B$ as desired. □

Proposition 7.2. *Every quotient of a divisible module is divisible.*

Proof. Let $\varepsilon: D \twoheadrightarrow E$ be an epimorphism and let D be divisible. For $e \in E$ and $0 \neq \lambda \in \Lambda$ there exists $d \in D$ with $\varepsilon(d) = e$ and $d' \in D$ with $\lambda d' = d$. Setting $e' = \varepsilon(d')$ we have $\lambda e' = \lambda\varepsilon(d') = \varepsilon(\lambda d') = \varepsilon(d) = e$. □

As a corollary we obtain the dual of Corollary 5.3.

Corollary 7.3. *Let Λ be a principal ideal domain. Every quotient of an injective Λ-module is injective.* □

Next we restrict ourselves temporarily to abelian groups and prove in that special case

Proposition 7.4. *Every abelian group may be embedded in a divisible (hence injective) abelian group.*

The reader may compare this Proposition to Proposition 4.3, which says that every Λ-module is a quotient of a free, hence projective, Λ-module.

Proof. We shall define a monomorphism of the abelian group A into a direct product of copies of $\mathbb{Q}/\mathbb{Z}$. By Proposition 6.3 this will suffice. Let $0 \neq a \in A$ and let (a) denote the subgroup of A generated by a. Define $\alpha : (a) \to \mathbb{Q}/\mathbb{Z}$ as follows: If the order of $a \in A$ is infinite choose $0 \neq \alpha(a)$ arbitrary. If the order of $a \in A$ is finite, say n, choose $0 \neq \alpha(a)$ to have order dividing n. Since $\mathbb{Q}/\mathbb{Z}$ is injective, there exists a map $\beta_a : A \to \mathbb{Q}/\mathbb{Z}$ such that the diagram

$$\begin{tikzcd} (a) \arrow[r, tail] \arrow[d, "\alpha"'] & A \arrow[dl, dotted, "\beta_a"] \\ \mathbb{Q}/\mathbb{Z} & \end{tikzcd}$$

is commutative. By the universal property of the product, the β_a define a unique homomorphism $\beta : A \to \prod_{\substack{a \in A \\ a \neq 0}} (\mathbb{Q}/\mathbb{Z})_a$. Clearly β is a monomorphism since $\beta_a(a) \neq 0$ if $a \neq 0$. □

For abelian groups, the additive group of the integers $\mathbb{Z}$ is projective and has the property that to any abelian group $G \neq 0$ there exists a nonzero homomorphism $\varphi : \mathbb{Z} \to G$. The group $\mathbb{Q}/\mathbb{Z}$ has the dual properties; it is injective and to any abelian group $G \neq 0$ there is a nonzero homomorphism $\psi : G \to \mathbb{Q}/\mathbb{Z}$. Since a direct sum of copies of $\mathbb{Z}$ is called free, we shall term a direct product of copies of $\mathbb{Q}/\mathbb{Z}$ *cofree*. Note that the two properties of $\mathbb{Z}$ mentioned above do *not* characterize $\mathbb{Z}$ entirely. Therefore "cofree" is *not* the exact dual of "free", it is dual only in certain respects. In Section 8 the generalization of this concept to arbitrary rings is carried through.

Exercises:

7.1. Prove the following proposition: The Λ-module I is injective if and only if for every left ideal $J \subset \Lambda$ and for every Λ-module homomorphism $\alpha : J \to I$ the diagram

$$\begin{tikzcd} J \arrow[r, tail] \arrow[d, "\alpha"'] & \Lambda \arrow[dl, dotted, "\beta"] \\ I & \end{tikzcd}$$

may be completed by a homomorphism $\beta : \Lambda \to I$ such that the resulting triangle is commutative. (Hint: Proceed as in the proof of Theorem 7.1.)

7.2. Let $0 \to R \to F \to A \to 0$ be a short exact sequence of abelian groups, with F free. By embedding F in a direct sum of copies of $\mathbb{Q}$, show how to embed A in a divisible group.

7.3. Show that every abelian group admits a unique maximal divisible subgroup.

7.4. Show that if A is a finite abelian group, then $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) \cong A$. Deduce that if there is a short exact sequence $0 \to A' \to A \to A'' \to 0$ of abelian groups with A finite, then there is a short exact sequence $0 \to A'' \to A \to A' \to 0$.

7.5. Show that a torsion-free divisible group D is a $\mathbb{Q}$-vector space. Show that $\mathrm{Hom}_{\mathbb{Z}}(A, D)$ is then also divisible. Is this true for any divisible group D?

7.6. Show that $\mathbb{Q}$ is a direct summand in a direct product of copies of $\mathbb{Q}/\mathbb{Z}$.

8. Cofree Modules

Let A be a right Λ-module and let G be an abelian group. Regarding A as an abelian group we can form the abelian group $\mathrm{Hom}_{\mathbf{Z}}(A, G)$ of homomorphisms from A into G. Using the *right* Λ-module structure of A we define in $\mathrm{Hom}_{\mathbf{Z}}(A, G)$ a left Λ-module structure as follows:

$$(\lambda\varphi)(a) = \varphi(a\lambda), \qquad a \in A,\ \lambda \in \Lambda,\ \varphi \in \mathrm{Hom}_{\mathbf{Z}}(A, G).$$

We leave it to the reader to verify the axioms. Similarly if A is a left Λ-module, $\mathrm{Hom}_{\mathbf{Z}}(A, G)$ acquires the structure of a right Λ-module.

Proposition 8.1. *Let A be a left Λ-module and let G be an abelian group. Regard $\mathrm{Hom}_{\mathbf{Z}}(\Lambda, G)$ as a left Λ-module via the right Λ-module structure of Λ. Then there is an isomorphism of abelian groups*

$$\eta = \eta_A : \mathrm{Hom}_\Lambda(A, \mathrm{Hom}_{\mathbf{Z}}(\Lambda, G)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Z}}(A, G).$$

Moreover, for every Λ-module homomorphism $\alpha: A \to B$ the diagram

$$\begin{tikzcd}
\mathrm{Hom}_\Lambda(B, \mathrm{Hom}_{\mathbf{Z}}(\Lambda, G)) \arrow[r, "\eta_B"] \arrow[d, "\alpha^*"] & \mathrm{Hom}_{\mathbf{Z}}(B, G) \arrow[d, "\alpha^*"] \\
\mathrm{Hom}_\Lambda(A, \mathrm{Hom}_{\mathbf{Z}}(\Lambda, G)) \arrow[r, "\eta_A"] & \mathrm{Hom}_{\mathbf{Z}}(A, G)
\end{tikzcd} \tag{8.1}$$

is commutative. (In this situation we shall say that η is natural.)

Proof. Let $\varphi: A \to \mathrm{Hom}_{\mathbf{Z}}(\Lambda, G)$ be a Λ-module homomorphism. We define a homomorphism of abelian groups $\varphi': A \to G$ by

$$\varphi'(a) = (\varphi(a))(1), \qquad a \in A.$$

Conversely, a homomorphism of abelian groups $\psi: A \to G$ gives rise to $\psi': A \to \mathrm{Hom}_{\mathbf{Z}}(\Lambda, G)$ by $(\psi'(a))(\lambda) = \psi(\lambda a)$, $a \in A$, $\lambda \in \Lambda$. Clearly ψ' is a homomorphism of abelian groups. We have to show that ψ' is a homomorphism of Λ-modules. Indeed, let $\zeta \in \Lambda$, then $(\psi'(\zeta a))(\lambda) = \psi(\lambda\zeta a)$; on the other hand $(\zeta(\psi'(a)))(\lambda) = (\psi'(a))(\lambda\zeta) = \psi(\lambda\zeta a)$. Clearly, $\varphi \mapsto \varphi'$ and $\psi \mapsto \psi'$ are homomorphisms of abelian groups. Finally, we claim $(\varphi')' = \varphi$ and $(\psi')' = \psi$. Indeed, $(\psi')'(a) = (\psi'(a))(1) = \psi(a)$, and

$$((\varphi')'(a))(\lambda) = \varphi'(\lambda a) = (\varphi(\lambda a))(1),$$

but

$$(\varphi(\lambda a))(1) = (\lambda(\varphi(a)))(1) = (\varphi(a))(1\lambda) = (\varphi(a))(\lambda),$$

since φ is a Λ-module homomorphism. Thus we define η by setting $\eta(\varphi) = \varphi'$, and η is an isomorphism. The naturality of η, i.e. the commutativity of the diagram (8.1), is evident. Notice that α^* on the right of the diagram (8.1) is a homomorphism of right Λ-modules. $\square$

We now look at $\Lambda^* = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Q}/\mathbb{Z})$, which is made into a left Λ-module using the right Λ-module structure of Λ. We claim that Λ^* has the property that to any nonzero Λ-module A there is a nonzero homomorphism $\varphi: A \to \Lambda^*$. Indeed, any nonzero homomorphism of abelian groups $\psi: A \to \mathbb{Q}/\mathbb{Z}$ will correspond by Proposition 8.1 to a nonzero $\varphi: A \to \Lambda^*$. Also, it will follow from Theorem 8.2 below that Λ^* is injective (set $G = \mathbb{Q}/\mathbb{Z}$). We therefore define

Definition. A Λ-module is *cofree* if it is the direct product of modules $\Lambda^* = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Q}/\mathbb{Z})$. Note that this is consistent with the description of $\mathbb{Q}/\mathbb{Z}$ as a cofree group, since $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$.

Theorem 8.2. *Let G be a divisible abelian group. Then $\bar{\Lambda} = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, G)$ is an injective Λ-module.*

Proof. Let $\mu: A \to B$ be a monomorphism of Λ-modules, and let $\alpha: A \to \bar{\Lambda}$ a homomorphism of Λ-modules. We have to show that there exists $\beta: B \to \bar{\Lambda}$ such that $\beta\mu = \alpha$. To prove this, we remark that $\alpha: A \to \bar{\Lambda}$ corresponds by Proposition 8.1 to a homomorphism of abelian groups $\alpha': A \to G$. Since G is injective, there exists $\beta': B \to G$ such that $\beta'\mu = \alpha'$. Under the inverse of η in Proposition 8.1 we obtain a homomorphism of Λ-modules $\beta: B \to \bar{\Lambda}$. Finally by the naturality of η, the diagram

$$\begin{tikzcd} A \arrow[r, tail, "\mu"] \arrow[d, "\alpha"'] & B \arrow[dl, dotted, "\beta"] \\ \bar{\Lambda} & \end{tikzcd}$$

is commutative. $\square$

We are now prepared to prove the dual of Proposition 4.3.

Proposition 8.3. *Every Λ-module A is a submodule of a cofree, hence injective, Λ-module.*

Proof. Let $0 \neq a \in A$ and let (a) denote the submodule of A generated by a. By the remarks preceeding Theorem 8.2 there exists a nonzero Λ-homomorphism $\alpha: (a) \to \Lambda^*$. Since Λ^* is injective there exists $\beta_a: A \to \Lambda^*$ such that the diagram

$$\begin{tikzcd} (a) \arrow[r, tail] \arrow[d, "\alpha"'] & A \arrow[dl, dotted, "\beta_a"] \\ \Lambda^* & \end{tikzcd}$$

is commutative. By the universal property of the direct product the β_a define a homomorphism $\beta: A \to \prod_{\substack{a \in A \\ a \neq 0}} (\Lambda_a^*)$, where $\Lambda_a^* = \Lambda^*$. Clearly β is monomorphic. $\square$

We conclude this section by dualizing Theorem 4.7.

Theorem 8.4. *For a Λ-module I the following statements are equivalent:*

(1) *I is injective;*

(2) *for every exact sequence $A \overset{\mu}{\rightarrowtail} B \overset{\varepsilon}{\twoheadrightarrow} C$ of Λ-modules the induced sequence*

$$0 \to \mathrm{Hom}_\Lambda(C, I) \xrightarrow{\varepsilon^*} \mathrm{Hom}_\Lambda(B, I) \xrightarrow{\mu^*} \mathrm{Hom}_\Lambda(A, I) \to 0$$

is exact;

(3) *if $\mu: I \rightarrowtail B$ is a monomorphism, then there exists $\beta: B \to I$ such that $\beta\mu = 1_I$;*

(4) *I is a direct summand in every module which contains I as submodule;*

(5) *I is a direct summand in a cofree module.*

The proof is dual to the proof of Theorem 4.7. For the step (3)$\Rightarrow$(4) one needs the dual of Lemma 4.6. The details are left to the reader. □

Note that, to preserve duality, one should really speak of "direct *factor*" in (4) and (5), rather than "direct summand". However, the two notions coincide!

Exercises:

8.1. Complete the proof of Theorem 8.4.

8.2. Let A be a Λ-module and let G be a divisible abelian group containing A. Show that we may embed A in an injective module by the scheme

$$A = \mathrm{Hom}_\Lambda(\Lambda, A) \subseteq \mathrm{Hom}_\mathbb{Z}(\Lambda, A) \subseteq \mathrm{Hom}_\mathbb{Z}(\Lambda, G).$$

(You should check that we obtain an embedding of Λ-modules.)

8.3. For any Λ-module A, let A^* be the right Λ-module $\mathrm{Hom}_\mathbb{Z}(A, \mathbb{Q}/\mathbb{Z})$. Show that A is naturally embedded in A^{**}. Use this embedding and a free presentation of A^* to embed A in a cofree module.

8.4. Suppose given $0 \to A \to I_1 \to J_1 \to 0$, $0 \to A \to I_2 \to J_2 \to 0$, with I_1, I_2 injective. Show that $I_1 \oplus J_2 \cong I_2 \oplus J_1$. To what statement is this dual?

8.5. State a property of injective modules which you suspect may not hold for arbitrary divisible modules.

9. Essential Extensions

In this section we shall show that to a given Λ-module A there exists an injective module E containing A such that every injective module containing A also contains an isomorphic copy of E. This property will define E up to isomorphism. E is called the *injective envelope* of A. We remark (see Exercise 9.5) that the dual of the injective envelope ("projective cover") does *not* exist in general.

Definition. A monomorphism $\mu: A \rightarrowtail B$ is called *essential* if for any submodule H of B, $H \neq 0$ implies $H \cap \mu A \neq 0$. If A is regarded as a submodule of B then B is called an *essential extension* of A (see [12]).

Examples. (a) As abelian group $\mathbb{Q}$ is an essential extension of $\mathbb{Z}$.

(b) The module $B = A \oplus C$ can never be an essential extension of A, unless $C = 0$. For $C \cap A = 0$.

Note that if B is an essential extension of A, and C is an essential extension of B, then C is an essential extension of A.

Proposition 9.1. *B is an essential extension of A if and only if, for every $0 \neq b \in B$, there exists $\lambda \in \Lambda$ such that $\lambda b \in A$ and $\lambda b \neq 0$.*

Proof. Let B be an essential extension of A, and let H be the submodule generated by $b \in B$. Since $H \neq 0$ it follows that $H \cap A \neq 0$, i.e. there exists $\lambda \in \Lambda$ such that $0 \neq \lambda b \in A$. Conversely, let H be a nontrivial submodule of B. For $0 \neq h \in H$ there exists $\lambda \in \Lambda$ such that $0 \neq \lambda h \in A$. Therefore $H \cap A \neq 0$, and B is an essential extension of A. $\square$

Let A be a submodule of a Λ-module M. Consider the set Φ of essential extensions of A, contained in M. Since A is an essential extension of itself, Φ is not empty. Under inclusion, Φ is inductive. Indeed, if $\{E_j\}$, $j \in J$, is a chain of essential extensions of A contained in M, then it follows easily from Proposition 9.1 that their union $\bigcup_{j \in J} E_j$ is again an essential extension of A contained in M. By Zorn's Lemma there exists a maximal essential extension E of A which is contained in M.

Theorem 9.2. *Let A be a submodule of the injective module I. Let E be a maximal essential extension of A contained in I. Then E is injective.*

Proof. First we show that E does not admit *any* nontrivial essential monomorphism.

Let $\mu: E \rightarrowtail X$ be an essential monomorphism. Since I is injective, there exists a homomorphism $\xi: X \to I$ completing the diagram

$$\begin{tikzcd} E \arrow[r, tail, "\mu"] \arrow[d, tail] & X \arrow[dl, dotted, "\xi"] \\ I & \end{tikzcd}$$

We show that ξ is monomorphic. Let H be the kernel of ξ. We then have $H \subseteq X$ and $H \cap \mu E = 0$. Hence $\ker \xi = H = 0$, for μ is essential. It follows that ξX is an essential extension of A contained in I. Since E is maximal, it follows that $X = E$.

Now consider the set Ψ of submodules $H \subseteq I$ such that $H \cap E = 0$. Since $0 \in \Psi$, Ψ is nonempty, it is ordered by inclusion and inductive. Hence by Zorn's Lemma there exists a maximal submodule $\bar{H}$ of I such that $\bar{H} \cap E = 0$. The canonical projection $\pi: I \twoheadrightarrow I/\bar{H}$ induces a monomorphism $\sigma = \pi|_E : E \rightarrowtail I/\bar{H}$. We shall show that σ is essential. Let $H/\bar{H}$ be a nontrivial submodule of $I/\bar{H}$, i.e. let $\bar{H} \subset H \subseteq I$ where the first inclusion is strict. By the maximality of $\bar{H}$ the intersection $H \cap E$ is nontrivial, hence $H/\bar{H} \cap \sigma E$ is nontrivial. It follows that σ is essential. By the first part of the proof E admits no proper essential monomorphism, whence it follows that $\sigma : E \xrightarrow{\sim} I/\bar{H}$ is an isomorphism. The sequence $\bar{H} \rightarrowtail I \xrightarrow{\sigma^{-1}\pi} E$ now splits by the embedding of E in I. Therefore E is a direct summand in I and is injective by Proposition 6.3. □

Corollary 9.3. *Let E_1, E_2 be two maximal essential extensions of A contained in injective modules I_1, I_2. Then $E_1 \cong E_2$ and every injective module I containing A also contains a submodule isomorphic to E_1.*

Definition. E_1 is called the *injective envelope* of A.

Proof. Consider the diagram

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, tail] & E_1 \\ E_2 & \end{tikzcd}$$

Since E_2 is injective there exists $\xi : E_1 \to E_2$ completing the diagram. As in the proof of Theorem 9.2 one shows that ξ is monomorphic. But then E_2, as an essential extension of A, is also an essential extension of E_1, which shows, again as in the proof of Theorem 9.2, that $\xi : E_1 \xrightarrow{\sim} E_2$. The proof of the second part is now trivial. □

Exercises:

9.1. Compute the injective envelope of $\mathbb{Z}$, $\mathbb{Z}_p$, p prime, $\mathbb{Z}_n$.

9.2. Show that if B_i is an essential extension of A_i, $i = 1, 2$, then $B_1 \oplus B_2$ is an essential extension of $A_1 \oplus A_2$. Extend this to direct sums over any index set J.

9.3. Given any abelian group A, let $T(A)$ be its torsion subgroup and $F(A) = A/T(A)$. Show that $\varphi : A \to B$ induces $T(\varphi) : T(A) \to T(B)$, $F(\varphi) : F(A) \to F(B)$, and that φ is a monomorphism if and only if $T(\varphi)$ and $F(\varphi)$ are monomorphisms. Show that the monomorphism φ is essential, if and only if $T(\varphi)$ and $F(\varphi)$ are essential. Now suppose given

$$\begin{tikzcd} 0 \arrow[r] & T(A) \arrow[r] \arrow[d, tail] & A \arrow[r] \arrow[d, tail] & F(A) \arrow[r] \arrow[d, tail] & 0 \\ 0 \arrow[r] & T(B) \arrow[r] \arrow[d, tail] & B \arrow[r] \arrow[d, tail] & F(B) \arrow[r] \arrow[d, tail] & 0 \\ 0 \arrow[r] & T(C) \arrow[r] & C \arrow[r] & F(C) \arrow[r] & 0 \end{tikzcd}$$

where $A \subseteq C$ is to be regarded as fixed, and B is an essential extension of A. Show that if $T(B)$ and $F(B)$ are maximal, so is B. Show that if B is maximal, so is $T(B)$, but that $F(B)$ may fail to be maximal.

Show that if C is divisible, so are $T(C)$ and $F(C)$. What does this tell us about the injective envelope of $T(A)$, A and $F(A)$?

9.4. Give a procedure for calculating the maximal essential extension of A in B, where B is a finitely generated abelian group.

9.5. Show that the dual of an injective envelope does not always exist. That is, given a Λ-module A, we cannot in general find $P \overset{\varepsilon}{\twoheadrightarrow} A, P$ projective, such that, given $Q \overset{\eta}{\twoheadrightarrow} A$, Q projective, we may factor η as $Q \overset{\varepsilon'}{\twoheadrightarrow} P \overset{\varepsilon}{\twoheadrightarrow} A$. (Hint: Take $\Lambda = \mathbb{Z}$, $A = \mathbb{Z}_5$.) Where does the dual argument fail?

II. Categories and Functors

In Chapter I we discussed various algebraic structures (rings, abelian groups, modules) and their appropriate transformations (homomorphisms). We also saw how certain constructions (for example, the formation of $\mathrm{Hom}_\Lambda(A, B)$ for given Λ-modules A, B) produced new structures out of given structures. Over and above this we introduced certain "universal" constructions (direct sum, direct product) and suggested that they constituted special cases of a general, and important, procedure. Our objective in this chapter is to establish the appropriate mathematical language for the general description of mathematical systems and of mappings of systems, insofar as that language is applicable to homological algebra.

The language of categories and functors was first introduced by Eilenberg and MacLane [13] to provide a precise description of the processes involved in algebraic topology. Since then an independent mathematical theory has grown up around the basic concepts of the language and today the development, elaboration and application of this theory constitute an extremely active area of mathematical research. It is not our intention to give a treatment of this developing theory; the reader who wishes to pursue the topic of categorical algebra is referred to the texts [6, 18, 35, 37–39] for further reading. Indeed, the reader familiar with the elements of categorical algebra may use this chapter simply as a source of relevant facts, terminology and notation.

1. Categories

To define a *category* $\mathfrak{C}$ we must give three pieces of data:

(1) a class of *objects* $A, B, C, \ldots$,

(2) to each pair of objects A, B of $\mathfrak{C}$, a set $\mathfrak{C}(A, B)$ of *morphisms from* A *to* B,

(3) to each triple of objects A, B, C of $\mathfrak{C}$, a *law of composition*

$$\mathfrak{C}(A, B) \times \mathfrak{C}(B, C) \to \mathfrak{C}(A, C).$$

Before giving the axioms which a category must satisfy we introduce some auxiliary notation: this should also serve to relate our terminology and notation with ideas which are already very familiar. If $f \in \mathfrak{C}(A, B)$ we may think of the morphism f as a generalized "function" from A to B and write

$$f : A \to B \quad \text{or} \quad A \xrightarrow{f} B ;$$

we call f a morphism from the *domain* A to the *codomain* (or *range*) B. The set $\mathfrak{C}(A, B) \times \mathfrak{C}(B, C)$ consists, of course, of pairs (f, g) where $f : A \to B$, $g : B \to C$ and we will write the composition of f and g as $g \circ f$ or, simply, gf. The rationale for this notation (see the Introduction) lies in the fact that if A, B, C are sets and f, g are functions then the composite function from A to C is the function h given by

$$h(a) = g(f(a)), \qquad a \in A .$$

Thus if the function symbol is written to the *left* of the argument symbol one is naturally led to write $h = fg$. (Of course it will turn out that sets, functions and function-composition do constitute a category.)

We are now ready to state the *axioms*. The first is really more of a convention, the latter two being much more substantial.

A 1: The sets $\mathfrak{C}(A_1, B_1)$, $\mathfrak{C}(A_2, B_2)$ are disjoint unless $A_1 = A_2$, $B_1 = B_2$.

A 2: Given $f : A \to B$, $g : B \to C$, $h : C \to D$, then

$$h(gf) = (hg)f \qquad (\textit{Associative} \text{ law of composition}).$$

A 3: To each object A there is a morphism $1_A : A \to A$ such that, for any $f : A \to B$, $g : C \to A$,

$$f 1_A = f, \qquad 1_A g = g \qquad (\text{Existence of } \textit{identities}).$$

It is easy to see that the morphism 1_A is uniquely determined by Axiom **A 3**. We call 1_A the *identity morphism* of A, and we will often suppress the suffix A, writing simply

$$f1 = f, \qquad 1g = g .$$

As remarked, and readily verified, the category $\mathfrak{S}$ of sets, functions and function-composition satisfies the axioms. We often refer to the *category of sets* $\mathfrak{S}$; indeed, more generally, in describing a category we omit reference to the law of composition when the morphisms are functions and composition is ordinary function-composition (or when, for some other reason, the law of composition is evident), and we even omit reference to the nature of the morphisms if the context, or custom, makes their nature obvious.

A word is necessary about the significance of Axiom **A 1.** Let us consider this axiom in $\mathfrak{S}$. It is standard practice today to distinguish two functions if their *domains* are distinct, even if they take the same values whenever they are both defined. Thus the sine function $\sin : \mathbb{R} \to \mathbb{R}$

is distinguished from its extension $\sin: \mathbb{C} \to \mathbb{C}$ to the complex field. However, the two functions

$$\sin: \mathbb{R} \to \mathbb{R}, \qquad \sin: \mathbb{R} \to [-1, 1]$$

would normally be regarded as the *same* function, although we have assigned to them different *codomains*. However we will see that it is useful – indeed, essential – in homological algebra to distinguish morphisms unless their (explicitly specified) domains *and* codomains coincide.

It is also crucial in topology. Suppose $f_1: X \to Y_1$, $f_2: X \to Y_2$ are two continuous functions which in fact take the same values, i.e., $f_1(x) = f_2(x)$, $x \in X$. Then it may well happen that one of those functions is contractible whereas the other is not. Take, as an example, $X = S^1$, the unit circle in $\mathbb{R}^2$, f_1 the embedding of X in $\mathbb{R}^2$ and f_2 the embedding of X in $\mathbb{R}^2$—(0). Then f_1 is contractible, while f_2 is not, so that certainly f_1 and f_2 should be distinguished.

Notice also that the composition gf is only defined if the codomain of f *coincides* with the domain of g.

We say that a morphism $f: A \to B$ in $\mathfrak{C}$ is *isomorphic* (or *invertible*) if there exists a morphism $g: B \to A$ in $\mathfrak{C}$ such that

$$gf = 1_A, \qquad fg = 1_B.$$

It is plain that g is then itself invertible and is uniquely determined by f; we write $g = f^{-1}$, so that

$$(f^{-1})^{-1} = f.$$

It is also plain that the composite of two invertible morphisms is again invertible and thus the relation

$$A \equiv B \quad \text{if there exists an invertible} \quad f: A \to B$$

(A is *isomorphic* to B) is an equivalence relation on the objects of the category $\mathfrak{C}$. This relation has special names in different categories (one-one correspondence of sets, isomorphism of groups, homeomorphism of spaces), but it is important to observe that it is a *categorical* concept.

We now list several examples of categories.

(a) The category $\mathfrak{S}$ of sets and functions;

(b) the category $\mathfrak{T}$ of topological spaces and continuous functions;

(c) the category $\mathfrak{G}$ of groups and homomorphisms;

(d) the category $\mathfrak{Ab}$ of abelian groups and homomorphisms;

(e) the category $\mathfrak{B}_F$ of vector spaces over the field F and linear transformations;

(f) the category $\mathfrak{G}_c$ of topological groups and continuous homomorphisms;

(g) the category $\mathfrak{R}$ of rings and ring-homomorphisms;

(h) the category $\mathfrak{R}_1$ of rings-with-unity-element and ring-homomorphisms preserving unity-element;

(i) the category $\mathfrak{M}^l_\Lambda$ of left Λ-modules, where Λ is an object of $\mathfrak{R}_1$, and module-homomorphisms;

(j) the category $\mathfrak{M}^r_\Lambda$ of right Λ-modules.

Plainly the list could be continued indefinitely. Plainly also each category carries its appropriate notion of invertible morphisms and isomorphic objects. In all the examples given the morphisms are structure-preserving *functions;* however, it is important to emphasize that the morphisms of a category need not be functions, even when the objects of the category are sets perhaps with additional structure. To give one example, consider the category $\mathfrak{T}_h$ of spaces and *homotopy classes* of continuous functions. Since the homotopy class of a composite function depends only on the homotopy classes of its factors it is evident that $\mathfrak{T}_h$ is a category – but the morphisms are not themselves functions. Other examples will be found in Exercises 1.1, 1.2.

Returning to our list of examples, we remark that in examples c, d, e, f, g, i, j the category $\mathfrak{C}$ in question possesses an object 0 with the property that, for any object X in $\mathfrak{C}$, the sets $\mathfrak{C}(X,0)$ and $\mathfrak{C}(0,X)$ both consist of precisely one element.

Thus in $\mathfrak{G}$ and $\mathfrak{Ab}$ we may take for 0 any one-element group. It is easy to prove that, if $\mathfrak{C}$ possesses such an object 0, called a *zero object*, then any two such objects are isomorphic and $\mathfrak{C}(X,Y)$ then possesses a distinguished morphism,

$$X \to 0 \to Y,$$

called the *zero morphism* and written 0_{XY}. For any $f: W \to X$, $g: Y \to Z$ in $\mathfrak{C}$ we have

$$0_{XY} f = 0_{WY}, \quad g 0_{XY} = 0_{XZ}.$$

As with the identity morphism, so with the zero morphism 0_{XY}, we will usually suppress the indices and simply write 0. If $\mathfrak{C}$ possesses zero objects it is called a *category with zero objects.*

If we turn to example (a) of the category $\mathfrak{S}$ then we notice that, given any set X, $\mathfrak{S}(\emptyset, X)$ consists of just one element (where $\emptyset$ is the empty set) and $\mathfrak{S}(X,(p))$ consists of just one element (where (p) is a one-element set). Thus in $\mathfrak{S}$ there is an *initial* object $\emptyset$ and a *terminal* (or *coinitial)* object (p), but no zero object. The reader should have no difficulty in providing precise definitions of initial and terminal objects in a category $\mathfrak{C}$, and will readily prove that all initial objects in a category $\mathfrak{C}$ are isomorphic and so, too, are all terminal objects.

The final notion we introduce in this section is that of a *subcategory* $\mathfrak{C}_0$ of a given category $\mathfrak{C}$. The reader will readily provide the explicit definition; of particular importance among the subcategories of $\mathfrak{C}$ are the *full* subcategories, that is, those subcategories $\mathfrak{C}_0$ of $\mathfrak{C}$ such that

$$\mathfrak{C}_0(A, B) = \mathfrak{C}(A, B)$$

for any objects A, B of $\mathfrak{C}_0$. For example, $\mathfrak{Ab}$ is a full subcategory of $\mathfrak{G}$, but $\mathfrak{R}_1$ is a subcategory of $\mathfrak{R}$ which is not full.

Exercises:

1.1. Show how to represent an ordered set as a category. (Hint: Regard the elements $a, b, \ldots$ of the set as objects in the category, and the instances $a \leqq b$ of the ordering relation as morphisms $a \to b$.) Express in categorical language the fact that the ordered set is directed [16]. Show that a subset of an ordered set, with its natural ordering, is a full subcategory.

1.2. Show how to represent a group as a category with a single object, all morphisms being invertible. Show that a subcategory is then precisely a submonoid. When is the subcategory full?

1.3. Show that the category of groups has a generator. (A *generator* U of a category $\mathfrak{C}$ is an object such that if $f, g : X \to Y$ in $\mathfrak{C}$, $f \neq g$, then there exists $u : U \to X$ with $fu \neq gu$.)

1.4. Show that, in the category of groups, there is a one-one correspondence between elements of G and morphisms $\mathbb{Z} \to G$.

1.5. Carry out exercises analogous to Exercises 1.3, 1.4 for the category of sets, the category of spaces, the category of pointed spaces (i.e. each space has a base-point and morphisms are to preserve base-points, see [21]).

1.6. Set out in detail the natural definition of the *Cartesian product* $\mathfrak{C}_1 \times \mathfrak{C}_2$ of two categories $\mathfrak{C}_1, \mathfrak{C}_2$.

1.7. Show that if a category has a zero object, then every initial object, and every terminal object, is isomorphic to that zero object. Deduce that the category of sets has no zero object.

2. Functors

Within a category $\mathfrak{C}$ we have the morphism sets $\mathfrak{C}(X, Y)$ which serve to establish connections between different objects of the category. Now the language of categories has been developed to delineate the various areas of mathematical theory; thus it is natural that we should wish to be able to describe connections between different categories. We now formulate the notion of a transformation from one category to another. Such a transformation is called a *functor;* thus, precisely, a functor $F : \mathfrak{C} \to \mathfrak{D}$ is a rule which associates with every object X of $\mathfrak{C}$ an object FX of $\mathfrak{D}$ and with every morphism f in $\mathfrak{C}(X, Y)$ a morphism Ff in $\mathfrak{D}(FX, FY)$, subject to the rules

$$F(fg) = (Ff)(Fg), \qquad F(1_A) = 1_{FA}. \tag{2.1}$$

The reader should be reminded, in studying (2.1), of rules governing homomorphisms of familiar algebraic systems. He should also observe that we have evidently the notion of an *identity* functor and of the composition of functors. Composition is associative and we may thus pass to *invertible* functors and *isomorphic* categories.

We now list several examples of functors. The reader will need to establish the necessary facts and complete the descriptions of the functors.

(a) The *embedding* of a subcategory $\mathfrak{C}_0$ in a category $\mathfrak{C}$ is a functor.

(b) Let G be any group and let G/G' be its *abelianized* group, i.e. the quotient of G by its commutator subgroup G'. Then $G \mapsto G/G'$ induces the abelianizing functor $\mathrm{Abel}: \mathfrak{G} \to \mathfrak{G}$. Of course this functor may also be regarded as a functor $\mathfrak{G} \to \mathfrak{Ab}$. This example enables us to exhibit, once more, the importance of being precise about specifying the codomain of a morphism. Consider the groups $G = C_3$, the cyclic group of order 3 generated by t, say, and $H = S_3$, the symmetric group on three symbols. Let $\varphi: G \to H$ be given by $\varphi(t) = (1\,2\,3)$, the cyclic permutation. Let H_0 be the subgroup of H generated by $(1\,2\,3)$ and let $\varphi_0: G \to H_0$ be given by $\varphi_0(t) = (1\,2\,3)$. It may well appear pedantic to distinguish φ_0 from φ but we justify the distinction when we apply the abelianizing functor $\mathrm{Abel}: \mathfrak{G} \to \mathfrak{G}$. For plainly $\mathrm{Abel}(G) = G$, $\mathrm{Abel}(H_0) = H_0$, $\mathrm{Abel}(\varphi_0) = \varphi_0$, which is an *isomorphism*. On the other hand, H_0 is the commutator subgroup of H, so that $\mathrm{Abel}(H) = H/H_0$ and so $\mathrm{Abel}(\varphi) = 0$, the constant homomorphism (or zero morphism) $G \to H/H_0$ ($\cong C_2$). Thus $\mathrm{Abel}(\varphi)$ and $\mathrm{Abel}(\varphi_0)$ are utterly different!

(c) Let S be a set and let $F(S)$ be the free abelian group on S as basis. This construction yields the *free functor* $F: \mathfrak{S} \to \mathfrak{Ab}$. Similarly there are free functors $\mathfrak{S} \to \mathfrak{G}$, $\mathfrak{S} \to \mathfrak{V}_F$, $\mathfrak{S} \to \mathfrak{M}_\Lambda^l$, $\mathfrak{S} \to \mathfrak{M}_\Lambda^r$, etc.

(d) Underlying every topological space there is a set. Thus we get an *underlying* functor $U: \mathfrak{T} \to \mathfrak{S}$. Similarly there are underlying functors from all the examples (a) to (j) of categories (in Section 1) to $\mathfrak{S}$. There are also underlying functors $\mathfrak{M}_\Lambda^l \to \mathfrak{Ab}$, $\mathfrak{M}_\Lambda^r \to \mathfrak{Ab}$, $\mathfrak{R} \to \mathfrak{Ab}$, etc., in which some structure is "forgotten" or "thrown away".

(e) The fundamental group may be regarded as a functor $\pi: \mathfrak{T}^0 \to \mathfrak{G}$, where $\mathfrak{T}^0$ is the category of spaces-with-base-point (see [21]). It may also be regarded as a functor $\bar{\pi}: \mathfrak{T}_h^0 \to \mathfrak{G}$, where the subscript h indicates that the morphisms are to be regarded as (based) homotopy classes of (based) continuous functions. Indeed there is an evident *classifying* functor $Q: \mathfrak{T}^0 \to \mathfrak{T}_h^0$ and then π factors as $\pi = \bar{\pi} Q$.

(f) Similarly the (singular) homology groups are functors $\mathfrak{T} \to \mathfrak{Ab}$ (or $\mathfrak{T}_h \to \mathfrak{Ab}$).

(g) We saw in Chapter I how the set $\mathfrak{M}_\Lambda^l(A, B) = \mathrm{Hom}_\Lambda(A, B)$ may be given the structure of an abelian group. If we hold A fixed and define

$\mathfrak{M}_\Lambda^l(A,-): \mathfrak{M}_\Lambda^l \to \mathfrak{Ab}$ by

$$\mathfrak{M}_\Lambda^l(A,-)(B) = \mathfrak{M}_\Lambda^l(A,B),$$

then $\mathfrak{M}_\Lambda^l(A,-)$ is a functor. More generally, for any category $\mathfrak{C}$ and object A of $\mathfrak{C}$, $\mathfrak{C}(A,-)$ is a functor from $\mathfrak{C}$ to $\mathfrak{S}$. We say that this functor is *represented* by A. It is an important question whether a given functor (usually to $\mathfrak{S}$) may be represented in this sense by an object of the category.

In viewing the last example the reader will have noted an asymmetry. We have recognized $\mathfrak{M}_\Lambda^l(A,-)$ as a functor $\mathfrak{M}_\Lambda^l \to \mathfrak{Ab}$, but if we look at the corresponding construct $\mathfrak{M}_\Lambda^l(-,B): \mathfrak{M}_\Lambda^l \to \mathfrak{Ab}$, we see that this is not a functor. For, writing F for $\mathfrak{M}_\Lambda^l(-,B)$, then F sends $f: A_1 \to A_2$ to $Ff: FA_2 \to FA_1$. This "reversal of arrows" turns up frequently in applications of categorical ideas and we now formalize the description.

Given any category $\mathfrak{C}$, we may form a new category $\mathfrak{C}^{\text{opp}}$, the category *opposite* to $\mathfrak{C}$. The objects of $\mathfrak{C}^{\text{opp}}$ are precisely those of $\mathfrak{C}$, but

$$\mathfrak{C}^{\text{opp}}(X,Y) = \mathfrak{C}(Y,X). \tag{2.2}$$

Then the composition in $\mathfrak{C}^{\text{opp}}$ is simply that which follows naturally from (2.2) and the law of composition in $\mathfrak{C}$. It is trivial to verify that $\mathfrak{C}^{\text{opp}}$ is a category with the same identity morphisms as $\mathfrak{C}$, and that if $\mathfrak{C}$ has zero objects, then the same objects are zero objects of $\mathfrak{C}^{\text{opp}}$. Moreover,

$$(\mathfrak{C}^{\text{opp}})^{\text{opp}} = \mathfrak{C}. \tag{2.3}$$

Of course the construction of $\mathfrak{C}^{\text{opp}}$ is merely a formal device. However it does enable us to express precisely the *contravariant* nature of $\mathfrak{M}_\Lambda^l(-,B)$ or, more generally, $\mathfrak{C}(-,B)$, and to formulate the concept of *categorical duality* (see Section 3).

Thus, given two categories $\mathfrak{C}$ and $\mathfrak{D}$ a *contravariant functor* from $\mathfrak{C}$ to $\mathfrak{D}$ is a functor from $\mathfrak{C}^{\text{opp}}$ to $\mathfrak{D}$. The reader should note that the effective difference between a functor as originally defined (often referred to as a *covariant* functor) and a contravariant functor is that, for a contravariant functor F from $\mathfrak{C}$ to $\mathfrak{D}$, F maps $\mathfrak{C}(X,Y)$ to $\mathfrak{D}(FY,FX)$ and (compare (2.1)) $F(fg) = F(g)F(f)$. We give the following examples of contravariant functors.

(a) $\mathfrak{C}(-,B)$, for B an object in $\mathfrak{C}$, is a contravariant functor from $\mathfrak{C}$ to $\mathfrak{S}$. Similarly, $\mathfrak{M}_\Lambda^l(-,B)$, $\mathfrak{M}_\Lambda^r(-,B)$ are contravariant functors from $\mathfrak{M}_\Lambda^l$, $\mathfrak{M}_\Lambda^r$ respectively to $\mathfrak{Ab}$. We say that these functors are *represented* by B.

(b) The (singular) cohomology groups are contravariant functors $\mathfrak{T} \to \mathfrak{Ab}$ (or $\mathfrak{T}_h \to \mathfrak{Ab}$).

(c) Let A be an object of $\mathfrak{M}_\Lambda^r$ and let G be an abelian group. We saw in Section I. 8 how to give $\operatorname{Hom}_{\mathbb{Z}}(A,G)$ the structure of a left Λ-module.

$\mathrm{Hom}_{\mathbb{Z}}(-, G)$ thus appears as a contravariant functor from $\mathfrak{M}^r_\Lambda$ to $\mathfrak{M}^l_\Lambda$. Further examples will appear as exercises.

Finally we make the following definitions. Recall from Section 1 the notion of a *full* subcategory. Consistent with that definition, we now define a functor $F: \mathfrak{C} \to \mathfrak{D}$ as *full* if F maps $\mathfrak{C}(A, B)$ *onto* $\mathfrak{D}(FA, FB)$ for all objects A, B in $\mathfrak{C}$, and as *faithful* if F maps $\mathfrak{C}(A, B)$ injectively to $\mathfrak{D}(FA, FB)$. Finally F is a *full embedding* if F is full and faithful and one-to-one on objects. Notice that then $F(\mathfrak{C})$ is a full subcategory of $\mathfrak{D}$ (in general, $F(\mathfrak{C})$ is not a category at all).

Exercises:

2.1. Regarding ordered sets as categories, identify functors from ordered sets to ordered sets, and to an arbitrary category $\mathfrak{C}$. Also interpret the opposite category. (See Exercise 1.1.)

2.2. Regarding groups as categories, identify functors from groups to groups. Show that the opposite of a group is isomorphic to the group.

2.3. Show that the center is *not* a functor $\mathfrak{G} \to \mathfrak{G}$ in any obvious way. Let $\mathfrak{G}_{\mathrm{epi}}$ be the subcategory of $\mathfrak{G}$ in which the morphisms are the *surjections*. Show that the center is a functor $\mathfrak{G}_{\mathrm{epi}} \to \mathfrak{G}$. Is it a functor $\mathfrak{G}_{\mathrm{epi}} \to \mathfrak{G}_{\mathrm{epi}}$?

2.4. Give examples of underlying functors.

2.5. Show that the composite of two functors is again a functor. (Discuss both covariant and contravariant functors.)

2.6. Let Φ associate with each commutative unitary ring R the set of its prime ideals. Show that Φ is a contravariant functor from the category of commutative unitary rings to the category of sets. Assign to the set of prime ideals of R the topology in which the closed sets are defined to be the sets of prime ideals containing a given ideal J, as J runs through the ideals of R. Show that Φ is then a contravariant functor to $\mathfrak{T}$.

2.7. Let $F: \mathfrak{C}_1 \times \mathfrak{C}_2 \to \mathfrak{D}$ be a functor from the Cartesian product $\mathfrak{C}_1 \times \mathfrak{C}_2$ to the category $\mathfrak{D}$ (see Exercise 1.6). F is then also called a *bifunctor* from $(\mathfrak{C}_1, \mathfrak{C}_2)$ to $\mathfrak{D}$. Show that, for each $C_1 \in \mathfrak{C}_1$, F determines a functor $F_{C_1}: \mathfrak{C}_2 \to \mathfrak{D}$ and, similarly, for each $C_2 \in \mathfrak{C}_2$, a functor $F_{C_2}: \mathfrak{C}_1 \to \mathfrak{D}$, such that, if $\varphi_1 : C_1 \to C_1'$, $\varphi_2 : C_2 \to C_2'$, then the diagram

$$\begin{tikzcd} F(C_1, C_2) \arrow[r, "F_{C_2}(\varphi_1)"] \arrow[d, "F_{C_1}(\varphi_2)"'] & F(C_1', C_2) \arrow[d, "F_{C_1'}(\varphi_2)"] \\ F(C_1, C_2') \arrow[r, "F_{C_2'}(\varphi_1)"] & F(C_1', C_2') \end{tikzcd} \qquad (*)$$

commutes. What is the diagonal of this diagram? Show conversely that if we have functors $F_{C_1}: \mathfrak{C}_2 \to \mathfrak{D}$, $F_{C_2}: \mathfrak{C}_1 \to \mathfrak{D}$, indexed by the objects of $\mathfrak{C}_1, \mathfrak{C}_2$ respectively, such that $F_{C_1}(C_2) = F_{C_2}(C_1)$ and $(*)$ commutes, then these families of functors determine a bifunctor $G: \mathfrak{C}_1 \times \mathfrak{C}_2 \to \mathfrak{D}$ such that $G_{C_1} = F_{C_1}, G_{C_2} = F_{C_2}$.

2.8. Show that $\mathfrak{C}(-, -): \mathfrak{C}^{\mathrm{opp}} \times \mathfrak{C} \to \mathfrak{S}$ is a bifunctor.

3. Duality

Our object in this section is to explain informally the duality principle in category theory. We first give an example taken from Section I. 6. We saw there that the *injective* homomorphisms in $\mathfrak{M}_\Lambda$ are precisely the *monomorphisms*, i.e. those morphisms μ such that for all α, β

$$\mu\alpha = \mu\beta \Rightarrow \alpha = \beta \,. \tag{3.1}$$

(The reader familiar with ring theory will notice the formal similarity with right-regularity.) Similarly the *surjective* homomorphisms in $\mathfrak{M}_\Lambda$ are precisely the *epimorphisms* in $\mathfrak{M}_\Lambda$, i.e. those morphisms ε such that for all α, β

$$\alpha\varepsilon = \beta\varepsilon \Rightarrow \alpha = \beta \,. \tag{3.2}$$

(The reader will notice that the corresponding concept in ring theory is left-regularity.) Now given any category, we *define* a monomorphism μ by (3.1) and an epimorphism ε by (3.2). It is then plain that, if φ is a morphism in $\mathfrak{C}$, then φ is a monomorphism in $\mathfrak{C}$ if and only if it is an epimorphism as a morphism of $\mathfrak{C}^{\mathrm{opp}}$. It then follows from (2.3) that a statement about epimorphisms and monomorphisms which is true in any category must remain true if the prefixes "epi-" and "mono-" are interchanged and "arrows are reversed". Let us take a trivial example. An easy argument establishes the fact that *if $\varphi\psi$ is monomorphic then ψ is monomorphic.* We may thus apply the "duality principle" to infer immediately that *if $\psi\varphi$ is epimorphic then ψ is epimorphic.* Indeed, the two italicized statements are logically equivalent – either stated for $\mathfrak{C}$ implies the other for $\mathfrak{C}^{\mathrm{opp}}$. It is superfluous to write down a proof of the second, once the first has been proved.

It is very likely that the reader will come better to appreciate the duality principle after meeting several examples of its applications. Nevertheless we will give a general statement of the principle; this statement will not be sufficiently formal to satisfy the canons of mathematical logic but will, we hope, be intelligible and helpful.

Let us consider a concept $\mathscr{C}$ (like monomorphism) which is meaningful in any category. Since the objects and morphisms of $\mathfrak{C}^{\mathrm{opp}}$ are those of $\mathfrak{C}$, it makes sense to apply the concept $\mathscr{C}$ to $\mathfrak{C}^{\mathrm{opp}}$ and then to *interpret the resulting statement in* $\mathfrak{C}$. This procedure leads to a new concept $\mathscr{C}^{\mathrm{opp}}$ which is related to $\mathscr{C}$ by the rule (writing $\mathscr{C}(\mathfrak{C})$ for the concept $\mathscr{C}$ applied to the category $\mathfrak{C}$)

$$\mathscr{C}^{\mathrm{opp}}(\mathfrak{C}) = \mathscr{C}(\mathfrak{C}^{\mathrm{opp}}) \quad \text{for any category } \mathfrak{C} \,.$$

Thus if $\mathscr{C}$ is the concept of monomorphism, $\mathscr{C}^{\mathrm{opp}}$ is the concept of epimorphism (compare (3.1), (3.2)). We may also say that $\mathscr{C}^{\mathrm{opp}}$ is obtained from $\mathscr{C}$ by "reversing arrows". This "arrow-reversing" procedure may thus be applied to definitions, axioms, statements, theorems ..., and hence also to proofs. Thus if one shows that a certain theorem $\mathscr{T}$ holds in any category $\mathfrak{C}$ satisfying certain additional axioms $A, B, \ldots$, then theorem $\mathscr{T}^{\text{opp}}$ holds in any category $\mathfrak{C}$ satisfying axioms $A^{\text{opp}}, B^{\text{opp}}, \ldots$. In particular if $\mathscr{T}$ holds in any category so does $\mathscr{T}^{\text{opp}}$.

This automatic process of dualizing is clearly extremely useful and convenient and will be much used in the sequel. However, the reader should be clear about the limitations in the scope of the duality principle. Suppose given a statement $\mathscr{S}_0$ about a *particular* category $\mathfrak{C}_0$, involving concepts $\mathscr{C}_{01}, \ldots, \mathscr{C}_{0k}$ expressed in terms of the objects and morphisms of $\mathfrak{C}_0$. For example, $\mathfrak{C}_0$ may be the category of groups and $\mathscr{S}_0$ may be the statement "A finite group of odd order is solvable". Now it may be possible to formulate a statement $\mathscr{S}$ about a general category $\mathfrak{C}$, and concepts $\mathscr{C}_1, \ldots, \mathscr{C}_k$, so that $\mathscr{S}(\mathfrak{C}_0), \mathscr{C}_1(\mathfrak{C}_0), \ldots, \mathscr{C}_k(\mathfrak{C}_0)$ are equivalent to $\mathscr{S}_0, \mathscr{C}_{01}, \ldots, \mathscr{C}_{0k}$ respectively. We may then dualize $\mathscr{S}, \mathscr{C}_1, \ldots, \mathscr{C}_k$, and interpret the resulting statement in the category $\mathfrak{C}_0$. Informally we may describe $\mathscr{S}^{\text{opp}}(\mathfrak{C}_0)$ as the dual of $\mathscr{S}_0$ but two warnings are in order:

(i) The passage from $\mathscr{S}_0$ to $\mathscr{S}$ is not single-valued; that is, there may well be several statements about a general category which specialize to the given statement $\mathscr{S}_0$ about the category $\mathfrak{C}_0$. Likewise of course, the concepts $\mathscr{C}_{01}, \mathscr{C}_{02}, \ldots, \mathscr{C}_{0k}$ may generalize in many different ways.

(ii) Even if $\mathscr{S}_0$ is provable in $\mathfrak{C}_0$, $\mathscr{S}^{\text{opp}}(\mathfrak{C}_0)$ may well be false in $\mathfrak{C}_0$.

However, *if $\mathscr{S}$ is provable*, then this constitutes a proof of $\mathscr{S}_0$ and of $\mathscr{S}^{\text{opp}}(\mathfrak{C}_0)$. (This does not prevent $\mathscr{S}^{\text{opp}}(\mathfrak{C}_0)$ from being vacuous, of course; we cannot guarantee that the dual in this informal sense is always interesting!)

As an example, consider the statement $\mathscr{S}_0$ "Every Λ-module is the quotient of a projective module". This is a statement about the category $\mathfrak{C}_0 = \mathfrak{M}_\Lambda^l$. Now there is a perfectly good concept of a projective object in any category $\mathfrak{C}$, based on the notion of an epimorphism. Thus (see Section 10) a projective object is an object P with the property that, given φ and ε,

$$\begin{tikzcd} & P \arrow[dl, dotted, "\theta"'] \arrow[d, "\varphi"] \\ A \arrow[r, "\varepsilon"] & B \end{tikzcd}$$

with ε epimorphic, there exists θ such that $\varepsilon\theta = \varphi$. We may formulate the statement $\mathscr{S}$, for any category $\mathfrak{C}$, whieh states that, given any object X in $\mathfrak{C}$ there is an epimorphism $\varepsilon: P \to X$ with P projective. Then $\mathscr{S}(\mathfrak{C}_0)$ is our original statement $\mathscr{S}_0$. We may now formulate $\mathscr{S}^{\text{opp}}$ which asserts that, given any object X in $\mathfrak{C}$ there is a monomorphism $\mu: X \to I$ with I injective (here "injective" is the evident concept dual to "projective"; the reader may easily formulate it explicitly). Then $\mathscr{S}^{\text{opp}}(\mathfrak{C}_0)$ is the statement "Every Λ-module may be embedded in an injective module". Now it happens (as we proved in Chapter I) that both $\mathcal{S}(\mathfrak{C}_0)$ and $\mathcal{S}^{\mathrm{opp}}(\mathfrak{C}_0)$ are true, but we cannot infer one from the other. For the right to do so would depend on our having a proof of $\mathcal{S}$ – and, *in general*, $\mathcal{S}$ is false.

We have said that, if $\mathcal{S}$ is provable then, of course, $\mathcal{S}(\mathfrak{C}_0)$ and $\mathcal{S}^{\mathrm{opp}}(\mathfrak{C}_0)$ are deducible. Clearly, though, this is usually too stringent a criterion; in other words, this principle does not permit us to deduce any but the most superficial of propositions about $\mathfrak{C}_0$, since it requires some statement to be true in any category. However, as suggested earlier, there is a refinement of the principle that does lead to practical results. Suppose we confine attention to categories satisfying certain conditions Q. Suppose moreover that these conditions are *self-dual* in the sense that, if any category $\mathfrak{C}$ satisfies Q, so does $\mathfrak{C}^{\mathrm{opp}}$, and suppose further that $\mathfrak{C}_0$ satisfies conditions Q. Suppose $\mathcal{S}$ is a statement meaningful for any category satisfying Q and suppose that $\mathcal{S}$ may be proved. Then we may infer both $\mathcal{S}(\mathfrak{C}_0)$ and $\mathcal{S}^{\mathrm{opp}}(\mathfrak{C}_0)$. This principle indicates the utility of proving $\mathcal{S}$ for the entire class of categories satisfying Q instead of merely for $\mathfrak{C}_0$. We will meet this situation in Section 9 when we come to discuss *abelian categories*.

Exercises:

3.1. Show that "epimorphic" means "surjective" and that "monomorphic" means "injective"
(i) in $\mathfrak{S}$, (ii) in $\mathfrak{T}$, (iii) in $\mathfrak{G}$.

3.2. Show that the inclusion $\mathbb{Z} \subsetneq \mathbb{Q}$ is an epimorphism in the category of integral domains. Generalize to other epimorphic non-surjections in this category.

3.3. Consider the underlying functor $U: \mathfrak{T} \to \mathfrak{S}$. Show that $j: X_0 \to X$ in $\mathfrak{T}$ is a homeomorphism of X_0 *into* X if and only if it is a monomorphism and, for any $f: Y \to X$ in $\mathfrak{T}$, a factorization $U(j)\, g_0 = U(f)$ in $\mathfrak{S}$ implies $j f_0 = f$ in $\mathfrak{T}$ with $g_0 = U(f_0)$. Dualize this categorical property of j and obtain a topological characterization of the dual categorical property.

3.4. Define the *kernel* of a morphism $\varphi: A \to B$ in a category with zero morphisms $\mathfrak{C}$ as a morphism $\mu: K \to A$ such that (i) $\varphi\mu = 0$, (ii) if $\varphi\psi = 0$, then $\psi = \mu\psi'$ and ψ' is unique. Identify the kernel, so defined, in $\mathfrak{Ab}$ and $\mathfrak{G}$. Dualize to obtain a definition of *cokernel* in $\mathfrak{C}$. Identify the cokernel in $\mathfrak{Ab}$ and $\mathfrak{G}$. Let $\mathfrak{S}^0$ be the category of sets with base points. Identify kernels and cokernels in $\mathfrak{S}^0$.

3.5. Generalize the definitions of kernel (and cokernel) above to *equalizers* (and *coequalizers*) of two morphisms $\varphi_1, \varphi_2: A \to B$. A morphism $\mu: E \to A$ is the *equalizer* of φ_1, φ_2 if (i) $\varphi_1\mu = \varphi_2\mu$, (ii) if $\varphi_1\psi = \varphi_2\psi$ then $\psi = \mu\psi'$ and ψ' is unique. Exhibit the kernel as an equalizer. Dualize.

4. Natural Transformations

We come now to the idea which deserves to be considered the original source of category theory, since it was in the (successful!) attempt to make precise the notion of a *natural transformation* that Eilenberg and MacLane were led to introduce the language of categories and functors (see [13]).

Let F, G be two functors from the category $\mathfrak{C}$ to the category $\mathfrak{D}$. Then a *natural transformation* t from F to G is a rule assigning to each object X in $\mathfrak{C}$ a morphism $t_X : FX \to GX$ in $\mathfrak{D}$ such that, for any morphism $f: X \to Y$ in $\mathfrak{C}$, the diagram

$$\begin{tikzcd} FX \arrow[r, "t_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "t_Y"] & GY \end{tikzcd}$$

commutes. If t_X is isomorphic for each X then t is called a *natural equivalence* and we write $F \simeq G$. It is plain that then $t^{-1} : G \simeq F$, where t^{-1} is given by $(t^{-1})_X = (t_X)^{-1}$. If $t : F \to G$, $u : G \to H$ are natural transformations then we may form the composition $ut : F \to H$, given by $(ut)_X = (u_X)(t_X)$; and the composition of natural transformations is plainly associative. Let $F : \mathfrak{C} \to \mathfrak{D}$, $G : \mathfrak{D} \to \mathfrak{C}$ be functors such that $GF \simeq I : \mathfrak{C} \to \mathfrak{C}$, $FG \simeq I : \mathfrak{D} \to \mathfrak{D}$, where I stands for the identity functor in any category. We then say that $\mathfrak{C}$ and $\mathfrak{D}$ are *equivalent categories*. Of course, isomorphic categories are equivalent, but equivalent categories need not be isomorphic (see Exercise 4.1). We now give some examples of natural transformations; we draw particular attention to the first example which refers to the first explicitly observed example of a natural transformation.

(a) Let V be a vector space over the field F, let V^* be the dual vector space and V^{**} the double dual. There is a linear map $\iota_V : V \to V^{**}$ given by $v \mapsto \tilde{v}$ where $\tilde{v}(\varphi) = \varphi(v)$, $v \in V$, $\varphi \in V^*$, $\tilde{v} \in V^{**}$. The reader will verify that ι is a natural transformation from the identity functor $I : \mathfrak{B}_F \to \mathfrak{B}_F$ to the double dual functor $** : \mathfrak{B}_F \to \mathfrak{B}_F$. Now let $\mathfrak{B}_F^f$ be the full subcategory of $\mathfrak{B}_F$ consisting of *finite-dimensional* vector spaces. It is then, of course, a basic theorem of linear algebra that ι, restricted to $\mathfrak{B}_F^f$, is a *natural equivalence*. (More accurately, the classical theorem says that ι_V is an isomorphism for each V in $\mathfrak{B}_F^f$.) The proof proceeds by observing that $V \cong V^*$ if V is finite-dimensional. However, this last isomorphism is not natural – to define it one needs to choose a basis for V and then to associate with this basis the dual basis of V^*. That is, the isomorphism between V and V^* depends on the choice of basis and lacks the canonical nature of the isomorphism ι_V between V and V^{**}.

(b) Let G be a group and let G/G' be its commutator factor group. There is an evident surjection $\kappa_G : G \to G/G'$ and κ is a natural transformation from the identity functor $\mathfrak{G} \to \mathfrak{G}$ to the abelianizing functor $\mathrm{Abel} : \mathfrak{G} \to \mathfrak{G}$.

(c) Let A be an abelian group and let A_F be the free abelian group on the *set* A as basis. There is an evident surjection $\tau_A : A_F \to A$, which maps the basis elements of A_F identically, and τ is a natural transformation from FU to I, where $U: \mathfrak{Ab} \to \mathfrak{S}$ is the underlying functor and $F: \mathfrak{S} \to \mathfrak{Ab}$ is the free functor.

(d) The Hurewicz homomorphism from homotopy groups to homology groups (see e.g. [21]) may be interpreted as a natural transformation of functors $\mathfrak{T}^0 \to \mathfrak{Ab}$ (or $\mathfrak{T}_h^0 \to \mathfrak{Ab}$).

We continue with the following important remark. Given two categories $\mathfrak{C}$, $\mathfrak{D}$, the reader is certainly tempted to regard the functors $\mathfrak{C} \to \mathfrak{D}$ as the objects of a new category with the natural transformations as morphisms. The one difficulty about this point of view is that it is not clear from a foundational viewpoint that the natural transformations of functors $\mathfrak{C} \to \mathfrak{D}$ form a *set*. This objection may be circumvented by adopting a set-theoretical foundation different from ours (see [32]) or simply by insisting that the collection of objects of $\mathfrak{C}$ form a set; such a category $\mathfrak{C}$ is called a *small* category. Thus if $\mathfrak{C}$ is small we may speak of the *category of functors* (or *functor category*) from $\mathfrak{C}$ to $\mathfrak{D}$ which we denote by $\mathfrak{D}^{\mathfrak{C}}$ or $[\mathfrak{C}, \mathfrak{D}]$. In keeping with this last notation we will denote the collection of natural transformations from the functor F to the functor G by $[F, G]$.

We illustrate the notion of the category of functors with the following example. Let $\mathfrak{C}$ be the category with two objects and identity morphisms only. A functor $F: \mathfrak{C} \to \mathfrak{D}$ is then simply a pair of objects in $\mathfrak{D}$, and a natural transformation $t: F \to G$ is a pair of morphisms in $\mathfrak{D}$. Thus it is seen that $\mathfrak{D}^{\mathfrak{C}} = [\mathfrak{C}, \mathfrak{D}]$ is the Cartesian product of the category $\mathfrak{D}$ with itself, that is the category $\mathfrak{D} \times \mathfrak{D}$ in the notation of Exercise 1.6.

We close this section with an important proposition. We have seen that, if A, B are objects of a category $\mathfrak{C}$, then $\mathfrak{C}(A, -)$ is a (covariant) functor $\mathfrak{C} \to \mathfrak{S}$ and $\mathfrak{C}(-, B)$ is a contravariant functor $\mathfrak{C} \to \mathfrak{S}$. If $\theta: B_1 \to B_2$ let us write θ_* for $\mathfrak{C}(A, \theta): \mathfrak{C}(A, B_1) \to \mathfrak{C}(A, B_2)$, so that

$$\theta_*(\varphi) = \theta\varphi\,, \qquad \varphi: A \to B_1\,,$$

and if $\psi: A_2 \to A_1$ let us write ψ^* for $\mathfrak{C}(\psi, B): \mathfrak{C}(A_1, B) \to \mathfrak{C}(A_2, B)$ so that

$$\psi^*(\varphi) = \varphi\psi\,, \qquad \varphi: A_1 \to B\,.$$

These notational simplifications should help the reader to understand the proof of the following proposition.

Proposition 4.1. *Let τ be a natural transformation from the functor $\mathfrak{C}(A, -)$ to the functor F from $\mathfrak{C}$ to $\mathfrak{S}$. Then $\tau \mapsto \tau_A(1_A)$ sets up a one-one correspondence between the set $[\mathfrak{C}(A, -), F]$ of natural transformations from $\mathfrak{C}(A, -)$ to F and the set $F(A)$.*

Proof. We show first that τ is entirely determined by the element $\tau_A(1_A) \in F(A)$. Let $\varphi: A \to B$ and consider the commutative diagram

$$\begin{array}{ccc} \mathfrak{C}(A,A) & \xrightarrow{\varphi_*} & \mathfrak{C}(A,B) \\ \downarrow{\scriptstyle \tau_A} & & \downarrow{\scriptstyle \tau_B} \\ FA & \xrightarrow{F\varphi} & FB \end{array}$$

Then $\tau_B(\varphi) = (\tau_B)(\varphi_*)(1_A) = (F\varphi)(\tau_A)(1_A)$, proving the assertion. The proposition is therefore established if we show that, for any $\kappa \in FA$, the rule

$$\tau_B(\varphi) = (F\varphi)(\kappa), \qquad \varphi \in \mathfrak{C}(A,B), \tag{4.1}$$

does define a natural transformation from $\mathfrak{C}(A,-)$ to F. Let $\theta: B_1 \to B_2$ and consider the diagram

$$\begin{array}{ccc} \mathfrak{C}(A,B_1) & \xrightarrow{\theta_*} & \mathfrak{C}(A,B_2) \\ \downarrow{\scriptstyle \tau_{B_1}} & & \downarrow{\scriptstyle \tau_{B_2}} \\ FB_1 & \xrightarrow{F\theta} & FB_2 \end{array}$$

We must show that this diagram commutes if τ_{B_1}, τ_{B_2} are defined as in (4.1). Now $(\tau_{B_2})\,\theta_*(\varphi) = (\tau_{B_2})(\theta\varphi) = F(\theta\varphi)(\kappa) = F(\theta)\,F(\varphi)(\kappa) = F(\theta)\,\tau_{B_1}(\varphi)$ for $\varphi: A \to B_1$. Thus the proposition is completely proved. □

By choosing $F = \mathfrak{C}(A',-)$ we obtain

Corollary 4.2. *The set of morphisms $\mathfrak{C}(A',A)$ and the set of natural transformations $[\mathfrak{C}(A,-), \mathfrak{C}(A',-)]$ are in one-to-one correspondence, the correspondence being given by $\psi \mapsto \psi^*$, $\psi: A' \to A$.*

Proof. If τ is such a natural transformation, let $\psi = \tau_A(1_A)$, so that $\psi: A' \to A$. Then, by (4.1) τ is given by

$$\tau_B(\varphi) = \varphi_*(\psi) = \varphi\psi = \psi^*(\varphi).$$

Thus $\tau_B = \psi^*$. Of course ψ is uniquely determined by τ and every ψ does induce a natural transformation $\mathfrak{C}(A,-) \to \mathfrak{C}(A',-)$. Thus the rule $\tau \mapsto \tau_A(1_A)$ sets up a one-one correspondence, which we write $\tau \mapsto \psi$, between the set of natural transformations $\mathfrak{C}(A,-) \to \mathfrak{C}(A'-)$ and the set $\mathfrak{C}(A',A)$. □

With respect to the correspondence $\tau \mapsto \psi$ we easily prove

Proposition 4.3. *Let* $\tau: \mathfrak{C}(A,-) \to \mathfrak{C}(A',-)$, $\tau': \mathfrak{C}(A',-) \to \mathfrak{C}(A'',-)$. *Then if* $\tau \mapsto \psi$, $\tau' \mapsto \psi'$, *where* $\psi: A' \to A$, $\psi': A'' \to A'$, *we have*

$$\tau'\tau \mapsto \psi\psi'.$$

In particular τ is a natural equivalence if and only if ψ is an isomorphism.

Proof. $(\tau'\tau)_B = (\tau'_B)(\tau_B) = \psi'^*\psi^* = (\psi\psi')^*$. □

Proposition 4.1 is often called the *Yoneda lemma*; it has many applications in algebraic topology and, as we shall see, in homological algebra.

If $\mathfrak{C}$ is a small category we may formulate the assertion of Corollary 4.2 in an elegant way in the functor category $\mathfrak{S}^{\mathfrak{C}}$. Then $A \mapsto \mathfrak{C}(A, -)$ is seen to be an embedding (called the *Yoneda embedding*) of $\mathfrak{C}^{\text{opp}}$ in $\mathfrak{S}^{\mathfrak{C}}$; and Corollary 4.2 asserts further that it is a *full* embedding.

Exercises:

4.1. A full subcategory $\mathfrak{C}_0$ of $\mathfrak{C}$ is said to be a *skeleton* of $\mathfrak{C}$ if, given any object A of $\mathfrak{C}$, there exists exactly one object A_0 of $\mathfrak{C}_0$ with $A_0 \cong A$. Show that every skeleton of $\mathfrak{C}$ is equivalent to $\mathfrak{C}$, and give an example to show that a skeleton of $\mathfrak{C}$ need not be isomorphic to $\mathfrak{C}$. Are all skeletons of $\mathfrak{C}$ isomorphic?

4.2. Represent the embedding of the commutator subgroup of G in G as a natural transformation.

4.3. Let $F, G: \mathfrak{C} \to \mathfrak{D}$, $E: \mathfrak{B} \to \mathfrak{C}$, $H: \mathfrak{D} \to \mathfrak{C}$ be functors, and let $t: F \to G$ be a natural transformation. Show how to define natural transformations $tE: FE \to GE$, and $Ht: HF \to HG$, and show that $H(tE) = (Ht)E$. Show that tE and Ht are natural equivalences if t is a natural equivalence.

4.4. Let $\mathfrak{C}$ be a category with zero object and kernels. Let $f: A \to B$ in $\mathfrak{C}$ with kernel $k: K \to A$. Then $f_*: \mathfrak{C}(-, A) \to \mathfrak{C}(-, B)$ is a natural transformation of contravariant functors from $\mathfrak{C}$ to $\mathfrak{S}_0$, the category of pointed sets. Show that $X \mapsto \ker(f_*)_X$ is a contravariant functor from $\mathfrak{C}$ to $\mathfrak{S}_0$ which is represented by K, and explain the sense in which k_* is the kernel of f_*.

4.5. Carry out an exercise similar to Exercise 4.4 replacing kernels in $\mathfrak{C}$ by cokernels in $\mathfrak{C}$.

4.6. Let $\mathfrak{A}$ be a small category and let $Y: \mathfrak{A} \to [\mathfrak{A}^{\text{opp}}, \mathfrak{S}]$ be the Yoneda embedding $Y(A) = \mathfrak{A}(-, A)$. Let $J: \mathfrak{A} \to \mathfrak{B}$ be a functor. Define $R: \mathfrak{B} \to [\mathfrak{A}^{\text{opp}}, \mathfrak{S}]$ on objects by $R(B) = \mathfrak{B}(J-, B)$. Show how to extend this definition to yield a functor R, and give reasonable conditions under which $Y = RJ$.

4.7. Let I be any set; regard I as a category with identity morphisms only. Describe $\mathfrak{C}^I$. What is $\mathfrak{C}^I$ if I is a set with 2 elements?

5. Products and Coproducts; Universal Constructions

The reader was introduced in Section I. 3 to the *universal property* of the direct product of modules. We can now state this property for a general category $\mathfrak{C}$.

Definition. Let $\{X_i\}$, $i \in I$, be a family of objects of the category $\mathfrak{C}$ indexed by the set I. Then a *product* $(X; p_i)$ of the objects X_i is an object X, together with morphisms $p_i: X \to X_i$, called *projections*, with the universal property: given any object Y and morphisms $f_i: Y \to X_i$, there exists a unique morphism $f = \{f_i\}: Y \to X$ with $p_i f = f_i$.

As we have said, in the category $\mathfrak{M}_\Lambda$ of (left) Λ-modules, we may take for X the direct product of the modules X_i (Section I. 3). In the category $\mathfrak{S}$ we have the ordinary Cartesian product, in the category $\mathfrak{T}$ we have the topological product (see [21]).

We cannot guarantee, of course, that the product always exists in $\mathfrak{C}$. However, we *can* guarantee that it is essentially unique – again the reader should recall the argument in Section I. 3.

Theorem 5.1. *Let $(X; p_i), (X'; p'_i)$ both be products of the family $\{X_i\}$, $i \in I$, in $\mathfrak{C}$. Then there exists a unique isomorphism $\xi: X \to X'$ such that $p'_i \xi = p_i$, $i \in I$.*

Proof. By the universal property for X there exists a (unique) morphism $\eta: X' \to X$ such that $p_i \eta = p'_i$. Similarly there exists a (unique) morphism $\xi: X \to X'$ such that $p'_i \xi = p_i$. Then

$$p_i \eta \xi = p_i = p_i 1 \quad \text{for all} \quad i \in I .$$

But by the uniqueness property of $(X; p_i)$, this implies that $\eta \xi = 1$. Similarly $\xi \eta = 1$. □

Of course, the uniqueness of $(X; p_i)$ expressed by Theorem 5.1 is as much as we can possibly expect. For if $(X; p_i)$ is a product and $\eta: X' \xrightarrow{\sim} X$, then $(X'; p_i \eta)$ is plainly also a product. Thus we allow ourselves to talk of *the* product of the X_i. We may write $X = \prod X_i, f = \{f_i\}$. By abuse we may even refer to X itself as the product of the X_i. If the indexing set is $I = (1, 2, \dots, n)$ we may write $X = X_1 \times X_2 \times \cdots \times X_n$ and $f = \{f_1, f_2, \dots, f_n\}$.

As we have said, such a product may not exist in a given category. Moreover, it is important to notice that the universal property of the product makes reference to the entire category. Thus it may well happen that not only the question of existence of a product of the objects X_i but even the nature of that product may depend on the category in question. However, before giving examples, let us state a few elementary propositions.

Proposition 5.2. *Let $\mathfrak{C}$ be a category in which $\mathfrak{C}(X, Y)$ is non-empty for all X, Y (e.g., a category with zero object). Then if $\prod_i X_i$ exists it admits each X_i as a retract. Thus, in particular, each p_i is an epimorphism.*

Proof. In the definition of $\prod_i X_i$, take $Y = X_j$, for a fixed $j \in I$, and $f_j = 1: X_j \to X_j$. For $i \neq j$ let f_i be arbitrary. Then $p_j f = 1: X_j \to X_j$ so that $\prod_i X_i$ retracts through p_j onto X_j. □

Proposition 5.3. *Given two families $\{X_i\}$, $\{Y_i\}$ of objects of $\mathfrak{C}$, indexed by the same indexing set I, then if the products $\prod_i X_i$, $\prod_i Y_i$ exist, and if $f_i: X_i \to Y_i$, $i \in I$, there is a uniquely determined morphism*

$$\prod_i f_i : \prod_i X_i \to \prod_i Y_i$$

such that

$$p_i\Big(\prod_i f_i\Big) = f_i p_i\,.$$

Moreover, if $\mathfrak{C}$ *admits products for all families indexed by* I, *then* $\prod_i$ *is a functor*

$$\prod_i : \mathfrak{C}^I \to \mathfrak{C}\,.$$

Proof. The first assertion is merely an application of the universal property of $\prod_i Y_i$. The proof of the second is left to the reader. (It should be clear what we understand by the category $\mathfrak{C}^I$; see Exercise 4.7.) □

If $I = (1, 2, \ldots, n)$ we naturally write $f_1 \times f_2 \times \cdots \times f_n$ for $\prod_i f_i$.

Proposition 5.4. *Let* $f_i: Z \to X_i$, $g: W \to Z$, $g_i: X_i \to Y_i$, $i \in I$. *Then, if the products in question exist,*

$$\text{(i)}\quad \{f_i\}\, g = \{f_i g\}\,, \qquad \text{(ii)}\quad \Big(\prod_i g_i\Big)\{f_i\} = \{g_i f_i\}\,.$$

Proof. We leave the proof to the reader, with the hint that it is sufficient to prove that each side projects properly onto the i-component under the projection p_i. □

Proposition 5.5. *Let* $\mathfrak{C}$ *be a category in which any two objects admit a product. Thus given objects* X, Y, Z *in* $\mathfrak{C}$ *we have projections*

$$\begin{aligned} p_1 &: X \times Y \to X\,, & q_1 &: (X \times Y) \times Z \to X \times Y\,,\\ p_2 &: X \times Y \to Y\,, & q_2 &: (X \times Y) \times Z \to Z\,. \end{aligned}$$

Then $((X \times Y) \times Z; p_1 q_1, p_2 q_1, q_2)$ *is the product of* X, Y, Z.

Proof. Given $f_1: W \to X, f_2: W \to Y, f_3: W \to Z$, we form $g: W \to X \times Y$ such that $p_1 g = f_1$, $p_2 g = f_2$. We then form $h: W \to (X \times Y) \times Z$ such that $q_1 h = g$, $q_2 h = f_3$. Then $p_1 q_1 h = f_1$, $p_2 q_1 h = f_2$. It remains to prove the uniqueness of h, so we suppose that $p_1 q_1 h = p_1 q_1 h'$, $p_2 q_1 h = p_2 q_1 h'$, $q_2 h = q_2 h'$. One application of uniqueness (to $X \times Y$) yields $q_1 h = q_1 h'$; and a second application yields $h = h'$. □

Proposition 5.6. *If any two objects in* $\mathfrak{C}$ *admit a product, so does any finite collection of objects.*

Proof. We argue by induction, using an obvious generalization of the proof of Proposition 5.5. □

Proposition 5.5 may be said also to exhibit the *associativity* of the product. Thus, there are canonical equivalences

$$(X \times Y) \times Z \cong X \times Y \times Z \cong X \times (Y \times Z)\,.$$

In an even stronger sense the product is *commutative;* for the very definition of $X \times Y$ is symmetric in X and Y.

The reader has already met many examples of products (in $\mathfrak{S}$, $\mathfrak{T}$, $\mathfrak{T}_h$, $\mathfrak{G}$, $\mathfrak{M}_\Lambda$, for example). There are, of course, many other examples familiar in mathematics. We now give a few examples to show what care must be taken in studying products in arbitrary categories.

Examples. (a) In the category $\mathfrak{S}(2)$ of two-element sets, no two objects admit a product. For let $B=(b_1, b_2)$, $C=(c_1, c_2)$ be two such sets and let us conjecture that $(D; p_1, p_2)$ is their product, $D=(d_1, d_2)$. This means that, given $A=(a_1, a_2)$, $f: A \to B$, $g: A \to C$, there exists (a unique) $h: A \to D$ with $p_1 h = f$, $p_2 h = g$. Now p_1 must be surjective since we may choose f surjective; similarly p_2 must be surjective. Without real loss of generality we may suppose $p_1(d_i)=b_i$, $p_2(d_i)=c_i$, $i=1,2$. Now if $f(A)=(b_1)$, $g(A)=(c_2)$, we have a contradiction since h must miss d_1 and d_2. Notice that the assertion of this example is *not* established merely by remarking that the Cartesian product of B and C is a 4-element set and hence not in $\mathfrak{S}(2)$.

(b) Consider the family of cyclic groups $\mathbb{Z}_{p^k}$, of order p^k, $k=1,2,\ldots$, where p is a fixed prime. Then

(i) in the category of cyclic groups no two groups of this family have a product;

(ii) in the category of finite abelian groups the family does not have a product;

(iii) in the category of torsion abelian groups, the family has a product which is not the direct product;

(iv) in the category of abelian groups, and in the category of groups, the direct product is the product.

We now prove these assertions.

(i) If $(\mathbb{Z}_m; q_1, q_2)$ is the product of $\mathbb{Z}_{p^k}$, $\mathbb{Z}_{p^l}$ then, as in the previous example, one immediately shows that q_1, q_2 are surjective. Suppose $k \geqq l$, then $m=p^k n$ and we may choose generators α, β_1, β_2 of $\mathbb{Z}_m$, $\mathbb{Z}_{p^k}$, $\mathbb{Z}_{p^l}$ so that $q_i(\alpha)=\beta_i$, $i=1,2$. Given $f_1 = 1: \mathbb{Z}_{p^k} \to \mathbb{Z}_{p^k}$, $f_2 = 0: \mathbb{Z}_{p^k} \to \mathbb{Z}_{p^l}$ suppose $f(\beta_1) = s\alpha$, where $f=\{f_1, f_2\}$. Then $s \equiv 1 \bmod p^k$, $s \equiv 0 \bmod p^l$, which is absurd.

(ii) If $(A; q_k, k=1,2,\ldots)$ were the product of the entire family, then, again, each q_k would be surjective. Thus the order of A would be divisible by p^k for every k, which is absurd. (This argument shows, of course, that the family has no product even in the category of finite groups.)

(iii) Let T be the torsion subgroup of the direct product P of the groups $\mathbb{Z}_{p^k}$. By virtue of the role of P in $\mathfrak{G}$ it is plain that $(T; q_k)$ is the product in the category of torsion abelian groups, where q_k is just the restriction to T of the projection $P \to \mathbb{Z}_{p^k}$.

(iv) Well-known.

We now turn briefly to coproducts. The duality principle enables us to make the following succinct definition:

Definition. Let $\{X_i\}$, $i \in I$, be a family of objects of the category $\mathfrak{C}$ indexed by the set I. Then $(X; q_i)$ is a *coproduct* of the objects X_i in $\mathfrak{C}$ if and only if it is a product of the objects X_i in $\mathfrak{C}^{\text{opp}}$.

This definition means, then, that in $\mathfrak{C}$, $q_i: X_i \to X$ and given morphisms $f_i: X_i \to Y$ there exists a unique $f: X \to Y$ with $fq_i = f_i$. The morphisms $q_i: X_i \to X$ are called *injections*. We write $X = \coprod_i X_i$, $f = \langle f_i \rangle$, and if $I = (1, 2, \dots, n)$, then $X = X_1 \amalg X_2 \amalg \cdots \amalg X_n$, $f = \langle f_1, f_2, \dots, f_n \rangle$. We need not state the duals of Proposition 5.2 through 5.6, leaving their enunciation as an exercise for the reader. We mention, however, a few examples.

Examples. (a) In $\mathfrak{S}$ the coproduct is the disjoint union with the evident injections q_i.

(b) In $\mathfrak{T}$ the coproduct is the disjoint union with the natural topology.

(c) In $\mathfrak{T}^0$ the coproduct is the disjoint union with base points identified.

(d) In $\mathfrak{G}$ the coproduct is the free product with the evident injections q_i, see [36].

(e) In $\mathfrak{M}_A$ the coproduct is the direct sum. In this case we shall write $\oplus$ instead of $\amalg$. We leave it to the reader to verify these assertions.

Exercises:

5.1. Let $\mathfrak{C}, \mathfrak{D}$ be categories admitting (finite) products. A functor $F: \mathfrak{C} \to \mathfrak{D}$ is said to be *product-preserving* if for any objects A_1, A_2 of $\mathfrak{C}$, $(F(A_1 \times A_2); Fp_1, Fp_2)$ is the product of FA_1 and FA_2 in $\mathfrak{D}$. Show that in the list of functors given in Section 2, b), d, e), g) are product-preserving, while c), f) are coproduct-preserving.

5.2. Show that a terminal (initial) object may be regarded as a (co-) product over an empty indexing set.

5.3. Show that A is the product of A_1 and A_2 in $\mathfrak{C}$ if and only if $\mathfrak{C}(X, A)$ is the product of $\mathfrak{C}(X, A_1)$ and $\mathfrak{C}(X, A_2)$ in $\mathfrak{S}$ for all X in $\mathfrak{C}$. (To make this statement precise, one should, of course, mention the morphisms p_1 and p_2.) Give a similar characterization of the coproduct.

5.4. Let $\mathfrak{C}$ be a category with zero object and finite products. A *group* in $\mathfrak{C}$ is a pair (A, m), where A is an object of $\mathfrak{C}$ and $m: A \times A \to A$ in $\mathfrak{C}$, subject to the axioms:

G1: *(Associativity)* $m(m \times 1) = m(1 \times m)$;
G2: *(Two-sided unity)* $m\{1, 0\} = 1 = m\{0, 1\}$;
G3: *(Two-sided inverse)* There exists $\sigma: A \to A$ such that

$$m\{1, \sigma\} = 0 = m\{\sigma, 1\}.$$

In other words the following diagrams are commutative

$$\begin{tikzcd}
A \times A \times A \arrow[r, "m \times 1"] \arrow[d, "1 \times m"'] & A \times A \arrow[d, "m"] \\
A \times A \arrow[r, "m"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times A \arrow[dr, "m"'] & A \arrow[l, "{\{1, 0\}}"'] \arrow[r, "{\{0, 1\}}"] \arrow[d, "1"] & A \times A \arrow[dl, "m"] \\
& A &
\end{tikzcd}$$

$$\begin{tikzcd}
A \times A \arrow[dr, "m"'] & A \arrow[l, "{\{1, \sigma\}}"'] \arrow[r, "{\{\sigma, 1\}}"] \arrow[d, "0"] & A \times A \arrow[dl, "m"] \\
& A &
\end{tikzcd}$$

Let $\mathfrak{S}'$ be the category of sets with specified basepoints. Show that a group in $\mathfrak{S}'$ is just a group in the usual sense. Show that (A, m) is a group in $\mathfrak{C}$ if and only if $\mathfrak{C}(X, A)$ is a group (in $\mathfrak{S}'$) for all X in $\mathfrak{C}$ under the obvious induced operation m_*. Show that, if B is an object of $\mathfrak{C}$ such that $\mathfrak{C}(X, B)$ is a group for all X in $\mathfrak{C}$ and if $f: X \to Y$ in $\mathfrak{C}$ induces a *homomorphism* $f^*: \mathfrak{C}(Y, B) \to \mathfrak{C}(X, B)$, then B admits a unique group structure m in $\mathfrak{C}$ such that m_* is the given group structure in $\mathfrak{C}(X, B)$.

5.5. Show that if (A, m) satisfies **G1** and the *one-sided* axioms

G2R: $m\{1, 0\} = 1$;

G3R: There exists $\sigma: A \to A$ such that $m\{1, \sigma\} = 1$;

then (A, m) is a group in $\mathfrak{C}$. Show also that σ is unique. (Hint: Use the argument of Exercise 5.4.)

5.6. Formulate the condition that the group (A, m) is commutative. Show that a product-preserving functor sends (commutative) groups to (commutative) groups.

5.7. Define the concept of a *cogroup*, the dual of a group. Show that in $\mathfrak{Ab}$ ($\mathfrak{M}^l_\Lambda$) every object is a cogroup.

5.8. Let $\mathfrak{C}$ be a category with products and coproducts. Let $f_{ij}: X_i \to Y_j$ in $\mathfrak{C}$, $i \in I$, $j \in J$. Show that $\langle\{f_{ij}\}_{j \in J}\rangle_{i \in I} = \{\langle f_{ij}\rangle_{i \in I}\}_{j \in J}: \coprod_{i \in I} X_i \to \prod_{j \in J} Y_j$. Hence, if $\mathfrak{C}$ has a zero object, establish a natural transformation from $\coprod_{i \in I}$ to $\prod_{i \in I}$.

6. Universal Constructions (Continued); Pull-backs and Push-outs

We are not yet ready to say precisely what is to be understood by a universal construction. Such a formulation will only become possible when we are armed with the language of adjoint functors (Section 7). However, we now propose to introduce a very important example of a universal construction and the reader should surely acquire an understanding of the essential nature of such constructions from this example (together with the examples of the *kernel*, and its dual, the *cokernel*; see final remark in Section I. 1).

It must already have been apparent that a basic concept in homological algebra, and, more generally, in category theory, is that of a *com-*

mutative diagram, and that the most fundamental of all commutative diagrams is the commutative square

$$\begin{tikzcd} \cdot \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \cdot \arrow[d, "\varphi"] \\ \cdot \arrow[r, "\psi"'] & \cdot \end{tikzcd} \qquad \varphi\alpha = \psi\beta\,. \tag{6.1}$$

There thus arises the natural question. Given φ, ψ in (6.1), is there a *universal* procedure for providing morphisms α, β to yield a commutative square? Of course, the dual question arises just as naturally, and may be regarded as being treated implicitly in what follows by the application of the duality principle. *Explicitly* we will only consider the question as posed and we immediately provide a precise definition.

Definition. Given $\varphi: A \to X$, $\psi: B \to X$ in $\mathfrak{C}$, a *pull-back* of (φ, ψ) is a pair of morphisms $\alpha: Y \to A$, $\beta: Y \to B$ such that $\varphi\alpha = \psi\beta$, and (6.1) has the following universal property: given $\gamma: Z \to A$, $\delta: Z \to B$ with $\varphi\gamma = \psi\delta$, there exists a unique $\zeta: Z \to Y$ with $\gamma = \alpha\zeta$, $\delta = \beta\zeta$.

$$\begin{tikzcd} Z \arrow[dr, dotted, "\zeta"] \arrow[drr, "\gamma"] \arrow[ddr, "\delta"'] & & \\ & Y \arrow[r, "\alpha"] \arrow[d, "\beta"] & A \arrow[d, "\varphi"] \\ & B \arrow[r, "\psi"'] & X \end{tikzcd} \tag{6.2}$$

Just as for the product, it follows readily that, if a pull-back exists, then it is essentially unique. Precisely, if (α', β') is also a pull-back of (φ, ψ), $\alpha': Y' \to A$, $\beta': Y' \to B$, then there exists a unique equivalence $\omega: Y \to Y'$ such that $\alpha'\omega = \alpha$, $\beta'\omega = \beta$. Thus we may permit ourselves to speak of *the* pull-back of φ and ψ.

We write $(Y; \alpha, \beta)$ for the pull-back of φ and ψ. Where convenient we may abbreviate this to (α, β) or to Y. We may also say that the square in (6.2) is a *pull-back square.*

Notice that the uniqueness of ζ in (6.2) may be expressed by saying that $\{\alpha, \beta\}: Y \to A \times B$ is a monomorphism, provided that $A \times B$ exists in $\mathfrak{C}$. In fact, there is a very close connection between pull-backs and products of two objects. On the one hand, if $\mathfrak{C}$ has a terminal object T and if $\varphi: A \to T$, $\psi: B \to T$ are the unique morphisms then the pull-back of φ and ψ consists of the projections $p_1: A \times B \to A$, $p_2: A \times B \to B$. On the other hand we may actually regard the pull-back as a product in a suitable category. Thus we fix the object X and introduce the category $\mathfrak{C}/X$ of *$\mathfrak{C}$-objects over X*. An object of $\mathfrak{C}/X$ is a morphism $\kappa: K \to X$ in $\mathfrak{C}$ and a morphism $\sigma: \kappa \to \lambda$ in $\mathfrak{C}/X$ is a morphism $\sigma: K \to L$ in $\mathfrak{C}$ making the diagram

$$\begin{tikzcd} K \arrow[r, "\sigma"] \arrow[dr, "\kappa"'] & L \arrow[d, "\lambda"] \\ & X \end{tikzcd}$$

commutative, $\lambda\sigma = \kappa$. Now let $\Delta = \varphi\alpha = \psi\beta$ be the diagonal of the square (6.2). Then the reader may easily prove

Proposition 6.1. *$(\Delta; \alpha, \beta)$ is the product of φ and ψ in $\mathfrak{C}/X$.* □

This means that α, β play the roles of p_1, p_2 in the definition of a product, when interpreted as morphisms $\alpha: \Delta \to \varphi$, $\beta: \Delta \to \psi$ in $\mathfrak{C}/X$.

From this proposition we may readily deduce, from the propositions of Section 5, propositions about the pull-back and its evident generalization to a family, instead of a pair, of morphisms in $\mathfrak{C}$ with codomains X. We will prove one theorem about pull-backs in categories with zero objects which applies to the categories of interest in homological algebra. We recall first (Exercise 3.4) how we define the *kernel* of a morphism $\sigma: K \to L$ in a category with zero objects by means of a universal property. We say that $\mu: J \to K$ is a kernel of σ if (i) $\sigma\mu = 0$ and (ii) if $\sigma\tau = 0$ then τ factorizes as $\tau = \mu\tau_0$, with τ_0 unique. As usual, the kernel is essentially unique; we (sometimes) call J the *kernel object*. Notice that μ is monic, by virtue of the uniqueness of τ_0.

Theorem 6.2. *Let* (6.1) *be a pull-back diagram in a category $\mathfrak{C}$ with zero object. Then*

(i) *if (J, μ) is the kernel of β, $(J, \alpha\mu)$ is the kernel of φ;*

(ii) *if (J, ν) is the kernel of φ, ν may be factored as $\nu = \alpha\mu$ where (J, μ) is the kernel of β.*

Note that (ii) is superfluous if we know that every morphism in $\mathfrak{C}$ has a kernel. We show here, in particular, that β has a kernel if and only if φ has a kernel, and the kernel objects coincide.

Proof. (i)

$$\begin{tikzcd} J \arrow[r, equal] \arrow[d, "\mu"] & J \arrow[d, "\nu"] \\ Y \arrow[r, "\alpha"] \arrow[d, "\beta"] & A \arrow[d, "\varphi"] \\ B \arrow[r, "\psi"] & X \end{tikzcd}$$

Set $\nu = \alpha\mu$. We first show that ν is monomorphic; for μ and $\{\alpha, \beta\}$ are monomorphic, so $\{\alpha, \beta\}\,\mu = \{\nu, 0\}: J \to A \times B$ is monomorphic and hence, plainly, ν is monomorphic. Next we observe that $\varphi\nu = \varphi\alpha\mu = \psi\beta\mu = 0$.

Finally we take $\tau: Z \to A$ and show that if $\varphi\tau = 0$ then $\tau = \nu\tau_0$ for some τ_0. Since $\psi 0 = 0$, the pull-back property shows that there exists $\sigma: Z \to Y$ such that $\alpha\sigma = \tau$, $\beta\sigma = 0$. Since (J, μ) is the kernel of β, $\sigma = \mu\tau_0$, so that $\tau = \alpha\mu\tau_0 = \nu\tau_0$.

(ii) Since $\varphi\nu = 0$ we argue as in (i) that there exists $\mu: J \to Y$ with $\alpha\mu = \nu$, $\beta\mu = 0$. Since ν is a monomorphism, so is μ and we show that (J, μ) is the kernel of β. Let $\beta\tau = 0$, $\tau: Z \to Y$. Then $\varphi\alpha\tau = \psi\beta\tau = 0$, so $\alpha\tau = \nu\tau_0 = \alpha\mu\tau_0$. But $\beta\tau = \beta\mu\tau_0 = 0$, so that, $\{\alpha, \beta\}$ being a monomorphism, $\tau = \mu\tau_0$. $\square$

In Chapter VIII we will refer back to this theorem as a very special case of a general result on commuting limits. We remark that the introduction of $A \times B$ in the proof was for convenience only. The argument is easily reformulated without invoking $A \times B$.

As examples of pull-backs, let us consider the categories $\mathfrak{S}$, $\mathfrak{T}$, $\mathfrak{G}$. In $\mathfrak{S}$, let φ, ψ be embeddings of A, B as subsets of X; then $Y = A \cap B$ and α, β are also embeddings. In $\mathfrak{T}$ we could cite an example similar to that given for $\mathfrak{S}$; however there is also an interesting example when φ, say, is a fiber-map. Then β is also a fiber-map and is often called the fiber-map *induced by* ψ *from* φ. (Indeed, in general, the pull-back is sometimes called the *fiber-product*.) In $\mathfrak{G}$ we again have an example similar to that given for $\mathfrak{S}$; however there is a nice general description of Y as the subgroup of $A \times B$ consisting of those elements (a, b) such that $\varphi(a) = \psi(b)$.

The dual notion to that of a pull-back is that of a *push-out*. Thus, in (6.1), (φ, ψ) is the push-out of (α, β) in $\mathfrak{C}$ if and only if it is the pull-back of (α, β) in $\mathfrak{C}^{\mathrm{opp}}$. The reader should have no difficulty in formulating an explicit universal property characterizing the push-out and dualizing the statements of this section. If α, β are embeddings (in $\mathfrak{S}$ or $\mathfrak{T}$) of $Y = A \cap B$ in A and B, then $X = A \cup B$. In $\mathfrak{G}$ we are led to the notion of *free product with amalgamations* [36].

We adopt for the push-out notational and terminological conventions analogous to those introduced for the pull-back.

Exercises:

6.1. Prove Proposition 6.1.

6.2. Given the commutative diagram in $\mathfrak{C}$

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, "\varphi_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "\varphi_2"] & A_3 \arrow[d, "\varphi_3"] \\
B_1 \arrow[r, "\beta_1"'] & B_2 \arrow[r, "\beta_2"'] & B_3
\end{tikzcd}$$

show that if both squares are pull-backs, so is the composite square. Show also that if the composite square is a pull-back and α_2 is monomorphic, then the left-hand square is a pull-back. Dualize these statements.

6.3. Recall the notion of *equalizer* of two morphisms $\varphi_1, \varphi_2 : A \to B$ in $\mathfrak{C}$ (see Exercise 3.5). Show that if $\mathfrak{C}$ admits finite products then $\mathfrak{C}$ admits pull-backs if and only if $\mathfrak{C}$ admits equalizers.

6.4. Show that the pull-back of

$$\begin{array}{ccc} & & A \\ & & \downarrow{\scriptstyle\varphi} \\ 0 & \longrightarrow & B \end{array}$$

in the category $\mathfrak{C}$ with zero object 0 is essentially just the kernel of φ. Generalize this to

$$\begin{array}{ccc} & & A \\ & & \downarrow{\scriptstyle\varphi} \\ C & \xrightarrow{\psi} & B \end{array}$$

where ψ is to be regarded as an embedding.

6.5. Identify the push-out in $\mathfrak{S}$, $\mathfrak{G}$ and $\mathfrak{M}_\Lambda$.

6.6. Show that the free module functor $\mathfrak{S} \to \mathfrak{M}_\Lambda$ preserves push-outs. Argue similarly for the free group functor.

6.7. Show that, in the category $\mathfrak{M}_\Lambda$, the pull-back square

$$\begin{array}{ccc} A_1 & \xrightarrow{\alpha} & A_2 \\ \downarrow{\scriptstyle\varphi_1} & & \downarrow{\scriptstyle\varphi_2} \\ B_1 & \xrightarrow[\beta]{} & B_2 \end{array}$$

is also a push-out if and only if $\langle \varphi_2, \beta \rangle : A_2 \oplus B_1 \to B_2$ is surjective.

6.8. Formulate a "dual" of the statement above – and prove it. Why is the word "dual" in inverted commas?

7. Adjoint Functors

One of the most basic notions of category theory, that of adjoint functors, was introduced by D. M. Kan [30]. We will illustrate it first by an example with which the reader is familiar from Chapter I. Let $F : \mathfrak{S} \to \mathfrak{M}_\Lambda$ be the *free functor*, which associates with every set the free Λ-module on that set as basis; and let $G : \mathfrak{M}_\Lambda \to \mathfrak{S}$ be the *underlying functor* which associates with every module its underlying set. We now define a transformation, natural in both S and A,

$$\eta = \eta_{SA} : \mathfrak{M}_\Lambda(FS, A) \to \mathfrak{S}(S, GA)$$

associating with a Λ-module homomorphism $\varphi : FS \to A$ the restriction of φ to the basis S of FS. The reader immediately sees that the universal property of free modules (Proposition I.4.2) is expressed by saying that

η is an equivalence. Abstracting from this situation we make the following definition:

Definition. Let $F:\mathfrak{C}\to\mathfrak{D}$, $G:\mathfrak{D}\to\mathfrak{C}$ be functors such that there is a natural equivalence

$$\eta=\eta_{XY}:\mathfrak{D}(FX,Y)\xrightarrow{\sim}\mathfrak{C}(X,GY)$$

of functors $\mathfrak{C}^{\text{opp}}\times\mathfrak{D}\to\mathfrak{S}$. We then say that F is *left adjoint* to G, G is *right adjoint* to F, and write $\eta:F\dashv G$. We call η the *adjugant equivalence* or, simply, *adjugant.*

In the example above we have seen that the free functor $F:\mathfrak{S}\to\mathfrak{M}_\Lambda$ is left adjoint to the underlying functor $G:\mathfrak{M}_\Lambda\to\mathfrak{S}$. The reader will readily verify that the concept of a free group (free object in the category of groups) and the concept of a polynomial algebra over the field K (free object in the category of commutative K-algebras) may also be formulated in terms of a free functor left adjoint to an underlying functor. From this, one is naturally led to a generalization of the concept of a free module (free group, polynomial algebra) to the notion of an object in a category which is *free* with respect to an "underlying" functor.

The theory of adjoint functors will find very frequent application in the sequel; various facts of homological algebra which were originally proved in an ad hoc fashion may be systematically explained by the use of adjoint functors. We now give some further examples of adjoint functors.

(a) In Proposition I.8.1 we have considered the functor $G:\mathfrak{Ab}\to\mathfrak{M}_\Lambda$ defined by

$$GC=\operatorname{Hom}_{\mathbb{Z}}(\Lambda,C),\quad C \text{ in } \mathfrak{Ab},$$

where the (left) Λ-module structure in GC is given by the right Λ-module structure of Λ. We denote by $F:\mathfrak{M}_\Lambda\to\mathfrak{Ab}$ the underlying functor, which forgets the Λ-module structure. Proposition I.8.1 then asserts that there is a natural equivalence

$$\eta:\operatorname{Hom}_\Lambda(A,GC)\xrightarrow{\sim}\operatorname{Hom}_{\mathbb{Z}}(FA,C)$$

for A in $\mathfrak{M}_\Lambda$ and C in $\mathfrak{Ab}$. Thus F is left adjoint to G and $\eta^{-1}:F\dashv G$ is the adjugant.

(b) Given a topological Hausdorff space X, we may give X a new topology by declaring $F\subseteq X$ to be closed if $F\cap K$ is closed in the original topology for every compact subset K of X. Write X_K for the set X furnished with this topology. Plainly X_K is a Hausdorff space and the obvious map $X_K\to X$ is continuous. Also, given $f:X\to Y$, a continuous map of Hausdorff spaces, then $f:X_K\to Y_K$ is also continuous. For if F is closed in Y_K and if L is compact in X, then

$$f^{-1}F\cap L=f^{-1}(F\cap fL)\cap L$$

is closed in X, so that $f^{-1}F$ is closed in X_K. We call a Hausdorff space a *Kelley space* if its closed sets are precisely those sets F such that $F \cap K$ is closed for every compact K. If X is a Kelley space then $X = X_K$; and X_K is a Kelley space for every Hausdorff space X. Summing up, we have the category $\mathfrak{H}$ of Hausdorff spaces, the category $\mathfrak{K}$ of Kelley spaces, the functor $K: \mathfrak{H} \to \mathfrak{K}$, given by $K(X) = X_K$, and the embedding functor $E: \mathfrak{K} \to \mathfrak{H}$. The facts adduced show that $E \dashv K$.

We will give later a theorem (Theorem 7.7) which provides additional motivation for studying adjoint functors. However, we now state some important propositions about adjoint functors.

Proposition 7.1. *Let $F: \mathfrak{C} \to \mathfrak{D}$, $F': \mathfrak{D} \to \mathfrak{E}$, $G: \mathfrak{D} \to \mathfrak{C}$, $G': \mathfrak{E} \to \mathfrak{D}$ be functors and let $\eta: F \dashv G$, $\eta': F' \dashv G'$ be adjugants. Then $\eta'': F'F \dashv GG'$, where $\eta'' = \eta \circ \eta'$.*

We leave the proof as an exercise. □

Next we draw attention to the relation which makes explicit the naturality of η. We again refer to the situation $\eta: F \dashv G$. Then this relation is

$$\eta(\beta \circ \varphi \circ F\alpha) = G\beta \circ \eta(\varphi) \circ \alpha\,, \tag{7.1}$$

$$\text{for all} \quad \alpha: X' \to X, \quad \varphi: FX \to Y, \quad \beta: Y \to Y'.$$

In particular, take $Y = FX$, $\varphi = 1_{FX}$, and set $\varepsilon_X = \eta(1_{FX}): X \to GFX$. Then (7.1) shows that ε is a natural transformation, $\varepsilon: 1 \to GF$. We call ε the *front adjunction* or *unit*. Similarly take $X = GY$, and set

$$\delta_Y = \eta^{-1}(1_{GY}): FGY \to Y.$$

Again (7.1) shows that δ is a natural transformation, $\delta: FG \to 1$, which we call the *rear adjunction* or *counit*. Further, (7.1) implies that

$$F \xrightarrow{F\varepsilon} FGF \xrightarrow{\delta F} F, \qquad G \xrightarrow{\varepsilon G} GFG \xrightarrow{G\delta} G$$

are identity transformations,

$$\delta F \circ F\varepsilon = 1\,, \qquad G\delta \circ \varepsilon G = 1\,. \tag{7.2}$$

For $\eta(\delta_{FX} \circ F\varepsilon_X) = \eta(\delta_{FX}) \circ \varepsilon_X = \varepsilon_X = \eta(1_{FX})$; and the second relation in (7.2) is proved similarly. Notice also that (7.1) implies that η is determined by ε, and that $\xi = \eta^{-1}$ is determined by δ, by the rules

$$\begin{aligned} &\eta(\varphi) = G\varphi \circ \varepsilon_X\,, \qquad \text{for} \quad \varphi: FX \to Y, \\ &\xi(\psi) = \eta^{-1}(\psi) = \delta_Y \circ F\psi\,, \qquad \text{for} \quad \psi: X \to GY. \end{aligned} \tag{7.3}$$

We now prove the converse of these results.

Proposition 7.2. *Let $F: \mathfrak{C} \to \mathfrak{D}$, $G: \mathfrak{D} \to \mathfrak{C}$ be functors and let $\varepsilon: 1 \to GF$, $\delta: FG \to 1$ be natural transformations such that $\delta F \circ F\varepsilon = 1$, $G\delta \circ \varepsilon G = 1$. Then $\eta: \mathfrak{D}(FX, Y) \to \mathfrak{C}(X, GY)$, defined by $\eta(\varphi) = G\varphi \circ \varepsilon_X$, for $\varphi: FX \to Y$,* *is a natural equivalence, so that $\eta : F \dashv G$. Moreover, ε, δ are the unit and counit of the adjugant η.*

Proof. First, η is natural. For

$$\begin{aligned}\eta(\beta \circ \varphi \circ F\alpha) &= G(\beta \circ \varphi \circ F\alpha) \circ \varepsilon_{X'} \\ &= G\beta \circ G\varphi \circ GF\alpha \circ \varepsilon_{X'} \\ &= G\beta \circ G\varphi \circ \varepsilon_X \circ \alpha\,, \qquad \text{since } \varepsilon \text{ is natural} \\ &= G\beta \circ \eta(\varphi) \circ \alpha\,.\end{aligned}$$

Define ξ by $\xi(\psi) = \delta_Y \circ F\psi$, for $\psi : X \to GY$. Again, ξ is natural and we will have established that $\eta : F \dashv G$ if we show that ξ is inverse to η. Now if $\varphi : FX \to Y$, then

$$\begin{aligned}\xi\eta(\varphi) &= \delta_Y \circ F\eta(\varphi) \\ &= \delta_Y \circ FG\varphi \circ F\varepsilon_X \\ &= \varphi \circ \delta_{FX} \circ F\varepsilon_X\,, \qquad \text{since } \delta \text{ is natural} \\ &= \varphi\,, \qquad \text{by (7.2).}\end{aligned}$$

Thus $\xi\eta = 1$ and similarly $\eta\xi = 1$. Finally we see that if ε', δ' are the unit and counit of η, then

$$\varepsilon'_X = \eta(1_{FX}) = 1_{GFX} \circ \varepsilon_X = \varepsilon_X\,,$$

and

$$\delta'_Y = \xi(1_{GY}) = \delta_Y \circ 1_{FGY} = \delta_Y\,. \quad \square$$

Proposition 7.3. *Suppose $F \dashv G$. Then F determines G up to natural equivalence and G determines F up to natural equivalence.*

Proof. It is plainly sufficient to establish the first assertion. Suppose then that $\eta : F \dashv G, \eta' : F \dashv G'$. Consider the natural equivalence of functors

$$\mathfrak{C}(-, GY) \xrightarrow[\sim]{\eta^{-1}} \mathfrak{D}(F-, Y) \xrightarrow[\sim]{\eta'} \mathfrak{C}(-, G'Y)\,.$$

By the dual of Corollary 4.2 and Proposition 4.3 such an equivalence is induced by an isomorphism $\theta_Y : GY \to G'Y$. Since $\eta' \circ \eta^{-1}$ is natural in Y, it readily follows that θ is a natural equivalence. $\square$

We remark that if $\varepsilon, \delta, \varepsilon', \delta'$ are unit and counit for η, η', then

$$\theta_Y = \eta'\eta^{-1}(1_{GY}) = \eta'(\delta_Y) = G'(\delta_Y) \circ \varepsilon'_{GY}$$

or, briefly,

$$\theta = G'\delta \circ \varepsilon' G\,. \tag{7.4}$$

It then immediately follows that the inverse $\bar{\theta}$ of θ is given by

$$\bar{\theta} = G\delta' \circ \varepsilon G'\,. \tag{7.5}$$

Proposition 7.4. *Under the same hypotheses as in Proposition 7.3, with $\theta, \bar{\theta}$ defined as in* (7.4), (7.5), *we have*

(i) $\theta F \circ \varepsilon = \varepsilon'$; $\delta \circ F\bar{\theta} = \delta'$;

(ii) $\theta_Y \circ \eta(\varphi) = \eta'(\varphi)$, *for any* $\varphi : FX \to Y$.

Conversely, let $\eta : F \dashv G$ *and let* $\theta : G \to G'$ *be a natural equivalence. Then* $\eta' : F \dashv G'$, *where* $\eta'(\varphi) = \theta_Y \circ \eta(\varphi)$. *Moreover, if* ε *and* δ *are the unit and counit for* η, *then* ε' *and* δ', *the unit and counit for* η', *are given by* (i) *above.*

Proof. (i)

$$\begin{aligned} \theta F \circ \varepsilon &= G'\delta F \circ \varepsilon' GF \circ \varepsilon \\ &= G'\delta F \circ G'F\varepsilon \circ \varepsilon', && \text{by the naturality of } \varepsilon', \\ &= \varepsilon'. \\ \delta \circ F\bar{\theta} &= \delta \circ FG\delta' \circ F\varepsilon G' \\ &= \delta' \circ \delta FG' \circ F\varepsilon G', && \text{by the naturality of } \delta, \\ &= \delta'. \end{aligned}$$

(ii)

$$\begin{aligned} \theta_Y \circ \eta(\varphi) &= \theta_Y \circ G\varphi \circ \varepsilon_X \\ &= G'\varphi \circ \theta_{FX} \circ \varepsilon_X, && \text{by the naturality of } \theta \\ &= G'\varphi \circ \varepsilon'_X, && \text{by (i)}, \\ &= \eta'(\varphi). \end{aligned}$$

The proof of the converse is left as an exercise to the reader. □

Proposition 7.5. *If* $F : \mathfrak{C} \to \mathfrak{D}$ *is full and faithful and if* $F \dashv G$, *then the unit* $\varepsilon : 1 \to GF$ *is a natural equivalence.*

Proof. Let $\delta : FG \to 1$ be the counit. Then $\delta F : FGF \to F$. Since F is full and faithful we may define a transformation $\varrho : GF \to 1$ by

$$F\varrho_X = \delta_{FX}$$

and it is plain that ϱ is natural. We show that ϱ is inverse to ε. First, $F\varrho \circ F\varepsilon = \delta F \circ F\varepsilon = 1$, so that $\varrho \circ \varepsilon = 1$, since F is faithful. Second, if η is the adjugant, then

$$\begin{aligned} \eta^{-1}(\varepsilon_X \circ \varrho_X) &= F\varrho_X, && \text{by (7.2) and (7.3)}, \\ &= \delta_{FX} \\ &= \eta^{-1}(1_{GFX}). \end{aligned}$$

Thus $\varepsilon \circ \varrho = 1$ and the proposition is proved. □

Proposition 7.6. *If* $F : \mathfrak{C} \to \mathfrak{D}$ *is a full embedding and if* $F \dashv G$, *then there exists* G' *with* $F \dashv G'$ *where the unit* $\varepsilon' : 1 \to G'F$ *is the identity.*

Proof. We construct the functor G' as follows

$$\begin{aligned} G'(Y) &= G(Y) \quad \text{if} \quad Y \notin \operatorname{Im} F, \\ G'F(X) &= X. \end{aligned}$$

For $\beta: Y_1 \to Y_2$

$$\begin{aligned} G'(\beta) &= G(\beta) && \text{if} \quad Y_1, Y_2 \notin \operatorname{Im} F, \\ &= F^{-1}(\beta) && \text{if} \quad Y_1, Y_2 \in \operatorname{Im} F, \\ &= G(\beta) \circ \varepsilon && \text{if} \quad Y_1 \in \operatorname{Im} F, \quad Y_2 \notin \operatorname{Im} F, \\ &= \varrho \circ G(\beta) && \text{if} \quad Y_1 \notin \operatorname{Im} F, \quad Y_2 \in \operatorname{Im} F, \end{aligned}$$

where ϱ is inverse to ε as in Proposition 7.5. A straightforward computation shows that G' is a functor.

We now define transformations $\theta: G \to G'$, $\bar{\theta}: G' \to G$ by

$$\begin{aligned} \theta_Y &= 1_{GY} && \text{if} \quad Y \notin \operatorname{Im} F, \\ &= \varrho_X && \text{if} \quad Y = FX; \\ \bar{\theta}_Y &= 1_{GY} && \text{if} \quad Y \notin \operatorname{Im} F, \\ &= \varepsilon_X && \text{if} \quad Y = FX. \end{aligned}$$

Again it is easy to show that $\theta, \bar{\theta}$ are natural and they are obviously mutual inverses. Thus, by Proposition 7.4, $F \dashv G'$ and the counit for this adjunction is given by

$$\varepsilon'_X = \theta_{FX} \circ \varepsilon_X = \varrho_X \circ \varepsilon_X = 1_X, \qquad \text{so that} \qquad \varepsilon' = 1. \quad \square$$

The reader should notice that where $F \dashv G$ with $GF = 1$ and $\varepsilon = 1$, then the adjointness is simply given by a counit $\delta: FG \to 1$, satisfying

$$\delta F = 1, \qquad G\delta = 1.$$

We close this section by relating adjoint functors to the universal constructions given in previous sections. The theorem below will be generalized in the next section.

Theorem 7.7. *If $G: \mathfrak{D} \to \mathfrak{C}$ has a left adjoint then G preserves products, pull-backs and kernels.*

Proof. We must show that if $\{Y; p_i\}$ is the product of objects Y_i in $\mathfrak{D}$, then $\{GY; G(p_i)\}$ is the product of the objects $G(Y_i)$ in $\mathfrak{C}$. Suppose given $f_i: X \to GY_i$. Let $\eta: F \dashv G$ with inverse ξ. Then $\xi(f_i): FX \to Y_i$ so that there exists a unique $g: FX \to Y$ with $p_i g = \xi(f_i)$. Then

$$G(p_i) \circ \eta(g) = \eta(p_i g) = f_i.$$

Moreover $\eta(g)$ is the unique morphism f such that $G(p_i) \circ f = f_i$; for every $f': X \to GY$ is of the form $f' = \eta(g')$ and g is uniquely determined by $p_i g = \xi(f_i)$.

Next we look at pull-backs. Given a pull-back

$$\begin{tikzcd} Y \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \arrow[d, "\varphi"] \\ B \arrow[r, "\psi"'] & X \end{tikzcd}$$

in $\mathfrak{D}$, we assert that

$$\begin{tikzcd} GY \arrow[r, "G\alpha"] \arrow[d, "G\beta"'] & GA \arrow[d, "G\varphi"] \\ GB \arrow[r, "G\psi"'] & GX \end{tikzcd}$$

is a pull-back in $\mathfrak{C}$. So suppose given $\gamma: Z \to GA$, $\delta: Z \to GB$ in $\mathfrak{C}$ with $G\varphi \circ \gamma = G\psi \circ \delta$. Applying ξ, we have $\varphi \circ \xi(\gamma) = \psi \circ \xi(\delta)$. Thus there exists a unique $\varrho: FZ \to Y$ such that $\alpha \circ \varrho = \xi(\gamma)$, $\beta \circ \varrho = \xi(\delta)$. Applying η, $G(\alpha) \circ \eta(\varrho) = \gamma$, $G(\beta) \circ \eta(\varrho) = \delta$, and, as for products, $\eta(\varrho)$ is the unique morphism satisfying these equations

We leave the proof that G preserves kernels to the reader. □

Exercises:

7.1. Prove Proposition 7.1.

7.2. Establish that G' in Proposition 7.6 is a functor.

7.3. Show that if $G: \mathfrak{D} \to \mathfrak{C}$ has a left adjoint, then G preserves equalizers. Deduce that G then preserves kernels.

7.4. Let ${}_m\mathfrak{Ab}$ be the full subcategory of $\mathfrak{Ab}$ consisting of those abelian groups A such that $mA = 0$. Show that ${}_m\mathfrak{Ab}$ admits kernels, cokernels, arbitrary products and arbitrary coproducts. Let $E: {}_m\mathfrak{Ab} \to \mathfrak{Ab}$ be the embedding and let $F: \mathfrak{Ab} \to {}_m\mathfrak{Ab}$ be given by $F(A) = A/mA$. Show that $F \dashv E$.

7.5. Show that it is possible to choose, for each Λ-module M, a surjection $P(M) \xrightarrow{\varepsilon_M} M$, where $P(M)$ is a free Λ-module, in such a way that P is a functor from $\mathfrak{M}_\Lambda$ to the category $\mathfrak{F}_\Lambda$ of free Λ-modules and ε_M is a natural transformation. If $E: \mathfrak{F}_\Lambda \to \mathfrak{M}_\Lambda$ is the embedding functor, is there an adjugant $\eta: E \dashv P$ such that ε is the counit?

7.6. Let $\mathfrak{C}$ be a category with products and let $D: \mathfrak{C} \to \mathfrak{C}$ be the functor $D(A) = A \times A$. Discuss the question of the existence of a left adjoint to D, and identify it, where it exists, in the cases $\mathfrak{C} = \mathfrak{S}$, $\mathfrak{C} = \mathfrak{T}$, $\mathfrak{C} = \mathfrak{T}^0$, $\mathfrak{C} = \mathfrak{G}$, $\mathfrak{C} = \mathfrak{M}_\Lambda$. What can we say in general?

8. Adjoint Functors and Universal Constructions

Theorem 7.7 established a connection between adjoint functors and universal constructions. We now establish a far closer connection which will enable us finally to give a *definition* of the notion of universal construction! At the same time it will allow us to place Theorem 7.7 in a far more general context.

As our first example of a universal construction we considered the case of a product. We recall that we mentioned in Proposition 5.3 that the construction of a product over the indexing set I could be regarded as a functor $\mathfrak{C}^I \to \mathfrak{C}$. Now there is a *constant functor* (or *diagonal* functor) $P: \mathfrak{C} \to \mathfrak{C}^I$, given by $P(B) = \{B_i\}$, $i \in I$, where $B_i = B$ for all $i \in I$. Suppose $P \dashv G$ and let $\delta: PG \to 1$ be the counit of the adjunction. Then if $\{X_i\}$ is an object of $\mathfrak{C}^I$, δ determines a family of morphisms $p_i: G(\{X_i\}) \to X_i$.

Proposition 8.1. *The product of the objects X_i is $(X; p_i)$ where $X = G(\{X_i\})$.*

Proof. Given $f_i: Y \to X_i$, we have a morphism $f = \{f_i\}: P(Y) \to \{X_i\}$. Then $\eta(f)$ is a morphism $Y \to X$ such that, by (7.3),

$$\delta \circ P(\eta(f)) = f .$$

But this simply means that $p_i \circ \eta(f) = f_i$ for all i. Moreover the equations $\delta \circ P(g) = f$ determines g, since then, again by (7.3), $g = \eta(f)$. $\square$

Thus we see that the product is given by a *right* adjoint to the constant functor $P: \mathfrak{C} \to \mathfrak{C}^I$, and the projections are given by the counit of the adjunction. Plainly the coproduct is given by a *left* adjoint to the constant functor P, the injections arising from the unit of the adjunction. We leave it to the reader to work out the details.

Generalizing the above facts, we *define* a *universal construction* (corresponding to a functor F) as a left adjoint (to F) together with the counit of the adjunction, or as a right adjoint (to F) together with the unit of the adjunction. Quite clearly we should really speak of universal and *co*universal constructions. However, we will adopt the usual convention of using the term "universal construction" in both senses.

We now give a couple of examples, to show just how universal constructions, already familiar to the reader, turn up as left or right adjoints. We first turn to the example of a pull-back.

Let $\mathfrak{L}$ be the category represented by the schema

$$\begin{tikzcd} & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

that is, $\mathfrak{L}$ consists of three objects and two morphisms in addition to the identity morphisms. We may write a functor $\mathfrak{L} \to \mathfrak{C}$ as a pair (φ, ψ) in $\mathfrak{C}$ and represent it as

$$\begin{tikzcd} & A \arrow[d, "\varphi"] \\ B \arrow[r, "\psi"'] & X \end{tikzcd}$$

There is a constant functor F from $\mathfrak{C}$ to the functor category $\mathfrak{C}^{\mathfrak{L}}$ which associates with Z the diagram

$$\begin{tikzcd} & Z \arrow[d, "1"] \\ Z \arrow[r, "1"'] & Z \end{tikzcd}$$

Notice that a morphism $(\gamma, \delta): F(Z) \to (\varphi, \psi)$ in $\mathfrak{C}^{\mathfrak{Q}}$ is really nothing but a pair of morphisms $\gamma: Z \to A$, $\delta: Z \to B$ in $\mathfrak{C}$ such that the square

$$\begin{tikzcd} Z \arrow[r, "\gamma"] \arrow[d, "\delta"'] & A \arrow[d, "\varphi"] \\ B \arrow[r, "\psi"'] & X \end{tikzcd}$$

commutes. Now let $F \dashv G$ and let $\pi: FG \to 1$ be the counit of the adjunction.

Proposition 8.2. *$\pi: FG(\varphi, \psi) \to (\varphi, \psi)$ is the pull-back of (φ, ψ).*

Proof. Let $G(\varphi, \psi) = Y$. Then $\pi: F(Y) \to (\varphi, \psi)$ is a pair of morphisms $\alpha: Y \to A$, $\beta: Y \to B$ such that $\varphi\alpha = \psi\beta$. Moreover, if $(\gamma, \delta): F(Z) \to (\varphi, \psi)$, then $\eta = \eta(\gamma, \delta): Z \to Y$ satisfies, by (7.3)

$$\pi \circ F(\eta) = (\gamma, \delta),$$

that is, $\alpha \circ \eta = \gamma$, $\beta \circ \eta = \delta$. Moreover the equation $\pi \circ F(\zeta) = (\gamma, \delta)$ determines ζ as $\eta(\gamma, \delta)$. $\square$

We remark that in this case (unlike that of Proposition 8.1) F is a full embedding. Thus we may suppose that the unit ε for the adjunction $F \dashv G$ is the identity. This means that the pull-back of $F(Z)$ consists of $(1_Z, 1_Z)$. To see that F is a full embedding, it is best to invoke a general theorem which will be used later. We call a category $\mathfrak{P}$ *connected* if, given any two objects A, B in $\mathfrak{P}$ there exists a (finite) sequence of objects $A_1, A_2, \ldots, A_n$ in $\mathfrak{P}$ such that $A_1 = A$, $A_n = B$ and, for any i, $1 \leqq i \leqq n-1$, $\mathfrak{P}(A_i, A_{i+1}) \cup \mathfrak{P}(A_{i+1}, A_i) \neq \emptyset$. This means that we can connect A to B by a chain of arrows, thus:

$$A \to \cdot \leftarrow \cdot \to \cdots \leftarrow \cdot \to B$$

Theorem 8.3. *Let $\mathfrak{P}$ be a small connected category and let $F: \mathfrak{C} \to \mathfrak{C}^{\mathfrak{P}}$ be the constant functor. Then F is a full embedding.*

Proof. The point at issue is that F is full. Let $f: X \to Y$ in $\mathfrak{C}$ and let P, Q be objects of $\mathfrak{P}$. We have a chain in $\mathfrak{P}$

$$P \to \cdot \leftarrow \cdot \to \cdots \leftarrow \cdot \to Q$$

and hence must show that, given a commutative diagram

$$\begin{tikzcd} X \arrow[r, "1"] \arrow[d, "f"'] & X \arrow[d] & X \arrow[l, "1"'] \arrow[r, "1"] \arrow[d] & \cdots & X \arrow[l, "1"'] \arrow[r, "1"] \arrow[d] & X \arrow[d, "f'"] \\ Y \arrow[r, "1"] & Y & Y \arrow[l, "1"'] \arrow[r, "1"] & \cdots & Y \arrow[l, "1"'] \arrow[r, "1"] & Y \end{tikzcd}$$

in $\mathfrak{C}$, then $f' = f$; but this is obvious. $\square$

Notice that an indexing *set* I, regarded as a category, is not connected (on the contrary, it is *discrete*) unless it is a singleton. On the other hand, directed sets are connected, so that our remarks are related to the classical theory of *inverse limits* (and by duality, *direct limits*). The reader is referred to Chapter VIII, Section 5, for details.

It is clear that the push-out is a universal construction which turns up as a left adjoint to the constant functor $F: \mathfrak{C} \to \mathfrak{C}^{\mathfrak{L}^{\mathrm{opp}}}$. Plainly also the formation of a free Λ-module on a given set is a universal construction corresponding to the underlying functor $U: \mathfrak{M}_\Lambda \to \mathfrak{S}$, which turns up as left adjoint to U.

We now discuss in greater detail another example of a universal construction which turns up as a left adjoint and which is of considerable independent interest: the *Grothendieck group*. Let S be an abelian semigroup. Then $S \times S$ is also, in an obvious way, an abelian semigroup. Introduce into $S \times S$ the homomorphic relation $(a, b) \sim (c, d)$ if and only if there exists $u \in S$ with $a + d + u = b + c + u$.

This is plainly an equivalence relation; moreover, $(S \times S)/\sim\; = \mathrm{Gr}(S)$ is clearly an abelian group since

$$[a, b] + [b, a] = [a + b, a + b] = [0, 0] = 0\,,$$

where square brackets denote equivalence classes. Further there is a homomorphism $\iota: S \to \mathrm{Gr}(S)$, given by $\iota(a) = [a, 0]$, and ι is injective if and only if S is a cancellation semigroup.

It is then easy to show that ι has the following universal property. Let A be an abelian group and let $\sigma: S \to A$ be a homomorphism. Then there exists a unique homomorphism $\bar{\sigma}: \mathrm{Gr}(S) \to A$ such that $\bar{\sigma}\iota = \sigma$,

$$\begin{tikzcd} S \arrow[r, "\iota"] \arrow[d, "\sigma"'] & \mathrm{Gr}(S) \arrow[dl, dotted, "\bar{\sigma}"] \\ A & \end{tikzcd} \tag{8.1}$$

Finally, one readily shows that this universal property determines $\mathrm{Gr}(S)$ up to canonical isomorphism; we call $\mathrm{Gr}(S)$ the *Grothendieck group* of S.

We now show how to express the construction of the Grothendieck group in terms of adjoint functors. Let $\mathfrak{Ab}$ be the category of abelian groups, let $\mathfrak{Abs}$ be the category of abelian semigroups and let $E: \mathfrak{Ab} \to \mathfrak{Abs}$ be the embedding (which is, of course, full). Suppose that $F \dashv E$ and let $\iota: 1 \to EF$ be the unit of the adjunction. Then the reader may readily show that $F(S)$ is the Grothendieck group of S, that ι_S coincides with ι in (8.1), and that $\bar{\sigma} = \eta^{-1}(\sigma)$ – note that σ in (8.1) is strictly a morphism $S \to E(A)$ in $\mathfrak{Abs}$.

The precise formulation of the notion of a universal construction serves to provide a general explanation of the facts asserted in Theorem

7.7. Given a functor $F: \mathfrak{C} \to \mathfrak{D}$ and a small category $\mathfrak{P}$ there is an obvious induced functor $F^{\mathfrak{P}}: \mathfrak{C}^{\mathfrak{P}} \to \mathfrak{D}^{\mathfrak{P}}$. The reader will readily prove the following lemmas.

Lemma 8.4. *If $F \dashv G$, then $F^{\mathfrak{P}} \dashv G^{\mathfrak{P}}$.* □

Lemma 8.5. *If $P: \mathfrak{C} \to \mathfrak{C}^{\mathfrak{P}}$ is the constant functor (for any $\mathfrak{C}$), then the diagram*

$$\begin{tikzcd} \mathfrak{C} \arrow[r, "P"] \arrow[d, "F"'] & \mathfrak{C}^{\mathfrak{P}} \arrow[d, "F^{\mathfrak{P}}"] \\ \mathfrak{D} \arrow[r, "P"'] & \mathfrak{D}^{\mathfrak{P}} \end{tikzcd}$$

commutes. □

We infer from Propositions 7.1, 7.3 and Lemmas 8.4, 8.5 the following basic theorem.

Theorem 8.6. *Let $F: \mathfrak{C} \to \mathfrak{D}$ and $F \dashv G$. Further let $P \dashv R$ (for $P: \mathfrak{C} \to \mathfrak{C}^{\mathfrak{P}}$ and $P: \mathfrak{D} \to \mathfrak{D}^{\mathfrak{P}}$). Then there is a natural equivalence $GR \to RG^{\mathfrak{P}}$ uniquely determined by the given adjugants.* □

This theorem may be described by saying that R commutes with right adjoints. In Chapter 8 we will use the terminology "limit" for such functors R right adjoint to constant functors. Its proof may be summed up in the vivid but slightly inaccurate phrase: if two functors commute so do their (left, right) adjoints. The percipient reader may note that Theorem 8.6 does not quite give the full force of Theorem 7.7. For Theorem 7.7 asserts for example that if a particular family $\{Y_i\}$ of objects of $\mathfrak{D}$ possess a product, so does the family $\{GY_i\}$ of objects of $\mathfrak{C}$; Theorem 8.10, on the other hand, addresses itself to the case where the appropriate universal constructions are known to exist over the *whole* of *both* categories. The reader is strongly advised to write out the proof of Theorem 8.6 in detail.

Exercises:

8.1. Write out in detail the proofs of Lemma 8.4, Lemma 8.5 and Theorem 8.6.

8.2. Express the *kernel* and the *equalizer* as a universal construction in the precise sense of this section.

8.3. Give examples of Theorem 8.6 in the categories $\mathfrak{S}$, $\mathfrak{G}$ and $\mathfrak{M}_A$.

8.4. Let S be an abelian semigroup. Let $F(S)$ be the free abelian group freely generated by the elements of S and let $R(S)$ be the subgroup of $F(S)$ generated by the elements

$$a \dotplus b - (a + b), a, b \in S;$$

here we write $\dotplus$ for the addition in $F(S)$ and $+$ for the addition in S. Establish a natural equivalence

$$Gr(S) \cong F(S)/R(S).$$

8.5. Show that if S is a (commutative) semiring (i.e., S satisfies all the ring axioms except for the existence of additive inverses), then $Gr(S)$ acquires, in a natural way, the structure of a (commutative) ring.

8.6. Show how the construction of the Grothendieck group of a semigroup S, given in Exercise 8.4 above, generalizes to yield the Grothendieck group of any small category with finite coproducts.

8.7. The Birkhoff-Witt Theorem asserts that every Lie algebra $\mathfrak{g}$ over the field K may be embedded in an associative K-algebra $U\mathfrak{g}$ in such a way that the Lie bracket $[x, y]$ coincides with $xy - yx$ in $U\mathfrak{g}$, $x, y \in \mathfrak{g}$, and such that to every associative K-algebra A and every K-linear map $f: \mathfrak{g} \to A$ with

$$f[x, y] = f(x) f(y) - f(y) f(x), \qquad x, y \in \mathfrak{g},$$

there exists a unique K-algebra homomorphism $f^*: U\mathfrak{g} \to A$ extending f. Express this theorem in the language of this section.

8.8. Consider in the category $\mathfrak{C}$ (for example, $\mathfrak{S}, \mathfrak{Ab}, \mathfrak{M}_\Lambda, \mathfrak{G}$) the situation

$$\cdots \rightarrowtail C_{-2} \overset{\gamma_{-2}}{\rightarrowtail} C_{-1} \overset{\gamma_{-1}}{\rightarrowtail} C_0 \overset{\gamma_0}{\rightarrowtail} C_1 \overset{\gamma_1}{\rightarrowtail} C_2 \rightarrowtail \cdots, C_i \text{ in } \mathfrak{C}.$$

Set $\varprojlim C_i = \bigcap_i C_i$ and $\varinjlim C_i = \bigcup_i C_i$, regarding the γ_i as embeddings. What are the universal properties satisfied by $\varprojlim C_i$ and $\varinjlim C_i$? Describe $\varprojlim$ as a right adjoint, and $\varinjlim$ as a left adjoint, to a constant functor. Use this description to suggest appropriate meanings for $\varprojlim C_i$ and $\varinjlim C_i$ if $\mathfrak{C} = \mathfrak{M}_\Lambda$ and each γ_i is epimorphic.

9. Abelian Categories

Certain of the categories we introduced in Section 1 possess significant additional structure. Thus in the categories $\mathfrak{Ab}, \mathfrak{M}_\Lambda^l, \mathfrak{M}_\Lambda^r$ the morphism sets all have abelian group structure and we have the notion of exact sequences. We proceed in this section to extract certain essential features of such categories and define the important notion of an *abelian category*: much of what we do in later chapters really consists of a study of the formal properties of abelian categories. It is a very important fact about such categories that the axioms which characterize them are self-dual, so that any theorem proved about abelian categories yields two dual theorems when applied to a particular abelian category such as $\mathfrak{M}_\Lambda^l$.

In fact, in a very precise sense, module categories are not so special in the totality of abelian categories. A result, called the *full embedding theorem* [37, p. 151] asserts that every small abelian category may be fully embedded in a category of modules over an appropriate ring, in such a way that exactness relations are preserved. This means, in effect, that, in any argument involving only a *finite* diagram, and such notions as kernel, cokernel, image, it is legitimate to suppose that we are operating in a category of modules. Usually, the point of such an assumption is to permit us to suppose that our objects are sets of elements, and to prove statements by "diagram-chases" with elements. The full embedding theorem does not permit us, however, to "argue with elements" if an infinite diagram (e.g., a countable product) is involved.

We begin by defining a notion more general than that of an abelian category.

Definition. An *additive category* $\mathfrak{A}$ is a category with zero object in which any two objects have a product and in which the morphism sets $\mathfrak{A}(A, B)$ are abelian groups such that the composition

$$\mathfrak{A}(A, B) \times \mathfrak{A}(B, C) \to \mathfrak{A}(A, C)$$

is bilinear.

Apart from the examples quoted there are, of course, very many examples of additive categories. We mention two which will be of particular importance to us.

Examples. (a) A *graded Λ-module A* (graded by the integers) is a family of Λ-modules $A = \{A_n\}, n \in \mathbb{Z}$. If A, B are graded Λ-modules, a morphism $\varphi : A \to B$ *of degree k* is a family of Λ-module homomorphisms $\{\varphi_n : A_n \to B_{n+k}\}, n \in \mathbb{Z}$. The category so defined is denoted by $\mathfrak{M}_\Lambda^{\mathbb{Z}}$. We obtain an *additive* category if we restrict ourselves to morphisms of degree 0. (The reader should note a slight abuse of notation: If $\mathbb{Z}$ is regarded as the discrete category consisting of the integers, then $\mathfrak{M}_\Lambda^{\mathbb{Z}}$ is the proper notation for the category with morphisms of degree 0.)

(b) We may replace the grading set $\mathbb{Z}$ in Example (a) by some other set. In particular we will be much concerned in Chapter VIII with modules graded by $\mathbb{Z} \times \mathbb{Z}$; such modules are said to be *bigraded*. If A and B are bigraded modules, a morphism $\varphi : A \to B$ *of bidegree* (k, l) is a family of module homomorphisms $\{\varphi_{n,m} : A_{n,m} \to B_{n+k,m+l}\}$. The category so defined is denoted by $\mathfrak{M}_\Lambda^{\mathbb{Z} \times \mathbb{Z}}$. If we restrict the morphisms to be of bidegree $(0, 0)$ we obtain an *additive* category.

Notice that, although $\mathfrak{M}_\Lambda^{\mathbb{Z}}, \mathfrak{M}_\Lambda^{\mathbb{Z} \times \mathbb{Z}}$ are not additive, they do admit kernels and cokernels. We will adopt the convention that *kernels and cokernels always have degree* 0 *(bidegree* $(0, 0)$*)*. If we define the *image* of a morphism as the kernel of the cokernel, then, of course, these categories also admit images (and *coimages*!).

Abelian categories are additive categories with extra structure. Before proceeding to describe that extra structure, we prove some results about additive categories. We write $A_1 \oplus A_2$ for the product of A_1 and A_2 in the additive category $\mathfrak{A}$. Before stating the first proposition we point out that the zero morphism of $\mathfrak{A}(A, B)$, in the sense of Section 1, *is* the zero element of the abelian group $\mathfrak{A}(A, B)$, so there is no confusion of terminology.

Our first concern is to make good our claim that the axioms are, in fact, self-dual. Apparently there is a failure of self-duality in that we have demanded (finite) products but not coproducts. We show that actually we can also guarantee the existence of coproducts. We prove the even stronger statement:

Proposition 9.1. *Let* $i_1 = \{1,0\} : A_1 \to A_1 \oplus A_2$, $i_2 = \{0,1\} : A_2 \to A_1 \oplus A_2$. *Then* $(A_1 \oplus A_2; i_1, i_2)$ *is the coproduct of* A_1 *and* A_2 *in the additive category* $\mathfrak{A}$.

We first need a basic lemma.

Lemma 9.2. $i_1 p_1 + i_2 p_2 = 1 : A_1 \oplus A_2 \to A_1 \oplus A_2$.

Proof. Now $p_1(i_1 p_1 + i_2 p_2) = p_1 i_1 p_1 + p_1 i_2 p_2 = p_1$, since $p_1 i_1 = 1$, $p_1 i_2 = 0$. Similarly $p_2(i_1 p_1 + i_2 p_2) = p_2$. Thus, by the uniqueness property of the product, $i_1 p_1 + i_2 p_2 = 1$. □

Proof of Proposition 9.1. Given $\varphi_i : A_i \to B$, $i = 1, 2$, define

$$\langle \varphi_1, \varphi_2 \rangle = \varphi_1 p_1 + \varphi_2 p_2 : A_1 \oplus A_2 \to B .$$

Then $\langle \varphi_1, \varphi_2 \rangle i_1 = (\varphi_1 p_1 + \varphi_2 p_2) i_1 = \varphi_1 p_1 i_1 + \varphi_2 p_2 i_1 = \varphi_1$, and similarly $\langle \varphi_1, \varphi_2 \rangle i_2 = \varphi_2$. We establish the uniqueness of $\langle \varphi_1, \varphi_2 \rangle$ by invoking Lemma 9.2. For if $\theta i_1 = \varphi_1$, $\theta i_2 = \varphi_2$, then

$$\theta = \theta(i_1 p_1 + i_2 p_2) = \theta i_1 p_1 + \theta i_2 p_2 = \varphi_1 p_1 + \varphi_2 p_2 = \langle \varphi_1, \varphi_2 \rangle . \quad \square$$

We use the term *sum* instead of *coproduct* in the case of an additive category. Of course, sums only coincide with products in an additive category if a *finite* number of objects is involved. We know from the example of $\mathfrak{Ab}$ that they do not coincide for infinite collections of objects.

Proposition 9.3. *Given*

$$A \xrightarrow{\{\varphi, \psi\}} B \oplus C \xrightarrow{\langle \gamma, \delta \rangle} D ,$$

we have

$$\langle \gamma, \delta \rangle \{\varphi, \psi\} = \gamma\varphi + \delta\psi$$

Proof.

$$\begin{aligned} \langle \gamma, \delta \rangle \{\varphi, \psi\} &= (\gamma p_1 + \delta p_2) \{\varphi, \psi\} = \gamma p_1 \{\varphi, \psi\} + \delta p_2 \{\varphi, \psi\} \\ &= \gamma\varphi + \delta\psi . \quad \square \end{aligned}$$

This proposition has the following interesting corollary.

Corollary 9.4. *The addition in the set* $\mathfrak{A}(A, B)$ *is determined by the category* $\mathfrak{A}$.

Proof. If $\varphi_1, \varphi_2 : A \to B$ then $\varphi_1 + \varphi_2 = \langle \varphi_1, \varphi_2 \rangle \{1, 1\}$. □

We may express this corollary as follows. Given a category with zero object and finite products, the defining property of an additive category asserts that the "morphism sets" functor $\mathfrak{A}^{\text{opp}} \times \mathfrak{A} \to \mathfrak{S}$ may be lifted to $\mathfrak{Ab}$,

$$\begin{tikzcd} & \mathfrak{Ab} \arrow[d, "U"] \\ \mathfrak{A}^{\mathrm{opp}} \times \mathfrak{A} \arrow[ur, dotted] \arrow[r, "\mathfrak{A}(-,-)"] & \mathfrak{S} \end{tikzcd}$$

where U is the "underlying set" functor. Then Corollary 9.4 asserts that the lifting is unique. We next discuss functors between additive categories. We prove

Proposition 9.5. *Let $F:\mathfrak{A}\to\mathfrak{B}$ be a functor from the additive category $\mathfrak{A}$ to the additive category $\mathfrak{B}$. Then the following conditions are equivalent:*

(i) *F preserves sums (of two objects);*

(ii) *F preserves products (of two objects);*

(iii) *for each A, A' in $\mathfrak{A}$, $F:\mathfrak{A}(A,A')\to\mathfrak{B}(FA,FA')$ is a homomorphism.*

Proof. (i)$\Rightarrow$(ii). This is not quite trivial since we are required to show that $F\langle 1,0\rangle=\langle 1,0\rangle$ and $F\langle 0,1\rangle=\langle 0,1\rangle$. Thus we must show that $F(0)=0$ and for this it is plainly sufficient to show that F maps zero objects to zero objects. Let 0 be a zero object of $\mathfrak{A}$. Then plainly, for any A in $\mathfrak{A}$, A is the sum of A and 0 with 1_A and 0 as canonical injections. Thus if $B=F(0)$, then FA is the sum of FA and B, with injections 1_{FA} and $\beta=F(0)$. Consider $0:FA\to B$ and $1:B\to B$. There is then a (unique) morphism $\theta:FA\to B$ such that $\theta 1=0$, $\theta\beta=1$. Thus $1=0:B\to B$ so that B is a zero object.

That (ii) $\Rightarrow$ (i) now follows by duality.

(i) $\Rightarrow$ (iii) If $\varphi_1,\varphi_2:A\to A'$ then $\varphi_1+\varphi_2=\langle\varphi_1,\varphi_2\rangle\{1,1\}$, so that

$$\begin{aligned} F(\varphi_1+\varphi_2) &= \langle F\varphi_1, F\varphi_2\rangle\{1,1\}, \quad \text{since } F \text{ preserves sums and products,} \\ &= F\varphi_1+F\varphi_2 . \end{aligned}$$

(iii) $\Rightarrow$ (ii) To show that F preserves products we must show that

$$\{Fp_1,Fp_2\}:F(A_1\oplus A_2)\to FA_1\oplus FA_2$$

is an isomorphism. We show that

$$F(i_1)\,p_1+F(i_2)\,p_2:FA_1\oplus FA_2\to F(A_1\oplus A_2)$$

is inverse to $\{Fp_1,Fp_2\}$. For

$$\begin{aligned} \{Fp_1,Fp_2\}\,(F(i_1)\,p_1+F(i_2)\,p_2) &= \{Fp_1,Fp_2\}\,F(i_1)\,p_1+\{Fp_1,Fp_2\}\,F(i_2)p_2 \\ &= \{F(p_1i_1),F(p_2i_1)\}\,p_1+\{F(p_1i_2),F(p_2i_2)\}\,p_2 \\ &= \{1,0\}\,p_1+\{0,1\}\,p_2, \quad \text{since} \quad F(0)=0, \\ &= i_1p_1+i_2p_2 \\ &= 1\,; \end{aligned}$$

and

$$\begin{aligned}(F(i_1)\,p_1 + F(i_2)\,p_2)\,\{Fp_1, Fp_2\} &= F(i_1)\,p_1\{Fp_1, Fp_2\} + F(i_2)\,p_2\{Fp_1, Fp_2\}\\ &= Fi_1Fp_1 + Fi_2Fp_2\\ &= F(i_1p_1 + i_2p_2)\,, \quad \text{since } F \text{ satisfies (iii)},\\ &= 1\,. \quad \square\end{aligned}$$

We call a functor satisfying any of the three conditions of Proposition 9.5 an *additive* functor. Such functors will play a crucial role in the sequel. However in order to be able to do effective homological algebra we need to introduce a richer structure into our additive categories; we want to have kernels, cokernels and images. Recall that kernels, if they exist, are always monomorphisms and (by duality) cokernels are always epimorphisms. In an additive category a monomorphism is characterized as having zero kernel, an epimorphism as having zero cokernel.

Definition. An *abelian category* is an additive category in which

(i) every morphism has a kernel and a cokernel;

(ii) every monomorphism is the kernel of its cokernel; every epimorphism is the cokernel of its kernel;

(iii) every morphism is expressible as the composite of an epimorphism and a monomorphism.

The reader will verify that all the examples given of additive categories are, in fact, examples of abelian categories. The category of finite abelian groups is abelian; the category of free abelian groups is additive but not abelian. We will be content in this section to prove a few fundamental properties of abelian categories and to define exact sequences. Notice however that the concept of an abelian category is certainly self-dual.

Proposition 9.6. *Given $\varphi: A \to B$ in the abelian category $\mathfrak{A}$, we may develop from φ the sequence*

$$(S_\varphi) \qquad K \overset{\mu}{\rightarrowtail} A \overset{\eta}{\twoheadrightarrow} I \overset{\nu}{\rightarrowtail} B \overset{\varepsilon}{\twoheadrightarrow} C\,,$$

where $\varphi = \nu\eta$, μ is the kernel of φ, ε is the cokernel of φ, η is the cokernel of μ, and ν is the kernel of ε. Moreover, the decomposition of φ as a composite of an epimorphism and a monomorphism is essentially unique.

We first prove a lemma.

Lemma 9.7. *Suppose $\nu\eta$ and η have the same kernel and η is an epimorphism. Then ν is a monomorphism.*

Proof. Use property (iii) of an abelian category to write $\nu = \varrho\sigma$, with σ epimorphic, ϱ monomorphic. Then $\nu\eta = \varrho\sigma\eta$ and if μ is the kernel of $\sigma\eta$, then μ is the kernel of $\varrho\sigma\eta = \nu\eta$ and hence also of η. Thus μ is the kernel of $\sigma\eta$ and of η so that, by property (ii), $\sigma\eta$ and η are both cokernels of μ. This means that there exists an isomorphism in $\mathfrak{A}$, say ω, such that $\sigma\eta = \omega\eta$, so that $\sigma = \omega$. Thus σ is an isomorphism so that v is a monomorphism. $\square$

Proof of Proposition 9.6. Let μ be the kernel of φ and let η be the cokernel of μ. Since $\varphi\mu = 0$, $\varphi = v\eta$. Since μ is the kernel of η, Lemma 9.7 assures us that v is a monomorphism. If ε is the cokernel of φ, then ε is the cokernel of v (since η is an epimorphism), so v is the kernel of ε and the existence of S_φ is proved.

Finally if $\varphi = v\eta = v_1\eta_1$, with η, η_1 epimorphic, v, v_1 monomorphic, then $\ker\varphi = \ker\eta = \ker\eta_1$ so that $\eta_1 = \omega\eta$ for some isomorphism ω and then $v = v_1\omega$. $\square$

We leave to the reader the proof of the following important corollary.

Corollary 9.8. *If the morphism α in the abelian category $\mathfrak{A}$ is a monomorphism and an epimorphism, then it is an isomorphism.* $\square$

We have shown that the sequence S_φ is, essentially, uniquely determined by the morphism φ. It is, of course, easy to show that the association is functorial in the sense that, given the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ A' & \xrightarrow{\varphi'} & B' \end{array}$$

there is a commutative diagram

$$\begin{array}{ccccccccc} K & \overset{\mu}{\rightarrowtail} & A & \overset{\eta}{\twoheadrightarrow} & I & \overset{v}{\rightarrowtail} & B & \overset{\varepsilon}{\twoheadrightarrow} & C \\ \downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\lambda} \\ K' & \underset{\mu'}{\rightarrowtail} & A' & \underset{\eta'}{\twoheadrightarrow} & I' & \underset{v'}{\rightarrowtail} & B' & \underset{\varepsilon'}{\twoheadrightarrow} & C' \end{array},$$

$$\varphi = v\eta\,, \qquad \varphi' = v'\eta'\,.$$

For since we construct μ, μ' as kernels and then η, η'; $\varepsilon, \varepsilon'$ as cokernels, we automatically obtain morphisms κ, ι, λ such that $\mu'\kappa = \alpha\mu$, $\eta'\alpha = \iota\eta$, $\varepsilon'\beta = \lambda\varepsilon$, and the only point at issue is to show that $v'\iota = \beta v$. But

$$v'\iota\eta = v'\eta'\alpha = \varphi'\alpha = \beta\varphi = \beta v\eta\,,$$

and so, since η is epimorphic, $v'\iota = \beta v$.

Definition. A *short exact sequence* in the abelian category $\mathfrak{A}$ is simply a sequence

$$\cdot \overset{\mu}{\rightarrowtail} \cdot \overset{\varepsilon}{\twoheadrightarrow} \cdot$$

in which μ is the kernel of ε, and ε is the cokernel of μ.

A *long exact sequence* in the abelian category $\mathfrak{A}$ is a sequence

$$\cdots \xrightarrow{\varphi_n} \cdot \xrightarrow{\varphi_{n+1}} \cdots,$$

$\varphi_n = \mu_n \varepsilon_n$, μ_n monomorphic, ε_n epimorphic, where, for each n, μ_n is the kernel of ε_{n+1} (and ε_{n+1} is the cokernel of μ_n).

Exercises:

9.1. Consider the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\alpha"] & B \arrow[r, "\psi"] \arrow[d, "\beta"] & C \arrow[d, "\gamma"] \\ A' \arrow[r, "\varphi'"'] & B' \arrow[r, "\psi'"'] & C' \end{tikzcd}$$

in the abelian category $\mathfrak{A}$. Show that if $A \xrightarrow{\beta\varphi} B' \xrightarrow{\psi'} C'$ is exact and β is a monomorphism, then $A \xrightarrow{\varphi} B \xrightarrow{\psi'\beta} C'$ is exact. What is the dual of this?

9.2. Show that the square

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\alpha"] & B \arrow[d, "\beta"] \\ A' \arrow[r, "\varphi'"'] & B' \end{tikzcd}$$

in the abelian category $\mathfrak{A}$ is commutative if and only if the

$$A \xrightarrow{\{\alpha,\varphi\}} A' \oplus B \xrightarrow{\langle -\varphi', \beta\rangle} B'$$

is differential, i.e., $\langle -\varphi', \beta\rangle \{\alpha, \varphi\} = 0$. Show further that

(i) the square is a pull-back if and only if $\{\alpha, \varphi\}$ is the kernel of $\langle -\varphi', \beta\rangle$,

(ii) the square is a push-out if and only if $\langle -\varphi', \beta\rangle$ is the cokernel of $\{\alpha, \varphi\}$.

9.3. Call the square in Exercise 9.2 above *exact* if the corresponding sequence is exact. Show that if the two squares in the diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & B \arrow[r, "\psi"] \arrow[d, "\beta"] & C \arrow[d, "\gamma"] \\ A' \arrow[r, "\varphi'"'] & B' \arrow[r, "\psi'"'] & C' \end{tikzcd}$$

are exact, so is the composite square.

9.4. In the abelian category $\mathfrak{A}$ the square

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ A' \arrow[r, "\varphi'"'] & B' \end{tikzcd}$$

is a pull-back and the square

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta_1"] \\ A' \arrow[r, "\varphi'_1"'] & B'_1 \end{tikzcd}$$

is a push-out. Show (i) that there exists $\omega: B'_1 \rightarrowtail B'$ such that $\omega\beta_1 = \beta, \omega\varphi'_1 = \varphi'$, and (ii) that the second square above is also a pull-back.

9.5. Let $\mathfrak{A}$ be an abelian category with arbitrary products and coproducts. Define the canonical sum-to-product morphism $\omega: \bigoplus_i A_i \to \prod_i A_i$, and prove that it is not true in general that ω is a monomorphism.

9.6. Let $\mathfrak{A}$ be an abelian category and $\mathfrak{C}$ a small category. Show that the functor category $\mathfrak{A}^{\mathfrak{C}}$ is also abelian. (Hint: Define kernels and cokernels componentwise).

9.7. Give examples of additive categories in which (i) not every morphism has a kernel, (ii) not every morphism has a cokernel.

9.8. Prove Corollary 9.8. Give a counterexample in a nonabelian category.

10. Projective, Injective, and Free Objects

Although our interest in projective and injective objects is confined, in this book, to abelian categories, we will define them in an arbitrary category since the elementary results we adduce in this section will have nothing to do with abelian, or even additive, categories. Our principal purpose in including this short section is to clarify the categorical connection between freeness and projectivity. However, Proposition 10.2 will be applied in Section IV.12, and again later in the book.

The reader will recall the notion of projective and injective modules in Chapter I. Abstracting these notions to an arbitrary category, we are led to the following definitions.

Definition. An object P of a category $\mathfrak{C}$ is said to be *projective* if given the diagram

$$\begin{tikzcd} & P \arrow[dl, dotted, "\psi"'] \arrow[d, "\varphi"] \\ A \arrow[r, two heads, "\varepsilon"'] & B \end{tikzcd}$$

in $\mathfrak{C}$ with ε epimorphic, there exists ψ with $\varepsilon\psi = \varphi$. An object J of $\mathfrak{C}$ is said to be *injective* if it is projective in $\mathfrak{C}^{\mathrm{opp}}$.

Much attention was given in Chapter I to the relation of projective modules to free modules. We now introduce the notion of a free object in an arbitrary category.

Definition. Let the category $\mathfrak{C}$ be equipped with an *underlying functor to sets*, that is, a functor $U: \mathfrak{C} \to \mathfrak{S}$ which is injective on morphisms, and let $Fr \dashv U$. Then, for any set S, $Fr(S)$ is called the *free object on* S (relative to U).

After the introduction to adjoint functors of Sections 7 and 8, the reader should have no difficulty in seeing that $Fr(S)$ has precisely the universal property we would demand of the free object on S. We will be concerned with two questions: (a) are free objects projective, (b) is every object the image of a free (or projective) object? We first note the following property of the category of sets.

Proposition 10.1. *In $\mathfrak{S}$ every object is both projective and injective.* $\square$

We now prove

Proposition 10.2. *Let $F:\mathfrak{C}\to\mathfrak{D}$ and $F\dashv G$. If G maps epimorphisms to epimorphisms, then F maps projectives to projectives.*

Proof. Let P be a projective object of $\mathfrak{C}$ and consider the diagram, in $\mathfrak{D}$,

$$\begin{tikzcd} & F(P) \arrow[d, "\varphi"] \\ A \arrow[r, two heads, "\varepsilon"] & B \end{tikzcd}$$

Applying the adjugant, this gives rise to a diagram

$$\begin{tikzcd} & P \arrow[d, "\eta(\varphi)"] \\ GA \arrow[r, two heads, "G\varepsilon"] & GB \end{tikzcd}$$

in $\mathfrak{C}$, where, by hypothesis, $G\varepsilon$ remains epimorphic. There thus exists $\psi':P\to GA$ in $\mathfrak{C}$ with $G\varepsilon\circ\psi'=\eta(\varphi)$, so that $\varepsilon\circ\psi=\varphi$, where $\eta(\psi)=\psi'$. $\square$

Corollary 10.3. *If the underlying functor $U:\mathfrak{C}\to\mathfrak{S}$ sends epimorphisms to surjections then every free object in $\mathfrak{C}$ is projective.* $\square$

This is the case, for example, for $\mathfrak{Ab}$, $\mathfrak{M}_A$, $\mathfrak{G}$; the hypothesis is false, however, for the category of integral domains, where, as the reader may show, the inclusion $\mathbb{Z}\subsetneqq\mathbb{Q}$ is an epimorphism (see Exercise 3.2).

We now proceed to the second question and show

Proposition 10.4. *Let $Fr\dashv U$, where $U:\mathfrak{C}\to\mathfrak{S}$ is the underlying functor. Then the counit $\delta:FrU(A)\to A$ is an epimorphism.*

Proof. Suppose $\alpha,\alpha':A\to B$ and $\alpha\circ\delta=\alpha'\circ\delta$. Applying the adjugant we find $U(\alpha)=U(\alpha')$. But U is injective on morphisms so $\alpha=\alpha'$. $\square$

Thus every object admits a free presentation by means of the free object on its underlying set and this free presentation is a projective presentation if U sends epimorphisms to surjections.

Proposition 10.5. (i) *Every retract of a projective object is projective.*

(ii) *If U sends epimorphisms to surjections, then every projective object is a retract of a free (projective) object.*

Proof. (i) Given $P\underset{\sigma}{\overset{\varrho}{\rightleftarrows}}Q$, $\varrho\sigma=1$, P projective, and

$$\begin{tikzcd} & Q \arrow[d, "\varphi"] \\ A \arrow[r, two heads, "\varepsilon"] & B, \end{tikzcd}$$

choose $\psi': P \to A$ so that $\varepsilon\psi' = \varphi\varrho$ and set $\psi = \psi'\sigma$. Then

$$\varepsilon\psi = \varepsilon\psi'\sigma = \varphi\varrho\sigma = \varphi\,.$$

(ii) Since δ is an epimorphism it follows that if A is projective there exists $\sigma: A \to FrU(A)$ with $\delta\sigma = 1$. Note that, even without the hypothesis on U, a projective P is a retract of $FrU(P)$; the force of the hypothesis is that then $FrU(P)$ is itself projective. □

Proposition 10.6. (i) *A coproduct of free objects is free.*
(ii) *A coproduct of projective objects is projective.*

Proof. (i) Since Fr has a right adjoint, it maps coproducts to coproducts. (Coproducts in $\mathfrak{S}$ are disjoint unions.)
(ii) Let $P = \coprod_i P_i$, P_i projective, and consider the diagram

$$\begin{tikzcd} & P \arrow[d, "\varphi"] \\ A \arrow[r, two heads, "\varepsilon"] & B \end{tikzcd}$$

Then $\varphi = \langle\varphi_i\rangle$, $\varphi_i: P_i \to B$ and, for each i, we have $\psi_i: P_i \to A$ with $\varepsilon\psi_i = \varphi_i$. Then if $\psi = \langle\psi_i\rangle$, we have $\varepsilon\psi = \varphi$. Notice that, if the morphism sets of $\mathfrak{C}$ are nonempty then if P is projective so is each P_i by Proposition 10.5 (i). □

We shall have nothing to say here about *injective* objects beyond those remarks which simply follow by dualization.

Exercises:

10.1. Use Proposition I. 8.1 to prove that if Λ is free as an abelian group, then every free Λ-module is a free abelian group. (Of course, there are other proofs!).

10.2. Verify in detail that $Fr(S)$ has the universal property we would demand of the free object on S in the case $\mathfrak{C} = \mathfrak{G}$.

10.3. Deduce by a categorical argument that if $\mathfrak{C} = \mathfrak{G}$, then $Fr(S \cup T)$ is the free product of $Fr(S)$ and $Fr(T)$ if $S \cap T = \emptyset$.

10.4. Dualize Proposition 10.5.

10.5. Show that $\mathbb{Z} \subseteqq \mathbb{Q}$ is an epimorphism (i) in the category of integral domains, (ii) in the category of commutative rings. Are there free objects in these categories which are not projective?

10.6. Let Λ be a ring not necessarily having a unity element. A (left) Λ-module is defined in the obvious way, simply suppressing the axiom $1a = a$. Show that Λ, as a (left) Λ-module, need not be free!

III. Extensions of Modules

In studying modules, as in studying any algebraic structures, the standard procedure is to look at submodules and associated quotient modules. The extension problem then appears quite naturally: given modules A, B (over a fixed ring Λ) what modules E may be constructed with submodule B and associated quotient module A? The set of equivalence classes of such modules E, written $E(A, B)$, may then be given an abelian group structure in a way first described by Baer [3]. It turns out that this group $E(A, B)$ is naturally isomorphic to a group $\operatorname{Ext}_\Lambda(A, B)$ obtained from A and B by the characteristic, indeed prototypical, methods of homological algebra. To be precise, $\operatorname{Ext}_\Lambda(A, B)$ is the value of the *first right derived* functor of $\operatorname{Hom}_\Lambda(-, B)$ on the module A, in the sense of Chapter IV.

In this chapter we study the homological and functorial properties of $\operatorname{Ext}_\Lambda(A, B)$. We show, in particular, that $\operatorname{Ext}_\Lambda(-, -)$ is balanced in the sense that $\operatorname{Ext}_\Lambda(A, B)$ is also the value of the first right derived functor of $\operatorname{Hom}_\Lambda(A, -)$ on the module B. Also, when $\Lambda = \mathbb{Z}$, so that A, B are abelian groups, we indicate how to compute the Ext groups; and prove a theorem of Stein-Serre showing how, for abelian groups of countable rank, the vanishing of $\operatorname{Ext}(A, \mathbb{Z})$ characterizes the free abelian groups A.

In view of the adjointness relation between the *tensor product* and Hom (see Theorem 7.2), it is natural to expect a similar theory for the tensor product and its first derived functors. This is given in the last two sections of the chapter.

1. Extensions

Let A, B be two Λ-modules. We want to consider all possible Λ-modules E such that B is a submodule of E and $E/B \cong A$. We then have a short exact sequence

$$B \overset{\kappa}{\rightarrowtail} E \overset{\nu}{\twoheadrightarrow} A$$

of Λ-modules; such a sequence is called an *extension* of A by B. We shall say that the extension $B \rightarrowtail E_1 \twoheadrightarrow A$ is *equivalent* to the extension $B \rightarrowtail E_2 \twoheadrightarrow A$ if there is a homomorphism $\xi : E_1 \to E_2$ such that the

diagram
$$\begin{tikzcd} B \arrow[r, tail] \arrow[d, equal] & E_1 \arrow[r, two heads] \arrow[d, "\xi"] & A \arrow[d, equal] \\ B \arrow[r, tail] & E_2 \arrow[r, two heads] & A \end{tikzcd}$$

is commutative. This relation plainly is transitive and reflexive. Since ξ is necessarily an isomorphism by Lemma I.1.1, it is symmetric, also.

The reader will notice that it would be possible to define an equivalence relation other than the one defined above: for example two extensions E_1, E_2 may be called equivalent if the modules E_1, E_2 are isomorphic, or they may be called equivalent if there exists a homomorphism $\xi: E_1 \to E_2$ inducing automorphisms in both A and B. In our definition of equivalence we insist that the homomorphism $\xi: E_1 \to E_2$ induces the *identity* in both A and B. We refer the reader to Exercise 1.1 which shows that the different definitions of equivalence are indeed different notions. The reason we choose our definition will become clear with Theorem 1.4 and Corollary 2.5.

We denote the set of equivalence classes of extensions of A by B by $E(A, B)$. Obviously $E(A, B)$ contains at least one element: The Λ-module $A \oplus B$, together with the maps ι_B, π_A, yields an extension

$$B \overset{\iota_B}{\rightarrowtail} A \oplus B \overset{\pi_A}{\twoheadrightarrow} A\,. \tag{1.1}$$

The map $\iota_A: A \to A \oplus B$ satisfies the equation $\pi_A \iota_A = 1_A$ and the map $\pi_B: A \oplus B \to B$ the equation $\pi_B \iota_B = 1_B$. Because of the existence of such maps we call any extension equivalent to (1.1) a *split extension* of A by B.

Our aim is now to make $E(-, -)$ into a functor; we therefore have to define induced maps. The main part of the work is achieved by the following lemmas.

Lemma 1.1. *The square*
$$\begin{tikzcd} Y \arrow[r, "\alpha"] \arrow[d, "\beta"] & A \arrow[d, "\varphi"] \\ B \arrow[r, "\psi"] & X \end{tikzcd} \tag{1.2}$$

is a pull-back diagram if and only if the sequence
$$0 \to Y \xrightarrow{\{\alpha, \beta\}} A \oplus B \xrightarrow{\langle \varphi, -\psi \rangle} X$$
is exact.

Proof. We have to show that the universal property of the pull-back of (φ, ψ) is the same as the universal property of the kernel of $\langle \varphi, -\psi \rangle$. But it is plain that two maps $\gamma: Z \to A$ and $\delta: Z \to B$ make the square

$$\begin{tikzcd} Z \arrow[r, "\gamma"] \arrow[d, "\delta"] & A \arrow[d, "\varphi"] \\ B \arrow[r, "\psi"] & X \end{tikzcd}$$

commutative if and only if they induce a map $\{\gamma, \delta\}: Z \to A \oplus B$ such that $\langle \varphi, -\psi \rangle \circ \{\gamma, \delta\} = 0$. The universal property of the kernel asserts the existence of a unique map $\zeta: Z \to Y$ with $\{\alpha, \beta\} \circ \zeta = \{\gamma, \delta\}$. The universal property of the pull-back asserts the existence of a unique map $\zeta: Z \to Y$ with $\alpha \circ \zeta = \gamma$ and $\beta \circ \zeta = \delta$. □

Lemma 1.2. *If the square* (1.2) *is a pull-back diagram, then*
(i) β *induces* $\ker \alpha \xrightarrow{\sim} \ker \psi$;
(ii) *if* ψ *is an epimorphism, then so is* α.

Proof. Part (i) has been proved in complete generality in Theorem II.6.2. For part (ii) we consider the sequence $0 \to Y \xrightarrow{\{\alpha,\beta\}} A \oplus B \xrightarrow{\langle \varphi, -\psi \rangle} X$, which is exact by Lemma 1.1. Suppose $a \in A$. Since ψ is epimorphic there exists $b \in B$ with $\varphi a = \psi b$, whence it follows that $(a, b) \in \ker \langle \varphi, -\psi \rangle = \operatorname{im} \{\alpha, \beta\}$ by exactness. Thus there exists $y \in Y$ with $a = \alpha y$ (and $b = \beta y$). Hence α is epimorphic. □

We now prove a partial converse of Lemma 1.2 (i).

Lemma 1.3. *Let*

$$\begin{tikzcd} B \arrow[r, tail, "\kappa'"] \arrow[d, equal] & E' \arrow[r, two heads, "\nu'"] \arrow[d, "\xi"] & A' \arrow[d, "\alpha"] \\ B \arrow[r, tail, "\kappa"] & E \arrow[r, two heads, "\nu"] & A \end{tikzcd}$$

be a commutative diagram with exact rows. Then the right-hand square is a pull-back diagram.

Proof. Let

$$\begin{tikzcd} P \arrow[r, "\varepsilon"] \arrow[d, "\varphi"] & A' \arrow[d, "\alpha"] \\ E \arrow[r, two heads, "\nu"] & A \end{tikzcd}$$

be a pull-back diagram. By Lemma 1.2 ε is epimorphic and φ induces an isomorphism $\ker \varepsilon \cong B$. Hence we obtain an extension

$$B \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A' .$$

By the universal property of P there exists a map $\zeta: E' \to P$, such that $\varphi\zeta = \xi$, $\varepsilon\zeta = \nu'$. Since ζ induces the identity in both A' and B, ζ is an isomorphism by Lemma I.1.1. □

We leave it to the reader to prove the duals of Lemmas 1.1, 1.2, 1.3. In the sequel we shall feel free to refer to these lemmas when we require either their statements or the dual statements.

Let $\alpha: A' \to A$ be a homomorphism and let $B \overset{\kappa}{\rightarrowtail} E \overset{\nu}{\twoheadrightarrow} A$ be a representative of an element in $E(A, B)$. Consider the diagram

$$\begin{tikzcd} & E^\alpha \arrow[r, dotted, "\nu'"] \arrow[d, dotted, "\xi"] & A' \arrow[d, "\alpha"] \\ B \arrow[r, tail, "\kappa"] & E \arrow[r, two heads, "\nu"] & A \end{tikzcd}$$

where $(E^\alpha; \nu', \xi)$ is the pull-back of (α, ν). By Lemma 1.2 we obtain an extension $B \rightarrowtail E^\alpha \overset{\nu'}{\twoheadrightarrow} A'$. Thus we can define our induced map

$$\alpha^*: E(A, B) \to E(A', B)$$

by assigning to the class of $B \rightarrowtail E \twoheadrightarrow A$ the class of $B \rightarrowtail E^\alpha \twoheadrightarrow A'$. Plainly this definition is independent of the chosen representative $B \rightarrowtail E \twoheadrightarrow A$.

We claim that this definition of $E(\alpha, B) = \alpha^*$ makes $E(-, B)$ into a contravariant functor. Indeed it is plain that for $\alpha = 1_A: A \to A$ the induced map is the identity in $E(A, B)$. Let $\alpha': A'' \to A'$ and $\alpha: A' \to A$. In order to show that $E(\alpha \circ \alpha', B) = E(\alpha', B) \circ E(\alpha, B)$, we have to prove that in the diagram

$$\begin{tikzcd} (E^\alpha)^{\alpha'} \arrow[r, dotted] \arrow[d, dotted] & A'' \arrow[d, "\alpha'"] \\ E^\alpha \arrow[r, dotted] \arrow[d, dotted] & A' \arrow[d, "\alpha"] \\ E \arrow[r, "\nu"] & A \end{tikzcd}$$

where each square is a pull-back, the composite square is the pull-back of $(\nu, \alpha \circ \alpha')$. But this follows readily from the universal property of the pull-back.

Now let $\beta: B \to B'$ be a homomorphism, and let $B \overset{\kappa}{\rightarrowtail} E \overset{\nu}{\twoheadrightarrow} A$ again be a representative of an element in $E(A. B)$. We consider the diagram

$$\begin{tikzcd} B \arrow[r, tail, "\kappa"] \arrow[d, "\beta"] & E \arrow[r, two heads, "\nu"] \arrow[d, dotted, "\xi"] & A \\ B' \arrow[r, dotted, "\kappa'"] & E_\beta & \end{tikzcd}$$

where $(E_\beta; \kappa', \xi)$ is the push-out of (β, κ). The dual of Lemma 1.2 shows that we obtain an extension $B' \rightarrowtail E_\beta \twoheadrightarrow A$. We then can define

$$\beta_*: E(A, B) \to E(A, B')$$

by assigning to the class of $B \rightarrowtail E \twoheadrightarrow A$ the class of $B' \rightarrowtail E_\beta \twoheadrightarrow A$. As above one easily proves that this definition of $E(A, \beta) = \beta_*$ makes $E(A, -)$ into a covariant functor. Indeed, we even assert:

Theorem 1.4. *$E(-, -)$ is a bifunctor from the category of Λ-modules to the category of sets. It is contravariant in the first and covariant in the second variable.*

Proof. It remains to check that $\beta_* \alpha^* = \alpha^* \beta_* : E(A, B) \to E(A', B')$. We can construct the following (3-dimensional) commutative diagram, using pull-backs and push-outs.

$$\begin{tikzcd}
B \arrow[rr, tail] \arrow[dr, equal] \arrow[dd, "\beta"'] & & E \arrow[rr, two heads] \arrow[dd] & & A \arrow[dd, equal] & \\
& B \arrow[rr, tail] \arrow[dd, "\beta"] & & E^\alpha \arrow[ul] \arrow[rr, two heads] \arrow[dd] \arrow[ddd, dashed, bend right] & & A' \arrow[ul, "\alpha"'] \arrow[dd, equal] \\
B' \arrow[rr, tail] \arrow[dr, equal] & & E_\beta \arrow[rr, two heads] & & A & \\
& B' \arrow[rr, tail] \arrow[drr, tail] & & (E^\alpha)_\beta \arrow[rr, two heads] \arrow[d, dashed] & & A' \arrow[ul, "\alpha"'] \\
& & & (E_\beta)^\alpha \arrow[uul] \arrow[urr, two heads] & &
\end{tikzcd}$$

We have to show the existence of $(E^\alpha)_\beta \to (E_\beta)^\alpha$ such that the diagram remains commutative. We first construct $E^\alpha \to (E_\beta)^\alpha$ satisfying the necessary commutativity relations. Since $E^\alpha \to E \to E_\beta \to A$ coincides with $E^\alpha \to A' \to A$, we do indeed find $E^\alpha \to (E_\beta)^\alpha$ such that $E^\alpha \to (E_\beta)^\alpha \to E_\beta$ coincides with $E^\alpha \to E \to E_\beta$ and $E^\alpha \to (E_\beta)^\alpha \to A'$ coincides with $E^\alpha \to A'$. It remains to check that $B \to E^\alpha \to (E_\beta)^\alpha$ coincides with $B \to B' \to (E_\beta)^\alpha$. By the uniqueness of the map into the pull-back it suffices to check that $B \to E^\alpha \to (E_\beta)^\alpha \to E_\beta$ coincides with $B \to B' \to (E_\beta)^\alpha \to E_\beta$ and $B \to E^\alpha \to (E_\beta)^\alpha \to A'$ coincides with $B \to B' \to (E_\beta)^\alpha \to A'$, and these facts follow from the known commutativity relations. Since $B \to E^\alpha \to (E_\beta)^\alpha$ coincides with $B \to B' \to (E_\beta)^\alpha$ we find $(E^\alpha)_\beta \to (E_\beta)^\alpha$ such that $B' \to (E^\alpha)_\beta \to (E_\beta)^\alpha$ coincides with $B' \to (E_\beta)^\alpha$ and $E^\alpha \to (E^\alpha)_\beta \to (E_\beta)^\alpha$ coincides with $E^\alpha \to (E_\beta)^\alpha$. It only remains to show that $(E^\alpha)_\beta \to (E_\beta)^\alpha \to A'$ coincides with $(E^\alpha)_\beta \to A'$. Again, uniqueness considerations allow us merely to prove that $B' \to (E^\alpha)_\beta \to (E_\beta)^\alpha \to A'$ coincides with $B' \to (E^\alpha)_\beta \to A'$, and $E^\alpha \to (E^\alpha)_\beta \to (E_\beta)^\alpha \to A'$ coincides with $E^\alpha \to (E^\alpha)_\beta \to A'$. Since these facts, too, follow from the known commutativity relations, the theorem is proved. $\square$

Exercises:

1.1. Show that the following two extensions are nonequivalent

$$\mathbb{Z} \overset{\mu}{\rightarrowtail} \mathbb{Z} \overset{\varepsilon}{\twoheadrightarrow} \mathbb{Z}_3, \qquad \mathbb{Z} \overset{\mu'}{\rightarrowtail} \mathbb{Z} \overset{\varepsilon'}{\twoheadrightarrow} \mathbb{Z}_3$$

where $\mu = \mu'$ is multiplication by 3, $\varepsilon(1) = 1 \pmod 3$ and $\varepsilon'(1) = 2 \pmod 3$.

1.2. Compute $E(\mathbb{Z}_p, \mathbb{Z})$, p prime.

1.3. Prove the duals of Lemmas 1.1, 1.2, 1.3.
1.4. Show that the class of the split extension in $E(A,B)$ is preserved under the induced maps.
1.5. Prove: If P is projective, $E(P,B)$ contains only one element.
1.6. Prove: If I is injective, $E(A,I)$ contains only one element.
1.7. Show that $E(A, B_1 \oplus B_2) \cong E(A, B_1) \times E(A, B_2)$. Is there a corresponding formula with respect to the first variable?
1.8. Prove Theorem 1.4 using explicit constructions of pull-back and push-out.

2. The Functor Ext

In the previous section we have defined a bifunctor $E(-,-)$ from the category of Λ-modules to the categories of sets. In this section we shall define another bifunctor $\mathrm{Ext}_\Lambda(-,-)$ to the category of abelian groups, and eventually compare the two.

A short exact sequence $R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ of Λ-modules with P projective is called a *projective presentation* of A. By Theorem I.2.2 such a presentation induces for a Λ-module B an exact sequence

$$\mathrm{Hom}_\Lambda(A,B) \overset{\varepsilon^*}{\rightarrowtail} \mathrm{Hom}_\Lambda(P,B) \xrightarrow{\mu^*} \mathrm{Hom}_\Lambda(R,B)\,. \tag{2.1}$$

To the modules A and B, and to the chosen projective presentation of A we therefore can associate the abelian group

$$\mathrm{Ext}^\varepsilon_\Lambda(A,B) = \operatorname{coker}(\mu^* : \mathrm{Hom}_\Lambda(P,B) \to \mathrm{Hom}_\Lambda(R,B))\,.$$

The superscript ε is to remind the reader that the group is defined via a particular projective presentation of A. An element in $\mathrm{Ext}^\varepsilon_\Lambda(A,B)$ may be represented by a homomorphism $\varphi : R \to B$. The element represented by $\varphi : R \to B$ will be denoted by $[\varphi] \in \mathrm{Ext}^\varepsilon_\Lambda(A,B)$. Then $[\varphi_1] = [\varphi_2]$ if and only if $\varphi_1 - \varphi_2$ extends to P.

Clearly a homomorphism $\beta : B \to B'$ will map the sequence (2.1) into the corresponding sequence for B'. We thus get an induced map $\beta_* : \mathrm{Ext}^\varepsilon_\Lambda(A,B) \to \mathrm{Ext}^\varepsilon_\Lambda(A,B')$, which is easily seen to make $\mathrm{Ext}^\varepsilon_\Lambda(A,-)$ into a functor.

Next we will show that for two different projective presentations of A we obtain the "same" functor. Let $R' \overset{\mu'}{\rightarrowtail} P' \overset{\varepsilon'}{\twoheadrightarrow} A'$ and $R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ be projective presentations of A', A respectively. Let $\alpha : A' \to A$ be a homomorphism. Since P' is projective, there is a homomorphism $\pi : P' \to P$, inducing $\sigma : R' \to R$ such that the following diagram is commutative:

$$\begin{tikzcd} R' \arrow[r, tail, "\mu'"] \arrow[d, "\sigma"] & P' \arrow[r, two heads, "\varepsilon'"] \arrow[d, "\pi"] & A' \arrow[d, "\alpha"] \\ R \arrow[r, tail, "\mu"] & P \arrow[r, two heads, "\varepsilon"] & A \end{tikzcd}$$

We sometimes say that π *lifts* α.

Clearly π, together with σ, will induce a map

$$\pi^* : \operatorname{Ext}_\Lambda^\varepsilon(A, B) \to \operatorname{Ext}_\Lambda^{\varepsilon'}(A', B)$$

which plainly is natural in B. Thus every π gives rise to a natural transformation from $\operatorname{Ext}_\Lambda^\varepsilon(A, -)$ into $\operatorname{Ext}_\Lambda^{\varepsilon'}(A', -)$. In the following lemma we prove that this natural transformation depends only on $\alpha : A' \to A$ and not on the chosen $\pi : P' \to P$ lifting α.

Lemma 2.1. *π^* does not depend on the chosen $\pi : P' \to P$ but only on $\alpha : A' \to A$.*

Proof. Let $\pi_i : P' \to P$, $i = 1, 2$, be two homomorphisms lifting α and inducing $\sigma_i : R' \to R$, so that the following diagram is commutative for $i = 1, 2$

$$\begin{tikzcd} R' \arrow[r, tail, "\mu'"] \arrow[d, "\sigma_i"] & P' \arrow[r, two heads, "\varepsilon'"] \arrow[d, "\pi_i"] & A' \arrow[d, "\alpha"] \\ R \arrow[r, tail, "\mu"] & P \arrow[r, two heads, "\varepsilon"] & A \end{tikzcd}$$

Consider $\pi_1 - \pi_2$; since π_1, π_2 induce the same map $\alpha : A' \to A$, $\pi_1 - \pi_2$ factors through a map $\tau : P' \to R$, such that $\pi_1 - \pi_2 = \mu\tau$. It follows that $\sigma_1 - \sigma_2 = \tau\mu'$. Thus, if $\varphi : R \to B$ is a representative of the element $[\varphi] \in \operatorname{Ext}_\Lambda^\varepsilon(A, B)$, we have $\pi_1^*[\varphi] = [\varphi\sigma_1] = [\varphi\sigma_2 + \varphi\tau\mu'] = [\varphi\sigma_2] = \pi_2^*[\varphi]$. □

To stress the independence from the choice of π we shall call the natural transformation $(\alpha; P', P) : \operatorname{Ext}_\Lambda^\varepsilon(A, -) \to \operatorname{Ext}_\Lambda^{\varepsilon'}(A', -)$, instead of π^*. Let $\alpha' : A'' \to A'$ and $\alpha : A' \to A$ be two homomorphisms and $R'' \rightarrowtail P'' \twoheadrightarrow A''$, $R' \rightarrowtail P' \twoheadrightarrow A'$, $R \rightarrowtail P \to A$ projective presentations of A'', A', A respectively. Let $\pi' : P'' \to P'$ lift α' and $\pi : P' \to P$ lift α. Then $\pi \circ \pi' : P'' \to P$ lifts $\alpha \circ \alpha'$; whence it follows that

$$(\alpha'; P'', P') \circ (\alpha; P', P) = (\alpha \circ \alpha'; P'', P). \tag{2.2}$$

Also, we have

$$(1_A; P, P) = 1. \tag{2.3}$$

This yields a proof of

Corollary 2.2. *Let $R \rightarrowtail P \overset{\varepsilon}{\twoheadrightarrow} A$ and $R' \rightarrowtail P' \overset{\varepsilon'}{\twoheadrightarrow} A$ be two projective presentations of A. Then*

$$(1_A; P', P) : \operatorname{Ext}_\Lambda^\varepsilon(A, -) \to \operatorname{Ext}_\Lambda^{\varepsilon'}(A, -)$$

is a natural equivalence.

Proof. Let $\pi : P \to P'$ and $\pi' : P' \to P$ both lift $1_A : A \to A$. By formulas (2.2) and (2.3) we obtain $(1_A; P, P') \circ (1_A; P', P) = (1_A; P, P) = 1 : \operatorname{Ext}_\Lambda^\varepsilon(A, -) \to \operatorname{Ext}_\Lambda^\varepsilon(A, -)$. Analogously, $(1_A; P', P) \circ (1_A; P, P') = 1$, whence the assertion. □

By this natural equivalence we are allowed to drop the superscript ε and to write, simply, $\operatorname{Ext}_\Lambda(A, B)$.

Of course, we want to make $\operatorname{Ext}_\Lambda(-, B)$ into a functor. It is obvious by now that given $\alpha : A' \to A$ we can define an induced map α^* as follows: Choose projective presentations $R' \rightarrowtail P' \overset{\varepsilon'}{\twoheadrightarrow} A'$ and $R \rightarrowtail P \overset{\varepsilon}{\twoheadrightarrow} A$ of A', A respectively, and let $\alpha^* = (\alpha; P', P) : \operatorname{Ext}^\varepsilon_\Lambda(A, B) \to \operatorname{Ext}^{\varepsilon'}_\Lambda(A', B)$. Formulas (2.2), (2.3) establish the facts that this definition is compatible with the natural equivalences of Corollary 2.2 and that $\operatorname{Ext}_\Lambda(-, B)$ becomes a (contravariant) functor. We leave it to the reader to prove the *bi*-functoriality part in the following theorem.

Theorem 2.3. $\operatorname{Ext}_\Lambda(-, -)$ *is a bifunctor from the category of Λ-modules to the category of abelian groups. It is contravariant in the first, and covariant in the second variable.* □

Instead of regarding $\operatorname{Ext}_\Lambda(A, B)$ as an abelian group, we clearly can regard it just as a set. We thus obtain a *set*-valued bifunctor which – for convenience – we shall still call $\operatorname{Ext}_\Lambda(-, -)$.

Theorem 2.4. *There is a natural equivalence of set-valued bifunctors* $\eta : E(-, -) \xrightarrow{\sim} \operatorname{Ext}_\Lambda(-, -)$.

Proof. We first define an isomorphism of sets

$$\eta : E(A, B) \xrightarrow{\sim} \operatorname{Ext}^\varepsilon_\Lambda(A, B),$$

natural in B, where $R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ is a fixed projective presentation of A. We will then show that η is natural in A.

Given an element in $E(A, B)$, represented by the extension $B \overset{\kappa}{\rightarrowtail} E \overset{\nu}{\twoheadrightarrow} A$, we form the diagram

$$\begin{tikzcd} R \arrow[r, tail, "\mu"] \arrow[d, dotted, "\psi"] & P \arrow[r, two heads, "\varepsilon"] \arrow[d, dotted, "\varphi"] & A \arrow[d, equal] \\ B \arrow[r, tail, "\kappa"] & E \arrow[r, two heads, "\nu"] & A \end{tikzcd}$$

The homomorphism $\psi : R \to B$ defines an element $[\psi] \in \operatorname{Ext}^\varepsilon_\Lambda(A, B) = \operatorname{coker}(\mu^* : \operatorname{Hom}_\Lambda(P, B) \to \operatorname{Hom}_\Lambda(R, B))$. We claim that this element does not depend on the particular $\varphi : P \to E$ chosen. Thus let $\varphi_i : P \to E$, $i = 1, 2$, be two maps inducing $\psi_i : R \to B$, $i = 1, 2$. Then $\varphi_1 - \varphi_2$ factors through $\tau : P \to B$, i.e., $\varphi_1 - \varphi_2 = \kappa\tau$. It follows that $\psi_1 - \psi_2 = \tau\mu$, whence $[\psi_1] = [\psi_2 + \tau\mu] = [\psi_2]$.

Since two representatives of the same element in $E(A, B)$ obviously induce the same element in $\operatorname{Ext}^\varepsilon_\Lambda(A, B)$, we have defined a map $\eta : E(A, B) \to \operatorname{Ext}^\varepsilon_\Lambda(A, B)$. We leave it to the reader to prove the naturality of η with respect to B.

Conversely, given an element in $\operatorname{Ext}^\varepsilon_\Lambda(A, B)$, we represent this element by a homomorphism $\psi : R \to B$. Taking the push-out of (ψ, μ) we obtain the diagram

$$\begin{tikzcd} R \arrow[r, tail, "\mu"] \arrow[d, "\psi"'] & P \arrow[r, two heads, "\varepsilon"] \arrow[d, dotted, "\varphi"] & A \arrow[d, equal] \\ B \arrow[r, dotted, "\kappa"] & E \arrow[r, dotted, "\nu"] & A \end{tikzcd}$$

By the dual of Lemma 1.2 the bottom row $B \overset{\kappa}{\rightarrowtail} E \overset{\nu}{\twoheadrightarrow} A$ is an extension. We claim that the equivalence class of this extension is independent of the particular representative $\psi : R \to B$ chosen. Indeed another representative $\psi' : R \to B$ has the form $\psi' = \psi + \tau\mu$ where $\tau : P \to B$. The reader may check that the diagram

$$\begin{tikzcd} R \arrow[r, tail, "\mu"] \arrow[d, "\psi'"'] & P \arrow[r, two heads, "\varepsilon"] \arrow[d, "\varphi'"] & A \arrow[d, equal] \\ B \arrow[r, tail, "\kappa"] & E \arrow[r, two heads, "\nu"] & A \end{tikzcd}$$

with $\varphi' = \varphi + \kappa\tau$ is commutative. By the dual of Lemma 1.3 the left hand square is a push-out diagram, whence it follows that the extension we arrive at does not depend on the representative. We thus have defined a map

$$\xi : \mathrm{Ext}^{\varepsilon}_{\Lambda}(A, B) \to E(A, B)$$

which is easily seen to be natural in B.

Using Lemma 1.3 it is easily proved that η, ξ are inverse to each other. We thus have an equivalence

$$\eta : E(A, B) \xrightarrow{\sim} \mathrm{Ext}^{\varepsilon}_{\Lambda}(A, B)$$

which is natural in B.

Note that η might conceivably depend upon the projective presentation of A. However we show that this cannot be the case by the following (3-dimensional) diagram, which shows also the naturality of η in A.

$$\begin{tikzcd}[column sep=small] R \arrow[rr, tail] \arrow[dd] & & P \arrow[rr, two heads, "\varepsilon"] \arrow[dd] & & A \arrow[dd, equal] & \\ & R' \arrow[ul] \arrow[rr, tail] \arrow[dd, dotted, "\psi"] & & P' \arrow[ul] \arrow[rr, two heads] \arrow[dd, dotted, "\varphi"] & & A' \arrow[ul, "\alpha"'] \arrow[dd, equal] \\ B \arrow[rr, tail] \arrow[dr, equal] & & E \arrow[rr, two heads] & & A & \\ & B \arrow[rr, tail] & & E^{\alpha} \arrow[ul] \arrow[rr, two heads] & & A' \arrow[ul, "\alpha"'] \end{tikzcd}$$

E^{α} is the pull-back of $E \to A$ and $A' \to A$. We have to show the existence of homomorphisms $\varphi : P' \to E^{\alpha}$, $\psi : R' \to B$ such that all faces are commutative. Since the maps $P' \to E \to A$ and $P' \to A' \to A$ agree they define a homomorphism $\varphi : P' \to E^{\alpha}$, into the pull-back. Then φ induces

$\psi: R' \to B$, and trivially all faces are commutative. (To see that $R' \to R \to B$ coincides with ψ, compose each with $B \rightarrowtail E$.) We therefore arrive at a commutative diagram

$$\begin{tikzcd} E(A,B) \arrow[r, "\alpha^*"] \arrow[d, shift right, "\eta"'] & E(A',B) \arrow[d, shift right, "\eta"'] \\ \operatorname{Ext}^{\varepsilon}_{\Lambda}(A,B) \arrow[r, "\alpha^*"] \arrow[u, shift right, "\xi"'] & \operatorname{Ext}^{\varepsilon'}_{\Lambda}(A',B) \arrow[u, shift right, "\xi"'] \end{tikzcd}$$

For $A' = A$, $\alpha = 1_A$ this shows that η is independent of the chosen projective presentation. In general it shows that η and ξ are natural in A. □

Corollary 2.5. *The set $E(A, B)$ of equivalence classes of extensions has a natural abelian group structure.*

Proof. This is obvious, since $\operatorname{Ext}_\Lambda(A, B)$ carries a natural abelian group structure and since $\eta: E(-,-) \xrightarrow{\sim} \operatorname{Ext}_\Lambda(-,-)$ is a natural equivalence. □

We leave as exercises (see Exercises 2.5 to 2.7) the direct description of the group structure in $E(A, B)$. However we shall exhibit here the neutral element of this group. Consider the diagram

$$\begin{tikzcd} R \arrow[r, tail, "\mu"] \arrow[d, "\psi"] & P \arrow[r, two heads, "\varepsilon"] \arrow[d, "\varphi"] & A \arrow[d, equal] \\ B \arrow[r, tail, "\kappa"] & E \arrow[r, two heads, "\nu"] & A \end{tikzcd}$$

The extension $B \rightarrowtail E \twoheadrightarrow A$ represents the neutral element in $E(A, B)$ if and only if $\psi: R \to B$ is the restriction of a homomorphism $\tau: P \to B$, i.e., if $\psi = \tau\mu$. The map $(\varphi - \kappa\tau)\mu: R \to E$ therefore is the zero map, so that $\varphi - \kappa\tau$ factors through A, defining a map $\sigma: A \to E$ with $\varphi - \kappa\tau = \sigma\varepsilon$. Since $\nu(\varphi - \kappa\tau) = \varepsilon$, σ is a right inverse to ν. Thus the extension $B \rightarrowtail E \twoheadrightarrow A$ splits. Conversely if $B \overset{\kappa}{\rightarrowtail} E \overset{\nu}{\twoheadrightarrow} A$ splits, the left inverse of κ is a map $E \to B$ which if composed with $\varphi: P \to E$ yields τ.

We finally note

Proposition 2.6. *If P is projective and I injective, then* $\operatorname{Ext}_\Lambda(P, B) = 0$ $= \operatorname{Ext}_\Lambda(A, I)$ *for all Λ-modules A, B.*

Proof. By Theorem 2.4 $\operatorname{Ext}_\Lambda(P, B)$ is in one-to-one correspondence with the set $E(P, B)$, consisting of classes of extensions of the form $B \rightarrowtail E \twoheadrightarrow P$. By Theorem I.4.7 short exact sequences of this form split. Hence $E(P, B)$ contains only one element, the zero element. For the other assertion one proceeds dually. □

Of course, we could prove this proposition directly, without involving Theorem 2.4.

Exercises:

2.1. Prove that $\mathrm{Ext}_\Lambda(-,-)$ is a bifunctor.

2.2. Suppose A is a right Γ-left Λ-bimodule. Show that $\mathrm{Ext}_\Lambda(A,B)$ has a left-Γ-module structure which is natural in B.

2.3. Suppose B is a right Γ-left Λ-bimodule. Show that $\mathrm{Ext}_\Lambda(A,B)$ has a right Γ-module structure, which is natural in A.

2.4. Suppose Λ commutative. Show that $\mathrm{Ext}_\Lambda(A,B)$ has a natural (in A and B) Λ-module structure.

2.5. Show that one can define an addition in $E(A,B)$ as follows: Let $B \rightarrowtail E_1 \twoheadrightarrow A$, $B \rightarrowtail E_2 \twoheadrightarrow A$ be representatives of two elements ξ_1, ξ_2 in $E(A,B)$. Let $\Delta_A: A \to A \oplus A$ be the map defined by $\Delta_A(a)=(a,a)$, $a \in A$, and let $\nabla_B: B \oplus B \to B$ be the map defined by $\nabla_B(b_1,b_2)=b_1+b_2$, $b_1, b_2 \in B$. Define the sum $\xi_1+\xi_2$ by

$$\xi_1+\xi_2=E(\Delta_A,\nabla_B)(B\oplus B \rightarrowtail E_1 \oplus E_2 \twoheadrightarrow A \oplus A).$$

2.6. Show that if $\alpha_1, \alpha_2: A' \to A$, then

$$(\alpha_1+\alpha_2)^* = \alpha_1^* + \alpha_2^*: E(A,B) \to E(A',B),$$

using the addition given in Exercise 2.5. Deduce that $E(A,B)$ admits additive inverses (without using Theorem 2.4).

2.7. Show that the addition defined in Exercise 2.5 is commutative and associative (without using Theorem 2.4). [Thus $E(A,B)$ is an abelian group.]

2.8. Let $\mathbb{Z}_4 \rightarrowtail \mathbb{Z}_{16} \twoheadrightarrow \mathbb{Z}_4$ be the evident exact sequence. Construct its inverse in $E(\mathbb{Z}_4, \mathbb{Z}_4)$.

2.9. Show the group table of $E(\mathbb{Z}_8, \mathbb{Z}_{12})$.

3. Ext Using Injectives

Given two Λ-modules A, B, we defined in Section 2 a group $\mathrm{Ext}_\Lambda(A,B)$ by using a projective presentation $R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ of A:

$$\mathrm{Ext}_\Lambda(A,B) = \operatorname{coker}(\mu^*: \mathrm{Hom}_\Lambda(P,B) \to \mathrm{Hom}_\Lambda(R,B)).$$

Here we consider the dual procedure: Choose an *injective presentation* of B, i.e. an exact sequence $B \overset{\nu}{\rightarrowtail} I \overset{\eta}{\twoheadrightarrow} S$ with I injective, and define the group $\overline{\mathrm{Ext}}{}^\nu_\Lambda(A,B)$ as the cokernel of the map $\eta_*: \mathrm{Hom}_\Lambda(A,I) \to \mathrm{Hom}_\Lambda(A,S)$. Dualizing the proofs of Lemma 2.1, Corollary 2.2, and Theorem 2.3 one could show that $\overline{\mathrm{Ext}}{}^\nu_\Lambda(A,B)$ does not depend upon the chosen injective presentation, and that $\overline{\mathrm{Ext}}_\Lambda(-,-)$ can be made into a bifunctor, covariant in the second, contravariant in the first variable. Also, by dualizing the proof of Theorem 2.4 one proves that there is a natural equivalence of set-valued bifunctors between $E(-,-)$ and $\overline{\mathrm{Ext}}_\Lambda(-,-)$.

Here we want to give a different proof of the facts mentioned above which has the advantage of yielding yet another description of $E(-,-)$. In contrast to $\mathrm{Ext}_\Lambda(-,-)$ and $\overline{\mathrm{Ext}}_\Lambda(-,-)$, the new description will be symmetric in A and B. Also, this proof establishes immediately that $\operatorname{Ext}_\Lambda(A, B)$ and $\overline{\operatorname{Ext}}_\Lambda(A, B)$ are isomorphic as abelian groups. First let us state the following lemma, due to J. Lambek (see [32]).

Lemma 3.1. *Let*

$$\begin{array}{ccccc} A' & \xrightarrow{\alpha_1} & A & \xrightarrow{\alpha_2} & A'' \\ \downarrow{\scriptstyle\psi} & \Sigma_1 & \downarrow{\scriptstyle\varphi} & \Sigma_2 & \downarrow{\scriptstyle\theta} \\ B' & \xrightarrow{\beta_1} & B & \xrightarrow{\beta_2} & B'' \end{array} \tag{3.1}$$

be a commutative diagram with exact rows. Then φ induces an isomorphism

$$\Phi : \ker\theta\alpha_2/(\ker\alpha_2 + \ker\varphi) \xrightarrow{\sim} (\operatorname{im}\varphi \cap \operatorname{im}\beta_1)/\operatorname{im}\varphi\alpha_1 .$$

Proof. First we show that φ induces a homomorphism of this kind. Let $x \in \ker\theta\alpha_2$; plainly $\varphi x \in \operatorname{im}\varphi$. Since $0 = \theta\alpha_2 x = \beta_2\varphi x$, $\varphi x \in \operatorname{im}\beta_1$. If $x \in \ker\alpha_2$, then $x \in \operatorname{im}\alpha_1$, and $\varphi x \in \operatorname{im}\varphi\alpha_1$. If $x \in \ker\varphi$, $\varphi x = 0$. Thus Φ is well-defined. Clearly Φ is a homomorphism. To show it is epimorphic, let $y \in \operatorname{im}\varphi \cap \operatorname{im}\beta_1$. There exists $x \in A$ with $\varphi x = y$. Since

$$\theta\alpha_2 x = \beta_2\varphi x = \beta_2 y = 0 ,$$

$x \in \ker\theta\alpha_2$. Finally we show that Φ is monomorphic. Suppose $x \in \ker\theta\alpha_2$, such that $\varphi x \in \operatorname{im}\varphi\alpha_1$, i.e. $\varphi x = \varphi\alpha_1 z$ for some $z \in A'$. Then $x = \alpha_1 z + t$, where $t \in \ker\varphi$. It follows that $x \in \ker\alpha_2 + \ker\varphi$. □

To facilitate the notation we introduce some terminology.

Definition. Let Σ be a commutative square of Λ-modules

$$\begin{array}{ccc} A' & \xrightarrow{\alpha} & A \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\varphi} \\ B' & \xrightarrow{\beta} & B \end{array}$$

We then write

$$\operatorname{Im}\Sigma = \operatorname{im}\varphi \cap \operatorname{im}\beta/\operatorname{im}\varphi\alpha ,$$

$$\operatorname{Ker}\Sigma = \ker\varphi\alpha/(\ker\alpha + \ker\psi) .$$

With this notation Lemma 3.1 may be stated in the following form:

If the diagram (3.1) *has exact rows, then φ induces an isomorphism* $\Phi : \operatorname{Ker}\Sigma_2 \xrightarrow{\sim} \operatorname{Im}\Sigma_1$.

Proposition 3.2. *For any projective presentation $R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ of A and any injective presentation $B \overset{\nu}{\rightarrowtail} I \overset{\eta}{\twoheadrightarrow} S$ of B, there is an isomorphism*

$$\sigma : \operatorname{Ext}_\Lambda^\varepsilon(A, B) \xrightarrow{\sim} \overline{\operatorname{Ext}}_\Lambda^\nu(A, B).$$

Proof. Consider the following commutative diagram with exact rows and columns

$$\begin{tikzcd}
\mathrm{Hom}_\Lambda(A, B) \arrow[r, tail] \arrow[d, tail] & \mathrm{Hom}_\Lambda(A, I) \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\Sigma_2"] & \mathrm{Hom}_\Lambda(A, S) \arrow[r, two heads] \arrow[d, tail] \arrow[dr, phantom, "\Sigma_1"] & \overline{\mathrm{Ext}}^v_\Lambda(A, B) \arrow[d] \\
\mathrm{Hom}_\Lambda(P, B) \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\Sigma_4"] & \mathrm{Hom}_\Lambda(P, I) \arrow[r] \arrow[d] \arrow[dr, phantom, "\Sigma_3"] & \mathrm{Hom}_\Lambda(P, S) \arrow[r] \arrow[d] & 0 \\
\mathrm{Hom}_\Lambda(R, B) \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\Sigma_5"] & \mathrm{Hom}_\Lambda(R, I) \arrow[r] \arrow[d] & \mathrm{Hom}_\Lambda(R, S) & \\
\mathrm{Ext}^\varepsilon_\Lambda(A, B) \arrow[r] & 0 & &
\end{tikzcd} \tag{3.2}$$

The reader easily checks that $\mathrm{Ker}\,\Sigma_1 = \overline{\mathrm{Ext}}^v_\Lambda(A, B)$ and $\mathrm{Ker}\,\Sigma_5 = \mathrm{Ext}^\varepsilon_\Lambda(A, B)$. Applying Lemma 3.1 repeatedly we obtain

$$\overline{\mathrm{Ext}}^v_\Lambda(A, B) = \mathrm{Ker}\,\Sigma_1 \cong \mathrm{Im}\,\Sigma_2 \cong \mathrm{Ker}\,\Sigma_3 \cong \mathrm{Im}\,\Sigma_4 \cong \mathrm{Ker}\,\Sigma_5 = \mathrm{Ext}^\varepsilon_\Lambda(A, B)\,. \quad \square$$

Thus for *any* injective presentation of B, $\overline{\mathrm{Ext}}^v_\Lambda(A, B)$ is isomorphic to $\mathrm{Ext}^\varepsilon_\Lambda(A, B)$. We thus are allowed to drop the superscript v and to write $\overline{\mathrm{Ext}}_\Lambda(A, B)$. Let $\beta: B \to B'$ be a homomorphism and let $B' \overset{v'}{\rightarrowtail} I' \twoheadrightarrow S'$ be an injective presentation. It is easily seen that if $\tau: I \to I'$ is a map inducing β the diagram (3.2) is mapped into the corresponding diagram for $B' \overset{v'}{\rightarrowtail} I' \twoheadrightarrow S'$. Therefore we obtain an induced homomorphism

$$\beta_*: \overline{\mathrm{Ext}}_\Lambda(A, B) \to \overline{\mathrm{Ext}}_\Lambda(A, B')$$

which agrees via the isomorphism defined above with the induced homomorphism $\beta_*: \mathrm{Ext}_\Lambda(A, B) \to \mathrm{Ext}_\Lambda(A, B')$.

Analogously one defines an induced homomorphism in the first variable. With these definitions of induced maps $\overline{\mathrm{Ext}}_\Lambda(-, -)$ becomes a bifunctor, and σ becomes a natural equivalence. We thus have

Corollary 3.3. $\overline{\mathrm{Ext}}_\Lambda(-, -)$ *is a bifunctor, contravariant in the first, covariant in the second variable. It is naturally equivalent to* $\mathrm{Ext}_\Lambda(-, -)$ *and therefore to* $E(-, -)$. $\square$

We sometimes express the natural equivalence between $\mathrm{Ext}_\Lambda(-, -)$ and $\overline{\mathrm{Ext}}_\Lambda(-, -)$ by saying that Ext is *balanced*.

Finally the above proof also yields a symmetric description of Ext from (3.2), namely:

Corollary 3.4. $\mathrm{Ext}_\Lambda(A, B) \cong \mathrm{Ker}\,\Sigma_3$. $\square$

In view of the above results we shall use only one notation, namely $\mathrm{Ext}_\Lambda(-, -)$ for the equivalent functors $E(-, -)$, $\mathrm{Ext}_\Lambda(-, -)$, $\overline{\mathrm{Ext}}_\Lambda(-, -)$.

Exercises:

3.1. Show that, if Λ is a principal ideal domain (p.i.d.), then an epimorphism $\beta: B \twoheadrightarrow B'$ induces an epimorphism $\beta_*: \mathrm{Ext}_\Lambda(A, B) \to \mathrm{Ext}_\Lambda(A, B')$. State and prove the dual.

3.2. Prove that $\mathrm{Ext}_{\mathbb{Z}}(A, \mathbb{Z}) \neq 0$ if A has elements of finite order.

3.3. Compute $\mathrm{Ext}_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z})$, using an *injective* presentation of $\mathbb{Z}$.

3.4. Show that $\mathrm{Ext}_{\mathbb{Z}}(A, \mathrm{Ext}_{\mathbb{Z}}(B, C)) \cong \mathrm{Ext}_{\mathbb{Z}}(B, \mathrm{Ext}_{\mathbb{Z}}(A, C))$ when A, B, C are finitely-generated abelian groups.

3.5. Let the natural equivalences $\eta: E(-,-) \to \mathrm{Ext}_\Lambda(-,-)$ be defined by Theorem 2.4, $\sigma: \overline{\mathrm{Ext}}_\Lambda(-,-) \to \mathrm{Ext}_\Lambda(-,-)$ by Proposition 3.2, and

$$\bar{\eta}: E(-,-) \to \overline{\mathrm{Ext}}_\Lambda(-,-)$$

by dualizing the proof of Theorem 2.4. Show that $\sigma \circ \eta = \bar{\eta}$.

4. Computation of some Ext-Groups

We start with the following

Lemma 4.1. (i) $\mathrm{Ext}_\Lambda\left(\bigoplus_i A_i, B\right) \cong \prod_i \mathrm{Ext}_\Lambda(A_i, B)$,

(ii) $\mathrm{Ext}_\Lambda\left(A, \prod_j B_j\right) \cong \prod_j \mathrm{Ext}_\Lambda(A, B_j)$.

Proof. We only prove assertion (i), leaving the other to the reader. For each i in the index set we choose a projective presentation $R_i \rightarrowtail P_i \twoheadrightarrow A_i$ of A_i. Then $\bigoplus_i R_i \rightarrowtail \bigoplus_i P_i \twoheadrightarrow \bigoplus_i A_i$ is a projective presentation of $\bigoplus_i A_i$. Using Proposition I.3.4 we obtain the following commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{Hom}_\Lambda\left(\bigoplus_i A_i, B\right) \arrow[r, tail] \arrow[d, "\wr"] & \mathrm{Hom}_\Lambda\left(\bigoplus_i P_i, B\right) \arrow[r] \arrow[d, "\wr"] & \mathrm{Hom}_\Lambda\left(\bigoplus_i R_i, B\right) \arrow[r, two heads] \arrow[d, "\wr"] & \mathrm{Ext}_\Lambda\left(\bigoplus_i A_i, B\right) \\
\prod_i \mathrm{Hom}_\Lambda(A_i, B) \arrow[r, tail] & \prod_i \mathrm{Hom}_\Lambda(P_i, B) \arrow[r] & \prod_i \mathrm{Hom}_\Lambda(R_i, B) \arrow[r, two heads] & \prod_i \mathrm{Ext}_\Lambda(A_i, B)
\end{tikzcd}$$

whence the result. $\square$

The reader may prefer to prove assertion (i) by using an injective presentation of B. Indeed in doing so it becomes clear that the two assertions of Lemma 4.1 are dual to each other.

In the remainder of this section we shall compute $\mathrm{Ext}_{\mathbb{Z}}(A, B)$ for A, B finitely-generated abelian groups. In view of Lemma 4.1 it is enough to consider the case where A, B are cyclic.

To facilitate the notation we shall write $\operatorname{Ext}(A, B)$ (for $\operatorname{Ext}_{\mathbb{Z}}(A, B)$) and $\operatorname{Hom}(A, B)$ (for $\operatorname{Hom}_{\mathbb{Z}}(A, B)$), whenever the ground ring is the ring of integers.

Since $\mathbb{Z}$ is projective, one has

$$\operatorname{Ext}(\mathbb{Z}, \mathbb{Z}) = 0 = \operatorname{Ext}(\mathbb{Z}, \mathbb{Z}_q)$$

by Proposition 2.6. To compute $\operatorname{Ext}(\mathbb{Z}_r, \mathbb{Z})$ and $\operatorname{Ext}(\mathbb{Z}_r, \mathbb{Z}_q)$ we use the projective presentation

$$\mathbb{Z} \overset{\mu}{\rightarrowtail} \mathbb{Z} \overset{\varepsilon}{\twoheadrightarrow} \mathbb{Z}_r$$

where μ is multiplication by r. We obtain the exact sequence

$$\begin{tikzcd}
\operatorname{Hom}(\mathbb{Z}_r, \mathbb{Z}) \arrow[r, rightarrowtail] \arrow[d, "\wr"] & \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}) \arrow[r, "\mu^*"] \arrow[d, "\wr"] & \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}) \arrow[r, twoheadrightarrow] \arrow[d, "\wr"] & \operatorname{Ext}(\mathbb{Z}_r, \mathbb{Z}) \\
0 \arrow[r, rightarrowtail] & \mathbb{Z} \arrow[r, "\mu^*"] & \mathbb{Z} &
\end{tikzcd}$$

Since μ^* is again multiplication by r we obtain

$$\operatorname{Ext}(\mathbb{Z}_r, \mathbb{Z}) \cong \mathbb{Z}_r .$$

Also the exact sequence

$$\begin{tikzcd}
\operatorname{Hom}(\mathbb{Z}_r, \mathbb{Z}_q) \arrow[r, rightarrowtail] \arrow[d, "\wr"] & \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}_q) \arrow[r, "\mu^*"] \arrow[d, "\wr"] & \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}_q) \arrow[r, twoheadrightarrow] \arrow[d, "\wr"] & \operatorname{Ext}(\mathbb{Z}_r, \mathbb{Z}_q) \\
\mathbb{Z}_{(r,q)} \arrow[r, rightarrowtail] & \mathbb{Z}_q \arrow[r, "\mu^*"] & \mathbb{Z}_q &
\end{tikzcd}$$

yields, since μ^* is multiplication by r,

$$\operatorname{Ext}(\mathbb{Z}_r, \mathbb{Z}_q) \cong \mathbb{Z}_{(r,q)}$$

where (r, q) denotes the greatest common divisor of r and q.

Exercises:

4.1. Show that there are p nonequivalent extensions $\mathbb{Z}_p \rightarrowtail E \twoheadrightarrow \mathbb{Z}_p$ for p a prime, but only two nonisomorphic groups E, namely $\mathbb{Z}_p \oplus \mathbb{Z}_p$ and $\mathbb{Z}_{p^2}$. How does this come about?

4.2. Classify the extension classes $[E]$, given by

$$\mathbb{Z}_m \rightarrowtail E \twoheadrightarrow \mathbb{Z}_n$$

under automorphisms of $\mathbb{Z}_m$ and $\mathbb{Z}_n$.

4.3. Show that if A is a finitely-generated abelian group such that $\operatorname{Ext}(A, \mathbb{Z}) = 0$, $\operatorname{Hom}(A, \mathbb{Z}) = 0$, then $A = 0$.

4.4. Show that $\operatorname{Ext}(A, \mathbb{Z}) \cong A$ if A is a finite abelian group.

4.5. Show that there is a natural equivalence of functors $\operatorname{Hom}(-, \mathbb{Q}/\mathbb{Z}) \cong \operatorname{Ext}(-, \mathbb{Z})$ if both functors are restricted to the category of torsion abelian groups.

4.6. Show that extensions of finite abelian groups of relatively prime order split.

5. Two Exact Sequences

Here we shall deduce two exact sequences connecting Hom and Ext. We start with the following very useful lemma.

Lemma 5.1. *Let the following commutative diagram have exact rows.*

$$\begin{tikzcd} & A \arrow[r, "\mu"] \arrow[d, "\alpha"] & B \arrow[r, "\varepsilon"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "\mu'"] & B' \arrow[r, "\varepsilon'"] & C' & \end{tikzcd}$$

Then there is a "connecting homomorphism" $\omega: \ker\gamma \to \operatorname{coker}\alpha$ *such that the following sequence is exact:*

$$\ker\alpha \xrightarrow{\mu_*} \ker\beta \xrightarrow{\varepsilon_*} \ker\gamma \xrightarrow{\omega} \operatorname{coker}\alpha \xrightarrow{\mu'_*} \operatorname{coker}\beta \xrightarrow{\varepsilon'_*} \operatorname{coker}\gamma\,. \quad (5.1)$$

If μ *is monomorphic, so is* μ_*; *if* ε' *is epimorphic, so is* ε'_*.

Proof. It is very easy to see – and we leave the verification to the reader – that the final sentence holds and that we have exact sequences

$$\ker\alpha \xrightarrow{\mu_*} \ker\beta \xrightarrow{\varepsilon_*} \ker\gamma\,,$$

$$\operatorname{coker}\alpha \xrightarrow{\mu'_*} \operatorname{coker}\beta \xrightarrow{\varepsilon'_*} \operatorname{coker}\gamma\,.$$

It therefore remains to show that there exists a homomorphism $\omega: \ker\gamma \to \operatorname{coker}\alpha$ "connecting" these two sequences. In fact, ω is defined as follows.

Let $c \in \ker\gamma$, choose $b \in B$ with $\varepsilon b = c$. Since $\varepsilon'\beta b = \gamma\varepsilon b = \gamma c = 0$ there exists $a' \in A'$ with $\beta b = \mu' a'$. Define $\omega(c) = [a']$, the coset of a' in $\operatorname{coker}\alpha$.

We show that ω is well defined, that is, that $\omega(c)$ is independent of the choice of b. Indeed, let $\bar{b} \in B$ with $\varepsilon\bar{b} = c$, then $\bar{b} = b + \mu a$ and

$$\beta(b + \mu a) = \beta b + \mu'\alpha a\,.$$

Hence $\bar{a}' = a' + \alpha a$, thus $[\bar{a}'] = [a']$. Clearly ω is a homomorphism.

Next we show exactness at $\ker\gamma$. If $c \in \ker\gamma$ is of the form εb for $b \in \ker\beta$, then $0 = \beta b = \mu' a'$, hence $a' = 0$ and $\omega(c) = 0$. Conversely, let $c \in \ker\gamma$ with $\omega(c) = 0$. Then $c = \varepsilon b$, $\beta b = \mu' a'$ and there exists $a \in A$ with $\alpha a = a'$. Consider $\bar{b} = b - \mu a$. Clearly $\varepsilon\bar{b} = c$, but

$$\beta\bar{b} = \beta b - \beta\mu a = \beta b - \mu' a' = 0\,,$$

hence $c \in \ker\gamma$ is of the form $\varepsilon\bar{b}$ with $\bar{b} \in \ker\beta$.

Finally we prove exactness at $\operatorname{coker}\alpha$. Let $\omega(c) = [a'] \in \operatorname{coker}\alpha$. Thus $c = \varepsilon b$, $\beta b = \mu' a'$, and $\mu'_*[a'] = [\mu' a'] = [\beta b] = 0$. Conversely, let $[a'] \in \operatorname{coker}\alpha$ with $\mu'_*[a'] = 0$. Then $\mu' a' = \beta b$ for some $b \in B$ and $c = \varepsilon b \in \ker\gamma$. Thus $[a'] = \omega(c)$. □

For an elegant proof of Lemma 5.1 using Lemma 3.1, see Exercise 5.1.

We remark that the sequence (5.1) is natural in the obvious sense: If we are given a commutative diagram with exact rows

$$\begin{tikzcd}
 & D \arrow[rr] \arrow[dd] & & E \arrow[rr] \arrow[dd] & & F \arrow[rr] \arrow[dd] & & 0 & \\
 & & A \arrow[ul] \arrow[rr] \arrow[dd] & & B \arrow[ul] \arrow[rr] \arrow[dd] & & C \arrow[ul] \arrow[rr] \arrow[dd] & & 0 \\
0 \arrow[r] & D' \arrow[rr] & & E' \arrow[rr] & & F' & & & \\
 & 0 \arrow[r] & A' \arrow[ul] \arrow[rr] & & B' \arrow[ul] \arrow[rr] & & C' \arrow[ul] & &
\end{tikzcd} \tag{5.2}$$

we obtain a mapping from the sequence stemming from the front diagram to the sequence stemming from the back diagram.

We use Lemma 5.1 to prove

Theorem 5.2. *Let A be a Λ-module and let $B' \overset{\varphi}{\rightarrowtail} B \overset{\psi}{\twoheadrightarrow} B''$ be an exact sequence of Λ-modules. There exists a "connecting homomorphism" $\omega: \mathrm{Hom}_\Lambda(A, B'') \to \mathrm{Ext}_\Lambda(A, B')$ such that the following sequence is exact and natural*

$$\begin{aligned}
0 \to \mathrm{Hom}_\Lambda(A, B') &\xrightarrow{\varphi_*} \mathrm{Hom}_\Lambda(A, B) \xrightarrow{\psi_*} \mathrm{Hom}_\Lambda(A, B'') \\
&\xrightarrow{\omega} \mathrm{Ext}_\Lambda(A, B') \xrightarrow{\varphi_*} \mathrm{Ext}_\Lambda(A, B) \xrightarrow{\psi_*} \mathrm{Ext}_\Lambda(A, B'') .
\end{aligned} \tag{5.3}$$

This sequence is called the *Hom-Ext-sequence (in the second variable).*

Proof. Choose any projective presentation $R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ of A and consider the following diagram with exact rows and columns

$$\begin{tikzcd}
 & \mathrm{Hom}_\Lambda(A, B') \arrow[r, dotted, "\varphi_*"] \arrow[d, tail] & \mathrm{Hom}_\Lambda(A, B) \arrow[r, dotted, "\psi_*"] \arrow[d, tail] & \mathrm{Hom}_\Lambda(A, B'') \arrow[d, tail] & \\
0 \arrow[r] & \mathrm{Hom}_\Lambda(P, B') \arrow[r] \arrow[d] & \mathrm{Hom}_\Lambda(P, B) \arrow[r] \arrow[d] & \mathrm{Hom}_\Lambda(P, B'') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}_\Lambda(R, B') \arrow[r] \arrow[d, two heads] & \mathrm{Hom}_\Lambda(R, B) \arrow[r] \arrow[d, two heads] & \mathrm{Hom}_\Lambda(R, B'') \arrow[d, two heads] & \\
 & \mathrm{Ext}_\Lambda(A, B') \arrow[r, dotted, "\varphi_*"] & \mathrm{Ext}_\Lambda(A, B) \arrow[r, dotted, "\psi_*"] & \mathrm{Ext}_\Lambda(A, B'') &
\end{tikzcd} \tag{5.4}$$

The second and third rows are exact by Theorem I.2.1. In the second row $\psi_*: \mathrm{Hom}_\Lambda(P, B) \to \mathrm{Hom}_\Lambda(P, B'')$ is epimorphic since P is projective (Theorem I.4.7). Applying Lemma 5.1 to the two middle rows of the diagram we obtain the homomorphism ω and the exactness of the resulting sequence.

Let $\alpha: A' \to A$ be a homomorphism and let $R' \rightarrowtail P' \twoheadrightarrow A'$ be a projective presentation of A'. Choose $\pi: P' \to P$ and $\sigma: R' \to R$ such that the diagram

$$\begin{tikzcd} R' \arrow[r, tail] \arrow[d, "\sigma"] & P' \arrow[r, two heads] \arrow[d, "\pi"] & A' \arrow[d, "\alpha"] \\ R \arrow[r, tail] & P \arrow[r, two heads] & A \end{tikzcd}$$

is commutative. Then α, π, σ induce a mapping from diagram (5.4) associated with $R \rightarrowtail P \twoheadrightarrow A$ to the corresponding diagram associated with $R' \rightarrowtail P' \twoheadrightarrow A'$. The two middle rows of these diagrams form a diagram of the kind (5.2). Hence the Hom-Ext sequence corresponding to A is mapped into the Hom-Ext sequence corresponding to A'. In particular – choosing $\alpha = 1_A: A \to A$ – this shows that ω is independent of the chosen projective presentation.

Analogously one proves that homomorphisms β', β, β'' which make the diagram

$$\begin{tikzcd} B' \arrow[r, tail] \arrow[d, "\beta'"] & B \arrow[r, two heads] \arrow[d, "\beta"] & B'' \arrow[d, "\beta''"] \\ C' \arrow[r, tail] & C \arrow[r, two heads] & C'' \end{tikzcd}$$

commutative induce a mapping from the Hom-Ext sequence associated with the short exact sequence $B' \rightarrowtail B \twoheadrightarrow B''$ to the Hom-Ext sequence associated with the short exact sequence $C' \rightarrowtail C \twoheadrightarrow C''$. In particular the following square is commutative.

$$\begin{tikzcd} \mathrm{Hom}_\Lambda(A, B'') \arrow[r, "\omega"] \arrow[d, "\beta''_*"] & \mathrm{Ext}_\Lambda(A, B') \arrow[d, "\beta'_*"] \\ \mathrm{Hom}_\Lambda(A, C'') \arrow[r, "\omega"] & \mathrm{Ext}_\Lambda(A, C') \end{tikzcd}$$

This completes the proof of Theorem 5.2. □

We make the following remark with respect to the connecting homomorphism $\omega: \mathrm{Hom}_\Lambda(A, B'') \to \mathrm{Ext}_\Lambda(A, B')$ as constructed in the proof of Theorem 5.2. Given $\alpha: A \to B''$ we define maps π, σ such that the diagram

$$\begin{tikzcd} R \arrow[r, tail] \arrow[d, "\sigma"] & P \arrow[r, two heads] \arrow[d, "\pi"] & A \arrow[d, "\alpha"] \\ B' \arrow[r, tail, "\varphi"] & B \arrow[r, two heads, "\psi"] & B'' \end{tikzcd}$$

is commutative. The construction of ω in diagram (5.4) shows that $\omega(\alpha) = [\sigma] \in \mathrm{Ext}_\Lambda(A, B')$. Now let E be the pull-back of (ψ, α). We then have a map $\pi': P \to E$ such that the diagram

$$\begin{tikzcd} R \arrow[r, tail] \arrow[d, "\sigma"] & P \arrow[r, two heads] \arrow[d, "\pi'"] & A \arrow[d, equal] \\ B' \arrow[r, tail] \arrow[d, equal] & E \arrow[r, two heads] \arrow[d] & A \arrow[d, "\alpha"] \\ B' \arrow[r, tail] & B \arrow[r, two heads] & B'' \end{tikzcd}$$

is commutative. By the definition of the equivalence

$$\xi : \mathrm{Ext}_\Lambda(A, B') \xrightarrow{\sim} E(A, B')$$

in Theorem 2.4 the element $\xi[\sigma]$ is represented by the extension $B' \rightarrowtail E \twoheadrightarrow A$.

We now introduce a *Hom-Ext-sequence in the first variable.*

Theorem 5.3. *Let B be a Λ-module and let $A' \overset{\varphi}{\rightarrowtail} A \overset{\psi}{\twoheadrightarrow} A''$ be a short exact sequence. Then there exists a connecting homomorphism*

$$\omega : \mathrm{Hom}_\Lambda(A', B) \to \mathrm{Ext}_\Lambda(A'', B)$$

such that the following sequence is exact and natural

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_\Lambda(A'', B) \xrightarrow{\psi^*} \mathrm{Hom}_\Lambda(A, B) \xrightarrow{\varphi^*} \mathrm{Hom}_\Lambda(A', B) \\ &\xrightarrow{\omega} \mathrm{Ext}_\Lambda(A'', B) \xrightarrow{\psi^*} \mathrm{Ext}_\Lambda(A, B) \xrightarrow{\varphi^*} \mathrm{Ext}_\Lambda(A', B)\,. \end{aligned} \tag{5.5}$$

The reader notes that, if Ext is identified with $\overline{\mathrm{Ext}}$, Theorem 5.3 becomes the dual of Theorem 5.2 and that it may be proved by proceeding dually to Theorem 5.2 (see Exercises 5.4, 5.5). We prefer, however, to give a further proof using only projectives and thus avoiding the use of injectives. For our proof we need the following lemma, which will be invoked again in Chapter IV.

Lemma 5.4. *To a short exact sequence $A' \overset{\varphi}{\rightarrowtail} A \overset{\psi}{\twoheadrightarrow} A''$ and to projective presentations $\varepsilon' : P' \twoheadrightarrow A'$ and $\varepsilon'' : P'' \twoheadrightarrow A''$ there exists a projective presentation $\varepsilon : P \twoheadrightarrow A$ and homomorphisms $\iota : P' \to P$ and $\pi : P \to P''$ such that the following diagram is commutative with exact rows*

$$\begin{tikzcd} P' \arrow[r, tail, "\iota"] \arrow[d, two heads, "\varepsilon'"] & P \arrow[r, two heads, "\pi"] \arrow[d, two heads, "\varepsilon"] & P'' \arrow[d, two heads, "\varepsilon''"] \\ A' \arrow[r, tail, "\varphi"] & A \arrow[r, two heads, "\psi"] & A'' \end{tikzcd}$$

Proof. Let $P = P' \oplus P''$, let $\iota : P' \to P' \oplus P''$ be the canonical injection, $\pi : P' \oplus P'' \to P''$ the canonical projection. We define ε by giving the components. The first component is $\varphi\varepsilon' : P' \to A$; for the second we use the fact that P'' is projective to construct a map $\chi : P'' \to A$ which makes the triangle

$$\begin{tikzcd} & P'' \arrow[dl, "\chi"'] \arrow[d, "\varepsilon''"] \\ A \arrow[r, two heads, "\psi"'] & A'' \end{tikzcd}$$

commutative, and take χ as the second component of ε. It is plain that with this definition the above diagram commutes. By Lemma I.1.1 ε is epimorphic. $\square$

Proof of Theorem 5.3. Using Lemma 5.4 projective presentations may be chosen such that the following diagram is commutative with short exact middle row

$$\begin{tikzcd} R' \arrow[r] \arrow[d, tail] & R \arrow[r] \arrow[d, tail] & R'' \arrow[d, tail] \\ P' \arrow[r, tail, "\iota"] \arrow[d, two heads, "\varepsilon'"] & P \arrow[r, two heads, "\pi"] \arrow[d, two heads, "\varepsilon"] & P'' \arrow[d, two heads, "\varepsilon''"] \\ A' \arrow[r, tail, "\varphi"] & A \arrow[r, two heads, "\psi"] & A'' \end{tikzcd}$$

By Lemma 5.1 applied to the second and third row the top row is short exact, also. Applying $\mathrm{Hom}_\Lambda(-, B)$ we obtain the following diagram

$$\begin{tikzcd} & \mathrm{Hom}_\Lambda(A'', B) \arrow[r, dotted, "\psi^*"] \arrow[d, tail] & \mathrm{Hom}_\Lambda(A, B) \arrow[r, dotted, "\varphi^*"] \arrow[d, tail] & \mathrm{Hom}_\Lambda(A', B) \arrow[d, tail] & \\ 0 \arrow[r] & \mathrm{Hom}_\Lambda(P'', B) \arrow[r] \arrow[d] & \mathrm{Hom}_\Lambda(P, B) \arrow[r, "\iota^*"] \arrow[d] & \mathrm{Hom}_\Lambda(P', B) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{Hom}_\Lambda(R'', B) \arrow[r] \arrow[d, two heads] & \mathrm{Hom}_\Lambda(R, B) \arrow[r] \arrow[d, two heads] & \mathrm{Hom}_\Lambda(R', B) \arrow[d, two heads] & \\ & \mathrm{Ext}_\Lambda(A'', B) \arrow[r, dotted, "\psi^*"] & \mathrm{Ext}_\Lambda(A, B) \arrow[r, dotted, "\varphi^*"] & \mathrm{Ext}_\Lambda(A', \mathrm{B}) & \end{tikzcd} \tag{5.6}$$

By Theorem I.2.2 the second and third rows are exact. In the second row $\iota^* : \mathrm{Hom}_\Lambda(P, B) \to \mathrm{Hom}_\Lambda(P', B)$ is epimorphic since $P = P' \oplus P''$, so $\mathrm{Hom}_\Lambda(P' \oplus P'', B) \cong \mathrm{Hom}_\Lambda(P', B) \oplus \mathrm{Hom}_\Lambda(P'', B)$. Lemma 5.1 now yields the Hom-Ext sequence claimed. As in the proof of Theorem 5.2 one shows that ω is independent of the chosen projective presentations. Also, one proves that the Hom-Ext sequence in the first variable is natural with respect to homomorphisms $\beta : B \to B'$ and with respect to maps γ', γ, γ'' making the diagram

$$\begin{tikzcd} A' \arrow[r, tail] \arrow[d, "\gamma'"] & A \arrow[r, two heads] \arrow[d, "\gamma"] & A'' \arrow[d, "\gamma''"] \\ C' \arrow[r, tail] & C \arrow[r, two heads] & C'' \end{tikzcd}$$

commutative. $\square$

If we try to describe the connecting homomorphism

$$\omega : \mathrm{Hom}_\Lambda(A', B) \to \mathrm{Ext}_\Lambda(A'', B)$$

in terms of extensions, it is natural to consider the push-out E of $\alpha : A' \to B$ and $\varphi : A' \to A$ and to construct the diagram

$$\begin{tikzcd} A' \arrow[r, tail] \arrow[d, "\alpha"] & A \arrow[r, two heads] \arrow[d] & A'' \arrow[d, equal] \\ B \arrow[r, tail] & E \arrow[r, two heads] & A'' \end{tikzcd}$$

We then consider the presentation $R'' \rightarrowtail P'' \twoheadrightarrow A''$ and note that the map $\chi : P'' \to A$ constructed in the proof of Lemma 5.4 induces σ such that the diagram

$$\begin{tikzcd} R'' \arrow[r, tail] \arrow[d, "\sigma"] & P'' \arrow[r, two heads] \arrow[d, "\chi"] & A'' \arrow[d, equal] \\ A' \arrow[r, tail] \arrow[d, "\alpha"] & A \arrow[r, two heads] \arrow[d] & A'' \arrow[d, equal] \\ B \arrow[r, tail] & E \arrow[r, two heads] & A'' \end{tikzcd}$$

is commutative. Now the definition of $\omega(\alpha)$ in diagram (5.6) is via the map $\varrho : R'' \to B$ which is obtained as $\varrho = \alpha\tau$ in

$$\begin{tikzcd} R \arrow[r, tail] \arrow[d] & P' \oplus P'' \arrow[r, "\pi_1"] & P' \arrow[r, "\varepsilon'"] & A' \arrow[r, "\alpha"] & B \\ R'' \arrow[urrr, "\tau"] \end{tikzcd}$$

But by the definition of $\varepsilon : P' \oplus P'' \to A$ in Lemma 5.4, the sum of the two maps

$$R \twoheadrightarrow R'' \xrightarrow{\sigma} A' \overset{\varphi}{\rightarrowtail} A$$

$$R \twoheadrightarrow R'' \xrightarrow{\tau} A' \overset{\varphi}{\rightarrowtail} A$$

is zero. Hence $\sigma = -\tau$, so that the element $-\omega(\alpha) = [-\tau]$ is represented by the extension $B \rightarrowtail E \twoheadrightarrow A''$.

Corollary 5.5. *The Λ-module A is projective if and only if* $\mathrm{Ext}_\Lambda(A, B) = 0$ *for all Λ-modules B.*

Proof. Suppose A is projective. Then $1 : A \xrightarrow{\sim} A$ is a projective presentation, whence $\mathrm{Ext}_\Lambda(A, B) = 0$ for all Λ-modules B. Conversely, suppose $\mathrm{Ext}_\Lambda(A, B) = 0$ for all Λ-modules B. Then for any short exact sequence $B' \rightarrowtail B \twoheadrightarrow B''$ the sequence

$$0 \to \mathrm{Hom}_\Lambda(A, B') \to \mathrm{Hom}_\Lambda(A, B) \to \mathrm{Hom}_\Lambda(A, B'') \to 0$$

is exact. By Theorem I.4.7 A is projective. $\square$

The reader may now easily prove the dual assertion.

Corollary 5.6. *The Λ-module B is injective if and only if* $\mathrm{Ext}_\Lambda(A, B) = 0$ *for all Λ-modules A.* □

In the special case where Λ is a principal ideal domain we obtain

Corollary 5.7. *Let Λ be a principal ideal domain. Then the homomorphisms* $\psi_* : \mathrm{Ext}_\Lambda(A, B) \to \mathrm{Ext}_\Lambda(A, B'')$ *in sequence* (5.3) *and*

$$\varphi^* : \mathrm{Ext}_\Lambda(A, B) \to \mathrm{Ext}_\Lambda(A', B)$$

in sequence (5.5) *are epimorphic.*

Proof. Over a principal ideal domain Λ submodules of projective modules are projective. Hence in diagram (5.4) R is projective; thus

$$\psi_* : \mathrm{Hom}_\Lambda(R, B) \to \mathrm{Hom}_\Lambda(R, B'')$$

is epimorphic, and hence $\psi_* : \mathrm{Ext}_\Lambda(A, B) \to \mathrm{Ext}_\Lambda(A, B'')$ is epimorphic. In diagram (5.6), R'' is projective. Hence the short exact sequence $R' \rightarrowtail R \twoheadrightarrow R''$ splits and it follows that $\varphi^* : \mathrm{Hom}_\Lambda(R, B) \to \mathrm{Hom}_\Lambda(R', B)$ is epimorphic. Hence $\varphi^* : \mathrm{Ext}_\Lambda(A, B) \to \mathrm{Ext}_\Lambda(A', B)$ is epimorphic. □

We remark, that if Λ is *not* a principal ideal domain the assertions of Corollary 5.7 are false in general (Exercise 5.3).

Exercises:

5.1. Consider the following diagram

$$\begin{tikzcd}
 & & \ker\beta \arrow[r, "\varepsilon_*"] \arrow[d, tail] \arrow[dr, phantom, "\Sigma_1"] & \ker\gamma \arrow[d, tail] & \\
 & A \arrow[r, "\mu"] \arrow[d, "\alpha"] & B \arrow[r, "\varepsilon"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\mu'"] \arrow[d, two heads] \arrow[dr, phantom, "\Sigma_4"] & B' \arrow[r, "\varepsilon'"] \arrow[d, two heads] & C' & \\
 & \operatorname{coker}\alpha \arrow[r, "\mu'_*"] & \operatorname{coker}\beta & &
\end{tikzcd}$$

with all sequences exact. Show that with the terminology of Lemma 3.1 we have $\mathrm{Im}\,\Sigma_1 \cong \operatorname{coker}\varepsilon_*$, $\mathrm{Ker}\,\Sigma_4 \cong \ker\mu'_*$. Show $\mathrm{Im}\,\Sigma_1 \cong \mathrm{Ker}\,\Sigma_4$ by a repeated application of Lemma 3.1. With that result prove Lemma 5.1.

5.2. Given $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ (not necessarily exact) deduce from Lemma 5.1 (or prove otherwise) that there is a natural exact sequence

$$0 \to \ker\alpha \to \ker\beta\alpha \to \ker\beta \to \operatorname{coker}\alpha \to \operatorname{coker}\beta\alpha \to \operatorname{coker}\beta \to 0\,.$$

5.3. Show that if R is not projective there exists a module B with $\mathrm{Ext}_\Lambda(R, B) \neq 0$. Suppose that in the projective presentation $R \overset{\varphi}{\rightarrowtail} P \overset{\psi}{\twoheadrightarrow} A$ of A the module R is *not* projective. Deduce that $\varphi^* : \mathrm{Ext}_\Lambda(P, B) \to \mathrm{Ext}_\Lambda(R, B)$ is not epimorphic. Compare with Corollary 5.7.

5.4. Prove Theorem 5.3 by using the definition of Ext by injectives and interpret the connecting homomorphism in terms of extensions. Does one get the same connecting homomorphism as in our proof of Theorem 5.3?

5.5. Prove Theorem 5.2 using the definition of Ext by injectives. (Use the dual of Lemma 5.4.) Does one get the same connecting homomorphism as in our proof of Theorem 5.2?

5.6. Establish equivalences of Ext_Λ and $\overline{\mathrm{Ext}}_\Lambda$ using (i) Theorem 5.2, (ii) Theorem 5.3. Does one get the same equivalences?

5.7. Evaluate the groups and homomorphisms in the appropriate sequences (of Theorems 5.2, 5.3) when

(i) $0 \to A' \to A \to A'' \to 0$ is $0 \to \mathbb{Z}_4 \to \mathbb{Z}_{16} \to \mathbb{Z}_4 \to 0$, B is $\mathbb{Z}_4$;
(ii) $0 \to A' \to A \to A'' \to 0$ is $0 \to \mathbb{Z}_4 \to \mathbb{Z}_8 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4 \to 0$, B is $\mathbb{Z}_4$;
(iii) A is $\mathbb{Z}_4$, $0 \to B' \to B \to B'' \to 0$ is $0 \to \mathbb{Z}_4 \to \mathbb{Z}_{16} \to \mathbb{Z}_4 \to 0$;
(iv) A is $\mathbb{Z}_4$, $0 \to B' \to B \to B'' \to 0$ is $0 \to \mathbb{Z}_4 \to \mathbb{Z}_8 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4 \to 0$;
(v) $0 \to A' \to A \to A'' \to 0$ is $0 \to \mathbb{Z}_4 \to \mathbb{Z}_{16} \to \mathbb{Z}_4 \to 0$, B is $\mathbb{Z}$.

5.8. For any abelian group A, let

$$mA = \{b \in A \mid b = ma, a \in A\},$$
$$_mA = \{a \in A \mid ma = 0\},$$
$$A_m = A/mA.$$

Show that there are exact sequences

$$0 \to \mathrm{Ext}(mA, \mathbb{Z}) \to \mathrm{Ext}(A, \mathbb{Z}) \to \mathrm{Ext}({}_mA, \mathbb{Z}) \to 0,$$

$$0 \to \mathrm{Hom}(A, \mathbb{Z}) \to \mathrm{Hom}(mA, \mathbb{Z}) \to \mathrm{Ext}(A_m, \mathbb{Z}) \to \mathrm{Ext}(A, \mathbb{Z}) \to \mathrm{Ext}(mA, \mathbb{Z}) \to 0,$$

and that $\mathrm{Hom}(A, \mathbb{Z}) \cong \mathrm{Hom}(mA, \mathbb{Z})$.
Prove the following assertions:
(i) ${}_mA = 0$ if and only if $\mathrm{Ext}(A, \mathbb{Z})_m = 0$;
(ii) if $A_m = 0$ then ${}_m\mathrm{Ext}(A, \mathbb{Z}) = 0$;
(iii) if ${}_m\mathrm{Ext}(A, \mathbb{Z}) = 0 = \mathrm{Hom}(A, \mathbb{Z})$, then $A_m = 0$.
Give a counterexample to show that the converse of (ii) is not true.
(Hint: an abelian group B such that $mB = 0$ is a direct sum of cyclic groups.)

6. A Theorem of Stein-Serre for Abelian Groups

By Corollary 5.5 A is projective if and only if $\mathrm{Ext}_\Lambda(A, B) = 0$ for all Λ-modules B. The question naturally arises as to whether it is necessary to use *all* Λ-modules B in $\mathrm{Ext}_\Lambda(A, B)$ to test whether A is projective; might it not happen that there exists a small family of Λ-modules B_i such that if $\mathrm{Ext}_\Lambda(A, B_i) = 0$ for every B_i in the family, then A is projective? Of course, as is easily shown, A is projective if $\mathrm{Ext}_\Lambda(A, R) = 0$ where $R \rightarrowtail P \twoheadrightarrow A$ is a projective presentation of A, but our intention is that the family B_i may be chosen independently of A.

For $\Lambda = \mathbb{Z}$ and A finitely-generated there is a very simple criterion for A to be projective (i.e. free): *If A is a finitely generated abelian group,* *then A is free if and only if* $\operatorname{Ext}(A, \mathbb{Z}) = 0$. This result immediately follows by using the fundamental theorem for finitely-generated abelian groups and the relations $\operatorname{Ext}(\mathbb{Z}, \mathbb{Z}) = 0$, $\operatorname{Ext}(\mathbb{Z}_r, \mathbb{Z}) = \mathbb{Z}_r$ of Section 4. Of course, if A is free, $\operatorname{Ext}(A, \mathbb{Z}) = 0$ but it is still an open question whether, for all abelian groups A, $\operatorname{Ext}(A, \mathbb{Z}) = 0$ implies A free.* However we shall prove the following theorem of Stein-Serre:

Theorem 6.1. *If A is an abelian group of countable rank, then* $\operatorname{Ext}(A, \mathbb{Z}) = 0$ *implies A free.*

Let us first remind the reader that the *rank* of an abelian group A, rank A, is the maximal number of linearly independent elements in A. For the proof of Theorem 6.1 we shall need the following lemma.

Lemma 6.2. *Let A be an abelian group of countable rank. If every subgroup of A of finite rank is free, then A is free.*

Proof. By hypothesis there is a maximal countable linearly independent set $T = (a_1, a_2, \ldots, a_n, \ldots)$ of elements of A. Let A_n be the subgroup of A consisting of all elements $a \in A$ linearly dependent on $(a_1, a_2, \ldots, a_n)$, i.e. such that $(a_1, a_2, \ldots, a_n, a)$ is linearly dependent. Since A is torsionfree, $A_0 = 0$. Plainly $A_{n-1} \subseteqq A_n$, and, since T is maximal, $A = \bigcup_n A_n$. Since A_n has finite rank n, it is free of rank n; in particular it is finitely-generated; hence A_n/A_{n-1} is finitely-generated, too. We claim that A_n/A_{n-1} is torsionfree. Indeed if for $a \in A$ the set $(a_1, a_2, \ldots, a_{n-1}, ka)$ is linearly dependent, $k \neq 0$, then $(a_1, a_2, \ldots, a_{n-1}, a)$ is linearly dependent, also. As a finitely-generated torsionfree group, A_n/A_{n-1} is free. Evidently its rank is one. Hence A_n/A_{n-1} is infinite cyclic. Let $b_n + A_{n-1}$, $b_n \in A_n$, be a generator of A_n/A_{n-1}. We claim that $S = (b_1, b_2, \ldots, b_n, \ldots)$ is a basis for A. Indeed, S is linearly independent, for if $\sum_{i=1}^{n} k_i b_i = 0$, $k_n \neq 0$, then $k_n b_n + A_{n-1} = A_{n-1}$ which is impossible since A_n/A_{n-1} is infinite cyclic on $b_n + A_{n-1}$ as generator. Also, S generates A; since $A = \bigcup_n A_n$ it is plainly sufficient to show that $(b_1, \ldots, b_n)$ generate A_n, and this follows by an easy induction on n. □

Proof of Theorem 6.1. We first make a couple of reductions. By Lemma 6.2 it suffices to show that every subgroup A' of finite rank is free. Since $\operatorname{Ext}(A, \mathbb{Z}) = 0$ implies $\operatorname{Ext}(A', \mathbb{Z}) = 0$ by Corollary 5.7, we have to show that for groups A of *finite* rank, $\operatorname{Ext}(A, \mathbb{Z}) = 0$ implies A free. If A'' is a finitely-generated subgroup of A with $\operatorname{Ext}(A, \mathbb{Z}) = 0$ it follows that $\operatorname{Ext}(A'', \mathbb{Z}) = 0$ and hence, by the remark at the beginning of the section, A'' is free. Since A is torsionfree if and only if every finitely generated subgroup of A is free, it will be sufficient to show that for any group *A of finite rank which is not free but torsionfree*, $\operatorname{Ext}(A, \mathbb{Z}) \neq 0$.

* This question has now been settled in a surprising way. See page 330.

Consider the sequence $\mathbb{Z} \rightarrowtail \mathbb{Q} \xrightarrow{\eta} \mathbb{Q}/\mathbb{Z}$ and the associated sequence

$$\mathrm{Hom}(A, \mathbb{Z}) \rightarrowtail \mathrm{Hom}(A, \mathbb{Q}) \xrightarrow{\eta_*} \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}) \twoheadrightarrow \mathrm{Ext}(A, \mathbb{Z}).$$

We have to prove that η_* is not epimorphic; we do that by showing that $\mathrm{card}(\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})) > \mathrm{card}(\mathrm{Hom}(A, \mathbb{Q}))$.

Let $(a_1, \ldots, a_n)$ be a maximal linearly independent set of elements of A. Let A_0 be the subgroup of A generated by $(a_1, \ldots, a_n)$. By hypothesis A_0 is free. Since A is not free, $A_0 \neq A$.

Since every homomorphism from A into $\mathbb{Q}$ is determined by its restriction to A_0, and since every homomorphism from A_0 to $\mathbb{Q}$ extends to A, we obtain

$$\mathrm{card}(\mathrm{Hom}(A, \mathbb{Q})) = \mathrm{card}(\mathrm{Hom}(A_0, \mathbb{Q})) = \aleph_0^n = \aleph_0 .$$

Take $b_1 \in A - A_0$ and let $k_1 \geqq 2$ be the smallest integer for which $k_1 b_1 \in A_0$. Let A_1 be the subgroup generated by $a_1, a_2, \ldots, a_n, b_1$. By hypothesis A_1 is free. But $A_1 \neq A$, since A is not free. Take $b_2 \in A - A_1$. Let $k_2 \geqq 2$ be the smallest integer with $k_2 b_2 \in A_1$. Let A_2 be generated by $(a_1, a_2, \ldots, a_n, b_1, b_2)$. A_2 is free, but $A_2 \neq A$ since A is not free. Continuing this way we obtain a sequence of elements of A, $b_1, b_2, \ldots, b_m, \ldots$, and a sequence of integers $k_1, k_2, \ldots, k_m, \ldots$, each of them $\geqq 2$, such that k_m is the smallest positive integer with $k_m b_m \in A_{m-1}$ where A_{m-1} is the subgroup of A generated by $(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_{m-1})$; $b_m \notin A_{m-1}$. Let $A_\infty = \bigcup_n A_n$. Since every homomorphism from A_∞ into $\mathbb{Q}/\mathbb{Z}$ may be extended to a homomorphism from A to $\mathbb{Q}/\mathbb{Z}$ one has

$$\mathrm{card}(\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})) \geqq \mathrm{card}(\mathrm{Hom}(A_\infty, \mathbb{Q}/\mathbb{Z})).$$

But $\mathrm{card}(\mathrm{Hom}(A_\infty, \mathbb{Q}/\mathbb{Z})) = \aleph_0 \cdot k_1 \cdot k_2 \cdot \cdots = 2^{\aleph_0}$. For one has $\aleph_0$ homomorphisms $\varphi_0 : A_0 \to \mathbb{Q}/\mathbb{Z}$, k_1 ways of extending φ_0 to $\varphi_1 : A_1 \to \mathbb{Q}/\mathbb{Z}$, k_2 ways of extending φ_1 to $\varphi_2 : A_2 \to \mathbb{Q}/\mathbb{Z}$, etc. We have shown that

$$\mathrm{card}(\mathrm{Hom}(A, \mathbb{Q})) < \mathrm{card}(\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})).$$

Hence $\mathrm{Ext}(A, \mathbb{Z}) \neq 0$. $\square$

From the proof of Theorem 6.1 one sees that $\mathrm{card}(\mathrm{Ext}(A, \mathbb{Z})) = 2^{\aleph_0}$ if A is of finite rank and *not* free but torsionfree. For our argument shows that $\mathrm{card}(\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})) \geqq 2^{\aleph_0}$; but if A is torsionfree of finite (or even countable) rank it is countable, hence the cardinality of the set of all functions from A to $\mathbb{Q}/\mathbb{Z}$ is $2^{\aleph_0}$; so that

$$\mathrm{card}(\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})) = 2^{\aleph_0} .$$

It follows that $\mathrm{card}(\mathrm{Ext}(A, \mathbb{Z})) = 2^{\aleph_0}$.

Exercises:

6.1. Show that if A is torsionfree, $\mathrm{Ext}(A,\mathbb{Z})$ is divisible, and that if A is divisible, $\mathrm{Ext}(A,\mathbb{Z})$ is torsionfree. Show conversely that if $\mathrm{Ext}(A,\mathbb{Z})$ is divisible, A is torsionfree and that if $\mathrm{Ext}(A,\mathbb{Z})$ is torsionfree and $\mathrm{Hom}(A,\mathbb{Z})=0$ then A is divisible. (See Exercise 5.8.)

6.2. Show that $\mathrm{Ext}(\mathbb{Q},\mathbb{Z})$ is divisible and torsionfree, and hence a $\mathbb{Q}$-vector space. (Compare Exercise 2.4.) Deduce that $\mathrm{Ext}(\mathbb{Q},\mathbb{Z})\cong\mathbb{R}$, $\mathrm{Hom}(\mathbb{Q},\mathbb{Q}/\mathbb{Z})\cong\mathbb{R}$. Compute $\mathrm{Ext}(\mathbb{R},\mathbb{Z})$.

6.3. Show that $\mathrm{Ext}(\mathbb{Q}/\mathbb{Z},\mathbb{Z})$ fits into exact sequences

$$0\to\mathbb{Z}\to\mathrm{Ext}(\mathbb{Q}/\mathbb{Z},\mathbb{Z})\to\mathbb{R}\to 0\,,$$

$$0\to\mathrm{Ext}(\mathbb{Q}/\mathbb{Z},\mathbb{Z})\to\mathbb{R}\to\mathbb{Q}/\mathbb{Z}\to 0\,.$$

6.4. Show that the simultaneous equations $\mathrm{Ext}(A,\mathbb{Z})=0$, $\mathrm{Hom}(A,\mathbb{Z})=0$ imply $A=0$.

6.5. Show that the simultaneous equations $\mathrm{Ext}(A,\mathbb{Z})=\mathbb{Q}$, $\mathrm{Hom}(A,\mathbb{Z})=0$ have no solution. Generalize this by replacing $\mathbb{Q}$ by a suitable $\mathbb{Q}$-vector space. What can you say of the solutions of $\mathrm{Ext}(A,\mathbb{Z})=\mathbb{R}$, $\mathrm{Hom}(A,\mathbb{Z})=0$?

7. The Tensor Product

In the remaining two sections of Chapter III we shall introduce two functors: the tensor product and the Tor-functor.

Let Λ again be a ring, A a right and B a left Λ-module.

Definition. The *tensor product* of A and B over Λ is the abelian group, $A\otimes_\Lambda B$, obtained as the quotient of the free abelian group on the set of all symbols $a\otimes b$, $a\in A$, $b\in B$, by the subgroup generated by

$$(a_1+a_2)\otimes b-(a_1\otimes b+a_2\otimes b),\, a_1,a_2\in A, b\in B;$$

$$a\otimes(b_1+b_2)-(a\otimes b_1+a\otimes b_2),\, a\in A, b_1,b_2\in B;$$

$$a\lambda\otimes b-a\otimes\lambda b,\, a\in A, b\in B, \lambda\in\Lambda\,.$$

In case $\Lambda=\mathbb{Z}$ we shall allow ourselves to write $A\otimes B$ for $A\otimes_{\mathbb{Z}}B$. For simplicity we shall denote the element of $A\otimes_\Lambda B$ obtained as canonical image of $a\otimes b$ in the free abelian group by the same symbol $a\otimes b$.

The ring Λ may be regarded as left or right Λ-module over Λ. It is easy to see that we have natural isomorphisms (of abelian groups)

$$\Lambda\otimes_\Lambda B\xrightarrow{\sim}B,\quad A\otimes_\Lambda\Lambda\xrightarrow{\sim}A$$

given by $\lambda\otimes b\mapsto\lambda b$ and $a\otimes\lambda\mapsto a\lambda$.

For any $\alpha:A\to A'$ we define an induced map $\alpha_*:A\otimes_\Lambda B\to A'\otimes_\Lambda B$ by $\alpha_*(a\otimes b)=(\alpha a)\otimes b$, $a\in A$, $b\in B$. Also, for $\beta:B\to B'$ we define

$\beta_*: A \otimes_\Lambda B \to A \otimes_\Lambda B'$ by $\beta_*(a \otimes b) = a \otimes (\beta b)$, $a \in A$, $b \in B$. With these definitions we obtain

Proposition 7.1. *For any left Λ-module B, $- \otimes_\Lambda B: \mathfrak{M}_\Lambda^r \to \mathfrak{Ab}$ is a covariant functor. For any right Λ-module A, $A \otimes_\Lambda - : \mathfrak{M}_\Lambda^l \to \mathfrak{Ab}$ is a covariant functor. Moreover, $- \otimes_\Lambda -$ is a bifunctor.*

The proof is left to the reader. □

If $\alpha: A \to A'$ and $\beta: B \to B'$ are homomorphisms we use the notation

$$\alpha \otimes \beta = \alpha_* \beta_* = \beta_* \alpha_* : A \otimes_\Lambda B \to A' \otimes_\Lambda B'.$$

The importance of the tensorproduct will become clear from the following assertion.

Theorem 7.2. *For any right Λ-module A, the functor $A \otimes_\Lambda - : \mathfrak{M}_\Lambda^l \to \mathfrak{Ab}$ is left adjoint to the functor* $\operatorname{Hom}_{\mathbb{Z}}(A, -): \mathfrak{Ab} \to \mathfrak{M}_\Lambda^l$.

Proof. The left-module structure of $\operatorname{Hom}_{\mathbb{Z}}(A, -)$ is induced by the right-module structure of A (see Section I.8). We have to show that there is a natural transformation η such that for any abelian group G and any left Λ-module B

$$\eta: \operatorname{Hom}_{\mathbb{Z}}(A \otimes_\Lambda B, G) \xrightarrow{\sim} \operatorname{Hom}_\Lambda(B, \operatorname{Hom}_{\mathbb{Z}}(A, G)).$$

Given $\varphi: A \otimes_\Lambda B \to G$ we define $\eta(\varphi)$ by the formula

$$((\eta(\varphi))(b))(a) = \varphi(a \otimes b).$$

Given $\psi: B \to \operatorname{Hom}_{\mathbb{Z}}(A, G)$ we define $\tilde{\eta}(\psi)$ by $(\tilde{\eta}(\psi))(a \otimes b) = (\psi(b))(a)$. We claim that $\eta, \tilde{\eta}$ are natural homomorphisms which are inverse to each other. We leave it to the reader to check the necessary details. □

Analogously we may prove that $- \otimes_\Lambda B: \mathfrak{M}_\Lambda^r \to \mathfrak{Ab}$ is left adjoint to $\operatorname{Hom}_{\mathbb{Z}}(B, -): \mathfrak{Ab} \to \mathfrak{M}_\Lambda^r$, where the right module structure of $\operatorname{Hom}_{\mathbb{Z}}(B, G)$ is given by the left module structure of B. We remark that the tensor-product-functor $A \otimes_\Lambda -$ is determined up to natural equivalence by the adjointness property of Theorem 7.2 (see Proposition II.7.3); a similar remark applies to the functor $- \otimes_\Lambda B$.

As an immediate consequence of Theorem 7.2 we have

Proposition 7.3. (i) *Let $\{B_j\}$, $j \in J$, be a family of left Λ-modules and let A be a right Λ-module. Then there is a natural isomorphism*

$$A \otimes_\Lambda \Big(\bigoplus_{j \in J} B_j\Big) \xrightarrow{\sim} \bigoplus_{j \in J} (A \otimes_\Lambda B_j).$$

(ii) *If $B' \xrightarrow{\beta'} B \xrightarrow{\beta''} B'' \to 0$ is an exact sequence of left Λ-modules, then for any right Λ-module A, the sequence*

$$A \otimes_\Lambda B' \xrightarrow{\beta'_*} A \otimes_\Lambda B \xrightarrow{\beta''_*} A \otimes_\Lambda B'' \to 0$$

is exact.

Proof. By the dual of Theorem II.7.7 a functor possessing a right adjoint preserves coproducts and cokernels. □

Of course there is a proposition analogous to Proposition 7.3 about the functor $-\otimes_\Lambda B$ for fixed B. The reader should note that, even if β' in Proposition 7.3 (ii) is monomorphic, β'_* will not be monomorphic in general: Let $\Lambda=\mathbb{Z}$, $A=\mathbb{Z}_2$, and consider the exact sequence $\mathbb{Z} \overset{\mu}{\rightarrowtail} \mathbb{Z} \twoheadrightarrow \mathbb{Z}_2$ where μ is multiplication by 2. Then

$$\mu_*(n\otimes m)=n\otimes 2m=2n\otimes m=0\otimes m=0\,,$$

$n\in\mathbb{Z}_2$, $m\in\mathbb{Z}$. Hence $\mu_*:\mathbb{Z}_2\otimes\mathbb{Z}\to\mathbb{Z}_2\otimes\mathbb{Z}$ is the zero map, while $\mathbb{Z}_2\otimes\mathbb{Z}\cong\mathbb{Z}_2$.

Definition. A left Λ-module B is called *flat* if for every short exact sequence $A' \overset{\mu}{\rightarrowtail} A \overset{\varepsilon}{\twoheadrightarrow} A''$ the induced sequence

$$0\to A'\otimes_\Lambda B \xrightarrow{\mu_*} A\otimes_\Lambda B\to A''\otimes_\Lambda B\to 0$$

is exact. This is to say that for every monomorphism $\mu:A'\to A$ the induced homomorphism $\mu_*:A'\otimes_\Lambda B\to A\otimes_\Lambda B$ is a monomorphism, also.

Proposition 7.4. *Every projective module is flat.*

Proof. A projective module P is a direct summand in a free module. Hence, since $A\otimes_\Lambda -$ preserves sums, it suffices to show that free modules are flat. By the same argument it suffices to show that Λ as a left module is flat. But this is trivial since $A\otimes_\Lambda\Lambda\cong A$. □

For abelian groups it turns out that "flat" is "torsionfree" (see Exercise 8.7). Since the additive group of the rationals $\mathbb{Q}$ is torsionfree but not free, one sees that flat modules are not, in general, projective.

Exercises:

7.1. Show that if A is a left Γ-right Λ-bimodule and B a left Λ-right Σ-bimodule then $A\otimes_\Lambda B$ may be given a left Γ-right Σ-bimodule structure.

7.2. Show that, if Λ is commutative, we can speak of the tensorproduct $A\otimes_\Lambda B$ of two left (!) Λ-modules, and that $A\otimes_\Lambda B$ has an obvious Λ-module structure. Also show that then $A\otimes_\Lambda B\cong B\otimes_\Lambda A$ and $(A\otimes_\Lambda B)\otimes_\Lambda C\cong A\otimes_\Lambda(B\otimes_\Lambda C)$ by canonical isomorphisms.

7.3. Prove the following generalization of Theorem 7.2. Let A be a left Γ-right Λ-bimodule, B a left Λ-module and C a left Γ-module. Then $A\otimes_\Lambda B$ can be given a left Γ-module structure, and $\mathrm{Hom}_\Gamma(A,C)$ a left Λ-module structure. Prove the adjointness relation

$$\eta:\mathrm{Hom}_\Gamma(A\otimes_\Lambda B,C)\xrightarrow{\sim}\mathrm{Hom}_\Lambda(B,\mathrm{Hom}_\Gamma(A,C))\,.$$

7.4. Show that, if A, B are Λ-modules and if $\sum_i a_i\otimes b_i=0$ in $A\otimes_\Lambda B$, then there are finitely generated submodules $A_0\subseteq A$, $B_0\subseteq B$ such that $a_i\in A_0$, $b_i\in B_0$ and $\sum_i a_i\otimes b_i=0$ in $A_0\otimes B_0$. Use this to show that $\mathbb{Q}$ is flat over $\mathbb{Z}$.

7.5. Show that if A, B are modules over a principal ideal domain and if $a \in A$, $b \in B$ are not torsion elements then $a \otimes b \neq 0$ in $A \otimes_\Lambda B$ and is not a torsion element.

7.6. Show that if A is a finitely generated module over a principal ideal domain and if $A \otimes_\Lambda A = 0$, then $A = 0$. Give an example of an abelian group $G \neq 0$ such that $G \otimes G = 0$.

7.7. Let $A \times B$ be the cartesian product of the sets underlying the right Λ-module A and the left Λ-module B. For G an abelian group call a function $f : A \times B \to G$ *bilinear* if

$$f(a_1 + a_2, b) = f(a_1, b) + f(a_2, b), \quad a_1, a_2 \in A, \ b \in B;$$
$$f(a, b_1 + b_2) = f(a, b_1) + f(a, b_2), \quad a \in A, \ b_1, b_2 \in B;$$
$$f(a\lambda, b) = f(a, \lambda b), \quad a \in A, \ b \in B, \ \lambda \in \Lambda.$$

Show that the tensor product has the following universal property. To every abelian group G and to every bilinear map $f : A \times B \to G$ there exists a unique homomorphism of abelian groups

$$g : A \otimes_\Lambda B \to G \quad \text{such that} \quad f(a, b) = g(a \otimes b).$$

7.8. Show that an associative algebra (with unity) over the commutative ring Λ may be defined as follows. An algebra A is a Λ-module together with Λ-module homomorphisms $\mu : A \otimes_\Lambda A \to A$ and $\eta : \Lambda \to A$ such that the following diagrams are commutative

$$\begin{tikzcd}
\Lambda \otimes_\Lambda A \arrow[r, "\sim"] \arrow[d, "\eta\otimes 1"'] & A \arrow[d, equal] & A \otimes_\Lambda \Lambda \arrow[l, "\sim"'] \arrow[d, "1\otimes\eta"] \\
A \otimes_\Lambda A \arrow[r, "\mu"] & A & A \otimes_\Lambda A \arrow[l, "\mu"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes_\Lambda A \otimes_\Lambda A \arrow[r, "\mu\otimes 1"] \arrow[d, "1\otimes\mu"] & A \otimes_\Lambda A \arrow[d, "\mu"] \\
A \otimes_\Lambda A \arrow[r, "\mu"'] & A.
\end{tikzcd}$$

(The first diagram shows that $\eta(1_\Lambda)$ is a left and a right unity for A, while the second diagram yields associativity of the product.) Show that if A and B are algebras over Λ then $A \otimes_\Lambda B$ may naturally be made into an algebra over Λ.

7.9. An algebra A over Λ is called *augmented* if a homomorphism $\varepsilon : A \to \Lambda$ of algebras is given. Show that the group algebra KG is augmented with $\varepsilon : KG \to K$ defined by $\varepsilon(x) = 1$, $x \in G$. Give other examples of augmented algebras.

8. The Functor Tor

Let A be a right Λ-module and let B be a left Λ-module. Given a projective presentation $R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ of A we define

$$\operatorname{Tor}_\varepsilon^\Lambda(A, B) = \ker(\mu_* : R \otimes_\Lambda B \to P \otimes_\Lambda B).$$

The sequence

$$0 \to \operatorname{Tor}_\varepsilon^\Lambda(A, B) \to R \otimes_\Lambda B \to P \otimes_\Lambda B \to A \otimes_\Lambda B \to 0$$

is exact. Obviously we can make $\operatorname{Tor}_\varepsilon^A(A,-)$ into a covariant functor by defining, for a map $\beta: B \to B'$, the associated map

$$\beta_*: \operatorname{Tor}_\varepsilon^A(A,B) \to \operatorname{Tor}_\varepsilon^A(A,B')$$

to be the homomorphism induced by $\beta_*: R \otimes_A B \to R \otimes_A B'$. To any projective presentation $S \overset{\nu}{\rightarrowtail} Q \overset{\eta}{\twoheadrightarrow} B$ of B we define

$$\overline{\operatorname{Tor}}_\eta^A(A,B) = \ker(\nu_*: A \otimes_A S \to A \otimes_A Q).$$

With this definition the sequence

$$0 \to \overline{\operatorname{Tor}}_\eta^A(A,B) \to A \otimes_A S \to A \otimes_A Q \to A \otimes_A B \to 0$$

is exact. Clearly, given a homomorphism $\alpha: A \to A'$, we can associate a homomorphism $\alpha_*: \overline{\operatorname{Tor}}_\eta^A(A,B) \to \overline{\operatorname{Tor}}_\eta^A(A',B)$, which is induced by $\alpha_*: A \otimes_A S \to A' \otimes_A S$. With this definition $\overline{\operatorname{Tor}}_\eta^A(-,B)$ is a covariant functor.

Proposition 8.1. *If A (or B) is projective, then*

$$\operatorname{Tor}_\varepsilon^A(A,B) = 0 = \overline{\operatorname{Tor}}_\eta^A(A,B).$$

Proof. Since A is projective, the short exact sequence $R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ splits, i.e. there is $\kappa: P \to R$ with $\kappa\mu = 1_R$. Hence

$$\kappa\mu \otimes 1 = (\kappa \otimes 1)(\mu \otimes 1) = 1_{R \otimes_A B},$$

and consequently $\mu \otimes 1$ is monomorphic. Thus $\operatorname{Tor}_\varepsilon^A(A,B) = 0$.

If A is projective, A is flat by Proposition 7.4. Hence

$$0 \to A \otimes_A S \to A \otimes_A Q \to A \otimes_A B \to 0$$

is exact. Thus $\overline{\operatorname{Tor}}_\eta^A(A,B) = 0$. The remaining assertions merely interchange left and right. □

Next we will use Lemma 5.1 to show that $\overline{\operatorname{Tor}}_\eta^A$ and $\operatorname{Tor}_\varepsilon^A$ denote the same functor. Again let $R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ and $S \overset{\nu}{\rightarrowtail} Q \overset{\eta}{\twoheadrightarrow} B$ be projective presentations. We then construct the commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\Sigma_5"] & \operatorname{Tor}_\varepsilon^A(A,B) \arrow[d, tail] \\
 & R \otimes_A S \arrow[r] \arrow[d] \arrow[dr, phantom, "\Sigma_3"] & R \otimes_A Q \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\Sigma_4"] & R \otimes_A B \arrow[d] \\
0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\Sigma_1"] & P \otimes_A S \arrow[r] \arrow[d] \arrow[dr, phantom, "\Sigma_2"] & P \otimes_A Q \arrow[r, two heads] \arrow[d, two heads] & P \otimes_A B \arrow[d, two heads] \\
\overline{\operatorname{Tor}}_\eta^A(A,B) \arrow[r] & A \otimes_A S \arrow[r] & A \otimes_A Q \arrow[r, two heads] & A \otimes_A B
\end{tikzcd} \tag{8.1}$$

By a repeated application of Lemma 3.1 we obtain

$$\overline{\mathrm{Tor}}_\eta^\Lambda(A, B) = \mathrm{Im}\,\Sigma_1 \cong \mathrm{Ker}\,\Sigma_2 \cong \mathrm{Im}\,\Sigma_3 \cong \mathrm{Ker}\,\Sigma_4 \cong \mathrm{Im}\,\Sigma_5 = \mathrm{Tor}_\varepsilon^\Lambda(A, B).$$

Now let $R' \overset{\mu'}{\rightarrowtail} P' \overset{\varepsilon'}{\twoheadrightarrow} A'$ be a projective presentation of A' and $\alpha: A \to A'$ a homomorphism. We can then find $\varphi: P \to P'$ and $\psi: R \to R'$ such that the following diagram commutes:

$$\begin{tikzcd} R \arrow[r, tail, "\mu"] \arrow[d, "\psi"] & P \arrow[r, two heads, "\varepsilon"] \arrow[d, "\varphi"] & A \arrow[d, "\alpha"] \\ R' \arrow[r, tail, "\mu'"] & P' \arrow[r, two heads, "\varepsilon'"] & A' \end{tikzcd} \tag{8.2}$$

These homomorphisms induce a map from the diagram (8.1) into the diagram corresponding to the presentation $R' \rightarrowtail P' \twoheadrightarrow A'$. Consequently we obtain a homomorphism

$$\mathrm{Tor}_\varepsilon^\Lambda(A, B) \xrightarrow{\sim} \overline{\mathrm{Tor}}_\eta^\Lambda(A, B) \xrightarrow{\alpha_*} \overline{\mathrm{Tor}}_\eta^\Lambda(A', B) \xrightarrow{\sim} \mathrm{Tor}_{\varepsilon'}^\Lambda(A', B)$$

which is visibly independent of the choice of φ in (8.2). Choosing $\alpha = 1_A$ we obtain an isomorphism $\mathrm{Tor}_\varepsilon^\Lambda(A, B) \xrightarrow{\sim} \overline{\mathrm{Tor}}_\eta^\Lambda(A, B) \xrightarrow{\sim} \mathrm{Tor}_{\varepsilon'}^\Lambda(A, B)$.

Collecting the information obtained, we have shown that there is a natural equivalence between the functors $\mathrm{Tor}_\varepsilon^\Lambda(A, -)$ and $\mathrm{Tor}_{\varepsilon'}^\Lambda(A, -)$, that we therefore can drop the subscript ε, writing $\mathrm{Tor}^\Lambda(A, -)$ from now on; further that $\mathrm{Tor}^\Lambda(-, B)$ can be made into a functor, which is equivalent to $\overline{\mathrm{Tor}}_\eta^\Lambda(-, B)$ for any η. We thus can use the notation $\mathrm{Tor}^\Lambda(A, B)$ for $\overline{\mathrm{Tor}}_\eta^\Lambda(A, B)$, also. We finally leave it to the reader to show that $\mathrm{Tor}^\Lambda(-, -)$ is a bifunctor. The fact that $\mathrm{Tor}^\Lambda(-, -)$ coincides with $\overline{\mathrm{Tor}}^\Lambda(-, -)$ is sometimes expressed by saying that Tor is *balanced.*

Similarly to Theorems 5.2 and 5.3, one obtains

Theorem 8.2. *Let A be a right Λ-module and $B' \overset{\kappa}{\rightarrowtail} B \overset{\nu}{\twoheadrightarrow} B''$ an exact sequence of left Λ-modules, then there exists a connecting homomorphism $\omega: \mathrm{Tor}^\Lambda(A, B'') \to A \otimes_\Lambda B'$ such that the following sequence is exact:*

$$\begin{aligned} &\mathrm{Tor}^\Lambda(A, B') \xrightarrow{\kappa_*} \mathrm{Tor}^\Lambda(A, B) \xrightarrow{\nu_*} \mathrm{Tor}^\Lambda(A, B'') \xrightarrow{\omega} A \otimes_\Lambda B' \\ &\xrightarrow{\kappa_*} A \otimes_\Lambda B \xrightarrow{\nu_*} A \otimes_\Lambda B'' \longrightarrow 0. \end{aligned} \tag{8.3}$$

Theorem 8.3. *Let B be a left Λ-module and let $A' \overset{\kappa}{\rightarrowtail} A \overset{\nu}{\twoheadrightarrow} A''$ be an exact sequence of right Λ-modules. Then there exists a connecting homomorphism $\omega: \mathrm{Tor}^\Lambda(A'', B) \to A' \otimes_\Lambda B$ such that the following sequence is exact:*

$$\begin{aligned} &\mathrm{Tor}^\Lambda(A', B) \xrightarrow{\kappa_*} \mathrm{Tor}^\Lambda(A, B) \xrightarrow{\nu_*} \mathrm{Tor}^\Lambda(A'', B) \xrightarrow{\omega} A' \otimes_\Lambda B \\ &\xrightarrow{\kappa_*} A \otimes_\Lambda B \xrightarrow{\nu_*} A'' \otimes_\Lambda B \longrightarrow 0. \end{aligned} \tag{8.4}$$

Proof. We only prove Theorem 8.2; the proof of Theorem 8.3 may be obtained by replacing Tor by $\overline{\mathrm{Tor}}$. Consider the projective presentation

$R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ and construct the diagram:

$$\begin{tikzcd}
 & \operatorname{Tor}^{\Lambda}(A, B') \arrow[r, dotted] \arrow[d] & \operatorname{Tor}^{\Lambda}(A, B) \arrow[r, dotted] \arrow[d] & \operatorname{Tor}^{\Lambda}(A, B) \arrow[d] & \\
 & R \otimes_{\Lambda} B' \arrow[r] \arrow[d] & R \otimes_{\Lambda} B \arrow[r] \arrow[d] & R \otimes_{\Lambda} B'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P \otimes_{\Lambda} B' \arrow[r] \arrow[d, two heads] & P \otimes_{\Lambda} B \arrow[r] \arrow[d, two heads] & P \otimes_{\Lambda} B'' \arrow[r] \arrow[d, two heads] & 0 \\
 & A \otimes_{\Lambda} B' \arrow[r, dotted] & A \otimes_{\Lambda} B \arrow[r, dotted] & A \otimes_{\Lambda} B'' &
\end{tikzcd} \tag{8.5}$$

By applying Lemma 5.1 we obtain the asserted sequence. □

We remark that like the Hom-Ext sequences the sequences (8.3) and (8.4) are natural. Notice that by contrast with the two sequences involving Ext we obtain only *one* kind of sequence involving Tor, since A, B play symmetric roles in the definition of Tor.

Corollary 8.4. *Let* Λ *be a principal ideal domain. Then the homomorphisms* $\kappa_* : \operatorname{Tor}^{\Lambda}(A, B') \to \operatorname{Tor}^{\Lambda}(A, B)$ *in sequence* (8.3) *and*

$$\kappa_* : \operatorname{Tor}^{\Lambda}(A', B) \to \operatorname{Tor}^{\Lambda}(A, B) \textit{ in sequence } (8.4) \textit{ are monomorphic.}$$

Proof. By Corollary I.5.3 R is a projective right Λ-module, hence the map $\kappa_* : R \otimes_{\Lambda} B' \to R \otimes_{\Lambda} B$ in diagram (8.5) is monomorphic, whence the first assertion. Analogously one obtains the second assertion. □

Exercises:

8.1. Show that, if A (or B) is flat, then $\operatorname{Tor}^{\Lambda}(A, B) = 0$.

8.2. Evaluate the exact sequences (8.3), (8.4) for the examples given in Exercise 5.7 (i), ..., (v).

8.3. Show that if A is a torsion group then $A \cong \operatorname{Tor}(A, \mathbb{Q}/\mathbb{Z})$; and that, in general, $\operatorname{Tor}(A, \mathbb{Q}/\mathbb{Z})$ embeds naturally as a subgroup of A. Identify this subgroup.

8.4. Show that if A and B are abelian groups and if $T(A)$, $T(B)$ are their torsion subgroups, then
$$\operatorname{Tor}(A, B) \cong \operatorname{Tor}(T(A), T(B)).$$
Show that $m \operatorname{Tor}(A, B) = 0$ if $m\, T(A) = 0$.

8.5. Show that Tor is additive in each variable.

8.6. Show that Tor respects direct limits over directed sets.

8.7. Show that the abelian group A is flat if and only if it is torsion-free.

8.8. Show that A' is pure in A if and only if $A' \otimes G \to A \otimes G$ is a monomorphism for all G (see Exercise I. 1.7).

8.9. Show that $\operatorname{Tor}^{\Lambda}(A, B)$ can be computed using a flat presentation of A; that is, if $R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ with P flat, then
$$\operatorname{Tor}^{\Lambda}(A, B) \cong \ker(\mu_* : R \otimes_{\Lambda} B \to P \otimes_{\Lambda} B).$$

IV. Derived Functors

In this chapter we go to the heart of homological algebra. Everything up to this point can be regarded as providing essential background for the theory of *derived functors*, and introducing the special cases of $\mathrm{Ext}_\Lambda(A, B)$, $\mathrm{Tor}^\Lambda(A, B)$. Subsequent chapters take up more sophisticated properties of derived functors and special features of the theory in various contexts* (cohomology of groups, cohomology of Lie algebras).

The basic definitions and properties of derived functors are given in this chapter. Given an additive functor $F: \mathfrak{A} \to \mathfrak{B}$ from the abelian category $\mathfrak{A}$ to the abelian category $\mathfrak{B}$, we may form its *left derived* functors $L_n F: \mathfrak{A} \to \mathfrak{B}$, $n \geqq 0$, provided $\mathfrak{A}$ has enough projectives, that is, provided every object of $\mathfrak{A}$ admits a projective presentation. Thus the theory is certainly applicable to the category $\mathfrak{M}_\Lambda$. We may regard $L_n F(A)$ as a "function" depending on the "variables" F and A, where A is an object of $\mathfrak{A}$; there are then two basic exact sequences, one arising from a variation of A and the other from a variation of F. We may, in particular, apply the theory to the tensor product; thus we may study $L_n F_B(A)$ where $F_B: \mathfrak{M}^r_\Lambda \to \mathfrak{Ab}$ is given by $F_B(A) = A \otimes_\Lambda B$, for fixed B in $\mathfrak{M}^l_\Lambda$, and $L_n G_A(B)$ where $G_A: \mathfrak{M}^l_\Lambda \to \mathfrak{Ab}$ is given by $G_A(B) = A \otimes_\Lambda B$, for fixed A in $\mathfrak{M}^r_\Lambda$. The two exact sequences then come into play to establish the natural isomorphism, for these two functors,

$$L_n F_B(A) \cong L_n G_A(B),$$

resulting in the balanced definition of $\mathrm{Tor}^\Lambda_n(A, B)$.

Similarly, we may form *right* derived functors of $F: \mathfrak{A} \to \mathfrak{B}$ if $\mathfrak{A}$ has enough injectives. We apply the resulting theory to Hom; thus we study $R_n F^B(A)$ where $F^B: \mathfrak{M}_\Lambda \to \mathfrak{Ab}$ is the contravariant functor given by $F^B(A) = \mathrm{Hom}_\Lambda(A, B)$ for fixed B in $\mathfrak{M}_\Lambda$, and $R_n G^A(B)$ where $G^A: \mathfrak{M}_\Lambda \to \mathfrak{Ab}$ is the (covariant) functor given by $G^A(B) = \mathrm{Hom}_\Lambda(A, B)$ for fixed A in $\mathfrak{M}_\Lambda$. Again we may prove with the help of the two basic exact sequences that

$$R_n F^B(A) \cong R_n G^A(B),$$

thus obtaining the balanced definition of $\mathrm{Ext}^n_\Lambda(A, B)$.

* Chapter VIII is somewhat special in this respect, in that it introduces a new tool in homological algebra, the theory of *spectral sequences*.

We also take up the question of how to define derived functors without using projective or injective resolutions. First we show that $\operatorname{Ext}_\Lambda^n(A,B)$ may be described in terms of *n-extensions* of A by B, generalizing the isomorphism $E(A,B)\cong \operatorname{Ext}_\Lambda(A,B)$ of Chapter III. Then we show that, for any *right exact* functor $F:\mathfrak{A}\to\mathfrak{B}$, the left derived functors of F may be characterized in terms of natural transformations into Ext; more precisely, one may give the collection $[F,\operatorname{Ext}_\Lambda^n(A,-)]$ of natural transformations from F to $\operatorname{Ext}_\Lambda^n(A,-)$ an abelian group structure and it is then isomorphic to $L_nF(A)$. The question of characterizing derived functors reappears in Chapter IX in a more general context.

The chapter closes with a discussion of the change-of-rings functor which is especially crucial in the cohomology theory of groups and Lie algebras.

1. Complexes

Let Λ be a fixed ring with 1. We remind the reader of the category $\mathfrak{M}_\Lambda^{\mathbb{Z}}$ of graded (left) modules (Example (a) in Section II.9). An object $\boldsymbol{M}$ in $\mathfrak{M}_\Lambda^{\mathbb{Z}}$ is a family $\{M_n\}$, $n\in\mathbb{Z}$, of Λ-modules, a morphism $\boldsymbol{\varphi}:\boldsymbol{M}\to\boldsymbol{M}'$ of degree p is a family $\{\varphi_n:M_n\to M'_{n+p}\}$, $n\in\mathbb{Z}$, of module homomorphisms.

Definition. A *chain complex* $\boldsymbol{C}=\{C_n,\partial_n\}$ over Λ is an object in $\mathfrak{M}_\Lambda^{\mathbb{Z}}$ together with an endomorphism $\partial:\boldsymbol{C}\to\boldsymbol{C}$ of degree -1 with $\partial\partial=0$. In other words we are given a family $\{C_n\}$, $n\in\mathbb{Z}$, of Λ-modules and a family of Λ-module homomorphisms $\{\partial_n:C_n\to C_{n-1}\}$, $n\in\mathbb{Z}$, such that $\partial_n\partial_{n+1}=0$:

$$\boldsymbol{C}:\cdots\to C_{n+1}\xrightarrow{\partial_{n+1}}C_n\xrightarrow{\partial_n}C_{n-1}\to\cdots.$$

The morphism ∂ (as well as its components ∂_n) is called the *differential* (or *boundary operator*).

A *morphism of complexes* or a *chain map* $\boldsymbol{\varphi}:\boldsymbol{C}\to\boldsymbol{D}$ is a morphism of degree 0 in $\mathfrak{M}_\Lambda^{\mathbb{Z}}$ such that $\boldsymbol{\varphi}\partial=\tilde{\partial}\boldsymbol{\varphi}$ where $\tilde{\partial}$ denotes the differential in $\boldsymbol{D}$. Thus a chain map $\boldsymbol{\varphi}$ is a family $\{\varphi_n:C_n\to D_n\}$, $n\in\mathbb{Z}$, of homomorphisms such that, for every n, the diagram

$$\begin{array}{ccc} C_n & \xrightarrow{\partial_n} & C_{n-1} \\ \downarrow{\scriptstyle\varphi_n} & & \downarrow{\scriptstyle\varphi_{n-1}} \\ D_n & \xrightarrow[\tilde{\partial}_n]{} & D_{n-1} \end{array} \tag{1.1}$$

is commutative. For simplicity we shall suppress the subscripts of the module homomorphisms ∂_n and φ_n when the meaning of the symbols is clear; so, for example, to express the commutativity of (1.1) we shall simply write $\varphi\partial=\tilde{\partial}\varphi$. We will usually not distinguish notationally between the differentials of various chain complexes, writing them all as ∂.

The reader will easily show that the collection of chain complexes over Λ and chain maps forms an abelian category. Also, if $F:\mathfrak{M}_\Lambda \to \mathfrak{M}_{\Lambda'}$ is an additive covariant functor and if $\boldsymbol{C}=\{C_n, \partial_n\}$ is a chain complex over Λ, then $F\boldsymbol{C}=\{FC_n, F\partial_n\}$ is a chain complex over Λ'. Thus F induces a functor on the category of chain complexes.

We shall now introduce the most important notion of *homology*. Let $\boldsymbol{C}=\{C_n, \partial_n\}$ be a chain complex. The condition $\partial\partial=0$ implies that $\operatorname{im}\partial_{n+1} \subseteqq \ker\partial_n$, $n\in\mathbb{Z}$. Hence we can associate with $\boldsymbol{C}$ the graded module

$$H(\boldsymbol{C})=\{H_n(\boldsymbol{C})\}, \quad \text{where } H_n(\boldsymbol{C})=\ker\partial_n/\operatorname{im}\partial_{n+1}, \quad n\in\mathbb{Z}.$$

Then $H(\boldsymbol{C})\,(H_n(\boldsymbol{C}))$ is called the (n-th) *homology module* of $\boldsymbol{C}$. (Of course if $\Lambda=\mathbb{Z}$ we shall speak of the (n-th) homology *group* of $\boldsymbol{C}$.) By diagram (1.1) a chain map $\boldsymbol{\varphi}:\boldsymbol{C}\to\boldsymbol{D}$ induces a well defined morphism, of degree zero, $H(\boldsymbol{\varphi})=\boldsymbol{\varphi}_*:H(\boldsymbol{C})\to H(\boldsymbol{D})$ of graded modules. It is clear that, with this definition, $H(-)$ becomes a functor, called the *homology functor*, from the category of chain complexes over Λ to the category of graded Λ-modules. Also, each $H_n(-)$ is a functor into $\mathfrak{M}_\Lambda$.

Often, in particular in applications to topology, elements of C_n are called *n-chains;* elements of $\ker\partial_n$ are called *n-cycles* and $\ker\partial_n$ is written $Z_n=Z_n(\boldsymbol{C})$; elements of $\operatorname{im}\partial_{n+1}$ are called *n-boundaries* and $\operatorname{im}\partial_{n+1}$ is written $B_n=B_n(\boldsymbol{C})$. Two n-cycles which determine the same element in $H_n(\boldsymbol{C})$ are called *homologous.* The element of $H_n(\boldsymbol{C})$ determined by the n-cycle c is called the *homology class* of c, and is denoted by $[c]$.

It will be clear to the reader that given a chain complex $\boldsymbol{C}$ a new chain complex $\boldsymbol{C}'$ may be constructed by replacing some or all of the differentials $\partial_n:C_n\to C_{n-1}$ by their negatives $-\partial_n:C_n\to C_{n-1}$. It is plain that $\boldsymbol{C}$ and $\boldsymbol{C}'$ are isomorphic in the category of chain complexes and that $Z(\boldsymbol{C})=Z(\boldsymbol{C}')$, $B(\boldsymbol{C})=B(\boldsymbol{C}')$, $H(\boldsymbol{C})=H(\boldsymbol{C}')$. Thus, in the homology theory of chain-complexes, we are free, if we wish, to change the signs of some of the differentials.

We finally make some remarks about the dual notion.

Definition. A *cochain complex* $\boldsymbol{C}=\{C^n,\delta^n\}$ is an object in $\mathfrak{M}_\Lambda^{\mathbb{Z}}$ together with an endomorphism $\boldsymbol{\delta}:\boldsymbol{C}\to\boldsymbol{C}$ of degree $+1$ with $\boldsymbol{\delta\delta}=0$. Again $\boldsymbol{\delta}$ is called the *differential (*or *coboundary operator). Morphisms of cochain complexes* or *cochain maps* are defined analogously to chain maps. Given a cochain complex $\boldsymbol{C}=\{C^n,\delta^n\}$ we define its *cohomology module* $H(\boldsymbol{C})=\{H^n(\boldsymbol{C})\}$ by

$$H^n(\boldsymbol{C})=\ker\delta^n/\operatorname{im}\delta^{n-1}, \qquad n\in\mathbb{Z}.$$

With the obvious definition of induced maps, $H(-)$ then becomes a functor, the *cohomology functor.* In case of a cochain complex we will speak of *co*chains, *co*boundaries, *co*cycles, *co*homologous *co*cycles, *co*-homology classes.

Of course the difference between the concepts "chain complex" and "cochain complex" is quite formal, so it will be unnecessary to deal with their theories separately. Indeed, given a chain complex $\boldsymbol{C}=\{C_n, \partial_n\}$ we obtain a cochain complex $\boldsymbol{D}=\{D^n, \delta^n\}$ by setting $D^n = C_{-n}$, $\delta^n = \partial_{-n}$. Conversely given a cochain complex we obtain a chain complex by this procedure.

Examples. (a) Let A, B be Λ-modules and let $R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ be a projective presentation of A. We define a cochain complex $\boldsymbol{C}$ of abelian groups as follows:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_\Lambda(P,B) \arrow[r, "\mu^*"] \arrow[d, equal] & \mathrm{Hom}_\Lambda(R,B) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & C^0 \arrow[r, "\delta^0"] & C^1 \arrow[r] & 0 \end{tikzcd}$$

and $C^n = 0$ for $n \neq 0, 1$. We immediately deduce

$$H^0(\boldsymbol{C}) = \mathrm{Hom}_\Lambda(A,B)\,,$$
$$H^1(\boldsymbol{C}) = \mathrm{Ext}_\Lambda(A,B)\,,$$
$$H^n(\boldsymbol{C}) = 0, n \neq 0, 1\,.$$

Consequently we obtain the groups $\mathrm{Hom}_\Lambda(A,B)$, $\mathrm{Ext}_\Lambda(A,B)$ as cohomology groups of an appropriate cochain complex $\boldsymbol{C}$. In Section 7 and 8 this procedure will lead us to an important generalisation of $\mathrm{Ext}_\Lambda(A,B)$.

(b) Let $B \overset{\kappa}{\rightarrowtail} I \overset{\nu}{\twoheadrightarrow} S$ be an injective presentation of B and form the cochain complex $\boldsymbol{C}'$, where $C'^n = 0$, $n \neq 0, 1$, and

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_\Lambda(A,I) \arrow[r, "\nu_*"] \arrow[d, equal] & \mathrm{Hom}_\Lambda(A,S) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & (C')^0 \arrow[r, "\delta^0"] & (C')^1 \arrow[r] & 0 \end{tikzcd}$$

One obtains

$$H^0(\boldsymbol{C}') = \mathrm{Hom}_\Lambda(A,B)\,,$$
$$H^1(\boldsymbol{C}') = \mathrm{Ext}_\Lambda(A,B)\,,$$
$$H^n(\boldsymbol{C}') = 0, n \neq 0, 1\,.$$

(c) Let A be a left Λ-module and B a right Λ-module. Take a projective presentation $R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ of A and form the chain complex $\boldsymbol{D}$,

$$\begin{tikzcd} 0 \arrow[r] & B \otimes_\Lambda R \arrow[r, "\mu_*"] \arrow[d, equal] & B \otimes_\Lambda P \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & D_1 \arrow[r, "\partial_1"] & D_0 \arrow[r] & 0\,. \end{tikzcd}$$

We easily see that

$$H_0(\boldsymbol{D}) = B \otimes_\Lambda A\,,$$
$$H_1(\boldsymbol{D}) = \operatorname{Tor}^\Lambda(B, A)\,,$$
$$H_n(\boldsymbol{D}) = 0,\ n \neq 0, 1\,.$$

In Section 11 this procedure will be generalized.

(d) We obtain yet another example by starting with a projective presentation of B and proceeding in a manner analogous to example (c). The homology of the complex $\boldsymbol{D}'$ so obtained is

$$H_0(\boldsymbol{D}') = B \otimes_\Lambda A\,,$$
$$H_1(\boldsymbol{D}') = \operatorname{Tor}^\Lambda(B, A)\,,$$
$$H_n(\boldsymbol{D}') = 0,\ n \neq 0, 1\,.$$

The reader should note that in all four of the above examples the homology does not depend on the particular projective or injective presentation that was chosen. This phenomenon will be clarified and generalized in Sections 4, 5.

We conclude with the following warning concerning the notations. Although we have so far adopted the convention that the dimension index n appears as a *sub*script for chain complexes and as a *super*script for cochain complexes, we may at times find it convenient to write the n as a subscript even in cochain complexes. This will prove particularly convenient in developing the theory of injective resolutions in Section 4.

Exercises:

1.1. Show that if $\boldsymbol{C}$ is a complex of abelian groups in which each C_n is free, then Z_n and B_n are also free, and that Z_n is a direct summand in C_n.

1.2. Given a chain map $\varphi: \boldsymbol{C} \to \boldsymbol{D}$, construct a chain complex as follows:

$$E_n = C_{n-1} \oplus D_n\,,$$
$$\partial(a, b) = (-\partial a, \varphi a + \partial b), \qquad a \in C_{n-1},\ b \in D_n\,.$$

Show that $\boldsymbol{E} = \{E_n, \partial_n\}$ is indeed a chain complex and that the inclusion $\boldsymbol{D} \subseteq \boldsymbol{E}$ is a chain map.

Write $\boldsymbol{E} = \boldsymbol{E}(\varphi)$, the *mapping cone* of φ. Show how a commutative diagram of chain maps

$$\begin{array}{ccc} \boldsymbol{C} & \xrightarrow{\varphi} & \boldsymbol{D} \\ \downarrow & & \downarrow \\ \boldsymbol{C}' & \xrightarrow{\varphi'} & \boldsymbol{D}' \end{array}$$

induces a chain map $\boldsymbol{E}(\varphi) \to \boldsymbol{E}(\varphi')$ and obtain in this way a suitable functor and a natural transformation.

1.3. Verify that the category of chain complexes is an abelian category.

1.4. Show that Z_n, B_n depend functorially on the complex.

1.5. Let C be a free abelian chain complex with $C_n = 0$, $n < 0, n > N$. Let ϱ_n be the rank of C_n and let p_n be the rank of $H_n(C)$. Show that

$$\sum_{n=0}^{N} (-1)^n \varrho_n = \sum_{n=0}^{N} (-1)^n p_n .$$

1.6. Given a chain complex C of right Λ-modules, a left Λ-module A and a right Λ-module B, suggest definitions for the chain complex $C \otimes_\Lambda A$, and the cochain complex $\mathrm{Hom}_\Lambda(C, B)$.

2. The Long Exact (Co)Homology Sequence

We have already remarked that the category of (co)chain complexes is abelian. Consequently we can speak of short exact sequences of (co)chain complexes. It is clear that the sequence $A \overset{\varphi}{\rightarrowtail} B \overset{\psi}{\twoheadrightarrow} C$ of complexes is short exact if and only if $0 \to A_n \xrightarrow{\varphi_n} B_n \xrightarrow{\psi_n} C_n \to 0$ is exact for all $n \in \mathbb{Z}$.

Theorem 2.1. *Given a short exact sequence* $A \overset{\varphi}{\rightarrowtail} B \overset{\psi}{\twoheadrightarrow} C$ *of chain complexes (cochain complexes) there exists a morphism of degree* -1 *(degree* $+1$*) of graded modules* $\omega : H(C) \to H(A)$ *such that the triangle*

$$\begin{tikzcd} H(A) \arrow[rr, "\varphi_*"] & & H(B) \arrow[dl, "\psi_*"] \\ & H(C) \arrow[ul, "\omega"] & \end{tikzcd}$$

is exact. (We call ω *the connecting homomorphism.)*

Explicitly the theorem claims that, in the case of chain complexes, the sequence

$$\cdots \xrightarrow{\omega_{n+1}} H_n(A) \xrightarrow{\varphi_*} H_n(B) \xrightarrow{\psi_*} H_n(C) \xrightarrow{\omega_n} H_{n-1}(A) \to \cdots \tag{2.1}$$

and, in the case of cochain complexes, the sequence

$$\cdots \xrightarrow{\omega^{n-1}} H^n(A) \xrightarrow{\varphi_*} H^n(B) \xrightarrow{\psi_*} H^n(C) \xrightarrow{\omega^n} H^{n+1}(A) \to \cdots \tag{2.2}$$

is exact.

We first prove the following lemma.

Lemma 2.2. $\partial_n : C_n \to C_{n-1}$ *induces* $\tilde{\partial}_n : \mathrm{coker}\, \partial_{n+1} \to \ker \partial_{n-1}$ *with* $\ker \tilde{\partial}_n = H_n(C)$ *and* $\mathrm{coker}\, \tilde{\partial}_n = H_{n-1}(C)$.

Proof. Since $\mathrm{im}\ \partial_{n+1} \subseteq \ker \partial_n$ and $\mathrm{im}\, \partial_n \subseteq \ker \partial_{n-1}$ the differential ∂_n induces a map $\tilde{\partial}_n$ as follows:

$$\mathrm{coker}\, \partial_{n+1} = C_n / \mathrm{im}\, \partial_{n+1} \twoheadrightarrow C_n / \ker \partial_n \cong \mathrm{im}\, \partial_n \subseteq \ker \partial_{n-1} .$$

One easily computes $\ker \tilde{\partial}_n = \ker \partial_n / \mathrm{im}\, \partial_{n+1} = H_n(C)$ and

$$\mathrm{coker}\ \tilde{\partial}_n = \ker \partial_{n-1} / \mathrm{im}\, \partial_n = H_{n-1}(C) . \quad \square$$

Proof of Theorem 2.1. We give the proof for chain complexes only, the proof for cochain complexes being analogous. We first look at the diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker \partial_n \arrow[r] \arrow[d, tail, dotted] & \ker \partial_n \arrow[r] \arrow[d, tail, dotted] & \ker \partial_n \arrow[d, tail, dotted] & \\
& A_n \arrow[r, tail, "\varphi_n"] \arrow[d, "\partial_n"] & B_n \arrow[r, two heads, "\psi_n"] \arrow[d, "\partial_n"] & C_n \arrow[d, "\partial_n"] & \\
& A_{n-1} \arrow[r, tail, "\varphi_{n-1}"] \arrow[d, two heads, dotted] & B_{n-1} \arrow[r, two heads, "\psi_{n-1}"] \arrow[d, two heads, dotted] & C_{n-1} \arrow[d, two heads, dotted] & \\
& \operatorname{coker} \partial_n \arrow[r] & \operatorname{coker} \partial_n \arrow[r] & \operatorname{coker} \partial_n \arrow[r] & 0
\end{tikzcd}$$

By Lemma III. 5.1 the sequence at the top and the sequence at the bottom are exact. Thus by Lemma 2.2 we obtain the diagram

$$\begin{tikzcd}
& H_n(\mathbf{A}) \arrow[r] \arrow[d, tail, dotted] & H_n(\mathbf{B}) \arrow[r] \arrow[d, tail, dotted] & H_n(\mathbf{C}) \arrow[d, tail, dotted] & \\
& \operatorname{coker} \partial_{n+1} \arrow[r] \arrow[d, "\tilde{\partial}_n"] & \operatorname{coker} \partial_{n+1} \arrow[r] \arrow[d, "\tilde{\partial}_n"] & \operatorname{coker} \partial_{n+1} \arrow[r] \arrow[d, "\tilde{\partial}_n"] & 0 \\
0 \arrow[r] & \ker \partial_{n-1} \arrow[r] \arrow[d, two heads, dotted] & \ker \partial_{n-1} \arrow[r] \arrow[d, two heads, dotted] & \ker \partial_{n-1} \arrow[d, two heads, dotted] & \\
& H_{n-1}(\mathbf{A}) \arrow[r] & H_{n-1}(\mathbf{B}) \arrow[r] & H_{n-1}(\mathbf{C}) &
\end{tikzcd}$$

Applying Lemma III. 5.1 again we deduce the existence of

$$\omega_n : H_n(\mathbf{C}) \to H_{n-1}(\mathbf{A})$$

such that the sequence (2.1) is exact. □

If we recall the explicit definition of ω_n, then it is seen to be equivalent to the following procedure. Let $c \in C_n$ be a representative cycle of the homology class $[c] \in H_n(\mathbf{C})$. Choose $b \in B_n$ with $\psi(b) = c$. Since (suppressing the subscripts) $\psi \partial b = \partial \psi b = \partial c = 0$ there exists $a \in A_{n-1}$ with $\varphi a = \partial b$. Then $\varphi \partial a = \partial \varphi a = \partial \partial b = 0$. Hence a is a cycle in $Z_{n-1}(\mathbf{A})$ and therefore determines an element $[a] \in H_{n-1}(\mathbf{A})$. The map ω_n is defined by $\omega_n[c] = [a]$.

We remark that the naturality of the ker-coker sequence of Lemma III. 5.1 immediately implies the naturality of sequences (2.1) and (2.2). If we are given a commutative diagram of chain complexes

$$\begin{tikzcd}
\mathbf{A} \arrow[r, tail] \arrow[d] & \mathbf{B} \arrow[r, two heads] \arrow[d] & \mathbf{C} \arrow[d] \\
\mathbf{A}' \arrow[r, tail] & \mathbf{B}' \arrow[r, two heads] & \mathbf{C}'
\end{tikzcd}$$

with exact rows, the homology sequence (2.1) will be mapped into the homology sequence arising from $\boldsymbol{A}' \rightarrowtail \boldsymbol{B}' \twoheadrightarrow \boldsymbol{C}'$, in such a way that the diagram is commutative.

Examples. (a) Let $R \overset{\mu}{\rightarrowtail} F \overset{\varepsilon}{\twoheadrightarrow} A$ be a free presentation of the abelian group A, and let $B' \overset{\beta'}{\rightarrowtail} B \overset{\beta''}{\twoheadrightarrow} B''$ be an exact sequence of abelian groups. We form the cochain complexes

$$\begin{tikzcd}
\boldsymbol{C}' : 0 \arrow[r] \arrow[d, rightarrowtail, "\beta'_*"'] & \operatorname{Hom}(F, B') \arrow[r, "\mu^*"] \arrow[d, rightarrowtail, "\beta'_*"'] & \operatorname{Hom}(R, B') \arrow[r] \arrow[d, rightarrowtail, "\beta'_*"'] & 0 \\
\boldsymbol{C} : 0 \arrow[r] \arrow[d, "\beta''_*"'] & \operatorname{Hom}(F, B) \arrow[r, "\mu^*"] \arrow[d, twoheadrightarrow, "\beta''_*"'] & \operatorname{Hom}(R, B) \arrow[r] \arrow[d, twoheadrightarrow, "\beta''_*"'] & 0 \\
\boldsymbol{C}'' : 0 \arrow[r] & \operatorname{Hom}(F, B'') \arrow[r, "\mu^*"] & \operatorname{Hom}(R, B'') \arrow[r] & 0
\end{tikzcd}$$

Since F and R are free abelian both columns of the diagram are short exact, i.e. $\boldsymbol{C}' \overset{\beta'_*}{\rightarrowtail} \boldsymbol{C} \overset{\beta''_*}{\twoheadrightarrow} \boldsymbol{C}''$ is a short exact sequence of cochain complexes. By Theorem 2.1 we obtain an exact sequence in cohomology

$$0 \longrightarrow \operatorname{Hom}(A, B') \xrightarrow{\beta'_*} \operatorname{Hom}(A, B) \xrightarrow{\beta''_*} \operatorname{Hom}(A, B'') \xrightarrow{\omega}$$

$$\xrightarrow{\omega} \operatorname{Ext}(A, B') \xrightarrow{\beta'_*} \operatorname{Ext}(A, B) \xrightarrow{\beta''_*} \operatorname{Ext}(A, B'') \longrightarrow 0 .$$

The reader may compare the above sequence with the sequence in Theorem III. 5.2.

(b) For a short exact sequence $A' \rightarrowtail A \twoheadrightarrow A''$ of abelian groups and an abelian group B, we choose an injective presentation of B. Proceeding analogously as in example (a) we obtain the sequence of Theorem III. 5.3.

Both sequences will be generalised in Sections 7, 8.

Exercises:

2.1. Use Theorem 2.1 to associate with a chain map $\varphi : \boldsymbol{C} \to \boldsymbol{D}$ an exact sequence

$$\cdots \to H_n(\boldsymbol{C}) \to H_n(\boldsymbol{D}) \to H_n(\boldsymbol{E}(\varphi)) \to H_{n-1}(\boldsymbol{C}) \to \cdots .$$

[Hint: Use the exact sequence $D_n \rightarrowtail E_n \twoheadrightarrow C_{n-1}$.] Deduce that $H(\boldsymbol{E}(\varphi)) = 0$ if and only if $\varphi_* : H(\boldsymbol{C}) \xrightarrow{\sim} H(\boldsymbol{D})$. Show that the association proved above is functorial.

2.2. Using a free presentation of the abelian group A and Theorem 2.1, deduce the sequence

$$0 \to \operatorname{Tor}(A, G') \to \operatorname{Tor}(A, G) \to \operatorname{Tor}(A, G'') \to A \otimes G' \to A \otimes G \to A \otimes G'' \to 0$$

associated with the short exact sequence $G' \rightarrowtail G \twoheadrightarrow G''$ (see Theorem III. 8.2.).

2.3. Let

$$\begin{tikzcd}
0 \arrow[r] & \boldsymbol{C}' \arrow[r] \arrow[d, "\varphi'"] & \boldsymbol{C} \arrow[r] \arrow[d, "\varphi"] & \boldsymbol{C}'' \arrow[r] \arrow[d, "\varphi''"] & 0 \\
0 \arrow[r] & \boldsymbol{D}' \arrow[r] & \boldsymbol{D} \arrow[r] & \boldsymbol{D}'' \arrow[r] & 0
\end{tikzcd}$$

be a map of short exact sequences of chain complexes. Show that if any two of φ', φ, φ'' induce isomorphisms in homology, so does the third.

2.4. Given a short exact sequence of chain complexes $A \overset{\varphi}{\rightarrowtail} B \overset{\psi}{\twoheadrightarrow} C$ in an *arbitrary* abelian category, prove the existence of a long exact homology sequence of the form (2.1).

3. Homotopy

Let C, D be two chain complexes and $\varphi, \psi: C \to D$ two chain maps. It is an important and frequently arising question when φ and ψ induce the same homomorphism between $H(C)$ and $H(D)$. To study this problem we shall introduce the notion of *homotopy*; that is, we shall describe a relation between φ and ψ which will be sufficient for

$$\varphi_* = \psi_* : H(C) \to H(D).$$

On the other hand, the relation is not necessary for $\varphi_* = \psi_*$, so that the notion of homotopy does not fully answer the above question; it is however most useful because of its good behavior with respect to chain maps and functors (see Lemmas 3.3, 3.4). In most cases where one is able to show that $\varphi_* = \psi_*$, this is proved as a consequence of the existence of a homotopy, in particular in all the cases we are concerned with in this book. We deal here with the case of chain complexes and leave to the reader the easy task of translating the results for cochain complexes.

Definition. A *homotopy* $\Sigma : \varphi \to \psi$ between two chain maps $\varphi, \psi : C \to D$ is a morphism of degree $+1$ of graded modules $\Sigma : C \to D$ such that $\psi - \varphi = \partial\Sigma + \Sigma\partial$, i.e., such that, for $n \in \mathbb{Z}$,

$$\psi_n - \varphi_n = \partial_{n+1} \Sigma_n + \Sigma_{n-1} \partial_n . \tag{3.1}$$

We say that φ, ψ are *homotopic*, and write $\varphi \simeq \psi$ if there exists a homotopy $\Sigma : \varphi \to \psi$.

The essential fact about homotopies is given in the following

Proposition 3.1. *If the two chain maps $\varphi, \psi : C \to D$ are homotopic, then $H(\varphi) = H(\psi) : H(C) \to H(D)$.*

Proof. Let $z \in \ker \partial_n$ be a cycle in C_n. If $\Sigma : \varphi \to \psi$, then

$$(\psi - \varphi)\, z = \partial \Sigma z + \Sigma \partial z = \partial \Sigma z ,$$

since $\partial z = 0$. Hence $\psi(z) - \varphi(z)$ is a boundary in D_n, i.e. $\psi(z)$ and $\varphi(z)$ are homologous. □

The reader is again warned that the converse of Proposition 3.1 is *not* true; at the end of this section we shall give an example of two chain maps which induce the same homomorphism in homology but are *not* homotopic. In the special case where $\boldsymbol{\varphi}=\mathbf{0}:\boldsymbol{C}\to\boldsymbol{C}$ and $\boldsymbol{\psi}=\mathbf{1}:\boldsymbol{C}\to\boldsymbol{C}$ a homotopy $\boldsymbol{\Sigma}:\mathbf{0}\to\mathbf{1}$ is called a *contracting homotopy* for $\boldsymbol{C}$. We are then given homomorphisms $\Sigma_n:C_n\to C_{n+1}$ with $\partial_{n+1}\Sigma_n+\Sigma_{n-1}\partial_n=1$, $n\in\mathbb{Z}$. By Proposition 3.1 an immediate consequence of the existence of such a contracting homotopy is $H(\boldsymbol{C})=0$, hence the complex $\boldsymbol{C}$ is exact. Indeed, very often where it is to be proved that a complex $\boldsymbol{C}$ is exact, this is achieved by constructing a contracting homotopy. We proceed with a number of results on the homotopy relation.

Lemma 3.2. *The homotopy relation "$\simeq$" is an equivalence relation.*

Proof. Plainly "$\simeq$" is reflexive and symmetric. To check transitivity, let $\psi-\varphi=\partial\Sigma+\Sigma\partial$ and $\chi-\psi=\partial T+T\partial$ (suppressing the subscripts). An easy calculation shows $\chi-\varphi=\partial(\Sigma+T)+(\Sigma+T)\,\partial$. □

Lemma 3.3. *Let $\boldsymbol{\varphi}\simeq\boldsymbol{\psi}:\boldsymbol{C}\to\boldsymbol{D}$ and $\boldsymbol{\varphi}'\simeq\boldsymbol{\psi}':\boldsymbol{D}\to\boldsymbol{E}$, then*

$$\boldsymbol{\varphi}'\boldsymbol{\varphi}\simeq\boldsymbol{\psi}'\boldsymbol{\psi}:\boldsymbol{C}\to\boldsymbol{E}\,.$$

Proof. Let $\psi-\varphi=\partial\Sigma+\Sigma\partial$; then

$$\varphi'\psi-\varphi'\varphi=\varphi'\partial\Sigma+\varphi'\Sigma\partial=\partial(\varphi'\Sigma)+(\varphi'\Sigma)\,\partial\,.$$

Also, from $\psi'-\varphi'=\partial T+T\partial$ we conclude

$$\psi'\psi-\varphi'\psi=\partial T\psi+T\partial\psi=\partial(T\psi)+(T\psi)\,\partial\,.$$

The result then follows by transitivity. □

Indeed we may say that if $\boldsymbol{\Sigma}:\boldsymbol{\varphi}\to\boldsymbol{\psi}$ is a homotopy, then

$$\boldsymbol{\varphi}'\boldsymbol{\Sigma}:\boldsymbol{\varphi}'\boldsymbol{\varphi}\to\boldsymbol{\varphi}'\boldsymbol{\psi}$$

is a homotopy; and if $\boldsymbol{T}:\boldsymbol{\varphi}'\to\boldsymbol{\psi}'$ is a homotopy then $\boldsymbol{T\psi}:\boldsymbol{\varphi}'\boldsymbol{\psi}\to\boldsymbol{\psi}'\boldsymbol{\psi}$ is a homotopy.

Lemma 3.4. *Let $F:\mathfrak{M}_\Lambda\to\mathfrak{M}_{\Lambda'}$ be an additive functor. If $\boldsymbol{C}$ and $\boldsymbol{D}$ are chain complexes of Λ-modules and $\boldsymbol{\varphi}\simeq\boldsymbol{\psi}:\boldsymbol{C}\to\boldsymbol{D}$, then $F\boldsymbol{\varphi}\simeq F\boldsymbol{\psi}:F\boldsymbol{C}\to F\boldsymbol{D}$.*

Proof. Let $\boldsymbol{\Sigma}:\boldsymbol{\varphi}\to\boldsymbol{\psi}$, then

$$F\psi-F\varphi=F(\psi-\varphi)=F(\partial\Sigma+\Sigma\partial)=F\partial F\Sigma+F\Sigma F\partial\,.$$

Hence $F\boldsymbol{\Sigma}:F\boldsymbol{\varphi}\to F\boldsymbol{\psi}$. □

Lemmas 3.3 and 3.4 show that the equivalence relation "$\simeq$" behaves nicely with respect to composition of chain maps and with respect to additive functors. Lemma 3.4 together with Proposition 3.1 now immediately yields.

Corollary 3.5. *If $\boldsymbol{\varphi}\simeq\boldsymbol{\psi}:\boldsymbol{C}\to\boldsymbol{D}$ and if F is an additive functor, then $H(F\boldsymbol{\varphi})=H(F\boldsymbol{\psi}):H(F\boldsymbol{C})\to H(F\boldsymbol{D})$.* □

We remark that Lemma 3.3 enables one to associate with the category of chain complexes and chain maps the category of chain complexes and *homotopy classes* of chain maps. The passage is achieved simply by identifying two chain maps if and only if they are homotopic. The category so obtained is called the *homotopy category*. By Lemma 3.4 an additive functor F will induce a functor between the homotopy categories and by Proposition 3.1 the homology functor will factor through the homotopy category.

We say that two complexes $\boldsymbol{C}, \boldsymbol{D}$ are of *the same homotopy type (*or *homotopic)* if they are isomorphic in the homotopy category, that is, if there exist chain maps $\boldsymbol{\varphi}: \boldsymbol{C} \to \boldsymbol{D}$ and $\boldsymbol{\psi}: \boldsymbol{D} \to \boldsymbol{C}$ such that $\boldsymbol{\psi\varphi} \simeq \mathbf{1_C}$ and $\boldsymbol{\varphi\psi} \simeq \mathbf{1_D}$. The chain map $\boldsymbol{\varphi}$ (or $\boldsymbol{\psi}$) is then called a *homotopy equivalence.*

We conclude this section with the promised example: Take $\Lambda = \mathbb{Z}$.

$$C_1 = \mathbb{Z} = (s_1)\,; \quad C_0 = \mathbb{Z} = (s_0)\,; \quad C_n = 0, n \neq 0, 1\,; \quad \partial s_1 = 2s_0\,;$$
$$D_1 = \mathbb{Z} = (t_1)\,; \quad D_n = 0, n \neq 1\,; \quad \varphi s_1 = t_1\,.$$

Clearly $\boldsymbol{\varphi}: \boldsymbol{C} \to \boldsymbol{D}$ and the zero chain map $\mathbf{0}: \boldsymbol{C} \to \boldsymbol{D}$ both induce the zero map in homology. To show that $\boldsymbol{\varphi}$ and $\mathbf{0}$ are not homotopic, we apply Corollary 3.5 to the functor $-\otimes \mathbb{Z}_2$; we obviously obtain

$$H_1(\boldsymbol{\varphi} \otimes \mathbb{Z}_2) \neq H_1(\mathbf{0} \otimes \mathbb{Z}_2)\,;$$

in fact,

$$H_1(\boldsymbol{\varphi} \otimes \mathbb{Z}_2) = 1 : \mathbb{Z}_2 \to \mathbb{Z}_2\,,$$
$$H_1(\mathbf{0} \otimes \mathbb{Z}_2) = 0 : \mathbb{Z}_2 \to \mathbb{Z}_2\,.$$

Exercises:

3.1. Show that if $\boldsymbol{\varphi} \simeq \boldsymbol{\psi}: \boldsymbol{C} \to \boldsymbol{D}$, then $\boldsymbol{E}(\boldsymbol{\varphi}) \cong \boldsymbol{E}(\boldsymbol{\psi})$ (see Exercise 1.2).

3.2. Show that, further, if $\boldsymbol{\varphi} \simeq \boldsymbol{\psi}: \boldsymbol{C} \to \boldsymbol{D}$, then the homology sequences for $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ of Exercise 2.1 are isomorphic.

3.3. Does $\boldsymbol{E}(\boldsymbol{\varphi})$ depend functorially on the homotopy class of $\boldsymbol{\varphi}$?

3.4. In the example given show directly that no homotopy $\boldsymbol{\Sigma}: \mathbf{0} \to \boldsymbol{\varphi}$ exists. Also, show that $\mathrm{Hom}(\boldsymbol{\varphi}, \mathbb{Z}_2) \neq \mathrm{Hom}(\mathbf{0}, \mathbb{Z}_2)$.

3.5. Suggest an appropriate definition for a homotopy between homotopies.

4. Resolutions

In this section we introduce a special kind of (co)chain complex which is a basic tool in developing the theory of derived functors. We shall restrict our attention for the moment to *positive* chain complexes, that is, chain complexes of the form

$$\boldsymbol{C}: \cdots \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to 0 \tag{4.1}$$

with $C_n = 0$ for $n < 0$.

Definition. The chain complex (4.1) is called *projective* if C_n is projective for all $n \geqq 0$; it is called *acyclic* if $H_n(\boldsymbol{C}) = 0$ for $n \geqq 1$.

Note that $\boldsymbol{C}$ is acyclic if and only if the sequence

$$\cdots \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to H_0(\boldsymbol{C}) \to 0$$

is exact. A projective and acyclic complex

$$\boldsymbol{P} : \cdots \to P_n \to P_{n-1} \to \cdots \to P_0$$

together with an isomorphism $H_0(\boldsymbol{P}) \xrightarrow{\sim} A$ is called a *projective resolution of A*. In the sequel we shall identify $H_0(\boldsymbol{P})$ with A via the given isomorphism.

Theorem 4.1. *Let $\boldsymbol{C} : \cdots \to C_n \to C_{n-1} \to \cdots \to C_0$ be projective and let $\boldsymbol{D} : \cdots \to D_n \to D_{n-1} \cdots \to D_0$ be acyclic. Then there exists, to every homomorphism $\varphi : H_0(\boldsymbol{C}) \to H_0(\boldsymbol{D})$, a chain map $\boldsymbol{\varphi} : \boldsymbol{C} \to \boldsymbol{D}$ inducing φ. Moreover two chain maps inducing φ are homotopic.*

Proof. The chain map $\boldsymbol{\varphi} : \boldsymbol{C} \to \boldsymbol{D}$ is defined recursively. Since $\boldsymbol{D}$ is positive, $D_0 \to H_0(\boldsymbol{D}) \to 0$ is exact. By the projectivity of C_0 there exists $\varphi_0 : C_0 \to D_0$ such that the diagram

$$\begin{tikzcd} C_0 \arrow[r] \arrow[d, dotted, "\varphi_0"] & H_0(\boldsymbol{C}) \arrow[d, "\varphi"] \\ D_0 \arrow[r, two heads] & H_0(\boldsymbol{D}) \end{tikzcd} \tag{4.2}$$

is commutative. Suppose $n \geqq 1$ and $\varphi_0, \varphi_1, \ldots, \varphi_{n-1}$ are defined. We consider the diagram

$$\begin{tikzcd} C_n \arrow[r, "\partial"] \arrow[d, dotted, "\varphi_n"] & C_{n-1} \arrow[r, "\partial"] \arrow[d, "\varphi_{n-1}"] & C_{n-2} \arrow[r] \arrow[d, "\varphi_{n-2}"] & \cdots \\ D_n \arrow[r, "\partial"] & D_{n-1} \arrow[r, "\partial"] & D_{n-2} \arrow[r] & \cdots \end{tikzcd}$$

(If $n=1$, set $C_{-1} = H_0(\boldsymbol{C})$, $D_{-1} = H_0(\boldsymbol{D})$, and the right-hand square above is just (4.2).) We have $\partial \varphi_{n-1} \partial = \varphi_{n-2} \partial \partial = 0$. Hence

$$\operatorname{im} \varphi_{n-1} \partial \subseteq \ker(\partial : D_{n-1} \to D_{n-2}).$$

Since $\boldsymbol{D}$ is acyclic, $\ker \partial_{n-1} = \operatorname{im}(\partial : D_n \to D_{n-1})$. The projectivity of C_n allows us to find $\varphi_n : C_n \to D_n$ such that $\varphi_{n-1} \partial = \partial \varphi_n$. This completes the inductive step.

Now let $\boldsymbol{\varphi} = \{\varphi_n\}$, $\boldsymbol{\psi} = \{\psi_n\}$ be two chain maps inducing the given $\varphi : H_0(\boldsymbol{C}) \to H_0(\boldsymbol{D})$. Recursively we shall define a homotopy $\boldsymbol{\Sigma} : \boldsymbol{\psi} \to \boldsymbol{\varphi}$. First consider the diagram

$$\begin{tikzcd} C_1 \arrow[r] \arrow[d, shift right, "\varphi_1"'] \arrow[d, shift left, "\psi_1"] & C_0 \arrow[r] \arrow[d, shift right, "\varphi_0"'] \arrow[d, shift left, "\psi_0"] \arrow[dl, dotted, "\Sigma_0"'] & H_0(\boldsymbol{C}) \arrow[r] \arrow[d, "\varphi"] & 0 \\ D_1 \arrow[r] & D_0 \arrow[r] & H_0(\boldsymbol{D}) \arrow[r] & 0 \end{tikzcd}$$

Since φ_0 and ψ_0 both induce φ, $\varphi_0 - \psi_0$ maps C_0 into

$$\ker(D_0 \to H_0(\boldsymbol{D})) = \operatorname{im}(D_1 \to D_0)\,.$$

Since $\boldsymbol{C}$ is projective, there exists $\Sigma_0 : C_0 \to D_1$ such that $\varphi_0 - \psi_0 = \partial \Sigma_0$.

Now suppose $n \geqq 1$ and suppose that $\Sigma_0, \ldots, \Sigma_{n-1}$ are defined in such a way that $\varphi_r - \psi_r = \partial \Sigma_r + \Sigma_{r-1} \partial$, $r \leqq n-1$ ($\Sigma_{-1}\partial$ being understood as zero). Consider the diagram

$$\begin{tikzcd}
C_{n+1} \arrow[r, "\partial"] \arrow[d, shift right, "\varphi_{n+1}"'] \arrow[d, shift left, "\psi_{n+1}"] & C_n \arrow[r, "\partial"] \arrow[d, shift right, "\varphi_n"'] \arrow[d, shift left, "\psi_n"] \arrow[dl, dotted, "\Sigma_n"] & C_{n-1} \arrow[d, shift right, "\varphi_{n-1}"'] \arrow[d, shift left, "\psi_{n-1}"] \arrow[dl, "\Sigma_{n-1}"] \\
D_{n+1} \arrow[r, "\partial"'] & D_n \arrow[r, "\partial"'] & D_{n-1}
\end{tikzcd}$$

We have

$$\begin{aligned}
\partial(\varphi_n - \psi_n - \Sigma_{n-1}\partial) &= \varphi_{n-1}\partial - \psi_{n-1}\partial - \partial\Sigma_{n-1}\partial \\
&= (\varphi_{n-1} - \psi_{n-1} - \partial\Sigma_{n-1})\,\partial = \Sigma_{n-2}\partial\partial = 0\,.
\end{aligned}$$

Hence $\varphi_n - \psi_n - \Sigma_{n-1}\partial$ maps C_n into

$$\ker(\partial : D_n \to D_{n-1}) = \operatorname{im}(\partial : D_{n+1} \to D_n)\,.$$

Since C_n is projective, there exists $\Sigma_n : C_n \to D_{n+1}$ such that

$$\varphi_n - \psi_n - \Sigma_{n-1}\partial = \partial\Sigma_n\,. \quad \square$$

Lemma 4.2. *To every Λ-module A there exists a projective resolution.*

Proof. Choose a projective presentation $R_1 \rightarrowtail P_0 \twoheadrightarrow A$ of A; then a projective presentation $R_2 \rightarrowtail P_1 \twoheadrightarrow R_1$ of R_1, etc. Plainly the complex

$$\boldsymbol{P} : \cdots \to P_n \xrightarrow{\partial_n} P_{n-1} \to \cdots \to P_0$$

where $\partial_n : P_n \to P_{n-1}$ is defined by $P_n \twoheadrightarrow R_n \rightarrowtail P_{n-1}$ is a projective resolution of A. For it is clearly projective and acyclic, and $H_0(\boldsymbol{P}) = A$. $\square$

Notice that every projective resolution arises in the manner described. Thus we see that the existence of projective resolutions is equivalent to the existence of projective presentations. In general we shall say that an abelian category $\mathfrak{A}$ has *enough projectives* if to every object A in $\mathfrak{A}$ there is at least one projective presentation of A. By the argument above every object in $\mathfrak{A}$ then has a projective resolution.

We also remark that in the category of abelian groups, we can take $P_1 = R_1$, $P_n = 0$, $n \geqq 2$, because an abelian group is projective if and only if it is free and a subgroup of a free group is free. We shall see later that for modules it may happen that no *finite* projective resolution exists, that is, there may be no projective resolution $\boldsymbol{P}$ of A such that $P_n = 0$ for n sufficiently large.

Proposition 4.3. *Two projective resolutions of A are canonically of the same homotopy type.*

Proof. Let C and D be two projective resolutions of A. By Theorem 4.1 there exist chain maps $\varphi: C \to D$ and $\psi: D \to C$ inducing the identity in $H_0(C) = A = H_0(D)$. The composition $\psi\varphi: C \to C$ as well as the identity $1: C \to C$ induce the identity in A. By Theorem 4.1 we have $\psi\varphi \simeq 1$. Analogously $\varphi\psi \simeq 1$. Hence C and D are of the same homotopy type. Since the homotopy class of the homotopy equivalence $\varphi: C \to D$ is uniquely determined, the resolutions C and D are canonically of the same homotopy type. □

We conclude this section with a remark on the dual situation. We look at *positive co*chain complexes, that is, cochain complexes of the form

$$C: 0 \to C_0 \to C_1 \to C_2 \to \cdots \to C_n \to C_{n+1} \to \cdots,$$

with $C_n = 0$ for $n < 0$.

We call C *injective* if each C_n is injective, and *acyclic* if $H^n(C) = 0$ for $n \neq 0$. We then can prove the dual of Theorem 4.1.

Theorem 4.4. *Let $C: C_0 \to C_1 \to \cdots \to C_n \to C_{n+1} \to \cdots$ be acyclic and $D: D_0 \to D_1 \to \cdots \to D_n \to D_{n+1} \to \cdots$ be injective. Then there exists, to every homomorphism $\varphi: H^0(C) \to H^0(D)$, a cochain map $\varphi: C \to D$ inducing φ. Moreover two cochain maps inducing φ are homotopic.* □

A complex $I: I_0 \to I_1 \to \cdots \to I_n \to I_{n+1} \to \cdots$ which is injective and acyclic with $H^0(I) = A$ is called an *injective resolution* of A.

Plainly an abelian category $\mathfrak{A}$, for example $\mathfrak{M}_A$, in which every object has an injective presentation will have injective resolutions, and conversely. Such a category will be said to have *enough injectives.* For later use we finally record the following consequence of Theorem 4.4.

Proposition 4.5. *Two injective resolutions of A are canonically of the same homotopy type.* □

Exercises:

4.1. Use Theorem 4.1 to show that if P is projective with $P_n = 0$, $n < 0$, then $H(P) = 0$ if and only if $1 \simeq 0: P \to P$.

4.2. Let $\varphi: C \to D$ be a chain map of the projective complex C into the projective complex D with $C_n = D_n = 0$, $n < 0$. Use the chain complex $E(\varphi)$ and Exercise 4.1 to show that φ is a homotopy equivalence if and only if $\varphi_*: H(C) \xrightarrow{\sim} H(D)$.

4.3. Dualize Exercises 4.1, 4.2 above.

4.4. Let $\varphi: C \to D$ be a chain map, where C is a free chain complex with $C_n = 0, n < 0$. Let $\{\gamma\}$ be a fixed homogeneous basis for C and write $\delta < \gamma$ ('δ is a face of γ') if δ appears in $\partial(\gamma)$ with non-zero coefficient. A function A from the basis $\{\gamma\}$ to the set of sub-complexes of D is called an *acyclic carrier* for φ if

(i) $\varphi(\gamma)$ is a chain of $A(\gamma)$,
(ii) $H(A(\gamma)) = 0$, all γ,
(iii) $\delta < \gamma \Rightarrow A(\delta) \subseteq A(\gamma)$.

Show that if φ admits an acyclic carrier then $\varphi \simeq \mathbf{0}$. [This is a crucial result in the homology theory of polyhedra.] Show that this result generalizes Theorem 4.1 as the latter applies to free chain complexes $\boldsymbol{C}$.

4.5. Let A' be a submodule of A and let

$$\begin{tikzcd} I'_0 \arrow[r] & \cdots \arrow[r] & I'_q \arrow[rr] \arrow[dr] & & I'_{q+1} \arrow[r] & \cdots \\ & & & K'_{q+1} \arrow[ur] & & \end{tikzcd}$$

be an injective resolution of A'. Show how to construct an injective resolution

$$\begin{tikzcd} I_0 \arrow[r] & \cdots \arrow[r] & I_q \arrow[rr] \arrow[dr] & & I_{q+1} \arrow[r] & \cdots \\ & & & K_{q+1} \arrow[ur] & & \end{tikzcd}$$

of A such that

(i) $I'_q \subseteq I_q$,

(ii) $K'_q \subseteq K_q$, all q,

(iii) $\begin{tikzcd} K'_q \arrow[r] \arrow[d, tail] & K_q \arrow[d, tail] \\ I'_q \arrow[r] & I_q \end{tikzcd}$ is a pullback.

Show that $I_0/I'_0 \to \cdots \to I_q/I'_q \to I_{q+1}/I'_{q+1} \to \cdots$ is then an injective resolution of A/A'.

5. Derived Functors

We are now prepared to tackle the main theme of homological algebra, that of derived functors. This theory may be regarded – and, indeed, historically arose – as a massive generalization of the theory of Tor and Ext, described in Chapter III.

We shall develop the theory in some generality and take as base functor an arbitrary additive and covariant functor $T: \mathfrak{M}_\Lambda \to \mathfrak{Ab}$. We shall carry out the definition of left derived functors in detail, while we restrict ourselves to some remarks on the definition of right derived functors. We leave even more details to the reader in translating the theory to that of an additive *contravariant* functor. The theory we present remains valid if the codomain of T is taken to be any abelian category; however in our principal applications the codomain is $\mathfrak{Ab}$.

Let $T: \mathfrak{M}_\Lambda \to \mathfrak{Ab}$ be an additive covariant functor. Our aim is to define a sequence of functors $L_n T: \mathfrak{M}_\Lambda \to \mathfrak{Ab}$, $n = 0, 1, 2, \ldots$ the so-called *left derived functors* of T. This definition is effected in several steps.

Given a Λ-module A and a projective resolution $\boldsymbol{P}$ of A we first define abelian groups $L_n^{\boldsymbol{P}} T(A)$, $n = 0, 1, \ldots$, as follows. Consider the complex of abelian groups $TP: \cdots \to TP_n \to TP_{n-1} \to \cdots \to TP_0 \to 0$ and define

$$L_n^{P}T(A) = H_n(TP), \qquad n = 0, 1, \ldots.$$

We shall show below that, if T is a given additive functor, $L_n^{P}T(A)$ does not depend on the resolution P, but only on A, and that for a given $\alpha: A \to A'$ it is possible to define an induced map $\alpha_*: L_n^{P}TA \to L_n^{P'}TA'$ making $L_n^{P}T(-)$ into a functor.

Let $\alpha: A \to A'$ be a homomorphism and let P, P' be projective resolutions of A, A' respectively. By Theorem 4.1 there exists a chain map $\boldsymbol{\alpha}: P \to P'$ inducing α, which is determined up to homotopy. By Corollary 3.5 we obtain a map

$$\alpha(P, P'): L_n^{P}TA \to L_n^{P'}TA', \qquad n = 0, 1, \ldots$$

which is independent of the choice of $\boldsymbol{\alpha}$.

Next consider $\alpha: A \to A'$, $\alpha': A' \to A''$ and projective resolutions P, P', P'' of A, A', A'' respectively. The composition $\alpha'\alpha: A \to A''$ induces, by the above, a map $\alpha'\alpha(P, P''): L_n^{P}TA \to L_n^{P''}TA''$ which may be constructed via a chain map $P \to P''$ inducing $\alpha'\alpha$. We choose for this chain map the composition of a chain map $\boldsymbol{\alpha}: P \to P'$ inducing α and a chain map $\boldsymbol{\alpha}': P' \to P''$ inducing α'. We thus obtain

$$(\alpha'\alpha)(P, P'') = \alpha'(P', P'') \circ \alpha(P, P'). \tag{5.1}$$

Also it is plain that $1_A: A \to A$ yields

$$1_A(P, P) = \text{identity of } L_n^{P}TA. \tag{5.2}$$

We are now prepared to prove

Proposition 5.1. *Let P, Q be two projective resolutions of A. Then there is a canonical isomorphism*

$$\eta = \eta_{P,Q}: L_n^{P}TA \xrightarrow{\sim} L_n^{Q}TA, \qquad n = 0, 1, \ldots.$$

Proof. Let $\boldsymbol{\eta}: P \to Q$ be a chain map inducing 1_A. Its homotopy class is uniquely determined; moreover it is clear from Proposition 4.3 that $\boldsymbol{\eta}$ is a homotopy equivalence. Hence we obtain a canonical isomorphism

$$\eta = 1_A(P, Q): L_n^{P}TA \xrightarrow{\sim} L_n^{Q}TA, \qquad n = 0, 1, \ldots$$

which may be computed via *any* chain map $\boldsymbol{\eta}: P \to Q$ inducing 1_A. $\square$

By (5.1) and (5.2) $\eta_{Q,R}\eta_{P,Q} = \eta_{P,R}$ for three resolutions P, Q, R of A, and $\eta_{P,P} = 1$. Thus we are allowed to identify the groups $L_n^{P}TA$ and $L_n^{Q}TA$ via the isomorphism η. Accordingly we shall drop the superscript P and write from now on L_nTA for $L_n^{P}TA$.

Finally we have to define, for a given $\alpha: A \to A'$, an induced homomorphism

$$\alpha_*: L_nTA \to L_nTA', \qquad n = 0, 1, \ldots.$$

Of course, we define

$$\alpha_* = \alpha(\boldsymbol{P}, \boldsymbol{P}') : L_n^{\boldsymbol{P}} TA \to L_n^{\boldsymbol{P}'} TA' .$$

Indeed, if we do so, then (5.1) and (5.2) will ensure that $L_n T$ is a functor. The only thing left to check is the fact that the definition of α_* is compatible with the identification made under η. This is achieved by the following computation. Let $\boldsymbol{P}, \boldsymbol{Q}$ be projective resolutions of A and $\boldsymbol{P}', \boldsymbol{Q}'$ projective resolutions of A'. Then by (5.1)

$$\begin{aligned} \eta' \circ \alpha(\boldsymbol{P}, \boldsymbol{P}') &= 1_{A'}(\boldsymbol{P}', \boldsymbol{Q}') \circ \alpha(\boldsymbol{P}, \boldsymbol{P}') = \alpha(\boldsymbol{P}, \boldsymbol{Q}') \\ &= \alpha(\boldsymbol{Q}, \boldsymbol{Q}') \circ 1_A(\boldsymbol{P}, \boldsymbol{Q}) = \alpha(\boldsymbol{Q}, \boldsymbol{Q}') \circ \eta . \end{aligned}$$

This completes the definition of the left derived functors. We may summarize the procedures as follows.

Definition. Let $T: \mathfrak{M}_\Lambda \to \mathfrak{Ab}$ be an additive covariant functor, then $L_n T: \mathfrak{M}_\Lambda \to \mathfrak{Ab}$, $n = 0, 1, \ldots$, is called the n-th *left derived functor* of T. The value of $L_n T$ on a Λ-module A is computed as follows. Take a projective resolution $\boldsymbol{P}$ of A, consider the complex $T\boldsymbol{P}$ and take homology; then $L_n TA = H_n(T\boldsymbol{P})$.

We first note the trivial but sometimes advantageous fact that in order to define the left derived functors $L_n T$ it is sufficient that T be given on projectives. In the rest of this section we shall discuss a number of basic results on left derived functors. More general properties will be discussed in Section 6.

Definition. The covariant functor $T: \mathfrak{M}_\Lambda \to \mathfrak{Ab}$ is called *right exact* if, for every exact sequence $A' \to A \to A'' \to 0$, the sequence

$$TA' \to TA \to TA'' \to 0$$

is exact. The reader may readily verify that a right exact functor is additive (see Exercise 5.8). An example of a right exact functor is $B \otimes_\Lambda -$ by Proposition III. 7.3.

Proposition 5.2. *Let $T: \mathfrak{M}_\Lambda \to \mathfrak{Ab}$ be right exact, then $L_0 T$ and T are naturally equivalent.*

Proof. Let $\boldsymbol{P}$ be a projective resolution of A. Then $P_1 \to P_0 \to A \to 0$ is exact. Hence $TP_1 \to TP_0 \to TA \to 0$ is exact. It follows that $H_0(T\boldsymbol{P}) \cong TA$. Plainly the isomorphism is natural. □

Proposition 5.3. *For P a projective Λ-module $L_n TP = 0$ for $n = 1, 2, \ldots$ and $L_0 TP = TP$.*

Proof. Clearly $\boldsymbol{P}: \cdots \to 0 \to P_0 \to 0$ with $P_0 = P$ is a projective resolution of P. □

Proposition 5.4. *The functors $L_nT: \mathfrak{M}_A \to \mathfrak{Ab}$, $n = 0, 1, \ldots$ are additive.*

Proof. Let $\boldsymbol{P}$ be a projective resolution of A and $\boldsymbol{Q}$ a projective resolution of B, then

$$\boldsymbol{P} \oplus \boldsymbol{Q}: \cdots \to P_n \oplus Q_n \to P_{n-1} \oplus Q_{n-1} \to \cdots \to P_0 \oplus Q_0 \to 0$$

is a projective resolution of $A \oplus B$. Since T is additive we obtain

$$L_nT(A \oplus B) = L_nTA \oplus L_nTB\,.$$

The reader may convince himself that $L_nT(\iota_A)$ and $L_nT(\iota_B)$ are the canonical injections. □

Proposition 5.5. *Let $K_q \overset{\mu}{\rightarrowtail} P_{q-1} \to P_{q-2} \to \cdots \to P_0 \twoheadrightarrow A$ be an exact sequence with $P_0, P_1, \ldots, P_{q-1}$ projective. Then if T is right exact, and $q \geqq 1$, the sequence*

$$0 \to L_qTA \to TK_q \xrightarrow{\mu_*} TP_{q-1}$$

is exact.

Proof. Let $\cdots \to P_{q+1} \to P_q \to K_q \to 0$ be an exact sequence with $P_q, P_{q+1}, \ldots$, projective. Then the complex

$$\begin{tikzcd}
\boldsymbol{P}: \cdots \arrow[r] & P_{q+1} \arrow[r, "\partial_{q+1}"] & P_q \arrow[rr, "\partial_q"] \arrow[dr] & & P_{q-1} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & 0 \\
& & & K_q \arrow[ur, tail, "\mu"'] & & & &
\end{tikzcd}$$

is a projective resolution of A. Since T is right exact the top row in the following commutative diagram is exact

$$\begin{tikzcd}
TP_{q+1} \arrow[r, "T(\partial_{q+1})"] \arrow[d] & TP_q \arrow[r] \arrow[d, "T(\partial_q)"'] & TK_q \arrow[r] \arrow[d, "\mu_*"'] & 0 \\
0 \arrow[r] 0 \arrow[r] & TP_{q-1} \arrow[r, "\sim"] & TP_{q-1} &
\end{tikzcd}$$

The ker-coker sequence of Lemma III. 5.1 yields the exact sequence

$$TP_{q+1} \xrightarrow{T(\partial_{q+1})} \ker T(\partial_q) \to \ker \mu_* \to 0\,.$$

But since $L_qTA = H_q(TP) = \ker T(\partial_q)/\operatorname{im} T(\partial_{q+1})$ we obtain $\ker \mu_* \cong L_qTA$, whence the result. □

Analogously one proves the following proposition, which does not, however, appear so frequently as Proposition 5.5 in applications.

Proposition 5.6. *Let $P_q \xrightarrow{\partial_q} P_{q-1} \to \cdots \to P_0 \twoheadrightarrow A$ be an exact sequence with $P_0, \ldots, P_{q-1}$ projective. Let $K_q = \operatorname{im} \partial_q$. Then if T is left exact, and $q \geqq 1$, the sequence*

$$T(P_q) \to T(K_q) \to L_{q-1}TA \to 0$$

is exact. □

(The definition of left exactness, if not already supplied by the reader, is given prior to Proposition 5.7.)

We conclude this section with some remarks on the definition of *right derived functors*. Let $T:\mathfrak{M}_\Lambda\to\mathfrak{Ab}$ again be an additive covariant functor. We define right derived functors $R^nT:\mathfrak{M}_\Lambda\to\mathfrak{Ab}$, $n=0,1,\ldots$ as follows: For any Λ-module A we obtain the abelian group R^nTA by taking an injective resolution $\boldsymbol{I}$ of A, forming the cochain complex $T\boldsymbol{I}$ and taking cohomology: $R^nTA=H^n(T\boldsymbol{I})$, $n=0,1,\ldots$. As in the case of left derived functors we prove that R^nTA is independent of the chosen resolution. Thus, given $\alpha:A\to A'$ and injective resolutions $\boldsymbol{I},\boldsymbol{I}'$ of A, A' respectively, we can find a cochain map $\boldsymbol{\alpha}:\boldsymbol{I}\to\boldsymbol{I}'$ inducing α. The cochain map $T\boldsymbol{\alpha}:T\boldsymbol{I}\to T\boldsymbol{I}'$ then induces a homomorphism between cohomology groups, thus

$$\alpha_*:R^nTA\to R^nTA',\qquad n=0,1,\ldots.$$

As in the case of left derived functors it is proved that α_* is independent of the chosen injective resolutions $\boldsymbol{I},\boldsymbol{I}'$ and also of the chosen cochain map $\boldsymbol{\alpha}$. Finally it is easy to see that with this definition of induced homomorphisms, R^nT becomes a functor. We define

Definition. The functor $T:\mathfrak{M}_\Lambda\to\mathfrak{Ab}$ is called *left exact* if, for every exact sequence $0\to A'\to A\to A''$ of Λ-modules, the sequence

$$0\to TA'\to TA\to TA''$$

is exact. Again, a left exact functor is additive (see Exercise 5.8). An example of a left-exact functor is $\mathrm{Hom}_\Lambda(B,-)$ (see Theorem I. 2.1).

Proposition 5.7. *For I an injective Λ-module $R^nTI=0$ for $n=1,2,\ldots$. If T is left-exact, then R^0T is naturally equivalent to T.* □

Again of course the functors R^nT are additive, and we also have results dual to Propositions 5.5, 5.6. We leave the actual formulation as well as the proofs to the reader.

In case of an additive but *contravariant* functor $S:\mathfrak{M}_\Lambda\to\mathfrak{Ab}$ the procedure is as follows. The right derived functors R^nS are obtained as the right derived functors of the covariant functor $S:\mathfrak{M}_\Lambda^{\mathrm{opp}}\to\mathfrak{Ab}$. So in order to compute R^nSA for a module A we choose a projective resolution $\boldsymbol{P}$ of A (i.e. an injective resolution in $\mathfrak{M}_\Lambda^{\mathrm{opp}}$), form the cochain complex $S\boldsymbol{P}$ and take cohomology

$$R^nSA=H^n(S\boldsymbol{P}),\qquad n=0,1,\ldots.$$

Analogously we obtain the *left* derived functors of contravariant functors via injective resolutions.

The *contravariant* functor S is called *left exact* if S, taken as covariant functor $\mathfrak{M}_A^{\mathrm{opp}} \to \mathfrak{Ab}$, is left exact, i.e. if for every exact sequence

$$A' \to A \to A'' \to 0$$

the sequence $0 \to SA'' \to SA \to SA'$ is exact. An instance of a left exact contravariant functor is $\mathrm{Hom}_A(-, B)$ (see Theorem I. 2.2). Analogously one defines right-exactness. In these cases too results similar to Propositions 5.2, 5.3, 5.4, 5.5, 5.6 may be proved. We leave the details to the reader, but would like to make explicit the result corresponding to Proposition 5.5.

Proposition 5.8. *Let $K_q \overset{\mu}{\rightarrowtail} P_{q-1} \to \cdots \to P_0 \twoheadrightarrow A$ be an exact sequence with $P_0, P_1, \ldots, P_{q-1}$ projective. Then if S is left exact contravariant and $q \geqq 1$, the sequence*

$$SP_{q-1} \xrightarrow{\mu^*} SK_q \to R^q SA \to 0$$

is exact. $\square$

Exercises:

[In Exercises 5.1, 5.2, 5.5 $T: \mathfrak{M}_A \to \mathfrak{Ab}$ is an additive functor and, in Exercise 5.5,

$$\begin{tikzcd}
\cdots \arrow[r] & P_q \arrow[rr] \arrow[dr, "\varepsilon"'] & & P_{q-1} \arrow[r] & \cdots \arrow[r] & P_0 \\
& & K_q \arrow[ur, "\mu"'] & & &
\end{tikzcd}$$

is a projective resolution of A.]

5.1. Show that $L_0 T$ is right exact.

5.2. Show that $L_m L_n T = \begin{cases} L_m T, & n = 0 \\ 0, & n > 0. \end{cases}$

5.3. Prove Proposition 5.6.

5.4. Dualize Propositions 5.5 and 5.6 to right derived functors.

5.5. Show that $0 \to L_q TA \to L_0 TK_q \xrightarrow{\mu_*} TP_{q-1}$ is exact, $q \geqq 1$, giving the appropriate interpretation of μ_*. Show also that

$$L_q TA \cong L_{q-1} TK_1 \cong L_{q-2} TK_2 \cong \cdots \cong L_1 TK_{q-1}, \quad q \geqq 1.$$

5.6. Give the contravariant forms of the statements of Proposition 5.6 and Exercises 5.4, 5.5.

5.7. Let $\mathrm{Phom}_A(A, B)$ consist of those homomorphisms $A \to B$ which factor through projectives. Show that $\mathrm{Phom}_A(A, B)$ is a subgroup of $\mathrm{Hom}_A(A, B)$. Let $\Pi P(A, B)$ be the quotient group. Show that if $0 \to B' \to B \to B'' \to 0$ is exact, then

$$\Pi P(A, B') \to \Pi P(A, B) \to \Pi P(A, B'')$$

is exact. Show that $\Pi P(A, -)$ is additive, and that it is left exact if A is a principal ideal domain. Dualize.

5.8. Prove that right (or left) exact functors are additive.

6. The Two Long Exact Sequences of Derived Functors

In this section we will establish the two basic long exact sequences associated with the concept of derived functors. In the first (Theorem 6.1), we vary the object in $\mathfrak{M}_\Lambda$ and keep the functor fixed; in the second (Theorem 6.3) we vary the functor and keep the object fixed.

Theorem 6.1. *Let $T: \mathfrak{M}_\Lambda \to \mathfrak{Ab}$ be an additive functor and let $A' \overset{\alpha'}{\rightarrowtail} A \overset{\alpha''}{\twoheadrightarrow} A''$ be a short exact sequence. Then there exist connecting homomorphisms*

$$\omega_n : L_n TA'' \to L_{n-1} TA', \qquad n = 1, 2, \ldots$$

such that the following sequence is exact:

$$\begin{aligned} \cdots \to L_n TA' \xrightarrow{\alpha'_*} L_n TA \xrightarrow{\alpha''_*} L_n TA'' \xrightarrow{\omega_n} L_{n-1}\, TA' \to \cdots \\ \cdots \to L_1\, TA'' \xrightarrow{\omega_1} L_0 TA' \xrightarrow{\alpha'_*} L_0 TA \xrightarrow{\alpha''_*} L_0 TA'' \to 0\,. \end{aligned} \tag{6.1}$$

Proof. By Lemma III. 5.4 we can construct a diagram with exact rows

$$\begin{array}{ccccc} P'_0 & \rightarrowtail & P_0 & \twoheadrightarrow & P''_0 \\ \downarrow{\scriptstyle \varepsilon'} & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon''} \\ A' & \overset{\alpha'}{\rightarrowtail} & A & \overset{\alpha''}{\twoheadrightarrow} & A'' \end{array}$$

with P'_0, P_0, P''_0 projective. Clearly, $P_0 = P'_0 \oplus P''_0$. By Lemma III. 5.1 the sequence of kernels

$$\ker \varepsilon' \rightarrowtail \ker \varepsilon \twoheadrightarrow \ker \varepsilon'' \tag{6.2}$$

is short exact. Repeating this procedure with the sequence (6.2) in place of $A' \rightarrowtail A \twoheadrightarrow A''$ and then proceeding inductively, we construct an exact sequence of complexes

$$\boldsymbol{P}' \overset{\alpha'}{\rightarrowtail} \boldsymbol{P} \overset{\alpha''}{\twoheadrightarrow} \boldsymbol{P}'',$$

where $\boldsymbol{P}', \boldsymbol{P}, \boldsymbol{P}''$ are projective resolutions of A', A, A'' respectively. Since T is additive and since $P_n = P'_n \oplus P''_n$ for every $n \geqq 0$, the sequence

$$0 \to T\boldsymbol{P}' \to T\boldsymbol{P} \to T\boldsymbol{P}'' \to 0$$

is short exact, also. Hence Theorem 2.1 yields the definition of

$$\omega_n : H_n(T\boldsymbol{P}'') \to H_{n-1}(T\boldsymbol{P}')$$

and the exactness of the sequence. We leave it to the reader to prove that the definition of ω_n is independent of the chosen resolutions $\boldsymbol{P}', \boldsymbol{P}, \boldsymbol{P}''$ and chain maps $\boldsymbol{\alpha}', \boldsymbol{\alpha}''$, and hence only depends on the given short exact sequence. □

Let $\tau: T \to T'$ be a natural transformation between additive covariant functors $T, T': \mathfrak{M}_\Lambda \to \mathfrak{Ab}$. For a projective resolution $\boldsymbol{P}$ of A we then obtain a chain map $\tau_{\boldsymbol{P}}: T\boldsymbol{P} \to T'\boldsymbol{P}$ defined by $(\tau_{\boldsymbol{P}})_n = \tau_{P_n}: TP_n \to T'P_n$, $n = 0, 1, 2, \ldots$. Clearly $\tau_{\boldsymbol{P}}$ induces a natural transformation of the left-derived functors, $\tau_A: L_n TA \to L_n T'A$, $n = 0, 1, \ldots$.

We may then express the naturality of (6.1), both with respect to T and with respect to the short exact sequence $A' \rightarrowtail A \twoheadrightarrow A''$, in the following portmanteau proposition.

Proposition 6.2. *Let $\tau: T \to T'$ be a natural transformation between additive covariant functors $T, T': \mathfrak{M}_\Lambda \to \mathfrak{Ab}$ and let the diagram*

$$\begin{tikzcd}
A' \arrow[r, tail, "\alpha'"] \arrow[d, "\varphi'"] & A \arrow[r, two heads, "\alpha''"] \arrow[d, "\varphi"] & A'' \arrow[d, "\varphi''"] \\
B' \arrow[r, tail, "\beta'"'] & B \arrow[r, two heads, "\beta''"'] & B''
\end{tikzcd}$$

be commutative with short exact rows. Then the following diagrams are commutative:

$$\begin{tikzcd}
\cdots \arrow[r] & L_n TA' \arrow[r, "\alpha'_*"] \arrow[d, "\tau_{A'}"] & L_n TA \arrow[r, "\alpha''_*"] \arrow[d, "\tau_A"] & L_n TA'' \arrow[r, "\omega_n"] \arrow[d, "\tau_{A''}"] & L_{n-1} TA' \arrow[r] \arrow[d, "\tau_{A'}"] & \cdots \\
\cdots \arrow[r] & L_n T'A' \arrow[r, "\alpha'_*"] & L_n T'A \arrow[r, "\alpha''_*"] & L_n T'A'' \arrow[r, "\omega'_n"] & L_{n-1} T'A' \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & L_n TA' \arrow[r, "\alpha'_*"] \arrow[d, "\varphi'_*"] & L_n TA \arrow[r, "\alpha''_*"] \arrow[d, "\varphi_*"] & L_n TA'' \arrow[r, "\omega_n"] \arrow[d, "\varphi''_*"] & L_{n-1} TA' \arrow[r] \arrow[d, "\varphi'_*"] & \cdots \\
\cdots \arrow[r] & L_n TB' \arrow[r, "\beta'_*"] & L_n TB \arrow[r, "\beta''_*"] & L_n TB'' \arrow[r, "\omega_n"] & L_{n-1} TB' \arrow[r] & \cdots
\end{tikzcd}$$

The proof is left to the reader. □

We now turn to the second long exact sequence.

Definition. A sequence $T' \xrightarrow{\tau'} T \xrightarrow{\tau''} T''$ of additive functors

$$T', T, T'': \mathfrak{M}_\Lambda \to \mathfrak{Ab}$$

and natural transformations τ', τ'' is called *exact on projectives* if, for every projective Λ-module P, the sequence

$$0 \to T'P \xrightarrow{\tau'_P} TP \xrightarrow{\tau''_P} T''P \to 0$$

is exact.

Theorem 6.3. *Let the sequence $T' \xrightarrow{\tau'} T \xrightarrow{\tau''} T''$ of additive functors $T', T, T'': \mathfrak{M}_\Lambda \to \mathfrak{Ab}$ be exact on projectives. Then, for every Λ-module A, there are connecting homomorphisms*

$$\omega_n: L_n T''A \to L_{n-1} T'A$$

such that the sequence

$$\begin{array}{l} \cdots \to L_n T'A \xrightarrow{\tau'} L_n TA \xrightarrow{\tau''} L_n T''A \xrightarrow{\omega_n} L_{n-1} T'A \to \cdots \\ \cdots \to L_1 T''A \xrightarrow{\omega_1} L_0 T'A \xrightarrow{\tau'} L_0 TA \xrightarrow{\tau''} L_0 T''A \to 0 \end{array} \tag{6.3}$$

is exact.

Proof. Choose a projective resolution $\boldsymbol{P}$ of A and consider the sequence of complexes

$$0 \to T'\boldsymbol{P} \xrightarrow{\tau'} T\boldsymbol{P} \xrightarrow{\tau''} T''\boldsymbol{P} \to 0$$

which is short exact since $T' \xrightarrow{\tau'} T \xrightarrow{\tau''} T''$ is exact on projectives. The long exact homology sequence (Theorem 2.1) then yields the connecting homomorphisms ω_n and the exactness of sequence (6.3). □

Of course the sequence (6.3) is natural, with respect to both A and the sequence $T' \to T \to T''$. In fact, we have

Proposition 6.4. *Let* $\alpha: A \to A'$ *be a homomorphism of* Λ*-modules and let*

$$\begin{array}{ccccc} T' & \xrightarrow{\tau'} & T & \xrightarrow{\tau''} & T'' \\ \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi''} \\ S' & \xrightarrow{\sigma'} & S & \xrightarrow{\sigma''} & S'' \end{array}$$

be a commutative diagram of additive functors and natural transformations such that the rows are exact on projectives. Then the following diagrams are commutative:

$$\begin{array}{ccccccccc} \cdots \to & L_n T'A & \xrightarrow{\tau'_A} & L_n TA & \xrightarrow{\tau''_A} & L_n T''A & \xrightarrow{\omega_n} & L_{n-1} T'A & \to \cdots \\ & \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha_*} & \\ \cdots \to & L_n T'A' & \xrightarrow{\tau'_{A'}} & L_n TA' & \xrightarrow{\tau''_{A'}} & L_n T''A' & \xrightarrow{\omega_n} & L_{n-1} T'A' & \to \cdots \end{array}$$

$$\begin{array}{ccccccccc} \cdots \to & L_n T'A & \xrightarrow{\tau'_A} & L_n TA & \xrightarrow{\tau''_A} & L_n T''A & \xrightarrow{\omega_n} & L_{n-1} T'A & \to \cdots \\ & \downarrow{\scriptstyle \varphi'_*} & & \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi''_*} & & \downarrow{\scriptstyle \varphi'_*} & \\ \cdots \to & L_n S'A & \xrightarrow{\sigma'_A} & L_n SA & \xrightarrow{\sigma''_A} & L_n S''A & \xrightarrow{\omega_n} & L_{n-1} S'A & \to \cdots \end{array}$$

The proof is left to the reader. □

Exercises:

6.1. Prove Proposition 6.2.

6.2. Prove Proposition 6.4.

6.3. Give an example of a sequence of functors $T' \to T \to T''$ which is exact on projectives, but not exact.

6.4. Use the exact sequences of this section to provide a solution of Exercise 5.5.

6.5. Give a direct proof of the exactness of $L_0TA' \to L_0TA \to L_0TA'' \to 0$ where $0 \to A' \to A \to A'' \to 0$ is exact and $T = \operatorname{Hom}_\Lambda(B, -)$.

6.6. Consider the category $\mathfrak{C}$ of short exact sequences in $\mathfrak{M}_\Lambda$ and consider the category $\mathfrak{D}$ of morphisms in $\mathfrak{Ab}$. Show that ω_n may be regarded as a functor $\omega_n : \mathfrak{C} \to \mathfrak{D}$.

7. The Functors $\operatorname{Ext}_\Lambda^n$ Using Projectives

The (contravariant) functor $\operatorname{Hom}_\Lambda(-, B)$ is additive. We therefore can define, in particular, right derived functors of $\operatorname{Hom}_\Lambda(-, B)$. These will be the $\operatorname{Ext}_\Lambda^n$ functors.

Definition. $\operatorname{Ext}_\Lambda^n(-, B) = R^n(\operatorname{Hom}_\Lambda(-, B))$, $n = 0, 1, \ldots$.

We recall that this means that the abelian group $\operatorname{Ext}_\Lambda^n(A, B)$ is computed by choosing a projective resolution $\boldsymbol{P}$ of A and taking cohomology in the cochain complex $\operatorname{Hom}_\Lambda(\boldsymbol{P}, B)$. Since $\operatorname{Hom}_\Lambda(-, B)$ is left exact it follows from Proposition 5.2 that $\operatorname{Ext}_\Lambda^0(A, B) = \operatorname{Hom}_\Lambda(A, B)$. The calculation of $\operatorname{Ext}_\Lambda^1(A, B)$ will justify our notation; we have

Proposition 7.1. $\operatorname{Ext}_\Lambda^1(A, B) \cong \operatorname{Ext}_\Lambda(A, B)$.

Proof. We consider the projective presentation $R_1 \overset{\mu}{\rightarrowtail} P_0 \overset{\varepsilon}{\twoheadrightarrow} A$ of A and apply Proposition 5.8. We obtain the exact sequence

$$\cdots \to \operatorname{Hom}_\Lambda(P_0, B) \to \operatorname{Hom}_\Lambda(R_1, B) \to \operatorname{Ext}_\Lambda^1(A, B) \to 0$$

whence it follows that $\operatorname{Ext}_\Lambda^1(A, B) \cong \operatorname{Ext}_\Lambda(A, B)$ by the definition of the latter (Section III. 2). $\square$

From the fact that $\operatorname{Ext}_\Lambda^n(-, B)$ is defined as a right derived functor the following is immediate by Theorem 6.1. Given a short exact sequence $A' \rightarrowtail A \twoheadrightarrow A''$ we obtain a long exact sequence

$$\cdots \to \operatorname{Ext}_\Lambda^n(A'', B) \to \operatorname{Ext}_\Lambda^n(A, B) \to \operatorname{Ext}_\Lambda^n(A', B) \xrightarrow{\omega_n} \operatorname{Ext}_\Lambda^{n+1}(A'', B) \to \cdots. \tag{7.1}$$

This sequence is called the *long exact* Ext-*sequence in the first variable*. By Proposition 6.2 this sequence is natural, i.e., if we are given a commutative diagram

$$\begin{tikzcd} A' \arrow[r, tail] \arrow[d, "\varphi'"] & A \arrow[r, two heads] \arrow[d, "\varphi"] & A'' \arrow[d, "\varphi''"] \\ \bar{A}' \arrow[r, tail] & \bar{A} \arrow[r, two heads] & \bar{A}'' \end{tikzcd}$$

then the diagram

$$\begin{array}{ccccccccc}
\cdots \to \operatorname{Ext}_\Lambda^n(\bar{A}'', B) & \to & \operatorname{Ext}_\Lambda^n(\bar{A}, B) & \to & \operatorname{Ext}_\Lambda^n(\bar{A}', B) & \xrightarrow{\omega_n} & \operatorname{Ext}_\Lambda^{n+1}(\bar{A}'', B) \to \cdots \\
\downarrow{\scriptstyle (\varphi'')^*} & & \downarrow{\scriptstyle \varphi^*} & & \downarrow{\scriptstyle (\varphi')^*} & & \downarrow{\scriptstyle (\varphi')^*} \\
\cdots \to \operatorname{Ext}_\Lambda^n(A'', B) & \to & \operatorname{Ext}_\Lambda^n(A, B) & \to & \operatorname{Ext}_\Lambda^n(A', B) & \xrightarrow{\omega_n} & \operatorname{Ext}_\Lambda^{n+1}(A'', B) \to \cdots
\end{array} \tag{7.2}$$

is commutative, also.

Proposition 7.2. *If P is projective and if I is injective, then*

$$\operatorname{Ext}_\Lambda^n(P, B) = 0 = \operatorname{Ext}_\Lambda^n(A, I) \quad \textit{for} \quad n = 1, 2, \ldots .$$

Proof. The first assertion is immediate by Proposition 5.3. To prove the second assertion, we merely remark that $\operatorname{Hom}_\Lambda(-, I)$ is an exact functor, so that its n^{th} derived functor is zero for $n \geqq 1$. □

Now let $\beta: B \to B'$ be a homomorphism of Λ-modules. Plainly β induces a natural transformation

$$\beta: \operatorname{Hom}_\Lambda(-, B) \to \operatorname{Hom}_\Lambda(-, B').$$

By Proposition 6.2 we have that, for any short exact sequence

$$A' \rightarrowtail A \twoheadrightarrow A'',$$

the diagram

$$\begin{array}{ccccccccc}
\cdots \to \operatorname{Ext}_\Lambda^n(A'', B) & \to & \operatorname{Ext}_\Lambda^n(A, B) & \to & \operatorname{Ext}_\Lambda^n(A', B) & \xrightarrow{\omega_n} & \operatorname{Ext}_\Lambda^{n+1}(A'', B) \to \cdots \\
\downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle \beta_*} \\
\cdots \to \operatorname{Ext}_\Lambda^n(A'', B') & \to & \operatorname{Ext}_\Lambda^n(A, B') & \to & \operatorname{Ext}_\Lambda^n(A', B') & \xrightarrow{\omega_n} & \operatorname{Ext}_\Lambda^{n+1}(A'', B') \to \cdots
\end{array} \tag{7.3}$$

is commutative. From (7.3) we easily deduce the following proposition.

Proposition 7.3. $\operatorname{Ext}_\Lambda^n(-, -)$, $n = 0, 1, \ldots$ *is a bifunctor.* □

Proposition 7.4. *Let $B' \overset{\beta'}{\rightarrowtail} B \overset{\beta''}{\twoheadrightarrow} B''$ be a short exact sequence, then the sequence $\operatorname{Hom}_\Lambda(-, B') \xrightarrow{\beta'_*} \operatorname{Hom}_\Lambda(-, B) \xrightarrow{\beta''_*} \operatorname{Hom}_\Lambda(-, B'')$ of left exact (contravariant) functors is exact on projectives.*

This is trivial. □

By Theorem 6.3 we now obtain

Proposition 7.5. *For any Λ-module A the short exact sequence*

$$B' \overset{\beta'}{\rightarrowtail} B \overset{\beta''}{\twoheadrightarrow} B''$$

gives rise to a long exact sequence

$$\cdots \to \operatorname{Ext}_\Lambda^n(A, B') \xrightarrow{\beta'_*} \operatorname{Ext}_\Lambda^n(A, B) \xrightarrow{\beta''_*} \operatorname{Ext}_\Lambda^n(A, B'') \xrightarrow{\omega_n} \operatorname{Ext}_\Lambda^{n+1}(A, B') \to \cdots . \ \square \tag{7.4}$$

Sequence (7.4) is called the *long exact* Ext-*sequence in the second variable.* By Proposition 6.4 sequence (7.4) is natural. Indeed, invoking the full force of Proposition 6.4, we infer

Proposition 7.6. *Let $\alpha : A \to A'$ be a homomorphism and let*

$$\begin{tikzcd} B' \arrow[r, tail] \arrow[d, "\psi'"] & B \arrow[r, two heads] \arrow[d, "\psi"] & B'' \arrow[d, "\psi''"] \\ \bar{B}' \arrow[r, tail] & \bar{B} \arrow[r, two heads] & \bar{B}'' \end{tikzcd}$$

be a commutative diagram with short exact rows. Then the following diagrams are commutative:

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{Ext}_\Lambda^n(A',B') \arrow[r] \arrow[d, "\alpha^*"] & \mathrm{Ext}_\Lambda^n(A',B) \arrow[r] \arrow[d, "\alpha^*"] & \mathrm{Ext}_\Lambda^n(A',B'') \arrow[r, "\omega_n"] \arrow[d, "\alpha^*"] & \mathrm{Ext}_\Lambda^{n+1}(A',B') \arrow[r] \arrow[d, "\alpha^*"] & \cdots \\ \cdots \arrow[r] & \mathrm{Ext}_\Lambda^n(A,B') \arrow[r] & \mathrm{Ext}_\Lambda^n(A,B) \arrow[r] & \mathrm{Ext}_\Lambda^n(A,B'') \arrow[r, "\omega_n"] & \mathrm{Ext}_\Lambda^{n+1}(A,B') \arrow[r] & \cdots \end{tikzcd} \tag{7.5}$$

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{Ext}_\Lambda^n(A,B') \arrow[r] \arrow[d, "\psi'_*"] & \mathrm{Ext}_\Lambda^n(A,B) \arrow[r] \arrow[d, "\psi_*"] & \mathrm{Ext}_\Lambda^n(A,B'') \arrow[r, "\omega_n"] \arrow[d, "\psi''_*"] & \mathrm{Ext}_\Lambda^{n+1}(A,B') \arrow[r] \arrow[d, "\psi'_*"] & \cdots \\ \cdots \arrow[r] & \mathrm{Ext}_\Lambda^n(A,\bar{B}') \arrow[r] & \mathrm{Ext}_\Lambda^n(A,\bar{B}) \arrow[r] & \mathrm{Ext}_\Lambda^n(A,\bar{B}'') \arrow[r, "\omega_n"] & \mathrm{Ext}_\Lambda^{n+1}(A,\bar{B}') \arrow[r] & \cdots \end{tikzcd} \tag{7.6}$$

□

Diagrams (7.2), (7.3), (7.5), (7.6) show that the long exact Ext-sequences are natural in every respect possible.

Exercises:

7.1. Let

$$\begin{tikzcd} \cdots \arrow[r] & P_q \arrow[rr] \arrow[dr, two heads] & & P_{q-1} \arrow[r] & \cdots \arrow[r] & P_0 \\ & & R_q \arrow[ur, tail] & & & \end{tikzcd}$$

be a projective resolution of A and let

$$\begin{tikzcd} I_0 \arrow[r] & I_1 \arrow[r] & \cdots \arrow[r] & I_{q-1} \arrow[rr] \arrow[dr, two heads] & & I_q \arrow[r] & \cdots \\ & & & & K_q \arrow[ur, tail] & & \end{tikzcd}$$

be an injective resolution of B. Establish isomorphisms

$$\mathrm{Ext}_\Lambda^n(A,B) \cong \mathrm{Ext}_\Lambda^{n-1}(R_1,B) \cong \cdots \cong \mathrm{Ext}_\Lambda^1(R_{n-1},B),$$
$$\mathrm{Ext}_\Lambda^n(A,B) \cong \mathrm{Ext}_\Lambda^{n-1}(A,K_1) \cong \cdots \cong \mathrm{Ext}_\Lambda^1(A,K_{n-1}), \quad n \geqq 1.$$

7.2. Suppose given the exact sequence

$$0 \to K_q \xrightarrow{\mu} P_{q-1} \to \cdots \to P_0 \to A \to 0$$

with $P_0, \dots, P_{q-1}$ projective. Prove that the sequence

$$\mathrm{Hom}_\Lambda(P_{q-1}, B) \to \mathrm{Hom}_\Lambda(K_q, B) \to \mathrm{Ext}^q_\Lambda(A, B) \to 0$$

is exact.

7.3. Let $M^* = \mathrm{Hom}_\Lambda(M, \Lambda)$ for any Λ-module M. Let $P_1 \to P_0 \to M \to 0$ be an exact sequence of Λ-modules with P_0, P_1 finitely generated projective. Let

$$D = \mathrm{coker}(P_0^* \to P_1^*).$$

Show that the sequence

$$0 \to \mathrm{Ext}^1_\Lambda(D, \Lambda) \to M \to M^{**} \to \mathrm{Ext}^2_\Lambda(D, \Lambda) \to 0$$

is exact. (Hint: Consider the diagram

$$\begin{tikzcd} & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K^* \arrow[r] & P_0^{**} \arrow[r] & M^{**} & \end{tikzcd}$$

where $K = \ker(P_1^* \to D) = \mathrm{coker}(M^* \to P_0^*)$; and show that $P_0 \to P_0^{**}$ is an isomorphism.)

7.4. Show that $\omega: \mathrm{Hom}_\Lambda(A, B'') \to \mathrm{Ext}^1_\Lambda(A, B')$ factors through $\Pi P(A, B'')$ (Exercise 5.7) and deduce that

$$\Pi P(A, B') \to \Pi P(A, B) \to \Pi P(A, B'') \to \mathrm{Ext}^1_\Lambda(A, B') \to \mathrm{Ext}^1_\Lambda(A, B) \to \cdots$$

is exact. What does this tell us about left derived functors of $\Pi P(A, -)$?

7.5. Establish the existence of an exact sequence

$$\cdots \to \Pi P_n(A, B') \to \Pi P_n(A, B) \to \Pi P_n(A, B'') \to \Pi P_{n-1}(A, B') \to \cdots$$

where $\Pi P_n(A, B) = \Pi P(A, S_n)$, and

$$\begin{tikzcd} \cdots \arrow[r] & Q_n \arrow[rr] \arrow[dr] & & Q_{n-1} \arrow[r] & \cdots \arrow[r] & Q_0 \\ & & S_n \arrow[ur] & & & \end{tikzcd}$$

is a projective resolution of B.

7.6. Show that $\Pi P_n(A, B) = L_{n-1}\mathrm{Hom}(A, -)(B)$, $n \geqq 2$. Does this hold for $n = 1$?

7.7. A Λ-module A is said to have *projective dimension* $\leqq m$, and we write

$$\mathrm{proj.dim.}\, A \leqq m,$$

if $\mathrm{Ext}^q_\Lambda(A, B) = 0$ for all $q > m$ and all Λ-modules B. Show that the following statements are equivalent:

(i) $\mathrm{proj.dim.}\, A \leqq m$;

(ii) $\mathrm{Ext}^{m+1}_\Lambda(A, B) = 0$ for all Λ-modules B;

(iii) There exists a projective resolution of A of length m, i.e., a resolution

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0$$

with

$$P_{m+1} = P_{m+2} = \cdots = 0.$$

(iv) In every projective resolution

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0$$

of A the image of $P_m \to P_{m-1}$ is projective, where $P_{-1} = A$.
(Of course, we write proj.dim.$A = m$ if proj.dim.$A \leqq m$ but proj.dim.$A \nleqq m-1$.)

8. The Functors $\overline{\mathrm{Ext}}_\Lambda^n$ Using Injectives

The covariant functor $\mathrm{Hom}_\Lambda(A, -)$ is additive. We therefore can define, in particular, right derived functors of $\mathrm{Hom}_\Lambda(A, -)$. These will be the $\overline{\mathrm{Ext}}_\Lambda^n$ functors.

Definition. $\overline{\mathrm{Ext}}_\Lambda^n(A, -) = R^n(\mathrm{Hom}_\Lambda(A, -))$, $n = 0, 1, \ldots$.

We recall that this means that the abelian group $\overline{\mathrm{Ext}}_\Lambda^n(A, B)$ is computed by choosing an injective resolution $\boldsymbol{I}$ of B and taking cohomology in the cochain complex $\mathrm{Hom}_\Lambda(A, \boldsymbol{I})$. Since $\mathrm{Hom}_\Lambda(A, -)$ is left exact

$$\overline{\mathrm{Ext}}_\Lambda^0(A, B) = \mathrm{Hom}_\Lambda(A, B)$$

(Proposition 5.7). In order to compute $\overline{\mathrm{Ext}}_\Lambda^1(A, B)$ we choose an injective presentation $B \rightarrowtail I \overset{\eta}{\twoheadrightarrow} S$ of B and apply the dual of Proposition 5.5. We obtain the exact sequence

$$\mathrm{Hom}_\Lambda(A, I) \xrightarrow{\eta_*} \mathrm{Hom}_\Lambda(A, S) \to \overline{\mathrm{Ext}}_\Lambda^1(A, B) \to 0\,. \tag{8.1}$$

By definitions made in III.3 it follows that

$$\overline{\mathrm{Ext}}_\Lambda^1(A, B) \cong \overline{\mathrm{Ext}}_\Lambda(A, B)\,.$$

This justifies our notation.

The fact that $\overline{\mathrm{Ext}}_\Lambda^n(A, -)$ is defined as a right derived functor immediately yields a number of results.

(1) For any injective Λ-module I,

$$\overline{\mathrm{Ext}}_\Lambda^n(A, I) = 0 \quad \text{for} \quad n = 1, 2, \ldots \tag{8.2}$$

(compare Proposition 7.2).

(2) A short exact sequence $B' \rightarrowtail B \twoheadrightarrow B''$ gives rise to a long exact $\overline{\mathrm{Ext}}$-sequence:

$$\cdots \to \overline{\mathrm{Ext}}_\Lambda^n(A, B') \to \overline{\mathrm{Ext}}_\Lambda^n(A, B) \to \overline{\mathrm{Ext}}_\Lambda^n(A, B'') \xrightarrow{\overline{\omega}} \overline{\mathrm{Ext}}_\Lambda^{n+1}(A, B') \to \cdots \tag{8.3}$$

(compare sequence (7.4)).

(3) Sequence (8.3) is natural with respect to the short exact sequence (compare diagram (7.6)).

(4) For any projective Λ-module P, $\overline{\mathrm{Ext}}_\Lambda^n(P, B) = 0$ for $n = 1, 2, \ldots$ (compare Proposition 7.2).

(5) A homomorphism $\alpha: A \to A'$ induces a natural transformation $\alpha^*: \mathrm{Hom}_\Lambda(A', -) \to \mathrm{Hom}_\Lambda(A, -)$, and sequence (8.3) is natural with respect to the first variable (compare diagram (7.5)). It follows that $\overline{\mathrm{Ext}}^n_\Lambda(-, -)$ is a bifunctor (compare Proposition 7.3).

(6) A short exact sequence $A' \overset{\alpha'}{\rightarrowtail} A \overset{\alpha''}{\twoheadrightarrow} A''$ induces a sequence of additive functors $\mathrm{Hom}_\Lambda(A'', -) \xrightarrow{\alpha''_*} \mathrm{Hom}_\Lambda(A, -) \xrightarrow{\alpha'_*} \mathrm{Hom}_\Lambda(A', -)$ which is exact on injectives and therefore gives rise to a long exact sequence

$$\cdots \to \overline{\mathrm{Ext}}^n_\Lambda(A'', B) \to \overline{\mathrm{Ext}}^n_\Lambda(A, B) \to \overline{\mathrm{Ext}}^n_\Lambda(A', B) \xrightarrow{\overline{\omega}} \overline{\mathrm{Ext}}^{n+1}_\Lambda(A'', B) \to \cdots \quad (8.4)$$

(compare sequence (7.1)).

(7) Sequence (8.4) is natural both with respect to the short exact sequence (compare diagram (7.2)) and with respect to the second variable (compare diagram (7.3)).

The conclusion of the reader from all these results must be that the functors $\overline{\mathrm{Ext}}^n$ and Ext^n are rather similar. Indeed we shall prove

Proposition 8.1. *The bifunctors* $\mathrm{Ext}^n_\Lambda(-, -)$ *and* $\overline{\mathrm{Ext}}^n_\Lambda(-, -)$, $n = 0, 1, \ldots$ *are naturally equivalent.*

Proof. We will define natural equivalences

$$\Phi^n : \mathrm{Ext}^n_\Lambda(-, -) \xrightarrow{\sim} \overline{\mathrm{Ext}}^n_\Lambda(-, -)$$

inductively.

The construction of Φ^n is trivial for $n = 0$; Φ^0 is the identity. Now let $B \overset{\nu}{\rightarrowtail} I \overset{\eta}{\twoheadrightarrow} S$ be an injective presentation. By Proposition 7.2 and (8.2) we have

$$\mathrm{Ext}^n_\Lambda(A, I) = 0 = \overline{\mathrm{Ext}}^n_\Lambda(A, I) \quad \text{for} \quad n = 1, 2, \ldots .$$

We then consider the long exact Ext-sequence (7.4) and the long exact $\overline{\mathrm{Ext}}$-sequence (8.3). We define $\Phi^1_{A,B}$ by requiring commutativity in the diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}_\Lambda(A, I) & \xrightarrow{\eta_*} & \mathrm{Hom}_\Lambda(A, S) & \xrightarrow{\omega_1} & \mathrm{Ext}^1_\Lambda(A, B) & \longrightarrow & 0 \\
\| & & \| & & \Big\downarrow{\scriptstyle \Phi^1_{A,B}} & & \\
\mathrm{Hom}_\Lambda(A, I) & \xrightarrow{\eta_*} & \mathrm{Hom}_\Lambda(A, S) & \xrightarrow{\overline{\omega}_1} & \overline{\mathrm{Ext}}^1_\Lambda(A, B) & \longrightarrow & 0
\end{array}$$

and, assuming Φ^n defined, we define $\Phi^{n+1}_{A,B}$ by requiring commutativity in the diagram

$$\begin{array}{ccc}
\mathrm{Ext}^n_\Lambda(A, S) & \xrightarrow[\sim]{\omega_n} & \mathrm{Ext}^{n+1}_\Lambda(A, B) \\
\Big\downarrow{\scriptstyle \Phi^n_{A,S}} & & \Big\downarrow{\scriptstyle \Phi^{n+1}_{A,B}} \\
\overline{\mathrm{Ext}}^n_\Lambda(A, S) & \xrightarrow[\sim]{\overline{\omega}_n} & \overline{\mathrm{Ext}}^{n+1}_\Lambda(A, B)
\end{array}$$

We obviously have to check that

1) the definition of $\Phi_{A,B}^{n+1}$ does not depend on the chosen presentation of B,

2) $\Phi_{A,B}^{n+1}$ is natural in B,

3) $\Phi_{A,B}^{n+1}$ is natural in A.

We shall deal in detail with points 1) and 2), but leave point 3) to the reader.

So suppose given the following diagram

$$\begin{tikzcd} B \arrow[r, tail] \arrow[d, "\beta"] & I \arrow[r, two heads] \arrow[d, "\varphi"] & S \arrow[d, "\psi"] \\ B' \arrow[r, tail] & I' \arrow[r, two heads] & S' \end{tikzcd}$$

with I, I' injective, and let us consider the cube

$$\begin{tikzcd} & \mathrm{Ext}_\Lambda^n(A,S') \arrow[rr, "\omega"] \arrow[dd, "\Phi^n"] & & \mathrm{Ext}_\Lambda^{n+1}(A,B') \arrow[dd, "\Phi^{n+1}"] \\ \mathrm{Ext}_\Lambda^n(A,S) \arrow[ur, "\psi_*"] \arrow[rr, "\omega"] \arrow[dd, "\Phi^n"] & & \mathrm{Ext}_\Lambda^{n+1}(A,B) \arrow[ur, "\beta_*"] \arrow[dd, "\Phi^{n+1}"] & \\ & \overline{\mathrm{Ext}}_\Lambda^n(A,S') \arrow[rr, "\bar\omega"] & & \overline{\mathrm{Ext}}_\Lambda^{n+1}(A,B') \\ \overline{\mathrm{Ext}}_\Lambda^n(A,S) \arrow[ur, "\psi_*"] \arrow[rr, "\bar\omega"'] & & \overline{\mathrm{Ext}}_\Lambda^{n+1}(A,B) \arrow[ur, "\beta_*"] & \end{tikzcd} \tag{8.6}$$

We claim that this diagram is commutative. The top square is commutative by naturality of the long Ext-sequence, the bottom square by analogous reasons for $\overline{\mathrm{Ext}}$. Front and back squares are commutative by definition, the left hand square by the inductive hypothesis that $\Phi_{A,S}^n$ is a natural transformation. It then follows that the right hand square also is commutative, since $\omega : \mathrm{Ext}_\Lambda^n(A,S) \to \mathrm{Ext}_\Lambda^{n+1}(A,B)$ is surjective.

To prove point 1) we now only have to set $\beta = 1_B : B \to B$; point 2) is proved by the fact that the right hand square of the diagram is commutative. $\square$

We also prove

Proposition 8.2. *For any A and any short exact sequence $B' \rightarrowtail B \twoheadrightarrow B''$ the following square is commutative*

$$\begin{tikzcd} \mathrm{Ext}_\Lambda^n(A,B'') \arrow[r, "\omega"] \arrow[d, "\Phi^n"] & \mathrm{Ext}_\Lambda^{n+1}(A,B') \arrow[d, "\Phi^{n+1}"] \\ \overline{\mathrm{Ext}}_\Lambda^n(A,B'') \arrow[r, "\bar\omega"'] & \overline{\mathrm{Ext}}_\Lambda^{n+1}(A,B') \end{tikzcd} \tag{8.7}$$

Proof. Choose an injective presentation $B' \rightarrowtail I \twoheadrightarrow S$ of B' and construct φ, ψ such that the diagram

$$\begin{tikzcd}
B' \arrow[r, tail] \arrow[d, equal] & B \arrow[r, two heads] \arrow[d, "\varphi"] & B'' \arrow[d, "\psi"] \\
B' \arrow[r, tail] & I \arrow[r, two heads] & S
\end{tikzcd}$$

is commutative. We then embed (8.7) as front square in the following cube

$$\begin{tikzcd}
& \mathrm{Ext}_A^n(A,S) \arrow[rr, "\omega"] \arrow[dd, "\Phi"] & & \mathrm{Ext}_A^{n+1}(A,B') \arrow[dd, "\wr\,\Phi"] \\
\mathrm{Ext}_A^n(A,B'') \arrow[ur, "\psi_*"] \arrow[rr, "\omega"] \arrow[dd, "\Phi"] & & \mathrm{Ext}_A^{n+1}(A,B') \arrow[ur, equal] \arrow[dd, "\Phi"] & \\
& \overline{\mathrm{Ext}}_A^n(A,S) \arrow[rr, "\bar\omega"] & & \overline{\mathrm{Ext}}_A^{n+1}(A,B') \\
\overline{\mathrm{Ext}}_A^n(A,B'') \arrow[ur, "\psi_*"] \arrow[rr, "\bar\omega"] & & \overline{\mathrm{Ext}}_A^{n+1}(A,B') \arrow[ur, equal] &
\end{tikzcd}$$

The right hand square trivially is commutative, the left hand square is commutative since Φ is a natural transformation. Top and bottom squares are commutative by naturality of the Ext-, resp. $\overline{\mathrm{Ext}}$-sequence. The back square is commutative by the definition of Φ. It follows that the front square is commutative, also. $\square$

By Proposition 8.2 the natural transformation Φ is compatible with the connecting homomorphism in the long exact Ext-sequence in the second variable. We remark that Φ as exhibited above is also compatible with the connecting homomorphism in the long exact sequence in the first variable (see Exercise 9.8). In view of the equivalence expressed in Propositions 8.1 and 8.2 we shall use only the notation Ext, even if we refer to the definition by injectives. We then may express the assertion of Proposition 8.1 by saying that the bifunctor Ext_A^n is *balanced*; it may be computed via a projective resolution of the first, or an injective resolution of the second variable, and is balanced in that the value of $\mathrm{Ext}_A^n(A,B)$ is obtained as the value of the n^{th} right derived functor of $\mathrm{Hom}_A(-,B)$ at A or the value of the n^{th} right derived functor of $\mathrm{Hom}_A(A,-)$ at B.

We finally point out the important fact that the steps in Section 7 necessary to define Ext^n and elicit its properties are possible in any abelian category with enough projective objects, and do not require any other particular property of the category $\mathfrak{M}_A$. Similarly, of course, the steps in Section 8 necessary to define $\overline{\mathrm{Ext}}^n$ and elicit its properties are possible in any abelian category with enough injective objects. Moreover, in a category with enough projectives and injectives, $\text{Ext}^n \cong \overline{\text{Ext}}^n$. However it may well happen that an abelian category has enough projectives but not enough injectives (for example the category of finitely generated abelian groups (see Exercise 8.1)); then clearly only the procedure using projectives will yield Ext-functors according to our definition. In the dual situation of course, that is, in a category with enough injective but not enough projective objects (for example in the category of torsion abelian groups (see Exercise 8.2)) only the procedure using injectives will yield Ext-functors. Actually, it may be shown that even in abelian categories with neither enough projectives nor enough injectives, functors having all the essential properties of Ext-functors may be defined (see Exercises 9.4 to 9.7).

Exercises:

8.1. Show that the category of finitely generated abelian groups has enough projectives but no non-zero injectives.

8.2. Show that the category of torsion abelian groups has enough injectives but no non-zero projectives.

8.3. Suppose given the exact sequence

$$0 \to B \to I_0 \to \cdots \to I_{q-1} \xrightarrow{\varepsilon} S_q \to 0$$

with $I_0, \ldots, I_{q-1}$ injective. Show that the sequence

$$\text{Hom}_\Lambda(A, I_{q-1}) \xrightarrow{\varepsilon_*} \text{Hom}_\Lambda(A, S_q) \to \text{Ext}_\Lambda^q(A, B) \to 0$$

is exact.

8.4. Dualize the definition of $\Pi P(A, B)$, to define $\Pi I(A, B)$.

8.5. Dualize Exercises 7.4, 7.5, 7.6.

8.6. Let us say that $\varphi: A \to B$ is a *fibre-map* if every homomorphism $I \to B$, I injective, factors through φ. Let $\iota: K \rightarrowtail A$ be the kernel of the fibre-map φ. Show that there is an exact sequence, for any X,

$$\cdots \to \Pi I_n(X, K) \xrightarrow{\iota_*} \Pi I_n(X, A) \xrightarrow{\varphi_*} \Pi I_n(X, B) \to \Pi I_{n-1}(X, K) \to \cdots.$$

Dualize.

8.7. A Λ-module B is said to have *injective dimension* $\leqq m$, and we write $\text{inj.dim.} B \leqq m$, if $\text{Ext}_\Lambda^q(A, B) = 0$ for all $q > m$ and for all Λ-modules A. Analogously to Exercise 7.7 give different characterisations for $\text{inj.dim.} B \leqq m$. (Of course, we write $\text{inj.dim.} B = m$ if $\text{inj.dim.} B \leqq m$ but $\text{inj.dim.} B \nleqq m - 1$.)

8.8. A ring Λ is said to have *global dimension* $\leqq m$, and we write $\text{gl.dim.} \Lambda \leqq m$, if $\text{Ext}_\Lambda^q(A, B) = 0$ for all $q > m$ and for all Λ-modules A, B. The smallest m with $\text{gl.dim} \Lambda \leqq m$ is called the *global dimension* of Λ. What is the global dimension of a field, of a semi-simple ring, of a p.i.d.? Characterize the global dimension of Λ in terms of the projective and injective dimension of Λ-modules.

9. Ext^n and n-Extensions

We recall that $\mathrm{Ext}_\Lambda(A,B)=\mathrm{Ext}_\Lambda^1(A,B)$ can be interpreted as the group of equivalence classes of extensions $B \rightarrowtail E \twoheadrightarrow A$. A generalization of this interpretation to Ext^n has been given by Yoneda. An exact sequence

$$\mathsf{E}: 0 \to B \to E_n \to \cdots \to E_1 \to A \to 0 \tag{9.1}$$

of Λ-modules is called an *n-extension* of A by B. Then an extension is a 1-extension. In the set of n-extensions of A by B we shall introduce an equivalence relation that generalizes the equivalence relation given in Section III.1 for 1-extensions. We shall say that the n-extensions E, E' satisfy the relation $\mathsf{E} \rightsquigarrow \mathsf{E}'$ if there is a commutative diagram

$$\begin{tikzcd}
\mathsf{E}: 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & E_n \arrow[r] \arrow[d] & \cdots \arrow[r] & E_1 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
\mathsf{E}': 0 \arrow[r] & B \arrow[r] & E'_n \arrow[r] & \cdots \arrow[r] & E'_1 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

It is easy to see that the relation $\rightsquigarrow$ is not symmetric for $n \geqq 2$, although it obviously is for $n=1$. However every relation *generates* an equivalence relation, which we now describe explicitly for the given relation $\rightsquigarrow$. Accordingly, we define E and E' to be equivalent, $\mathsf{E} \sim \mathsf{E}'$, if and only if there exists a chain $\mathsf{E}_0=\mathsf{E}, \mathsf{E}_1, \ldots, \mathsf{E}_k=\mathsf{E}'$ with

$$\mathsf{E}_0 \rightsquigarrow \mathsf{E}_1 \leftsquigarrow \mathsf{E}_2 \rightsquigarrow \cdots \leftsquigarrow \mathsf{E}_k .$$

By $[\mathsf{E}]$ we denote the equivalence class of the n-extension E, and by $\mathrm{Yext}_\Lambda^n(A,B)$, $n \geqq 1$, we denote the set of all equivalence classes of n-extensions of A by B. In order to make $\mathrm{Yext}_\Lambda^n(-,-)$ into a bifunctor we shall define induced maps as follows.

First let B be fixed and let $\alpha: A' \to A$ be a homomorphism. Let $\mathsf{E}: 0 \to B \to E_n \to \cdots \to E_1 \xrightarrow{\eta} A \to 0$ be a representative of an element in $\mathrm{Yext}_\Lambda^n(A,B)$. Define E_1^α as the pull-back of (α,η),

$$\begin{tikzcd}
E_1^\alpha \arrow[r, dotted, "\eta'"] \arrow[d, dotted] & A' \arrow[d, "\alpha"] \\
E_1 \arrow[r, two heads, "\eta"'] & A
\end{tikzcd}$$

By Lemma III.1.2 η' is epimorphic and has the same kernel as η. We therefore obtain an exact sequence

$$\mathsf{E}^\alpha: 0 \to B \to E_n \to \cdots \to E_2 \to E_1^\alpha \xrightarrow{\eta'} A' \to 0$$

which determines an element in $\mathrm{Yext}_\Lambda^n(A',B)$. It is to be proved that two different representatives of an element in $\mathrm{Yext}_\Lambda^n(A,B)$ define the same element in $\mathrm{Yext}^n_\Lambda(A', B)$. This is achieved by proving that the relation $E \rightsquigarrow \tilde{E}$ implies the relation $E^\alpha \rightsquigarrow \tilde{E}^\alpha$. We concentrate on the right hand end of the sequences. Setting $K = \ker \eta$, $\tilde{K} = \ker \tilde{\eta}$ we obtain the following diagram

$$\begin{tikzcd}
& K \arrow[rr, tail] \arrow[dl] \arrow[dd, equal] & & E_1^\alpha \arrow[rr, two heads, "\eta'"] \arrow[dd] & & A' \arrow[dl, equal] \arrow[dd, "\alpha"] \\
\tilde{K} \arrow[rr, tail, crossing over] \arrow[dd, equal] & & \tilde{E}_1^\alpha \arrow[rr, two heads, "\tilde{\eta}'", crossing over] & & A' & \\
& K \arrow[rr, tail] \arrow[dl] & & E_1 \arrow[rr, two heads, "\eta"] \arrow[dl] & & A \arrow[dl, equal] \\
\tilde{K} \arrow[rr, tail] & & \tilde{E}_1 \arrow[rr, two heads, "\tilde{\eta}"] \arrow[from=uu, crossing over] & & A \arrow[from=uu, "\alpha", crossing over] &
\end{tikzcd} \tag{9.2}$$

where E_1^α is the pull-back of (α, η) and $\tilde{E}_1^\alpha$ is the pull-back of $(\alpha, \tilde{\eta})$. We have to show the existence of a map $\xi: E_1^\alpha \to \tilde{E}_1^\alpha$ making the diagram commutative. The maps $E_1^\alpha \xrightarrow{\eta'} A' \to A$ and $E_1^\alpha \to E_1 \to \tilde{E}_1 \xrightarrow{\tilde{\eta}} A$ agree. Since $\tilde{E}_1^\alpha$ is a pull-back there is a (unique) map $\xi: E_1^\alpha \to \tilde{E}_1^\alpha$ making the right hand cube commutative. The reader will show easily that the left hand cube also is commutative (see the proof of Theorem III.1.4), so that ξ establishes the relation $E^\alpha \rightsquigarrow \tilde{E}^\alpha$.

Thus $\alpha^*[E] = [E^\alpha]$ defines a map $\alpha^*: \mathrm{Yext}^n_\Lambda(A, B) \to \mathrm{Yext}^n_\Lambda(A', B)$. It is plain that $1^* = 1$. Also, using the fact that the composite of two pull-back squares is a pull-back square, we have $(\alpha\alpha')^* = \alpha'^*\alpha^*$. These facts combine to show that $\mathrm{Yext}^n_\Lambda(-, B)$ is a contravariant functor.

Given $\beta: B \to B'$ we define an induced map

$$\beta_*: \mathrm{Yext}^n_\Lambda(A, B) \to \mathrm{Yext}^n_\Lambda(A, B')$$

by the dual process. Thus let $E: 0 \to B \to E_n \to \cdots \to E_1 \to A \to 0$ be a representative of an element in $\mathrm{Yext}^n_\Lambda(A, B)$. Let

$$\begin{tikzcd}
B \arrow[r, tail] \arrow[d, "\beta"] & E_n \arrow[d, dotted] \\
B' \arrow[r, dotted] & (E_n)_\beta
\end{tikzcd}$$

be a push-out square. We obtain a sequence

$$E_\beta: 0 \to B' \to (E_n)_\beta \to E_{n-1} \to \cdots \to E_1 \to A \to 0$$

which determines an element in $\mathrm{Yext}^n_\Lambda(A, B')$. As above one proves that $\beta_*[E] = [E_\beta]$ yields a map $\beta_*: \mathrm{Yext}^n_\Lambda(A, B) \to \mathrm{Yext}^n_\Lambda(A, B')$ which makes $\mathrm{Yext}^n_\Lambda(A, -)$ into a covariant functor.

It is immediate that

$$\operatorname{Yext}^1_\Lambda(A,B)=E(A,B)\cong\operatorname{Ext}^1_\Lambda(A,B) \tag{9.3}$$

naturally in both A and B (see Theorem III. 2.4). Since $E(-,-)$ is a bifunctor (Theorem III. 1.4) $\operatorname{Yext}^1_\Lambda(-,-)$ is a bifunctor, also. Indeed, this is the only non-trivial case of the proposition that $\operatorname{Yext}^n_\Lambda(-,-)$ is a bifunctor for $n\geqq 1$. Generalizing (9.3) we have

Theorem 9.1. *There is a natural equivalence of set-valued bifunctors* $\theta_n:\operatorname{Yext}^n_\Lambda(-,-)\xrightarrow{\sim}\operatorname{Ext}^n_\Lambda(-,-)$, $n=1,2,\ldots$.

Note that since $\operatorname{Ext}^n_\Lambda(A,B)$ carries a natural abelian group structure the equivalence θ_n, once established, introduces a natural abelian group structure into $\operatorname{Yext}^n_\Lambda(A,B)$.

Proof. We proceed by a method analogous to the one used in the proof of Theorem III.2.4. We first choose a projective resolution

$$\boldsymbol{P}:\cdots\to P_n\xrightarrow{\partial_n}P_{n-1}\to\cdots\to P_0$$

of A. Proposition 5.8 applied to the functor $\operatorname{Hom}_\Lambda(-,B)$ yields the exact sequence

$$\operatorname{Hom}_\Lambda(P_{n-1},B)\xrightarrow{\iota^*}\operatorname{Hom}_\Lambda(R_n,B)\xrightarrow{[\;]}\operatorname{Ext}^n_\Lambda(A,B)\to 0 \tag{9.4}$$

where $\iota:R_n\rightarrowtail P_{n-1}$ is the embedding of $R_n=\operatorname{im}\partial_n$ in P_{n-1}. We define $\theta:\operatorname{Yext}^n_\Lambda(A,B)\to\operatorname{Ext}^n_\Lambda(A,B)$ as follows.

Given the n-extension $\boldsymbol{E}:0\to B\to E_n\to\cdots\to E_1\to A\to 0$ we consider the acyclic complex $\boldsymbol{D}:0\to B\to E_n\to\cdots\to E_2\to E_1\to 0$ with $D_0=E_1,\ldots,D_{n-1}=E_n$, $D_n=B$, $D_k=0$ for $k\geqq n+1$. Plainly $H_0(\boldsymbol{D})=A$. By Theorem 4.1 there exists a chain map $\boldsymbol{\varphi}=\{\varphi_0,\ldots,\varphi_n\}$ such that the following diagram is commutative

$$\begin{tikzcd}
 & R_n \arrow[dr, "\iota"] & & & & & \\
P_{n+1} \arrow[r] \arrow[d] & P_n \arrow[u, "\eta"] \arrow[r, "\partial_n"] \arrow[d, "\varphi_n"] & P_{n-1} \arrow[r] \arrow[d, "\varphi_{n-1}"] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, "\varphi_0"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & E_n \arrow[r] & \cdots \arrow[r] & E_1 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

Clearly $\varphi_n:P_n\to B$ factors as $\varphi_n=\varphi\eta$ where $\varphi:R_n\to B$. We define $\theta(\boldsymbol{E})=[\varphi]$. We have to show that this definition is independent of the chain map $\boldsymbol{\varphi}$. By Theorem 4.1 it follows that if $\boldsymbol{\psi}=\{\psi_0,\ldots,\psi_n\}$ is another chain map there exists a chain homotopy $\boldsymbol{\Sigma}:\boldsymbol{\varphi}\to\boldsymbol{\psi}$. In particular we have $\psi_n-\varphi_n=\Sigma_{n-1}\partial_n$, so that $\psi-\varphi=\Sigma_{n-1}\iota$. It follows by (9.4) that $[\psi]=[\varphi+\Sigma_{n-1}\iota]=[\varphi]$. Finally it is obvious that if $\boldsymbol{E}\rightsquigarrow\boldsymbol{E}'$ then

$\theta(E) = [\varphi] = \theta(E')$. This completes the definition of the map

$$\theta : \operatorname{Yext}^n_\Lambda(A, B) \to \operatorname{Ext}^n_\Lambda(A, B).$$

Next we define a map $\tilde{\theta} : \operatorname{Ext}^n_\Lambda(A, B) \to \operatorname{Yext}^n_\Lambda(A, B)$. Let $\varphi : R_n \to B$ represent the element $[\varphi] \in \operatorname{Ext}^n_\Lambda(A, B)$. We associate with φ the equivalence class of the n-extension C_φ in the diagram

$$\begin{tikzcd}
 & & & R_{n-1} \arrow[dr] \arrow[dd, equal] & & & & & \\
C : 0 \arrow[r] & R_n \arrow[r, "\iota"] \arrow[dd, "\varphi"] & P_{n-1} \arrow[ur] \arrow[rr] \arrow[dd, "\chi"] & & P_{n-2} \arrow[r] \arrow[dd, equal] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[dd, equal] & A \arrow[r] \arrow[dd, equal] & 0 \\
 & & & R_{n-1} \arrow[dr] & & & & & \\
C_\varphi : 0 \arrow[r] & B \arrow[r, "\kappa"] & E \arrow[ur] \arrow[rr] & & P_{n-2} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

where E is the push-out of (ι, φ). If φ is replaced by $\varphi' = \varphi + \Sigma\iota$ then it is easy to see that, if χ is replaced by $\chi' = \chi + \kappa\Sigma$, the diagram is again commutative. It then follows that E is also the push-out of (ι, φ'). Thus if we set $\tilde{\theta}[\varphi] = [C_\varphi]$, we indeed have defined a map

$$\tilde{\theta} : \operatorname{Ext}^n_\Lambda(A, B) \to \operatorname{Yext}^n(A, B).$$

Plainly $\theta\tilde{\theta} = 1$ and the diagram

$$\begin{tikzcd}
C : 0 \arrow[r] & R_n \arrow[r] \arrow[d, "\varphi"] & P_{n-1} \arrow[r] \arrow[d] & P_{n-2} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, equal] & 0 \\
C_\varphi : 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\varphi'_{n-1}"] & P_{n-2} \arrow[r] \arrow[d, "\varphi_{n-2}"] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, "\varphi_0"] & A \arrow[r] \arrow[d, equal] & 0 \\
E : 0 \arrow[r] & B \arrow[r] & E_n \arrow[r] & E_{n-1} \arrow[r] & \cdots \arrow[r] & E_1 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

where φ'_{n-1} is defined by the push-out property of E shows that $\tilde{\theta}\theta = 1$.

It remains to prove that θ is a natural transformation. First let $\beta : B \to B'$ be given; then the diagram

$$\begin{tikzcd}
C : 0 \arrow[r] & R_n \arrow[r] \arrow[d, "\varphi"] & P_{n-1} \arrow[r] \arrow[d, "\varphi_{n-1}"] & P_{n-2} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, "\varphi_0"] & A \arrow[r] \arrow[d, equal] & 0 \\
E : 0 \arrow[r] & B \arrow[r] \arrow[d, "\beta"] & E_n \arrow[r] \arrow[d] & E_{n-1} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & E_1 \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, equal] & 0 \\
E_\beta : 0 \arrow[r] & B' \arrow[r] & (E_n)_\beta \arrow[r] & E_{n-1} \arrow[r] & \cdots \arrow[r] & E_1 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

shows that $\theta\beta_*[E] = \theta[E_\beta] = [\beta\varphi] = \beta_*[\varphi] = \beta_*\theta[E]$. Finally let $\alpha: A' \to A$ be given. We then have to look at the following diagram

$$\begin{tikzcd}[column sep=small]
 & 0 \arrow[r] & R'_n \arrow[r] \arrow[d, "\chi"'] \arrow[dd, "\varphi\chi"] & P'_{n-1} \arrow[r] \arrow[d] \arrow[dd] & \cdots \arrow[r] & P'_1 \arrow[r] \arrow[d] \arrow[dd] & P'_0 \arrow[r] \arrow[d] \arrow[dd, dotted] & A' \arrow[r] \arrow[d, "\alpha"'] \arrow[dd, equal] & 0 \\
 & 0 \arrow[r] & R_n \arrow[r] \arrow[dd, "\varphi"', bend right] & P_{n-1} \arrow[r] \arrow[dd, bend right] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[dd, "\varphi_1"', bend right] & P_0 \arrow[r] \arrow[dd, "\varphi_0"', bend right] & A \arrow[r] \arrow[dd, equal, bend right] & 0 \\
E^\alpha: & 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & E_n \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & E_2 \arrow[r] \arrow[d, equal] & E_1^\alpha \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d, "\alpha"] & 0 \\
E: & 0 \arrow[r] & B \arrow[r] & E_n \arrow[r] & \cdots \arrow[r] & E_2 \arrow[r] & E_1 \arrow[r, "\eta"'] & A \arrow[r] & 0
\end{tikzcd}$$

where E_1^α is the pull-back of (η, α). Since the maps $P'_0 \to P_0 \to E_1 \to A$ and $P'_0 \to A' \to A$ coincide we obtain a (unique) map $P'_0 \to E_1^\alpha$ which makes the diagram commutative. (There is, as usual, the extra argument establishing that $P'_1 \to P'_0 \to E_1^\alpha$ coincides with $P'_1 \to E_2 \to E_1^\alpha$.) Thus we obtain

$$\theta\alpha^*[E] = \theta[E^\alpha] = [\varphi\chi] = \alpha^*[\varphi] = \alpha^*\theta[E].$$

This completes the proof that θ is a natural transformation. □

We now give a description of the connecting homomorphisms of the Ext-sequences in terms of n-extensions. We first consider the long exact Ext-sequences in the *second* variable (7.4). Let $B' \rightarrowtail B \twoheadrightarrow B''$ be a short exact sequence and let $\boldsymbol{P}: \cdots \to P_n \xrightarrow{\partial_n} P_{n-1} \to \cdots \to P_1 \to P_0$ be a projective resolution of A. Set $R_n = \operatorname{im}\partial_n$, $n \geqq 1$, $R_0 = A$. An element in $\operatorname{Ext}_\Lambda^n(A, B'')$ is represented by a homomorphism $\varphi: R_n \to B''$ (see (9.4)). By construction the connecting homomorphism

$$\omega: \operatorname{Ext}_\Lambda^n(A, B'') \to \operatorname{Ext}_\Lambda^{n+1}(A, B')$$

associates with $\varphi: R_n \to B''$ the homomorphism $\psi: R_{n+1} \to B'$ in the commutative diagram

$$\begin{tikzcd}
R_{n+1} \arrow[r, tail] \arrow[d, "\psi"'] & P_n \arrow[r, two heads] \arrow[d] & R_n \arrow[d, "\varphi"] \\
B' \arrow[r, tail] & B \arrow[r, two heads] & B''
\end{tikzcd}$$

(see the remark after the proof of Theorem 2.1). It follows that the diagram

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & R_{n+1} \arrow[r] \arrow[dd, "\psi"'] & P_n \arrow[r, "\partial_n"] \arrow[d] \arrow[dd, bend right] & P_{n-1} \arrow[r] \arrow[dd] & P_{n-2} \arrow[r] \arrow[dd, equal] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[dd, equal] & A \arrow[r] \arrow[dd, equal] & 0 \\
 & & R_n \arrow[ur, "\iota"'] \arrow[dd, "\varphi", bend left] & & & & & & \\
0 \arrow[r] & B' \arrow[r] & B \arrow[r] \arrow[d] & E \arrow[r] & P_{n-2} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & A \arrow[r] & 0 \\
 & & B'' \arrow[ur] & & & & & &
\end{tikzcd}$$

is commutative, where E is the push-out of (ι, φ). Hence if

$$E: 0 \to B'' \to E_n \to E_{n-1} \to \cdots \to E_1 \to A \to 0$$

represents the element $[\varphi] \in \operatorname{Ext}^n_\Lambda(A, B'')$, then

$$E': 0 \to B' \to B \to E_n \to E_{n-1} \cdots \to E_1 \to A \to 0 \tag{9.5}$$

represents the element $\omega[\varphi] \in \operatorname{Ext}^{n+1}_\Lambda(A, B')$. We continue with an analysis of the connecting homomorphism in the *first* variable, that is, in sequence (7.1). Let $A' \rightarrowtail A \twoheadrightarrow A''$ be a short exact sequence and let $\boldsymbol{P}' \rightarrowtail \boldsymbol{P} \twoheadrightarrow \boldsymbol{P}''$ be the short exact sequence of resolutions of A', A, A'' respectively, as constructed in the proof of Theorem 6.1. We recall that the resolution $\boldsymbol{P}$ is constructed by induction using diagrams of the following form

$$\begin{tikzcd}
R'_{n+1} \arrow[r, tail] \arrow[d, tail] & R_{n+1} \arrow[r, two heads] \arrow[d, tail, "\iota_n"'] & R''_{n+1} \arrow[d, tail] \\
P'_n \arrow[r, tail] \arrow[d, two heads] & P'_n \oplus P''_n \arrow[r, two heads] \arrow[d, two heads, "\varepsilon_n"'] & P''_n \arrow[d, two heads] \arrow[dl, "\chi_n"] \\
R'_n \arrow[r, tail] & R_n \arrow[r, two heads] & R''_n
\end{tikzcd}$$

where $R'_n = \operatorname{im} \partial'_n$, $R_n = \operatorname{im} \partial_n$, $R''_n = \operatorname{im} \partial''_n$, $n \geqq 1$, $R'_0 = A'$, $R_0 = A$, $R''_0 = A''$ and ε_n is constructed via χ_n. Now let $\varphi: R'_n \to B$ represent an element of $\operatorname{Ext}^n_\Lambda(A', B)$. Then, by construction of the connecting homomorphism $\omega: \operatorname{Ext}^n_\Lambda(A', B) \to \operatorname{Ext}^{n+1}_\Lambda(A'', B)$, the element $\omega[\varphi]$ is represented by the map $\varphi\sigma$ in the diagram

$$\begin{tikzcd}
R_{n+1} \arrow[r] \arrow[d, two heads] & P'_n \oplus P''_n \arrow[r, two heads, "\pi"] & P'_n \arrow[r, two heads] & R'_n \arrow[r, "\varphi"] & B \\
R''_{n+1} \arrow[urrr, "\sigma = \sigma_{n+1}"'] & & & &
\end{tikzcd}$$

On the other hand it is clear that $\chi_n: P''_n \to R_n$ induces $\tau = \tau_{n+1}: R''_{n+1} \to R'_n$. From the construction of ε_n one sees that the sum of the two maps

$$R_{n+1} \twoheadrightarrow R''_{n+1} \xrightarrow{\sigma} R'_n \rightarrowtail R_n$$

$$R_{n+1} \twoheadrightarrow R''_{n+1} \xrightarrow{\tau} R'_n \rightarrowtail R_n$$

is zero, so that $\sigma = -\tau$.

Also, the following diagram is commutative

$$
\begin{tikzcd}
R''_{n+1} \arrow[r, tail] \arrow[d, "\tau_{n+1}"] & P''_n \arrow[r, two heads] \arrow[d, "\chi_n"] & R''_n \arrow[d, equal] \\
R'_n \arrow[r, tail] \arrow[dd, equal] & R_n \arrow[r, two heads] \arrow[d, tail, "\iota_{n-1}"] & R''_n \arrow[dd, "\sigma_n"] \\
 & P'_{n-1} \oplus P''_{n-1} \arrow[d, two heads, "\pi_{n-1}"] & \\
R'_n \arrow[r, tail] & P'_{n-1} \arrow[r, two heads] & R'_{n-1}
\end{tikzcd}
\tag{9.6}
$$

Set $\theta_n = \pi_{n-1} \circ \iota_{n-1} \circ \chi_n$. We may compose the diagrams of the form (9.6) to yield the top two rows in the following commutative diagram

$$
\begin{tikzcd}
 & & & & & & R''_1 \arrow[dr] \arrow[dd, "-\sigma_1"] & & & \\
0 \arrow[r] & R''_{n+1} \arrow[r] \arrow[dd, "(-1)^{n+1}\sigma_{n+1}"] & P''_n \arrow[r] \arrow[dd, "(-1)^n\theta_n"] & P''_{n-1} \arrow[r] \arrow[dd, "(-1)^{n-1}\theta_{n-1}"] & \cdots \arrow[r] & P''_1 \arrow[rr] \arrow[ur] \arrow[dd, "(-1)\theta_1"] & & P''_0 \arrow[r] \arrow[dd, "\chi_0"] & A'' \arrow[r] \arrow[dd, equal] & 0 \\
 & & & & & & A' \arrow[dr] & & & \\
0 \arrow[r] & R'_n \arrow[r] \arrow[d, "\varphi"] & P'_{n-1} \arrow[r] \arrow[d] & P'_{n-2} \arrow[r] \arrow[d] & \cdots \arrow[r] & P'_0 \arrow[rr] \arrow[ur] \arrow[d] & & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \arrow[r] & E \arrow[r] & P'_{n-2} \arrow[r] & \cdots \arrow[r] & P'_0 \arrow[rr] & & A \arrow[r] & A'' \arrow[r] & 0
\end{tikzcd}
$$

It follows that if

$$E' : 0 \to B \to E_n \to E_{n-1} \to \cdots \to E_1 \to A' \to 0$$

represents the element $[\varphi] \in \mathrm{Ext}^n_\Lambda(A', B)$ then $(-1)^{n+1}\omega[\varphi] \in \mathrm{Ext}^{n+1}_\Lambda(A'', B)$ is represented by

$$E'' : 0 \to B \to E_n \to E_{n-1} \to \cdots \to E_1 \to A \to A'' \to 0\,. \tag{9.7}$$

It is clear that using injectives instead of projectives one can construct a natural equivalence $\overline{\mathrm{Ext}}^n_\Lambda(-,-) \cong \mathrm{Yext}^n_\Lambda(-,-)$, An analysis of the connecting homomorphisms in the long exact sequences of $\overline{\mathrm{Ext}}$ shows that the connecting homomorphism in the first variable is simply given by the composition (9.7) whereas the connecting homomorphism in the second variable is given by the composition (9.5) together with the sign $(-1)^{n+1}$.

Finally we would like to draw the reader's attention to Exercises 9.4 to 9.7 where a direct description of the addition in $\mathrm{Yext}^n_\Lambda(A, B)$ is given. As a consequence it is then possible to construct functors $\mathrm{Yext}^n(-,-)$

possessing the properties of $\mathrm{Ext}^n(-,-)$ (i.e. the usual long exact sequences) even in abelian categories with neither enough projectives nor enough injectives.

Exercises:

9.1. Give the detailed proof that $\mathrm{Yext}^n(-,-)$ is a bifunctor, $n \geqq 2$.

9.2. Given the diagram

$$\begin{array}{ccccccccccccc} E: 0 & \to & B & \to & E_n & \to & \cdots & \to & E_1 & \to & A & \to & 0 \\ & & \downarrow{\scriptstyle\beta} & & \downarrow & & & & \downarrow & & \| & & \\ E': 0 & \to & B' & \to & E'_n & \to & \cdots & \to & E'_1 & \to & A & \to & 0 \end{array}$$

Show that E' and E_β are in the same class. Dualize.

9.3. Define the *Yoneda product*

$$\sigma: \mathrm{Ext}^n_\Lambda(A,B) \otimes \mathrm{Ext}^m_\Lambda(B,C) \to \mathrm{Ext}^{n+m}_\Lambda(A,C)$$

by "splicing" an n-sequence starting with B with an m-sequence ending with B. Show bilinearity, associativity, and existence of a unity.

9.4. Define addition in $\mathrm{Yext}^m_\Lambda(A,B)$, independently of the equivalence $\mathrm{Ext}^m_\Lambda \cong \mathrm{Yext}^m_\Lambda$. Describe a representative of $0 \in \mathrm{Yext}^m_\Lambda(A,B)$, $m \geqq 2$ and show that $\xi + 0 = \xi$, $\xi \in \mathrm{Yext}^m_\Lambda(A,B)$.

9.5. Show that if $\alpha_1, \alpha_2: A' \to A$, then

$$(\alpha_1+\alpha_2)^* = \alpha_1^* + \alpha_2^*: \mathrm{Yext}^m_\Lambda(A,B) \to \mathrm{Yext}^m_\Lambda(A',B),$$

using your definition of addition in Exercise 9.4 above, but without invoking the equivalence $\mathrm{Yext}^m_\Lambda \cong \mathrm{Ext}^m_\Lambda$. Using this property show that $\mathrm{Yext}^m_\Lambda(A,B)$ admits additive inverses.

9.6. Prove that the addition given in Exercise 9.4 above is compatible with the equivalence $\mathrm{Ext}^m_\Lambda \cong \mathrm{Yext}^m_\Lambda$ of Theorem 9.1.

9.7. Given $B: B' \rightarrowtail B \twoheadrightarrow B''$, define a homomorphism

$$\omega: \mathrm{Yext}^m_\Lambda(A,B'') \to \mathrm{Yext}^{m+1}_\Lambda(A,B')$$

by setting $\omega[E] = [E']$ where

$$E': 0 \to B' \to B \to E_m \to \cdots \to E_1 \to A \to 0,$$

(i.e. $[E'] = \sigma([E],[B])$ in terms of the Yoneda product).
Prove directly that the sequence

$$\cdots \to \mathrm{Yext}^m_\Lambda(A,B') \to \mathrm{Yext}^m_\Lambda(A,B) \to \mathrm{Yext}^m_\Lambda(A,B'') \xrightarrow{\omega} \mathrm{Yext}^{m+1}_\Lambda(A,B') \to \cdots$$

is exact. Does this sequence coincide with (7.4)? Also, define a connecting homomorphism for Yext in the first variable and deduce an exact sequence corresponding to (7.1).

9.8. Construct an equivalence $\bar{\theta}_n: \mathrm{Yext}^n_\Lambda \xrightarrow{\sim} \overline{\mathrm{Ext}}^n_\Lambda$ by proceeding dually to the proof of Theorem 9.1. Identify the connecting homomorphisms in the exact sequences (8.3) and (8.4) in terms of n-extensions. Express the equivalence

$\Phi^n: \mathrm{Ext}^n \xrightarrow{\sim} \overline{\mathrm{Ext}}^n$ as constructed in the proof of Proposition 8.1 in terms of θ^n (Theorem 9.1) and $\bar\theta^n$. Show that Φ^n is compatible with the connecting homomorphism in the first variable.

9.9. Let $A' \rightarrowtail A \to A''$ and $B' \rightarrowtail B \twoheadrightarrow B''$ be two short exact sequences. Show that

$$\begin{tikzcd} \mathrm{Ext}^n_\Lambda(A', B'') \arrow[r, "\omega"] \arrow[d, "\omega"] & \mathrm{Ext}^{n+1}_\Lambda(A', B') \arrow[d, "\omega"] \\ \mathrm{Ext}^{n+1}_\Lambda(A'', B'') \arrow[r, "\omega"] & \mathrm{Ext}^{n+2}_\Lambda(A'', B') \end{tikzcd}$$

is anticommutative.

9.10. Show that the diagram of abelian group homomorphisms

$$\begin{tikzcd} A \arrow[r, two heads] & B \arrow[d, tail] \\ & C \end{tikzcd}$$

may always be embedded in a square, which is both a pull-back and a push-out. Does this property remain valid if we replace abelian groups by Λ-modules?

10. Another Characterization of Derived Functors

We have defined the functor $\mathrm{Ext}^q_\Lambda(A, -)$ as the q-th right derived functor of $\mathrm{Hom}_\Lambda(A, -)$. We now show how the left derived functors of any right exact functor may be obtained by means of the Ext-functors (Corollary 10.2).

We use, as before, the symbol $[S, S']$ to denote the set (or class) of natural transformations of the functor S into the functor S'. Clearly for natural transformations of functors into an additive category one has a well defined notion of addition.

Let $R_q \overset{\mu}{\rightarrowtail} P_{q-1} \to P_{q-2} \to \cdots \to P_0 \twoheadrightarrow A$ be an exact sequence with $P_0, P_1, \ldots, P_{q-1}$ projective. To the additive functor $T: \mathfrak{M}_\Lambda \to \mathfrak{Ab}$ we define abelian groups $\tilde L_q TA$ as follows

$$\begin{aligned} \tilde L_q TA &= \ker(\mu_*: TR_q \to TP_{q-1}), \qquad q = 1, 2, \ldots, \\ \tilde L_0 TA &= TA. \end{aligned}$$

For T a right exact functor we know by Proposition 5.5 that the sequence

$$0 \to L_q TA \to TR_q \xrightarrow{\mu_*} TP_{q-1}$$

is exact. Hence in this case we conclude

$$\tilde L_q TA = L_q TA, \qquad q = 0, 1, \ldots . \tag{10.1}$$

In particular this shows that $\tilde L_q TA$ does not depend upon the choice of the modules $P_0, P_1, \ldots, P_{q-1}$ in case T is right exact. It is an immediate corollary of the following result that this assertion holds true for arbitrary T.

Theorem 10.1. *Let $T: \mathfrak{M}_\Lambda \to \mathfrak{Ab}$ be an additive covariant functor and let A be a Λ-module. Then there are natural isomorphisms*

$$\Gamma: [\mathrm{Ext}^q_\Lambda(A, -), T] \xrightarrow{\sim} \tilde{L}_q TA, \qquad q = 0, 1, \ldots .$$

In case $T: \mathfrak{M}_\Lambda \to \mathfrak{Ab}$ is right exact the assertion (10.1) immediately yields the following characterization of left derived functors.

Corollary 10.2. *Let $T: \mathfrak{M}_\Lambda \to \mathfrak{Ab}$ be a right exact functor and let A be a Λ-module. Then there are natural isomorphisms*

$$\Gamma: [\mathrm{Ext}^q_\Lambda(A, -), T] \xrightarrow{\sim} L_q TA, \qquad q = 0, 1, \ldots . \quad \square$$

Proof of Theorem 10.1. Since $\mathrm{Hom}_\Lambda(-, B)$ is left exact we may apply Proposition 5.8 to obtain the following commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{Hom}_\Lambda(P_{q-1}, R_q) \arrow[r] \arrow[d, "\mu_*"] & \mathrm{Hom}_\Lambda(R_q, R_q) \arrow[r, "\Delta"] \arrow[d, "\mu_*"] & \mathrm{Ext}^q_\Lambda(A, R_q) \arrow[r] \arrow[d, "\mu_*"] & 0 \\
\mathrm{Hom}_\Lambda(P_{q-1}, P_{q-1}) \arrow[r] & \mathrm{Hom}_\Lambda(R_q, P_{q-1}) \arrow[r, "\Delta"] & \mathrm{Ext}^q_\Lambda(A, P_{q-1}) \arrow[r] & 0
\end{tikzcd}$$

Consider the element $\eta \in \mathrm{Ext}^q_\Lambda(A, R_q)$ defined by $1: R_q \to R_q$. Since $\mu: R_q \to P_{q-1}$ extends to $1: P_{q-1} \to P_{q-1}$ we have $\mu_*(\eta) = 0$. Now let $\Phi: \mathrm{Ext}^q(A, -) \to T$ be a natural transformation. We look at the diagram

$$\begin{tikzcd}
\mathrm{Ext}^q_\Lambda(A, R_q) \arrow[r, "\Phi_{R_q}"] \arrow[d, "\mu_*"] & T(R_q) \arrow[d, "\mu_*"] \\
\mathrm{Ext}^q_\Lambda(A, P_{q-1}) \arrow[r, "\Phi_{P_{q-1}}"] & T(P_{q-1})
\end{tikzcd}$$

Since $\mu_*(\eta) = 0$, the element $\xi = \Phi_{R_q}(\eta)$ is in the kernel of

$$\mu_*: T(R_q) \to T(P_{q-1}),$$

hence an element of $\tilde{L}_q TA$. Thus, given the natural transformation Φ, we have assigned to Φ an element $\xi = \Gamma(\Phi) \in \tilde{L}_q TA$. Clearly this map $\Gamma: [\mathrm{Ext}^q_\Lambda(A, -), T] \to \tilde{L}_q TA$ is a homomorphism.

Conversely, suppose the element $\xi \in \tilde{L}_q TA$ is given. We have to define a natural transformation $\Phi = \Phi^\xi$, such that

$$\Phi_{R_q}(\eta) = \xi \in T(R_q). \tag{10.2}$$

We first show that this rule determines Φ, if Φ is to be a natural transformation. For let M be an arbitrary Λ-module and let $\varrho \in \mathrm{Ext}^q_\Lambda(A, M)$. Since

$$\mathrm{Hom}_\Lambda(P_{q-1}, M) \to \mathrm{Hom}_\Lambda(R_q, M) \to \mathrm{Ext}^q_\Lambda(A, M) \to 0$$

is exact, $\varrho \in \mathrm{Ext}_\Lambda^q(A, M)$ can be represented by a homomorphism

$$\sigma : R_q \to M .$$

We consider the square

$$\begin{tikzcd}
\mathrm{Ext}_\Lambda^q(A, R_q) \arrow[r, "\Phi_{R_q}"] \arrow[d, "\sigma_*"] & T(R_q) \arrow[d, "\sigma_*"] \\
\mathrm{Ext}_\Lambda^q(A, M) \arrow[r, "\Phi_M"] & T(M)
\end{tikzcd}$$

Then commutativity forces $\Phi_M(\varrho) = \Phi_M \sigma_*(\eta) = \sigma_* \Phi_{R_q}(\eta) = \sigma_*(\xi)$. We next show that we obtain the same value for $\Phi_M(\varrho)$, if we choose another representative σ' of ϱ. Consider $\sigma - \sigma' : R_q \to M$; it must factor through P_{q-1}. Hence $(\sigma - \sigma')_* : TR_q \xrightarrow{\mu_*} TP_{q-1} \to TM$; but since $\xi \in \tilde{L}_q TA = \ker \mu_*$, we have $(\sigma - \sigma')_*(\xi) = 0$, so that $\sigma_*(\xi) = \sigma'_*(\xi)$. Finally we show that the Φ we have defined is indeed a natural transformation. Let $\varphi : M \to N$ be a homomorphism then the diagram

$$\begin{tikzcd}
\mathrm{Ext}_\Lambda^q(A, M) \arrow[r, "\Phi_M"] \arrow[d, "\varphi_*"] & TM \arrow[d, "\varphi_*"] \\
\mathrm{Ext}_\Lambda^q(A, N) \arrow[r, "\Phi_N"] & TN
\end{tikzcd}$$

must be shown to be commutative. Since $\varphi\sigma : R_q \to N$ represents

$$\varphi_*(\varrho) \in \mathrm{Ext}_\Lambda^q(A, N)$$

we have $\Phi_N(\varphi_*(\varrho)) = (\varphi\sigma)_*(\xi) = \varphi_* \sigma_*(\xi) = \varphi_* \Phi_M(\varrho)$. □

We remark that the assertion of Theorem 10.1 for $q = 0$ is nothing but the Yoneda Lemma (Proposition II. 4.1) applied to additive functors.

We may apply Theorem 10.1 to find a description of the natural transformations $\Phi : \mathrm{Ext}_\Lambda^1(A, -) \to \mathrm{Ext}_\Lambda^1(A', -)$. It is clear that any homomorphism $\alpha : A' \to A$ will induce such a natural transformation. Proposition 10.3 says that *all* natural transformations are of this kind.

Proposition 10.3. *Every natural transformation*

$$\Phi : \mathrm{Ext}_\Lambda^1(A, -) \to \mathrm{Ext}_\Lambda^1(A', -)$$

is induced by a homomorphism $\alpha : A' \to A$.

Proof. Since $\mathrm{Ext}_\Lambda^1(A', -)$ is additive we may apply Theorem 10.1. We have that $[\mathrm{Ext}_\Lambda^1(A, -), \mathrm{Ext}_\Lambda^1(A', -)] \xrightarrow{\sim} \tilde{L}_1(\mathrm{Ext}_\Lambda^1(A', -))(A)$. Let $R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ be a projective presentation; then

$$\tilde{L}_1(\mathrm{Ext}_\Lambda^1(A', -))(A) = \ker(\mu_* : \mathrm{Ext}_\Lambda^1(A', R) \to \mathrm{Ext}_\Lambda^1(A', P)) .$$

On the other hand the long exact Ext-sequence (7.4) yields

$$\cdots \to \mathrm{Hom}(A', P) \to \mathrm{Hom}(A', A) \xrightarrow{\omega} \mathrm{Ext}_\Lambda^1(A', R) \xrightarrow{\mu_*} \mathrm{Ext}_\Lambda^1(A', P) \to \cdots \quad (10.3)$$

whence it follows that any natural transformation Φ may be described by a homomorphism $\alpha: A' \to A$. It is to be shown that the natural transformation Φ *described* by α is indeed the one *induced* by α. Let $R' \rightarrowtail P' \twoheadrightarrow A'$ be a projective presentation of A' and consider the following diagram

$$\begin{tikzcd}
 & \mathrm{Hom}_\Lambda(A',R) \arrow[r] \arrow[d, tail] & \mathrm{Hom}_\Lambda(A',P) \arrow[r] \arrow[d, tail] & \mathrm{Hom}_\Lambda(A',A) \arrow[d, tail] \\
 & \mathrm{Hom}_\Lambda(P',R) \arrow[r] \arrow[d] & \mathrm{Hom}_\Lambda(P',P) \arrow[r, two heads] \arrow[d] & \mathrm{Hom}_\Lambda(P',A) \arrow[d] \\
1 \in \mathrm{Hom}_\Lambda(R,R) \arrow[r, "\theta^*"] \arrow[d] \arrow[d, squiggly, shift right=8] & \mathrm{Hom}_\Lambda(R',R) \arrow[r] \arrow[d, two heads] & \mathrm{Hom}_\Lambda(R',P) \arrow[r] \arrow[d, two heads] & \mathrm{Hom}_\Lambda(R',A) \arrow[d, two heads] \\
\eta \in \mathrm{Ext}^1_\Lambda(A,R) \arrow[r, "\alpha^*"] & \mathrm{Ext}^1_\Lambda(A',R) \arrow[r] & \mathrm{Ext}^1_\Lambda(A',P) \arrow[r] & \mathrm{Ext}^1_\Lambda(A',A),
\end{tikzcd}$$

where θ^* is explained below.

We have to show that the natural transformation α^* induced by α has the property that $\alpha^*(\eta) = \xi = \omega(\alpha)$. By the remark after (10.2) this is sufficient. In order to describe α^* we choose χ, θ such that the following diagram commutes

$$\begin{tikzcd}
R' \arrow[r, tail] \arrow[d, "\theta"] & P' \arrow[r, two heads] \arrow[d, "\chi"] & A' \arrow[d, "\alpha"] \\
R \arrow[r, tail] & P \arrow[r, two heads] & A
\end{tikzcd}$$

By construction of the connecting homomorphism ω we then have $\omega(\alpha) = [\theta] = [\theta^*(1_R)] = \alpha^*(\eta) = \xi$. $\square$

It follows from the exact sequence (10.3) that the homomorphism $\alpha: A' \to A$ is determined up to a homomorphism factoring through P.

Theorem 10.4. *For $\alpha: A' \to A$ the induced natural transformation $\alpha^*: \mathrm{Ext}^1_\Lambda(A, -) \to \mathrm{Ext}^1_\Lambda(A', -)$ is a natural equivalence if and only if α is of the form $\alpha = \pi\sigma\iota$,*

$$\alpha: A' \overset{\iota_{A'}}{\rightarrowtail} A' \oplus Q \xrightarrow[\sim]{\sigma} A \oplus P \overset{\pi_A}{\twoheadrightarrow} A \tag{10.4}$$

where P, Q are projective.

In case (10.4) holds we say that α is an *isomorphism modulo projectives.*

Proof. If α is of the given form, then α^* is clearly a natural equivalence. To prove the converse first note that if $\alpha^*: \mathrm{Ext}^1_\Lambda(A, -) \to \mathrm{Ext}^1_\Lambda(A', -)$ is an equivalence, then $\alpha^*: \mathrm{Ext}^q_\Lambda(A, -) \to \mathrm{Ext}^q_\Lambda(A', -)$ is an equivalence for all $q \geq 1$. Now suppose that $\alpha: A' \to A$ is epimorphic. Let $\ker\alpha = A''$ and consider the short exact sequence $A'' \rightarrowtail A' \twoheadrightarrow A$. For any B we have a long exact sequence

$$\begin{aligned}0 \to \operatorname{Hom}(A,B) \to \operatorname{Hom}(A',B) \to \operatorname{Hom}(A'',B) \to \operatorname{Ext}^1_\Lambda(A,B) \xrightarrow[\sim]{\alpha^*} \operatorname{Ext}^1_\Lambda(A',B)\\ \to \operatorname{Ext}^1_\Lambda(A'',B) \to \operatorname{Ext}^2_\Lambda(A,B) \xrightarrow[\sim]{\alpha^*} \operatorname{Ext}^2_\Lambda(A',B) \to \cdots .\end{aligned} \tag{10.5}$$

Thus $\operatorname{Ext}^1_\Lambda(A'',B)=0$ for all B, so that A'' is projective (Corollary III. 5.5).

Now choose $B=A''$ in sequence (10.5) and observe that

$$\operatorname{Hom}(A',A'') \to \operatorname{Hom}(A'',A'')$$

is surjective. The identity of A'' therefore is induced by a map $A' \to A''$. Hence the exact sequence $A'' \rightarrowtail A' \twoheadrightarrow A$ splits, i.e. $A'=A\oplus A''$. We have $\alpha: A' \xrightarrow[\sim]{\sigma} A\oplus P \xrightarrow{\pi_A} \!\!\!\!\rightarrow A$ with $P=A''$ projective.

For the general case take a projective presentation $\varepsilon: Q \twoheadrightarrow A$ and consider the epimorphism

$$\bar{\alpha}=\langle \alpha, \varepsilon\rangle : A'\oplus Q \twoheadrightarrow A .$$

Then $\bar{\alpha}\iota=\alpha$, where $\iota=\iota_{A'}: A' \rightarrowtail A'\oplus Q$; moreover ι^* is obviously a natural equivalence, so that $\bar{\alpha}^*: \operatorname{Ext}^1_\Lambda(A,-) \to \operatorname{Ext}^1_\Lambda(A'\oplus Q,-)$ is a natural equivalence. By the previous argument $\bar{\alpha}$ is of the form

$$\bar{\alpha}: A'\oplus Q \xrightarrow[\sim]{\sigma} A\oplus P \xrightarrow{\pi_A} A ,$$

so that $\alpha=\pi\sigma\iota$ as required. $\square$

Exercises:

10.1. Dualize the theorems of this section.

10.2. Prove that $\beta: B \to B''$ induces isomorphisms $\beta_*: \Pi P(-,B) \xrightarrow{\sim} \Pi P(-,B'')$, $\beta_{*1}: \Pi P_1(-,B) \xrightarrow{\sim} \Pi P_1(-,B'')$ if and only if β is an isomorphism modulo projectives. Strengthen this result by removing the condition on β_{*1}. Dualize.

10.3. Show that $[\operatorname{Ext}^q_\Lambda(A,-), L_0 T] \cong L_q T(A)$, for any additive functor T.

10.4. Show that $[\operatorname{Ext}^q_\Lambda(A,-), \operatorname{Ext}^1_\Lambda(A',-)] = \Pi P_{q-1}(A',A)$, $q \geqq 1$. Dualize.

11. The Functor $\operatorname{Tor}^\Lambda_n$.

Here we generalise the bifunctor $\operatorname{Tor}^\Lambda(-,-)$ defined in Section III. 8. Let A be a right Λ-module and B a left Λ-module.

By Proposition III. 7.3 the functor $A\otimes_\Lambda -$ is additive, indeed right exact. We therefore can define

Definition. $\operatorname{Tor}^\Lambda_n(A,-)=L_n(A\otimes_\Lambda -)$, $n=0,1,\ldots$.

We briefly recall how the abelian group $\operatorname{Tor}^\Lambda_n(A,B)$ is calculated. Choose any projective resolution $\boldsymbol{P}$ of B, form the chain complex $A\otimes_\Lambda \boldsymbol{P}$

and then take homology,

$$\mathrm{Tor}_n^A(A,B)=H_n(A\otimes_A \boldsymbol{P}).$$

It follows from Proposition 5.2 that

$$\mathrm{Tor}_0^A(A,B)=A\otimes_A B.$$

Similarly to Proposition 7.1, we prove that

$$\mathrm{Tor}_1^A(A,B)\cong \mathrm{Tor}^A(A,B),$$

as defined in Chapter III. 8.

Given a short exact sequence $B'\rightarrowtail B\twoheadrightarrow B''$ we obtain the *long exact* Tor-*sequence in the second variable*

$$\begin{aligned}\cdots\to\mathrm{Tor}_n^A(A,B')\to\mathrm{Tor}_n^A(A,B)\to\mathrm{Tor}_n^A(A,B'')\xrightarrow{\omega_n}\mathrm{Tor}_{n-1}^A(A,B)\to\cdots\\ \cdots\to\mathrm{Tor}_1^A(A,B'')\xrightarrow{\omega_1}A\otimes_A B'\to A\otimes_A B\to A\otimes_A B''\to 0\end{aligned}\tag{11.1}$$

by Theorem 6.1. Sequence (11.1) is natural with respect to the short exact sequence. By Proposition 5.3 it follows that, for P projective, $\mathrm{Tor}_n^A(A,P)=0$ for $n=1,2,\ldots$.

A homomorphism $\alpha:A\to A'$ clearly induces a map

$$\alpha_*:\mathrm{Tor}_n^A(A,B)\to\mathrm{Tor}_n^A(A',B),$$

which makes $\mathrm{Tor}_n^A(-,B)$, $n=0,1,\ldots$, into a functor. Indeed we have – as the reader may show – that $\mathrm{Tor}_n^A(-,-)$, $n=0,1,\ldots$, is a bifunctor (compare Proposition 7.3). Applying Proposition 6.2 to the natural transformation $\alpha_*:A\otimes_A-\to A'\otimes_A-$ we deduce that sequence (11.1) is natural with respect to the first variable.

Given the short exact sequence $A'\rightarrowtail A\twoheadrightarrow A''$ the sequence of functors $A'\otimes_A-\to A\otimes_A-\to A''\otimes_A-$ is exact on projectives. It follows then by Theorem 6.3 that there exists a *long exact* Tor-*sequence in the first variable*,

$$\begin{aligned}\cdots\to\mathrm{Tor}_n^A(A',B)\to\mathrm{Tor}_n^A(A,B)\to\mathrm{Tor}_n^A(A'',B)\xrightarrow{\omega_n}\mathrm{Tor}_{n-1}^A(A',B)\to\cdots\\ \cdots\to\mathrm{Tor}_1^A(A'',B)\xrightarrow{\omega_1}A'\otimes_A B\to A\otimes_A B\to A''\otimes_A B\to 0.\end{aligned}\tag{11.2}$$

This sequence is natural both with respect to the short exact sequence and with respect to B.

In Section 8 we have shown that Ext_A^n may also be obtained as a derived functor in the second variable. Similarly we have

Proposition 11.1. $\mathrm{Tor}_n^A(A,B)=L_n(-\otimes_A B)(A)$, $n=0,1,\ldots$.

We may express the assertion of Proposition 11.1 by saying that the bifunctor $\mathrm{Tor}(-,-)$ is *balanced;* it may be computed via a projective resolution of the first or a projective resolution of the second variable.

We leave the proof, which is analogous to that of Proposition 8.1, to the reader. As a consequence, we have that $\operatorname{Tor}_n^\Lambda(P, B) = 0$ for P projective and $n = 1, 2, \ldots$, though this, of course, follows from the first definition of Tor and the fact that $P \otimes_\Lambda -$ is exact.

We finally give another characterization of $\operatorname{Tor}_n^\Lambda$ using Corollary 10.2.

Theorem 11.2. *There are natural isomorphisms*

$$\Gamma : [\operatorname{Ext}_\Lambda^n(B, -), A \otimes_\Lambda -] \xrightarrow{\sim} \operatorname{Tor}_n^\Lambda(A, B) \quad \textit{for} \quad n = 0, 1, \ldots .$$

Proof. We only have to observe that $A \otimes_\Lambda -$ is right exact and that $\operatorname{Tor}_n^\Lambda(A, B) = L_n(A \otimes_\Lambda -)(B)$. The assertion then follows from Corollary 10.2. □

Exercises:

11.1. Write out a complete proof that $\operatorname{Tor}_n^\Lambda(-, -)$ is a bifunctor.

11.2. Prove Proposition 11.1.

11.3. Show that $\operatorname{Tor}_n^\Lambda(A, B) = \ker(R_n \otimes_\Lambda B \to P_{n-1} \otimes_\Lambda B)$, where

$$\begin{tikzcd} \cdots \arrow[r] & P_n \arrow[rr] \arrow[dr] & & P_{n-1} \arrow[r] & \cdots \arrow[r] & P_0 \\ & & R_n \arrow[ur] & & & \end{tikzcd}$$

is a projective resolution of A.

11.4. Show that, if P is flat, then $\operatorname{Tor}_n^\Lambda(P, A) = 0 = \operatorname{Tor}_n^\Lambda(A, P)$ for all A and $n \geqq 1$.

11.5. Let $Q : \cdots \to Q_n \to Q_{n-1} \to \cdots \to Q_0$ be a flat resolution of A (i.e. the sequence is exact in dimensions $n \geqq 1$, $H_0(Q) = A$, and each Q_n is flat). Show that $\operatorname{Tor}_n^\Lambda(A, B) = H_n(Q \otimes_\Lambda B)$.

11.6. Let A be a fixed Λ-module. Show that if $\operatorname{Ext}_\Lambda^n(A, -) = 0$, then $\operatorname{Tor}_n^\Lambda(A, -) = 0$, $n > 0$. Show that the converse is false.

12. Change of Rings

In this final section of Chapter IV we study the effect of a change of rings on the functor $\operatorname{Ext}_\Lambda^n$. However, we will make further applications of a change of rings in Chapters VI and VII, and hence we record in this section certain results for future use.

Let Λ, Λ' be two rings and let $U : \mathfrak{M}_{\Lambda'} \to \mathfrak{M}_\Lambda$ be a functor. Then we may restate Proposition II. 10.2 (and its dual) in this context as follows.

Theorem 12.1. (i) *If U has a left adjoint $F : \mathfrak{M}_\Lambda \to \mathfrak{M}_{\Lambda'}$ and if U preserves surjections (i.e., if U is exact), then F sends projectives to projectives.*

(ii) *If U has a right adjoint $\bar{F} : \mathfrak{M}_\Lambda \to \mathfrak{M}_{\Lambda'}$ and if U preserves injections (i.e., if U is exact), then $\bar{F}$ sends injectives to injectives.* □

Now let U satisfy the hypotheses of Theorem 12.1 (i), let A be a Λ-module and let B' be a Λ'-module. Choose a projective resolution

$$\boldsymbol{P}: \cdots \to P_n \to P_{n-1} \to \cdots \to P_0$$

of A, and consider

$$F\boldsymbol{P}: \cdots \to FP_n \to FP_{n-1} \to \cdots \to FP_0 .$$

By Theorem 12.1, $F\boldsymbol{P}$ is a projective complex (of Λ'-modules), but it is not in general acyclic. However, since F is right exact,

$$H_0(F\boldsymbol{P}) = FA .$$

Let $\boldsymbol{P}'$ be a projective resolution of FA. By Theorem 4.1 there exists a chain map $\varphi: F\boldsymbol{P} \to \boldsymbol{P}'$, determined up to homotopy, inducing the identity on FA. Combining this with the adjugant η: $F \dashv U$, we obtain a cochain map

$$\mathrm{Hom}_{\Lambda'}(\boldsymbol{P}', B') \xrightarrow{\varphi^*} \mathrm{Hom}_{\Lambda'}(F\boldsymbol{P}, B') \xrightarrow{\tilde{\eta}} \mathrm{Hom}_{\Lambda}(\boldsymbol{P}, UB')$$

which gives rise to homomorphisms

$$\Phi^n: \mathrm{Ext}^n_{\Lambda'}(FA, B') \to \mathrm{Ext}^n_{\Lambda}(A, UB'), \qquad n = 0, 1, 2, \ldots \tag{12.1}$$

which are easily seen to be natural in A and B'. Thus Φ is a natural transformation, uniquely determined by the adjugant η.

We remark that if F preserves injections (i.e., if F is exact), then $F\boldsymbol{P}$ is a projective resolution of FA and Φ is then a natural equivalence.

Now let $\delta: FU \to 1$ be the co-unit of the adjugant η (see Proposition II. 7.2). By (12.1) we obtain homomorphisms $\theta^n = \Phi^n \delta^{*n}$,

$$\begin{gathered}\theta^n: \mathrm{Ext}^n_{\Lambda'}(A', B') \xrightarrow{\delta^{*n}} \mathrm{Ext}^n_{\Lambda'}(FUA', B') \xrightarrow{\Phi^n} \mathrm{Ext}^n_{\Lambda}(UA', UB'), \\ n = 0, 1, 2, \ldots, \end{gathered} \tag{12.2}$$

for any Λ'-module A', and θ^n is plainly natural in A' and B'. If F is exact, then θ^n is equivalent to δ^{*n}. The reader is referred to Exercise 12.5 for a different description of θ^n.

The theory described may be applied to the following specific pair of adjoint functors. Let $\gamma: \Lambda \to \Lambda'$ be a ring-homomorphism. Then any Λ'-module M' may be given the structure of a Λ-module *via* γ by defining

$$\lambda m' = (\gamma\lambda)\, m', \qquad \lambda \in \Lambda, \qquad m' \in M'. \tag{12.3}$$

We denote this Λ-module by $U^\gamma M'$ so that $U^\gamma: \mathfrak{M}_{\Lambda'} \to \mathfrak{M}_\Lambda$ is a functor called the *change-of-rings functor induced by* $\gamma: \Lambda \to \Lambda'$. Then U^γ obviously preserves surjections and it is easily verified that U^γ has a left adjoint $F: \mathfrak{M}_\Lambda \to \mathfrak{M}_{\Lambda'}$, given by

$$FM = \Lambda' \otimes_\Lambda M, \qquad M \text{ in } \mathfrak{M}_\Lambda ; \tag{12.4}$$

here Λ' is regarded as a right Λ-module via γ, and FM acquires the structure of a Λ'-module through the Λ'-module structure on Λ'. Thus, for U^γ, we have natural transformations

$$\begin{aligned} &\Phi: \operatorname{Ext}^n_{\Lambda'}(\Lambda' \otimes_\Lambda A, B') \to \operatorname{Ext}^n_\Lambda(A, U^\gamma B'), \\ &\theta: \operatorname{Ext}^n_{\Lambda'}(A', B') \to \operatorname{Ext}^n_\Lambda(U^\gamma A', U^\gamma B'). \end{aligned} \tag{12.5}$$

We record for future reference

Proposition 12.2. *If Λ' is flat as a right Λ-module via γ, then*

$$\Phi: \operatorname{Ext}^n_{\Lambda'}(\Lambda' \otimes_\Lambda A, B') \xrightarrow{\sim} \operatorname{Ext}^n_\Lambda(A, U^\gamma B')$$

is a natural equivalence.

Proof. This is clear since the functor F given by (12.4) is then exact. □

We may apply Theorem 12.1 (ii) in essentially the same way. We leave the details to the reader and simply assert that, with U and $\bar{F}$ satisfying the hypothesis of Theorem 12.1 (ii), we get natural transformations

$$\begin{aligned} &\bar{\Phi}: \operatorname{Ext}^n_{\Lambda'}(A', \bar{F}B) \to \operatorname{Ext}^n_\Lambda(UA', B), \\ &\bar{\theta}: \operatorname{Ext}^n_{\Lambda'}(A', B') \to \operatorname{Ext}^n_\Lambda(UA', UB'). \end{aligned} \tag{12.6}$$

Moreover, $\bar{\Phi}$ is a natural equivalence if $\bar{F}$ preserves surjections (i.e., if $\bar{F}$ is exact).

The case of special interest to us involves the same functor

$$U^\gamma: \mathfrak{M}_{\Lambda'} \to \mathfrak{M}_\Lambda$$

as above. For U^γ obviously preserves injections and, as the reader will readily verify, U^γ admits the right adjoint $\bar{F}: \mathfrak{M}_\Lambda \to \mathfrak{M}_{\Lambda'}$, given by

$$\bar{F}M = \operatorname{Hom}_\Lambda(\Lambda', M), \qquad M \text{ in } \mathfrak{M}_\Lambda; \tag{12.7}$$

here Λ' is regarded as a left Λ-module via γ, and $\bar{F}M$ acquires the structure of a Λ'-module through the right Λ'-module structure on Λ'. Thus, we have natural transformations

$$\begin{aligned} &\bar{\Phi}: \operatorname{Ext}^n_{\Lambda'}(A', \operatorname{Hom}_\Lambda(\Lambda', B)) \to \operatorname{Ext}^n_\Lambda(U^\gamma A', B), \\ &\bar{\theta}: \operatorname{Ext}^n_{\Lambda'}(A', B') \to \operatorname{Ext}^n_\Lambda(U^\gamma A', U^\gamma B'). \end{aligned} \tag{12.8}$$

Again we record for future reference

Proposition 12.3. *If Λ' is projective as a left Λ-module via γ, then $\bar{\Phi}: \operatorname{Ext}^n_{\Lambda'}(A', \operatorname{Hom}_\Lambda(\Lambda', B)) \xrightarrow{\sim} \operatorname{Ext}^n_\Lambda(U^\gamma A', B)$ is a natural equivalence.* □

We also remark that the natural transformations $\theta, \bar{\theta}$ of (12.2), (12.6) are, in fact, defined whenever U is exact and do not depend on the existence of adjoints to U (though the descriptions we have given, in terms of adjoints, facilitate their study). Indeed they have a very obvious definition in terms of the Yoneda description of Ext^n. Thus, in particular, we have (see Exercises 12.5, 12.6)

Proposition 12.4. *The natural transformation* θ *of* (12.5) *coincides with the natural transformation* $\bar{\theta}$ *of* (12.8). □

Finally, we record for application in Chapters VI and VII the following further consequences of Proposition II. 10.2.

Theorem 12.5. *Let* $U^\gamma: \mathfrak{M}_{\Lambda'} \to \mathfrak{M}_\Lambda$ *be the change-of-rings functor induced by* $\gamma: \Lambda \to \Lambda'$. *Then* (i) *if* Λ' *is a projective (left)* Λ*-module via* γ, U^γ *sends projectives to projectives;* (ii) *if* Λ' *is a flat right* Λ*-module via* γ, U^γ *sends injectives to injectives.*

Proof. (i) The hypothesis implies that $\bar{F}$ preserves epimorphisms, so U^γ sends projectives to projectives. (ii) The hypothesis implies that F preserves monomorphisms, so U^γ sends injectives to injectives. □

Exercises:

12.1. Apply the theory of this section to a discussion of Tor in place of Ext.

12.2. Let K be a field and let $K[\ ]$ be the *group algebra over* K functor. Let $\varphi: \pi \rightarrowtail \pi'$ be a monomorphism of groups and let $\gamma = K[\varphi]: K[\pi] \to K[\pi']$. Show that $K[\pi']$ is free as a left or right $K[\pi]$-module via γ.

12.3. Show that $\delta: FU^\gamma A' \to A'$ is surjective, where F, U^γ are given by (12.4), (12.3). Hence embed θ of (12.5) in an exact sequence when Λ' is flat as a right Λ-module via γ.

12.4. Give details of the definitions of $\bar{\Phi}, \bar{\theta}$ in (12.6). Carry out the exercise corresponding to Exercise 12.3.

12.5. Identify θ in (12.2) with the homomorphism described as follows (U being *exact*). Let $\boldsymbol{P}'$ be a projective resolution of A' and let $\boldsymbol{P}$ be a projective resolution of UA'. Then we have a chain map $\psi: \boldsymbol{P} \to U\boldsymbol{P}'$ inducing the identity on UA' and we form

$$\mathrm{Hom}_{\Lambda'}(\boldsymbol{P}', B') \to \mathrm{Hom}_\Lambda(U\boldsymbol{P}', UB') \xrightarrow{\psi^*} \mathrm{Hom}_\Lambda(\boldsymbol{P}, UB').$$

Pass to cohomology.

12.6. Carry out a similar exercise for $\bar{\theta}$ in (12.6). Deduce that $\theta = \bar{\theta}$.

12.7. Show that θ (12.2) and $\bar{\theta}$ (12.6) are compatible with the connecting homomorphisms.

V. The Künneth Formula

The Künneth formula has its historic origin in algebraic topology. Given two topological spaces X and Y, we may ask how the (singular) homology groups of their topological product $X \times Y$ is related to the homology groups of X and Y. This question may be answered by separating the problem into two parts. If $\boldsymbol{C}(X)$, $\boldsymbol{C}(Y)$, $\boldsymbol{C}(X \times Y)$ stand for the singular chain complexes of $X, Y, X \times Y$ respectively, then a theorem due to Eilenberg-Zilber establishes that the chain complex $\boldsymbol{C}(X \times Y)$ is canonically homotopy-equivalent to the *tensor product* of the chain complexes $\boldsymbol{C}(X)$ and $\boldsymbol{C}(Y)$,

$$\boldsymbol{C}(X \times Y) \simeq \boldsymbol{C}(X) \otimes \boldsymbol{C}(Y) ;$$

(for the precise definition of the tensor product of two chain complexes, see Section 1, Example (a)). Thus the problem is reduced to the purely algebraic problem of relating the homology groups of the tensor product of $\boldsymbol{C}(X)$ and $\boldsymbol{C}(Y)$ to the homology groups of $\boldsymbol{C}(X)$ and $\boldsymbol{C}(Y)$. This relation is furnished by the *Künneth formula*, whose validity we establish under much more general circumstances than would be required by the topological situation. For, in that case, we are concerned with free chain complexes of $\mathbb{Z}$-modules; the argument we give permits arbitrary chain complexes $\boldsymbol{C}$, $\boldsymbol{D}$ of Λ-modules, where Λ is any p.i.d., provided only that one of $\boldsymbol{C}$, $\boldsymbol{D}$ is flat. This generality allows us then to subsume under the same theory not only the Künneth formula in its original context but also another important result drawn from algebraic topology, the *universal coefficient theorem in homology*.

When the Künneth formula is viewed in a purely algebraic context, it is natural to ask whether there is a similar ("dual") formula relating to Hom instead of the tensor product. It turns out that this is the case, and we give such a development in Section 3. Here the topological motivation is not so immediate, but we do get, by specialization, the *universal coefficient theorem in cohomology*.

Applications are given in Section 4. Others will be found in Chapter VI.

1. Double Complexes

Definition. A *double complex of chains* $\boldsymbol{B}$ over Λ is an object in $\mathfrak{M}_\Lambda^{\mathbb{Z}\times\mathbb{Z}}$, together with two endomorphisms $\partial' : \boldsymbol{B} \to \boldsymbol{B}$, $\partial'' : \boldsymbol{B} \to \boldsymbol{B}$ of degree $(-1, 0)$ and $(0, -1)$ respectively, called the *differentials*, such that

$$\partial'\partial' = 0\,, \qquad \partial''\partial'' = 0\,, \qquad \partial''\partial' + \partial'\partial'' = 0\,. \tag{1.1}$$

In other words, we are given a bigraded family of Λ-modules $\{B_{p,q}\}$, $p, q \in \mathbb{Z}$ and two families of Λ-module homomorphisms

$$\{\partial'_{p,q} : B_{p,q} \to B_{p-1,q}\}\,, \quad \{\partial''_{p,q} : B_{p,q} \to B_{p,q-1}\}\,,$$

such that (1.1) holds. As in Chapter IV we shall suppress the subscripts of the differentials when the meaning of the symbols is clear.

We leave to the reader the obvious definition of a morphism of double complexes. We now describe two ways to construct a chain complex out of $\boldsymbol{B}$.

First we define a graded module $\operatorname{Tot} \boldsymbol{B}$ by

$$(\operatorname{Tot} \boldsymbol{B})_n = \bigoplus_{p+q=n} B_{p,q}\,.$$

Notice that $\partial'(\operatorname{Tot} \boldsymbol{B})_n \subseteq (\operatorname{Tot} \boldsymbol{B})_{n-1}$, $\partial''(\operatorname{Tot} \boldsymbol{B})_n \subseteq (\operatorname{Tot} \boldsymbol{B})_{n-1}$, and that

$$(\partial' + \partial'') \circ (\partial' + \partial'') = \partial'\partial' + \partial''\partial' + \partial'\partial'' + \partial''\partial'' = 0\,.$$

Thus $\operatorname{Tot} \boldsymbol{B}$ becomes a chain complex if we set

$$\partial = \partial' + \partial'' : (\operatorname{Tot} \boldsymbol{B})_n \to (\operatorname{Tot} \boldsymbol{B})_{n-1}\,,$$

for all n. We call $\operatorname{Tot} \boldsymbol{B}$ the *(first) total chain complex of* $\boldsymbol{B}$. Second, we define a graded module $\operatorname{Tot}' \boldsymbol{B}$ by

$$(\operatorname{Tot}' \boldsymbol{B})_n = \prod_{p+q=n} B_{p,q}\,.$$

Then if $b = \{b_{p,q}\} \in (\operatorname{Tot}' \boldsymbol{B})_n$, we define ∂b by

$$(\partial b)_{p,q} = \partial' b_{p+1,q} + \partial'' b_{p,q+1}\,.$$

Again, the relations (1.1) guarantee that ∂ is a differential, so we obtain $\operatorname{Tot}' \boldsymbol{B}$, the *(second) total chain complex of* $\boldsymbol{B}$. Note that $\operatorname{Tot} \boldsymbol{B} \subseteq \operatorname{Tot}' \boldsymbol{B}$, the inclusion being an equality if for example $B_{p,q} = 0$ for p or q negative (positive).

Of course, given a double complex $\boldsymbol{B}$, we may form the *partial* chain complexes $(\boldsymbol{B}, \partial')$ and $(\boldsymbol{B}, \partial'')$ (of graded Λ-modules). If $H(\boldsymbol{B}, \partial')$ is the homology module of $(\boldsymbol{B}, \partial')$, then ∂'' plainly induces a differential ∂''_* in $H(\boldsymbol{B}, \partial')$ by the rule

$$\partial''_*[z] = [\partial'' z]$$

where z is a cycle of $(\boldsymbol{B}, \partial')$; for by (1.1) $\partial'' z$ is a ∂'-cycle and if $z = \partial' b$ then $\partial'' z = \partial'' \partial' b = -\partial' \partial'' b$ is a ∂'-boundary. Writing ∂'' for ∂''_* by abuse of notation we may thus form

$$H(H(\boldsymbol{B}, \partial'), \partial'') \tag{1.2}$$

and, similarly, we may form

$$H(H(\boldsymbol{B}, \partial''), \partial'). \tag{1.3}$$

A principal object of the study of double complexes is to establish a connection between (1.2), (1.3) and the homology of $\operatorname{Tot} \boldsymbol{B}$ or $\operatorname{Tot}' \boldsymbol{B}$. In general, this connection is given by a spectral sequence (see Section VIII.9), but there are cases, some of which will be discussed in detail in this chapter, in which the connection is much simpler to describe.

Examples. (a) Given two chain complexes $\boldsymbol{C}$, $\boldsymbol{D}$ of right and left Λ-modules, respectively, we define $\boldsymbol{B}$, a double complex of abelian groups, by

$$B_{p,q} = C_p \otimes_\Lambda D_q,$$

$$\partial'(c \otimes d) = \partial c \otimes d, \quad \partial''(c \otimes d) = (-1)^p c \otimes \partial d, \quad c \in C_p, \quad d \in D_q.$$

(1.1) is then easily verified; we remark that the sign $(-1)^p$ is inserted into the definition of ∂'' to guarantee that $\partial'' \partial' + \partial' \partial'' = 0$. Other devices would also imply this relation, but the device employed is standard. We call $\operatorname{Tot} \boldsymbol{B}$ the *tensor product of $\boldsymbol{C}$ and $\boldsymbol{D}$*, and write

$$\boldsymbol{B} = \boldsymbol{C} \otimes_\Lambda \boldsymbol{D},$$

$$\operatorname{Tot} \boldsymbol{B} = \boldsymbol{C} \otimes_\Lambda \boldsymbol{D}.$$

We record explicitly the differential in $\operatorname{Tot} \boldsymbol{B} = \boldsymbol{C} \otimes_\Lambda \boldsymbol{D}$,

$$\partial^\otimes (c \otimes d) = \partial c \otimes d + (-1)^p c \otimes \partial d, \quad c \in C_p, \quad d \in D_q. \tag{1.4}$$

It will be convenient in the sequel to write $\eta : \boldsymbol{C} \to \boldsymbol{C}$ for the involution given by $\eta(c) = (-1)^p c$, $c \in C_p$, so that (1.4) asserts that $\partial^\otimes = \partial \otimes 1 + \eta \otimes \partial$. Moreover

$$\eta \partial = -\partial \eta, \quad \eta^2 = 1. \tag{1.5}$$

(b) For our second example we consider two chain complexes of Λ-modules $\boldsymbol{D}$, $\boldsymbol{E}$ and define a double complex $\boldsymbol{B}$, of abelian groups, by

$$B_{p,q} = \operatorname{Hom}_\Lambda(D_{-p}, E_q),$$

$$(\partial' f)(d) = (-1)^{p+q+1} f(\partial d), \quad d \in D_{-p+1}, f : D_{-p} \to E_q,$$

$$(\partial'' f)(d) = \partial(f d), \quad d \in D_{-p}, f : D_{-p} \to E_q.$$

Obviously $\partial'\partial' = 0$, $\partial''\partial'' = 0$; also

$$(\partial''\partial' f)(d) = \partial(\partial' f)(d) = (-1)^{p+q+1}\,\partial(f(\partial d)),$$
$$(\partial'\partial'' f)(d) = (-1)^{p+q}(\partial'' f)(\partial d) = (-1)^{p+q}\,\partial(f(\partial d)),$$

$d \in D_{-p+1}, f: D_{-p} \to E_q$, so that $\partial''\partial' + \partial'\partial'' = 0$.

The presence of the term D_{-p} in the definition of $B_{p,q}$ is dictated by the requirement that ∂' have degree $(-1, 0)$. Other conventions may also, of course, be used. In particular if $\boldsymbol{D}$ is a chain complex and $\boldsymbol{E}$ a cochain complex, then by taking $B^*_{p,q} = \mathrm{Hom}_\Lambda(D_p, E_q)$, we obviously obtain a *double complex of cochains*. This convention is often the appropriate one; however, for our purposes in this chapter it is better to adopt the stated convention, whereby $\boldsymbol{B}$ is always a double complex of chains, but the translation to $\boldsymbol{B}^*$ is automatic.

We call $\mathrm{Tot}'\,\boldsymbol{B}$ the *chain complex of homomorphisms from* $\boldsymbol{D}$ *to* $\boldsymbol{E}$, and write

$$\boldsymbol{B} = \mathbf{Hom}_\Lambda(\boldsymbol{D}, \boldsymbol{E}),$$
$$\mathrm{Tot}'\,\boldsymbol{B} = \mathrm{Hom}_\Lambda(\boldsymbol{D}, \boldsymbol{E}).$$

We record explicitly the differential in $\mathrm{Tot}'\,\boldsymbol{B} = \mathrm{Hom}_\Lambda(\boldsymbol{D}, \boldsymbol{E})$, namely,

$$(\partial^H f)_{p,q} = (-1)^{p+q} f_{p+1,q}\,\partial + \partial f_{p,q+1}, \tag{1.6}$$

where $f = \{f_{p,q}\}$, $f_{p,q}: D_{-p} \to E_q$.

Our reason for preferring the *second* total complex in this example is made clear in the following basic adjointness relation. Note first however that if $\boldsymbol{E}$ is a chain complex of abelian groups, then we may give $\mathrm{Hom}_{\mathbb{Z}}(D_{-p}, E_q)$ the structure of a left (right) Λ-module if $\boldsymbol{D}$ is a chain complex of right (left) Λ-modules; and then $\mathrm{Hom}_{\mathbb{Z}}(\boldsymbol{D}, \boldsymbol{E})$ is a chain-complex of left (right) Λ-modules, since ∂' and ∂'' are plainly Λ-module homomorphisms. We write Hom for $\mathrm{Hom}_{\mathbb{Z}}$ and state the adjointness theorem as follows.

Theorem 1.1. *Let* $\boldsymbol{C}$ *be a chain complex of right* Λ*-modules,* $\boldsymbol{D}$ *a chain complex of left* Λ*-modules and* $\boldsymbol{E}$ *a chain complex of abelian groups. Then there is a natural isomorphism of chain complexes of abelian groups*

$$\mathrm{Hom}(\boldsymbol{C} \otimes_\Lambda \boldsymbol{D}, \boldsymbol{E}) \xrightarrow{\sim} \mathrm{Hom}_\Lambda(\boldsymbol{C}, \mathrm{Hom}(\boldsymbol{D}, \boldsymbol{E}))$$

Proof. We have already observed the basic adjointness relation (Theorem III.7.2)

$$\mathrm{Hom}(C_{-p} \otimes_\Lambda D_{-q}, E_r) \cong \mathrm{Hom}_\Lambda(C_{-p}, \mathrm{Hom}(D_{-q}, E_r)). \tag{1.7}$$

This induces a natural isomorphism

$$\mathrm{Hom}(\boldsymbol{C} \otimes_\Lambda \boldsymbol{D}, \boldsymbol{E}) \cong \mathrm{Hom}_\Lambda(\boldsymbol{C}, \mathrm{Hom}(\boldsymbol{D}, \boldsymbol{E})) \tag{1.8}$$

as graded abelian groups, and it remains to check the compatibility with the differentials. Note that we achieve this last isomorphism precisely because we have chosen the *second* total complex as the definition of the Hom complex.

Now let $f_{(p,q),r}$ correspond to $f_{p,(q,r)}$ under the isomorphism (1.7), and let $f=\{f_{(p,q),r}\}$, $f'=\{f_{p,(q,r)}\}$, so that f corresponds to f' under (1.8). Then, if $c\in C_{-p}$, $d\in D_{-q}$,

$$\begin{aligned}&(\partial^H f)_{(p,q),r}(c\otimes d)\\ &=(-1)^{p+q+r}(f_{(p+1,q),r}(\partial c\otimes d)+(-1)^p f_{(p,q+1),r}(c\otimes \partial d))+\partial f_{(p,q),r+1}(c\otimes d);\end{aligned}$$

on the other hand

$$\begin{aligned}&(\partial^H f')_{p,(q,r)}(c)(d)\\ &=(-1)^{p+q+r}(f_{p+1,(q,r)}\partial c)(d)+(\partial^H f'(c))_{p,r}(d)\\ &=(-1)^{p+q+r}(f_{p+1,(q,r)}\partial c)(d)+(-1)^{q+r}f_{p,(q+1,r)}(c)(\partial d)+\partial(f_{p,(q,r+1)}(c))(d).\end{aligned}$$

This calculation shows that $\partial^H f$ corresponds to $\partial^H f'$ under (1.8) and completes the proof of the theorem. □

This theorem will be used in Section 4 to obtain connections between the functors Hom, Ext, $\otimes$, and Tor in the category of abelian groups.

We close this section with some remarks on the homotopy relation in the chain complexes $\boldsymbol{C}\otimes_\Lambda \boldsymbol{D}$, $\mathrm{Hom}_\Lambda(\boldsymbol{C},\boldsymbol{D})$. First we remark that a chain map $\boldsymbol{\varphi}:\boldsymbol{C}\to\boldsymbol{C}'$ plainly induces chain maps

$$\begin{aligned}&\boldsymbol{\varphi}_\#:\boldsymbol{C}\otimes_\Lambda\boldsymbol{D}\to\boldsymbol{C}'\otimes_\Lambda\boldsymbol{D},\\ &\boldsymbol{\varphi}^\#:\mathrm{Hom}_\Lambda(\boldsymbol{C}',\boldsymbol{D})\to\mathrm{Hom}_\Lambda(\boldsymbol{C},\boldsymbol{D})\end{aligned}$$

while a chain map $\boldsymbol{\psi}:\boldsymbol{D}\to\boldsymbol{D}'$ induces chain maps

$$\begin{aligned}&\boldsymbol{\psi}_\#:\boldsymbol{C}\otimes_\Lambda\boldsymbol{D}\to\boldsymbol{C}\otimes_\Lambda\boldsymbol{D}',\\ &\boldsymbol{\psi}_\flat:\mathrm{Hom}_\Lambda(\boldsymbol{C},\boldsymbol{D})\to\mathrm{Hom}_\Lambda(\boldsymbol{C},\boldsymbol{D}').\end{aligned}$$

Moreover, we have the commutation laws

$$\boldsymbol{\psi}_\#\boldsymbol{\varphi}_\#=\boldsymbol{\varphi}_\#\boldsymbol{\psi}_\#,\ \boldsymbol{\psi}_\flat\boldsymbol{\varphi}^\#=\boldsymbol{\varphi}^\#\boldsymbol{\psi}_\flat,$$

so that the tensor product complex and the homomorphism complex are both bifunctors. Now suppose that $\boldsymbol{\Gamma}$ is a chain homotopy from $\boldsymbol{\varphi}$ to $\boldsymbol{\varphi}'$, where $\boldsymbol{\varphi},\boldsymbol{\varphi}':\boldsymbol{C}\to\boldsymbol{C}'$. Thus $\varphi'-\varphi=\partial\Gamma+\Gamma\partial$. It then follows easily that $\boldsymbol{\Gamma}_\#$, defined in the obvious way, is a chain homotopy from $\boldsymbol{\varphi}_\#$ to $\boldsymbol{\varphi}'_\#$. For plainly

$$\begin{aligned}\varphi'_\#-\varphi_\#=\varphi'\otimes 1-\varphi\otimes 1&=(\varphi'-\varphi)\otimes 1\\ &=(\partial\Gamma+\Gamma\partial)\otimes 1\\ &=(\partial\otimes 1)(\Gamma\otimes 1)+(\Gamma\otimes 1)(\partial\otimes 1)\\ &=(\partial\otimes 1+\eta\otimes\partial)(\Gamma\otimes 1)+(\Gamma\otimes 1)(\partial\otimes 1+\eta\otimes\partial),\\ &\quad\text{since } \eta\Gamma+\Gamma\eta=0,\\ &=\partial^\otimes\Gamma_\#+\Gamma_\#\partial^\otimes.\end{aligned}$$

Similarly one shows that $\boldsymbol{\Gamma}$ induces a chain homotopy, which we call $\boldsymbol{\Gamma}^{\#}$, between $\varphi'^{\#}$ and $\varphi^{\#}$ and that if $\boldsymbol{\Delta}$ is a chain homotopy from ψ to ψ', where $\psi, \psi' : D \to D'$, then $\boldsymbol{\Delta}$ induces a chain homotopy $\boldsymbol{\Delta}_{\#}$ from $\psi_{\#}$ to $\psi'_{\#}$ and $\boldsymbol{\Delta}_{\flat}$ from $\psi_{\flat}$ to $\psi'_{\flat}$. The reader will, in fact, easily prove the following generalization.

Proposition 1.2. *Chain maps* $\varphi : \boldsymbol{C} \to \boldsymbol{C}'$, $\psi : \boldsymbol{D} \to \boldsymbol{D}'$ *induce chain maps*

$$\varphi \otimes \psi : \boldsymbol{C} \otimes_{\Lambda} \boldsymbol{D} \to \boldsymbol{C}' \otimes_{\Lambda} \boldsymbol{D}',$$

$$\operatorname{Hom}(\varphi, \psi) : \operatorname{Hom}_{\Lambda}(\boldsymbol{C}', \boldsymbol{D}) \to \operatorname{Hom}_{\Lambda}(\boldsymbol{C}, \boldsymbol{D}').$$

Moreover if $\varphi \simeq \varphi'$, $\psi \simeq \psi'$, *then* $\varphi \otimes \psi \simeq \varphi' \otimes \psi'$,

$$\operatorname{Hom}(\varphi, \psi) \simeq \operatorname{Hom}(\varphi', \psi'). \ \square$$

Corollary 1.3. *If* $\boldsymbol{C} \simeq \boldsymbol{C}'$, $\boldsymbol{D} \simeq \boldsymbol{D}'$, *then* $\boldsymbol{C} \otimes_{\Lambda} \boldsymbol{D} \simeq \boldsymbol{C}' \otimes_{\Lambda} \boldsymbol{D}'$,

$$\operatorname{Hom}_{\Lambda}(\boldsymbol{C}, \boldsymbol{D}) \simeq \operatorname{Hom}_{\Lambda}(\boldsymbol{C}', \boldsymbol{D}'). \ \square$$

We note finally that we obtain special cases $\boldsymbol{C} \otimes_{\Lambda} \boldsymbol{D}$, $\operatorname{Hom}_{\Lambda}(\boldsymbol{C}, \boldsymbol{D})$ by allowing one of the chain complexes $\boldsymbol{C}, \boldsymbol{D}$ to degenerate to a single Λ-module, regarded as a chain complex concentrated in dimension 0. We shall feel free to speak of these special cases, and to refer to them by the notation indicated, in the sequel, without further discussion. Here, however, one remark is in order with regard to $\operatorname{Hom}_{\Lambda}(\boldsymbol{C}, B)$. It is natural to regard $\operatorname{Hom}_{\Lambda}(\boldsymbol{C}, B)$ as a *co*chain complex, in which

$$C^n = \operatorname{Hom}_{\Lambda}(C_n, B)$$

and $\delta^n : C^n \to C^{n+1}$ is induced by ∂_{n+1}. Thus

$$C^n = (\operatorname{Hom}_{\Lambda}(\boldsymbol{C}, B))_{-n}. \tag{1.9}$$

Study of (1.6) shows that δ^n differs from ∂^H_{-n} only in sign. Thus there is no real harm in identifying $\operatorname{Hom}_{\Lambda}(\boldsymbol{C}, B)$, as a special case of $\operatorname{Hom}_{\Lambda}(\boldsymbol{C}, \boldsymbol{D})$, with the cochain complex $\operatorname{Hom}_{\Lambda}(\boldsymbol{C}, B)$, by means of (1.9). We exploit this observation later in Sections 3 and 4.

Exercises:

1.1. Show that if Λ is a commutative ring, then $\boldsymbol{C} \otimes_{\Lambda} \boldsymbol{D}$, $\operatorname{Hom}_{\Lambda}(\boldsymbol{C}, \boldsymbol{D})$ acquire naturally the structure of chain complexes over Λ and that there are then natural isomorphisms of chain-complexes over Λ

$$\boldsymbol{C} \otimes_{\Lambda} \boldsymbol{D} \cong \boldsymbol{D} \otimes_{\Lambda} \boldsymbol{C},$$

$$\boldsymbol{C}' \otimes_{\Lambda} (\boldsymbol{C} \otimes_{\Lambda} \boldsymbol{C}'') \cong (\boldsymbol{C}' \otimes_{\Lambda} \boldsymbol{C}) \otimes_{\Lambda} \boldsymbol{C}'',$$

$$\operatorname{Hom}_{\Lambda}(\boldsymbol{C}' \otimes_{\Lambda} \boldsymbol{C}, \boldsymbol{C}'') \cong \operatorname{Hom}_{\Lambda}(\boldsymbol{C}', \operatorname{Hom}_{\Lambda}(\boldsymbol{C}, \boldsymbol{C}'')).$$

1.2. Prove Proposition 1.2 in detail.

1.3. Propose other "rules of sign" for the differentials in $\boldsymbol{C} \otimes_{\Lambda} \boldsymbol{D}$, $\operatorname{Hom}_{\Lambda}(\boldsymbol{C}, \boldsymbol{D})$ which preserve the adjointness relation of Theorem 1.1.

1.4. We define a *differential right Λ-module with involution* to be a right Λ-module A equipped with an endomorphism $d: A \to A$ such that $d^2 = 0$ and an involution $\eta: A \to A$ such that $d\eta = -\eta d$. Given such an object A, show how to introduce a differential into $A \otimes_\Lambda B$ where B is a differential left Λ-module, and into $\mathrm{Hom}_\Lambda(A, B)$ where B is a differential right Λ-module. Suggest a definition of a chain map of differential Λ-modules with involution, and of a chain homotopy between such chain maps.

1.5. Let Λ be a commutative ring and let $\boldsymbol{A}$ be a chain complex over Λ. Show that there are natural isomorphisms of chain complexes

$$\Lambda \otimes_\Lambda \boldsymbol{A} \xrightarrow{\sim} \boldsymbol{A}, \quad \boldsymbol{A} \otimes_\Lambda \Lambda \xrightarrow{\sim} \boldsymbol{A}$$

(considering Λ as a chain-complex concentrated in dimension zero). Define a *differential graded algebra* $\boldsymbol{A}$ as a chain complex $\boldsymbol{A}$ together with a chain map $\boldsymbol{\eta}: \Lambda \to \boldsymbol{A}$ *(unity)* and a chain map $\boldsymbol{\mu}: \boldsymbol{A} \otimes_\Lambda \boldsymbol{A} \to \boldsymbol{A}$ *(product)* such that the diagrams of Exercise III.7.8 are commutative. (If the differential in $\boldsymbol{A}$ is trivial we simply speak of a *graded algebra* over Λ.) If $\boldsymbol{A}, \boldsymbol{B}$ are differential graded algebras over Λ, show how to give $\boldsymbol{A} \otimes_\Lambda \boldsymbol{B}$ the structure of a differential graded algebra.

1.6. Show that if $\{A_i\}$, $-\infty < i < +\infty$, is a graded algebra over the commutative ring Λ then $A = \bigoplus_i A_i$ is an algebra over Λ. (We then call A *internally* graded.)

2. The Künneth Theorem

The Künneth theorem expresses, under certain restrictive hypotheses, the homology of the tensor product $\boldsymbol{C} \otimes_\Lambda \boldsymbol{D}$ in terms of the homology of $\boldsymbol{C}$ and $\boldsymbol{D}$. Our main restriction will be to insist that Λ *be a principal ideal domain* (p.i.d.). Of course, the most important case is that of $\Lambda = \mathbb{Z}$, but we do not gain any simplicity by committing ourselves to the domain $\mathbb{Z}$.

However even the restriction to the case when Λ is a p.i.d. is not enough, as the following example shows.

Example. Let $\Lambda = \mathbb{Z}$. Let $\boldsymbol{C} = \mathbb{Z}_2$, concentrated in dimension 0, and let $\boldsymbol{D} = \boldsymbol{C}$. Let $C'_0 = \mathbb{Z} = (t)$, $C'_1 = \mathbb{Z} = (s)$, $C'_p = 0$, $p \neq 0, 1$, and let $\partial s = 2t$. Then plainly

$$H_p(\boldsymbol{C}) = H_p(\boldsymbol{C}'),$$
$$H_1(\boldsymbol{C}' \otimes \boldsymbol{D}) = \mathbb{Z}_2, \quad H_1(\boldsymbol{C} \otimes \boldsymbol{D}) = 0.$$

Thus the homology of $\boldsymbol{C} \otimes \boldsymbol{D}$ is not determined by that of $\boldsymbol{C}$, $\boldsymbol{D}$.

To eliminate this counterexample we make a further hypothesis, namely that one of $\boldsymbol{C}, \boldsymbol{D}$ is *flat* (a chain complex is flat if its constituent modules are flat). We may then prove

Theorem 2.1. *Let $\boldsymbol{C}, \boldsymbol{D}$ be chain complexes over the p.i.d. Λ, and suppose that one of $\boldsymbol{C}, \boldsymbol{D}$ is flat. Then there is a natural short exact sequence*

$$\bigoplus_{p+q=n} H_p(\boldsymbol{C}) \otimes_\Lambda H_q(\boldsymbol{D}) \overset{\zeta}{\rightarrowtail} H_n(\boldsymbol{C} \otimes_\Lambda \boldsymbol{D}) \twoheadrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}_1^\Lambda(H_p(\boldsymbol{C}), H_q(\boldsymbol{D})), \quad (2.1)$$

where ζ is induced by the inclusion mapping

$$Z_p(\boldsymbol{C}) \otimes_\Lambda Z_q(\boldsymbol{D}) \to Z_{p+q}(\boldsymbol{C} \otimes_\Lambda \boldsymbol{D}),$$

of representative cycles.
Moreover the sequence splits, but not naturally.

Proof. We recall the boundary operator $\partial^\otimes$ in $\boldsymbol{C} \otimes_\Lambda \boldsymbol{D}$, given by

$$\partial^\otimes(c \otimes d) = \partial c \otimes d + (-1)^p c \otimes \partial d, \quad c \in C_p, \quad d \in D_q. \tag{2.2}$$

Now it is plain that (over any commutative ring Λ), there is a natural isomorphism

$$\boldsymbol{C} \otimes_\Lambda \boldsymbol{D} \cong \boldsymbol{D} \otimes_\Lambda \boldsymbol{C} \tag{2.3}$$

given by $c \otimes d \mapsto (-1)^{pq} d \otimes c$, $c \in C_p$, $d \in D_q$, so that it is sufficient to prove the theorem when $\boldsymbol{C}$ is flat.

We next introduce the notation

$$\begin{aligned} Z_p &= Z_p(\boldsymbol{C}), & B_p &= B_p(\boldsymbol{C}); \\ \bar{Z}_p &= Z_p(\boldsymbol{D}), & \bar{B}_p &= B_p(\boldsymbol{D}). \end{aligned} \tag{2.4}$$

We consider $\boldsymbol{Z} = \{Z_p\}$, $\boldsymbol{B} = \{B_p\}$ as complexes with trivial differentials. We also introduce the notation $B'_p = B_{p-1}(\boldsymbol{C})$ and the complex $\boldsymbol{B}' = \{B'_p\}$, where the grading is chosen precisely so that the differential in $\boldsymbol{C}$ may be regarded as a chain map $\partial: \boldsymbol{C} \to \boldsymbol{B}'$. We then consider the exact sequence of chain complexes

$$0 \to \boldsymbol{Z} \xrightarrow{\iota} \boldsymbol{C} \xrightarrow{\partial} \boldsymbol{B}' \to 0.$$

Since Λ is a p.i.d. and since $\boldsymbol{C}$ is flat, it follows that $\boldsymbol{Z}, \boldsymbol{B}$, and $\boldsymbol{B}'$ are flat also. Thus we obtain an exact sequence

$$0 \to \boldsymbol{Z} \otimes_\Lambda \boldsymbol{D} \xrightarrow{\iota \otimes 1} \boldsymbol{C} \otimes_\Lambda \boldsymbol{D} \xrightarrow{\partial \otimes 1} \boldsymbol{B}' \otimes_\Lambda \boldsymbol{D} \to 0 \tag{2.5}$$

of chain complexes. We apply Theorem IV. 2.1 to (2.5) to obtain the exact triangle

$$\begin{tikzcd} H(\boldsymbol{Z} \otimes_\Lambda \boldsymbol{D}) \arrow[rr, "(\iota \otimes 1)_*"] & & H(\boldsymbol{C} \otimes_\Lambda \boldsymbol{D}) \arrow[dl, "(\partial \otimes 1)_*"] \\ & H(\boldsymbol{B}' \otimes_\Lambda \boldsymbol{D}) \arrow[ul, "\omega"] & \end{tikzcd} \tag{2.6}$$

Note that $(\partial \otimes \boldsymbol{1})_*$ has degree 0 and that ω has degree -1. If we replace $H(\boldsymbol{B}' \otimes_\Lambda \boldsymbol{D})$ by $H(\boldsymbol{B} \otimes_\Lambda \boldsymbol{D})$ then $(\partial \otimes \boldsymbol{1})_*$ has degree -1 and ω has degree 0.

We now analyse $H(\boldsymbol{B}' \otimes_\Lambda \boldsymbol{D})$. We first remark that since the differential in $\boldsymbol{B}'$ is trivial, the differential in $\boldsymbol{B}' \otimes \boldsymbol{D}$ is $\boldsymbol{1} \otimes \partial$ up to a sign. Hence we may compute $H(\boldsymbol{B}' \otimes_\Lambda \boldsymbol{D})$ using the differential $\boldsymbol{1} \otimes \partial$. So consider the complex

$$\cdots \longrightarrow (\boldsymbol{B}' \otimes_{\Lambda} \boldsymbol{D})_{n+1} \xrightarrow{(1\otimes\partial)_{n+1}} (\boldsymbol{B}' \otimes_{\Lambda} \boldsymbol{D})_n \xrightarrow{(1\otimes\partial)_n} (\boldsymbol{B}' \otimes_{\Lambda} \boldsymbol{D})_{n-1} \longrightarrow \cdots .$$

Since $\boldsymbol{B}'$ is flat, we obtain

$$\ker(\mathbf{1}\otimes\partial)_n = (\boldsymbol{B}' \otimes_{\Lambda} \bar{\boldsymbol{Z}})_n = (\boldsymbol{B} \otimes_{\Lambda} \bar{\boldsymbol{Z}})_{n-1},$$

$$\operatorname{im}(\mathbf{1}\otimes\partial)_{n+1} = (\boldsymbol{B}' \otimes_{\Lambda} \bar{\boldsymbol{B}})_n = (\boldsymbol{B} \otimes_{\Lambda} \bar{\boldsymbol{B}})_{n-1},$$

so that

$$H_n(\boldsymbol{B}' \otimes_{\Lambda} \boldsymbol{D}) = (\boldsymbol{B} \otimes_{\Lambda} H(\boldsymbol{D}))_{n-1} . \tag{2.7}$$

Similarly, since $\boldsymbol{Z}$ is flat,

$$H_n(\boldsymbol{Z} \otimes_{\Lambda} \boldsymbol{D}) = (\boldsymbol{Z} \otimes_{\Lambda} H(\boldsymbol{D}))_n . \tag{2.8}$$

Thus (2.6) becomes

$$\begin{tikzcd} \boldsymbol{Z} \otimes_{\Lambda} H(\boldsymbol{D}) \arrow[rr, "(\iota\otimes\mathbf{1})_*"] & & H(\boldsymbol{C} \otimes_{\Lambda} \boldsymbol{D}) \arrow[dl, "(\partial\otimes\mathbf{1})_*"] \\ & \boldsymbol{B} \otimes_{\Lambda} H(\boldsymbol{D}) \arrow[ul, "\omega"] & \end{tikzcd} \tag{2.9}$$

Moreover, it is plain that $(\boldsymbol{\iota}\otimes\mathbf{1})_*$ induces ζ in the statement of the theorem. We next analyse $\boldsymbol{\omega}$. We revert to (2.5) and pick a representative $\partial c \otimes z$ of a generator $\partial c \otimes [z]$ of $H(\boldsymbol{B} \otimes_{\Lambda} \boldsymbol{D}) = \boldsymbol{B} \otimes_{\Lambda} H(\boldsymbol{D})$. Then

$$\partial c \otimes z = (\partial \otimes 1)(c \otimes z)$$

and $\partial^{\otimes}(c\otimes z) = \partial c \otimes z$. Thus $\omega(\partial c \otimes [z])$ is the homology class in $H(\boldsymbol{Z} \otimes_{\Lambda} \boldsymbol{D}) = \boldsymbol{Z} \otimes_{\Lambda} H(\boldsymbol{D})$, of $\partial c \otimes z$. This means that $\boldsymbol{\omega}$ in (2.9) is simply induced by the inclusion $\boldsymbol{B} \subseteq \boldsymbol{Z}$. Finally, since $\boldsymbol{Z}$ is flat, we obtain the exact sequence

$$0 \to \operatorname{Tor}_1^{\Lambda}(H(\boldsymbol{C}), H(\boldsymbol{D})) \to \boldsymbol{B} \otimes_{\Lambda} H(\boldsymbol{D}) \xrightarrow{\omega} \boldsymbol{Z} \otimes_{\Lambda} H(\boldsymbol{D}) \to H(\boldsymbol{C}) \otimes_{\Lambda} H(\boldsymbol{D}) \to 0$$

where $\operatorname{Tor}_1^{\Lambda}(-,-)$ has the obvious meaning on graded modules. Hence (2.9) yields the sequence

$$0 \to H(\boldsymbol{C}) \otimes_{\Lambda} H(\boldsymbol{D}) \xrightarrow{\zeta} H(\boldsymbol{C} \otimes_{\Lambda} \boldsymbol{D}) \to \operatorname{Tor}_1^{\Lambda}(H(\boldsymbol{C}), H(\boldsymbol{D})) \to 0 .$$

This sequence is, however, precisely the sequence (2.1).

It is plain, without going into details, that every step in the argument is natural, so that the Künneth sequence is itself natural.

We prepare for the proof that the Künneth sequence splits by demonstrating some basic lemmas related to free chain complexes over p.i.d.'s.

Lemma 2.2. *Let $\boldsymbol{H}$ be a graded module over the p.i.d. Λ. Then there exists a free chain complex $\boldsymbol{C}$ over Λ such that $H(\boldsymbol{C}) \cong \boldsymbol{H}$.*

Proof. Let $0 \to R_p \to F_p \to H_p \to 0$ be a free presentation of H_p. Set

$$C_p = F_p \oplus R_{p-1},$$
$$\partial(x, y) = (y, 0), \qquad x \in F_p, \qquad y \in R_{p-1}.$$

Then $\partial\partial = 0$, $Z_p(C) = F_p$, $B_p(C) = R_p$, so that $H_p(C) \cong H_p$. □

Lemma 2.3. *Let $\boldsymbol{C}, \boldsymbol{D}$ be chain complexes over the p.i.d. Λ and let $\boldsymbol{C}$ be free. Let $\psi : H(\boldsymbol{C}) \to H(\boldsymbol{D})$ be a homomorphism. Then there exists a chain map $\boldsymbol{\varphi} : \boldsymbol{C} \to \boldsymbol{D}$ such that $\boldsymbol{\varphi}_* = \psi$.*

Proof. Consider $0 \to B_p \to Z_p \to H_p \to 0$, $C_p \overset{\partial_p}{\twoheadrightarrow} B_{p-1}$, where everything relates to the chain complex $\boldsymbol{C}$. Since B_{p-1} is free, it follows that $C_p = Z_p \oplus Y_p$, where $\partial_p | Y_p : Y_p \xrightarrow{\sim} B_{p-1}$. Using barred letters to refer to $\boldsymbol{D}$, we have the diagrams

$$\begin{tikzcd} 0 \arrow[r] & B_p \arrow[r] \arrow[d, dotted, "\theta_p"] & Z_p \arrow[r] \arrow[d, dotted, "\varphi_p^1"] & H_p \arrow[r] \arrow[d, "\psi_p"] & 0 \\ 0 \arrow[r] & \bar{B}_p \arrow[r] & \bar{Z}_p \arrow[r] & \bar{H}_p \arrow[r] & 0, \end{tikzcd} \tag{2.10}$$

$$\begin{tikzcd} Y_p \arrow[r, "\partial_p", "\sim"'] \arrow[d, dotted, "\varphi_p^2"] & B_{p-1} \arrow[d, "\theta_{p-1}"] \\ D_p \arrow[r, two heads, "\bar{\partial}_p"'] & \bar{B}_{p-1} \end{tikzcd}\ ; \tag{2.11}$$

here we use the fact that Z_p is free (projective) to lift ψ_p to $\varphi_p^1 : Z_p \to \bar{Z}_p$, inducing $\theta_p : B_p \to \bar{B}_p$, and then we use the fact that Y_p is free (projective) to lift θ_{p-1} to $\varphi_p^2 : Y_p \to D_p$. The reader will now easily verify that $\boldsymbol{\varphi} = \{\varphi_p\}$, where

$$\varphi_p = \langle \varphi_p^1, \varphi_p^2 \rangle : C_p = Z_p \oplus Y_p \to D_p\ ;$$

is a chain map inducing ψ in homology. □

We will need a refinement of Lemma 2.3 in the next section. For our present purposes we record the following immediate consequence of Lemmas 2.2 and 2.3.

Proposition 2.4. *Let $\boldsymbol{C}$ be a chain complex over the p.i.d. Λ. Then there exists a free chain complex $\boldsymbol{F}$ over Λ and a chain map $\boldsymbol{\varphi} : \boldsymbol{F} \to \boldsymbol{C}$ such that $\boldsymbol{\varphi}_* : H(\boldsymbol{F}) \xrightarrow{\sim} H(\boldsymbol{C})$.* □

We are now ready to prove that the Künneth sequence (2.1) splits. We first make the simplifying assumption that $\boldsymbol{C}$ and $\boldsymbol{D}$ are free. Then we have projections $\boldsymbol{\kappa} : \boldsymbol{C} \to \boldsymbol{Z}$, $\bar{\boldsymbol{\kappa}} : \boldsymbol{D} \to \bar{\boldsymbol{Z}}$ and plainly

$$\boldsymbol{\kappa} \otimes \bar{\boldsymbol{\kappa}} : \boldsymbol{C} \otimes_\Lambda \boldsymbol{D} \to \boldsymbol{Z} \otimes_\Lambda \bar{\boldsymbol{Z}}$$

maps a boundary of $\boldsymbol{C}\otimes_\Lambda \boldsymbol{D}$ to $\boldsymbol{B}\otimes_\Lambda \bar{\boldsymbol{Z}}+\boldsymbol{Z}\otimes_\Lambda \bar{\boldsymbol{B}}$. It follows that $\boldsymbol{\kappa}\otimes\bar{\boldsymbol{\kappa}}$ induces $\boldsymbol{\theta}: H(\boldsymbol{C}\otimes_\Lambda \boldsymbol{D})\to H(\boldsymbol{C})\otimes_\Lambda H(\boldsymbol{D})$ such that $\boldsymbol{\theta}\zeta=1$ on $H(\boldsymbol{C})\otimes_\Lambda H(\boldsymbol{D})$. Thus the sequence (2.1) splits if $\boldsymbol{C}$ and $\boldsymbol{D}$ are free.

We now return to the general case when $\boldsymbol{C}$ or $\boldsymbol{D}$ is flat, so that we have a Künneth sequence (2.1) natural in $\boldsymbol{C}$ and $\boldsymbol{D}$. By Proposition 2.4 we may find free chain complexes $\boldsymbol{F},\boldsymbol{G}$ and chain maps $\varphi:\boldsymbol{F}\to\boldsymbol{C}$, $\psi:\boldsymbol{G}\to\boldsymbol{D}$ such that $\varphi_*: H(\boldsymbol{F})\xrightarrow{\sim} H(\boldsymbol{C})$, $\psi_*: H(\boldsymbol{G})\xrightarrow{\sim} H(\boldsymbol{D})$. In view of the naturality of (2.1) we have a commutative diagram

$$\begin{array}{ccccccccc}
0\to & H(\boldsymbol{F})\otimes_\Lambda H(\boldsymbol{G}) & \longrightarrow & H(\boldsymbol{F}\otimes_\Lambda \boldsymbol{G}) & \longrightarrow & \mathrm{Tor}_1^\Lambda(H(\boldsymbol{F}),H(\boldsymbol{G})) & \to 0 \\
 & \downarrow{\scriptstyle \varphi_*\otimes\psi_*} & & \downarrow{\scriptstyle (\varphi\otimes\psi)_*} & & \downarrow{\scriptstyle \mathrm{Tor}(\varphi_*,\psi_*)} & \\
0\to & H(\boldsymbol{C})\otimes_\Lambda H(\boldsymbol{D}) & \longrightarrow & H(\boldsymbol{C}\otimes_\Lambda \boldsymbol{D}) & \longrightarrow & \mathrm{Tor}_1^\Lambda(H(\boldsymbol{C}),H(\boldsymbol{D})) & \to 0
\end{array} \tag{2.12}$$

However, since φ_*,ψ_* are isomorphisms, so are $\varphi_*\otimes\psi_*$, $\mathrm{Tor}(\varphi_*,\psi_*)$. Thus $(\varphi\otimes\psi)_*$ is an isomorphism and (2.12) exhibits an isomorphism between two exact sequences. Since the top sequence splits, so does the bottom one.

The only assertion of Theorem 2.1 remaining to be proved is that the splitting of (2.1) is not natural. Were it natural, we would have, for any $\varphi:\boldsymbol{C}\to\boldsymbol{C}'$, $\psi:\boldsymbol{D}\to\boldsymbol{D}'$, that $(\varphi\otimes\psi)_*=0$ if $\varphi_*\otimes\psi_*=0$ and $\mathrm{Tor}(\varphi_*,\psi_*)=0$. We will give a counter-example to this implication. Take $\Lambda=\mathbb{Z}$, $C_1=\mathbb{Z}=(s_1)$, $C_0=\mathbb{Z}=(s_0)$, $C_n=0$, $n\neq 0,1$, $\partial s_1=2s_0$; $C_1'=\mathbb{Z}=(s_1')$, $C_n'=0$, $n\neq 1$; $\varphi_1(s_1)=s_1'$; $D_0=\mathbb{Z}_2=(t_0)$, $D_n=0$, $n\neq 0$; $\boldsymbol{D}'=\boldsymbol{D}$; $\psi=\mathbf{1}$. Plainly $\varphi_*=0$, so that, were the splitting to be natural, we would have $(\varphi\otimes\psi)_*=0$. But $H_1(\boldsymbol{C}\otimes\boldsymbol{D})=\mathbb{Z}_2=(s_1\otimes t_0)$, $H_1(\boldsymbol{C}'\otimes\boldsymbol{D}')=\mathbb{Z}_2=(s_1'\otimes t_0)$, so that $(\varphi\otimes\psi)_*: H_1(\boldsymbol{C}\otimes\boldsymbol{D})\xrightarrow{\sim} H_1(\boldsymbol{C}'\otimes\boldsymbol{D})$. This completes the proof of Theorem 2.1. □

A particularly important special case of the Künneth sequence occurs when $\boldsymbol{D}$ is just a Λ-module A regarded as a chain complex concentrated in dimension 0. We then obtain a slightly stronger result.

Theorem 2.5. *(Universal coefficient theorem in homology.) Let Λ be a p.i.d., let $\boldsymbol{C}$ be a flat chain complex over Λ and let A be a Λ-module. Then there is a natural short exact sequence*

$$0\to H_n(\boldsymbol{C})\otimes_\Lambda A\xrightarrow{\zeta} H_n(\boldsymbol{C}\otimes_\Lambda A)\to \mathrm{Tor}_1^\Lambda(H_{n-1}(\boldsymbol{C}),A)\to 0\,. \tag{2.13}$$

Moreover, (2.13) *splits; the splitting is unnatural in $\boldsymbol{C}$ but natural in A.*

Proof. The only part of the assertion requiring proof is the final phrase. That the splitting is unnatural in $\boldsymbol{C}$ is attested by the example given to prove the unnaturality of the splitting of (2.1). Thus it remains to prove the naturality of the splitting of (2.13) in the variable A. If $\boldsymbol{C}$ is free, the splitting is given by $\boldsymbol{\kappa}\otimes 1:\boldsymbol{C}\otimes_\Lambda A\to Z(\boldsymbol{C})\otimes_\Lambda A$. Thus, once $\boldsymbol{\kappa}$ is chosen, we get a left inverse θ, to ζ, which is plainly natural in A. If $\boldsymbol{C}$

is an arbitrary flat chain complex over Λ, then, as demonstrated in the proof of Theorem 2.1, there is a free chain complex $\boldsymbol{F}$ and a chain map $\varphi: \boldsymbol{F} \to \boldsymbol{C}$ which induces an isomorphism of the universal coefficient sequence for $\boldsymbol{F}$ with the universal coefficient sequence for $\boldsymbol{C}$ which is natural in A. Since the splitting of the sequence for $\boldsymbol{F}$ is natural in A, so is the splitting of the sequence for $\boldsymbol{C}$. $\square$

Exercises:

2.1. Let $\boldsymbol{C}$ be a resolution of $\mathbb{Z}_k$; thus $C_0 = F$, $C_1 = R$, ∂_1 is the inclusion, where $0 \to R \to F \to \mathbb{Z}_k \to 0$ is a presentation of $\mathbb{Z}_k$. Similarly let $\boldsymbol{D}$ be a resolution of $\mathbb{Z}_l$. Compute $H(\boldsymbol{C} \otimes \boldsymbol{D})$, $H(\boldsymbol{C} \otimes \boldsymbol{D} \otimes \mathbb{Z}_m)$.

2.2. State and prove a Künneth formula for the tensor product of *three* chain complexes over a p.i.d.

2.3. What does the Künneth formula become for tensor products over a field?

2.4. Show that if $\boldsymbol{A}$ is a differential graded algebra over the commutative ring Λ, then $H(\boldsymbol{A})$ is a graded algebra over Λ (see Exercise 1.5).

2.5. How may we weaken the hypothesis on Λ and still retain the validity of the Künneth formula?

3. The Dual Künneth Theorem

In this section we obtain a sequence which enables us to analyse the homology of $\mathrm{Hom}_\Lambda(\boldsymbol{C}, \boldsymbol{D})$, in the sense in which the Künneth sequence provides an analysis of the homology of $\boldsymbol{C} \otimes_\Lambda \boldsymbol{D}$. Again we suppose throughout that Λ is a p.i.d.

Theorem 3.1. *Let $\boldsymbol{C}, \boldsymbol{D}$ be chain complexes over the p.i.d. Λ, with $\boldsymbol{C}$ free. Then there is a natural short exact sequence*

$$\begin{aligned} &\prod_{q-p=n+1} \mathrm{Ext}^1_\Lambda(H_p(\boldsymbol{C}), H_q(\boldsymbol{D})) \rightarrowtail H_n(\mathrm{Hom}_\Lambda(\boldsymbol{C}, \boldsymbol{D})) \\ &\overset{\zeta}{\twoheadrightarrow} \prod_{q-p=n} \mathrm{Hom}_\Lambda(H_p(\boldsymbol{C}), H_q(\boldsymbol{D})) \end{aligned} \tag{3.1}$$

where ζ associates with $f \in Z_n(\mathrm{Hom}_\Lambda(\boldsymbol{C}, \boldsymbol{D}))$ the induced homomorphism $f_*: H(\boldsymbol{C}) \to H(\boldsymbol{D})$. Moreover, the sequence splits non-naturally.

Proof. The reader should be able to provide the details of the proof of the exactness and naturality of (3.1) by retracing, with suitable modifications, the argument establishing (2.1). It is pertinent to comment that, if we define a *chain map of degree n* from $\boldsymbol{C}$ to $\boldsymbol{D}$ to be a collection f of morphisms $f_p: C_p \to D_{p+n}$ such that

$$f\partial = (-1)^n \partial f \tag{3.2}$$

then plainly (see (1.6)) such an f is just a cycle of dimension n of $\mathrm{Hom}_A(\boldsymbol{C}, \boldsymbol{D})$ and f induces $f_* \in \prod_{q-p=n} \mathrm{Hom}(H_p(\boldsymbol{C}), H_q(\boldsymbol{D}))$.

This clarifies the definition of ζ; for it is plain from (1.6) that a boundary of $\mathrm{Hom}_A(\boldsymbol{C}, \boldsymbol{D})$ maps a cycle of $\boldsymbol{C}$ to a boundary of $\boldsymbol{D}$. Further, we replace (2.5) by

$$0 \to \mathrm{Hom}_A(\boldsymbol{B}', \boldsymbol{D}) \to \mathrm{Hom}_A(\boldsymbol{C}, \boldsymbol{D}) \to \mathrm{Hom}_A(\boldsymbol{Z}, \boldsymbol{D}) \to 0. \tag{3.3}$$

Here exactness is guaranteed by the fact that $\boldsymbol{B}'$ is free; since we are concerned with the functor $\mathrm{Hom}_A(\boldsymbol{C}, -)$ rather than $\boldsymbol{C} \otimes_A -$ (similarly for $\boldsymbol{Z}, \boldsymbol{B}'$) we must demand that $\boldsymbol{C}$ be free rather than merely flat.

We now enter into detail with regard to the splitting of (3.1). Again we imitate the argument for the splitting of (2.1) by first assuming $\boldsymbol{D}$ is also free. Reverting to the argument of Lemma 2.3 we see how to adapt it to the case of a homomorphism $\psi: H(\boldsymbol{C}) \to H(\boldsymbol{D})$ *of degree n*. The only essential modification is that we must take

$$\varphi_p = \langle \varphi_p^1, (-1)^n \varphi_p^2 \rangle : C_p = Z_p \oplus Y_p \to D_{p+n} \tag{3.4}$$

in order to achieve $\varphi\partial = (-1)^n \partial\varphi$ (3.2). However, an additional point arises if $\boldsymbol{D}$ is also free; namely, we choose a fixed splitting

$$D_{p+n} = \bar{Z}_{p+n} \oplus \bar{Y}_{p+n},$$

for each p, with $\bar{\partial} | \bar{Y}_{p+n}: \bar{Y}_{p+n} \xrightarrow{\sim} \bar{B}_{p+n-1}$. Then the lifting of θ_{p-1} to $\varphi_p^2: Y_p \to \bar{Y}_{p+n}$ in (2.11) becomes canonical and the only choice exercised in the construction of $\boldsymbol{\varphi}$ from $\boldsymbol{\psi}$ is in the lifting of ψ_p to $\varphi_p^1: Z_p \to \bar{Z}_{p+n}$. We now prove

Lemma 3.2. *If $\boldsymbol{D}$ is free, the construction of $\boldsymbol{\varphi}$ from $\boldsymbol{\psi}$ in Lemma 2.3 induces a homomorphism*

$$\theta: \prod_{q-p=n} \mathrm{Hom}_A(H_p(\boldsymbol{C}), H_q(\boldsymbol{D})) \to H_n(\mathrm{Hom}_A(\boldsymbol{C}, \boldsymbol{D})).$$

Proof. It is plain that the only assertion to be established is that the homology class of $\boldsymbol{\varphi}$ in $H_n(\mathrm{Hom}_A(\boldsymbol{C}, \boldsymbol{D}))$ is independent of the choice of $\boldsymbol{\varphi}^1$. Consider therefore a family of morphisms

$$\alpha_{-p,q}: Z_p \to \bar{B}_{p+n}, \qquad q = p + n.$$

(The indexing is consistent with our rule in Example (b) of Section 1, whereby $\mathrm{Hom}_A(C_{-p}, D_q)$ is indexed as (p, q).) We may lift $\alpha_{-p,q}$ to

$$\gamma_{-p,q+1}: Z_p \to \bar{Y}_{p+n+1}$$

so that

$$\bar{\partial}\boldsymbol{\gamma} = \boldsymbol{\alpha}. \tag{3.5}$$

Now extend $\gamma_{-p,q+1}$ to $\gamma_{-p,q+1}: C_p \to D_{p+n+1}$ by defining $\gamma_{-p,q+1} | Y_p = 0$.

According to the rule (3.4), and incorporating the canonical lifting of θ_{p-1} in (2.11), we see that α gives rise to the family of morphisms

$$\beta_{-p,q} = \langle \alpha_{-p,q}, (-1)^n \bar{\partial}^{-1} \circ \alpha_{-p+1,q-1} \circ \partial \rangle : C_p = Z_p \oplus Y_p \to D_{p+n}. \tag{3.6}$$

Thus our assertion is proved if we can show that $\boldsymbol{\beta}$ is a boundary. In fact we show that

$$\partial^H(\gamma) = \boldsymbol{\beta}. \tag{3.7}$$

For we find, by (1.6),

$$(\partial^H \gamma)_{-p,q} = (-1)^n \gamma_{-p+1,q} \partial + \bar{\partial} \gamma_{-p,q+1}.$$

Thus, on Z_p,

$$(\partial^H \gamma)_{-p,q} = \bar{\partial} \gamma_{-p,q+1} = \alpha_{-p,q}, \qquad \text{by (3.5)};$$

and, on Y_p,

$$(\partial^H \gamma)_{-p,q} = (-1)^n \gamma_{-p+1,q} \partial = (-1)^n \bar{\partial}^{-1} \circ \alpha_{-p+1,q-1} \circ \partial,$$

again by (3.5).

This proves (3.7) and hence the lemma. □

We now return to the proof of Theorem 3.1. It is plain that $\zeta\theta$ is the identity on $\prod \operatorname{Hom}(H_p(\boldsymbol{C}), H_q(\boldsymbol{D}))$, so that we have indeed proved that (3.1) splits if $\boldsymbol{D}$ is free. We now complete the proof exactly as for the sequence (2.1); that is, we use Proposition 2.4 to find a free chain complex $\boldsymbol{G}$ and a chain map $\varphi: \boldsymbol{G} \to \boldsymbol{D}$ inducing an isomorphism in homology; and then prove that the Künneth sequence for $\operatorname{Hom}_\Lambda(\boldsymbol{C}, \boldsymbol{G})$ is isomorphic to that for $\operatorname{Hom}_\Lambda(\boldsymbol{C}, \boldsymbol{D})$. The reader is now invited to construct an example to show that the splitting is not natural. As in the case of (2.1) such an example is easily constructed with $\boldsymbol{D}$ concentrated in dimension 0. This completes the proof of the theorem. □

We may apply Theorem 3.1 to the case when $\boldsymbol{D}$ is a Λ-module B, regarded as a chain complex concentrated in dimension 0. Let us then write $H^n(\operatorname{Hom}_\Lambda(\boldsymbol{C}, B))$ for $H_{-n}(\operatorname{Hom}_\Lambda(\boldsymbol{C}, B))$. We obtain

Theorem 3.3. *(Universal coefficient theorem in cohomology.) Let Λ be a p.i.d., let $\boldsymbol{C}$ be a free chain complex over Λ, and let B be a Λ-module. Then there is a natural short exact sequence*

$$0 \to \operatorname{Ext}_\Lambda(H_{n-1}(\boldsymbol{C}), B) \to H^n(\operatorname{Hom}_\Lambda(\boldsymbol{C}, B)) \xrightarrow{\zeta} \operatorname{Hom}_\Lambda(H_n(\boldsymbol{C}), B) \to 0. \tag{3.8}$$

Moreover, (3.8) *splits; the splitting is unnatural in $\boldsymbol{C}$ but natural in B.*

Proof. Only a few remarks are required. First, the notation

$$H^n(\operatorname{Hom}_\Lambda(\boldsymbol{C}, B))$$

is unambiguous, since the cohomology modules of the *cochain* complex $(\operatorname{Hom}_\Lambda(\boldsymbol{C}, B), \operatorname{Hom}(\partial, 1))$ are given precisely by

$$H^n(\operatorname{Hom}_\Lambda(\boldsymbol{C}, B)) = H_{-n}(\operatorname{Hom}_\Lambda(\boldsymbol{C}, \boldsymbol{B})),$$

where $\boldsymbol{B}$ is the chain-complex consisting just of B in dimension 0. Second, the example (which the reader should have constructed!) to show that the splitting of (3.1) is not natural shows that the splitting of (3.8) is not natural in $\boldsymbol{C}$. That the splitting is natural in $\boldsymbol{B}$ is evident from the fact that, when $\boldsymbol{D}=\boldsymbol{B}$, we construct a canonical right inverse to ζ based on a splitting of $\boldsymbol{C}$ as $\boldsymbol{Z}\oplus\boldsymbol{Y}$. $\square$

Exercises:

3.1. Compute $H(\mathrm{Hom}(\boldsymbol{C},\boldsymbol{D}))$ where $\boldsymbol{C},\boldsymbol{D}$ are as in Exercise 2.1.

3.2. Prove that if $\boldsymbol{C}$ is a free chain complex of abelian groups, then

$$\mathrm{Hom}(\boldsymbol{C},G)\cong\mathrm{Hom}(\boldsymbol{C},\mathbb{Z})\otimes G$$

provided $\boldsymbol{C}$ is finitely generated in each dimension *or* G is finitely generated. Deduce that $\mathrm{Hom}(\boldsymbol{C},G)\simeq\mathrm{Hom}(\boldsymbol{C},\mathbb{Z})\otimes G$ if $H(\boldsymbol{C})$ is finitely generated in each dimension. How may we generalize this to chain complexes over a ring Λ?

3.3. Use the result of the exercise above to obtain an alternative universal coefficient theorem for $\mathrm{Hom}(\boldsymbol{C},G)$ under suitable hypotheses. May we obtain in a similar way an alternative to the dual Künneth formula?

3.4. Reformulate the Künneth formula, regarding $\mathrm{Hom}(\boldsymbol{C},\boldsymbol{D})$ as a cochain complex.

3.5. Obtain a Künneth formula for $\mathrm{Tot}\,\boldsymbol{B}$, where $\boldsymbol{B}=\mathrm{Hom}_\Lambda(\boldsymbol{D},\boldsymbol{E})$ (we worked with $\mathrm{Tot}'\,\boldsymbol{B}$!). Prove the splitting property.

4. Applications of the Künneth Formulas

Since we are concerned to give here some fairly concrete applications, we will be content to state our results for the case $\Lambda=\mathbb{Z}$; we will propose in exercises the evident generalization to the case when Λ is an arbitrary p.i.d. The following proposition is evident.

Proposition 4.1. *Let $\boldsymbol{C}',\boldsymbol{C},\boldsymbol{C}''$ be chain complexes of abelian groups. Then there is a natural isomorphism*

$$(\boldsymbol{C}'\otimes\boldsymbol{C})\otimes\boldsymbol{C}''\cong\boldsymbol{C}'\otimes(\boldsymbol{C}\otimes\boldsymbol{C}''). \quad\square \tag{4.1}$$

We are going to exploit (4.1) together with the companion formula (which is just Theorem 1.1 in the case $\Lambda=\mathbb{Z}$)

$$\mathrm{Hom}(\boldsymbol{C}'\otimes\boldsymbol{C},\boldsymbol{C}'')\cong\mathrm{Hom}(\boldsymbol{C}',\mathrm{Hom}(\boldsymbol{C},\boldsymbol{C}'')). \tag{4.2}$$

First, we consider (4.1). We take $\boldsymbol{C}',\boldsymbol{C},\boldsymbol{C}''$ to be resolutions of abelian groups A',A,A''. Thus, for example $C_1=R$, $C_0=F$, $C_p=0$, $p\neq 0,1$, and ∂_1 is the inclusion $R\subseteq F$, where

$$0\to R\to F\to A\to 0$$

is a free presentation of A. If we compute homology by means of the Künneth formula on either side of (4.1), we find

$$\begin{aligned}
&H_0((\boldsymbol{C}'\otimes\boldsymbol{C})\otimes\boldsymbol{C}'') = (A'\otimes A)\otimes A'',\\
&H_1((\boldsymbol{C}'\otimes\boldsymbol{C})\otimes\boldsymbol{C}'') \cong \mathrm{Tor}(A',A)\otimes A''\oplus\mathrm{Tor}(A'\otimes A, A''),\\
&H_2((\boldsymbol{C}'\otimes\boldsymbol{C})\otimes\boldsymbol{C}'') = \mathrm{Tor}(\mathrm{Tor}(A',A),A'');\\
&H_0(\boldsymbol{C}'\otimes(\boldsymbol{C}\otimes\boldsymbol{C}'')) = A'\otimes(A\otimes A''),\\
&H_1(\boldsymbol{C}'\otimes(\boldsymbol{C}\otimes\boldsymbol{C}'')) \cong A'\otimes\mathrm{Tor}(A,A'')\oplus\mathrm{Tor}(A', A\otimes A''),\\
&H_2(\boldsymbol{C}'\otimes(\boldsymbol{C}\otimes\boldsymbol{C}'')) = \mathrm{Tor}(A',\mathrm{Tor}(A,A'')),
\end{aligned}$$

where Tor means $\mathrm{Tor}_1^{\mathbb{Z}}$. We readily infer

Theorem 4.2. *Let A', A, A'' be abelian groups. There is then an unnatural isomorphism*

$$\mathrm{Tor}(A',A)\otimes A''\oplus\mathrm{Tor}(A'\otimes A,A'')\cong A'\otimes\mathrm{Tor}(A,A'')\oplus\mathrm{Tor}(A',A\otimes A''), \tag{4.3}$$

and a natural isomorphism

$$\mathrm{Tor}(\mathrm{Tor}(A',A),A'')\cong\mathrm{Tor}(A',\mathrm{Tor}(A,A'')). \tag{4.4}$$

Proof. We simply show why (4.4) is natural. A homomorphism $\varphi: A\to B$ induces a unique homotopy class of chain maps $\boldsymbol{\varphi}: \boldsymbol{C}(A)\to\boldsymbol{C}(B)$, where $\boldsymbol{C}(A), \boldsymbol{C}(B)$ are resolutions of A, B. Thus from $\varphi': A'\to B'$, $\varphi: A\to B$, $\varphi'': A''\to B''$, we obtain $\boldsymbol{\varphi}': \boldsymbol{C}(A')\to\boldsymbol{C}(B')$, $\boldsymbol{\varphi}: \boldsymbol{C}(A)\to\boldsymbol{C}(B)$, $\boldsymbol{\varphi}'': \boldsymbol{C}(A'')\to\boldsymbol{C}(B'')$. Then Proposition 1.2 guarantees unique homotopy classes

$$\begin{aligned}
&(\boldsymbol{\varphi}'\otimes\boldsymbol{\varphi})\otimes\boldsymbol{\varphi}'': (\boldsymbol{C}(A')\otimes\boldsymbol{C}(A))\otimes\boldsymbol{C}(A'')\to(\boldsymbol{C}(B')\otimes\boldsymbol{C}(B))\otimes\boldsymbol{C}(B''),\\
&\boldsymbol{\varphi}'\otimes(\boldsymbol{\varphi}\otimes\boldsymbol{\varphi}''): \boldsymbol{C}(A')\otimes(\boldsymbol{C}(A)\otimes\boldsymbol{C}(A''))\to\boldsymbol{C}(B')\otimes(\boldsymbol{C}(B)\otimes\boldsymbol{C}(B'')),
\end{aligned}$$

compatible with (4.1). Since the calculation of $H_2((\boldsymbol{C}'\otimes\boldsymbol{C})\otimes\boldsymbol{C}'')$, $H_2(\boldsymbol{C}'\otimes(\boldsymbol{C}\otimes\boldsymbol{C}''))$ does not involve the splitting of the Künneth sequence (2.1), this proves naturality. □

We now turn to (4.2) and use the same chain complexes $\boldsymbol{C}', \boldsymbol{C}, \boldsymbol{C}''$ as in the proof of Theorem 4.2. Computing either side of (4.2) by means of the dual Künneth formula, we find

$$\begin{aligned}
&H_0(\mathrm{Hom}(\boldsymbol{C}'\otimes\boldsymbol{C},\boldsymbol{C}'')) = \mathrm{Hom}(A'\otimes A, A''),\\
&H_{-1}(\mathrm{Hom}(\boldsymbol{C}'\otimes\boldsymbol{C},\boldsymbol{C}'')) = \mathrm{Hom}(\mathrm{Tor}(A',A),A'')\oplus\mathrm{Ext}(A'\otimes A, A''),\\
&H_{-2}(\mathrm{Hom}(\boldsymbol{C}'\otimes\boldsymbol{C},\boldsymbol{C}'')) = \mathrm{Ext}(\mathrm{Tor}(A',A),A'');\\
&H_0(\mathrm{Hom}(\boldsymbol{C}',\mathrm{Hom}(\boldsymbol{C},\boldsymbol{C}''))) = \mathrm{Hom}(A',\mathrm{Hom}(A,A'')),\\
&H_{-1}(\mathrm{Hom}(\boldsymbol{C}',\mathrm{Hom}(\boldsymbol{C},\boldsymbol{C}''))) = \mathrm{Hom}(A',\mathrm{Ext}(A,A''))\oplus\mathrm{Ext}(A',\mathrm{Hom}(A,A'')),\\
&H_{-2}(\mathrm{Hom}(\boldsymbol{C}',\mathrm{Hom}(\boldsymbol{C},\boldsymbol{C}''))) = \mathrm{Ext}(A',\mathrm{Ext}(A,A'')),
\end{aligned}$$

where Tor means $\mathrm{Tor}_1^{\mathbb{Z}}$ and Ext means $\mathrm{Ext}_{\mathbb{Z}}^1$. We readily infer, leaving all details to the reader,

Theorem 4.3. *Let A', A, A'' be abelian groups. There is then an unnatural isomorphism*

$$\begin{aligned}&\mathrm{Hom}(\mathrm{Tor}(A', A), A'') \oplus \mathrm{Ext}(A' \otimes A, A'') \\ &\quad \cong \mathrm{Hom}(A', \mathrm{Ext}(A, A'')) \oplus \mathrm{Ext}(A', \mathrm{Hom}(A, A'')),\end{aligned} \tag{4.5}$$

and a natural isomorphism

$$\mathrm{Ext}(\mathrm{Tor}(A', A), A'') \cong \mathrm{Ext}(A', \mathrm{Ext}(A, A'')). \quad \square \tag{4.6}$$

We may draw some immediate inferences from Theorem 4.3.

Corollary 4.4. *If A is torsion-free, then* $\mathrm{Ext}(A, B)$ *is divisible, for all B.*

Proof. It follows from (4.6) that, if A is torsion-free, then

$$\mathrm{Ext}(A', \mathrm{Ext}(A, B)) = 0$$

for all A', B. This means that $\mathrm{Ext}(A, B)$ is injective, that is, divisible, for all B. $\square$

Corollary 4.5. *If A' is torsion-free, then* $\mathrm{Ext}(A', \mathrm{Ext}(A, A'')) = 0$ *for all A, A''.* $\square$

Corollary 4.6. (i) *There is a natural isomorphism*

$$\mathrm{Ext}(A', \mathrm{Ext}(A, A'')) \cong \mathrm{Ext}(A, \mathrm{Ext}(A', A'')).$$

(ii) *There is an unnatural isomorphism*

$$\begin{aligned}&\mathrm{Hom}(A', \mathrm{Ext}(A, A'')) \oplus \mathrm{Ext}(A', \mathrm{Hom}(A, A'')) \\ &\quad \cong \mathrm{Hom}(A, \mathrm{Ext}(A', A'')) \oplus \mathrm{Ext}(A, \mathrm{Hom}(A', A'')). \quad \square\end{aligned}$$

Less immediate consequences are the following; the reader should recall that $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z}) = \mathbb{R}$ (see Chapter III, Exercise 6.2).

Corollary 4.7. *If* $\mathrm{Ext}(A, \mathbb{Z}) = 0$, $\mathrm{Hom}(A, \mathbb{Z}) = 0$, *then* $A = 0$.

Proof. By (4.5) we infer $\mathrm{Ext}(A' \otimes A, \mathbb{Z}) = 0$ for all A'. Now, since $\mathrm{Ext}(A, \mathbb{Z}) = 0$, A is torsion-free. Thus if $A \neq 0$, take $A' = \mathbb{Q}$. Then $\mathbb{Q} \otimes A$ is a non-zero vector space over $\mathbb{Q}$, so that $\mathrm{Ext}(\mathbb{Q} \otimes A, \mathbb{Z}) \neq 0$. $\square$

Corollary 4.8. *There is no abelian group A such that* $\mathrm{Ext}(A, \mathbb{Z}) = \mathbb{Q}$, $\mathrm{Hom}(A, \mathbb{Z}) = 0$.

Proof. Since $\mathrm{Ext}(A, \mathbb{Z}) = \mathbb{Q}$, A is a non-zero torsion-free group. Again by (4.5) we infer

$$\mathrm{Ext}(\mathbb{Q} \otimes A, \mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Q}, \mathbb{Q}).$$

But $\mathrm{Hom}(\mathbb{Q}, \mathbb{Q}) \cong \mathbb{Q}$ and $\mathbb{Q} \otimes A$ is a non-zero vectorspace over $\mathbb{Q}$. Since $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z}) = \mathbb{R}$, we have a contradiction. $\square$

Remark. Theorems 4.2 and 4.3 really express certain *associativity* relations between the bifunctors $\otimes$, Tor, Hom and Ext. Their true nature is masked by the traditional notation, adopted here, whereby $\otimes$ is written *between* the two arguments, while Tor, Hom and Ext are written to the left of their arguments. If we were to write

$$
\begin{array}{lll}
A * B & \text{for} & \operatorname{Tor}(A, B), \\
A \pitchfork B & \text{for} & \operatorname{Hom}(A, B), \\
A \dagger B & \text{for} & \operatorname{Ext}(A, B),
\end{array}
$$

then (4.3)–(4.6) would assume the form

$$
\begin{gathered}
(A' * A) \otimes A'' \oplus (A' \otimes A) * A'' \cong A' * (A \otimes A'') \oplus A' \otimes (A * A''), \\
(A' * A) * A'' \cong A' * (A * A''), \\
(A' * A) \pitchfork A'' \oplus (A' \otimes A) \dagger A'' \cong A' \dagger (A \pitchfork A'') \oplus A' \pitchfork (A \dagger A''), \\
(A' * A) \dagger A'' \cong A' \dagger (A \dagger A'').
\end{gathered}
$$

These forms are surely more perspicuous.

Exercises:

4.1. Extend the results of this section to modules over arbitrary p.i.d.'s.

4.2. Show that all isomorphisms obtained by considering tensor products of *four* chain complexes may be deduced from (4.3), (4.4) and the associativity of $\otimes$ by using functorial properties of $\otimes$ and Tor.

4.3. Prove (by a suitable counterexample) that (4.3) is not natural.

4.4. Similarly, prove that (4.5) is not natural.

4.5. Generalize Corollary 4.8 in the following sense: Find a family $\mathfrak{F}$ of abelian groups such that $\mathbb{Q} \in \mathfrak{F}$ and such that the relations $\operatorname{Ext}(A, \mathbb{Z}) \in \mathfrak{F}$, $\operatorname{Hom}(A, \mathbb{Z}) = 0$ have no solution.

VI. Cohomology of Groups

In this chapter we shall apply the theory of derived functors to the important special case where the ground ring Λ is the group ring $\mathbb{Z}G$ of an abstract group G over the integers. This will lead us to a definition of cohomology groups $H^n(G, A)$ and homology groups $H_n(G, B)$, $n \geqq 0$, where A is a left and B a right G-module (we speak of "G-modules" instead of "$\mathbb{Z}G$-modules"). In developing the theory we shall attempt to deduce as much as possible from general properties of derived functors. Thus, for example, we shall give a proof of the fact that $H^2(G, A)$ classifies extensions which is not based on a particular (i.e. standard) resolution.

The scope of the book (and of this chapter) clearly allows us to present the most fundamental results only. The interested reader is referred to the books [20, 33, 49; 41], for further material relating to the cohomology of groups.

In this introduction we first give a survey of the content of this chapter and will then discuss the historical origins of the theory in algebraic topology.

In Sections 1, 2 we introduce the group ring and define the (co)homology groups. Then we exhibit the nature of these groups in dimensions 0 and 1 in Sections 3,4. Section 5 consists of a discussion of the fundamental interplay between the augmentation ideal, derivations, and the semi-direct product. Section 6 is devoted to a short exact sequence associated with an extension of groups. We then apply this in Section 7 to compute the (co)homology of cyclic groups and in Section 8 to deduce the so called 5-term exact sequence which connects the (co)homology in dimensions 1 and 2. The 5-term sequence is then used in Section 9 to exhibit relations between the homology of a group and its lower central series; and it is the main tool for the proof, in the next section, of the fact that $H^2(G, A)$ classifies extensions with abelian kernel.

We present in Section 11 the theory of relative injective and projective modules as far as it is necessary to give a proof of the reduction theorems (Section 12) and a description of various standard resolutions (Section 13). In Sections 14 and 15 we discuss the behavior of (co)homology with respect to free and direct products of groups. Also, we state the universal coefficient theorems. We conclude the chapter with the definition of various important maps in (co)homology and finally apply the cohomology theory of groups to give a proof of Maschke's Theorem in the representation theory of groups.

Homological algebra has profited greatly from interaction with algebraic topology. Indeed, at a very superficial level, it is obvious that the homology theory of chain-complexes is just an algebraic abstraction (via, e.g., the singular chain-complex functor) of the homology theory of topological spaces. However, at a deeper level, the mathematical discipline known as homological algebra may be held to have originated with the homology theory of groups. This theory itself arose out of an observation of the topologist Witold Hurewicz in 1935 about aspherical spaces. An aspherical space is a topological space X such that all the higher homotopy groups of X, $\pi_i(X)$, $i \geqq 2$, are trivial. Hurewicz pointed out that the homology groups of a path-connected aspherical space X are entirely determined by its fundamental group. It was natural, therefore, to inquire precisely how this determination was effected, and a solution was given independently by Hopf and Freudenthal in the years 1945 to 1946. Hopf based himself on his own study of the influence of the fundamental group on the second homology group of a space. Indeed, Hopf had shown earlier that if one considers the quotient group of the second homology group by the subgroup consisting of spherical cycles, then this group can be explicitly determined in terms of a given presentation of the fundamental group. The resulting formula has come to be known as Hopf's formula for $H_2(\pi)$, where π is the fundamental group (see Section 10). Hopf generalized this result and defined higher homology groups of the group π in terms of a certain standard resolution associated with the group π. These groups are then the homology groups of a path-connected aspherical space X with $\pi_1(X) = \pi$.

At about the same time (actually, in the case of Eilenberg and MacLane, a little earlier) certain cohomology groups of the group π were being introduced and investigated by Eilenberg and MacLane and independently by Eckmann.

Actually, we know now that in a certain sense the second homology group H_2 had been invented earlier, for back in 1904 Schur had introduced the notion of the *multiplicator* of a group. This group was studied by Schur in connection with the question of projective representations of a group. It turns out that Schur's multiplicator is canonically isomorphic to the second integral homology group, so that one could say that Schur's introduction of the multiplicator was, in a sense, the precursor of the theory.

The techniques employed by Hopf, Freudenthal, and Eckmann were all, in their initial phases, very strongly influenced by the topological application. If X is an aspherical space, then its universal covering space

$\tilde{X}$ is contractible. Moreover, it is a space upon which the fundamental group acts freely. Thus the chain complex of $\tilde{X}$ is, in modern terminology, a free $\pi_1(X)$-resolution of the integers. If we take a group B upon which $\pi_1(X)$ operates, that is to say, a $\pi_1(X)$-module B, then we may form the tensor product of the chain complex $C(\tilde{X})$ with B over the group ring of $\pi_1(X)$, and this chain complex will yield the homology groups of X with coefficients in the $\pi_1(X)$-module B, or, in other words, the homology groups of X with local coefficients B. In particular, if $\pi_1(X)$ operates trivially on B we will get the usual homology groups of X with coefficients in B. If, instead of taking the tensor product we take the cochain-complex $\mathrm{Hom}_\pi(C(\tilde{X}), A)$, where $\pi = \pi_1(X)$ and A is a π-module, then we obtain the cohomology groups in the sense of Eckmann and Eilenberg-MacLane.

We now know, following Cartan, Eilenberg and MacLane, precisely how to interpret this entire program in a purely algebraic manner and it is this purely algebraic treatment that we give in this chapter.

1. The Group Ring

Let G be a group written multiplicatively. The *integral group ring* $\mathbb{Z}G$ of G is defined as follows. Its underlying abelian group is the free abelian group on the set of elements of G as basis; the product of two basis elements is given by the product in G. Thus the elements of the group ring $\mathbb{Z}G$ are sums $\sum_{x \in G} m(x)\,x$, where m is a function from G to $\mathbb{Z}$ which takes the value zero except on a finite number of elements of G. The multiplication is given by

$$\Big(\sum_{x \in G} m(x)\,x\Big) \cdot \Big(\sum_{y \in G} m'(y)\,y\Big) = \sum_{x,y \in G} (m(x) \cdot m'(y))\,xy\,. \tag{1.1}$$

The group ring is characterised by the following universal property. Let $i: G \to \mathbb{Z}G$ be the obvious embedding.

Proposition 1.1. *Let R be a ring. To any function $f: G \to R$ with $f(xy) = f(x) \cdot f(y)$ and $f(1) = 1_R$ there exists a unique ring homomorphism $f': \mathbb{Z}G \to R$ such that $f'i = f$.*

Proof. We define $f'\Big(\sum_{x \in G} m(x)\,x\Big) = \sum_{x \in G} m(x) f(x)$ which obviously is the only ring homomorphism for which $f'i = f$. $\square$

A *(left) G-module* is an abelian group A together with a group homomorphism $\sigma: G \to \mathrm{Aut}\,A$. In other words the group elements act as automorphisms on A. We shall denote the image of $a \in A$ under the automorphism $\sigma(x)$, $x \in G$, by $x \circ a$ or simply by xa if this notation cannot cause any confusion.

Since $\operatorname{Aut} A \subseteq \operatorname{End} A$, the universal property of the group ring yields a ring homomorphism $\sigma': \mathbb{Z}G \to \operatorname{End} A$, making A into a (left) module over $\mathbb{Z}G$. Conversely, if A is a (left) module over $\mathbb{Z}G$ then A is a (left) G-module, since any ring homomorphism takes invertible elements into invertible elements, and since the group elements in $\mathbb{Z}G$ are invertible. Thus we need not retain any distinction between the concepts of G-module and $\mathbb{Z}G$-module.

We leave it to the reader to word the definition of *a right G-module*. A (left) G-module is called *trivial* if the structure map $\sigma: G \to \operatorname{Aut} A$ is trivial, i.e. if every group element of G acts as the identity in A. Every abelian group may be regarded as a trivial left or right G-module for any group G.

The trivial map from G into the integers $\mathbb{Z}$, sending every $x \in G$ into $1 \in \mathbb{Z}$, gives rise to a unique ring homomorphism $\varepsilon: \mathbb{Z}G \to \mathbb{Z}$. This map is called the *augmentation* of $\mathbb{Z}G$. If $\sum_{x \in G} m(x)\, x$ is an arbitrary element in $\mathbb{Z}G$, then

$$\varepsilon\Big(\sum_{x \in G} m(x)\, x\Big) = \sum_{x \in G} m(x)\,. \tag{1.2}$$

The kernel of ε, denoted by IG, is called the *augmentation ideal* of G. It will play a key role in this chapter. First we note

Lemma 1.2. (i) *As an abelian group IG is free on the set*

$$W = \{x - 1 \mid 1 \neq x \in G\}\,.$$

(ii) *Let S be a generating set for G. Then, as G-module, IG is generated by $S - 1 = \{s - 1 \mid s \in S\}$.*

Proof. (i) Clearly, the set W is linearly independent. We have to show that it generates IG. Let $\sum_{x \in G} m(x)\, x \in IG$, then $\sum_{x \in G} m(x) = 0$. Hence $\sum_{x \in G} m(x)\, x = \sum_{x \in G} m(x)\,(x - 1)$, and (i) is proved.

(ii) It is sufficient to show that if $x \in G$, then $x - 1$ belongs to the module generated by $S - 1$. Since $xy - 1 = x(y - 1) + (x - 1)$, and

$$x^{-1} - 1 = -x^{-1}(x - 1)\,,$$

this follows easily from the representation of x as $x = s_1^{\pm 1} s_2^{\pm 1} \dots s_k^{\pm 1}$, $s_i \in S$. □

Lemma 1.3. *Let U be a subgroup of G. Then $\mathbb{Z}G$ is free as left (or right) U-module.*

Proof. Choose $\{x_i\}$, $x_i \in G$, a system of representatives of the left cosets of U in G. The underlying set of G may be regarded as the disjoint union of the sets $x_i U$. Clearly, the part of $\mathbb{Z}G$ linearly spanned by $x_i U$ for fixed i is a right U-module isomorphic to $\mathbb{Z}U$. Hence the right module $\mathbb{Z}G$ is a direct sum of submodules isomorphic to $\mathbb{Z}U$. □

With Lemma 1.3 we deduce immediately from Theorem IV. 12.5

Corollary 1.4. *Every projective (injective) G-module is a projective (injective) U-module for any subgroup U of G.* $\square$

Exercises:

1.1. Let Λ be a ring with unit and let $U(\Lambda)$ be the set of units of Λ. Show that U is a functor $\mathfrak{R}_1 \to \mathfrak{G}$ from rings with unity to groups, and that U is right adjoint to the group ring functor $\mathbb{Z}(\ )$. Deduce that if G is the free product of the groups G_1 and G_2, then $\mathbb{Z}G$ is the coproduct of $\mathbb{Z}G_1$ and $\mathbb{Z}G_2$ in the category of rings with unity.

1.2. Interpret the augmentation $\varepsilon: \mathbb{Z}G \to \mathbb{Z}$ (i) as a G-module homomorphism, (ii) as a morphism in the image of the functor $\mathbb{Z}(\ )$.

1.3. Set up an isomorphism between the category of left G-modules and the category of right G-modules.

1.4. Propose a definition of ΛG where Λ is a ring with unity and G is a group. This is the *group ring* of G over Λ. Develop the concepts related to G-modules as in this section, replacing "abelian groups" by "Λ-modules". What is a ΛG-module when Λ is the field K?

1.5. Prove Corollary 1.4 without appealing to the theory of adjoint functors.

1.6. Show that the functor $-\otimes_{\mathbb{Z}G}\mathbb{Z}$ is left adjoint to the functor which assigns to an abelian group the structure of a trivial G-module. Deduce that if P is a projective G-module, then $P_G = P \otimes_{\mathbb{Z}G} \mathbb{Z}$ is a free abelian group.

2. Definition of (Co)Homology

For convenience we shall use $A, A', A'', \dots$ only to denote *left* G-modules, and $B, B', B'', \dots$ only to denote *right* G-modules. Moreover we shall write $B \otimes_G A$, $\operatorname{Hom}_G(A, A')$, $\operatorname{Tor}_n^G(B, A)$, $\operatorname{Ext}_G^n(A, A')$ for

$$B \otimes_{\mathbb{Z}G} A\,, \quad \operatorname{Hom}_{\mathbb{Z}G}(A, A')\,, \quad \operatorname{Tor}_n^{\mathbb{Z}G}(B, A)\,, \quad \operatorname{Ext}_{\mathbb{Z}G}^n(A, A')\,,$$

respectively.

We define the *n-th cohomology group* of G with coefficients in the left G-module A by

$$H^n(G, A) = \operatorname{Ext}_G^n(\mathbb{Z}, A)\,, \tag{2.1}$$

where $\mathbb{Z}$ is to be regarded as trivial G-module. The *n-th homology group* of G with coefficients in the right G-module B is defined by

$$H_n(G, B) = \operatorname{Tor}_n^G(B, \mathbb{Z})\,, \tag{2.2}$$

where again $\mathbb{Z}$ is to be regarded as trivial G-module.

Clearly both $H^n(G, -)$ and $H_n(G, -)$ are covariant functors. The following is obviously an economical method of computing these groups: Take a G-projective resolution $\boldsymbol{P}$ of the trivial (left) G-module $\mathbb{Z}$, form the complexes $\mathrm{Hom}_G(\boldsymbol{P}, A)$ and $B \otimes_G \boldsymbol{P}$, and compute their homology. In Section 13 we shall give a standard procedure of constructing such a resolution $\boldsymbol{P}$ from the group G. Unfortunately even for groups of a very simple structure the actual computation of the (co)homology groups by resolutions is very hard. We therefore put the emphasis here rather on general results about the (co)homology than on actual computations. Indeed, we shall give a complete description of the (co)homology only for cyclic groups (Section 7) and for free groups (Section 5).

Some properties of $H^n(G, A)$, $H_n(G, B)$ immediately follow from their definition. We list the following:

(2.3) To a short exact sequence $A' \rightarrowtail A \twoheadrightarrow A''$ of G-modules there is a long exact cohomology sequence

$$0 \to H^0(G, A') \to H^0(G, A) \to H^0(G, A'') \to H^1(G, A') \to \cdots$$
$$\cdots \to H^n(G, A') \to H^n(G, A) \to H^n(G, A'') \to H^{n+1}(G, A') \to \cdots.$$

To a short exact sequence $B' \rightarrowtail B \twoheadrightarrow B''$ there is a long exact homology sequence

$$\cdots \to H_n(G, B') \to H_n(G, B) \to H_n(G, B'') \to H_{n-1}(G, B') \to \cdots$$
$$\cdots \to H_1(G, B'') \to H_0(G, B') \to H_0(G, B) \to H_0(G, B'') \to 0.$$

(2.4) If A is injective, then $H^n(G, A) = 0$ for all $n \geqq 1$. If B is flat (in particular if B is projective), then $H_n(G, B) = 0$ for all $n \geqq 1$.

(2.5) If $A \rightarrowtail I \twoheadrightarrow A'$ is an injective presentation of A, then

$$H^{n+1}(G, A) \cong H^n(G, A')$$

for $n \geqq 1$. If $B' \rightarrowtail P \twoheadrightarrow B$ is a projective (or flat) presentation of B, then $H_{n+1}(G, B) \cong H_n(G, B')$ for $n \geqq 1$.

(2.6) Let $0 \to K \to P_k \to \cdots \to P_0 \to \mathbb{Z} \to 0$ be an exact sequence of (left) G-modules, with $P_0, \ldots, P_k$ projective. Then the following sequences are exact and specify the (co)homology groups of G:

$$\mathrm{Hom}_G(P_k, A) \to \mathrm{Hom}_G(K, A) \to H^{k+1}(G, A) \to 0,$$
$$0 \to H_{k+1}(G, B) \to B \otimes_G K \to B \otimes_G P_k.$$

Under the same hypotheses as in (2.6) we have, for $n \geqq k+2$,

$$\begin{aligned} H^n(G, A) &\cong \mathrm{Ext}_G^{n-k-1}(K, A), \\ H_n(G, B) &\cong \mathrm{Tor}^G_{n-k-1}(B, K). \end{aligned} \tag{2.7}$$

In particular, using $0 \to IG \to \mathbb{Z}G \to \mathbb{Z} \to 0$, we get, for $n \geqq 2$,

$$\begin{aligned} H^n(G, A) &\cong \mathrm{Ext}_G^{n-1}(IG, A), \\ H_n(G, B) &\cong \mathrm{Tor}^G_{n-1}(B, IG). \end{aligned} \tag{2.8}$$

The proofs of these simple facts (2.3), ..., (2.8) are left to the reader. Next we make some remarks on the functoriality of the (co)homology groups.

Let $f: G \to G'$ be a group homomorphism; clearly f induces a ring homomorphism $\mathbb{Z}f: \mathbb{Z}G \to \mathbb{Z}G'$, which we shall also write as f. By (IV. 12.3), f gives rise to a functor $U^f: \mathfrak{M}_{\mathbb{Z}G'} \to \mathfrak{M}_{\mathbb{Z}G}$. If A' is a G'-module then $x \in G$ acts on $a' \in A' = U^f A'$ by $x \circ a' = f(x) \circ a'$. By (IV. 12.4) the functor U^f has a left adjoint $F: \mathfrak{M}_{\mathbb{Z}G} \to \mathfrak{M}_{\mathbb{Z}G'}$ defined by $FA = \mathbb{Z}G' \otimes_G A$. By (IV. 12.5) this situation gives rise to a natural homomorphism

$$\theta: H^n(G', A') \to H^n(G, U^f A'). \tag{2.9}$$

If we proceed similarly for right modules and if we use the statement for the functor Tor analogous to (IV. 12. 5), we obtain a natural homomorphism

$$\tilde{\theta}: H_n(G, U^f B') \to H_n(G', B'). \tag{2.10}$$

For convenience we shall omit the functor U^f in the statements (2.9), (2.10), whenever it is clear from the context that the G'-modules A', B' are to be regarded as G-modules via f.

The above suggests that we regard $H^n(-,-)$ as a functor on the category $\mathfrak{G}^*$ whose objects are pairs (G, A) with G a group and A a G-module. A morphism $(f, \alpha): (G, A) \to (G', A')$ in this category consists of a group homomorphism $f: G \to G'$ and a homomorphism $\alpha: U^f A' \to A$ (backwards!) of G-modules. It is obvious from (2.9) that (f, α) induces a homomorphism

$$(f, \alpha)^* = \alpha_* \circ \theta: H^n(G', A') \to H^n(G, A) \tag{2.11}$$

which makes $H^n(-,-)$ into a contravariant functor on the category $\mathfrak{G}^*$. We leave it to the reader to define a category $\mathfrak{G}_*$ on which $H_n(-,-)$ is a (covariant) functor.

We finally note the important fact that for trivial G-modules A, B we may regard $H^n(-, A)$ and $H_n(-, B)$ as functors on the category of groups.

Exercises:

2.1. Compute $H^n(G, A)$, $H_n(G, B)$ where G is the trivial group.

2.2. Show that $H^n(G, -)$, $H_n(G, -)$ are additive functors.

2.3. Prove the statements (2.3), ..., (2.8).

2.4. Check explicitly that (2.11) indeed makes $H^n(-,-)$ into a functor. Similarly for $H_n(-,-)$.

2.5. Let $f: G \to G'$ be a group homomorphism. Show that for a G'-module A the change-of-rings map $(f, 1)^*: H^n(G', A) \to H^n(G, A)$, $n \geqq 0$, may be obtained by the following procedure. Let $\boldsymbol{P}$ be a G-projective resolution of $\mathbb{Z}$ and $\boldsymbol{Q}$ a G'-projective resolution of $\mathbb{Z}$. By the comparison theorem (Theorem IV. 4.1) there exists a (G-module) chain map $\boldsymbol{\varphi}: \boldsymbol{P} \to \boldsymbol{Q}$ lifting $1: \mathbb{Z} \to \mathbb{Z}$. Then $(f, 1)^*$ is induced by $\boldsymbol{\varphi}$. Proceed similarly to obtain the change-of-rings map in homology.

3. H^0, H_0

Let A be a G-module. By definition we have $H^0(G, A) = \mathrm{Hom}_G(\mathbb{Z}, A)$. Now a homomorphism $\varphi: \mathbb{Z} \to A$ is entirely given by the image of $1 \in \mathbb{Z}$, $\varphi(1) = a \in A$. The fact that φ is a *G-module* homomorphism implies that $x \circ a = \varphi(x \circ 1) = \varphi(1) = a$ for all $x \in G$. Indeed one sees that φ is a G-module homomorphism if and only if $\varphi(1) = a$ remains *fixed* under the action of G. Thus, if we write

$$A^G = \{a \in A \mid x \circ a = a \quad \text{for all} \quad x \in G\} \tag{3.1}$$

for the subgroup of *invariant elements* in A, we have

$$H^0(G, A) = \mathrm{Hom}_G(\mathbb{Z}, A) = A^G\,. \tag{3.2}$$

Let B be a right G-module. By definition $H_0(G, B) = B \otimes_G \mathbb{Z}$. Thus $H_0(G, B)$ is the quotient of the abelian group $B \cong B \otimes \mathbb{Z}$ by the subgroup generated by the elements of the form $bx - b = b(x-1)$, $b \in B$, $x \in G$. Since the elements $x - 1 \in \mathbb{Z}G$ precisely generate the augmentation ideal IG (Lemma 1.2), this subgroup may be expressed as $B \circ IG$. Thus if we write

$$B_G = B/B \circ IG = B/\{b(x-1) \mid b \in B, x \in G\} \tag{3.3}$$

we have

$$H_0(G, B) = B \otimes_G \mathbb{Z} = B_G\,. \tag{3.4}$$

We may summarize our results in

Proposition 3.1. *Let A, B be G-modules. Then*

$$H^0(G, A) = A^G\,, \qquad H_0(G, B) = B_G\,.$$

If A, B are trivial G-modules, then

$$H^0(G, A) = A\,, \qquad H_0(G, B) = B\,.$$

Proof. It is immediate that, in case the G-action is trivial, $A^G = A$ and $B_G = B$. $\square$

Exercises:

3.1. Express the isomorphism $\mathrm{Hom}_G(\mathbb{Z}, A) \cong A^G$ as an equivalence of functors.

3.2. Show that $A^G = {}_{IG}A = \{a \mid \lambda a = 0, \lambda \in IG\}$.

3.3. Let $F^l: \mathfrak{Ab} \to \mathfrak{M}_G$ assign to each abelian group A the trivial left G-module with underlying abelian group A. Show that F^l is left adjoint to the functor $-^G$. Similarly show that F^r (obvious definition) is right adjoint to the functor $-_G$.

3.4. Prove, without appeal to homology theory, that if $0 \to A' \to A \xrightarrow{\varepsilon} A'' \to 0$ is a short exact sequence of G-modules, then

$$0 \longrightarrow A'^G \longrightarrow A^G \xrightarrow{\varepsilon_*} A''^G$$

is exact. Give an example where ε_* is not surjective. Carry out a similar exercise for the short exact sequence $0 \to B' \to B \to B'' \to 0$ of right G-modules, and the functor $-_G$.

3.5. Express the functorial dependence of A^G, B_G on G.

4. H^1, H_1 with Trivial Coefficient Modules

It turns out to be natural to begin with a study of H_1. By definition we have $H_1(G,B) = \mathrm{Tor}_1^G(B,\mathbb{Z})$. If we take the obvious $\mathbb{Z}G$-free presentation of $\mathbb{Z}$, i.e.,

$$IG \overset{\iota}{\rightarrowtail} \mathbb{Z}G \overset{\varepsilon}{\twoheadrightarrow} \mathbb{Z}\,, \tag{4.1}$$

we get the exact sequence

$$0 \to H_1(G,B) \to B \otimes_G IG \xrightarrow{\iota_*} B \otimes_G \mathbb{Z}G \to H_0(G,B) \to 0\,.$$

We therefore obtain, for an arbitrary G-module B,

$$H_1(G,B) = \ker(\iota_* : B \otimes_G IG \to B) \tag{4.2}$$

where $\iota_*(b \otimes (x-1)) = bx - b$, $b \in B$, $x \in G$. In order to compute the first homology group for B a *trivial* G-module we remark that then ι_* is the zero homomorphism and hence $H_1(G,B) \cong B \otimes_G IG$. To compute $B \otimes_G IG$ when B is trivial, we have to consider the subgroup of $B \otimes IG$ generated by $b \otimes y(x-1) - by \otimes (x-1)$. But $by \otimes (x-1) = b \otimes (x-1)$; hence the subgroup is generated by $b \otimes (y-1)(x-1)$ and so, if B is a trivial G-module, $B \otimes_G IG \cong B \otimes IG/(IG)^2$.

Finally, let $G_{ab} = G/G'$ denote the quotient of G by its commutator subgroup $G' = [G,G]$, i.e., the subgroup of G generated by all elements of the form $x^{-1}y^{-1}xy$, $x, y \in G$. By Lemma 4.1 below we obtain, for a trivial G-module B,

$$H_1(G,B) \cong B \otimes IG/(IG)^2 \cong B \otimes G/G'\,. \tag{4.3}$$

In particular we note the result (well known to topologists!)

$$H_1(G,\mathbb{Z}) \cong G/G' = G_{ab}\,. \tag{4.4}$$

Lemma 4.1. $\mathbb{Z} \otimes_G IG = IG/(IG)^2 \cong G_{ab}$.

Proof. The first equality is already proved, so we have only to show that $IG/(IG)^2 \cong G_{ab}$. By Lemma 1.2 the abelian group IG is free on $W = \{x - 1 \mid 1 \neq x \in G\}$. The function $\psi : W \to G/G'$ defined by

$$\psi(x-1) = xG'$$

extends uniquely to $\psi' : IG \to G/G'$. Since

$$(x-1)(y-1) = (xy-1) - (x-1) - (y-1)\,,$$

ψ' factors through $\psi'' : IG/(IG)^2 \to G/G'$.

On the other hand, the definition $\varphi(x) = (x-1) + (IG)^2$ yields (by the same calculation as above) a group homomorphism $\varphi' : G \to IG/(IG)^2$ inducing $\varphi'' : G/G' \to IG/(IG)^2$. It is trivial that φ'' and ψ'' are inverse to each other. □

We now turn to cohomology. Again by definition we have

$$H^1(G, A) = \operatorname{Ext}^1_G(\mathbb{Z}, A),$$

and (4.1) yields the exact sequence

$$0 \to H^0(G, A) \to \operatorname{Hom}_G(\mathbb{Z}G, A) \xrightarrow{\iota^*} \operatorname{Hom}_G(IG, A) \to H^1(G, A) \to 0.$$

We obtain for an arbitrary G-module A,

$$H^1(G, A) = \operatorname{coker}(\iota^* : A \to \operatorname{Hom}_G(IG, A)) \tag{4.5}$$

where $\iota^*(a)(x-1) = xa - a$, $a \in A$, $x \in G$. For A a trivial G-module we remark that ι^* is the zero homomorphism; hence

$$H^1(G, A) \cong \operatorname{Hom}_G(IG, A).$$

Moreover, $\varphi : IG \to A$ is a homomorphism of G-modules if and only if $\varphi(x(y-1)) = x\varphi(y-1) = \varphi(y-1)$, $x, y \in G$; hence if and only if

$$\varphi((x-1)(y-1)) = 0.$$

Using Lemma 4.1 we therefore obtain, for A a trivial G-module,

$$H^1(G, A) \cong \operatorname{Hom}(IG/(IG)^2, A) \cong \operatorname{Hom}(G_{ab}, A). \tag{4.6}$$

The relation of (4.6) to (4.3) which asserts that, for a trivial G-module A,

$$H^1(G, A) \cong \operatorname{Hom}(H_1(G, \mathbb{Z}), A)$$

is a special case of the *universal coefficient theorem* (see Theorem V. 3.3), to be discussed in detail later (Section 15).

Exercises:

4.1. Use the adjointness of Exercise 3.3 to prove $\operatorname{Hom}_G(IG, A) \cong \operatorname{Hom}(IG/(IG)^2, A)$ for A a trivial G-module.

4.2. Let H, G be two groups, let A be a right H-module, let B be a left H-right G-bimodule, and let C be a left G-module. Prove

$$(A \otimes_H B) \otimes_G C \cong A \otimes_H (B \otimes_G C).$$

Use this to show that,for a trivial right G-module M

$$M \otimes_G IG \cong (M \otimes \mathbb{Z}) \otimes_G IG \cong M \otimes (\mathbb{Z} \otimes_G IG) \cong M \otimes IG/(IG)^2.$$

4.3. Show that the isomorphisms

$$H_1(G, B) \cong B \otimes G_{ab},$$
$$H^1(G, A) \cong \operatorname{Hom}(G_{ab}, A),$$

where A, B are trivial modules, are natural in A, B and G.

4.4. Let $0 \to B' \to B \to B'' \to 0$ be a short exact sequence of abelian groups. Show that the connecting homomorphism $\omega: H_1(G, B'') \to H_0(G, B')$ is trivial. Does the conclusion follow if $0 \to B' \to B \to B'' \to 0$ is a short exact sequence of G-modules?

4.5. Carry out an exercise similar to Exercise 4.4 above in cohomology.

5. The Augmentation Ideal, Derivations, and the Semi-Direct Product

In the previous section we evaluated $H^1(G, A)$ for a trivial G-module A. Here we give an interpretation of $H^1(G, A)$ in the non-trivial case. (The analogous interpretation of $H_1(G, A)$ is possible, but does not seem to have any interesting applications.)

Definition. A function $d: G \to A$ from the group G into the G-module A with the property

$$d(x \cdot y) = dx + x \circ (dy), \quad x, y \in G, \tag{5.1}$$

is called a *derivation (*or *crossed homomorphism)* from G into A.

Notice that, if d is a derivation, $d(1) = 0$. The set of all derivations $d: G \to A$ may be given an obvious abelian group structure; this abelian group will be denoted by $\operatorname{Der}(G, A)$. Note that for a G-module homomorphism $\alpha: A \to A'$ and a derivation $d: G \to A$ the composition

$$\alpha \circ d: G \to A'$$

again is a derivation. With this $\operatorname{Der}(G, -): \mathfrak{M}_G \to \mathfrak{Ab}$ becomes a functor. For A a trivial G-module a derivation $d: G \to A$ is simply a group homomorphism.

Next we relate the derivations to the augmentation ideal.

Theorem 5.1. *The homomorphism* $\eta: \operatorname{Der}(G, A) \to \operatorname{Hom}_G(IG, A)$ *defined by*

$$(\eta(d))(y - 1) = d(y), \quad y \in G \tag{5.2}$$

is a natural isomorphism.

The theorem claims that the augmentation ideal IG *represents* the functor $\operatorname{Der}(G, -)$.

Proof. Given a derivation $d: G \to A$, we claim that the group homomorphism $\eta(d) = \varphi_d: IG \to A$ defined by $\varphi_d(y - 1) = dy$, $y \in G$, is a G-

module homomorphism. Indeed

$$\begin{aligned}\varphi_d(x(y-1)) &= \varphi_d((xy-1)-(x-1)) = d(xy)-dx\\ &= dx + x(dy) - dx = x \circ \varphi_d(y-1).\end{aligned}$$

Conversely, given a G-module homomorphism $\varphi : IG \rightarrow A$, we define a map $d_\varphi : G \rightarrow A$ by $d_\varphi(y) = \varphi(y-1)$. We claim that d_φ is a derivation. Indeed

$$\begin{aligned}d_\varphi(xy) &= \varphi(xy-1) = \varphi(x(y-1)+(x-1))\\ &= x\varphi(y-1) + \varphi(x-1) = xd_\varphi(y) + d_\varphi(x).\end{aligned}$$

It is quite obvious that η is a homomorphism of abelian groups and that $\varphi \mapsto d_\varphi$ is inverse to η. □

The above theorem now allows us to give a description of the first cohomology group in terms of derivations. By (4.5) $H^1(G, A)$ is the quotient of $\mathrm{Hom}_G(IG, A)$ by the subgroup of homomorphisms $\varphi : IG \rightarrow A$ of the form $\varphi(x-1) = xa - a$ for some $a \in A$. The derivation $d_\varphi : G \rightarrow A$ associated with this map φ has the form

$$d_\varphi(x) = (x-1)\, a \tag{5.3}$$

for some $a \in A$.

Derivations of this kind are called *inner derivations* (or *principal crossed homomorphisms*). The subgroup of $\mathrm{Der}(G, A)$ of inner derivations is denoted by $\mathrm{Ider}(G, A)$. We then can state

Corollary 5.2. $H^1(G, A) \cong \mathrm{Der}(G, A)/\mathrm{Ider}(G, A)$. □

Definition. Given a group G and a G-module A, we define their *semi-direct-product* $A \times G$ in the following way. The underlying set of $A \times G$ is the set of ordered pairs (a, x), $a \in A$, $x \in G$. The product is given by

$$(a, x) \cdot (a', x') = (a + xa', xx'). \tag{5.4}$$

This product is easily shown to be associative, to have a neutral element $(0, 1)$, and an inverse $(a, x)^{-1} = (-x^{-1}a, x^{-1})$. There is an obvious monomorphism of groups $i : A \rightarrow A \times G$, given by $i(a) = (a, 1)$, $a \in A$. Also, there is an obvious epimorphism of groups $p : A \times G \rightarrow G$, defined by $p(a, x) = x$, $a \in A$, $x \in G$. It is easy to see that iA is normal in $A \times G$ with quotient G, the canonical projection being p; thus the sequence

$$A \overset{i}{\rightarrowtail} A \times G \overset{p}{\twoheadrightarrow} G \tag{5.5}$$

is exact. We say that $A \times G$ is an extension of G by A (see Section 10 for the precise definition of the term *extension*). It follows that $A \times G$ acts by conjugation in iA; we denote this action by $\circ$. We have

$$(a', x) \circ (a, 1) = (a', x) \cdot (a, 1) \cdot (-x^{-1}a', x^{-1}) = (xa, 1), \tag{5.6}$$
$$a, a' \in A, \qquad x \in G.$$

In other words, the element $(a', x) \in A \times G$ acts in iA in the same way as the element $x \in G$ acts by the given G-module structure in A. Thus we may regard A itself as an $(A \times G)$-module by $(a', x) \circ a = xa$.

We finally note that in (5.5) there is a group homomorphism $s: G \to A \times G$, given by $sx = (0, x)$, $x \in G$, which is a one-sided inverse to p, $ps = 1_G$. It is because of the existence of the map s that we shall refer – by analogy with the abelian case – to the extension (5.5) as the *split* extension; s is called a *splitting*.

In contrast with the abelian case however the splitting s does *not* force $A \times G$ to be the *direct* (but only the semi-direct) product of A and G. The projection $q: A \times G \to A$, given by $q(a, x) = a$, is not a group homomorphism; however it *is* a derivation:

$$q((a, x) \cdot (a', x')) = q(a + xa', xx') = a + xa' = q(a, x) + (a, x) \circ q(a', x). \quad (5.7)$$

We now easily deduce the following universal property of the semi-direct product:

Proposition 5.3. *Suppose given a group G and a G-module A. To every group homomorphism $f: X \to G$ and to every f-derivation $d: X \to A$ (i.e. d is a derivation if A is regarded as an X-module via f), there exists a unique group homomorphism $h: X \to A \times G$ such that the following diagram is commutative:*

$$\begin{tikzcd}
& X \arrow[dl, "d"'] \arrow[d, dotted, "h"] \arrow[dr, "f"] & \\
A & A \times G \arrow[l, "q"'] \arrow[r, "p"] & G
\end{tikzcd}$$

Conversely, every group homomorphism $h: X \to A \times G$ determines a homomorphism $f = ph: X \to G$ and an f-derivation $qh = d: X \to A$.

The proof is obvious; h is defined by $hx = (dx, fx)$, $x \in X$, and it is straightforward to check that h is a homomorphism. $\square$

By taking $X = G$ and $f = 1_G$ we obtain:

Corollary 5.4. *The set of derivations from G into A is in one-to-one correspondence with the set of group homomorphisms $f: G \to A \times G$ for which $pf = 1_G$.* $\square$

As an application we shall prove the following result on the augmentation ideal of a free group.

Theorem 5.5. *The augmentation ideal IF of a group F which is free on the set S is the free $\mathbb{Z}F$-module on the set $S - 1 = \{s - 1 \mid s \in S\}$.*

Proof. We show that any function f from the set $\{s - 1 \mid s \in S\}$ into an F-module M may be uniquely extended to an F-module homomorphism $f': IF \to M$. First note that uniqueness is clear, since $\{s - 1 \mid s \in S\}$ generates IF as F-module by Lemma 1.2 (ii). Using the fact that F is free on S we define a group homomorphism $\bar{f}: F \to M \times F$ by $\bar{f}(s) = (f(s-1), s)$. By Corollary 5.4 $\bar{f}$ defines a derivation $d: F \to M$ with $d(s) = f(s-1)$. By Theorem 5.1 d corresponds to an F-module homomorphism $f': IF \to M$ with $f'(s-1) = f(s-1)$. $\square$

Corollary 5.6. *For a free group F, we have*

$$H^n(F, A) = 0 = H_n(F, B)$$

for all F-modules A, B and all $n \geqq 2$.

Proof. $IF \rightarrowtail \mathbb{Z}F \twoheadrightarrow \mathbb{Z}$ is an F-free resolution of $\mathbb{Z}$. $\square$

Exercises:

5.1. Let $d: G \to A$ be a derivation. Interpret and prove the following relation

$$d(x^n) = \left(\frac{x^n - 1}{x - 1}\right) dx, \quad n \in \mathbb{Z}, \quad x \in G.$$

5.2. Let $A \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} G$ be an exact sequence of groups with A abelian. Let $s: G \to E$ be a one-sided inverse of p, $ps = 1_G$. Show that $E \cong A \times G$.

5.3. Let the (multiplicative) cyclic group of order 2, C_2, operate on $\mathbb{Z}$ by

$$x \circ n = -n,$$

where x generates C_2. Use Corollary 5.2 to compute $H^1(C_2, \mathbb{Z})$, for this action of C_2 on $\mathbb{Z}$.

5.4. Carry out a similar exercise to Exercise 5.3, replacing C_2 by C_{2k}.

5.5. Let C_m operate on $\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$ (m copies) by

$$x a_i = a_{i+1}, \quad i = 1, \ldots m, \quad (a_{m+1} = a_1)$$

where x generates C_m and a_i generates the i^{th} copy of $\mathbb{Z}_2$. Compute

$$H^1(C_m, \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2),$$

for this action of C_m on $\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$.

5.6. For a fixed group Q consider the category $\mathfrak{G}/Q$ of $\mathfrak{G}$-objects over Q. Consider the functors $F: \mathfrak{G}/Q \to \mathfrak{M}_Q$ and $U: \mathfrak{M}_Q \to \mathfrak{G}/Q$ defined by $F(G \to Q) = IG \otimes_G \mathbb{Z}Q$ and $U(A) = (A \times Q \overset{p}{\to} Q)$, where $G \to Q$ is a group homomorphism, A is a $\mathbb{Z}Q$-module and $A \times Q$ is the semi-direct product. Show that $F \dashv U$. Deduce Proposition 5.3 and Corollary 5.4.

6. A Short Exact Sequence

In this section we shall assign to any extension of groups $N \rightarrowtail G \twoheadrightarrow Q$ a short exact sequence of Q-modules. We shall later apply this exact sequence to compute the (co)homology of cyclic groups (Section 7), and to deduce a 5-term exact sequence (Section 8) which will be basic for our treatment of extension theory. We start with the following two lemmas.

Lemma 6.1. *If* $N \overset{\iota}{\rightarrowtail} G \overset{p}{\twoheadrightarrow} Q$ *is an exact sequence of groups, then* $\mathbb{Z} \otimes_N \mathbb{Z}G \cong \mathbb{Z}Q$ *as right G-modules.*

Proof. As abelian group $\mathbb{Z} \otimes_N \mathbb{Z}G$ is free on the set of right cosets $G/N \cong Q$. It is easy to see that the right action of G induced by the product in $\mathbb{Z}G$ is the right G-action in $\mathbb{Z}Q$ via p. $\square$

Lemma 6.2. *If* $N \rightarrowtail G \twoheadrightarrow Q$ *is an exact sequence of groups and if A is a left G-module, then* $\operatorname{Tor}_n^N(\mathbb{Z}, A) \cong \operatorname{Tor}_n^G(\mathbb{Z}Q, A)$.

Proof. The argument that follows applies, in generalized form, to a change of rings (see Proposition IV. 12.2). Let X be a G-projective resolution of A, hence by Corollary 1.4 also an N-projective resolution of A. By Lemma 6.1, $\mathbb{Z} \otimes_N X \cong \mathbb{Z} \otimes_N \mathbb{Z}G \otimes_G X \cong \mathbb{Z}Q \otimes_G X$; which proves Lemma 6.2. $\square$

Consider now the sequence of G-modules $IG \overset{\iota}{\rightarrowtail} \mathbb{Z}G \overset{\varepsilon}{\twoheadrightarrow} \mathbb{Z}$. Tensoring with $\mathbb{Z}Q$ over G we obtain

$$0 \to \operatorname{Tor}_1^G(\mathbb{Z}Q, \mathbb{Z}) \to \mathbb{Z}Q \otimes_G IG \xrightarrow{\iota_*} \mathbb{Z}Q \otimes_G \mathbb{Z}G \xrightarrow{\varepsilon_*} \mathbb{Z}Q \otimes_G \mathbb{Z} \to 0\,. \tag{6.1}$$

Note that each term in (6.1) has a natural Q-module structure, and that (6.1) is a sequence of Q-modules. It is easy to see that the map

$$\mathbb{Z}Q \cong \mathbb{Z}Q \otimes_G \mathbb{Z}G \xrightarrow{\varepsilon_*} \mathbb{Z}Q \otimes_G \mathbb{Z} \cong \mathbb{Z}$$

is the augmentation of $\mathbb{Z}Q$. By Lemma 6.2,

$$\operatorname{Tor}_1^G(\mathbb{Z}Q, \mathbb{Z}) \cong \operatorname{Tor}_1^N(\mathbb{Z}, \mathbb{Z}) = H_1(N, \mathbb{Z}) \cong N/N'\,.$$

Hence we get the following important result.

Theorem 6.3. *Let* $N \rightarrowtail G \twoheadrightarrow Q$ *be an exact sequence of groups. Then*

$$0 \to N_{ab} \xrightarrow{\kappa} \mathbb{Z}Q \otimes_G IG \xrightarrow{\nu} IQ \to 0 \tag{6.2}$$

is an exact sequence of Q-modules. $\square$

For our applications of (6.2) we shall need an explicit description of the Q-module structure in $N_{ab} = N/N'$, as well as of the map

$$\kappa : N_{ab} \to \mathbb{Z}Q \otimes_G IG\,.$$

For that we compute $\operatorname{Tor}_1^N(\mathbb{Z},\mathbb{Z})$ by the N-free presentation $IN \rightarrowtail \mathbb{Z}N \twoheadrightarrow \mathbb{Z}$ of $\mathbb{Z}$ and by the G-free (hence N-free) presentation $IG \rightarrowtail \mathbb{Z}G \twoheadrightarrow \mathbb{Z}$. We obtain the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Tor}_1^N(\mathbb{Z},\mathbb{Z}) \arrow[r] \arrow[d, "\wr"] & \mathbb{Z} \otimes_N IN \arrow[r] \arrow[d] & \mathbb{Z} \otimes_N \mathbb{Z}N \arrow[r] \arrow[d] & \mathbb{Z} \otimes_N \mathbb{Z} \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \operatorname{Tor}_1^N(\mathbb{Z},\mathbb{Z}) \arrow[r] \arrow[d, "\wr"] & \mathbb{Z} \otimes_N IG \arrow[r] \arrow[d, "\wr"] & \mathbb{Z} \otimes_N \mathbb{Z}G \arrow[r] \arrow[d, "\wr"] & \mathbb{Z} \otimes_N \mathbb{Z} \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \operatorname{Tor}_1^G(\mathbb{Z}Q,\mathbb{Z}) \arrow[r] & \mathbb{Z}Q \otimes_G IG \arrow[r] & \mathbb{Z}Q \otimes_G \mathbb{Z}G \arrow[r] & \mathbb{Z}Q \otimes_G \mathbb{Z} \arrow[r] & 0
\end{tikzcd} \tag{6.3}$$

The vertical maps in the top half are induced by the embedding $N \rightarrowtail G$, in the bottom half they are given as in Lemma 6.2. If we now trace the map $\kappa: N_{ab} \xrightarrow{\sim} \mathrm{Tor}_1^N(\mathbb{Z}, \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z} \otimes_N IN \to \mathbb{Z} \otimes_N IG \xrightarrow{\sim} \mathbb{Z}Q \otimes_G IG$, we see that κ is given by

$$\kappa(nN') = 1_Q \otimes (n-1) \in \mathbb{Z}Q \otimes_G IG, \qquad n \in N. \tag{6.4}$$

As a consequence we shall prove that the Q-module structure in N_{ab} is (as expected) induced by conjugation in the group G, that is,

$$y \circ nN' = (xnx^{-1})N', \tag{6.5}$$

where $n \in N$ and $x \in G$ is a representative of $y \in Q$, i.e., $y = px$ (see Lemma 6.1). To prove this we proceed as follows, using the fact that κ is a Q-module monomorphism. Then

$$\kappa(y \circ nN') = y \otimes (n-1) = 1 \otimes x(n-1) \in \mathbb{Z}Q \otimes_G IG.$$

Since $xnx^{-1} \in N$, it follows that $1 \otimes (xnx^{-1} - 1)(x-1) = 0$ in $\mathbb{Z}Q \otimes_G IG$, so we get $1 \otimes x(n-1) = 1 \otimes (xnx^{-1} - 1)$ which obviously is the κ-image of $xnx^{-1}N' \in N/N'$, proving (6.5).

Corollary 6.4. *Let $R \rightarrowtail F \twoheadrightarrow Q$ be an exact sequence of groups with F a free group, i.e. a free presentation of Q. Then*

$$0 \to R_{ab} \xrightarrow{\kappa} \mathbb{Z}Q \otimes_F IF \xrightarrow{\nu} IQ \to 0 \tag{6.6}$$

is a Q-free presentation of IQ.

Proof. By Theorem 5.5 IF is F-free, therefore $\mathbb{Z}Q \otimes_F IF$ is Q-free. □

Corollary 6.5. *Let $R \rightarrowtail F \twoheadrightarrow Q$ be a free presentation of Q. Then for any Q-modules A, B and all $n \geq 3$*

$$\begin{aligned} H_n(Q, B) &\cong \mathrm{Tor}_{n-1}^Q(B, IQ) \cong \mathrm{Tor}_{n-2}^Q(B, R_{ab}), \\ H^n(Q, A) &\cong \mathrm{Ext}_Q^{n-1}(IQ, A) \cong \mathrm{Ext}_Q^{n-2}(R_{ab}, A). \end{aligned} \tag{6.7}$$

Proof. The exact sequences $IQ \rightarrowtail \mathbb{Z}Q \twoheadrightarrow \mathbb{Z}$ and (6.6) together with (2.7) give the result. □

Exercises:

6.1. Establish the naturality of the isomorphisms in Corollary 6.5.

6.2. Generalize Corollary 6.5 to establish natural homomorphisms

$$H_n(Q, B) \to \mathrm{Tor}_{n-2}^Q(B, N/N'),$$
$$\mathrm{Ext}_Q^{n-2}(N/N', A) \to H^n(Q, A)$$

associated with $N \rightarrowtail G \twoheadrightarrow Q$, $n \geq 3$.

6.3. Let $R \rightarrowtail F \twoheadrightarrow Q$ be the free presentation of Q, free abelian on 2 generators, by F, free on two generators. Describe R_{ab} as a Q-module.

7. The (Co)Homology of Finite Cyclic Groups

We denote by C_k the (multiplicatively written) cyclic group of order k with generator τ, by C the (multiplicatively written) infinite cyclic group with generator t. Given C_k, we consider the exact sequence of groups $C \overset{\mu}{\rightarrowtail} C \overset{\varepsilon}{\twoheadrightarrow} C_k$ where $\mu(t) = t^k$, $\varepsilon(t) = \tau$. By Corollary 6.4 we have a C_k-free presentation

$$\mathbb{Z} \overset{\kappa}{\rightarrowtail} \mathbb{Z}C_k \otimes_C IC \overset{\nu}{\twoheadrightarrow} IC_k, \tag{7.1}$$

where the domain $\mathbb{Z}$ of κ is C_{ab}, the infinite cyclic group generated by t, written *additively* and regarded as a trivial C_k-module. For $n \geqq 3$ and for a C_k-module A, Corollary 6.5 yields

$$H^n(C_k, A) \cong \operatorname{Ext}^{n-2}_{C_k}(\mathbb{Z}, A) = H^{n-2}(C_k, A). \tag{7.2}$$

Hence we obtain for $n = 1, 2, \ldots$

$$\begin{aligned} H^{2n-1}(C_k, A) &\cong H^1(C_k, A), \\ H^{2n}(C_k, A) &\cong H^2(C_k, A). \end{aligned} \tag{7.3}$$

Since $H^0(C_k, A) = A^{C_k}$ by (3.2), the cohomology of C_k is known, once it is computed in dimensions 1 and 2. The higher dimensional cohomology groups then are determined by (7.3) which says that the cohomology of C_k is *periodic* with period 2.

By Theorem 5.5 the augmentation ideal IC is C-free on $t-1$; hence $\mathbb{Z}C_k \otimes_C IC \cong \mathbb{Z}C_k$. The sequence (7.1) therefore becomes

$$\mathbb{Z} \overset{\alpha}{\rightarrowtail} \mathbb{Z}C_k \overset{\beta}{\twoheadrightarrow} IC_k \tag{7.4}$$

Now by (6.4) κ sends the generator t of $\mathbb{Z}$ into $1 \otimes_C (t^k - 1) \in \mathbb{Z}C_k \otimes_C IC$. Since $1 \otimes_C (t^k - 1) = (\tau^{k-1} + \tau^{k-2} + \cdots + \tau + 1) \otimes_C (t-1)$, the map α is described by $\alpha(t) = \tau^{k-1} + \tau^{k-2} + \cdots + \tau + 1 \in \mathbb{Z}C_k$. The map β clearly is multiplication in $\mathbb{Z}C_k$ by $\tau - 1$, whence it follows from (7.4) that

$$IC_k = \mathbb{Z}C_k/(\tau^{k-1} + \tau^{k-2} + \cdots + \tau + 1). \tag{7.5}$$

Using the remark (2.6) we obtain

$$\begin{aligned} H^2(C_k, A) &= \operatorname{coker}(\alpha^* : \operatorname{Hom}_{C_k}(\mathbb{Z}C_k, A) \to \operatorname{Hom}_{C_k}(\mathbb{Z}, A)) \\ &= \{a \in A \mid \tau a = a\}/(\tau^{k-1} + \tau^{k-2} + \cdots + \tau + 1)A \\ H^1(C_k, A) &= \operatorname{coker}(\iota^* : \operatorname{Hom}_{C_k}(\mathbb{Z}C_k, A) \to \operatorname{Hom}_{C_k}(IC_k, A)) \\ &= \{a \in A \mid (\tau^{k-1} + \tau^{k-2} + \cdots + \tau + 1)a = 0\}/(\tau - 1)A, \end{aligned}$$

the latter using (7.5). Proceeding analogously for homology, one obtains the homology of C_k (see Proposition 7.1).

If we define C_k-homomorphisms $\varphi, \psi : A \to A$ by

$$\varphi a = (\tau - 1)a, \quad \psi a = (\tau^{k-1} + \tau^{k-2} + \cdots + \tau + 1)a, \quad a \in A, \tag{7.6}$$

and similar maps φ, ψ for the right C_k-module B, we can state our results as follows:

Proposition 7.1. *Let C_k be a cyclic group of order k with generator τ, and let A, B be C_k-modules. Then, for $n \geqq 1$,*

$$\begin{aligned} H^{2n-1}(C_k, A) &= \ker\psi/\mathrm{im}\,\varphi\,, \quad & H^{2n}(C_k, A) &= \ker\varphi/\mathrm{im}\,\psi\,; \\ H_{2n-1}(C_k, B) &= \ker\varphi/\mathrm{im}\,\psi\,, \quad & H_{2n}(C_k, B) &= \ker\psi/\mathrm{im}\,\varphi\,. \end{aligned} \tag{7.7}$$

while $H^0(C_k, A) = \ker\varphi$, $H_0(C_k, A) = \mathrm{coker}\,\varphi$. For A, B trivial C_k-modules we have

$$\begin{aligned} H^{2n-1}(C_k, A) &= \ker k\,, \quad & H^{2n}(C_k, A) &= \mathrm{coker}\,k\,; \\ H_{2n-1}(C_k, B) &= \mathrm{coker}\,k\,, \quad & H_{2n}(C_k, B) &= \ker k \end{aligned} \tag{7.8}$$

where $\psi = k : A \to A$ (resp. $k : B \to B$) is multiplication by k.

It follows readily from these results that a (non-trivial) finite cyclic group C_k has $H^n(C_k, \mathbb{Z}) \neq 0$ for infinitely many n. Hence there cannot exist a *finite* C_k-projective resolution of $\mathbb{Z}$.

Exercises:

7.1. Prove the following statement: To a group G containing an element $x \neq 1$ of finite order there cannot exist a finite G-projective resolution of $\mathbb{Z}$.

7.2. Describe explicitly a periodic free resolution of $\mathbb{Z}$ as C_k-module.

7.3. Compute $H^n(C_k, \mathbb{Z})$, $H_n(C_k, \mathbb{Z})$ explicitly.

7.4. Use Exercise 2.5 and the periodic resolution of Exercise 7.2 to compute explicitly the change-of-rings map in integral homology for $f: C_m \to C_n$ where $f(t) = s^r$, t is the generator of C_m, s is the generator of C_n, and $n \mid rm$.

7.5. Let C_m be generated by t, and C_{m^2} by s. Define an action of C_m on C_{m^2} by $t \circ s = s^{m+1}$. Using Exercise 7.4, compute the resulting C_m-module structure on $H_j(C_{m^2})$, $j \geqq 0$, the integral homology of C_{m^2}.

7.6. Under the same hypotheses as in Exercise 7.5 compute $H_i(C_m, H_j(C_{m^2}))$, where m is an odd prime.

7.7. Let G be a group with one defining rèlator, i.e. there exists a free group F and an element $r \in F$ such that $G \cong F/R$ where R is the smallest normal subgroup of F containing r. It has been shown that the relator r may be written in a unique way as $r = w^q$, where w cannot be written as a proper power of any other element in F. Note that q may be 1. Denote by C the cyclic subgroup generated by the image of w in G. R. C. Lyndon has proved the deep result that $R_{ab} \cong \mathbb{Z} \otimes_C \mathbb{Z}G$. Using this and Corrollary 6.5 show that, for G- modules A, B, we have $H^n(G, A) \cong H^n(C, A)$ and $H_n(G, B) \cong H_n(C, B)$ for $n \geqq 3$. Deduce that if r cannot be written as a proper power (i.e., if $q = 1$) then G is torsion-free.

8. The 5-Term Exact Sequences

Theorem 8.1. *Let $N \rightarrowtail G \twoheadrightarrow Q$ be an exact sequence of groups. For Q-modules A, B the following sequences are exact (and natural)*

$$\begin{gathered} H_2(G,B) \to H_2(Q,B) \to B \otimes_Q N_{ab} \to B \otimes_G IG \to B \otimes_Q IQ \to 0\,; \\ 0 \to \mathrm{Der}(Q,A) \to \mathrm{Der}(G,A) \to \mathrm{Hom}_Q(N_{ab},A) \to H^2(Q,A) \to H^2(G,A)\,. \end{gathered} \tag{8.1}$$

Proof. We only prove the first of the two sequences, the cohomology sequence being proved similarly, using in addition the natural isomorphisms $\mathrm{Der}(G,A) \cong \mathrm{Hom}_G(IG,A)$, $\mathrm{Der}(Q,A) \cong \mathrm{Hom}_Q(IQ,A)$.

By Theorem 6.3 $N_{ab} \overset{\kappa}{\rightarrowtail} \mathbb{Z}Q \otimes_G IG \twoheadrightarrow IQ$ is exact. Tensoring with B over $\mathbb{Z}Q$ yields the exact sequence

$$\mathrm{Tor}_1^Q(B, \mathbb{Z}Q \otimes_G IG) \to \mathrm{Tor}_1^Q(B, IQ) \to B \otimes_Q N_{ab} \to B \otimes_G IG \to B \otimes_Q IQ \to 0$$

since, plainly, $B \otimes_Q \mathbb{Z}Q \otimes_G IG \cong B \otimes_G IG$. Moreover, by (2.8)

$$H_2(Q,B) \cong \mathrm{Tor}_1^Q(B, IQ)\,.$$

It therefore suffices to find a (natural) map

$$\mathrm{Tor}_1^G(B, IG) \to \mathrm{Tor}_1^Q(B, \mathbb{Z}Q \otimes_G IG)$$

and to show that it is epimorphic. To do so, we choose a Q-projective presentation $M \rightarrowtail P \twoheadrightarrow B$ of B. Applying the functors $- \otimes_G IG$ and $- \otimes_Q (\mathbb{Z}Q \otimes_G IG)$ we obtain the commutative diagram, with exact rows,

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{Tor}_1^G(P, IG) \arrow[r] \arrow[d] & \mathrm{Tor}_1^G(B, IG) \arrow[r] \arrow[d] & M \otimes_G IG \arrow[r] \arrow[d, equal] & P \otimes_G IG \arrow[r] \arrow[d, equal] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & \mathrm{Tor}_1^Q(B, \mathbb{Z}Q \otimes_G IG) \arrow[r] & M \otimes_G IG \arrow[r] & P \otimes_G IG \arrow[r] & \cdots \end{tikzcd}$$

which proves immediately that the map in question is epimorphic. Naturality of the sequence is left as an exercise. □

We remark that the sequences (8.1) can be altered to

$$\begin{gathered} H_2(G,B) \to H_2(Q,B) \to B \otimes_Q N_{ab} \to H_1(G,B) \to H_1(Q,B) \to 0\,, \\ 0 \to H^1(Q,A) \to H^1(G,A) \to \mathrm{Hom}_Q(N_{ab},A) \to H^2(Q,A) \to H^2(G,A)\,. \end{gathered} \tag{8.2}$$

Again we concentrate on the homology sequence. Using (4.2) we obtain the following commutative diagram, with exact rows and columns,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & H_1(G,B) \arrow[r, "p_*"] \arrow[d] & H_1(Q,B) \arrow[d] & \\
\cdots \arrow[r] & B\otimes_Q N_{ab} \arrow[r, "\kappa_*"] & B\otimes_G IG \arrow[r, "p_*"] \arrow[d, "\iota_*"] & B\otimes_Q IQ \arrow[r] \arrow[d, "\iota_*"] & 0 \\
 & & B \arrow[r, equal] \arrow[d] & B \arrow[d] & \\
 & & \vdots & \vdots &
\end{tikzcd}$$

It is obvious now that $p_*: H_1(G,B) \to H_1(Q,B)$ is epimorphic. Furthermore we have $0 = \iota_* p_* \kappa_* = \iota_* \kappa_* : B\otimes_Q N_{ab} \to B$, whence it follows that κ_* factors through $H_1(G,B)$. Exactness of (8.2) is then trivial.

We remark that the sequences (8.2) coincide with the sequences (8.1) in case A, B are trivial Q-modules.

Finally, with a view to application in the next section, we write down explicitly the sequence in the case of integral homology. For short we write $H_n(G)$ for $H_n(G,\mathbb{Z})$, and analogously for Q. By (4.4) we have $H_1(G)\cong G_{ab}$, $H_1(Q)\cong Q_{ab}$. Also, $\mathbb{Z}\otimes_Q N_{ab}$ is isomorphic to the quotient of N_{ab} by the subgroup generated by the elements $(y-1)\circ(nN')$ where $y\in Q$, $n\in N$, and $\circ$ denotes the Q-action. By (6.5) we see that $\mathbb{Z}\otimes_Q N_{ab}$ is therefore isomorphic to the quotient of N by the normal subgroup generated by $xnx^{-1}n^{-1}$ with $x\in G$, $n\in N$. This subgroup is normally denoted by $[G,N]$, so that

$$\mathbb{Z}\otimes_Q N_{ab} \cong N/[G,N]. \tag{8.3}$$

With these preparations we get the following

Corollary 8.2. *Let $N \rightarrowtail G \twoheadrightarrow Q$ be an exact sequence of groups. Then the following sequence is exact*

$$H_2(G)\to H_2(Q)\to N/[G,N]\to G_{ab}\to Q_{ab}\to 0. \tag{8.4}$$

Exercises:

8.1. Prove without homological algebra the exactness of

$$N/[G,N]\to G_{ab}\to Q_{ab}\to 0.$$

8.2. Use Theorem 8.1 to compute $H_2(C_k, B)$ and $H^2(C_k, A)$.
8.3. Prove the exactness of the cohomology sequence in Theorem 8.1 in detail.
8.4. Prove that the 5-term sequences of this section are natural.
8.5. Prove that the maps $H_2(G,B)\to H_2(Q,B)$ and $H_1(G,B)\to H_1(Q,B)$ of (8.2) are the maps given by (2.10). Similarly in cohomology.
8.6. Prove that if H is a normal subgroup of K of prime index, then

$$H/[K,H]\to H[K,K]/[K,K]$$

is monomorphic.

9. H_2, Hopf's Formula, and the Lower Central Series

Let $R\rightarrowtail F\twoheadrightarrow G$ be an exact sequence of groups with F free, i.e., a presentation of the group G. For B a G-module, Theorem 8.1 provides us with the exact sequence

$$H_2(F,B)\to H_2(G,B)\to B\otimes_G R_{ab}\to B\otimes_F IF\to B\otimes_G IG\to 0\,.$$

By Corollary 5.6 we have $H_2(F,B)=0$, whence

$$H_2(G,B)\cong \ker(B\otimes_G R_{ab}\to B\otimes_F IF)\,. \tag{9.1}$$

In case $B=\mathbb{Z}$ Corollary 8.2 leaves us with

$$H_2(G)\cong \ker(R/[F,R]\to F/[F,F])\,,$$

and we obtain *Hopf's formula* for the second integral homology group

$$H_2(G)\cong R\cap[F,F]/[F,R]\,. \tag{9.2}$$

As an immediate consequence we deduce that the group given by the formula on the right hand side of (9.2) is independent of the choice of presentation of G.

Next we state a result which relates the homology theory of a group to its lower central series.

Definition. Given a group G, we define a series of subgroups G_n, $n\geqq 0$, by

$$G_0=G\,,\qquad G_{n+1}=[G,G_n]\,. \tag{9.3}$$

This series is called the *lower central series* of G. A group G with $G_n=\{1\}$ is called *nilpotent of class* $\leqq n$.

It is easily proved by induction on n that the groups G_n are normal in G. Also, the quotients G_n/G_{n+1} are plainly abelian. A homomorphism $f: G\to H$ maps G_n into H_n for every $n\geqq 0$.

Theorem 9.1. *Let $f: G\to H$ be a group homomorphism such that the induced homomorphism $f_*: G_{ab}\to H_{ab}$ is an isomorphism, and that*

$$f_*: H_2(G)\to H_2(H)$$

is an epimorphism. Then f induces isomorphisms

$$f_n : G/G_n \xrightarrow{\sim} H/H_n\,, \qquad n \geqq 0\,.$$

Proof. We proceed by induction. For $n=0, 1$ the assertion is trivial or part of the hypotheses. For $n \geqq 2$ consider the exact sequences

$$G_{n-1} \rightarrowtail G \twoheadrightarrow G/G_{n-1}\,, \qquad H_{n-1} \rightarrowtail H \twoheadrightarrow H/H_{n-1}$$

and the associated 5-term sequences in homology (Corollary 8.2):

$$\begin{tikzcd} H_2(G) \arrow[r] \arrow[d, "\alpha_1"] & H_2(G/G_{n-1}) \arrow[r] \arrow[d, "\alpha_2"] & G_{n-1}/G_n \arrow[r] \arrow[d, "\alpha_3"] & G_{ab} \arrow[r] \arrow[d, "\alpha_4"] & (G/G_{n-1})_{ab} \arrow[r] \arrow[d, "\alpha_5"] & 0 \\ H_2(H) \arrow[r] & H_2(H/H_{n-1}) \arrow[r] & H_{n-1}/H_n \arrow[r] & H_{ab} \arrow[r] & (H/H_{n-1})_{ab} \arrow[r] & 0 \end{tikzcd} \tag{9.4}$$

Note that $[G, G_{n-1}] = G_n$, $[H, H_{n-1}] = H_n$ by definition. By naturality the map f induces homomorphisms α_i, $i=1, \ldots, 5$, such that (9.4) is commutative. By hypothesis α_1 is epimorphic and α_4 is isomorphic. By induction α_2 and α_5 are isomorphic. Hence by the generalized five Lemma (Exercise I. 1.2) α_3 is isomorphic. Next consider the diagram

$$\begin{tikzcd} G_{n-1}/G_n \arrow[r, tail] \arrow[d, "\alpha_3"] & G/G_n \arrow[r, two heads] \arrow[d, "f_n"] & G/G_{n-1} \arrow[d, "f_{n-1}"] \\ H_{n-1}/H_n \arrow[r, tail] & H/H_n \arrow[r, two heads] & H/H_{n-1}\,. \end{tikzcd}$$

The map $f: G \to H$ induces α_3, f_n, f_{n-1}. By the above α_3 is isomorphic, by induction f_{n-1} is isomorphic, hence f_n is isomorphic. □

Corollary 9.2. *Let $f: G \to H$ satisfy the hypotheses of Theorem* 9.1. *Suppose further that G, H are nilpotent. Then f is an isomorphism, $f: G \xrightarrow{\sim} H$.*

Proof. The assertion follows from Theorem 9.1 and the remark that there exists $n \geqq 0$ such that $G_n = \{1\}$ and $H_n = \{1\}$. □

Exercises:

9.1. Suppose $f: G \to H$ satisfies the hypotheses of Theorem 9.1. Prove that f induces a monomorphism $f: G\Big/\bigcap\limits_{n=0}^{\infty} G_n \rightarrowtail H\Big/\bigcap\limits_{n=0}^{\infty} H_n$.

9.2. Let $R \rightarrowtail F \twoheadrightarrow G$ be a free presentation of the group G. Let $\{x_i\}$ be a set of generators of F and $\{r_j\}$ a set of elements of F generating R as a normal subgroup. Then the data $P = (\{x_i\}; \{r_j\})$ is called a *group presentation* of G, x_i are called *generators*, r_j are called *relators*. The group presentation P is called *finite* if both sets $\{x_i\}$, $\{r_j\}$ are finite. A group G is called *finitely presentable* if there exists a finite group presentation for G. The *deficiency* of a finite group presentation, def$P$, is the integer given by def$P$ = (number of generators − number of relators). The *deficiency* of a finitely presentable group, def$G$, is defined as the maximum deficiency of finite group presentations for G. Prove that $\operatorname{def} G \leqq \operatorname{rank} G_{ab} - sH_2(G)$, where sM denotes the minimum number of generators of the abelian group M.

9.3. Let G have a presentation with $n+r$ generators and r relators. Suppose $s(G_{ab}) \leqq n$. Prove that $H_2(G)=0$ and conclude by Exercise 9.1 that G contains a free group F on n generators such that the embedding $i: F \subseteq G$ induces isomorphisms $i_k: F/F_k \xrightarrow{\sim} G/G_k$, $k \geqq 0$. Conclude also that if G can be generated by n elements, then G is isomorphic to the free group F on n generators (Magnus). (Hint: Use the fact that $\bigcap_{k=0}^{\infty} F_k = \{1\}$ for a free group F.)

9.4. Prove that the right hand side of (9.2) depends only on G without using Hopf's formula.

9.5. Deduce (8.4) from Hopf's formula.

10. H^2 and Extensions

Let $A \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} G$ be an exact sequence of groups, with A abelian. It will be convenient to write the group operation in A as addition, in G and E as multiplication, so that i transfers sums into products. Let the function *(section)* $s: G \to E$ assign to every $x \in G$ a representative sx of x, i.e., $ps = 1_G$. Given such a section s, we can define a G-module structure in iA, and hence in A, by the following formula

$$x \circ (ia) = (sx)(ia)(sx)^{-1}, \qquad x \in G, \qquad a \in A \tag{10.1}$$

where the multiplication on the right hand side is in E. It must be shown that $(xy) \circ (ia) = x \circ (y \circ ia)$ but this follows immediately from the remark that $s(xy) = (sx)(sy)(ia')$ for some $a' \in A$ and the fact that A is abelian. Similarly we see that $1 \circ (ia) = ia$. Also, again since A is abelian, different sections $s, s': G \to E$ yield the same G-module structure in A, because $s'x = (sx)(ia')$ for some $a' \in A$.

We define an *extension* of the group G by the G-module A as an exact sequence of groups

$$A \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} G \tag{10.2}$$

such that the G-module structure on A defined by (10.1) is the given G-module structure.

We proceed in this section to classify extensions of the form (10.2), and we will of course be guided by the classification theory for abelian extensions presented in Chapter III.

We shall call the extension $A \rightarrowtail E \twoheadrightarrow G$ *equivalent* to $A \rightarrowtail E' \twoheadrightarrow G$, if there exists a group homomorphism $f: E \to E'$ such that

$$\begin{array}{ccccc} A & \rightarrowtail & E & \twoheadrightarrow & G \\ \| & & \downarrow{\scriptstyle f} & & \| \\ A & \rightarrowtail & E' & \twoheadrightarrow & G \end{array}$$

is commutative. Note that then f must be an isomorphism. We denote the *set of equivalence classes of extensions* of G by A by $M(G, A)$, and the element of $M(G, A)$ containing the extension $A \rightarrowtail E \twoheadrightarrow G$ by $[E]$. The reader notes that in case G is commutative, and operates trivially on A, we have $E(G, A) \subseteq M(G, A)$, where $E(G, A)$ was defined in III. 1.

The set $M(G, A)$ always contains at least one element, namely, the equivalence class of the split extension $A \rightarrowtail A \times G \twoheadrightarrow G$, where $A \times G$ is the semi-direct product (see (5.5)).

We now will define a map $\Delta : M(G, A) \to H^2(G, A)$. Given an extension (10.2) then Theorem 8.1 yields the exact sequence

$$0 \to \mathrm{Der}(G,A) \to \mathrm{Der}(E,A) \to \mathrm{Hom}_G(A,A) \xrightarrow{\theta} H^2(G,A) \to H^2(E,A). \tag{10.3}$$

We then associate with the extension $A \rightarrowtail E \twoheadrightarrow G$ the element

$$\Delta[E] = \theta(1_A) \in H^2(G, A). \tag{10.4}$$

The naturality of (10.3) immediately shows that $\theta(1_A) \in H^2(G, A)$ does not depend on the extension but only on its equivalence class in $M(G, A)$. Hence Δ is well-defined,

$$\Delta : M(G, A) \to H^2(G, A).$$

We shall prove below that Δ is both one-to-one and surjective. The analogous result in the abelian case $E(A, B) \cong \mathrm{Ext}_\Lambda(A, B)$ has been proved using prominently a projective presentation of A, the quotient group in the extension (see Theorem III. 2.4). If we try to imitate this procedure here, we are naturally led to consider a free presentation $R \rightarrowtail F \twoheadrightarrow G$ of G. We then can find a map $f: F \to E$ such that the following diagram commutes

$$\begin{tikzcd} R \arrow[r, tail] \arrow[d, "\bar{f}"] & F \arrow[r, two heads] \arrow[d, "f"] & G \arrow[d, equal] \\ A \arrow[r, tail, "i"] & E \arrow[r, two heads, "p"] & G, \end{tikzcd} \tag{10.5}$$

where $\bar{f}$ is induced by f. Clearly $\bar{f}$ induces a homomorphism of G-modules $\varphi : R_{ab} \to A$. Diagram (10.5) now yields the commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{Der}(E, A) \arrow[r] \arrow[d, "f^*"] & \mathrm{Hom}_G(A, A) \arrow[r, "\theta"] \arrow[d, "\varphi^*"] & H^2(G, A) \arrow[r] \arrow[d, equal] & H^2(E, A) \arrow[d] \\ \cdots \arrow[r] & \mathrm{Der}(F, A) \arrow[r, "\tau"] & \mathrm{Hom}_G(R_{ab}, A) \arrow[r, "\sigma"] & H^2(G, A) \arrow[r] & 0 \end{tikzcd} \tag{10.6}$$

It follows that $\Delta[E] = \theta(1_A) = \sigma\varphi^*(1_A) = \sigma(\varphi)$. We are now prepared to prove

Proposition 10.1. *The map $\Delta : M(G, A) \to H^2(G, A)$ is surjective.*

Proof. Since σ in (10.6) is surjective, it suffices to show that every G-module homomorphism $\varphi: R_{ab} \to A$ arises from a diagram of the form (10.5). In other words we have to fill in the diagram

$$\begin{tikzcd} R \arrow[r, tail, "h"] \arrow[d, "\bar f"] & F \arrow[r, two heads, "q"] \arrow[d, dotted, "f"] & G \arrow[d, equal] \\ A \arrow[r, dotted, "i"] & E \arrow[r, dotted, "p"] & G \end{tikzcd} \tag{10.7}$$

where $\bar f$ induces φ. We construct E as follows. Regard A as an F-module via q and form the semi-direct product $V = A \times F$. The set

$$U = \{(\bar f r, hr^{-1}) \mid r \in R\}$$

is easily seen to be a normal subgroup in V. Define $E = V/U$. The map $i: A \to E$ is induced by the embedding $A \to A \times F$ and $p: E \to G$ is induced by $A \times F \to F$ followed by $q: F \to G$. Finally $f: F \to E$ is induced by $F \to A \times F$. The sequence $A \to E \to G$ is easily seen to be an extension of G by A, and (10.7) is plainly commutative. $\square$

Proposition 10.2. *If two extension have the same image under Δ, they are equivalent, in other words, the map $\Delta: M(G, A) \to H^2(G, A)$ is injective.*

Proof. Let the two extensions be denoted by $A \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} G$ and $A \overset{i'}{\rightarrowtail} E' \overset{p'}{\twoheadrightarrow} G$. Choose a presentation $R \overset{h}{\rightarrowtail} F \overset{q}{\twoheadrightarrow} G$ and (see (10.5)) lifting maps $f: F \to E$, $f': F \to E'$, lifting the identity on G, in such a way that f and f' are both *surjective*. (This can easily be achieved; but it would suffice that f be surjective.) Let f, f' induce $\varphi, \varphi': R_{ab} \to A$. Note that φ, φ' are surjective if and only if f, f' are surjective.

Since $\Delta[E] = \Delta[E']$, it follows that $\sigma(\varphi) = \sigma(\varphi')$ in (10.6). Thus, by the exactness of the lower row in (10.6), there exists a derivation $d: F \to A$ such that $\varphi = \varphi' + \tau(d)$. Consider now $f'': F \to E'$ defined by

$$f''x = (i'dx)(f'x), \qquad x \in F.$$

We claim that (i) f'' is a group homomorphism, and (ii) f'' is surjective. We remark that once the first assertion has been proved, the second is immediate, since plainly f'' induces $\varphi'' = \tau(d) + \varphi' = \varphi: R_{ab} \to A$, which is surjective by hypothesis.

For the proof of (i) let $x, y \in F$. Consider

$$\begin{aligned} f''(xy) &= i'd(xy) \cdot f'(xy) = i'(dx + xdy) \cdot (f'x) \cdot (f'y) \\ &= (i'dx) \cdot (x \circ i'dy) \cdot (f'x) \cdot (f'y) \end{aligned}$$

where $\cdot$ denotes the multiplication in E' and $\circ$ the action of F on A which is given via $q: F \to G$. Since this action is defined by conjugation in E' we obtain

$$x \circ (i'dy) = (f'x) \cdot (i'dy) \cdot (f'x)^{-1}$$

whence it follows that $f''(xy) = (f''x)\cdot(f''y)$. Hence f'' is indeed a homomorphism.

We now have the following commutative diagram

$$\begin{tikzcd}
& R \arrow[rr, tail] \arrow[dl, equal] \arrow[dd, "\bar{f}''"] & & F \arrow[rr, two heads] \arrow[dl, equal] \arrow[dd, "f''"] & & G \arrow[dl, equal] \arrow[dd, equal] \\
R \arrow[rr, tail] \arrow[dd, "\bar{f}"'] & & F \arrow[rr, two heads] \arrow[dd, "f"] & & G \arrow[dd, equal] & \\
& A \arrow[rr, tail] \arrow[dl, equal] & & E' \arrow[rr, two heads] & & G \arrow[dl, equal] \\
A \arrow[rr, tail] & & E \arrow[rr, two heads] & & G &
\end{tikzcd}$$

where $\bar{f}''$ induces $\varphi'': R_{ab} \to A$. Since $\varphi = \varphi''$, it follows that $\bar{f} = \bar{f}'': R \to A$; hence f and f'' have the same kernel, namely, the kernel of $\bar{f}$. It then follows that there is an isomorphism $E \to E'$ inducing the identity in A and G. $\square$

Propositions 10.1 and 10.2 yield the following theorem.

Theorem 10.3. *There is a one-to-one correspondence between $H^2(G, A)$ and the set $M(G, A)$ of equivalence classes of extensions of G by A. The set $M(G, A)$ has therefore a natural abelian group structure and*

$$M(G, -): \mathfrak{M}_G \to \mathfrak{Ab}$$

is a (covariant) functor. $\square$

Note that, if A is a trivial G-module, then $M(G, A)$ is the set of equivalence classes of *central* extensions of G by A, i.e., extensions $A \rightarrowtail E \twoheadrightarrow G$ with A a central subgroup of E.

We conclude this section with the observation that the neutral element in the abelian group $M(G, A)$ is represented by the split extension $A \rightarrowtail A \times G \twoheadrightarrow G$. By Proposition 10.2 it is enough to show that Δ maps the class of the split extension into the neutral element of $H^2(G, A)$, i.e., one has to show that $\theta(1_A) = 0$ in 10.3. By exactness this comes to showing that there is a derivation $d: E \to A$ which, when restricted to A, is the identity. But, for $E = A \times G$, such a derivation is given by $d(a, x) = a$, $a \in A$, $x \in G$.

Exercises:

10.1. Show that an extension $A \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} G$ may be described by a "factor set", as follows. Let $s: G \to E$ be a section, so that $ps = 1_G$. Every element in E is of the form $i(a)\cdot s(x)$ with a, x uniquely determined. The multiplication in E

determines a function $f: G\times G \to A$ by

$$s(x)\cdot s(x') = if(x,x')\cdot s(xx'), \qquad x,x'\in G.$$

Show that associativity of multiplication in E implies

(i) $xf(y,z) - f(xy,z) + f(x,yz) - f(x,y) = 0, \quad x,y,z\in G.$

A function f satisfying (i) is called a *factor set.*

Show that if $s, s': G\to E$ are two sections and f, f' the corresponding factor sets, then there is a function $g: G\to A$ with

(ii) $f'(x,y) = f(x,y) + g(xy) - g(x) - xg(y), \quad x,y\in G.$

[In fact, every factor set can be realized by means of a suitable extension equipped with a suitable section. For an indirect argument, see Exercise 13.7.]

10.2. Show directly that $M(G,-)$ is a functor.

10.3. Proceeding analogously to Exercise 2.5, 2.6, 2.7 of Chapter III describe an addition in $M(G,A)$. Show that with this addition Δ becomes a group isomorphism.

10.4. Using the universal property of free groups, show that $M(F,A)$, with F free, consists of one element only, the class containing the semi-direct product.

10.5. Given the group extension $E: A \rightarrowtail G \twoheadrightarrow Q$ with abelian kernel, show that we may associate with E the 2-extension of Q-modules

$$0\to A\to \mathbb{Z}Q\otimes_G IG \to \mathbb{Z}Q \to \mathbb{Z}\to 0,$$

(called the *characteristic class* of E). Interpret this in terms of $H^2(Q,A)$ and $\operatorname{Ext}^2_Q(\mathbb{Z},A)$.

11. Relative Projectives and Relative Injectives

It is clear (see (2.4)) that $H^n(G,A)=0$ for $n\geqq 1$, whenever A is injective, and that $H_n(G,B)=0$ for $n\geqq 1$, whenever B is projective (or flat). We shall see in this section that the class of modules for which the (co)homology groups become trivial in higher dimensions is much wider.

Definition. The right G-module B is called *induced,* if there is an abelian group X, such that $B\cong X\otimes \mathbb{Z}G$ as G-modules.

It is easy to see that any G-module B is a quotient of an induced G-module. For let us denote by B_0 the underlying abelian group of B; then $\varphi: B_0\otimes\mathbb{Z}G\to B$ defined by $\varphi(b\otimes x) = bx$, $b\in B$, $x\in G$ is an epimorphism of G-modules. We remark that the map φ is even functorially dependent on B, for $B \rightsquigarrow B_0$ is easily seen to be a functor.

Proposition 11.1. *If B is an induced G-module, then $H_n(G,B)=0$ for $n\geqq 1$.*

Proof. Let $\boldsymbol{P}$ be a G-projective resolution of $\mathbb{Z}$. The homology of G with coefficients in B is the homology of the complex $B\otimes_G \boldsymbol{P}$. Since $B\cong X\otimes\mathbb{Z}G$ for a certain abelian group X, we have $B\otimes_G\boldsymbol{P} = X\otimes\boldsymbol{P}$.

Since the underlying abelian group of a G-projective module is free, the homology of $X \otimes P$ is $\operatorname{Tor}_n^{\mathbb{Z}}(X, \mathbb{Z})$ which is trivial for $n \geqq 1$. $\square$

Definition. A direct summand of an induced module is called *relative projective.*

Since the module B is a quotient of $B_0 \otimes \mathbb{Z}G$, every module has a relative projective presentation.

The reader may turn to Exercise 11.2 to learn of a different characterisation of relative projective modules. This other characterisation also explains the terminology. We next state the following elementary propositions.

Proposition 11.2. *A direct sum* $B = \bigoplus_{i \in I} B_i$ *is relative projective if and only if each* B_i, $i \in I$, *is relative projective.*

The proof is immediate from the definition. $\square$

Since $H_n(G, -)$ is an additive functor, we have

Proposition 11.3. *If B is a relative projective G-module, then*

$$H_n(G, B) = 0$$

for $n \geqq 1$. $\square$

We now turn to the "dual" situation:

Definition. A left G-module A is called *coinduced,* if there is an abelian group X such that $A \cong \operatorname{Hom}(\mathbb{Z}G, X)$ as G-modules, where here the G-module structure in $\operatorname{Hom}(\mathbb{Z}G, X)$ is defined by $(y\varphi)(x) = \varphi(y^{-1}x)$, $x, y \in G$, $\varphi: \mathbb{Z}G \to X$. Any G-module A may be embedded functorially in a coinduced module. For let A_0 denote the underlying abelian group of A; then the map $\psi: A \to \operatorname{Hom}(\mathbb{Z}G, A_0)$, defined by $\psi(a)(x) = x^{-1}a$, $x \in G$, $a \in A$ is a monomorphism of G-modules. The functoriality follows easily from the fact that $A \rightsquigarrow A_0$ is a functor.

Proposition 11.4. *If A is a coinduced G-module then $H^n(G, A) = 0$ for* $n \geqq 1$.

The proof is left to the reader. $\square$

Definition. A direct summand of a coinduced module is called *relative injective.*

Again it is clear that every module has a relative injective presentation.

Proposition 11.5. *A direct product* $A = \prod_{i \in I} A_i$ *is relative injective if and only if each* A_i, $i \in I$ *is relative injective.* $\square$

Since $H^n(G, -)$ is an additive functor, we have:

Proposition 11.6. *If A is a relative injective G-module, then $H^n(G, A) = 0$ for* $n \geqq 1$. $\square$

For the next two sections the following remarks will be crucial.

Let A_1, A_2 be left G-modules. We define in $A_1 \otimes A_2$ (the tensor product over $\mathbb{Z}$) a G-module structure by

$$x(a_1 \otimes a_2) = xa_1 \otimes xa_2, \qquad x \in G, \qquad a_1 \in A_1, \qquad a_2 \in A_2. \tag{11.1}$$

The module axioms are easily verified. We shall say that G acts by *diagonal action*.

It should be noted that the definition (11.1) is *not* possible if we replace $\mathbb{Z}G$ by an arbitrary ring Λ. It depends upon the fact that the map $\Delta : \mathbb{Z}G \to \mathbb{Z}G \otimes \mathbb{Z}G$ given by $\Delta(x) = x \otimes x$, $x \in G$, is a *ring* homomorphism. Generally, one can define an analogous module action for an augmented K-algebra Λ, K a commutative ring, if one is given a homomorphism of augmented K-algebras $\Delta : \Lambda \to \Lambda \otimes_K \Lambda$, called the *diagonal*. Given Λ-modules A_1 and A_2, there is an obvious action of $\Lambda \otimes_K \Lambda$ on $A_1 \otimes_K A_2$ and Λ then acts on $A_1 \otimes_K A_2$ by *diagonal action*, that is,

$$\lambda(a_1 \otimes a_2) = (\Delta\lambda)(a_1 \otimes a_2).$$

Such an algebra Λ, together with the diagonal Δ, is usually called a *Hopf algebra*.

Henceforth we will adhere to the following two conventions.

(11.2) If A is a G-module, we will regard its underlying abelian group A_0 as a trivial G-module.

(11.3) Whenever we form the tensor product over $\mathbb{Z}$ of two G-modules it is understood to be endowed with a G-module structure by diagonal action.

With these conventions, our enunciations become much simplified.

Lemma 11.7. *Let A be a G-module. Then the G-modules $A' = \mathbb{Z}G \otimes A$ and $A'' = \mathbb{Z}G \otimes A_0$ are isomorphic.*

Proof. We define a homomorphism $\varphi : A' \to A''$ by

$$\varphi(x \otimes a) = x \otimes (x^{-1}a), \qquad x \in G, \qquad a \in A.$$

Plainly, φ respects the G-module structures and has a two-sided inverse $\psi : A'' \to A'$, defined by $\psi(x \otimes a) = x \otimes xa$. $\square$

Corollary 11.8. *$A' = \mathbb{Z}G \otimes A$ is relative projective.* $\square$

We note for future reference that if A_0 is a free abelian group, $\mathbb{Z}G \otimes A_0$ and, hence, $\mathbb{Z}G \otimes A$ are even free G-modules.

We now turn to the "dual" situation.

Let A_1, A_2 be left G-modules. We define a G-module structure in $\mathrm{Hom}(A_1, A_2)$ by

$$(y\alpha)(a) = y(\alpha(y^{-1}a)), \qquad y \in G, \qquad a \in A_1, \qquad \alpha : A_1 \to A_2. \tag{11.4}$$

Again the module axioms are easily checked. We shall say that G acts by *diagonal action* on $\mathrm{Hom}(A_1, A_2)$. Also, we shall adopt the following convention which is analogous to (11.3).

(11.5) $\mathrm{Hom}(A_1, A_2)$ is understood to be endowed with a G-module structure by diagonal action.

Lemma 11.9. *Let A be a left G-module. Then the G-modules*

$$A' = \mathrm{Hom}(\mathbb{Z}G, A)$$

and $A'' = \mathrm{Hom}(\mathbb{Z}G, A_0)$ *are isomorphic.*

Proof. We define $\varphi : A' \to A''$ by $(\varphi(\alpha))(x) = x^{-1}(\alpha(x)), x \in G, \alpha : \mathbb{Z}G \to A$. We verify that φ is a homomorphism of G-modules:

$$(\varphi(y \circ \alpha))(x) = x^{-1}((y \circ \alpha)(x)) = x^{-1}(y(\alpha(y^{-1}x))),$$
$$(y \circ (\varphi\alpha))(x) = (\varphi\alpha)(y^{-1}x) = (x^{-1}y)(\alpha(y^{-1}x)), \qquad x, y \in G.$$

The map $\psi : A'' \to A'$ defined by $(\psi\alpha)(x) = x(\alpha(x))$ is easily checked to be a two-sided inverse of φ. □

Corollary 11.10. $A' = \mathrm{Hom}(\mathbb{Z}G, A)$ *is relative injective.* □

Exercises:

11.1. Show that the functor $-\otimes\mathbb{Z}G$ is left-adjoint to the functor $B \rightsquigarrow B_0$.

11.2. Prove that a G-module P is relative projective if and only if it has the following property: If $A \rightarrowtail B \twoheadrightarrow P$ is any short exact sequence of G-modules which splits as a sequence of abelian groups, then it also splits as a sequence of G-modules. (See also Exercise IX.1.7.)

11.3. Characterise relative injective G-modules by a property dual to the property stated in Exercise 11.2.

11.4. Show (by induction) that $H^n(G, A)$, may be computed by using a relative injective resolution of A and $H_n(G, B)$ by using a relative projective resolution of B.

11.5. Show that $\Delta : \mathbb{Z}G \to \mathbb{Z}G \otimes \mathbb{Z}G$ defined by $\Delta(x) = x \otimes x$, $x \in G$ is a homomorphism of augmented algebras over $\mathbb{Z}$; hence $\mathbb{Z}G$ is a Hopf algebra.

11.6. Show that the tensor algebra TV over the K-vectorspace V is a Hopf algebra, Δ being defined by $\Delta(v) = v \otimes 1 + 1 \otimes v$, $v \in V$.

11.7. Show that with the conventions (11.3) and (11.5) $\mathrm{Hom}(-,-)$ and $-\otimes-$ are bifunctors to the category of G-modules.

11.8. Let $A_1, \ldots, A_n$ be G-modules. Let $A_1 \otimes \cdots \otimes A_n$ be given a G-module structure by diagonal action, i.e., $x(a_1 \otimes \cdots \otimes a_n) = xa_1 \otimes xa_2 \otimes \cdots \otimes xa_n$, $x \in G$, $a_i \in A_i$, $i = 1, \ldots, n$. Show that $\mathbb{Z}G \otimes A_1 \otimes \cdots \otimes A_n \cong \mathbb{Z}G \otimes A_{1\,0} \otimes \cdots \otimes A_{n\,0}$.

12. Reduction Theorems

Theorem 12.1. *For $n \geqq 2$ we have*

$$H_n(G, B) \cong H_{n-1}(G, B \otimes IG),$$
$$H^n(G, A) \cong H^{n-1}(G, \mathrm{Hom}(IG, A)),$$

where $B \otimes IG$ and $\mathrm{Hom}(IG, A)$ *are G-modules by diagonal action.*

Proof. We only prove the cohomology part of this theorem. Consider the short exact sequence of G-module homomorphisms (see Exercise 11.7)

$$\operatorname{Hom}(\mathbb{Z}, A) \rightarrowtail \operatorname{Hom}(\mathbb{Z}G, A) \twoheadrightarrow \operatorname{Hom}(IG, A).$$

By Corollary 11.10, $\operatorname{Hom}(\mathbb{Z}G, A)$ is relative injective, so that the above sequence is a relative injective presentation of $\operatorname{Hom}(\mathbb{Z}, A) \cong A$. By the long exact cohomology sequence and Proposition 11.6, we obtain the result. $\square$

Theorem 12.2. *Let $G \cong F/R$ with F free. For $n \geqq 3$, we have*

$$H_n(G, B) \cong H_{n-2}(G, B \otimes R_{ab}),$$
$$H^n(G, A) \cong H^{n-2}(G, \operatorname{Hom}(R_{ab}, A)),$$

where $B \otimes R_{ab}$ and $\operatorname{Hom}(R_{ab}, A)$ are G-modules by diagonal action.

Proof. Again we only prove the cohomology part. By Corollary 6.4 we have the following short exact sequence of G-module homomorphisms

$$\operatorname{Hom}(IG, A) \rightarrowtail \operatorname{Hom}(\mathbb{Z}G \otimes_F IF, A) \twoheadrightarrow \operatorname{Hom}(R_{ab}, A).$$

Now $\mathbb{Z}G \otimes_F IF$ is G-free, hence $\operatorname{Hom}(\mathbb{Z}G \otimes_F IF, A)$ is relative injective by Proposition 11.5 and Corollary 11.10. The long exact cohomology sequence together with Theorem 12.1 yields the desired result. $\square$

Exercises:

12.1. Show that Theorem 12.2 generalizes the periodicity theorem for cyclic groups.
12.2. Prove the homology statements of Theorems 12.1, 12.2.

13. Resolutions

Both for theoretical and for computational aspects of the homology theory of groups, it is often convenient to have an explicit description of a resolution of $\mathbb{Z}$ over the given group. In this section we shall present four such resolutions. The first three will turn out to be, in fact, equivalent descriptions of one and the same resolution, called the *(normalized) standard resolution* or *bar resolution.* This resolution is entirely described in terms of the group G itself, and indeed, depends functorially on G; it is the resolution used, almost exclusively, in the pioneering work in the homology theory of groups described in the introduction to this chapter. The fourth resolution, on the other hand, depends on a chosen free presentation of the group G. Throughout this section G will be a fixed group.

(a) *The Homogeneous Bar Resolution.* We first describe the *non-normalized* bar resolution. Let $\bar{B}_n$, $n \geqq 0$, be the free abelian group on the set of all $(n+1)$-tuples $(y_0, y_1, \ldots, y_n)$ of elements of G. Define a *left G-module* structure in $\bar{B}_n$ *by*

$$y(y_0, y_1, \ldots, y_n) = (yy_0, yy_1, \ldots, yy_n), \qquad y \in G. \tag{13.1}$$

It is clear that $\bar{B}_n$ is a free G-module, a basis being given by the $(n+1)$-tuples $(1, y_1, \ldots, y_n)$. We define the differential in the sequence

$$\bar{\boldsymbol{B}}: \cdots \longrightarrow \bar{B}_n \xrightarrow{\partial_n} \bar{B}_{n-1} \longrightarrow \cdots \longrightarrow \bar{B}_1 \xrightarrow{\partial_1} \bar{B}_0 \tag{13.2}$$

by the simplicial boundary formula

$$\partial_n(y_0, y_1, \ldots, y_n) = \sum_{i=0}^{n} (-1)^i (y_0, \ldots, \hat{y}_i, \ldots, y_n), \tag{13.3}$$

where the symbol $\hat{y}_i$ indicates that y_i is to be omitted; and the augmentation $\varepsilon: \bar{B}_0 \to \mathbb{Z}$ by

$$\varepsilon(y) = 1. \tag{13.4}$$

Plainly ∂_n, ε are G-module homomorphisms. Moreover, an elementary calculation, very familiar to topologists, shows that

$$\partial_{n-1}\partial_n = 0, \quad n \geqq 2; \quad \varepsilon\partial_1 = 0.$$

We claim that $\bar{\boldsymbol{B}}$ is a free G-resolution of $\mathbb{Z}$; this, too, is a translation into algebraic terms of a fact familiar to topologists, but we will give the proof. We regard

$$\cdots \longrightarrow \bar{B}_n \xrightarrow{\partial_n} \bar{B}_{n-1} \longrightarrow \cdots \longrightarrow \bar{B}_1 \xrightarrow{\partial_1} \bar{B}_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

as a chain-complex of abelian groups and, as such, it may readily be seen to admit a contracting homotopy $\bar{\varDelta}$, given by

$$\bar{\varDelta}_{-1}(1) = 1, \quad \bar{\varDelta}_n(y_0, \ldots, y_n) = (1, y_0, \ldots, y_n).$$

We leave the reader to verify that $\bar{\varDelta}$ is indeed a contracting homotopy, that is, that

$$\varepsilon\bar{\varDelta}_{-1} = 1, \quad \partial_1\bar{\varDelta}_0 + \bar{\varDelta}_{-1}\varepsilon = 1, \quad \partial_{n+1}\bar{\varDelta}_n + \bar{\varDelta}_{n-1}\partial_n = 1, \quad n \geqq 1. \tag{13.5}$$

The complex $\bar{\boldsymbol{B}}$ is called the *(non-normalized) standard (*or *bar) resolution in homogeneous form.* Now let $D_n \subseteqq \bar{B}_n$ be the subgroup generated by the $(n+1)$-tuples $(y_0, y_1, \ldots, y_n)$ such that $y_i = y_{i+1}$ for at least one value of i, $i = 0, 1, \ldots, n-1$; such an $(n+1)$-tuple will be called *degenerate*, and plainly D_n is a submodule of $\bar{B}_n$, generated by the degenerate $(n+1)$-tuples with $y_0 = 1$. We claim that $\partial D_n \subseteqq D_{n-1}$. For, if $(y_0, y_1, \ldots, y_n)$ is degenerate, let $y_j = y_{j+1}$. Then $\partial_n(y_0, y_1, \ldots, y_n)$ is a linear combination of degenerate n-tuples, together with the term

$$\begin{aligned} &(-1)^j (y_0, \ldots, y_{j-1}, y, y_{j+2}, \ldots, y_n) \\ &\quad + (-1)^{j+1} (y_0, \ldots, y_{j-1}, y, y_{j+2}, \ldots, y_n), \qquad y = y_j = y_{j+1}, \end{aligned}$$

which is clearly zero. Thus the submodules D_n yield a subcomplex $\boldsymbol{D}$, called the *degenerate* subcomplex of $\bar{\boldsymbol{B}}$. (Of course, we could choose other definitions of degeneracy; for example, we could merely require that *any* two of $y_0, y_1, \ldots, y_n$ be the same.) We remark that $D_0 = 0$. We also notice that the contracting homotopy $\bar{\Delta}$ has the property that $\bar{\Delta}_n D_n \subseteqq D_{n+1}$, $n \geqq 0$. Thus we see that, passing to the quotient complex $\boldsymbol{B} = \bar{\boldsymbol{B}}/\boldsymbol{D}$, each G-module B_n is free (on the $(n+1)$-tuples $(y_0 = 1, y_1, \ldots, y_n)$ for which $y_i = y_{i+1}$ for *no* value of i, $i = 0, 1, \ldots, n-1$), and $\boldsymbol{B}$ is a G-free resolution of $\mathbb{Z}$, the contracting homotopy Δ being induced by $\bar{\Delta}$. The complex $\boldsymbol{B}$ is called the *(normalized) standard (*or *bar) resolution in homogeneous form.* It is customary in homological algebra to use the normalized form with precisely this definition of degeneracy.

(b) *The Inhomogeneous Bar Resolution.* Let $\bar{B}'_n$, $n \geqq 0$, be the free left G-module on the set of all n-tuples $[x_1|x_2|\ldots|x_n]$ of elements of G. We define the differential in the sequence

$$\bar{\boldsymbol{B}}' : \cdots \longrightarrow \bar{B}'_n \xrightarrow{\partial_n} \bar{B}'_{n-1} \longrightarrow \cdots \longrightarrow \bar{B}'_1 \xrightarrow{\partial_1} \bar{B}'_0 \tag{13.6}$$

by the formula

$$\begin{aligned} \partial_n[x_1|x_2|\ldots|x_n] &= x_1[x_2|\ldots|x_n] \\ &+ \sum_{i=1}^{n-1} (-1)^i [x_1|x_2|\ldots|x_i x_{i+1}|\ldots|x_n] \\ &+ (-1)^n [x_1|x_2|\ldots|x_{n-1}]\,; \end{aligned} \tag{13.7}$$

and the augmentation $\varepsilon : \bar{B}'_0 \to \mathbb{Z}$ by

$$\varepsilon[\;] = 1\,. \tag{13.8}$$

The reader is advised to give a direct proof that $\bar{\boldsymbol{B}}'$ is a G-free resolution of $\mathbb{Z}$, using the hint that the contracting homotopy is given by

$$\bar{\Delta}_{-1}(1) = [\;]\,, \quad \bar{\Delta}_n(x[x_1|\ldots|x_n]) = [x|x_1|\ldots|x_n]\,, \quad n \geqq 0\,, \tag{13.9}$$

(recall that $\bar{\Delta}_n$ is a homomorphism of abelian groups). However, we avoid this direct proof by establishing an isomorphism between $\bar{\boldsymbol{B}}'$ and $\bar{\boldsymbol{B}}$, compatible with the augmentations. Thus we define $\varphi_n : \bar{B}_n \to \bar{B}'_n$ by

$$\varphi_n(1, y_1, \ldots, y_n) = [y_1|y_1^{-1}y_2|\ldots|y_{n-1}^{-1}y_n]$$

and $\psi_n : \bar{B}'_n \to \bar{B}_n$ by

$$\psi_n[x_1|\ldots|x_n] = (1, x_1, x_1x_2, \ldots, x_1x_2 \ldots x_n)\,.$$

It is easy to see that φ_n, ψ_n are mutual inverses, and that they are compatible with the differentials and the augmentations. Moreover, if $D'_n = \varphi_n D_n$, then D'_n is the submodule of $\bar{B}'_n$ generated by the n-tuples $[x_1|x_2|\ldots|x_n]$ with at least one x_i equal to 1. The modules D'_n constitute the *degenerate* subcomplex $\boldsymbol{D}'$ of $\bar{\boldsymbol{B}}'$ and the quotient complex $\boldsymbol{B}' = \bar{\boldsymbol{B}}'/\boldsymbol{D}'$ is a G-free resolution of $\mathbb{Z}$, isomorphic to $\boldsymbol{B}$, and called the *(normalized) standard (*or *bar) resolution in inhomogeneous form.*

(c) *Alternative Description of the Bar Resolution.* Here and in (d) below we shall construct a resolution step by step. First we recall that

$$IG \rightarrowtail \mathbb{Z}G \overset{\varepsilon}{\twoheadrightarrow} \mathbb{Z}$$

is a G-free presentation of $\mathbb{Z}$. Tensor with the free abelian group IG to obtain the exact sequence of G-modules

$$IG \otimes IG \rightarrowtail \mathbb{Z}G \otimes IG \twoheadrightarrow IG\,.$$

By Corollary 11.8 and the remark following it, this is a G-free presentation of IG. In general write IG^n for the n-fold tensor product of IG, and give IG^n a G-module-structure by diagonal action (see Exercise 11.8). Clearly

$$IG^{n+1} \rightarrowtail \mathbb{Z}G \otimes IG^n \twoheadrightarrow IG^n \tag{13.10}$$

is a G-free presentation of IG^n. Putting the short exact sequences (13.10) together, we obtain a G-free resolution of $\mathbb{Z}$

$$\boldsymbol{C}: \cdots \to \mathbb{Z}G \otimes IG^n \xrightarrow{\partial_n} \mathbb{Z}G \otimes IG^{n-1} \to \cdots \to \mathbb{Z}G \tag{13.11}$$

In each $\mathbb{Z}G \otimes IG^n$ the G-action is given by the diagonal action

$$x(y \otimes (z_1 - 1) \otimes \cdots \otimes (z_n - 1)) = xy \otimes x(z_1 - 1) \otimes \cdots \otimes x(z_n - 1)\,,$$
$$x, y, z_1, \ldots, z_n \in G\,.$$

The differential $\partial_n : \mathbb{Z}G \otimes IG^n \to \mathbb{Z}G \otimes IG^{n-1}$ is defined by

$$\begin{aligned} &\partial_n(x \otimes (z_1 - 1) \otimes \cdots \otimes (z_n - 1)) \\ &\quad = (z_1 - 1) \otimes \cdots \otimes (z_n - 1)\,, \qquad x, z_1, \ldots, z_n \in G\,. \end{aligned} \tag{13.12}$$

One can prove that the resolution $\boldsymbol{C}$ is isomorphic to the resolution $\boldsymbol{B}$. The isomorphism $\theta_n : B_n \to \mathbb{Z}G \otimes IG^n$ is defined by

$$\begin{aligned} &\theta_n(y_0, y_1, \ldots, y_n) \\ &\quad = y_0 \otimes (y_1 - y_0) \otimes \cdots \otimes (y_n - y_{n-1})\,, \qquad y_0, \ldots, y_n \in G\,. \end{aligned} \tag{13.13}$$

Details are left to the reader (see Exercises 13.1 through 13.5). It is also plain that a homomorphism $f: G \to \bar{G}$ induces a chain map

$$\boldsymbol{B}f : \boldsymbol{B}(G) \to \boldsymbol{B}(\bar{G}) \qquad (\boldsymbol{B}'f : \boldsymbol{B}'(G) \to \boldsymbol{B}'(\bar{G}),\ \boldsymbol{C}f : \boldsymbol{C}(G) \to \boldsymbol{C}(\bar{G}))\,,$$

which is even a chain map of G-complexes if $\boldsymbol{B}(\bar{G})$ is given the structure of a G-complex via f. Thus the bar construction is evidently functorial, and the isomorphisms $\varphi_n, \psi_n, \theta_n$ of (b) and (c) yield natural equivalences of functors.

(d) *The Gruenberg Resolution.* Here we shall present a resolution, which, unlike the bar resolution, depends on a chosen free presentation of the group G. Let G be presented as $G \cong F/R$ with F free. We recall that

$$IG \rightarrowtail \mathbb{Z}G \twoheadrightarrow \mathbb{Z} \tag{13.14}$$

is a G-free presentation of $\mathbb{Z}$. By Corollary 6.4 the short exact sequence

$$R_{ab} \rightarrowtail \mathbb{Z}G \otimes_F IF \twoheadrightarrow IG \tag{13.15}$$

is a G-free presentation of IG. Tensoring (13.14), (13.15) with the n-fold tensor product R_{ab}^n of the free abelian group R_{ab} endowed with the G-module structure by diagonal action ($R_{ab}^0 = \mathbb{Z}$), we obtain G-free presentations

$$IG \otimes R_{ab}^n \rightarrowtail \mathbb{Z}G \otimes R_{ab}^n \twoheadrightarrow R_{ab}^n, \quad n \geqq 1, \tag{13.16}$$

$$R_{ab}^{n+1} \rightarrowtail (\mathbb{Z}G \otimes_F IF) \otimes R_{ab}^n \twoheadrightarrow IG \otimes R_{ab}^n, \quad n \geqq 0. \tag{13.17}$$

Thus we obtain a G-free resolution of $\mathbb{Z}$,

$$\boldsymbol{D}: \cdots \to D_{2n+1} \to D_{2n} \to \cdots \to D_0,$$

where

$$D_{2n} = \mathbb{Z}G \otimes R_{ab}^n, \qquad D_{2n+1} = (\mathbb{Z}G \otimes_F IF) \otimes R_{ab}^n.$$

The differentials are given by combining (13.15), (13.16); thus

$$\partial_{2n+1}: D_{2n+1} \to D_{2n}$$

is induced by $\mathbb{Z}G \otimes_F IF \twoheadrightarrow IG \rightarrowtail \mathbb{Z}G$ and $\partial_{2n}: D_{2n} \to D_{2n-1}$ by

$$\mathbb{Z}G \otimes R_{ab} \twoheadrightarrow R_{ab} \rightarrowtail \mathbb{Z}G \otimes_F IF.$$

We conclude with the remark that, if we take F to be the free group on the set $S = \{x \in G \mid x \neq 1\}$, then we obtain the resolution $\boldsymbol{C}$, described under (c), hence a resolution isomorphic to the standard resolution $\boldsymbol{B}$. The only thing to prove is that the two short exact sequences

$$R_{ab} \rightarrowtail \mathbb{Z}G \otimes_F IF \twoheadrightarrow IG, \qquad IG \otimes IG \rightarrowtail \mathbb{Z}G \otimes IG \twoheadrightarrow IG$$

are isomorphic. Indeed, the map $\alpha: \mathbb{Z}G \otimes_F IF \to \mathbb{Z}G \otimes IG$ defined by $\alpha(x \otimes (y-1)) = x \otimes x(y-1)$, $y \neq 1$, is an isomorphism and induces the identity in IG. Hence it also induces an isomorphism

$$\beta: R_{ab} \xrightarrow{\sim} IG \otimes IG.$$

We summarize this last result in

Proposition 13.1. *Let $G \cong F/R$ with F free on all non-unity elements in G. Then $R_{ab} \cong IG \otimes IG$ as G-modules.* $\square$

Exercises:

13.1. Show that the functions Δ'_n given by

$$\Delta'_n(y_0 \otimes (y_1 - 1) \otimes \cdots \otimes (y_n - 1)) = 1 \otimes (y_0 - 1) \otimes \cdots \otimes (y_n - 1),$$

yield a contracting homotopy in the augmented complex $C \overset{\varepsilon}{\twoheadrightarrow} \mathbb{Z}$ of (c).

13.2. Show that $\theta_n : B_n \to C_n$ as defined in (13.13) is a G-module homomorphism. Show that $\Delta'_n \theta_n = \theta_{n+1} \Delta_n$.

13.3. Define $\zeta_n : C_n \to B_n$ inductively by $\zeta_0 = \theta_0^{-1}$,

$$\begin{aligned}&\zeta_n(x \otimes (y_1 - 1) \otimes (y_2 - 1) \otimes \cdots \otimes (y_n - 1)\\ &= x\Delta_{n-1}\zeta_{n-1}(x^{-1}y_1 \otimes \cdots \otimes (x^{-1}y_n - x^{-1}) - x^{-1} \otimes \cdots \otimes (x^{-1}y_n - x^{-1})).\end{aligned}$$

Show that ζ_n is a G-module homomorphism.

13.4. Show (inductively) that ζ_n is a two-sided inverse of θ_n.

13.5. Show that θ_n respects the differential, either directly or inductively by using the fact that it is enough to prove $\partial \theta_n \Delta_{n-1} = \theta_{n-1} \partial \Delta_{n-1}$, since $\Delta_{n-1} B_{n-1} \subset B_n$ generates B_n as G-module.

13.6. Let $\boldsymbol{B}$ denote the homogeneous bar resolution of the group G. Consider cochains with coefficients in a ring R, regarded as a trivial G-module. To a p-cochain $f: B_p \to R$ and a q-cochain $g: B_q \to R$ associate a $(p+q)$-cochain $f \cup g : B_{p+q} \to R$ by defining

$$(f \cup g)(x_0, \ldots, x_{p+q}) = f(x_0, \ldots, x_p) \cdot g(x_p, \ldots, x_{p+q}).$$

Show that this definition makes $\operatorname{Hom}_G(\boldsymbol{B}, R)$ into a differential graded algebra (see Exercise V. 1.5), and hence, by Exercise V. 2.4, that $H^*(G, R)$ becomes a graded ring. This ring is called the *cohomology ring of G with coefficients in R*, and the product induced by $\cup$ is called the *cup-product*. Show that the ring structure in $H^*(G, R)$ is natural in both variables. (Harder:) Show that if R is commutative, $H^*(G, R)$ is commutative in the graded sense.

13.7. Compare formulas (i), (ii) of Exercise 10.1 with the formulas for 2-cocycles and 1-coboundaries in the inhomogeneous description of the bar construction. Conclude that $M(G, A) \cong H^2(G, A)$ (compare Theorem 10.3).

13.8. Show that if G is finite, and if A, B are finitely-generated G-modules, then $H^n(G, A)$, $H_n(G, B)$ are finitely-generated.

14. The (Co)Homology of a Coproduct

Let G_1, G_2 be two groups. Denote as usual their coproduct (free product) by $G = G_1 * G_2$. Let A, B be G-modules. By (2.9), (2.10) the coproduct injections $\iota_i : G_i \to G_1 * G_2$ yield maps

$$H^n(G, A) \to H^n(G_1, A) \oplus H^n(G_2, A), \qquad n \geqq 0,$$

$$H_n(G_1, B) \oplus H_n(G_2, B) \to H_n(G, B), \qquad n \geqq 0.$$

In this section we shall prove that these maps are isomorphisms for $n \geqq 2$. So, loosely speaking, $H^n(-, A)$, $H_n(-, B)$ are coproduct-preserving. We start with the following lemma.

Lemma 14.1. *Let $G = G_1 * G_2$. Then there is a natural isomorphism*

$$IG \cong (\mathbb{Z}G \otimes_{G_1} IG_1) \oplus (\mathbb{Z}G \otimes_{G_2} IG_2). \tag{14.1}$$

Proof. First we claim that for all G-modules A there is a natural isomorphism

$$\mathrm{Der}(G, A) \cong \mathrm{Der}(G_1, A) \oplus \mathrm{Der}(G_2, A). \tag{14.2}$$

Clearly, by restriction, a derivation $d: G \to A$ gives rise to derivations $d_i: G_i \to A, i = 1, 2$. On the other hand a derivation, $d_i: G_i \to A$ corresponds by Corollary 5.4 to a group homomorphism $f_i: G_i \to A \times G_i \subseteq A \times G$ such that the composition with projection onto G is the injection $\iota_i: G_i \to G$. By the universal property of the coproduct the homomorphisms $G_i \to A \times G$ give rise to a group-homomorphism $f: G \to A \times G$. Composition of f with projection onto G clearly yields the identity. So f gives rise to a derivation $d: G \to A$, whose restriction to G_i is $d_i: G_i \to A$. This proves (14.2). Finally we have $\mathrm{Der}(G, A) \cong \mathrm{Hom}_G(IG, A)$ and

$$\mathrm{Der}(G_i, A) \cong \mathrm{Hom}_{G_i}(IG_i, A) \cong \mathrm{Hom}_G(\mathbb{Z}G \otimes_{G_i} IG_i, A)$$

(see (IV. 12.4)). Together with (14.2) this proves Lemma 14.1. □

Theorem 14.2. *Let $G = G_1 * G_2$, A a left G-module, B a right G-module. Then for $n \geqq 2$*

$$H^n(G, A) \cong H^n(G_1, A) \oplus H^n(G_2, A),$$
$$H_n(G_1, B) \oplus H_n(G_2, B) \cong H_n(G, B).$$

Proof. We only prove the cohomology part of the assertion. For $n \geqq 2$ we have, by (6.7),

$$\begin{aligned} H^n(G, A) &\cong \mathrm{Ext}_G^{n-1}(IG, A) \\ &\cong \mathrm{Ext}_G^{n-1}(\mathbb{Z}G \otimes_{G_1} IG_1, A) \oplus \mathrm{Ext}_G^{n-1}(\mathbb{Z}G \otimes_{G_2} IG_2, A), \end{aligned}$$

by Lemma 14.1. But $\mathrm{Ext}_G^{n-1}(\mathbb{Z}G \otimes_{G_i} IG_i, A) \cong \mathrm{Ext}_{G_i}^{n-1}(IG_i, A)$ by Proposition IV. 12.2. □

The conclusion of Theorem 14.2 is clearly false for $n = 0$; for $n = 1$ and *trivial* coefficient modules the conclusion is true, the cohomology part being a restatement of (14.2), and the homology part following easily from (14.1). However, in general, it is false for $n = 1$, as we now show by a counterexample. Let G be the free group on two elements x_1, x_2, and let A be an infinite cyclic group on which x_1, x_2 act non-trivially; $x_1 a = -a = x_2 a$, $a \in A$. Now consider the exact sequence

$$\mathrm{Hom}_G(\mathbb{Z}, A) \rightarrowtail \mathrm{Hom}_G(\mathbb{Z}G, A) \to \mathrm{Hom}_G(IG, A) \twoheadrightarrow H^1(G, A).$$

Since $\mathrm{Hom}_G(\mathbb{Z}, A) = A^G = 0$ and since IG is G-free on two elements it follows that rank $H^1(G, A) = 1$. On the other hand $G = G_1 * G_2$ where G_i is infinite cyclic on x_i, $i = 1, 2$. Thus rank$(H^1(G_1, A) \oplus H^1(G_2, A))$ is even.

Exercises:

14.1. Compute $H^1(G, A)$, $H^1(G_i, A)$, $i = 1, 2$ for G, G_i, A as in the counterexample at the end of Section 14.

14.2. Let

$$\begin{tikzcd} U \arrow[r, tail, "\iota_1"] \arrow[d, tail, "\iota_2"] & G_1 \arrow[d, "\kappa_1"] \\ G_2 \arrow[r, "\kappa_2"] & G \end{tikzcd}$$

be a pushout diagram in the category of groups with $\iota_i : U \rightarrowtail G_i$ monomorphic for $i = 1, 2$. The group G is usually called the *free product of* G_1 *and* G_2 *with amalgamated subgroup* U(see [36]). Prove that for every G-module A the sequence *(Mayer-Vietoris-sequence)*

$$0 \longrightarrow \mathrm{Der}(G, A) \xrightarrow{\kappa^*} \mathrm{Der}(G_1, A) \oplus \mathrm{Der}(G_2, A) \xrightarrow{\iota^*} \mathrm{Der}(U, A) \longrightarrow H^2(G, A) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^n(G, A) \xrightarrow{\kappa^*} H^n(G_1, A) \oplus H^n(G_2, A) \xrightarrow{\iota^*} H^n(U, A) \longrightarrow H^{n+1}(G, A) \longrightarrow \cdots$$

is exact, where $\kappa^* = \{\kappa_1^*, \kappa_2^*\}$ and $\iota^* = \langle \iota_1^*, -\iota_2^* \rangle$. (Hint: Use the fact that κ_1, κ_2 are monomorphic to prove first that the square

$$\begin{tikzcd} \mathbb{Z}G \otimes_U IU \arrow[r, tail, "(\iota_1)_*"] \arrow[d, tail, "(\iota_2)_*"] & \mathbb{Z}G \otimes_{G_1} IG_1 \arrow[d, "(\kappa_1)_*"] \\ \mathbb{Z}G \otimes_{G_2} IG_2 \arrow[r, "(\kappa_2)_*"] & IG \end{tikzcd}$$

is a pushout diagram in the category of G-modules.)

14.3. Show that the Mayer-Vietoris sequence may be started in dimension 0, i.e. that

$$0 \longrightarrow H^0(G, A) \xrightarrow{\kappa^*} H^0(G_1, A) \oplus H^0(G_2, A) \xrightarrow{\iota^*} H^0(U, A) \longrightarrow H^1(G, A) \xrightarrow{\kappa^*}$$
$$\longrightarrow H^1(G_1, A) \oplus H^1(G_2, A) \xrightarrow{\iota^*} H^1(U, A) \longrightarrow H^2(G, A) \longrightarrow \cdots$$

is exact.

14.4. Using Exercise 14.3 show that the conclusion of Theorem 14.2 fails to be true in dimensions 0, 1. What happens if A is a trivial G-module?

14.5. Compute the cohomology with integer coefficients of the group G given by the presentation $(x, y; x^2 y^{-3})$.

15. The Universal Coefficient Theorem and the (Co)Homology of a Product

In the previous section the (co)homology of a coproduct of groups was computed. It may be asked, whether the (co)homology of a (direct) product of groups can be computed similarly from the (co)homology of its factors. We will not discuss this question in general, but restrict ourselves to the case where the coefficient modules are trivial. We will see that then the answer may be given using the Künneth theorem (Theorem V. 2.1).

As a first step we deduce the universal coefficient theorems which allows us to compute the (co)homology with trivial coefficient modules from the *integral* homology. As before we shall write $H_n(G)$ instead of $H_n(G, \mathbb{Z})$.

Theorem 15.1. *Let G be a group and let C be an abelian group considered as a trivial G-module. Then the following sequences are exact and natural, for every* $n \geqq 0$,

$$H_n(G) \otimes C \rightarrowtail H_n(G, C) \twoheadrightarrow \mathrm{Tor}(H_{n-1}(G), C),$$

$$\mathrm{Ext}(H_{n-1}(G), C) \rightarrowtail H^n(G, C) \twoheadrightarrow \mathrm{Hom}(H_n(G), C).$$

Moreover both sequences split by an unnatural splitting.

Proof. Let $\boldsymbol{P}$ be a G-free (or G-projective) resolution of $\mathbb{Z}$. Tensoring over G with $\mathbb{Z}$ yields $\boldsymbol{P}_G = \boldsymbol{P} \otimes_G \mathbb{Z}$, which is a complex of free abelian groups. Also, plainly, $\boldsymbol{P} \otimes_G C \cong \boldsymbol{P}_G \otimes C$ and $\mathrm{Hom}_G(\boldsymbol{P}, C) \cong \mathrm{Hom}(\boldsymbol{P}_G, C)$. Theorem V. 2.5 establishes the homology part, Theorem V. 3.3 the cohomology part of the assertion. $\square$

By Theorem 15.1 the question about the (co)homology with trivial coefficients of a product is reduced to a discussion of the integral homology.

Now let G_1, G_2 be two groups, and $G = G_1 \times G_2$ their (direct) product. Let $\boldsymbol{P}^{(i)}$, $i = 1, 2$, be a G_i-free (or G_i-projective) resolution of $\mathbb{Z}$. Since the complexes $\boldsymbol{P}^{(i)}$ are complexes of free abelian groups, we may apply the Künneth theorem (Theorem V. 2.1) to compute the homology of the complex $\boldsymbol{P}^{(1)} \otimes \boldsymbol{P}^{(2)}$. We obtain

$$H_0(\boldsymbol{P}^{(1)} \otimes \boldsymbol{P}^{(2)}) = \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}; \qquad H_n(\boldsymbol{P}^{(1)} \otimes \boldsymbol{P}^{(2)}) = 0, \qquad n \geqq 1.$$

Furthermore we can regard $\boldsymbol{P}^{(1)} \otimes \boldsymbol{P}^{(2)}$ as a complex of G-modules, the G-module structure being given by

$$(x_1, x_2)(a^{(1)} \otimes a^{(2)}) = x_1 a^{(1)} \otimes x_2 a^{(2)}, \qquad x_i \in G_i, \qquad a^{(i)} \in \boldsymbol{P}^{(i)}, \qquad i = 1, 2.$$

The reader may verify that this action is compatible with the differential in $\boldsymbol{P}^{(1)} \otimes \boldsymbol{P}^{(2)}$. Also, $P_k^{(1)} \otimes P_l^{(2)}$ is a projective G-module. To see this, one only has to prove that $\mathbb{Z}G_1 \otimes \mathbb{Z}G_2 \cong \mathbb{Z}G$, which we leave to the reader. Thus $\boldsymbol{P}^{(1)} \otimes \boldsymbol{P}^{(2)}$ is a G-projective resolution of $\mathbb{Z}$. Finally

$$H_n(G) = H_n((\boldsymbol{P}^{(1)} \otimes \boldsymbol{P}^{(2)}) \otimes_G \mathbb{Z}) = H_n((\boldsymbol{P}^{(1)} \otimes \boldsymbol{P}^{(2)})_G) = H_n(\boldsymbol{P}^{(1)}_{G_1} \otimes \boldsymbol{P}^{(2)}_{G_2}).$$

Since the complexes $P_G^{(i)}$, $i=1,2$, are complexes of free abelian groups we may apply the Künneth theorem again. This proves the following Künneth theorem in the homology of groups.

Theorem 15.2. *Let G_i, $i=1,2$ be two groups, and let $G=G_1\times G_2$ be their direct product. Then the following sequence is exact:*

$$\bigoplus_{p+q=n} H_p(G_1)\otimes H_q(G_2) \rightarrowtail H_n(G) \twoheadrightarrow \bigoplus_{p+q=n-1} \operatorname{Tor}(H_p(G_1), H_q(G_2))\,.$$

Moreover the sequence splits by an unnatural splitting. □

We finally note that the two theorems of this section allow us to compute the (co)homology groups of any finitely generated abelian group with trivial coefficient module (see Exercises 15.1, 15.3).

Exercises:

15.1. Compute the integral (co)homology of $C_n\times C_m$.

15.2. Show that the integral (co)homology groups of a finitely generated commutative group G are finitely generated. (An interesting example of Stallings [43] shows that this is not true if G is an arbitrary finitely presentable group.)

15.3. Find a formula for the integral homology of a finitely generated commutative group.

15.4. What information do we obtain about the homology of a group G by computing its (co)homology with rational coefficients?

15.5. Show that the splitting in the universal coefficient theorem in homology (Theorem 15.1) is unnatural in G, but may be made natural in C.

15.6. *A* group G is said to be of *cohomological dimension* $\leqq m$, $\operatorname{cd} G\leqq m$, if $H^q(G,A)=0$ for every $q>m$ and every G-module A. It is said to be of *cohomological dimension* m, if $\operatorname{cd} G\leqq m$ but $\operatorname{cd} G\nleqq m-1$. Show that $\operatorname{cd} G\leqq m$, $m\geqq 1$, if and only if, for every G-projective resolution,

$$\cdots\to P_n\to P_{n-1}\to\cdots\to P_1\to P_0$$

of $\mathbb{Z}$, the image of $P_m\to P_{m-1}$ is projective. Show that

(i) for G a free group we have $\operatorname{cd} G=1$;

(ii) if $\operatorname{cd} G_1=m_1$, $\operatorname{cd} G_2=m_2$, then $\operatorname{cd}(G_1 * G_2)=\max(m_1,m_2)$, and

$$\operatorname{cd}(G_1\times G_2)\leqq m_1+m_2\,.$$

(iii) Compute $\operatorname{cd} G$ for G finitely-generated free abelian.

16. Groups and Subgroups

In this section we shall introduce certain maps which are very significant in a detailed study of (co)homology, especially of finite groups. We restrict ourselves entirely to cohomology and leave to the reader the translation of the results to the "dual" situation.

In (2.11) it was shown that $H^n(-,-)$ may be regarded as a contravariant functor on the category $\mathfrak{G}^*$ of pairs (G, A), with G a group and A a G-module. A morphism $(f, \alpha): (G, A) \to (G_1, A_1)$ in $\mathfrak{G}^*$ consists of a group homomorphism $f: G \to G_1$ and a map $\alpha: A_1 \to A$ which is a homomorphism of G-modules if A_1 is regarded as a G-module via f. Thus

$$\alpha(f(x)\, a_1) = x\alpha(a_1), \qquad a_1 \in A_1, \qquad x \in G. \tag{16.1}$$

The maps in cohomology to be defined in the sequel will be obtained by choosing specified maps f, α.

(a) *The Restriction Map.* Consider a group G and a G-module A. Let U be a subgroup of G. Regard A as a U-module via the embedding $\iota: U \rightarrowtail G$. Clearly $(\iota, 1_A): (U, A) \to (G, A)$ is a morphism in $\mathfrak{G}^*$. We define the *restriction* (from G to U) by

$$\mathrm{Res} = (\iota, 1_A)^* : H^n(G, A) \to H^n(U, A), \qquad n \geqq 0.$$

The following considerations allow us to make a more detailed study of the restriction map. Let $\varepsilon: \mathbb{Z}G \to \mathbb{Z}$ be the augmentation; tensor it with $\mathbb{Z}$ over $\mathbb{Z}U$. We obtain the short exact sequence

$$K \rightarrowtail \mathbb{Z}G \otimes_U \mathbb{Z} \overset{\varepsilon'}{\twoheadrightarrow} \mathbb{Z} \tag{16.2}$$

where K is the kernel of ε'. Next we apply the functor $\mathrm{Hom}_G(-, A)$ to (16.2). By Proposition IV. 12.2 we obtain

$$\mathrm{Ext}_G^n(\mathbb{Z}G \otimes_U \mathbb{Z}, A) \cong \mathrm{Ext}_U^n(\mathbb{Z}, A) = H^n(U, A).$$

Hence we have proved

Proposition 16.1. *Let U be a subgroup of G, and let A be a G-module. Denote by K the kernel of $\varepsilon': \mathbb{Z}G \otimes_U \mathbb{Z} \to \mathbb{Z}$ in* (16.2). *Then the following sequence is exact:*

$$\cdots \to \mathrm{Ext}_G^{n-1}(K, A) \to H^n(G, A) \xrightarrow{\mathrm{Res}} \mathrm{H}^n(U, A) \to \mathrm{Ext}_G^n(K, A) \to \cdots. \ \square$$

Note that, in case U is *normal* in G with quotient group Q, the module $\mathbb{Z}G \otimes_U \mathbb{Z}$ is isomorphic to $\mathbb{Z}Q$ by Lemma 6.1. Hence $K \cong IQ$, the augmentation ideal of Q.

(b) *The Inflation Map.* Let $N \rightarrowtail G \overset{p}{\twoheadrightarrow} Q$ be an exact sequence of groups, and let A be a G-module. Consider A^N, the subgroup of A consisting of those elements which remain invariant under the action of N. Then A^N admits an obvious Q-module structure. Denote the embedding of A^N in A by $\alpha: A^N \to A$. Then $(p, \alpha): (G, A) \to (Q, A^N)$ is easily seen to be a morphism in $\mathfrak{G}^*$. We define the *inflation map* (from Q to G) by

$$\mathrm{Inf} = (p, \alpha)^* : H^n(Q, A^N) \to H^n(G, A), \qquad n \geqq 0.$$

In Proposition 16.1 the restriction map has been embedded in a long exact sequence. We remark that an analogous embedding for the inflation map exists; but since it is of no apparent use in the study of the inflation map we refrain from stating it here.

(c) *Conjugation.* Let $x \in G$ be a fixed but arbitrary element, and let A be a G-module. Define $f: G \to G$ and $\alpha: A \to A$ by

$$f(y) = x^{-1}yx, \quad y \in G; \quad \alpha(a) = xa, \quad a \in A. \tag{16.3}$$

It is easily seen that $(f, \alpha): (G, A) \to (G, A)$ satisfies the condition (16.1), and therefore is a morphism in $\mathfrak{G}^*$. Moreover (f, α) is invertible in $\mathfrak{G}^*$, hence the induced map

$$(f, \alpha)^*: H^n(G, A) \to H^n(G, A), \quad n \geqq 0$$

is an isomorphism. However, we prove more, namely

Proposition 16.2. *Let* $(f, \alpha): (G, A) \to (G, A)$ *be defined as in* (16.3). *Then* $(f, \alpha)^*: H^n(G, A) \to H^n(G, A)$, $n \geqq 0$, *is the identity.*

Proof. We proceed by induction on n. For $n = 0$, $H^0(G, A) = A^G$, and the assertion is trivial. If $n \geqq 1$ we choose an injective presentation $A \rightarrowtail I \twoheadrightarrow A'$, and consider the long exact cohomology sequence

$$\begin{array}{ccccc} \cdots \to H^{n-1}(G, A') & \longrightarrow & H^n(G, A) & \to & 0 \\ \downarrow{\scriptstyle (f,\alpha')^*} & & \downarrow{\scriptstyle (f,\alpha)^*} & & \\ \cdots \to H^{n-1}(G, A') & \longrightarrow & H^n(G, A) & \to & 0 \end{array}$$

where of course $\alpha' a = xa'$, $a' \in A'$. By induction $(f, \alpha')^*$ is the identity, hence so is $(f, \alpha)^*$. □

(d) *The Corestriction Map.* Let A be a G-module, and let U be a subgroup of finite index m in G. Suppose $G = \bigcup_{i=1}^{m} Ux_i$ is a coset decomposition of G. We then define a map $\theta: \mathrm{Hom}_U(\mathbb{Z}G, A) \to A$, by

$$\theta(\varphi) = \sum_{i=1}^{m} x_i^{-1} \varphi x_i, \quad \varphi: \mathbb{Z}G \to A. \tag{16.4}$$

We claim that θ is independent of the chosen coset decomposition. Indeed, if $G = \bigcup_{i=1}^{m} Uy_i$ is another coset decomposition, then we may assume that the enumeration is such that there exist $u_i \in U$ with $x_i = u_i y_i$, $i = 1, \ldots, m$. But then clearly

$$\Sigma x_i^{-1} \varphi x_i = \Sigma x_i^{-1} \varphi(u_i y_i) = \Sigma x_i^{-1} u_i \varphi y_i = \Sigma y_i^{-1} \varphi y_i.$$

Furthermore we claim that θ is a G-module homomorphism. To show this let $y \in G$, and define a permutation π of $(1, \ldots, m)$ and elements

$v_i \in U$ by the equations

$$x_i y = v_i x_{\pi i}, \qquad i = 1, \dots, m. \tag{16.5}$$

We then have

$$\begin{aligned}\theta(y\varphi) &= \Sigma x_i^{-1}\varphi(x_i y) = \Sigma x_i^{-1}\varphi(v_i x_{\pi i}) = \Sigma x_i^{-1} v_i \varphi(x_{\pi i}) \\ &= \Sigma y x_{\pi i}^{-1}\varphi(x_{\pi i}) = y\theta(\varphi).\end{aligned}$$

Finally we claim that θ is epimorphic. Let $a \in A$, and define φ by $\varphi(x_i) = 0$ if $i \neq 1$, $\varphi(x_1) = x_1 a$; then $\theta(\varphi) = a$. We summarize our results in the following proposition.

Proposition 16.3. *Let U be a subgroup of finite index m in G, and let*

$$G = \bigcup_{i=1}^{m} U x_i$$

be a coset decomposition. Then the map $\theta: \mathrm{Hom}_U(\mathbb{Z}G, A) \to A$, defined by

$$\theta(\varphi) = \sum_{i=1}^{m} x_i^{-1}\varphi x_i$$

is an epimorphism of G-modules. □

Now since, by Proposition IV.12.3, $H^n(G, \mathrm{Hom}_U(\mathbb{Z}G, A)) \cong H^n(U, A)$, $n \geqq 0$, we may define the *corestriction map* (from U to G)

$$\mathrm{Cor}: H^n(U, A) \to H^n(G, A)$$

by

$$H^n(U, A) \cong H^n(G, \mathrm{Hom}_U(\mathbb{Z}G, A)) \xrightarrow{\theta_*} H^n(G, A), \qquad n \geqq 0. \tag{16.6}$$

Using the fact that θ is epimorphic, the reader may easily embed the corestriction map in a long exact sequence (compare Proposition 16.1).

Theorem 16.4. *Let U be a subgroup of finite index m in the group G, and let A be a G-module. Then $\mathrm{Cor} \circ \mathrm{Res}: H^n(G, A) \to H^n(G, A)$, $n \geqq 0$, is just multiplication by m.*

Proof. We proceed by induction on n. For $n = 0$ the restriction $\mathrm{Res}: H^0(G, A) \to H^0(U, A)$ simply embeds A^G in A^U. The corestriction $\mathrm{Cor}: A^U \xrightarrow{\sim} (\mathrm{Hom}_U(\mathbb{Z}G, A))^G \to A^G$ sends a G-invariant (!) element $a \in A$ first into the U-module homomorphism $\varphi: \mathbb{Z}G \to A$ given by

$$\varphi(x_i) = x_i \circ a = a$$

and then into

$$\theta(\varphi) = \sum_{i=1}^{m} x_i^{-1}\varphi x_i = ma.$$

For $n \geqq 1$ let $A \rightarrowtail I \twoheadrightarrow A'$ be a G-injective presentation of A. Note that I is U-injective, also. Then the diagram

$$\begin{array}{ccccc} \cdots \longrightarrow H^{n-1}(G, A') & \longrightarrow & H^n(G, A) & \longrightarrow & 0 \\ \Big\downarrow {\scriptstyle \mathrm{Cor} \circ \mathrm{Res}} & & \Big\downarrow {\scriptstyle \mathrm{Cor} \circ \mathrm{Res}} & & \\ \cdots \longrightarrow H^{n-1}(G, A') & \longrightarrow & H^n(G, A) & \longrightarrow & 0 \end{array}$$

is commutative, and the assertion follows by induction. $\square$

Corollary 16.5. *Let G be a finite group of order m. Then $mH^n(G, A) = 0$ for all $n \geqq 1$.*

Proof. Use Theorem 16.4 with $U = \{1\}$ and observe that $H^n(\{1\}, A) = 0$ for $n \geqq 1$. $\square$

We close this section by applying Corollary 16.5 to yield a proof of a celebrated theorem in the theory of group representations. We have seen that K-representations of G are in one-to-one correspondence with KG-modules (Example (c) in Section I. 1). The K-representations of G are said to be *completely reducible* if every KG-module is *semi-simple*, i.e., if every short exact sequence of KG-modules splits.

Theorem 16.6 (Maschke). *Let G be a group of order m, and let K be a field, whose characteristic does not divide m. Then the K-representations of G are completely reducible.*

Proof. We have to show that every short exact sequence

$$V' \overset{\alpha}{\rightarrowtail} V \overset{\beta}{\twoheadrightarrow} V'' \tag{16.7}$$

of KG-modules splits. This is equivalent to the assertion that the induced sequence

$$0 \longrightarrow \mathrm{Hom}_G(V'', V') \xrightarrow{\beta^*} \mathrm{Hom}_G(V, V') \xrightarrow{\alpha^*} \mathrm{Hom}_G(V', V') \longrightarrow 0 \tag{16.8}$$

is exact. In order to prove this, we first look at the short exact sequence of K-vector spaces of K-linear maps

$$0 \longrightarrow \mathrm{Hom}_K(V'', V') \xrightarrow{\beta^*} \mathrm{Hom}_K(V, V') \xrightarrow{\alpha^*} \mathrm{Hom}_K(V', V') \longrightarrow 0. \tag{16.9}$$

We remark that these vector spaces may be given a KG-module structure by diagonal action (compare (11.4)) as follows. If, for instance, $\sigma: V \to V'$ is a K-linear map, we define

$$(x\sigma)\, v = x\sigma(x^{-1} v), \qquad x \in G, \qquad v \in V. \tag{16.10}$$

It is easily checked that, with this G-module structure, (16.9) becomes an exact sequence of G-modules. In terms of the module structure (16.10), the K-linear map $\sigma: V \to V'$ is a G-module homomorphism if and only if σ is an invariant element in the G-module $\mathrm{Hom}_K(V, V')$. It therefore remains to prove that

$$0 \to H^0(G, \mathrm{Hom}_K(V'', V')) \to H^0(G, \mathrm{Hom}_K(V, V)) \to H^0(G, \mathrm{Hom}_K(V', V')) \to 0$$

is exact. This clearly is the case if $H^1(G, \mathrm{Hom}_K(V'', V')) = 0$, which is proved in Lemma 16.7. In fact we shall prove more, namely

Lemma 16.7. *Under the hypotheses of Theorem* 16.6 *we have*

$$H^n(G, W) = 0$$

for $n \geqq 1$ *and any* KG*-module* W.

Proof. Consider the map $m: W \to W$, multiplication by m. This clearly is a G-module homomorphism. Since the characteristic of K does not divide m, the map $m: W \to W$ is in fact an isomorphism, having $1/m: W \to W$ as its inverse. Hence the induced map $m_*: H^n(G, W) \to H^n(G, W)$ is an isomorphism, also. On the other hand it follows from the additivity of $H^n(G, -)$ that m_* is precisely multiplication by m. But by Corollary 16.5 we have $mH^n(G, W) = 0$ for all $n \geqq 1$, whence $H^n(G, W) = 0$ for $n \geqq 1$. □

Exercises:

16.1. Define Res, Inf, Cor for homology and prove results analogous to Propositions 16.1, 16.3, Theorem 16.4, and Corollary 16.5.

16.2. Prove that the (co)homology groups of a finite group with coefficients in a finitely generated module are finite.

16.3. Let A be a G-module and let A, G be of coprime order. Show that every extension $A \rightarrowtail E \twoheadrightarrow G$ splits.

16.4. Let U be of finite index in G. Compute explicitely Cor: $H_1(G, \mathbb{Z}) \to H_1(U, \mathbb{Z})$. Show that this is the classical *transfer* [28].

16.5. Let G be a group with cd $G = m$ (see Exercise 15.6). Let U be of finite index in G. Show that cd $U = m$. (Hint: The functor $H^m(G, -)$ is right exact. Let A be a G-module with $H^m(G, A) \neq 0$. Then Cor: $H^m(U, A) \to H^m(G, A)$ is surjective.)

16.6. Prove the following theorem due to Schur. If Z denotes the center of G and if G/Z is finite, then $G' = [G, G]$ is finite, also. (Hint: First show that $G'/G' \cap Z$ is finite. Then use sequence (8.4) in homology for $N = Z$.)

VII. Cohomology of Lie Algebras

In this Chapter we shall give a further application of the theory of derived functors. Starting with a Lie algebra $\mathfrak{g}$ over the field K, we pass to the universal enveloping algebra $U\mathfrak{g}$ and define cohomology groups $H^n(\mathfrak{g}, A)$ for every (left) $\mathfrak{g}$-module A, by regarding A as a $U\mathfrak{g}$-module. In Sections 1 through 4 we will proceed in a way parallel to that adopted in Chapter VI in presenting the cohomology theory of groups. We therefore allow ourselves in those sections to leave most of the proofs to the reader. Since our primary concern is with the homological aspects of Lie algebra theory, we will *not* give proofs of two deep results of Lie algebra theory although they are fundamental for the development of the cohomology theory of Lie algebras; namely, we shall not give a proof for the Birkhoff-Witt Theorem (Theorem 1.2) nor of Theorem 5.2 which says that the bilinear form of certain representations of semi-simple Lie algebras is non-degenerate. Proofs of both results are easily accessible in the literature.

As in the case of groups, we shall attempt to deduce as much as possible from general properties of derived functors. For example (compare Chapter VI) we shall prove the fact that $H^2(\mathfrak{g}, A)$ classifies extensions without reference to a particular resolution.

Again, a brief historical remark is in order. As for groups, the origin of the cohomology theory of Lie algebras lies in algebraic topology. Chevalley-Eilenberg [8] have shown that the real cohomology of the underlying topological space of a compact connected Lie group is isomorphic to the real cohomology of its Lie algebra, computed from the complex $\mathrm{Hom}_{\mathfrak{g}}(C, \mathbb{R})$, where C is the resolution of Section 4. Subsequently the cohomology theory of Lie algebras has, however, developed as a purely algebraic discipline, as outlined in the main Introduction.

1. Lie Algebras and their Universal Enveloping Algebra

Let K be a field. A *Lie algebra* $\mathfrak{g}$ over K is a vectorspace over K together with a bilinear map $[\,,\,]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the *Lie bracket*, satisfying the following two identities

$$[x,x]=0, \qquad x \in \mathfrak{g}; \tag{1.1}$$

$$[[x,y],z]+[[y,z],x]+[[z,x],y]=0, \qquad x,y,z \in \mathfrak{g}. \tag{1.2}$$

(1.2) is called the *Jacobi identity*. Note that (1.1) and the bilinearity of the bracket imply $[x,y]=-[y,x]$, $x,y \in \mathfrak{g}$.

A *Lie algebra homomorphism* $f:\mathfrak{g}\to\mathfrak{h}$ is a K-linear map with $f[x,y]=[fx,fy]$, $x,y\in\mathfrak{g}$. A *Lie subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$ is a subspace of $\mathfrak{g}$ closed under $[\,,\,]$. A Lie subalgebra $\mathfrak{h}$ is called a *Lie ideal* of $\mathfrak{g}$, if $[x,y]\in\mathfrak{h}$ for all $x\in\mathfrak{g}$ and $y\in\mathfrak{h}$. If $\mathfrak{h}$ is a Lie ideal of $\mathfrak{g}$ then the quotient space $\mathfrak{g}/\mathfrak{h}$ has a natural Lie algebra structure induced by the Lie bracket in $\mathfrak{g}$.

A Lie algebra $\mathfrak{g}$ is called *abelian* if $[x,y]=0$ for all $x,y\in\mathfrak{g}$. To any Lie algebra $\mathfrak{g}$ we can associate its "largest abelian quotient" $\mathfrak{g}_{ab}$; clearly the kernel of the projection map from $\mathfrak{g}$ to $\mathfrak{g}_{ab}$ must contain the Lie subalgebra $[\mathfrak{g},\mathfrak{g}]$ generated by all $[x,y]$ with $x,y\in\mathfrak{g}$. It is easy to see that $[\mathfrak{g},\mathfrak{g}]$ is an ideal, so that $\mathfrak{g}_{ab}=\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$. Any K-vector space may be regarded as an abelian Lie algebra. Given any K-algebra Λ we can associate (functorially) a Lie algebra $L\Lambda$ with the same underlying vector space as Λ, the Lie bracket being defined by

$$[x,y]=xy-yx, \qquad x,y\in\Lambda.$$

We leave it to the reader to verify the Lie algebra axioms for $L\Lambda$.

Next we ask whether there exists a construction for a Lie algebra analogous to the construction of the group ring for a group. We remind the reader that the group ring functor is determined by the fact that it is a left adjoint to the unit functor, from rings to groups, which assigns to every ring Λ its group of units (see Exercise VI.1.1). Now, our functor L from algebras to Lie algebras will correspond to the unit functor, so that we have to discuss the existence of a left adjoint to L. Such a left adjoint indeed exists; the image of the Lie algebra $\mathfrak{g}$ under that functor is called the *universal enveloping algebra* of $\mathfrak{g}$ and is denoted by $U\mathfrak{g}$. (We follow here the usual notational convention of denoting the universal enveloping algebra by $U\mathfrak{g}$, despite the fact that U is *left* adjoint to L.)

We now proceed to give the explicit construction of $U\mathfrak{g}$, state its adjoint property in Proposition 1.1, and discuss additional properties in the remainder of the section. For the construction of $U\mathfrak{g}$ we need the notion of the *tensor algebra* TM over the K-vector space M. Denote, for $n\geqq 1$, the n-fold tensor product of M by T_nM,

$$T_nM=M\otimes_K M\otimes_K\cdots\otimes_K M, \qquad n\text{-fold}.$$

Set $T_0 M = K$. Then the tensor algebra TM is $\bigoplus_{n=0}^{\infty} T_n M$, with the multiplication induced by

$$(m_1 \otimes m_2 \otimes \cdots \otimes m_{p'}) \cdot (m'_1 \otimes m'_2 \otimes \cdots \otimes m'_q)$$
$$= m_1 \otimes m_2 \otimes \cdots \otimes m_p \otimes m'_1 \otimes m'_2 \otimes \cdots \otimes m'_q,$$

where $m_i, m'_j \in M$ for $1 \leqq i \leqq p$, $1 \leqq j \leqq q$. Note that TM is the *free* K-algebra over M; more precisely: To any K-algebra Λ and any K-linear map $f: M \to \Lambda$ there exists a unique algebra homomorphism $f_0: TM \to \Lambda$ extending f. In other words the functor T is left adjoint to the underlying functor to K-vector spaces which forgets the algebra structure. This assertion is easily proved by observing that $f_0(m_1 \otimes \cdots \otimes m_p)$ may, and in fact must, be defined by

$$f(m_1) \cdot f(m_2) \cdot \ldots \cdot f(m_p).$$

Definition. Given a K-Lie algebra $\mathfrak{g}$; we define *the universal enveloping algebra* $U\mathfrak{g}$ of $\mathfrak{g}$ to be the quotient of the tensor algebra $T\mathfrak{g}$ by the ideal I generated by the elements of the form

$$x \otimes y - y \otimes x - [x, y], \qquad x, y \in \mathfrak{g};$$

thus

$$U\mathfrak{g} = T\mathfrak{g}/(x \otimes y - y \otimes x - [x, y]).$$

Clearly we have a canonical mapping of K-vector spaces $i: \mathfrak{g} \to U\mathfrak{g}$ defined by $\mathfrak{g} \subseteq T\mathfrak{g} \xrightarrow{p} U\mathfrak{g}$, which plainly is a Lie algebra homomorphism $i: \mathfrak{g} \to LU\mathfrak{g}$. It is now easy to see that any Lie algebra homomorphism $f: \mathfrak{g} \to L\Lambda$ induces a unique K-algebra homomorphism $f_1: U\mathfrak{g} \to \Lambda$, since plainly the homomorphism $f_0: T\mathfrak{g} \to \Lambda$ vanishes on the ideal I. Thus U is seen to be left adjoint to L. We further remark that the Lie algebra map $i: \mathfrak{g} \to LU\mathfrak{g}$ is nothing else but the unit of the adjoint pair $U \dashv L$.

Proposition 1.1. *The universal enveloping algebra functor U is a left adjoint to the functor L.* $\square$

Next we state without proof the famous *Birkhoff-Witt Theorem* which is a structure theorem for $U\mathfrak{g}$.

Let $\{e_i\}$, $i \in J$, be a K-basis of $\mathfrak{g}$ indexed by a simply-ordered set J. Let $I = (i_1, i_2, \ldots, i_k)$ denote an increasing sequence of elements in J, i.e., $i_l \in J$ for $1 \leqq l \leqq k$, and $i_1 \leqq i_2 \leqq \cdots \leqq i_k$ under the given order relation in J. Then we define $e_I = e_{i_1} e_{i_2} \ldots e_{i_k} \in U\mathfrak{g}$ to be the projection of $e_{i_1} \otimes \cdots \otimes e_{i_k} \in T\mathfrak{g}$.

Theorem 1.2 (Birkhoff-Witt). *Let $\{e_i\}$, $i \in J$, be a K-basis of $\mathfrak{g}$. Then the elements e_I corresponding to all finite increasing sequences I (including the empty one) form a K-basis of $U\mathfrak{g}$.*

For a proof of this theorem we refer the reader to N. Jacobson [29, p. 159]; J.-P. Serre [42, LA. 3]. As an immediate corollary we note

Corollary 1.3. *The unit $i: \mathfrak{g} \to LU\mathfrak{g}$ is an embedding.* □

Consequently we see that every Lie algebra $\mathfrak{g}$ over K is isomorphic to a Lie subalgebra of a Lie algebra of the form $L\Lambda$ for some K-algebra Λ.

Before we state further corollaries of Theorem 1.2 we introduce the notion of a (left) $\mathfrak{g}$-module.

Definition. A *left* $\mathfrak{g}$*-module* A is a K-vector space A together with a homomorphism of Lie algebras $\varrho: \mathfrak{g} \to L(\mathrm{End}_K A)$.

We may therefore think of the elements of $\mathfrak{g}$ as acting on A and write $x \circ a$ for $\varrho(x)(a)$, $x \in \mathfrak{g}$, $a \in A$, so that $x \circ a \in A$. Then A is a (left) $\mathfrak{g}$-module if $x \circ a$ is K-linear in x and a and

$$[x, y] \circ a = x \circ (y \circ a) - y \circ (x \circ a), \quad x, y \in \mathfrak{g}, a \in A. \tag{1.3}$$

By the universal property of $U\mathfrak{g}$ the map ϱ induces a unique algebra homomorphism $\varrho_1: U\mathfrak{g} \to \mathrm{End}_K A$, thus making A into a left $U\mathfrak{g}$-module. Conversely, if A is a left $U\mathfrak{g}$-module, so that we have a structure map $\sigma: U\mathfrak{g} \to \mathrm{End}_K A$, it is also a $\mathfrak{g}$-module by $\varrho = \sigma i$. Thus the notions of a $\mathfrak{g}$-module and a $U\mathfrak{g}$-module effectively coincide. We leave to the reader the obvious definition of a *right* $\mathfrak{g}$-module. As in the case of Λ-modules we shall use the term $\mathfrak{g}$-module to mean *left* $\mathfrak{g}$-module.

An important phenomenon in the theory of Lie algebras is that the Lie algebra $\mathfrak{g}$ itself may be regarded as a left (or right) $\mathfrak{g}$-module. The structure map is written $\mathrm{ad}: \mathfrak{g} \to L(\mathrm{End}_K \mathfrak{g})$ and is defined by

$$(\mathrm{ad}\, x)(z) = [x, z], \quad x, z \in \mathfrak{g}. \tag{1.4}$$

It is easy to verify that ad does give $\mathfrak{g}$ the structure of a $\mathfrak{g}$-module. For $[x, z]$ is certainly K-bilinear and (1.3) in this case is essentially just the Jacobi identity (1.2).

A $\mathfrak{g}$-module A is called *trivial*, if the structure map $\varrho: \mathfrak{g} \to L(\mathrm{End}_K A)$ is trivial, i.e. if $x \circ a = 0$ for all $x \in \mathfrak{g}$. It follows that a trivial $\mathfrak{g}$-module is just a K-vector space. Conversely, any K-vector space may be regarded as a trivial $\mathfrak{g}$-module for any Lie algebra $\mathfrak{g}$.

The structure map of K, regarded as a trivial $\mathfrak{g}$-module, sends every $x \in \mathfrak{g}$ into zero. The associated (unique) algebra homomorphism $\varepsilon: U\mathfrak{g} \to K$ is called the *augmentation* of $U\mathfrak{g}$. The kernel $I\mathfrak{g}$ of ε is called the *augmentation ideal* of $\mathfrak{g}$. The reader will notice that $I\mathfrak{g}$ is just the ideal of $U\mathfrak{g}$ generated by $i(\mathfrak{g})$.

Corollary 1.4. *Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. Then $U\mathfrak{g}$ is free as an $\mathfrak{h}$-module.*

Proof. Choose $\{e'_i\}$, $i \in J'$, a basis in $\mathfrak{h}$ and expand it by $\{e_j\}$, $j \in J$, to a basis in $\mathfrak{g}$. Let both J', J be simply ordered. Make $J' \cup J$ simply ordered by setting

$$i \leqq j \begin{cases} \text{if} & i,j \in J' & \text{and} & i \leqq j & \text{in} & J', \\ \text{if} & i \in J' & \text{and} & j \in J, \\ \text{if} & i,j \in J & \text{and} & i \leqq j & \text{in} & J. \end{cases}$$

It follows from Theorem 1.2 that the elements e_I for all finite increasing sequences in J form a basis of $U\mathfrak{g}$ as $\mathfrak{h}$-module. □

The reader may compare Corollary 1.4 with the corresponding result for groups (Lemma VI.1.3). We note explicitly the following consequence of Corollary 1.4 and Theorem IV.12.5.

Corollary 1.5. *Every* $\mathfrak{g}$*-projective (injective) module is* $\mathfrak{h}$*-projective (injective).* □

If $\mathfrak{n}$ is a Lie ideal of $\mathfrak{g}$ with quotient $\mathfrak{h}$, we say that the sequence $\mathfrak{n} \rightarrowtail \mathfrak{g} \twoheadrightarrow \mathfrak{h}$ is *exact*.

Corollary 1.6. *If* $\mathfrak{n} \rightarrowtail \mathfrak{g} \twoheadrightarrow \mathfrak{h}$ *is an exact sequence of Lie algebras, then* $K \otimes_{U\mathfrak{n}} U\mathfrak{g} \cong U\mathfrak{h}$ *as right* $\mathfrak{g}$*-modules.*

The proof is left to the reader. □

Exercises:

1.1. Show that the following are examples of Lie algebras over K, under a suitable bracket operation.
(a) the skew-symmetric $n \times n$ matrices over K,
(b) the $n \times n$ matrices over K with trace 0.

1.2. Show that the following are examples of Lie algebras over $\mathbb{C}$, under a suitable bracket operation.
(a) the skew-hermitian $n \times n$ matrices over $\mathbb{C}$,
(b) the skew-hermitian $n \times n$ matrices with trace 0.

1.3. Show that the set of all elements $x \in \mathfrak{g}$ with $[x, y] = 0$ for all $y \in \mathfrak{g}$ is an ideal. (This ideal is called the *center* of $\mathfrak{g}$. Clearly the center is an abelian ideal.)

1.4. Show that, for $f: \mathfrak{g} \twoheadrightarrow \mathfrak{h}$ surjective, the induced map $Uf: U\mathfrak{g} \to U\mathfrak{h}$ is surjective, also.

1.5. Prove that $I\mathfrak{g}$ is generated by $i\mathfrak{g}$ as an ideal of $U\mathfrak{g}$.

1.6. Prove Corollaries 1.5, 1.6.

1.7. Let A be a (non-trivial) left $\mathfrak{g}$-module. Define in A a (non-trivial) right $\mathfrak{g}$-module structure. (Hint: Define $ax = -xa$)

1.8. Let $\mathfrak{g}_1, \mathfrak{g}_2$ be two Lie algebras over K. Show that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ has a natural Lie algebra structure, which makes $\mathfrak{g}$ the product of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ in the category of Lie algebras over K.

1.9. Prove that the product in TM makes $\{T_n M\}$, $n = 0, 1, \ldots$ into a graded K-algebra (see Exercise V.1.5).

2. Definition of Cohomology; H^0, H^1

For notational convenience we shall write $\mathrm{Hom}_{\mathfrak{g}}(-,-)$ $\mathrm{Ext}^n_{\mathfrak{g}}(-,-)$, etc., for $\mathrm{Hom}_{U\mathfrak{g}}(-,-)$, $\mathrm{Ext}^n_{U\mathfrak{g}}(-,-)$, etc.

Definition. Given a Lie algebra $\mathfrak{g}$ over K and a $\mathfrak{g}$-module A, we define the n^{th} *cohomology group of* $\mathfrak{g}$ *with coefficients in* A by

$$H^n(\mathfrak{g}, A) = \mathrm{Ext}^n_{\mathfrak{g}}(K, A), \qquad n = 0, 1, \ldots$$

where K is, of course, regarded as a trivial $\mathfrak{g}$-module.

We note that each $H^n(\mathfrak{g}, A)$ is actually a K-vector space. Nevertheless we shall continue to use the term cohomology *group*. Plainly, the cohomology theory of Lie algebras has properties closely analogous to those listed in Section VI.2 for the cohomology theory of groups. We therefore shall abstain from listing them again here (see Exercise 2.2).

We shall compute H^0, H^1. For any $\mathfrak{g}$-module A, $H^0(\mathfrak{g}, A)$ is by definition $\mathrm{Hom}_{\mathfrak{g}}(K, A)$. By arguments similar to those used for groups in Section VI.3 we obtain

$$H^0(\mathfrak{g}, A) = \{a \in A \mid x \circ a = 0, \quad \text{for all} \quad x \in \mathfrak{g}\}\,; \tag{2.1}$$

we call this the subspace of *invariant* elements in A and denote it by $A^{\mathfrak{g}}$.

In order to exhibit the nature of $H^1(\mathfrak{g}, A)$ we introduce the notion of Lie algebra derivations.

Definition. A *derivation* from a Lie algebra $\mathfrak{g}$ into a $\mathfrak{g}$-module A is a K-linear map $d: \mathfrak{g} \to A$ such that

$$d([x, y]) = x \circ d(y) - y \circ d(x), \qquad x, y \in \mathfrak{g}\,. \tag{2.2}$$

Notice that this property of d is compatible with (1.1) and the Jacobi identity (1.2). It is plain that the set of all derivations $d: \mathfrak{g} \to A$ has a K-vector space structure; we shall denote this vector space by $\mathrm{Der}(\mathfrak{g}, A)$. Note that if A is a trivial $\mathfrak{g}$-module, a derivation is simply a Lie algebra homomorphism where A is regarded as an abelian Lie algebra.

For $a \in A$ fixed we obtain a derivation $d_a: \mathfrak{g} \to A$ by setting $d_a(x) = x \circ a$. Derivations of this kind are called *inner*. The inner derivations in $\mathrm{Der}(\mathfrak{g}, A)$ clearly form a K-subspace, which we denote by $\mathrm{Ider}(\mathfrak{g}, A)$.

The reader should compare the following two results with Theorem VI.5.1 and Corollary VI.5.2.

Theorem 2.1. *The functor* $\mathrm{Der}(\mathfrak{g}, -)$ *is represented by the* $\mathfrak{g}$*-module* $I\mathfrak{g}$, *that is, for any* $\mathfrak{g}$*-module* A *there is a natural isomorphism between the* K*-vector spaces* $\mathrm{Der}(\mathfrak{g}, A)$ *and* $\mathrm{Hom}_{\mathfrak{g}}(I\mathfrak{g}, A)$.

Proof. Given a derivation $d: \mathfrak{g} \to A$, we define a K-linear map $f'_d: T\mathfrak{g} \to A$ by sending $K = T^0\mathfrak{g} \subseteq T\mathfrak{g}$ into zero and $x_1 \otimes \cdots \otimes x_n$ into $x_1 \circ (x_2 \circ \cdots \circ (x_{n-1} \circ dx_n) \ldots)$. Since d is a derivation f'_d vanishes on all elements of the form $t \otimes (x \otimes y - y \otimes x - [x, y])$, $x, y \in \mathfrak{g}$, $t \in T\mathfrak{g}$. Since A

is a $\mathfrak{g}$-module, f'_d vanishes on all elements of the form

$$t_1 \otimes (x \otimes y - y \otimes x - [x, y]) \otimes t_2,$$

$x, y \in \mathfrak{g}$, $t_1, t_2 \in T\mathfrak{g}$. Thus f'_d defines a map $f_d : I\mathfrak{g} \to A$, which is easily seen to be a $\mathfrak{g}$-module homomorphism.

On the other hand, if $f : I\mathfrak{g} \to A$ is given, we extend f to $U\mathfrak{g}$ by setting $f(K) = 0$ and then we define a derivation $d_f : \mathfrak{g} \to A$ by $d_f = f i$, where $i : \mathfrak{g} \to U\mathfrak{g}$ is the canonical embedding. It is easy to check that $f_{(d_f)} = f$ and $d_{(f_d)} = d$, and also that the map $f \mapsto d_f$ is K-linear. $\square$

If we take the obvious free presentation of K

$$I\mathfrak{g} \rightarrowtail U\mathfrak{g} \twoheadrightarrow K,$$

then, given a $\mathfrak{g}$-module A, we obtain

$$H^1(\mathfrak{g}, A) = \operatorname{coker}(\operatorname{Hom}_{\mathfrak{g}}(U\mathfrak{g}, A) \to \operatorname{Hom}_{\mathfrak{g}}(I\mathfrak{g}, A)). \tag{2.3}$$

Hence $H^1(\mathfrak{g}, A)$ is isomorphic to the vector space of derivations from $\mathfrak{g}$ into A modulo those that arise from $\mathfrak{g}$-module homomorphisms $f : U\mathfrak{g} \to A$. If $f(1_{U\mathfrak{g}}) = a$, then clearly $d_f(x) = x \circ a$, so that these are precisely the inner derivations. We obtain

Proposition 2.2. $H^1(\mathfrak{g}, A) \cong \operatorname{Der}(\mathfrak{g}, A)/\operatorname{Ider}(\mathfrak{g}, A)$. *If A is a trivial $\mathfrak{g}$-module,* $H^1(\mathfrak{g}, A) \cong \operatorname{Hom}_K(\mathfrak{g}_{ab}, A)$.

Proof. Only the second assertion remains to be proved. Since A is trivial, there are no non-trivial inner derivations, and a derivation $d : \mathfrak{g} \to A$ is simply a Lie algebra homomorphism, A being regarded as an abelian Lie algebra. $\square$

Next we show that, as in the case of groups, derivations are related to split extensions, i.e., semi-direct products.

Definition. Given a Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module A we define the *semi-direct product* $A \times \mathfrak{g}$ to be the following Lie algebra. The underlying vector space of $A \times \mathfrak{g}$ is $A \oplus \mathfrak{g}$. For $a, b \in A$ and $x, y \in \mathfrak{g}$ we define $[(a, x), (b, y)] = (x \circ b - y \circ a, [x, y])$. We leave it to the reader to show that $A \times \mathfrak{g}$ is a Lie algebra, and that, if A is given the structure of an abelian Lie algebra, then the canonical embeddings $i_A : A \to A \times \mathfrak{g}$, $i_{\mathfrak{g}} : \mathfrak{g} \to A \times \mathfrak{g}$ as well as the canonical projection $p_{\mathfrak{g}} : A \times \mathfrak{g} \to \mathfrak{g}$ are Lie algebra homomorphisms. The semi-direct product therefore gives rise to an extension of Lie algebras, with abelian kernel,

$$A \xrightarrow{i_A} A \times \mathfrak{g} \xrightarrow{p_{\mathfrak{g}}} \mathfrak{g} \tag{2.4}$$

which splits by $i_{\mathfrak{g}} : \mathfrak{g} \to A \times \mathfrak{g}$. The study of extensions with abelian kernel will be undertaken systematically in Section 3. Here we use the split extensions (2.4) to prove the analogue of Corollary VI.5.4.

Proposition 2.3. *The vector space* $\mathrm{Der}(\mathfrak{g}, A)$ *is naturally isomorphic to the vector space of Lie algebra homomorphisms* $f: \mathfrak{g} \to A \times \mathfrak{g}$ *for which* $p_\mathfrak{g} f = 1_\mathfrak{g}$.

Proof. First we note that A may be regarded as an $A \times \mathfrak{g}$-module via $p_\mathfrak{g}: A \times \mathfrak{g} \to \mathfrak{g}$, and that then the canonical projection $d' = p_A : A \times \mathfrak{g} \to A$ becomes a derivation. A Lie algebra homomorphism $f: \mathfrak{g} \to A \times \mathfrak{g}$, inducing the identity on $\mathfrak{g}$, now clearly gives rise to a derivation $d_f = d' f : \mathfrak{g} \to A$. On the other hand, given a derivation $d: \mathfrak{g} \to A$, we define a Lie algebra homomorphism $f_d: \mathfrak{g} \to A \times \mathfrak{g}$ by $f_d(x) = (dx, x)$, $x \in \mathfrak{g}$. The two maps $f \mapsto d_f$, $d \mapsto f_d$ are easily seen to be inverse to each other, to be K-linear, and to be natural in A. ▯

We conclude this section by establishing the analogue of Corollary VI.5.6, which asserts that the cohomology of a free group is trivial in dimensions $\geqq 2$. First we introduce the notion of a free Lie algebra.

Definition. Given a K-vectorspace V, the *free K-Lie algebra* $\mathfrak{f} = \mathfrak{f}(V)$ *on* V is a Lie algebra over K containing V as a subspace, such that the following universal property holds: To any K-linear map $f: V \to \mathfrak{g}$ of V into a Lie algebra $\mathfrak{g}$ over K there exists a unique Lie algebra map $\tilde{f}: \mathfrak{f} V \to \mathfrak{g}$ extending f. In other words, $\mathfrak{f}$ is left adjoint to the underlying functor from Lie algebras to vector spaces which forgets the Lie algebra structure. The existence of $\mathfrak{f}(V)$ is proved in Proposition 2.4. Note that its uniqueness follows, of course, from purely categorical arguments.

Proposition 2.4. *Let TV denote the tensor algebra over the K-vector space V. The free Lie algebra $\mathfrak{f}(V)$ over K is the Lie subalgebra of LTV generated by V.*

Proof. Suppose given $f: V \to \mathfrak{g}$. By the universal property of the tensor algebra the map $if: V \to \mathfrak{g} \rightarrowtail U\mathfrak{g}$ extends to an algebra homomorphism $TV \to U\mathfrak{g}$. Clearly the Lie subalgebra of LTV generated by V is mapped into $\mathfrak{g} \subseteqq LU\mathfrak{g}$. The uniqueness of the extension is trivial. ▯

Theorem 2.5. *The augmentation ideal $I\mathfrak{f}$ of a free Lie algebra $\mathfrak{f}$ is a free $\mathfrak{f}$-module.*

Proof. Let $\mathfrak{f} = \mathfrak{f}(V)$ and let $\{e\}$ be a K-basis of V, and let $f: \{e\} \to M$ be a function into the $\mathfrak{f}$-module M. We shall show that f may be extended uniquely to a $\mathfrak{f}$-module homomorphism $f': I\mathfrak{f} \to M$. First note that uniqueness is clear since f extends uniquely to a K-linear map $\tilde{f}: V \to M$ and $V \subseteqq I\mathfrak{f}$ generates $I\mathfrak{f}$. Using the fact that $\mathfrak{f}$ is free on V, we define a Lie algebra homomorphism $\bar{f}': \mathfrak{f} \to M \times \mathfrak{f}$ by extending $\bar{f}(v) = (\tilde{f}(v), v)$, $v \in V$. By Proposition 2.3 $\bar{f}'$ determines a derivation $d: \mathfrak{f} \to M$ with $d(v) = \tilde{f}(v)$, $v \in V$. By Theorem 2.1 d corresponds to an $\mathfrak{f}$-module homomorphism $f': I\mathfrak{f} \to M$ with $f'(v) = \tilde{f}(v)$, $v \in V$. Thus $\{e\}$ is an $\mathfrak{f}$-basis for $I\mathfrak{f}$. ▯

Corollary 2.6. *For a free Lie algebra* $\mathfrak{f}$, *we have* $H^n(\mathfrak{f}, A)=0$ *for all* $\mathfrak{f}$*-modules* A *and all* $n \geqq 2$. $\square$

Exercises:

2.1. For a Lie algebra $\mathfrak{g}$ over K and a right $\mathfrak{g}$-module B, define homology groups of $\mathfrak{g}$ by

$$H_n(\mathfrak{g}, B) = \operatorname{Tor}_n^{\mathfrak{g}}(B, K), \qquad n \geqq 0.$$

Show that $H_0(\mathfrak{g}, B) = B/B\mathfrak{g}$, where $B\mathfrak{g}$ stands for the submodule of B generated by $b \circ x$; $b \in B$, $x \in \mathfrak{g}$. Show that $H_1(\mathfrak{g}, B) = \ker(B \otimes_{\mathfrak{g}} I\mathfrak{g} \to B \otimes_{\mathfrak{g}} U\mathfrak{g})$. Finally show that for B a trivial $\mathfrak{g}$-module, $H_1(\mathfrak{g}, B) \cong B \otimes_K \mathfrak{g}_{ab}$.

2.2. List the properties of $H^n(\mathfrak{g}, A)$ and $H_n(\mathfrak{g}, B)$ analogous to the properties stated in Section VI.2 for the (co)homology of groups.

2.3. Regard $\mathfrak{g}$ as a $\mathfrak{g}$-module. Show that $\operatorname{Der}(\mathfrak{g}, \mathfrak{g})$ has the structure of a Lie algebra.

3. H^2 and Extensions

In order to interpret the second cohomology group, $H^2(\mathfrak{g}, A)$, we shall also proceed in the same way as for groups. The relation of this section to Sections 6, 8, 10 of Chapter VI will allow us to leave most of the proofs to the reader.

Let $\mathfrak{n} \rightarrowtail \mathfrak{g} \twoheadrightarrow \mathfrak{h}$ be an exact sequence of Lie algebras over K. Consider the short exact sequence of $\mathfrak{g}$-modules $I\mathfrak{g} \rightarrowtail U\mathfrak{g} \twoheadrightarrow K$. Tensoring with $U\mathfrak{h}$ yields

$$0 \to \operatorname{Tor}_1^{\mathfrak{g}}(U\mathfrak{h}, K) \to U\mathfrak{h} \otimes_{\mathfrak{g}} I\mathfrak{g} \to U\mathfrak{h} \otimes_{\mathfrak{g}} U\mathfrak{g} \to U\mathfrak{h} \otimes_{\mathfrak{g}} K \to 0,$$

with each term having a natural $\mathfrak{h}$-module structure. Using Corollaries 1.5, 1.6 and the results of Section IV.12 we obtain

$$\operatorname{Tor}_1^{\mathfrak{g}}(U\mathfrak{g} \otimes_{\mathfrak{n}} K, K) = \operatorname{Tor}_1^{\mathfrak{g}}(U\mathfrak{h}, K) \cong \operatorname{Tor}_1^{\mathfrak{n}}(K, K).$$

Since $\operatorname{Tor}_1^{\mathfrak{n}}(K, K) \cong \mathfrak{n}_{ab}$ by Exercise 2.1 we obtain

Theorem 3.1. *If* $\mathfrak{n} \rightarrowtail \mathfrak{g} \twoheadrightarrow \mathfrak{h}$ *is an exact sequence of Lie algebras, then* $0 \to \mathfrak{n}_{ab} \to U\mathfrak{h} \otimes_{\mathfrak{g}} I\mathfrak{g} \to I\mathfrak{h} \to 0$ *is an exact sequence of* $\mathfrak{h}$*-modules.* $\square$

From this result we deduce, exactly as in the case of groups,

Theorem 3.2. *If* $\mathfrak{n} \rightarrowtail \mathfrak{g} \twoheadrightarrow \mathfrak{h}$ *is an exact sequence of Lie algebras and if* A *is an* $\mathfrak{h}$*-module, then the following sequence is exact*

$$0 \to \operatorname{Der}(\mathfrak{h}, A) \to \operatorname{Der}(\mathfrak{g}, A) \to \operatorname{Hom}_{\mathfrak{g}}(\mathfrak{n}_{ab}, A) \to H^2(\mathfrak{h}, A) \to H^2(\mathfrak{g}, A). \tag{3.1}$$

The proof is analogous to the proof of Theorem VI.8.1 and is left to the reader. $\square$

Let $A \overset{i}{\rightarrowtail} \mathfrak{g} \overset{p}{\twoheadrightarrow} \mathfrak{h}$ be an extension of Lie algebras over K, with abelian kernel A. If $s: \mathfrak{h} \to \mathfrak{g}$ is a section, that is, a K-linear map such that $ps = 1_{\mathfrak{g}}$, we can define in iA, and hence in A, an $\mathfrak{h}$-module structure by $x \circ ia = [sx, ia]$, $a \in A$, $x \in \mathfrak{h}$, where [,] denotes the bracket in $\mathfrak{g}$. It is easily verified that, since A is abelian, the $\mathfrak{h}$-action thus defined on A does not depend upon the choice of section s. This $\mathfrak{h}$-module structure on A is called the $\mathfrak{h}$-module structure *induced by the extension.*

An *extension* of $\mathfrak{h}$ by an $\mathfrak{h}$-module A is an extension of Lie algebras $A \rightarrowtail \mathfrak{g} \twoheadrightarrow \mathfrak{h}$, with abelian kernel, such that the given $\mathfrak{h}$-module structure in A agrees with the one induced by the extension. Notice that the split extension (2.4) is an extension of $\mathfrak{g}$ by the $\mathfrak{g}$-module A.

We shall call two extensions $A \rightarrowtail \mathfrak{g} \twoheadrightarrow \mathfrak{h}$ and $A \rightarrowtail \mathfrak{g}' \twoheadrightarrow \mathfrak{h}$ *equivalent*, if there is a Lie algebra homomorphism $f: \mathfrak{g} \to \mathfrak{g}'$ such that the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, equal] & \mathfrak{g} \arrow[r, two heads] \arrow[d, "f"] & \mathfrak{h} \arrow[d, equal] \\ A \arrow[r] & \mathfrak{g}' \arrow[r, two heads] & \mathfrak{h} \end{tikzcd}$$

is commutative. Note that, if it exists, f is automatically an isomorphism. We denote the set of equivalence classes of extensions of $\mathfrak{h}$ by A by $M(\mathfrak{h}, A)$. By the above, $M(\mathfrak{h}, A)$ contains at least one element, the equivalence class containing the semi-direct product $A \xrightarrow{i_A} A \times \mathfrak{h} \xrightarrow{p_{\mathfrak{h}}} \mathfrak{h}$. With these definitions one proves, formally just as for groups (Section VI.10), the following characterization of $H^2(\mathfrak{h}, A)$.

Theorem 3.3. *There is a one-to-one correspondence between $H^2(\mathfrak{h}, A)$ and the set $M(\mathfrak{h}, A)$ of equivalence classes of extensions of $\mathfrak{h}$ by A. The set $M(\mathfrak{h}, A)$ therefore has a natural K-vector space structure and $M(\mathfrak{h}, -)$ is a (covariant) functor from $\mathfrak{h}$-modules to K-vector spaces.*

The proof is left to the reader; also we leave it to the reader to show that the zero element in $H^2(\mathfrak{h}, A)$ corresponds to the equivalence class of the semi-direct product. □

Exercises:

3.1. Let $\mathfrak{n} \rightarrowtail \mathfrak{g} \twoheadrightarrow \mathfrak{h}$ be an exact sequence of Lie algebras and let B be a right $\mathfrak{h}$-module. Show that the following sequence is exact

$$H_2(\mathfrak{g}, B) \to H_2(\mathfrak{h}, B) \to B \otimes_{\mathfrak{g}} \mathfrak{n}_{ab} \to H_1(\mathfrak{g}, B) \to H_1(\mathfrak{h}, B) \to 0 .$$

3.2. Assume $\mathfrak{g} = \mathfrak{f}/\mathfrak{r}$ where $\mathfrak{f}$ is a free Lie algebra. Show that $H_2(\mathfrak{g}, K) = [\mathfrak{f}, \mathfrak{f}] \cap \mathfrak{r}/[\mathfrak{f}, \mathfrak{r}]$, where $[\mathfrak{f}, \mathfrak{r}]$ denotes the Lie ideal of $\mathfrak{f}$ generated by all $[f, r]$ with $f \in \mathfrak{f}$, $r \in \mathfrak{r}$.

3.3. Prove the following result: Let $f: \mathfrak{g} \to \mathfrak{h}$ be a homomorphism of Lie algebras, such that $f_*: \mathfrak{g}_{ab} \to \mathfrak{h}_{ab}$ is an isomorphism and $f_*: H_2(\mathfrak{g}, K) \to H_2(\mathfrak{h}, K)$ is surjective. Then f induces isomorphisms

$$f_n : \mathfrak{g}/\mathfrak{g}_n \xrightarrow{\sim} \mathfrak{h}/\mathfrak{h}_n, \qquad n=0,1,\dots ;$$

where $\mathfrak{g}_n$ and $\mathfrak{h}_n$ denote the n-th terms of the lower central series ($\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{g}_n = [\mathfrak{g}, \mathfrak{g}_{n-1}]$).

4. A Resolution of the Ground Field K

By definition of the cohomology of Lie algebras, $H^n(\mathfrak{g}, A)$ may be computed via any $\mathfrak{g}$-projective resolution of the trivial $\mathfrak{g}$-module K. For actual computations it is desirable to have some standard procedure for constructing such a resolution. We remark that copying Section VI.13 yields such standard resolutions. However, for Lie algebras a much simpler, i.e., smaller resolution is available. In order to give a comprehensive description of it we proceed as follows.

For any K-vector space V, and $n \geqq 1$, we define $E_n V$ to be the quotient of the n-fold tensor product of V, that is, $T_n V$, by the subspace generated by

$$x_1 \otimes x_2 \otimes \cdots \otimes x_n - (\operatorname{sig} \sigma) x_{\sigma 1} \otimes x_{\sigma 2} \otimes \cdots \otimes x_{\sigma n},$$

for $x_1, \dots, x_n \in V$, and all permutations σ of the set $\{1, 2, \dots, n\}$. The symbol $\operatorname{sig}\sigma$ denotes the parity of the permutation σ. We shall use $\langle x_1, \dots, x_n \rangle$ to denote the element of $E_n V$ corresponding to $x_1 \otimes \cdots \otimes x_n$. Clearly we have

$$\langle x_1, \dots, x_i, \dots, x_j, \dots, x_n \rangle = -\langle x_1, \dots, x_j, \dots, x_i, \dots, x_n \rangle .$$

Note that $E_1 V \cong V$, and set $E_0 V = K$. Then $E_n V$ is called the n^{th} *exterior power* of V and the (internally graded) K-algebra $EV = \bigoplus_{n=0}^{\infty} E_n V$, with multiplication induced by that in TV, is called the *exterior algebra* on the vector space V.

Now let $\mathfrak{g}$ be a Lie algebra over K, and let V be the underlying vector space of $\mathfrak{g}$. Denote by C_n the $\mathfrak{g}$-module $U\mathfrak{g} \otimes_K E_n V$, $n = 0, 1, \dots$. For short we shall write $u\langle x_1, \dots, x_n \rangle$ for $u \otimes \langle x_1, \dots, x_n \rangle$, $u \in U\mathfrak{g}$. We shall prove that differentials $d_n : C_n \to C_{n-1}$ may be defined such that

$$\cdots \to C_n \xrightarrow{d_n} C_{n-1} \to \cdots \to C_1 \to C_0 \tag{4.1}$$

is a $\mathfrak{g}$-projective resolution of K. Of course $C_0 = U\mathfrak{g}$, and $\varepsilon : C_0 \to K$ is just the augmentation. Notice that plainly C_n, $n = 0, 1, \dots$, is $\mathfrak{g}$-free, since $E_n V$ is K-free.

We first show that (4.1) is a complex. This will be achieved in the 5 steps (a), (b), ..., (e), below. It then remains to prove that the augmented complex $\cdots \to C_n \to C_{n-1} \to \cdots \to C_0 \xrightarrow{\varepsilon} K \to 0$ is exact. This will be a consequence of Lemma 4.1 below.

(a) We define, for every $y \in \mathfrak{g}$, a $\mathfrak{g}$-module homomorphism $\theta(y): C_n \to C_n$, $n = 0, 1, \ldots$, by

$$\theta(y)\langle x_1, \ldots, x_n\rangle = -y\langle x_1, \ldots, x_n\rangle + \sum_{i=1}^{n} (-1)^{i+1} \langle [y, x_i], x_1, \ldots, \hat{x}_i, \ldots, x_n\rangle ,$$

where the symbol $\hat{x}_i$ indicates that x_i is to be omitted. Note that $(-1)^{i+1}\langle [y, x_i], x_1, \ldots, \hat{x}_i, \ldots, x_n\rangle = \langle x_1, \ldots, [y, x_i], \ldots, x_n\rangle$. We use this remark to prove that

$$\theta([x, y]) = \theta(x)\,\theta(y) - \theta(y)\,\theta(x) . \tag{4.2}$$

Proof of (4.2)
We have

$$\begin{aligned}\theta(x)\,\theta(y)\langle x_1, \ldots, x_n\rangle &= yx\langle x_1, \ldots, x_n\rangle \\ &- \sum_{i=1}^{n} x\langle x_1, \ldots, [y, x_i], \ldots, x_n\rangle - \sum_{i=1}^{n} y\langle x_1, \ldots, [x, x_i], \ldots, x_n\rangle \\ &+ \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \langle x_1, \ldots, [x, x_i], \ldots, [y, x_j], \ldots, x_n\rangle \\ &+ \sum_{i=1}^{n} \langle x_1, \ldots, [x, [y, x_i]], \ldots, x_n\rangle .\end{aligned}$$

Using the Jacobi identity we obtain

$$\begin{aligned}(\theta(x)\,\theta(y) - \theta(y)\,\theta(x))\langle x_1, \ldots, x_n\rangle &= [y, x]\langle x_1, \ldots, x_n\rangle \\ + \sum_{i=1}^{n} \langle x_1, \ldots [[x, y], x_i], \ldots, x_n\rangle &= \theta([x, y])\langle x_1, \ldots, x_n\rangle . \quad \square\end{aligned}$$

(b) We define $\mathfrak{g}$-module homomorphisms $\sigma(y): C_n \to C_{n+1}$, $n = 0, 1, \ldots$, by

$$\sigma(y)\langle x_1, \ldots, x_n\rangle = \langle y, x_1, \ldots, x_n\rangle .$$

We claim that

$$\theta(x)\,\sigma(y) - \sigma(y)\,\theta(x) = \sigma([x, y]) . \tag{4.3}$$

Proof of (4.3)

$$\begin{aligned}(\theta(x)\,\sigma(y) - \sigma(y)\,\theta(x))\langle x_1, \ldots, x_n\rangle &= -x\langle y, x_1, \ldots, x_n\rangle \\ &+ \langle [x, y], x_1, \ldots, x_n\rangle + \sum_{i=1}^{n} \langle y, x_1, \ldots, [x, x_i], \ldots, x_n\rangle \\ &+ x\langle y, x_1, \ldots, x_n\rangle - \sum_{i=1}^{n} \langle y, x_1, \ldots, [x, x_i], \ldots, x_n\rangle \\ &= \sigma([x, y])\langle x_1, \ldots, x_n\rangle . \quad \square\end{aligned}$$

(c) Next we define g-module homomorphisms $d_n: C_n \to C_{n-1}$, $n=0, 1, 2, \ldots$, such that, for all $y \in \mathfrak{g}$,

$$\sigma(y)\, d_{n-1} + d_n\, \sigma(y) = -\theta(y), \qquad n = 1, 2, \ldots . \tag{4.4}$$

We set $d_0 = 0$. We then proceed inductively. Assume $d_{n-1}: C_{n-1} \to C_{n-2}$ is defined. Since $\langle x_1, \ldots, x_n \rangle = \sigma(x_1) \langle x_2, \ldots, x_n \rangle$, we are forced by (4.4) to define d_n by

$$\begin{aligned} d_n\langle x_1, \ldots, x_n\rangle &= d_n\, \sigma(x_1)\, \langle x_2, \ldots, x_n\rangle \\ &= (-\theta(x_1) - \sigma(x_1) d_{n-1})\, \langle x_2, \ldots, x_n\rangle . \end{aligned}$$

We remark that d_n is given explicitly by

$$\begin{aligned} d_n\langle x_1, \ldots, x_n\rangle = &\sum_{i=1}^{n} (-1)^{i+1} x_i \langle x_1, \ldots, \hat{x}_i, \ldots, x_n\rangle \\ &+ \sum_{1 \leqq i < j \leqq n} (-1)^{i+j} \langle [x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n\rangle , \end{aligned} \tag{4.5}$$

since this d_n obviously satisfies our requirements.

(d) We claim that

$$\theta(y) d_n - d_n \theta(y) = 0 \tag{4.6}$$

for $n = 0, 1, 2, \ldots$.

Proof of (4.6)

We proceed by induction on n. For $n=0$, (4.6) is trivial. For $n \geqq 1$,

$$(\theta(y) d_n - d_n \theta(y))\, \langle x_1, \ldots, x_n\rangle = (\theta(y) d_n \sigma(x_1) - d_n \theta(y) \sigma(x_1))\, \langle x_2, \ldots, x_n\rangle .$$

Thus it is sufficient to show that

$$\theta(y) d_n \sigma(x) - d_n \theta(y)\, \sigma(x) = 0 .$$

But

$$\begin{aligned} &\theta(y) d_n \sigma(x) - d_n \theta(y) \sigma(x) \\ &= -\theta(y)\theta(x) - \theta(y)\sigma(x) d_{n-1} - d_n \sigma(x) \theta(y) - d_n \sigma[y, x], \quad \text{by (4.4) and (4.3)}, \\ &= -\theta(y)\theta(x) - \theta(y)\sigma(x) d_{n-1} + \theta(x)\theta(y) + \sigma(x) d_{n-1} \theta(y) + \theta[y, x] \\ &\qquad\qquad + \sigma[y, x] d_{n-1}, \quad \text{by (4.4)}, \\ &= -\theta(y)\sigma(x) d_{n-1} + \sigma(x)\theta(y) d_{n-1} + \sigma[y, x] d_{n-1}, \quad \text{by (4.2) and the inductive hypothesis}, \\ &= 0 \quad \text{by (4.3)}. \quad \square \end{aligned}$$

(e) Finally, we prove that $d_{n-1} d_n = 0$, whence it will follow that (4.1) is a complex. Clearly $d_0 d_1 = 0$. To prove $d_{n-1} d_n = 0$ we proceed by induction. We have, for $n \geqq 2$,

$$d_{n-1} d_n \langle x_1, \ldots, x_n\rangle = d_{n-1} d_n \sigma(x_1)\, \langle x_2, \ldots, x_n\rangle ;$$

but by (4.4) we obtain

$$\begin{aligned} d_{n-1} d_n \sigma(x_1) &= -d_{n-1}(\theta(x_1)+\sigma(x_1)d_{n-1}) \\ &= -d_{n-1}\theta(x_1)+\theta(x_1)d_{n-1}+\sigma(x_1)d_{n-2}d_{n-1}=0 \end{aligned}$$

by (4.6) and the induction hypothesis. □

It remains to prove that the complex

$$C: \cdots \to C_n \to C_{n-1} \to \cdots \to C_0 \xrightarrow{\varepsilon} K \to 0\,, \tag{4.7}$$

where $\varepsilon: C_0 \to K$ is the augmentation $\varepsilon: U\mathfrak{g} \to K$, is exact. This is achieved by regarding (4.7) as a complex of K-vector spaces and proving that its homology is trivial. Our tactics here are entirely different from those adopted in proving that (4.7) is a complex. (We use (4.5) which has *not* been used previously!)

Let $\{e_i\}$, $i \in J$, be a K-basis of $\mathfrak{g}$, and assume the index set J simply ordered. By Theorem 1.2 (Birkhoff-Witt) the elements

$$e_{k_1} \cdots e_{k_m} \langle e_{l_1}, \ldots, e_{l_n} \rangle \tag{4.8}$$

with

$$k_1 \leqq k_2 \leqq \cdots \leqq k_m \quad \text{and} \quad l_1 < l_2 < \cdots < l_n$$

form a K-basis of C_n. We define a family of subcomplexes F_pC of C, $p=0,1,\ldots$, as follows: $(F_pC)_{-1}=K$, and $(F_pC)_n$, $n \geqq 0$, is the subspace of C_n generated by the basis elements (4.8) with $m+n \leqq p$. Plainly, the differential d_n, $n \geqq 0$, maps $(F_pC)_n$ into $(F_pC)_{n-1}$, so that F_pC is indeed a subcomplex of C. Plainly also, $F_{p+1}C \supseteqq F_pC$ and $\bigcup F_pC = C$. For every $p \geqq 1$ we define a complex W^p by $W^p_n = (F_pC)_n/(F_{p-1}C)_n$ for $n \geqq 0$ and $W^p_{-1} = K$. It is immediate from (4.5) that the differential d^p in W^p is given by

$$\begin{aligned} &d^p_n(e_{k_1} e_{k_2} \cdots e_{k_m} \langle e_{l_1}, \ldots, e_{l_n} \rangle) \\ &\equiv \sum_{i=1}^{n} (-1)^{i+1} e_{k_1} \cdots e_{k_m} e_{l_i} \langle e_{l_1}, \ldots, \hat{e}_{l_i}, \ldots, e_{l_n} \rangle \bmod (F_{p-1}C)_{n-1} \end{aligned} \tag{4.9}$$

Note that the summands on the right hand side are not necessarily of the form (4.8), since we cannot guarantee $k_m \leqq l_i$. However, it follows from the proof of Theorem 1.2 (Birkhoff-Witt) that the class in $W^p \bmod F_{p-1}C$ represented by an element of the form (4.8) remains the same when the order in which $e_{k_1}, \ldots, e_{k_m}$ are written is changed. We remark that, in the terminology of Section VIII.2, $\{W^p\}$ is the *graded object associated with the object C filtered by F_pC.*

Lemma 4.1. *The complex W^p is exact.*

We postpone the proof of Lemma 4.1 in order to show how it implies the desired result on C.

It follows from Lemma 4.1 that $H_n(\boldsymbol{W}^p)=0$ for all $p \geqq 1$ and all n. We then consider the short exact sequence of complexes

$$F_{p-1}\boldsymbol{C} \rightarrowtail F_p \boldsymbol{C} \twoheadrightarrow \boldsymbol{W}^p .$$

The associated long exact homology sequence then shows that $H_n(F_{p-1}\boldsymbol{C}) \cong H_n(F_p\boldsymbol{C})$ for all n, and all $p \geqq 1$. Since $F_0\boldsymbol{C}$ is the complex $0 \to K \to K \to 0$, we have $H_n(F_0\boldsymbol{C})=0$, for all n. Hence, by induction, $H_n(F_p\boldsymbol{C})=0$ for all n and all $p \geqq 0$. Since $\boldsymbol{C}=\bigcup_{p \geqq 0} F_p\boldsymbol{C}$ it follows easily that $H_n(\boldsymbol{C})=0$.

Proof of Lemma 4.1. In order to show that $\boldsymbol{W}^p$ is exact, we define a K-linear contracting homotopy Σ as follows. $\Sigma_{-1}: K \to W_0^p$ is given by $\Sigma_{-1}(1_K)=1\langle\ \rangle$, and, for $n \geqq 0$, we define $\Sigma_n: W_n^p \to W_{n+1}^p$ by

$$\Sigma_n(e_{k_1}\dots e_{k_m}\langle e_{l_1},\dots,e_{l_n}\rangle)\begin{cases} =0, & \text{if } k_m \leqq l_n \text{ in } J, \text{ in particular if } m=0\,; \\ =(-1)^n e_{k_1}\dots e_{k_{m-1}}\langle e_{l_1},\dots,e_{l_n},e_{k_m}\rangle, & \text{if } k_m > l_n . \end{cases}$$

One readily verifies that $\varepsilon\Sigma_{-1}=1$, $d_1^p\Sigma_0+\Sigma_{-1}\varepsilon=1$ and

$$d_{n+1}^p\Sigma_n+\Sigma_{n-1}d_n^p=1\,. \quad \square$$

We now summarize our results in a single statement.

Theorem 4.2. *Let $C_n=U\mathfrak{g}\otimes_K E_n V$ where V is the vectorspace underlying $\mathfrak{g}$, and let $d_n: C_n \to C_{n-1}$ be the $\mathfrak{g}$-module maps defined by*

$$\begin{aligned} d_n(\langle x_1,\dots,x_n\rangle) &= \sum_{i=1}^{n}(-1)^{i+1}x_i\langle x_1,\dots,\hat{x}_i,\dots,x_n\rangle \\ &+ \sum_{1\leqq i<j\leqq n}(-1)^{i+j}\langle [x_i,x_j],x_1,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,x_n\rangle\,. \end{aligned}$$

Then the sequence

$$\boldsymbol{C}: \cdots \to C_n \xrightarrow{d_n} C_{n-1} \to \cdots \to C_0$$

is a $\mathfrak{g}$-free resolution of the trivial $\mathfrak{g}$-module K. $\square$

We finally note the following important corollary.

Corollary 4.3. *Let $\mathfrak{g}$ be a Lie algebra of dimension n over K. Then for any $\mathfrak{g}$-module A, $H^k(\mathfrak{g},A)=0$ for $k \geqq n+1$.*

Proof. For $k \geqq n+1$ we have $E_k V=0$. $\square$

Exercises:

4.1. Show that the product in the tensor algebra TV induces a product in $EV=\bigoplus_{n=0}^{\infty} E_n V$, which makes EV into a K-algebra.

4.2. Suppose that the characteristic of K is different from 2. Show that $EV \cong TV/(v^2)$, where (v^2) denotes the ideal in TV generated by all squares in TV.

4.3. Let $A \rightarrowtail \mathfrak{g} \overset{p}{\twoheadrightarrow} \mathfrak{h}$ be an extension of Lie algebras over K. Let $s: \mathfrak{h} \to \mathfrak{g}$ be a section, that is, a K-linear map with $ps = 1_{\mathfrak{h}}$, so that, as K-vector-spaces, $\mathfrak{g} \cong A \oplus \mathfrak{h}$. Show that the Lie algebra structure of $\mathfrak{g}$ may be described by a K-bilinear function $h: \mathfrak{h} \times \mathfrak{h} \to A$ defined by $[sx, sy] = s[x, y] + h(x, y)$, $x, y \in \mathfrak{h}$. Show that h is a 2-cocycle in $\mathrm{Hom}_{\mathfrak{h}}(\boldsymbol{C}, A)$ where $\boldsymbol{C}$ is the resolution of Theorem 4.2 for the Lie algebra $\mathfrak{h}$. Also, show that two different sections s_1, s_2, yield two cohomologous cocycles h_1, h_2.

4.4. Using Exercise 4.3, show directly that $H^2(\mathfrak{h}, A) \cong M(\mathfrak{h}, A)$.

5. Semi-simple Lie Algebras

In the next two sections of Chapter VII we shall give cohomological proofs of two main theorems in the theory of Lie algebras over a field of *characteristic* 0.

The first is that the finite-dimensional representations of a semi-simple Lie algebra are completely reducible. The main step in that proof will be to show that the *first* cohomology group of a semi-simple Lie algebra with arbitrary finite-dimensional coefficient module is trivial. This is known as the *first Whitehead Lemma* (Proposition 6.1). Secondly we shall prove that every finite dimensional Lie algebra $\mathfrak{g}$ is the split extension of a semi-simple Lie algebra by the radical of $\mathfrak{g}$. The main step in the proof of this result will be to show that the *second* cohomology group of a semi-simple Lie algebra with arbitrary finite-dimensional coefficient module is trivial. This is known as the *second Whitehead Lemma* (Proposition 6.3). Since this section is preparatory for Section 6, we will postpone exercises till the end of that section.

In the whole of this section $\mathfrak{g}$ will denote a *finite-dimensional Lie algebra over a field K of characteristic* 0. Also, A will denote a *finite-dimensional* $\mathfrak{g}$*-module.*

Definition. To any Lie algebra $\mathfrak{g}$ and any $\mathfrak{g}$-module A we define an *associated bilinear form* β from $\mathfrak{g}$ to K as follows. Let $\varrho: \mathfrak{g} \to L(\mathrm{End}_K A)$ be the structure map of A. If $x, y \in \mathfrak{g}$ then ϱx, ϱy are K-linear endomorphisms of A. We define $\beta(x, y)$ to be the trace of the endomorphism $(\varrho x)(\varrho y)$,

$$\beta(x, y) = \mathrm{Tr}((\varrho x)(\varrho y)), \quad x, y \in \mathfrak{g}. \tag{5.1}$$

The proof that β is bilinear is straightforward and will be left to the reader. Trivially $\beta(x, y) = \beta(y, x)$, β is symmetric.

If $A = \mathfrak{g}$, i.e., if $\mathfrak{g}$ is regarded as $\mathfrak{g}$-module, then the associated bilinear form is called the *Killing form* of $\mathfrak{g}$; thus, the Killing form is $\mathrm{Tr}((\mathrm{ad}\, x)(\mathrm{ad}\, y))$.

Lemma 5.1.

$$\beta([x, y], z) = \beta(x, [y, z]), \quad x, y, z \in \mathfrak{g}.$$

Proof. Since the trace function is additive and $\operatorname{Tr}(\varphi\psi) = \operatorname{Tr}(\psi\varphi)$, for $\varphi, \psi \in \operatorname{End}_K A$, we have

$$\beta([x,y],z) = \operatorname{Tr}((\varrho x\, \varrho y - \varrho y\, \varrho x)\varrho z) = \operatorname{Tr}(\varrho x(\varrho y\, \varrho z - \varrho z\, \varrho y)) = \beta(x,[y,z]). \quad \square$$

Definition. A Lie algebra $\mathfrak{g}$ is called *semi-simple* if $\{0\}$ is the only abelian ideal of $\mathfrak{g}$.

We now cite a key theorem from the theory of semi-simple Lie algebras.

Theorem 5.2. *Let $\mathfrak{g}$ be semi-simple (over a field of characteristic 0), and let A be a $\mathfrak{g}$-module. If the structure map ϱ is injective, then the bilinear form β corresponding to A is non-degenerate.*

The fact that ϱ is injective is usually expressed by the phrase that ϱ is a *faithful representation* of $\mathfrak{g}$ in A.

We do *not* attempt to give a proof of this rather deep result, which is closely related to Cartan's criterion for solvability of Lie algebras. Elementary proofs are easily accessible in the literature (G. Hochschild [25, p. 117–122]; J.-P. Serre [42, LA. 5.14–LA. 5.20]).

Corollary 5.3. *The Killing form of a semi-simple Lie algebra is non-degenerate.*

Proof. The structure map $\operatorname{ad}: \mathfrak{g} \to L(\operatorname{End}_K \mathfrak{g})$ of the $\mathfrak{g}$-module $\mathfrak{g}$ has the center of $\mathfrak{g}$ as kernel (see Exercise 1.2). Since the center is an abelian ideal, it is trivial. Hence ad is injective. $\square$

Corollary 5.4. *Let $\mathfrak{a}$ be an ideal in the semi-simple Lie algebra $\mathfrak{g}$. Then there exists an ideal $\mathfrak{b}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$, as Lie algebras.*

Proof. Define $\mathfrak{b}$ to be the orthogonal complement of $\mathfrak{a}$ with respect to the Killing form β. Clearly it is sufficient to show (i) that $\mathfrak{b}$ is an ideal and (ii) that $\mathfrak{a} \cap \mathfrak{b} = \{0\}$. To prove (i) let $x \in \mathfrak{g}$, $b \in \mathfrak{b}$, $a \in \mathfrak{a}$. We have $\beta(a, [x, b]) = \beta([a, x], b) = \beta(a', b) = 0$, where $[a, x] = a' \in \mathfrak{a}$. Hence with $b \in \mathfrak{b}$, $[x, b] \in \mathfrak{b}$ and $\mathfrak{b}$ is an ideal. To prove (ii) let $x, y \in \mathfrak{a} \cap \mathfrak{b}$, $z \in \mathfrak{g}$; then $\beta([x, y], z) = \beta(x, [y, z]) = 0$, since $[y, z] \in \mathfrak{b}$ and $x \in \mathfrak{a}$. Since β is non-degenerate it follows that $[x, y] = 0$. Thus $\mathfrak{a} \cap \mathfrak{b}$ is an abelian ideal of $\mathfrak{g}$, hence trivial. $\square$

Corollary 5.5. *If $\mathfrak{g}$ is semi-simple, then every ideal $\mathfrak{a}$ in $\mathfrak{g}$ is semi-simple also.*

Proof. Since $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ by Corollary 5.4, every ideal $\mathfrak{a}'$ in $\mathfrak{a}$ is also an ideal in $\mathfrak{g}$. In particular if $\mathfrak{a}'$ is an abelian ideal, it follows that $\mathfrak{a}' = 0$. $\square$

We now return to the cohomology theory of Lie algebras. Recall that the ground field K is assumed to have characteristic 0.

Proposition 5.6. *Let A be a (finite-dimensional) simple module over the semi-simple Lie algebra $\mathfrak{g}$ with non-trivial $\mathfrak{g}$-action. Then $H^q(\mathfrak{g}, A) = 0$ for all $q \geqq 0$.*

Proof. Let the structure map $\varrho: g \to L(\mathrm{End}_K A)$ have kernel $\mathfrak{h}'$. By Corollary 5.4, $\mathfrak{h}'$ has a complement $\mathfrak{h}$ in $\mathfrak{g}$, which is non-zero because A is non-trivial. Since $\mathfrak{h}$ is semi-simple by Corollary 5.5, and since ϱ restricted to $\mathfrak{h}$ is injective, the associated bilinear form β is non-degenerate by Theorem 5.2. Note that β is the restriction to $\mathfrak{h}$ of the bilinear form on $\mathfrak{g}$ associated with ϱ. By linear algebra we can choose K-bases $\{e_i\}$, $i=1,\dots,m$, and $\{e'_j\}$, $j=1,\dots,m$, of $\mathfrak{h}$ such that $\beta(e_i, e'_j)=\delta_{ij}$. We now prove the following assertions:

(a) If $x \in \mathfrak{g}$ and if $[e_i, x] = \sum_{k=1}^{m} c_{ik} e_k$ and $[x, e'_j] = \sum_{l=1}^{m} d_{jl} e'_l$, then

$$c_{ij} = d_{ji}\,. \tag{5.2}$$

Proof. $\beta([e_i, x], e'_j) = \beta(\Sigma\, c_{ik} e_k, e'_j) = c_{ij}$; but

$$\beta([e_i, x], e'_j) = \beta(e_i, [x, e'_j]) = \beta(e_i, \Sigma\, d_{jl} e'_l) = d_{ji}\,. \quad \square$$

(b) The element $\sum_{i=1}^{m} e_i e'_i \in U\mathfrak{g}$ is in the center of $U\mathfrak{g}$; hence for any g-module B the map $t = t_B : B \to B$ defined by $t(b) = \sum_{i=1}^{m} e_i \circ (e'_i \circ b)$ is a $\mathfrak{g}$-module homomorphism.

Proof. Let $x \in \mathfrak{g}$, then

$$\begin{aligned} x\Big(\sum_i e_i e'_i\Big) &= \sum_i ([x, e_i] e'_i + e_i x e'_i) = -\sum_{i,k} c_{ik} e_k e'_i + \sum_i e_i x e'_i \\ &= -\sum_{i,k} d_{ki} e_k e'_i + \sum_k e_k x e'_k = -\sum_k e_k [x, e'_k] + \sum_k e_k x e'_k \\ &= \Big(\sum_k e_k e'_k\Big) x\,. \quad \square \end{aligned}$$

It is clear that, if $\varphi: B_1 \to B_2$ is a homomorphism of $\mathfrak{g}$-modules, then $t\varphi = \varphi t$.

(c) Consider the resolution $\boldsymbol{C}: \cdots \to C_n \to C_{n-1} \to \cdots \to C_0$ of Theorem 4.2. The homomorphisms t_{C_n} define a chain map $\boldsymbol{\tau}$ of $\boldsymbol{C}$ into itself. We claim that $\boldsymbol{\tau}$ is homotopic to the zero map.

Proof. We have to find maps $\Sigma_n: C_n \to C_{n+1}$, $n=0,1,\dots$, such that $d_1 \Sigma_0 = \tau_0$ and $d_{n+1}\Sigma_n + \Sigma_{n-1} d_n = \tau_n$, $n \geqq 1$. Define Σ_n to be the $\mathfrak{g}$-module homomorphism given by

$$\Sigma_n \langle x_1, \dots, x_n \rangle = \sum_{k=1}^{m} e_k \langle e'_k, x_1, \dots, x_n \rangle\,.$$

The assertion is then proved by the following computation (k varies from 1 to m; i, j vary from 1 to n):

$$
\begin{aligned}
(d_{n+1}\Sigma_n + \Sigma_{n-1} d_n)\langle x_1, \dots, x_n\rangle = {} & \sum_k e_k e_{k'} \langle x_1, \dots, x_n\rangle \\
& + \sum_{i,k} (-1)^i e_k x_i \langle e'_k, x_1, \dots, \hat{x}_i, \dots, x_n\rangle \\
& + \sum_{i,k} (-1)^i e_k \langle [e'_k, x_i], x_1, \dots, \hat{x}_i, \dots, x_n\rangle \\
& + \sum_{k,i<j} (-1)^{i+j} e_k \langle [x_i, x_j], e'_k, x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n\rangle \\
& + \sum_{i,k} (-1)^{i+1} x_i e_k \langle e'_k, x_1, \dots, \hat{x}_i, \dots, x_n\rangle \\
& + \sum_{k,i<j} (-1)^{i+j} e_k \langle e_{k'}, [x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n\rangle \\
& = \tau_n \langle x_1, \dots, x_n\rangle + \sum_{i,k} (-1)^i [e_k, x_i] \langle e'_k, x_1, \dots, \hat{x}_i, \dots, x_n\rangle \\
& + \sum_{i,k} (-1)^i e_k \langle [e'_k, x_i], x_1, \dots, \hat{x}_i, \dots, x_n\rangle .
\end{aligned}
$$

Using (5.2) the two latter sums cancel each other, and thus assertion (c) is proved. □

Consider now the map $t = t_A : A \to A$ and the induced map

$$t_* : H^q(\mathfrak{g}, A) \to H^q(\mathfrak{g}, A).$$

By the nature of t_A (see the final remark in (b)), it is clear that t_* may be computed as the map induced by $\boldsymbol{\tau}: \boldsymbol{C} \to \boldsymbol{C}$. Hence, by assertion (c), t_* is the zero map. On the other hand $t : A \to A$ must either be an automorphism or the zero map, since A is simple. But it cannot be the zero map, because the trace of the linear transformation t equals $\sum_{i=1}^{m} \beta(e_i, e'_i) = m \neq 0$. Hence, it follows that $H^q(\mathfrak{g}, A) = 0$ for all $q \geqq 0$. □

We do not offer exercises on this section, but we do recommend the reader to study a proof of Theorem 5.2!

6. The two Whitehead Lemmas

Again let $\mathfrak{g}$ be a finite dimensional Lie algebra and let A be a finite dimensional $\mathfrak{g}$-module. We prove the first Whitehead Lemma:

Proposition 6.1. *Let* $\mathfrak{g}$ *be semi-simple, then* $H^1(\mathfrak{g}, A) = 0$.

Proof. Suppose there is a $\mathfrak{g}$-module A with $H^1(\mathfrak{g}, A) \neq 0$. Then there is such a $\mathfrak{g}$-module A with minimal K-dimension. If A is not simple, then there is a proper submodule $0 \neq A' \subset A$. Consider $0 \to A' \to A \to A/A' \to 0$

and the associated long exact cohomology sequence

$$\cdots \to H^1(\mathfrak{g}, A') \to H^1(\mathfrak{g}, A) \to H^1(\mathfrak{g}, A/A') \to \cdots.$$

Since $\dim_K A' < \dim_K A$ and $\dim_K A/A' < \dim_K A$ it follows that

$$H^1(\mathfrak{g}, A') = H^1(\mathfrak{g}, A/A') = 0.$$

Hence $H^1(\mathfrak{g}, A) = 0$, which is a contradiction. It follows that A has to be simple. But then A has to be a trivial $\mathfrak{g}$-module by Proposition 5.6. (Indeed it has to be K; but we make no use of this fact.) We then have $H^1(\mathfrak{g}, A) \cong \mathrm{Hom}_K(\mathfrak{g}_{ab}, A)$ by Proposition 2.2. Now consider

$$[\mathfrak{g}, \mathfrak{g}] \rightarrowtail \mathfrak{g} \twoheadrightarrow \mathfrak{g}_{ab}.$$

By Corollary 5.4 the ideal $[\mathfrak{g}, \mathfrak{g}]$ has a complement which plainly must be isomorphic to $\mathfrak{g}_{ab}$, in particular it must be abelian. Since $\mathfrak{g}$ is semi-simple, $\mathfrak{g}_{ab} = 0$. Hence $H^1(\mathfrak{g}, A) \cong \mathrm{Hom}_K(\mathfrak{g}_{ab}, A) = 0$, which is a contradiction. It follows that $H^1(\mathfrak{g}, A) = 0$ for all $\mathfrak{g}$-modules A. $\square$

Theorem 6.2 (Weyl). *Every (finite-dimensional) module A over a semi-simple Lie algebra $\mathfrak{g}$ is a direct sum of simple $\mathfrak{g}$-modules.*

Proof. Using induction on the K-dimension of A, we have only to show that every non-trivial submodule $0 \neq A' \subset A$ is a direct summand in A. To that end we consider the short exact sequence

$$A' \rightarrowtail A \twoheadrightarrow A'' \tag{6.1}$$

and the induced sequence

$$0 \to \mathrm{Hom}_K(A'', A') \to \mathrm{Hom}_K(A, A') \to \mathrm{Hom}_K(A', A') \to 0, \tag{6.2}$$

which is exact since K is a field. We remark that each of the vector spaces in (6.2) is finite-dimensional and can be made into a $\mathfrak{g}$-module by the following procedure. Let B, C be $\mathfrak{g}$-modules; then $\mathrm{Hom}_K(B, C)$ is a $\mathfrak{g}$-module by $(xf)(b) = xf(b) - f(xb)$, $x \in \mathfrak{g}$, $b \in B$. With this understanding, (6.2) becomes an exact sequence of $\mathfrak{g}$-modules. Note that the invariant elements in $\mathrm{Hom}_K(B, C)$ are precisely the $\mathfrak{g}$-module homomorphisms from B to C. Now consider the long exact cohomology sequence arising from (6.2)

$$\begin{aligned} 0 &\to H^0(\mathfrak{g}, \mathrm{Hom}_K(A'', A')) \to H^0(\mathfrak{g}, \mathrm{Hom}_K(A, A')) \\ &\to H^0(\mathfrak{g}, \mathrm{Hom}_K(A', A')) \to H^1(\mathfrak{g}, \mathrm{Hom}_K(A'', A')) \to \cdots \end{aligned} \tag{6.3}$$

By Proposition 6.1, $H^1(\mathfrak{g}, \mathrm{Hom}_K(A'', A'))$ is trivial. Passing to the interpretation of H^0 as the group of invariant elements, we obtain an epimorphism

$$\mathrm{Hom}_{\mathfrak{g}}(A, A') \twoheadrightarrow \mathrm{Hom}_{\mathfrak{g}}(A', A').$$

It follows that there is a $\mathfrak{g}$-module homomorphism $A \to A'$ inducing the identity in A'; hence (6.1) splits. $\square$

The reader should compare this argument with the proof of Maschke's Theorem (Theorem VI.16.6).

We proceed with the second Whitehead Lemma.

Proposition 6.3. *Let* $\mathfrak{g}$ *be a semi-simple Lie algebra and let* A *be a (finite-dimensional)* $\mathfrak{g}$*-module. Then* $H^2(\mathfrak{g}, A) = 0$.

Proof. We begin as in the proof of Proposition 6.1. Suppose there is a $\mathfrak{g}$-module A with $H^2(\mathfrak{g}, A) \neq 0$. Then there is such a $\mathfrak{g}$-module A with minimal K-dimension. If A is not simple, then there is a proper submodule $0 \neq A' \subset A$. Consider $0 \to A' \to A \to A/A' \to 0$ and the associated long exact cohomology sequence

$$\cdots \to H^2(\mathfrak{g}, A') \to H^2(\mathfrak{g}, A) \to H^2(\mathfrak{g}, A/A') \to \cdots .$$

Since A' is a proper submodule, the minimality property of A leads to the contradiction $H^2(\mathfrak{g}, A) = 0$. Hence A has to be simple. But then A has to be a trivial $\mathfrak{g}$-module by Proposition 5.6. Since K is the only simple trivial $\mathfrak{g}$-module, we have to show that $H^2(\mathfrak{g}, K) = 0$. This will yield the desired contradiction.

By the interpretation of H^2 given in Theorem 3.3, we have to show that every central extension

$$K \overset{i}{\rightarrowtail} \mathfrak{h} \overset{p}{\twoheadrightarrow} \mathfrak{g} \tag{6.4}$$

of the Lie algebra $\mathfrak{g}$ splits.

Let $s: \mathfrak{g} \to \mathfrak{h}$ be a K-linear section of (6.4), so that $ps = 1_{\mathfrak{g}}$. Using the section s, we define, in the K-vector space underlying $\mathfrak{h}$, a $\mathfrak{g}$-module structure by

$$x \circ y = [sx, y], \quad x \in \mathfrak{g}, \quad y \in \mathfrak{h}$$

where the bracket is in $\mathfrak{h}$. The module axioms are easily verified once one notes that $s([x, x']) = [sx, sx'] + k$, where $k \in K$. Clearly K is a submodule of the $\mathfrak{g}$-module $\mathfrak{h}$ so defined.

Now regard $\mathfrak{h}$ as a $\mathfrak{g}$-module. By Theorem 6.2 K is a direct summand in $\mathfrak{h}$, say $\mathfrak{h} = K \oplus \mathfrak{h}'$. It is easily seen that this $\mathfrak{h}'$ is in fact a Lie subalgebra of $\mathfrak{h}$; it is then isomorphic to $\mathfrak{g}$. Hence $\mathfrak{h} \cong K \oplus \mathfrak{g}$ as Lie algebras and the extension (6.4) splits. □

We shall shortly use Proposition 6.3 to prove Theorem 6.7 below (the Levi-Malcev Theorem). However, in order to be able to state that theorem, we shall need some additional definitions. First we shall introduce the notion of derived series, derived length, solvability, etc. for Lie algebras. It will be quite obvious to the reader that these notions as well as certain basic results are merely translations from group theory.

Definition. Given a Lie algebra $\mathfrak{g}$, we define its *derived series* $\mathfrak{g}_0, \mathfrak{g}_1, \cdots$ inductively by

$$\mathfrak{g}_0 = \mathfrak{g}, \mathfrak{g}_{n+1} = [\mathfrak{g}_n, \mathfrak{g}_n], \qquad n = 0, 1, \ldots ,$$

where $[S, T]$, for any subsets S and T of $\mathfrak{g}$, denotes the Lie subalgebra generated by all $[s, t]$ for $s \in S$, $t \in T$.

We leave it to the reader to prove that $\mathfrak{g}_n$ is automatically an ideal in $\mathfrak{g}$.

Definition. A Lie algebra $\mathfrak{g}$ is called *solvable*, if there is an integer $n \geqq 0$ with $\mathfrak{g}_n = \{0\}$. The first integer n for which $\mathfrak{g}_n = \{0\}$ is called the *derived length* of $\mathfrak{g}$. The (easy) proofs of the following two lemmas are left to the reader.

Lemma 6.4. *In the exact sequence* $\mathfrak{h} \rightarrowtail \mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{h}$ *of Lie algebras,* $\mathfrak{g}$ *is solvable if and only if* $\mathfrak{h}$ *and* $\mathfrak{g}/\mathfrak{h}$ *are solvable.* □

Lemma 6.5. *If the ideals* $\mathfrak{a}$ *and* $\mathfrak{b}$ *of* $\mathfrak{g}$ *are solvable then the ideal* $\mathfrak{a} + \mathfrak{b}$ *generated by* $\mathfrak{a}$ *and* $\mathfrak{b}$ *is solvable.* □

An immediate consequence of Lemma 6.5 is the important fact that every finite-dimensional Lie algebra $\mathfrak{g}$ has a unique maximal solvable ideal $\mathfrak{r}$. Indeed take $\mathfrak{r}$ to be the ideal generated by all solvable ideals of $\mathfrak{g}$.

Definition. The unique maximal solvable ideal $\mathfrak{r}$ of $\mathfrak{g}$ is called the *radical* of $\mathfrak{g}$.

Proposition 6.6. $\mathfrak{g}/\mathfrak{r}$ *is semi-simple.*

Proof. Let $\mathfrak{a}/\mathfrak{r}$ be an abelian ideal of $\mathfrak{g}/\mathfrak{r}$; then the sequence $\mathfrak{r} \rightarrowtail \mathfrak{a} \twoheadrightarrow \mathfrak{a}/\mathfrak{r}$ has both ends solvable, hence $\mathfrak{a}$ is solvable by Lemma 6.4. By the maximality of $\mathfrak{r}$, it follows that $\mathfrak{a} = \mathfrak{r}$, whence $\mathfrak{g}/\mathfrak{r}$ is semi-simple. □

Theorem 6.7 (Levi-Malcev). *Every (finite-dimensional) Lie algebra* $\mathfrak{g}$ *is the split extension of a semi-simple Lie algebra by the radical* $\mathfrak{r}$ *of* $\mathfrak{g}$.

Proof. We proceed by induction on the derived length of $\mathfrak{r}$. If $\mathfrak{r}$ is abelian, then it is a $\mathfrak{g}/\mathfrak{r}$-module and $H^2(\mathfrak{g}/\mathfrak{r}, \mathfrak{r}) = 0$ by Proposition 6.3. Since H^2 classifies extensions with abelian kernel the extension $\mathfrak{r} \rightarrowtail \mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{r}$ splits. If $\mathfrak{r}$ is non-abelian with derived length $n \geqq 2$, we look at the following diagram

$$\begin{tikzcd}
\mathfrak{r} \arrow[r, tail] \arrow[d] & \mathfrak{g} \arrow[r, two heads] \arrow[d] & \mathfrak{g}/\mathfrak{r} \arrow[d, equal] \\
\mathfrak{r}/[\mathfrak{r}, \mathfrak{r}] \arrow[r, tail] & \mathfrak{g}/[\mathfrak{r}, \mathfrak{r}] \arrow[r, two heads] & \mathfrak{g}/\mathfrak{r}
\end{tikzcd} \tag{6.5}$$

The bottom sequence splits by the first part of the proof, say by $s : \mathfrak{g}/\mathfrak{r} \to \mathfrak{g}/[\mathfrak{r}, \mathfrak{r}]$. Let $\mathfrak{h}/[\mathfrak{r}, \mathfrak{r}]$ be the image of $\mathfrak{g}/\mathfrak{r}$ under s; clearly $s : \mathfrak{g}/\mathfrak{r} \xrightarrow{\sim} \mathfrak{h}/[\mathfrak{r}, \mathfrak{r}]$ and $[\mathfrak{r}, \mathfrak{r}]$ must be the radical of $\mathfrak{h}$. Now consider the extension $[\mathfrak{r}, \mathfrak{r}] \rightarrowtail \mathfrak{h} \twoheadrightarrow \mathfrak{h}/[\mathfrak{r}, \mathfrak{r}]$. Since $[\mathfrak{r}, \mathfrak{r}]$ has derived length $n-1$, it follows, by the inductive hypothesis, that the extension must split, say by $q : \mathfrak{h}/[\mathfrak{r}, \mathfrak{r}] \to \mathfrak{h}$. Finally it is easy to see that the top sequence of (6.5) splits by $t = qs$, $t : \mathfrak{g}/\mathfrak{r} \xrightarrow{\sim} \mathfrak{h}/[\mathfrak{r}, \mathfrak{r}] \to \mathfrak{h} \subset \mathfrak{g}$. □

Exercises:

6.1. Let $\mathfrak{g}$ be a Lie algebra, finite-dimensional over a field of characteristic 0. Use the exact sequence of Exercise 3.1 and the Whitehead Lemmas to prove that $\mathfrak{r} \cap [\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}, \mathfrak{r}]$, where $\mathfrak{r}$ is the radical of $\mathfrak{g}$. ($[\mathfrak{g}, \mathfrak{r}]$ is called the *nilpotent radical*).

6.2. Let $\beta: \mathfrak{g} \times \mathfrak{g} \to K$ be the Killing form of the semi-simple Lie algebra $\mathfrak{g}$ over the field K of characteristic 0. Show that

$$f(x, y, z) = \beta([x, y], z), \qquad x, y, z \in \mathfrak{g}$$

defines a 3-cocycle in $\operatorname{Hom}_{\mathfrak{g}}(\boldsymbol{C}, K)$, where $\boldsymbol{C}$ denotes the resolution of Theorem 4.2. In fact (see [8, p. 113]), f is not a coboundary. Deduce that $H^3(\mathfrak{g}, K) \neq 0$.

6.3. Using Exercise 6.2 show that, for $\mathfrak{g}$ semi-simple, $H^2(\mathfrak{g}, I\mathfrak{g}) \neq 0$. ($I\mathfrak{g}$ is not finite dimensional!)

6.4. Prove Lemmas 6.4, 6.5.

6.5. Establish the step in the proof of Theorem 6.7 which asserts that $[\mathfrak{r}, \mathfrak{r}]$ is the radical of $\mathfrak{h}$.

7. Appendix: Hilbert's Chain-of-Syzygies Theorem

In this appendix we prove a famous theorem due to Hilbert. We choose to insert this theorem at this point because we have made the *Koszul resolution* available in this chapter. Of course, Hilbert's original formulation did not refer explicitly to the concepts of homological algebra! However, it is easy to translate his formulation, in terms of presentations of polynomial ideals, into that adopted below.

Definition. Let Λ be a ring. We say that the *global dimension* of Λ is less than or equal to m (gl. dim. $\Lambda \leqq m$) if for all Λ-modules A and all projective resolutions $\boldsymbol{P}$ of A, $K_m(A) = \ker(P_{m-1} \to P_{m-2})$ is projective. Thus gl.dim.$\Lambda \leqq m$ if and only if $\operatorname{Ext}^n_\Lambda(A, B) = 0$ for all A, B and $n > m$. Of course we say that gl.dim.$\Lambda = m$ if gl.dim.$\Lambda \leqq m$ but gl.dim.$\Lambda \nleqq m-1$. (See Exercise IV.8.8.)

Let K be a field and let $P = K[x_1, \ldots, x_m]$ be the ring of polynomials over K in the indeterminates $x_1, \ldots, x_m$. We denote by P_j the subspace of homogeneous polynomials of degree j, and consider $P = \bigoplus_{j=0}^{\infty} P_j$ as *internally graded* (see Exercise V.1.6). Consequently, we consider (internally) graded P-modules; an *(internally) graded* P-module A is a P-module A which is a direct sum $A = \bigoplus_{j=0}^{\infty} A_j$ of abelian groups A_j, such that $x_i A_j \subseteq A_{j+1}$, $i = 1, 2, \ldots, m$. The elements of A_j are called *homogeneous* of degree j. It is clear how to extend the definition of global dimension from rings to internally graded rings; the definition employs, of course, the concept of (internally) graded modules. Then Hilbert's theorem reads:

Theorem 7.1. *The global dimension of* $P = K[x_1, \ldots, x_m]$ *is* m. *Moreover, every projective graded* P*-module is free.*

The proof of this theorem will be executed in several steps; not all these steps require that K be a field, so we will specify the assumptions at each stage. Now K can be considered as a graded P-module through the augmentation $\varepsilon: P \to K$, which associates with each polynomial its constant term. Thus K is concentrated in degree 0.

Note that P may be regarded as the universal enveloping algebra of the abelian Lie algebra $\mathfrak{a}$, where $x_1, \ldots, x_m$ form a K-basis of $\mathfrak{a}$. The Lie algebra resolution for $\mathfrak{a}$ of Theorem 4.2 is known as the *Koszul resolution*, or *Koszul complex*. By Theorem 4.2 we have

Proposition 7.2 (Koszul). *Let* $D_n = P \otimes_K E_n \mathfrak{a}$, *and let* $d_n: D_n \to D_{n-1}$ *be defined by*

$$d_n(p \otimes \langle x_{j_1}, \ldots, x_{j_n} \rangle) = \sum_{i=1}^{n} (-1)^{i+1} p x_{j_i} \otimes \langle x_{j_1}, \ldots, \hat{x}_{j_i}, \ldots, x_{j_n} \rangle$$

where $j_1 < j_2 < \cdots < j_n$. *Then* $\mathbf{D}: 0 \to D_m \to D_{m-1} \to \cdots \to D_0$ *is a* P*-free resolution of* K *regarded as graded* P*-module.* $\square$

We note in passing that Proposition 7.2 admits a fairly easy direct proof (see Exercise 7.1).

Proposition 7.3. *Let* M *be a graded* P*-module. If* K *is a commutative ring and if* $M \otimes_P K = 0$ *then* $M = 0$.

Proof. Let I be the *augmentation ideal* in P, that is,

$$I \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} K$$

is exact. Then the homogeneous non-zero elements of I all have positive degree. Now the sequence

$$M \otimes_P I \xrightarrow{\mu_*} M \otimes_P P \xrightarrow{\varepsilon_*} M \otimes_P K \to 0$$

is exact so that the hypothesis implies that

$$\mu_*: M \otimes_P I \to M \otimes_P P$$

is surjective. Moreover, $M \otimes_P P \cong M$ and, when the codomain of μ_* is interpreted as M, μ_* takes the form

$$\mu_*(m \otimes f) = mf, \quad m \in M, \quad f \in I. \tag{7.1}$$

Now suppose $M \neq 0$, and let $m \neq 0$ be an element of M of minimal degree. Then (7.1) leads to an immediate contradiction with the statement that μ_* is surjective. For if

$$m = \mu_* \sum_{i=1}^{s} (m_i \otimes f_i),$$

where we may assume m_i, f_i homogeneous and non-zero, then $m = \sum_{i=1}^{s} m_i f_i$, $\deg f_i \geqq 1$, so $\deg m_i < \deg m$, contrary to the minimality of the degree of m. $\square$

This proposition leads to the key theorem.

Theorem 7.4. *Let B be a graded P-module. If K is a field and if $\mathrm{Tor}_1^P(B, K) = 0$ then B is free.*

Proof. It is plain that every element of $B \otimes_P K$ can be expressed as $b \otimes 1$, $b \in B$, where 1 is the unity element of K. Thus we may select a basis $\{b_i \otimes 1\}$, $i \in I$, for $B \otimes_P K$ as vector space over K. Let F be the free graded P-module generated by $\{b_i\}$, $i \in I$, and let $\varphi : F \to B$ be the homomorphism of graded P-modules given by $\varphi(b_i) = b_i$, $i \in I$. Notice that φ induces the identity φ_*,

$$\varphi_* = 1 : F \otimes_P K \to B \otimes_P K.$$

We show that φ is an isomorphism. First φ is *surjective*. For, given the exact sequence $F \xrightarrow{\varphi} B \twoheadrightarrow C$, we obtain the exact sequence

$$F \otimes_P K \xrightarrow{1} B \otimes_P K \twoheadrightarrow C \otimes_P K$$

so that $C \otimes_P K = 0$ and hence, by Proposition 7.3, $C = 0$. Next φ is *injective*. For, given the exact sequence $R \rightarrowtail F \overset{\varphi}{\twoheadrightarrow} B$, we obtain the exact sequence

$$\cdots \to \mathrm{Tor}_1^P(B, K) \to R \otimes_P K \to F \otimes_P K \xrightarrow{1} B \otimes_P K.$$

Since $\mathrm{Tor}_1^P(B, K) = 0$ it follows that $R \otimes_P K = 0$ and, by a second application of Proposition 7.3, $R = 0$. $\square$

Proof of Theorem 7.1. Let B be any projective graded P-module. Then $\mathrm{Tor}_1^P(B, K) = 0$ and B is free by Theorem 7.4, thus proving the second assertion of Theorem 7.1. By Proposition 7.2 it follows that $\mathrm{Tor}_{m+1}^P(A, K) = 0$ for every graded P-module A. But

$$\mathrm{Tor}_{m+1}^P(A, K) = \mathrm{Tor}_1^P(K_m(A), K).$$

Hence Theorem 7.4 implies that $K_m(A)$ is free, so that gl.dim.$P \leqq m$. To complete the proof of the theorem it remains to exhibit modules A, B such that $\mathrm{Tor}_m^P(A, B) \neq 0$. In fact, we show that

$$\mathrm{Tor}_m^P(K, K) \cong K. \tag{7.2}$$

For, reverting to the Koszul resolution, we observe that $E_m \mathfrak{a} = K$, so that $D_m \otimes_P K \cong K$; and that $d_m \otimes 1 : D_m \otimes_P K \to D_{m-1} \otimes_P K$ is the zero homomorphism since $d_m\langle x_1, \dots, x_m\rangle = \Sigma \pm x_i \langle x_1, \dots, \hat{x}_i, \dots, x_m\rangle$ and $x_i K = 0$. Thus (7.2) is established and the Hilbert chain-of-syzygies theorem is completely proved. $\square$

Exercises:

7.1. Give a direct proof of Proposition 7.2 by constructing a homotopy similar to the one used in the proof of Lemma 4.1.

7.2. Let J be a graded ideal in the polynomial algebra $P = K[x_1, \dots, x_n]$, i.e. an internally graded submodule of the P-module P. Prove that proj.dim.$J \leqq n - 1$. (Hint: Consider a projective resolution of J and extend it by $J \rightarrowtail P \twoheadrightarrow P/J$. Then use Exercise IV.7.7.)

7.3. Show that a flat graded P-module is free.

VIII. Exact Couples and Spectral Sequences

In this chapter we develop the theory of *spectral sequences*: applications will be found in Section 9 and in Chapter IX. Our procedure will be to base the theory on the study of *exact couples*, but we do not claim, of course, that this is the unique way to present the theory; indeed, an alternative approach is to be found e.g. in [7]. Spectral sequences themselves frequently arise from *filtered differential objects* in an abelian category – for example, filtered chain complexes. In such cases it is naturally quite possible to pass directly from the filtered differential object to the spectral sequence without the intervention of the exact couple. However, we believe that the explicit study of the exact couple illuminates the nature of the spectral sequence and of its *limit*.

Also, we do not wish to confine ourselves to those spectral sequences which arise from filtrations although our actual applications will be concerned with such a situation. For many of the spectral sequences of great importance in algebraic topology refer to geometric situations which naturally give rise to exact couples, but not to filtered chain complexes. Thus we may fairly claim that the approach to spectral sequences via exact couples is not only illuminating but also of rather universal significance.

We do not introduce grading into the exact couple until we come to discuss convergence questions. In this way we simplify the description of the algebraic machinery, and exploit the grading (or, as will be the case, bigrading) precisely where it plays a key role in the theory. We distinguish carefully between two aspects of the convergence question for a given spectral sequence $\boldsymbol{E}=\{(E_n, d_n)\}$. First, we may ask whether the spectral sequence converges *finitely*; that is, whether E_∞, the limit term, is reached after a finite number of steps through the spectral sequence. Second, we may ask whether E_∞ is what we want it to be; thus, in the case of a filtered chain complex $\boldsymbol{C}$, we would want E_∞ to be related to $H(\boldsymbol{C})$ in a perfectly definite way. Now the first question may be decided by consulting the exact couple; the second question involves entities not represented in the exact couple. Thus it is necessary to enrich the algebraic system, and replace exact couples by *Rees systems* (Section 6), in order to be able to discuss both aspects of the convergence question. We emphasize that those topological situations referred to above – as well, of course, as the study of filtered differential objects – which lead naturally to the study of exact couples lead just as naturally to the study of Rees systems.

We admit that our discussion of convergence questions is more general than would be required by the applications we make of spectral sequences. However, this appears to us to be justified, first, by the expectation that the reader will wish to apply spectral sequences beyond the explicit scope of this book – and even, perhaps, develop the theory itself further – and, second, by the important concepts of (categorical) *limits* and *colimits* thereby thrown into prominence, together with their relation to general properties of adjoint functors (see II.7). However the reader only interested in the applications made in this book may omit Sections 6, 7, 8.

Although we use the language of abelian categories in stating our results, we encourage the reader, if he would thereby feel more comfortable, to think of categories of (graded, bigraded) modules. Indeed, many of our arguments are formulated in a manner appropriate to this concrete setting. Those readers who prefer entirely "categorical" proofs are referred to [10] for those they cannot supply themselves. The *embedding theorem* for abelian categories [37, p. 151] would actually permit us to think of the objects of our category as sets, thus possessing elements, insofar as arguments not involving limiting processes are concerned; however, many of the arguments are clearer when expressed in purely categorical language, expecially those involving categorical duality.

We draw the reader's attention to a divergence of notation between this text and many others in respect of the indexing of terms in the spectral sequence associated with a filtered chain complex. Details of this notation are given at the end of Section 2, where we also offer a justification of the conventions we have adopted.

1. Exact Couples and Spectral Sequences

Let $\mathfrak{A}$ be an abelian category. A *differential object* in $\mathfrak{A}$ is a pair (A, d) consisting of an object A of $\mathfrak{A}$ and an endomorphism $d: A \to A$ such that $d^2 = 0$. We may construct a category $(\mathfrak{A}, d)$ of differential objects of $\mathfrak{A}$ in the obvious way; moreover, we may construct the homology object associated with (A, d), namely, $H(A, d) = \ker d/\operatorname{im} d$, so that H is an additive functor $H: (\mathfrak{A}, d) \to \mathfrak{A}$. We may abbreviate (A, d) to A and simply write $H(A)$ for the homology object. We will also talk of the *cycles* and *boundaries* of A, writing $Z(A)$, $B(A)$ for the appropriate objects of $\mathfrak{A}$. Thus $H(A) = Z(A)/B(A)$.

Definition. A *spectral sequence* in $\mathfrak{A}$ is a sequence of differential objects of $\mathfrak{A}$,

$$\boldsymbol{E} = \{(E_n, d_n)\}, \qquad n = 0, 1, 2, \dots,$$

such that $H(E_n, d_n) = E_{n+1}$, $n = 0, 1, \dots$. A *morphism* $\varphi : \boldsymbol{E} \to \boldsymbol{E}'$ of spectral sequences is a sequence of morphisms $\varphi_n : E_n \to E'_n$ of $(\mathfrak{A}, d)$ such that $H(\varphi_n) = \varphi_{n+1}$, $n = 0, 1, \dots$. We write $\mathfrak{E}$, or $\mathfrak{E}(\mathfrak{A})$, for the category of spectral sequences in $\mathfrak{A}$.

Instead of showing directly how spectral sequences arise in homological algebra, we will introduce the category of *exact couples* in $\mathfrak{A}$ and a functor from this category to $\mathfrak{E}$; we will then show how exact couples arise.

Definition. An *exact couple* $\boldsymbol{EC} = \{D, E, \alpha, \beta, \gamma\}$ in $\mathfrak{A}$ is an exact triangle of morphisms in $\mathfrak{A}$,

$$\begin{tikzcd} D \arrow[rr, "\alpha"] & & D \arrow[dl, "\beta"] \\ & E \arrow[ul, "\gamma"] & \end{tikzcd} \tag{1.1}$$

A *morphism* Φ from $\boldsymbol{EC}$ to $\boldsymbol{EC}' = \{D', E', \alpha', \beta', \gamma'\}$ is a pair of morphisms $\kappa : D \to D'$, $\lambda : E \to E'$, such that

$$\alpha' \kappa = \kappa \alpha, \;\; \beta' \kappa = \lambda \beta, \;\; \gamma' \lambda = \kappa \gamma .$$

We write $\mathfrak{E}\mathfrak{C}(\mathfrak{A})$, or briefly $\mathfrak{E}\mathfrak{C}$, for the category of exact couples in $\mathfrak{A}$. We now define the *spectral sequence functor* $SS : \mathfrak{E}\mathfrak{C} \to \mathfrak{E}$. In this section we will give a very direct description of this functor; in Section 4 we will adopt a more categorical viewpoint and exhibit quite explicitly the way in which the spectral sequence is contained in the entire *ladder* of an exact couple.

We proceed, then, to define the spectral sequence functor. Thus, given the exact couple (1.1), we define $d_0 : E \to E$ by $d_0 = \beta\gamma$. Since $\gamma\beta = 0$, it is plain that $d_0^2 = 0$, so that (E, d_0) is a differential object of $\mathfrak{A}$. We will show how to construct a spectral sequence (E_n, d_n) such that $(E_0, d_0) = (E, d_0)$. Set $D_1 = \alpha D$, $E_1 = H(E_0, d_0)$ and define morphisms $\alpha_1, \beta_1, \gamma_1$ as follows:

$$\left.\begin{aligned} &\alpha_1 : D_1 \to D_1 && \text{is induced by } \alpha , \\ &\beta_1 : D_1 \to E_1 && \text{is induced by } \beta\alpha^{-1} , \\ &\gamma_1 : E_1 \to D_1 && \text{is induced by } \gamma . \end{aligned}\right\} \tag{1.2}$$

These descriptions are adapted to a concrete abelian category (e.g., a category of graded modules). The reader who wishes to express the argument in a manner appropriate to any abelian category may either turn to Section 4 or may assiduously translate the arguments presented below.

Thus the meaning of α_1 presents no problem. As to β_1, we mean that we set

$$\beta_1(\alpha x) = [\beta x],$$

where $[z]$ refers to the homology class of the cycle z; and, as to γ_1, we mean that

$$\gamma_1[z] = \gamma(z).$$

To justify the description of β_1 we must first show that βx is a cycle; but $(\beta\gamma)\beta = 0$. We must then show that $[\beta x]$ depends only on αx or, equivalently, that βx is a boundary if $\alpha x = 0$. But if $\alpha x = 0$, then $x = \gamma y$, $y \in E$, so that $\beta x = \beta\gamma y = d_0 y$ and is a boundary. To justify the description of γ_1 we must first show that $\gamma(z)$ belongs to D_1. But $D_1 = \ker\beta$ and $\beta\gamma(z) = 0$ since z is a cycle. We must then show that $\gamma(z)$ depends only on $[z]$ or, equivalently, that $\gamma(z) = 0$ if z is a boundary. But if $z = \beta\gamma(y)$, then $\gamma(z) = \gamma\beta\gamma(y) = 0$. Thus the definitions (1.2) make sense and, plainly, β_1 and γ_1 and, of course, α_1 are homomorphisms.

Theorem 1.1. *The couple*

$$\begin{tikzcd} D_1 \arrow[rr, "\alpha_1"] & & D_1 \arrow[dl, "\beta_1"] \\ & E_1 \arrow[ul, "\gamma_1"] & \end{tikzcd} \tag{1.3}$$

is exact.

Proof. Exactness at top left D_1: $\alpha_1\gamma_1[z] = \alpha\gamma(z) = 0$. Conversely, if $x \in D_1 = \alpha D$ and $\alpha x = 0$, then $x \in \ker\beta$ and $x = \gamma y$, $y \in E$. Thus $d_0 y = \beta\gamma y = 0$, y is a cycle of E, and $x = \gamma_1[y]$.

Exactness at top right D_1: $\beta_1\alpha_1(x) = \beta_1(\alpha x) = [\beta x]$; but $x \in D_1 = \ker\beta$, so $\beta x = 0$, so $\beta_1\alpha_1 = 0$. Conversely, if $\beta_1(\alpha x) = 0$ then $\beta x = \beta\gamma y$, $y \in E$, so $x = \gamma y + x_0$, where $x_0 \in \ker\beta = D_1$. Thus $\alpha x = \alpha x_0 = \alpha_1(x_0)$.

Exactness at E_1: $\gamma_1\beta_1(\alpha x) = \gamma_1[\beta x] = \gamma\beta(x) = 0$. Conversely, if $\gamma_1[z] = 0$, then $z = \beta x$, so $[z] = \beta_1(\alpha x)$. $\square$

We call (1.3) the *derived couple* of (1.1). We draw attention to the relation

$$E_1 = \gamma^{-1}(\alpha D)/\beta(\alpha^{-1}(0)) \tag{1.4}$$

which follows immediately from the fact that $E_1 = H(E) = Z(E)/B(E)$.

By iterating the process of passing to the derived couple, we obtain a sequence of exact couples $\boldsymbol{EC}(=\boldsymbol{EC}_0)$, $\boldsymbol{EC}_1, \dots, \boldsymbol{EC}_n, \dots$, where

$$\boldsymbol{EC}_n = \{D_n, E_n; \alpha_n, \beta_n, \gamma_n\},$$

and thus a spectral sequence (E_n, d_n), $n = 0, 1, \dots$, where $d_n = \beta_n\gamma_n$. This defines the *spectral sequence functor*

$$SS : \mathfrak{EC} \to \mathfrak{E};$$

we leave to the reader the verification that SS is indeed a functor. An easy induction establishes the following theorem.

Theorem 1.2. *$E_n = \gamma^{-1}(\alpha^n D)/\beta(\alpha^{-n}(0))$, and $d_n : E_n \to E_n$ is induced by $\beta\alpha^{-n}\gamma$.* □

Indeed we have, for each n, the exact sequence

$$\alpha^n D \xrightarrow{\alpha_n} \alpha^n D \xrightarrow{\beta_n} E_n \xrightarrow{\gamma_n} \alpha^n D \xrightarrow{\alpha_n} \alpha^n D\,, \tag{1.5}$$

where α_n is induced by α, β_n is induced by $\beta\alpha^{-n}$, and γ_n is induced by γ. Thus we have a short exact sequence

$$0 \to \operatorname{coker}\alpha_n \xrightarrow{\bar{\beta}_n} E_n \xrightarrow{\bar{\gamma}_n} \ker\alpha_n \to 0 \tag{1.6}$$

where $\bar{\beta}_n$, $\bar{\gamma}_n$ are induced by β_n, γ_n, and

$$\operatorname{coker}\alpha_n = \alpha^n D/\alpha^{n+1} D \cong D/\alpha D \cup \alpha^{-n}(0)\,, \tag{1.7}$$

$$\ker\alpha_n = \alpha^n D \cap \alpha^{-1}(0)\,. \tag{1.8}$$

We close this introductory section by giving a description of the *limit term* of a spectral sequence. We will explain later in just what sense E_∞ is a limit (Section 5). We will also explain how, under reasonable conditions frequently encountered in practice, the limit is often achieved by a *finite* convergence process, so that E_∞ is actually to be found within the spectral sequence itself (Section 3). However it does seem advisable to present the construction of E_∞ at this stage in order to be able to explain the basic rationale for spectral sequences; in applications it is usually the case that we have considerable (even, perhaps, complete) knowledge of the early terms of a spectral sequence and E_∞ is closely related to an object which we wish to study; sometimes, conversely, we use our knowledge of E_∞ to shed light on the early terms of the spectral sequence. An important special case of the relation of E_0 to E_∞ is given in the next section. The relation between E_∞ and the spectral sequence itself will be discussed further in Sections 3 and 6.

Our description of E_∞ will again be predicated on the assumption that $\mathfrak{A}$ is a category of modules. Since, in general, E_∞ involves a limiting process over an infinite set, this is really a loss of generality, which will be made good in Section 5. However, the description of E_∞ in the concrete case is much easier to understand, and shows us clearly how E_∞ is a sort of glorified homology object for the entire spectral sequence.

Let us write $E_{n,n+1}$ for the subobject of E_n consisting of those elements of E_n which are cycles for d_n, thus

$$E_{n,n+1} = Z(E_n)\,.$$

(The notations of this paragraph will be justified in Section 4.) There is then an epimorphism $\sigma = \sigma_{n,n+1} : E_{n,n+1} \twoheadrightarrow E_{n+1}$, and we may consider the subobject $E_{n,n+2}$ of $E_{n,n+1}$ consisting of those elements x of $E_{n,n+1}$

such that $\sigma(x)$ is a cycle for d_{n+1},

$$E_{n,n+2} = \{x \in E_{n,n+1} \mid \sigma(x) \in E_{n+1,n+2} = Z(E_{n+1})\}.$$

If $x \in E_{n,n+2}$, then $\sigma^2(x) \in E_{n+2}$ and we define $E_{n,n+3}$ by

$$E_{n,n+3} = \{x \in E_{n,n+2} \mid \sigma^2(x) \in E_{n+2,n+3} = Z(E_{n+2})\}.$$

By an abuse of language, we say that $x \in E_{n,n+1}$ if it is a cycle for d_n; $x \in E_{n,n+2}$ if it is a cycle for d_n, d_{n+1}; $x \in E_{n,n+3}$ if it is a cycle for $d_n, d_{n+1}, d_{n+2}; \ldots$. We may thus construct the subobject $E_{n,\infty}$ of E_n consisting of those x in E_n which are cycles for every d_r, $r \geqq n$. Plainly $\sigma = \sigma_{n,n+1}$ maps $E_{n,\infty}$ onto $E_{n+1,\infty}$ so we get the sequence

$$\cdots \twoheadrightarrow E_{n,\infty} \overset{\sigma}{\twoheadrightarrow} E_{n+1,\infty} \overset{\sigma}{\twoheadrightarrow} E_{n+2,\infty} \twoheadrightarrow \cdots$$

and we define

$$E_\infty = \varinjlim_n (E_{n,\infty}, \sigma). \tag{1.9}$$

Explicitly, an element of E_∞ is represented by an element x of some $E_{n,\infty}$; and x represents 0 if and only if it is a boundary for some d_r $(r \geqq n)$. Thus we may say that $x \in E_n$, $x \neq 0$, *survives to infinity* if it is a cycle for *every* d_r, $r \geqq n$, and a boundary for *no* d_r, $r \geqq n$; and E_∞ consists, as a set, precisely of 0 and equivalence classes of elements which survive to infinity.

It is again plain that E_∞ depends functorially on the spectral sequence $\boldsymbol{E}$, yielding a functor

$$\lim : \mathfrak{E} \to \mathfrak{A}, \tag{1.10}$$

at least in the case when $\mathfrak{A}$ is a category of modules. For an arbitrary abelian category $\mathfrak{A}$ we need, of course, to rephrase our description of the functor lim and to put some conditions on the category $\mathfrak{A}$ guaranteeing the existence of this functor. Such considerations will not, however, be our immediate concern, since we will first be interested in the case of finite convergence. This is also the case of primary concern for us in view of the applications we wish to make.

Exercises:

1.1. Prove Theorem 1.2.

1.2. Establish (1.5).

1.3. Show that E_n is determined, up to module extension, by $\alpha : D \to D$.

1.4. Show that $SS : \mathfrak{E}\mathfrak{C} \to \mathfrak{E}$ is an additive functor of additive categories. Are the categories $\mathfrak{E}\mathfrak{C}$, $\mathfrak{E}$ abelian?

1.5. Let C be a free abelian chain complex and let $\mathbb{Z} \overset{\mu}{\rightarrowtail} \mathbb{Z} \twoheadrightarrow \mathbb{Z}_m$ be the obvious exact sequence. Obtain an exact couple

$$\begin{tikzcd} H(C) \arrow[rr] & & H(C) \arrow[dl] \\ & H(C\otimes\mathbb{Z}_m) \arrow[ul] & \end{tikzcd}$$

and interpret the differentials of the resulting spectral sequence.

1.6. Carry out a similar exercise with $\mathrm{Hom}(C, \mathbb{Z}_m)$ replacing $C\otimes\mathbb{Z}_m$.

2. Filtered Differential Objects

In this section we describe one of the commonest sources of exact couples. We consider an object C of the category $(\mathfrak{A}, d)$ and suppose it is *filtered* by subobjects (of $(\mathfrak{A}, d)$)

$$\cdots \subseteqq C^{(p-1)} \subseteqq C^{(p)} \subseteqq \cdots \subseteqq C, \qquad -\infty < p < \infty\,. \tag{2.1}$$

Thus each $C^{(p)}$ is closed under the differential d on C, that is, $dC^{(p)} \subseteqq C^{(p)}$.

Denote the category of filtered differential objects in $\mathfrak{A}$ by $(\mathfrak{A}, d, f)$. Clearly morphisms in $(\mathfrak{A}, d, f)$ respect filtration and commute with differentials.

If we consider the short exact sequence

$$0 \rightarrow C^{(p-1)} \rightarrow C^{(p)} \rightarrow C^{(p)}/C^{(p-1)} \rightarrow 0\,,$$

we obtain a homology exact triangle

$$\begin{tikzcd} H(C^{(p-1)}) \arrow[rr, "\alpha"] & & H(C^{(p)}) \arrow[dl, "\beta"] \\ & H(C^{(p)}/C^{(p-1)})\,. \arrow[ul, "\gamma"] & \end{tikzcd} \tag{2.2}$$

Now let D be the *graded* object such that $D^p = H(C^{(p)})$ and let E be the graded object such that $E^p = H(C^{(p)}/C^{(p-1)})$. Then we may subsume (2.2), for all p, in the exact couple

$$\begin{tikzcd} D \arrow[rr, "\alpha"] & & D \arrow[dl, "\beta"] \\ & E \arrow[ul, "\gamma"] & \end{tikzcd} \tag{2.3}$$

in the graded category $\mathfrak{A}^{\mathbb{Z}}$, where $\deg\alpha = 1$, $\deg\beta = 0$, $\deg\gamma = -1$. This process describes a functor

$$\bar{H} : (\mathfrak{A}, d, f) \rightarrow \mathfrak{E}\mathfrak{C}(\mathfrak{A}^{\mathbb{Z}}) \tag{2.4}$$

from the category of filtered differential objects of $\mathfrak{A}$ to the category of exact couples of $\mathfrak{A}^{\mathbb{Z}}$. Notice that if we simply extract from the exact couple the E-term we have a functor $E:(\mathfrak{A},d,f)\to\mathfrak{A}^{\mathbb{Z}}$. This functor may be factorized in the following important way.

Given any abelian category $\mathfrak{B}$, we may form the category $(\mathfrak{B},f)$ of *filtered objects* of $\mathfrak{B}$,

$$\cdots \subseteqq B^{(p-1)} \subseteqq B^{(p)} \subseteqq \cdots \subseteqq B, \quad -\infty < p < \infty\,. \tag{2.5}$$

A morphism $\varphi: B\to B'$ of filtered objects then sends $B^{(p)}$ to $B'^{(p)}$ for all p. From (2.5) we construct the graded object whose p^{th} component is $B^{(p)}/B^{(p-1)}$. Then we plainly have a functor

$$Gr:(\mathfrak{B},f)\to\mathfrak{B}^{\mathbb{Z}}\,.$$

which is said to attach to a filtered object of $\mathfrak{B}$ the *associated graded object in* $\mathfrak{B}^{\mathbb{Z}}$. Now if $\mathfrak{B}=(\mathfrak{A},d)$, and if $X\in(\mathfrak{A},d,f)$, $Gr(X)\in(\mathfrak{A},d)^{\mathbb{Z}}=(\mathfrak{A}^{\mathbb{Z}},d)$ and so we may apply the homology functor H to $Gr(X)$ to get an object of $\mathfrak{A}^{\mathbb{Z}}$. Plainly

$$E=H\circ Gr:(\mathfrak{A},d,f)\to\mathfrak{A}^{\mathbb{Z}}\,.$$

On the other hand, starting from (2.1) we may pass to homology and obtain a filtration of $M=H(C)$ by

$$\cdots \subseteqq M^{(p-1)} \subseteqq M^{(p)} \subseteqq \cdots \subseteqq M\,, \tag{2.6}$$

where

$$M^{(p)}=H(C)^{(p)}=\operatorname{im}H(C^{(p)})\subseteqq H(C)\,. \tag{2.7}$$

By abuse of notation let us also write H for the functor associating (2.6) with (2.1). Thus, now,

$$H:(\mathfrak{A},d,f)\to(\mathfrak{A},f)\,.$$

so that we get a functor

$$Gr\circ H:(\mathfrak{A},d,f)\to\mathfrak{A}^{\mathbb{Z}}\,.$$

The functors H and Gr do not "commute"; indeed, in the cases we will be considering, passage through the spectral sequence will provide us with a measure of the failure of commutativity. Thus, as pointed out, $H\circ Gr$ yields $E=E_0$ from a filtered object in $(\mathfrak{A},d)$; and, by imposing certain reasonable conditions on the filtration, $Gr\circ H$ will yield E_∞ as we shall see; moreover these reasonable conditions will also ensure that E_∞ is reached after a finite number of steps through the spectral sequence, so that no sophisticated limiting process will be involved.

The assumption then is that we are interested in determining $H(C)$ and that we can, to a significant extent, determine $H(C^{(p)}/C^{(p-1)})$. *The spectral sequence is then designed to yield us information about the graded object associated with $H(C)$ filtered by its subobjects* im $H(C^{(p)})$. The question then arises as to how much information we can recover about $H(C)$ from the associated graded object. In this informal discussion let us again revert to the language of concrete categories. Then two conditions which we would obviously wish the filtration of $M = H(C)$ (2.6) to fulfil in order that the quotients $M^{(p)}/M^{(p-1)}$ adequately represent M are

$$\text{(i)} \quad \bigcap_p M^{(p)} = 0, \qquad \text{(ii)} \quad \bigcup_p M^{(p)} = M. \tag{2.8}$$

For if (i) fails there will be non-zero elements of M in every $M^{(p)}$ and thus lost in $Gr(M)$; and if (ii) fails there will be non-zero elements of M in no $M^{(p)}$ and thus unrepresented in $Gr(M)$. If both these conditions hold, then, for every $x \in M$, $x \neq 0$, there exists precisely one integer p such that $x \notin M^{(r)}$, $r < p$, $x \in M^{(r)}$, $r \geqq p$. Thus every $x \in M$, $x \neq 0$, is represented by a unique homogeneous non-zero element in $Gr(M)$ and, of course, conversely, every such homogeneous non-zero element represents a non-zero element of M. Thus all that is lost in the passage from M to $Gr(M)$ is information about the module-extensions involved; we do not know just how $M^{(p-1)}$ is embedded in $M^{(p)}$ from our knowledge of $Gr(M)$.

Evidently then we will be concerned also to impose conditions under which (2.8) (i) and (ii) hold. We thus add these requirements to our earlier criterion, for a "good" spectral sequence, that $Gr \circ H(C) = E_\infty$. Such requirements are often fulfilled in the case when $\mathfrak{A}$ is itself a graded category so that (2.1) is a *filtered chain complex* of $\mathfrak{A}$; we then suppose, of course, that the differential d lowers degree by 1. Then the associated exact couple

$$\begin{tikzcd} D \arrow[rr, "\alpha"] & & D \arrow[dl, "\beta"] \\ & E \arrow[ul, "\gamma"] & \end{tikzcd} \tag{2.9}$$

where

$$\begin{aligned} D &= \{D^{p,q}\}, \ D^{p,q} = H_q(C^{(p)}), \\ E &= \{E^{p,q}\}, \ E^{p,q} = H_q(C^{(p)}/C^{(p-1)}), \end{aligned}$$

is an exact couple in $\mathfrak{A}^{\mathbb{Z} \times \mathbb{Z}}$, and the bidegrees of α, β, γ are given by

$$\deg\alpha = (1,0), \ \deg\beta = (0,0), \ \deg\gamma = (-1,-1). \tag{2.10}$$

It then follows from the remark following Theorem 1.2 that, in the n^{th} derived couple and the associated spectral sequence, we have the bidegrees

$$\begin{aligned} &\deg\alpha_n = (1,0), \ \deg\beta_n = (-n,0), \\ &\deg\gamma_n = (-1,-1), \ \deg d_n = (-n-1,-1). \end{aligned} \tag{2.11}$$

In the next section we will use (2.11) to obtain conditions under which $E_\infty \cong Gr \circ H(C)$, and (2.8) holds for $M = H(C)$. Of course, a similar story is available in cohomology; the reader will readily amend (2.10), (2.11) to refer to the case of a filtered cochain complex.

Remark on Notational Conventions. We have indexed spectral sequences to begin with E_0; and have accordingly identified the E-term of the exact couple (2.9) with E_0. Conventions adopted in several other texts effectively enumerate the terms of the spectral sequence starting with E_1. Thus, if we write E_n for this rival convention, it is related to our convention by

$$E_n = \mathsf{E}_{n+1}\,.$$

This difference of convention should be particularly borne in mind when the reader meets, elsewhere, a reference to the E_2-term. For it often happens that this term (i.e., our E_1-term) has a special significance in the context of a given spectral sequence; see, for example, Theorem 9.3. A justification for our convention consists in the vital statement in Theorem 1.2 that d_n is induced by $\beta\alpha^{-n}\gamma$; it is surely convenient that the index n is precisely the power of α^{-1}.

A further matter of notational convention arises in indexing the terms of (2.9). Some texts put emphasis on what is called the *complementary degree*, so that what we have called $D^{p,q}$, $E^{p,q}$ would appear as $\mathsf{D}^{p,q-p}$, $\mathsf{E}^{p,q-p}$. Where this convention is to be found, in addition to that already referred to, the relation to our convention is given by

$$E_n^{p,q} = \mathsf{E}_{n+1}^{p,q-p}\,;$$

of course, translation from one convention to another is quite automatic. We defend our convention here with the claim that we set in evidence the degree, q, of the object being filtered in the case of a spectral sequence arising from a filtered chain complex; it is then obvious that any differential lowers the q-degree by 1. Some authors call the q-degree the *total degree.*

Exercises:

2.1. Show that the category $(\mathfrak{A}, d)$ is abelian.

2.2. Assign degrees corresponding to (2.11) for a filtered cochain complex. What should we understand by a *cofiltration*? Assign degrees to the exact couple associated with a cofiltered cochain complex.

2.3. Assign degrees in the exact couples of Exercises 1.5, 1.6.

2.4. Interpret E_∞ for the exact couples of Exercises 1.5, 1.6 when m is a prime and $\boldsymbol{C}$ is of finite type, i.e., each C_n is finitely-generated.

2.5. Show that, in the spectral sequence associated with (2.1),

$$E_r^p = \operatorname{im} H(C^{(p-r)}/C^{(p-r-1)}) \subseteq H(C^{(p)}/C^{(p-1)})\,.$$

2.6. Let $\varphi, \psi : C \to C'$ be morphisms of the category $(\mathfrak{A}, d, f)$ and suppose $\varphi \simeq \psi$ under a chain-homotopy Σ such that $\Sigma(C^{(p)}) \subseteq C'^{(p+k)}$, for fixed k and all p. Show that $\varphi_k \simeq \psi_k : E_k \to E'_k$.

3. Finite Convergence Conditions for Filtered Chain Complexes

In this section we will give conditions on the filtered chain complex (2.1) which simultaneously ensure that $Gr \circ H(C) \cong E_\infty$, that (2.8) (i) and (ii) hold, and that E_∞ is reached after only a finite number of steps through the spectral sequence (in a "local" sense to be explained in Theorem 3.1). At the same time, of course, the conditions must be such as to be fulfilled in most applications. A deeper study of convergence questions will be made in Section 7.

Insofar as mere finite convergence of the spectral sequence is concerned we can proceed from the bigraded exact couple (2.9). However, if we also wish to infer that the E_∞ term is indeed the graded object associated with $H(C)$, suitably filtered, and that conditions (2.8) (i) and (ii) for the filtration of $M = H(C)$ hold, then we will obviously have to proceed from the filtration (2.1) of C. First then we consider finite convergence of the spectral sequence.

Definition. We say that $\alpha: D \to D$ in (2.9) is *positively stationary* if, given q, there exists p_0 such that $\alpha: D^{p,q} \xrightarrow{\sim} D^{p+1,q}$ for $p \geqq p_0$. Similarly we define *negative stationarity*. If α is both positively and negatively stationary, it is *stationary*.

Theorem 3.1. *If α is stationary, the spectral sequence associated with the exact couple* (2.9) *converges finitely; that is, given* (p, q), *there exists r such that* $E_r^{p,q} = E_{r+1}^{p,q} = \cdots = E_\infty^{p,q}$.

Proof. Consider the exact sequence

$$D^{p-1,q} \xrightarrow{\alpha} D^{p,q} \xrightarrow{\beta} E^{p,q} \xrightarrow{\gamma} D^{p-1,q-1} \xrightarrow{\alpha} D^{p,q-1}. \tag{3.1}$$

Fix q. Since α is positively stationary, it follows that each α in (3.1) is an isomorphism for p sufficiently large. Thus $E^{p,q} = 0$ for p sufficiently large. Similarly $E^{p,q} = 0$ for p sufficiently small*. Now fix p, q and consider

$$E_r^{p+r+1,q+1} \xrightarrow{d_r} E_r^{p,q} \xrightarrow{d_r} E_r^{p-r-1,q-1}. \tag{3.2}$$

By what we have proved it follows that, for r sufficiently large,

$$E^{p+r+1,q+1} = 0, \qquad E^{p-r-1,q-1} = 0$$

so that $E_n^{p+r+1,q+1} = 0$, $E_n^{p-r-1,q-1} = 0$ for all $n \geqq 0$. Thus, for r sufficiently large, $E_r^{p,q} = E_{r+1}^{p,q}$. With our interpretation of E_∞ it follows also, of course, that $E_r^{p,q} = E_{r+1}^{p,q} = \cdots = E_\infty^{p,q}$, since the whole of $E_r^{p,q}$ is a cycle for every d_s, $s \geqq r$, and only 0 is a boundary for some d_s, $s \geqq r$. □

We next consider conditions on (2.1) which will guarantee that $E_\infty^{p,q} = \operatorname{im} H_q(C^{(p)}) / \operatorname{im} H_q(C^{(p-1)})$, while also guaranteeing that α is stationary so that the spectral sequence converges finitely. We proceed by obtaining from (2.1) a second exact couple.

* "Small" means, of course, "large and negative"!

Let $\bar{D}$ be the bigraded object given by

$$\bar{D}^{p,q} = H_q(C/C^{(p-1)}). \tag{3.3}$$

(Our reason for adopting this convention is explained in Section 6; but we remark that it leads to a symmetry between (2.10) and (3.5) below.)

Then the exact sequence of chain-complexes

$$0 \to C^{(p)}/C^{(p-1)} \to C/C^{(p-1)} \to C/C^{(p)} \to 0$$

gives rise to an exact couple of bigraded objects

$$\begin{tikzcd} \bar{D} \arrow[rr, "\bar{\alpha}"] & & \bar{D} \arrow[dl, "\bar{\beta}"] \\ & E \arrow[ul, "\bar{\gamma}"] & \end{tikzcd} \tag{3.4}$$

where

$$\deg \bar{\alpha} = (1,0), \quad \deg \bar{\beta} = (-1,-1), \quad \deg \bar{\gamma} = (0,0). \tag{3.5}$$

We now make a definition which will be applied to D, E and $\bar{D}$.

Definition. The bigraded object A is said to be *positively graded* if, given q there exists p_0 such that $A^{p,q} = 0$ if $p < p_0$. Similarly we define a *negative grade.*

We have the trivial proposition

Proposition 3.2. *If D (or $\bar{D}$) is positively (negatively) graded, then α (or $\bar{\alpha}$) is negatively (positively) stationary.* □

Theorem 3.3. *The following conditions are equivalent:*

(i) *α is positively stationary,*

(ii) *E is negatively graded,*

(iii) *$\bar{\alpha}$ is positively stationary.*

Of course, we can interchange "positive" and "negative" in this theorem.

Proof. In the course of proving Theorem 3.1 we showed that (i)$\Rightarrow$(ii). Conversely, consider the exact sequence

$$E^{p,q+1} \xrightarrow{\gamma} D^{p-1,q} \xrightarrow{\alpha} D^{p,q} \xrightarrow{\beta} E^{p,q}. \tag{3.6}$$

If E is negatively graded, then, given q, $E^{p,q+1} = 0$, $E^{p,q} = 0$, for p sufficiently large. Thus α is an isomorphism for p sufficiently large, so that (ii)$\Rightarrow$(i).

The implications (ii)$\Leftrightarrow$(iii) are derived similarly from the exact couple (3.4). □

We now complete our preparations for proving the main theorem of this section. We will say that the filtration

$$\cdots \subseteq C^{(p-1)} \subseteq C^{(p)} \subseteq \cdots \subseteq C, \quad -\infty < p < \infty \tag{3.7}$$

of the chain complex C is *finite*, if, for each q, there exist p_0, p_1 with

$$\begin{aligned} &\text{(i)} \quad C_q^{(p)} = 0 \quad \text{for} \quad p \leqq p_0\,, \\ &\text{(ii)} \quad C_q^{(p)} = C_q \quad \text{for} \quad p \geqq p_1\,. \end{aligned} \tag{3.8}$$

We will say that (3.7) is *homologically finite*, if, for each q, there exist p_0, p_1 with

$$\begin{aligned} &\text{(i)} \quad H_q(C^{(p)}) = 0 \qquad \text{for} \quad p \leqq p_0\,, \\ &\text{(ii)} \quad H_q(C^{(p)}) = H_q(C) \quad \text{for} \quad p \geqq p_1\,. \end{aligned} \tag{3.9}$$

Proposition 3.4. *If the filtration of the chain complex C is finite, it is homologically finite.*

Proof. Plainly (3.8) (i) implies (3.9) (i). Also (3.8) (ii) implies that, given q,

$$C_q^{(p)} = C_q, \;\; C_{q+1}^{(p)} = C_{q+1}, \quad \text{for } p \text{ large}\,.$$

Thus $H_q(C^{(p)}) = H_q(C)$ for p large. □

Theorem 3.5. *If the filtration of the chain complex C is homologically finite, then*

(i) *the associated spectral sequence converges finitely;*
(ii) *the induced filtration of $H(C)$ is finite;*
(iii) $E_\infty \cong Gr \circ H(C)$; *precisely,*

$$E_\infty^{p,q} \cong (Gr \circ H_q(C))_p = \operatorname{im} H_q(C^{(p)}) / \operatorname{im} H_q(C^{(p-1)})\,.$$

Remark. In the case where the conclusions of Theorem 3.5 hold we say that the *spectral sequence converges finitely to the graded object associated with $H(C)$, suitably filtered.* We will abbreviate this by saying that the *spectral sequence converges finitely to $H(C)$*, or simply by the symbol

$$E_1^{p,q} \Rightarrow H_q(C)\,.$$

Proof. (i) Plainly (3.9) (i) asserts that D is positively graded; and (3.9) (ii) is equivalent to the statement that $\bar{D}$ is negatively graded, as is seen immediately by applying homology to the sequence

$$C^{(p)} \rightarrowtail C \twoheadrightarrow C/C^{(p)}\,.$$

By Proposition 3.2, α is negatively stationary and $\bar{\alpha}$ is positively stationary. By Theorem 3.3, α is positively stationary, hence α is stationary and we apply Theorem 3.1 to obtain (i).

(ii) This is trivial, but we note that (3.9) is a stronger statement than conclusion (ii).

(iii) Consider the following extract from the n^{th} derived couple of the exact couple (2.9) – see (2.11) for the bidegrees of the maps –

$$D_n^{p+n-1,q} \xrightarrow{\alpha_n} D_n^{p+n,q} \xrightarrow{\beta_n} E_n^{p,q} \xrightarrow{\gamma_n} D_n^{p-1,q-1}\,. \tag{3.10}$$

We fix p, q and suppose n large so that $E_n^{p,q} = E_\infty^{p,q}$ by (i). Now $D_n^{p+n,q} = \alpha^n D^{p,q} = \operatorname{im} H_q(C^{(p)}) \subseteqq H_q(C^{(p+n)})$. It follows from (3.9) (ii) that, for n large, $D_n^{p+n,q} = H_q(C)^{(p)} = \operatorname{im} H_q(C^{(p)}) \subseteqq H_q(C)$. Similarly, for n large, $D_n^{p+n-1,q} = H_q(C)^{(p-1)} = \operatorname{im} H_q(C^{(p-1)}) \subseteqq H_q(C)$; and α_n is then just the inclusion $H_q(C)^{(p-1)} \subseteqq H_q(C)^{(p)}$. Also

$$D_n^{p-1,q-1} = \alpha^n D^{p-n-1,q-1} = \operatorname{im} H_{q-1}(C^{(p-n-1)}) \subseteqq H_{q-1}(C^{(p-1)})$$

and, for n large, this is zero by (3.9) (i). Thus for n large, β_n induces

$$(Gr \circ H_q(C))_p \cong E_\infty^{p,q},$$

completing the proof of (iii) and of the theorem. $\square$

Again the reader is invited to formulate Theorem 3.5 for a filtered cochain complex. Notice that we may formally obtain a "translation" by the sign-reversing trick of replacing (p,q) by $(-p, -q)$. We will feel free in the sequel to quote Theorem 3.5 in its dual form, that is, for cochain complexes.

Exercises:

3.1. Show that the validity of Theorems 3.1, 3.3 depends only on $\deg\alpha$ and $\deg\beta\gamma$ $(=\deg d)$, and not on the individual degrees of β and γ.

3.2. Adapt Theorems 3.1, 3.3, 3.5 to the case of cochain complexes.

3.3. Show that the spectral sequences of the exact couples (2.9), (3.4) coincide.

3.4. *(Comparison Theorem).* Let $\varphi: C \to C'$ be a morphism of homologically finite filtered chain-complexes. Show that if $\varphi_*: E_r \xrightarrow{\sim} E'_r$ for any r then

$$\varphi_*: H(C) \xrightarrow{\sim} H(C').$$

3.5. Let C be a filtered chain complex of abelian groups, in which

$$C^{(p)} = 0, \quad p < 0$$

$$C_q^{(p)} = C_q, \quad p \geqq q.$$

Show that if C satisfies the following conditions (i), (ii), (iii), then $E_1^{p,q}$ is finitely generated for all p, q.

(i) $H_q(C)$ is finitely generated for all q.

(ii) $E_1^{0,0}$ is finitely generated.

(iii) For all p, if $E_1^{p,p}$ is finitely generated, then $E_1^{p,q}$ is finitely generated for all q.

Also show that if C satisfies the conditions (i′), (ii′), (iii′), then $E_1^{p,q}$ is finitely generated for all p, q.

(i′) $H_q(C)$ is finitely generated for all q.

(ii′) $E_1^{0,0}$ is finitely generated.

(iii′) For all r, if $E_1^{0,r}$ is finitely generated, then $E_1^{p,p+r}$ is finitely generated for all p.

4. The Ladder of an Exact Couple

In this section we give a more categorical approach to the process of deriving an exact couple and hence, by iteration, obtaining the associated spectral sequence. Although this alternative viewpoint does, we believe, illuminate the arguments given in Section 1 and explains more precisely the nature of the E_∞ term, we present it here primarily in order to facilitate the discussion of spectral sequences and their convergence in the absence of the type of strong finiteness condition imposed in Section 3.

The basic idea of this section is the following. Suppose given a diagram in $\mathfrak{A}$,

$$\begin{tikzcd} A_1 & & C_1 \arrow[d, tail, "\varrho"] \\ A \arrow[u, two heads, "\sigma"] \arrow[r, "\beta"] & B \arrow[r, "\gamma"] & C \end{tikzcd} \tag{4.1}$$

in which $\gamma\beta$ factors through ϱ and through σ. We may then take the pull-back of (γ, ϱ) which, since ϱ is monic, simply amounts to taking $\gamma^{-1}(C_1)$. If the pull-back is

$$\begin{tikzcd} B_{0,1} \arrow[r, "\gamma_{0,1}"] \arrow[d, tail, "\varrho_{0,1}"] & C_1 \arrow[d, tail, "\varrho"] \\ B \arrow[r, "\gamma"'] & C \end{tikzcd}$$

then, since $\gamma\beta$ factors through ϱ, there exists a unique morphism $\beta_{0,1}: A \to B_{0,1}$ such that $\varrho_{0,1}\beta_{0,1} = \beta$. Thus we have the diagram

$$\begin{tikzcd} A_1 & & \\ A \arrow[u, two heads, "\sigma"] \arrow[r, "\beta_{0,1}"] \arrow[d, equal] & B_{0,1} \arrow[r, "\gamma_{0,1}"] \arrow[d, tail, "\varrho_{0,1}"] \arrow[dr, phantom, "\mathrm{PB}"] & C_1 \arrow[d, tail, "\varrho"] \\ A \arrow[r, "\beta"'] & B \arrow[r, "\gamma"'] & C \end{tikzcd} \qquad (\mathrm{PB} = \text{pull-back}) \tag{4.2}$$

and, plainly, $\gamma_{0,1}\beta_{0,1}$ factors through σ. For if κ is the kernel of σ, then

$\gamma\beta$ factors through $\sigma \Leftrightarrow \gamma\beta\kappa = 0 \Leftrightarrow \gamma_{0,1}\beta_{0,1}\kappa = 0 \Leftrightarrow \gamma_{0,1}\beta_{0,1}$ factors through σ.

We say that the sequence $(\beta_{0,1}, \gamma_{0,1})$ is obtained from the sequence (β, γ) by the *Q^ϱ-process*, and we write

$$Q^\varrho(\beta, \gamma) = (\beta_{0,1}, \gamma_{0,1}) \, . \tag{4.3}$$

Now we may apply the dual process to (4.2); that is, we take the push-out of $(\beta_{0,1}, \sigma)$ which, since σ is epic, amounts to constructing $\operatorname{coker} \beta_{0,1}\kappa$

where, as before, κ is the kernel of σ. If the push-out is

$$\begin{tikzcd}
A_1 \arrow[r, "\beta_1"] & B_1 \\
A \arrow[u, two heads, "\sigma"] \arrow[r, "\beta_{0,1}"] & B_{0,1} \arrow[u, two heads, "\sigma_{0,1}"']
\end{tikzcd}$$

then, since $\gamma_{0,1}\beta_{0,1}$ factors through σ, there exists a unique morphism $\gamma_1 : B_1 \to C_1$ such that $\gamma_1\sigma_{0,1} = \gamma_{0,1}$. Thus we have the diagram

$$\begin{tikzcd}
A_1 \arrow[r, "\beta_1"] & B_1 \arrow[r, "\gamma_1"] & C_1 \\
A \arrow[u, two heads, "\sigma"] \arrow[r, "\beta_{0,1}"] \arrow[ur, phantom, "\mathrm{PO}"] \arrow[d, equal] & B_{0,1} \arrow[u, two heads, "\sigma_{0,1}"'] \arrow[r, "\gamma_{0,1}"] \arrow[d, tail, "\varrho_{0,1}"'] \arrow[dr, phantom, "\mathrm{PB}"] & C_1 \arrow[u, equal] \arrow[d, tail, "\varrho"] \\
A \arrow[r, "\beta"'] & B \arrow[r, "\gamma"'] & C
\end{tikzcd} \tag{4.4}$$

We say that (β_1, γ_1) is obtained from $(\beta_{0,1}, \gamma_{0,1})$ by the *Q_σ-process*, and we write

$$Q_\sigma(\beta_{0,1}, \gamma_{0,1}) = (\beta_1, \gamma_1) \tag{4.5}$$

so that

$$(\beta_1, \gamma_1) = Q_\sigma Q^\varrho(\beta, \gamma). \tag{4.6}$$

On the other hand we may plainly reverse the order in which we apply the two processes. We then obtain the diagram

$$\begin{tikzcd}
A_1 \arrow[r, "\bar\beta_1"] \arrow[d, equal] & \bar B_1 \arrow[r, "\bar\gamma_1"] \arrow[d, tail, "\varrho_{1,0}"] \arrow[dr, phantom, "\mathrm{PB}"] & C_1 \arrow[d, tail, "\varrho"] & (\mathrm{PB} = \text{pull-back}) \\
A_1 \arrow[r, "\beta_{1,0}"] & B_{1,0} \arrow[r, "\gamma_{1,0}"] & C & \\
A \arrow[u, two heads, "\sigma"] \arrow[r, "\beta"] \arrow[ur, phantom, "\mathrm{PO}"] & B \arrow[u, two heads, "\sigma_{1,0}"'] \arrow[r, "\gamma"] & C \arrow[u, equal] & (\mathrm{PO} = \text{push-out})
\end{tikzcd} \tag{4.7}$$

and

$$(\beta_{1,0}, \gamma_{1,0}) = Q_\sigma(\beta, \gamma), \quad (\bar\beta_1, \bar\gamma_1) = Q^\varrho(\beta_{1,0}, \gamma_{1,0}) = Q^\varrho Q_\sigma(\beta, \gamma). \tag{4.8}$$

Theorem 4.1. $Q^\varrho Q_\sigma = Q_\sigma Q^\varrho$.

Proof. It is clear, in fact, that B_1 is obtained from B by first cutting down to the subobject $\gamma^{-1}(C_1)$ and then factoring out $\beta\sigma^{-1}(0)$, whereas $\bar B_1$ is obtained by the opposite process, that is, first factoring out $\beta\sigma^{-1}(0)$ and then cutting down to the subobject corresponding to $\gamma^{-1}(C_1)$. Thus

$$B_1 = \bar B_1 = \gamma^{-1}(C_1)/\beta\sigma^{-1}(0). \tag{4.9}$$

Moreover, $\bar\beta_1$ and β_1 are induced on $A_1 = A/\sigma^{-1}(0)$ by β, and $\bar\gamma_1, \gamma_1$ are induced by γ. $\square$

The reader requiring a more category-theoretical argument will find it in [10], to which he should refer for a detailed careful approach to the arguments of this and subsequent sections. (Actually the categorical argument appears in the proof of Theorem III. 1.4 in the case when $A \to B \to C$ is short exact.)

We may now eliminate the bars over β_1, γ_1, B_1 in (4.7). This enables us to enunciate the next proposition.

Proposition 4.2. *The square*

$$\begin{tikzcd} B_{0,1} \arrow[r, two heads, "\sigma_{0,1}"] \arrow[d, tail, "\varrho_{0,1}"] & B_1 \arrow[d, tail, "\varrho_{1,0}"] \\ B \arrow[r, two heads, "\sigma_{1,0}"] & B_{1,0} \end{tikzcd}$$

is bicartesian (i.e., a pull-back and push-out).

Proof. The square may be written

$$\begin{tikzcd} \gamma^{-1}(C_1) \arrow[r, two heads] \arrow[d, tail] & \gamma^{-1}(C_1)/\beta\sigma^{-1}(0) \arrow[d, tail] \\ B \arrow[r, two heads] & B/\beta\sigma^{-1}(0) \end{tikzcd}$$

with the obvious morphisms, and this is plainly bicartesian. $\square$

Proposition 4.3. *Consider the diagram*

$$\begin{tikzcd} & & C_2 \arrow[d, tail, "\varrho_1"] \\ & & C_1 \arrow[d, tail, "\varrho"] \\ A \arrow[r, "\beta"] & B \arrow[r, "\gamma"] & C \end{tikzcd}$$

where $\gamma\beta$ factors through $\varrho\varrho_1$. Then, if $(\beta', \gamma') = Q^{\varrho}(\beta, \gamma)$, $\gamma'\beta'$ factors through ϱ_1 and

$$Q^{\varrho\varrho_1} = Q^{\varrho_1} Q^{\varrho}. \tag{4.10}$$

Proof. Let $\gamma\beta = \varrho\varrho_1\delta$. Then $\varrho\gamma'\beta' = \varrho\varrho_1\delta$ so that $\gamma'\beta' = \varrho_1\delta$. Then (4.10) follows either by observing that the juxtaposition of two pull-back squares is again a pull-back, or that restricting to $\gamma^{-1}(C_2)$ is equivalent to first restricting to $\gamma^{-1}(C_1)$ and then restricting to $\gamma^{-1}(C_2)$! $\square$

Proposition 4.3 has a dual which we enunciate simply as follows: if $\gamma\beta$ factors through σ_1, σ, $A \overset{\sigma}{\twoheadrightarrow} A_1 \overset{\sigma_1}{\twoheadrightarrow} A_2$, then

$$Q_{\sigma_1\sigma} = Q_{\sigma_1} Q_{\sigma}. \tag{4.11}$$

Since we will be principally concerned with the case when $\gamma\beta=0$, the factorization hypothesis present in the construction of Q^ϱ, Q_σ will not usually detain us. In the light of Theorem 4.1 we may write

$$Q_\sigma^\varrho = Q^\varrho Q_\sigma = Q_\sigma Q^\varrho \tag{4.12}$$

and then (4.10), (4.11) imply

$$Q_{\sigma_1}^{\varrho_1} Q_\sigma^\varrho = Q_{\sigma_1\sigma}^{\varrho\varrho_1}. \tag{4.13}$$

Getting closer to the situation of our exact couple, we prove

Theorem 4.4. *If the bottom row of* (4.4) *is exact, so are all rows of* (4.4) *and* (4.7).

Proof. That the middle row of (4.4) is exact is plain, since

$$\gamma^{-1}(0) \subseteq \gamma^{-1}(C_1).$$

The rest of the statement of the theorem follows by duality. □

We now apply the processes Q^ϱ, Q_σ to the study of exact couples. Given $\alpha: D \to D$ we split α as an epimorphism σ followed by a monomorphism ϱ,

$$D \overset{\sigma}{\twoheadrightarrow} D_1 \overset{\varrho}{\rightarrowtail} D, \qquad \alpha = \varrho\sigma.$$

Inductively, we set $\varrho_0 = \varrho$, $\sigma_0 = \sigma$, and, having defined

$$D_{n-1} \overset{\sigma_{n-1}}{\twoheadrightarrow} D_n \overset{\varrho_{n-1}}{\rightarrowtail} D_{n-1}, \qquad \alpha_{n-1} = \varrho_{n-1}\sigma_{n-1},$$

we define $\alpha_n = \sigma_{n-1}\varrho_{n-1}: D_n \to D_n$ and split α_n as

$$D_n \overset{\sigma_n}{\twoheadrightarrow} D_{n+1} \overset{\varrho_n}{\rightarrowtail} D_n, \qquad \alpha_n = \varrho_n\sigma_n. \tag{4.14}$$

Of course, $D_n = \alpha^n D \cong D/\alpha^{-n}(0)$ and α_n is obtained by restricting α. We further set

$$\nu_n = \varrho\varrho_1 \dots \varrho_{n-1}: D_n \rightarrowtail D, \qquad \eta_n = \sigma_{n-1} \dots \sigma_1\sigma: D \twoheadrightarrow D_n.$$

Then $\alpha^n: D \to D$ splits as

$$D \overset{\eta_n}{\twoheadrightarrow} D_n \overset{\nu_n}{\rightarrowtail} D, \qquad \alpha^n = \nu_n\eta_n. \tag{4.15}$$

Remark. The description above is not quite adequate to the (bi)-graded case. We will explain the requisite modifications at the end of this section.

Consider now the exact couple (1.1) which we write as

$$D \xrightarrow{\alpha} D \xrightarrow{\beta} E \xrightarrow{\gamma} D \xrightarrow{\alpha} D. \tag{4.16}$$

Carrying out the $Q_{\eta_m}^{\nu_n}$-process we obtain

$$D_m \xrightarrow{\alpha_m} D_m \xrightarrow{\beta_{m,n}} E_{m,n} \xrightarrow{\gamma_{m,n}} D_n \xrightarrow{\alpha_n} D_n, \qquad (\beta_{m,n}, \gamma_{m,n}) = Q_{\eta_m}^{\nu_n}(\beta, \gamma). \tag{4.17}$$

Theorem 4.5. *The sequence* (4.17) *is exact*

Proof. We have already shown exactness at $E_{m,n}$ (Theorem 4.4). To show that $E_{m,n} \xrightarrow{\gamma_{m,n}} D_n \xrightarrow{\alpha_n} D_n$ is exact, first take $m=0$ and consider

$$\begin{tikzcd} E_{0,n} \arrow[r, "\gamma_{0,n}"] \arrow[d, tail] \arrow[dr, phantom, "\mathrm{PB}"] & D_n \arrow[r, "\alpha_n"] \arrow[d, tail, "\nu_n"] & D_n \arrow[d, tail, "\nu_n"] \\ E \arrow[r, "\gamma"] & D \arrow[r, "\alpha"] & D \end{tikzcd} \quad (\mathrm{PB} = \text{pull-back})$$

Then $E_{0,n} = \gamma^{-1}(D_n)$ so that $\gamma_{0,n} E_{0,n} = \gamma E \cap D_n = \alpha^{-1}(0) \cap D_n = \alpha_n^{-1}(0)$, since α_n is the restriction of α.

The general case now follows. For the diagram

$$\begin{tikzcd} D_m \arrow[r, "\beta_{m,n}"] & E_{m,n} \arrow[r, "\gamma_{m,n}"] & D_n \\ D \arrow[u, two heads, "\eta_m"] \arrow[r, "\beta_{0,n}"'] \arrow[ur, phantom, "\mathrm{PO}"] & E_{0,n} \arrow[u, two heads, "\eta_{m,n}"] \arrow[r, "\gamma_{0,n}"'] & D_n \arrow[u, equal] \end{tikzcd} \quad (\mathrm{PO} = \text{push-out})$$

shows that $\gamma_{m,n} E_{m,n} = \gamma_{0,n} E_{0,n}$; and the remaining exactness assertion of the theorem follows by duality. □

Note that

$$E_{m,n} = \gamma^{-1}(D_n)/\beta \eta_m^{-1}(0) = \gamma^{-1}(\alpha^n D)/\beta \alpha^{-m}(0). \tag{4.18}$$

In particular, $E_{n,n} = E_n$ and (4.17) in the case $m=n$ is just the n^{th} derived couple. Moreover, we may apply the Q^{ϱ_n}- and Q_{σ_m}-processes to (4.17). Then Proposition 4.2 implies

Theorem 4.6. *The square*

$$\begin{tikzcd} E_{m,n+1} \arrow[r, two heads, "\sigma_{m,n+1}"] \arrow[d, tail, "\varrho_{m,n}"] & E_{m+1,n+1} \arrow[d, tail, "\varrho_{m+1,n}"] \\ E_{m,n} \arrow[r, two heads, "\sigma_{m,n}"] & E_{m+1,n} \end{tikzcd}$$

is bicartesian. □

The notation of this theorem enables us to describe E_∞ as a double limit in a very precise way. For we will find that, in the notation of Section 1,

$$E_{m,\infty} = \varprojlim_n (E_{m,n}; \varrho_{m,n}) \tag{4.19}$$

(see Exercise II.8.8), and thus $E_\infty = \varinjlim_m E_{m,\infty}$, or

$$E_\infty = \varinjlim_m \varprojlim_n (E_{m,n}; \varrho_{m,n}, \sigma_{m,n}). \tag{4.20}$$

We will explain (4.19) and (4.20) more fully in the next section, where we will also see that

$$E_\infty = \varprojlim_n \varinjlim_m (E_{m,n}; \varrho_{m,n}, \sigma_{m,n}). \tag{4.21}$$

However, we will here break temporarily with our severely categorical formulations to give descriptions of $E_{m,\infty}$, E_∞ appropriate to a concrete category. We then observe that $E_{m,\infty} = \bigcap_n E_{m,n}$ and that $E_{m,\infty}$, E_∞ have descriptions analogous to $E_{m,n}$ in (4.18) namely,

$$E_{m,\infty} = \gamma^{-1}(\alpha^\infty D)/\beta\alpha^{-m}(0), \tag{4.22}$$

$$E_\infty = \gamma^{-1}(\alpha^\infty D)/\beta\alpha^{-\infty}(0), \tag{4.23}$$

where we define

$$\alpha^\infty D = \bigcap_n \alpha^n D, \qquad \alpha^{-\infty}(0) = \bigcup_m \alpha^{-m}(0). \tag{4.24}$$

These descriptions follow from the characterization of $E_{m,\infty}$, E_∞ in Section 1. Also we point out that if we define

$$E_{\infty,n} = \varinjlim_m (E_{m,n}; \sigma_{m,n}) = E_{m,n}/\bigcup_k \sigma^{-k}(0), \tag{4.25}$$

where

$$\sigma^k = \sigma_{m+k-1,n} \cdots \sigma_{m+1,n}\sigma_{m,n},$$

then $E_\infty = \varprojlim_n E_{\infty,n}$, and

$$E_{\infty,n} = \gamma^{-1}(\alpha^n D)/\beta\alpha^{-\infty}(0). \tag{4.26}$$

Of course, (4.22), (4.23), (4.26) may be formulated categorically; we need to note that

$$\alpha^\infty D = \varprojlim (D_n, \varrho_n), \qquad D/\alpha^{-\infty}(0) = \varinjlim (D_n, \sigma_n), \tag{4.27}$$

where the meaning of the limit ($\varinjlim$ or $\varprojlim$) in a general category will be explained in the next section.

The n^{th} rung of the ladder of an exact couple is, as we have said, just the n^{th} derived couple. We have seen that there is actually an $(m,n)^{\text{th}}$ *rung*, connected to the original couple by the $Q_{\eta_m}^{\nu_n}$-process; thus

$$\begin{tikzcd}
D_m \arrow[r, "\alpha_m"] & D_m \arrow[r, "\beta_{m,n}"] & E_{m,n} \arrow[r, "\gamma_{m,n}"] & D_n \arrow[r, "\alpha_n"] & D_n \\
D \arrow[u, "\eta_m"] \arrow[r, "\alpha"] & D \arrow[u, "\eta_m"] \arrow[r, "\beta"] & E \arrow[r, "\gamma"] & D \arrow[u, leftarrow, "\nu_n"] \arrow[r, "\alpha"] & D \arrow[u, leftarrow, "\nu_n"]
\end{tikzcd} \tag{4.28}$$

$$\beta_{n,n} = \beta_n, \quad \gamma_{n,n} = \gamma_n, \quad E_{n,n} = E_n.$$

We will see in the next section how to extend this to include $m=\infty$, or $n=\infty$, or both. Meanwhile we describe, as promised, the modifications necessary to cover the case of a (bi)graded couple.

Our point of view is that *the vertical morphisms of a ladder should always be degree-preserving*, so that the morphisms in any vertical family all carry the same degree. To achieve this we must complicate our procedure in obtaining the ladder, precisely in the factorization of α as $\varrho\sigma$. For we will want ϱ and σ to be degree-preserving and thus we factorize α as

$$\alpha=\varrho\omega\sigma,\quad D \overset{\sigma}{\twoheadrightarrow} D' \overset{\omega}{\underset{\sim}{\rightarrow}} D'' \overset{\varrho}{\rightarrowtail} D\,. \tag{4.29}$$

where ω is an isomorphism carrying the (bi)degree of α. More generally,

$$\alpha^n=\nu_n\omega_n\eta_n\,,$$

where $\omega_n: D_n' \xrightarrow{\sim} D_n''$ carries the degree of α^n.

Thus (4.28) is replaced by

$$\begin{tikzcd}
D_m' \arrow[r, "\alpha_m'"] & D_m' \arrow[r, "\beta_{m,n}'"] & E_{m,n} \arrow[r, "\gamma_{m,n}''"] & D_n'' \arrow[r, "\alpha_n''"] & D_n'' \arrow[d, tail, "\nu_n"] \\
D \arrow[u, two heads, "\eta_m"] \arrow[r, "\alpha"] & D \arrow[u, two heads, "\eta_m"] \arrow[r, "\beta"] & E \arrow[r, "\gamma"] & D \arrow[u, phantom] \arrow[r, "\alpha"] & D
\end{tikzcd} \tag{4.30}$$

and, of course,

$$\alpha_n''\omega_n=\omega_n\alpha_n'\,. \tag{4.31}$$

If we wish to obtain an exact couple from the n^{th} rung, we have to decide (arbitrarily, from the category-theoretical point of view) which of D_n', D_n'' is to be regarded as D_n. It is standard practice, in view of classical procedures, to choose $D_n=D_n''$. Then we set

$$\alpha_n=\alpha_n'',\quad \beta_n=\beta_n'\omega_n^{-1},\quad \gamma_n=\gamma_n''$$

and thus obtain the rules

$$\deg\alpha_n=\deg\alpha,\quad \deg\beta_n=\deg\beta-n\cdot\deg\alpha,\quad \deg\gamma_n=\deg\gamma\,, \tag{4.32}$$

$$\deg d_n=\deg\beta_n\gamma_n=\deg\beta+\deg\gamma-n\cdot\deg\alpha\,, \tag{4.33}$$

agreeing with (2.11). Of course, the degree of d_n is independent of whether we regard D_n' or D_n'' as D_n.

Exercises:

4.1. Prove directly, without appeal to duality, (i) (4.11), (ii) Theorem 4.4, (iii) the exactness of (4.17).

4.2. Give detailed proofs of (4.18), (4.22), (4.23).

4.3. Describe $E_{\infty,n}$ in a way analogous to the description of $E_{m,\infty}$ in Section 1.

4.4. Show that $E_{m,n}$ is determined by $\alpha: D \to D$ up to a module extension.

4.5. Use Theorem 4.6 to show that $E_{m,n}(m \leqq \infty, n \leqq \infty)$ is entirely determined by the spectral sequence.

4.6. Consider (4.1). Describe the Q^ϱ-process when γ factors through ϱ. Dualize.

5. Limits

In this section we formulate the theory of limits and colimits in so far as it is necessary to establish the crucial Theorem 5.3 below. Our general discussion is, of course, based on the material of Section II.8.

Let $\mathfrak{C}$ be an arbitrary category and I a small index category which we may assume here to be connected. We have the diagonal, or constant, functor $P: \mathfrak{C} \to \mathfrak{C}^I$ and we suppose that P has a right adjoint $R: \mathfrak{C}^I \to \mathfrak{C}$. Then, according to Proposition II.7.6 and Theorem II.8.3, we may suppose $RP = 1$, and the counit $\delta: PR \to 1$ satisfies $\delta P = 1$, $R\delta = 1$. Then, for any functor $F: I \to \mathfrak{C}$, the *limit* of F, $\varprojlim F$, is defined by

$$\varprojlim F = R(F), \tag{5.1}$$

and δ yields the morphisms $\delta_i: R(F) \to F_i$ completing the description of the limit.

We point out that the universal property of $\varprojlim F$ is as follows. Let morphisms $\varphi_i: X \to F(i)$, $i \in I$, be given such that, for all $\alpha: i \to j$ in I, the diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi_i} & F(i) \\ \| & & \downarrow{\scriptstyle F(\alpha)} \\ X & \xrightarrow[\varphi_j]{} & F(j) \end{array}$$

is commutative. Then there exists a unique morphism $\varphi: X \to \varprojlim F$ such that $\delta_i \varphi = \varphi_i: X \to F(i)$, $i \in I$. Indeed φ is given as $R(\tilde{\varphi})$ where $\tilde{\varphi}: PX \to F$ is the morphism of $\mathfrak{C}^I$ corresponding to the set of morphisms φ_i.

Similarly, a left adjoint L to P yields the *colimit*;

$$\varinjlim F = L(F), \quad F: I \to \mathfrak{C}, \tag{5.2}$$

and the unit $\varepsilon: 1 \to PL$ yields the morphisms $\varepsilon_i: F_i \to L(F)$ completing the description of the colimit.

According to Theorem II.8.6 any right adjoint functor preserves limits. Thus, in particular, *limits commute*. We proceed to make this assertion precise and explicit.

Consider a functor $F: I \times J \to \mathfrak{C}$, where I, J are two (connected) index categories. We may regard F as a functor $I \to \mathfrak{C}^J$ or as a functor

$J \to \mathfrak{C}^I$; in other words, there are canonical identifications

$$\mathfrak{C}^{I \times J} \cong (\mathfrak{C}^J)^I \cong (\mathfrak{C}^I)^J \,, \tag{5.3}$$

and we will henceforth make these identifications. Let $P: \mathfrak{C} \to \mathfrak{C}^{I \times J}$, $P_1: \mathfrak{C} \to \mathfrak{C}^I$, $P_2: \mathfrak{C} \to \mathfrak{C}^J$ be the diagonal functors and suppose that

$$P_i \dashv R_i, \qquad i = 1, 2\,. \tag{5.4}$$

There is a commutative diagram

$$\begin{tikzcd} \mathfrak{C} \arrow[r, "P_1"] \arrow[d, "P_2"'] \arrow[dr, "P"] & \mathfrak{C}^I \arrow[d, "P_2^I"] \\ \mathfrak{C}^J \arrow[r, "P_1^J"'] & \mathfrak{C}^{I \times J} \end{tikzcd} \tag{5.5}$$

with diagonal P.

Theorem 5.1. *There is a natural equivalence $R_2 R_1^J \cong R_1 R_2^I$. Setting either equal to R, we have $P \dashv R$, $RP = 1$ and the counit $\delta: PR \to 1$ is given by $\delta = \delta_1^J \circ P_1^J \delta_2 R_1^J$ if $R = R_2 R_1^J$, or $\delta = \delta_2^I \circ P_2^I \delta_1 R_2^I$ if $R = R_1 R_2^I$.*

Proof. The first assertion is a special case of Theorem II.8.6. The rest follows readily from Proposition II.7.1 and we leave the details to the reader. □

This theorem asserts then that limits commute; similarly, of course, colimits commute. However, it is not true in general that limits commute with colimits (see Exercise 5.4). Nevertheless, since the pull-back is a limit and the push-out is a colimit, Theorem 4.1 constitutes an example where this phenomenon does in fact occur.

We now consider the following situation. We suppose given the diagram

$$\begin{tikzcd} \cdots \arrow[r] & A_{n+1} \arrow[r, "\alpha_n"] \arrow[d, "\varphi_{n+1}"] & A_n \arrow[r, "\alpha_{n-1}"] \arrow[d, "\varphi_n"] & \cdots \arrow[r] & A_1 \arrow[r, "\alpha_0"] \arrow[d, "\varphi_1"] & A_0 \arrow[d, "\varphi_0"] \\ \cdots \arrow[r] & B_{n+1} \arrow[r, "\beta_n"] & B_n \arrow[r, "\beta_{n-1}"] & \cdots \arrow[r] & B_1 \arrow[r, "\beta_0"] & B_0 \end{tikzcd} \tag{5.6}$$

in $\mathfrak{C}$ and let $A_\infty = \varprojlim(A_n, \alpha_n)$, $B_\infty = \varprojlim(B_n, \beta_n)$. Then there is a limit diagram (where $\bar{\alpha}$, $\bar{\beta}$ are given by the counit (see (5.1)))

$$\begin{tikzcd} A_\infty \arrow[r, "\bar{\alpha}"] \arrow[d, "\varphi_\infty"] & A_0 \arrow[d, "\varphi_0"] \\ B_\infty \arrow[r, "\bar{\beta}"] & B_0 \end{tikzcd} \tag{5.7}$$

Theorem 5.2. *If each square in* (5.6) *is a pull-back, then* (5.7) *is a pull-back.*

This theorem can be regarded as a special case of Theorem 5.1. However, we prefer to give a direct proof.

Proof. We have to show that (5.7) is a pull-back diagram. Suppose then given $\psi: X \to A_0$ and $\chi: X \to B_\infty$, with $\varphi_0 \psi = \bar{\beta}\chi$. We then obtain morphisms $\psi_0 = \psi: X \to A_0$ and $\chi_1 = \delta_1 \chi$, where $\delta_1: B_\infty \to B_1$ is given by the counit (see (5.1)). Clearly $\varphi_0 \psi_0 = \beta_0 \chi_1$; hence, since each square in (5.6) is a pull-back, there exists a unique $\psi_1: X \to A_1$ satisfying the usual commutativity relations. Proceeding by induction we obtain a family of morphisms $\{\psi_i: X \to A_i\}$ with $\alpha_i \psi_i = \psi_{i-1}$ for $i \geqq 1$. Hence there exists a unique morphism $\psi_\infty: X \to A_\infty$ with $\delta_i \psi_\infty = \psi_i$, where $\delta_i: A_\infty \to A_i$ are given by the counit, and $\bar{\alpha}\psi_\infty = \psi$. Similarly $\{\varphi_i \psi_i: X \to B_i\}$ give rise to the map $\chi: X \to B_\infty$, so that, by the universal property we then have $\varphi_\infty \psi_\infty = \chi$. Hence it follows that ψ_∞ satisfies the required conditions. We leave it to the reader to prove the uniqueness of ψ_∞ satisfying these conditions. □

Notice that this result applies to an arbitrary category, provided only that the limits exist.

We use this theorem, and its dual, to prove the basic result on exact couples in an abelian category and the limit of the associated spectral sequence.

We recall from Section 4 the notations (see (4.14), (4.15))

$$\begin{aligned} &\sigma_n: D_n \twoheadrightarrow D_{n+1}, && \varrho_n: D_{n+1} \rightarrowtail D_n, \\ &\eta_n: D \twoheadrightarrow D_n, && \nu_n: D_n \rightarrowtail D. \end{aligned} \tag{5.8}$$

Then we set

$$I = \varprojlim (D_n, \varrho_n), \qquad U = \varinjlim (D_n, \sigma_n) \tag{5.9}$$

and let

$$\nu: I \rightarrowtail D, \qquad \eta: D \twoheadrightarrow U \tag{5.10}$$

be the canonical morphisms. We apply the Q_η^ν-process to the exact couple (1.1).

Theorem 5.3. *In the notation of Theorem 4.6 we have*

$$E_\infty = \varinjlim_m \varprojlim_n (E_{m,n}; \varrho_{m,n}, \sigma_{m,n}) = \varprojlim_n \varinjlim_m (E_{m,n}; \varrho_{n,n}, \sigma_{m,n}). \tag{5.11}$$

The Q_η^ν-process yields

$$\begin{tikzcd} U \arrow[r, "\alpha'"] & U \arrow[r, "\beta_\infty"] & E_\infty \arrow[r, "\gamma_\infty"] & I \arrow[r, "\alpha''"] \arrow[d, tail, "\nu"] & I \arrow[d, tail, "\nu"] \\ D \arrow[r, "\alpha"] \arrow[u, two heads, "\eta"] & D \arrow[r, "\beta"] \arrow[u, two heads, "\eta"] & E \arrow[r, "\gamma"] & D \arrow[r, "\alpha"] & D \end{tikzcd} \tag{5.12}$$

where α', α'' are induced by α and the top row is exact.

Notice that in the concrete setting of Section 4 we have (see 4.27)

$$I = \bigcap_n \alpha^n D = \alpha^\infty D, \qquad U = D/\bigcup_m \alpha^{-m}(0) = D/\alpha^{-\infty}(0). \tag{5.13}$$

Thus Theorem 5.3 effectively establishes all the facts given in Section 4, relating to E_∞, since we may, of course pass to the limit starting from any derived couple of the given exact couple.

Proof. Let us execute $Q_\eta^\vee$ as $Q_\eta Q^\vee$. We thus obtain

$$\begin{tikzcd} U \arrow[r, "\alpha'"] & U \arrow[r, "\beta_{\infty,\infty}"] & E_{\infty,\infty} \arrow[r, "\gamma_{\infty,\infty}"] & I \arrow[r, "\alpha''"] & I \\ D \arrow[u, two heads, "\eta"] \arrow[r, "\alpha"] \arrow[d, equal] & D \arrow[u, two heads, "\eta"] \arrow[r, "\beta_{0,\infty}"] \arrow[d, equal] \arrow[ur, phantom, "\mathrm{PO}"] & E_{0,\infty} \arrow[u, two heads, "\bar\eta"] \arrow[r, "\gamma_{0,\infty}"] \arrow[d, tail, "\bar\nu"] \arrow[dr, phantom, "\mathrm{PB}"] & I \arrow[u, equal] \arrow[r, "\alpha''"] \arrow[d, tail, "\nu"] & I \arrow[u, equal] \arrow[d, tail, "\nu"] \\ D \arrow[r, "\alpha"] & D \arrow[r, "\beta"] & E \arrow[r, "\gamma"] & D \arrow[r, "\alpha"] & D \end{tikzcd} \tag{5.14}$$

However, by Theorem 5.2, $E_{0,\infty} = \varprojlim E_{0,n} = \bigcap_n E_{0,n}$, and so is, in fact, the subobject of E designated as $E_{0,\infty}$ in Section 1. Of course, the identical argument would establish that if we pulled back from the m^{th} derived couple we would obtain $E_{m,\infty}$, as defined in (4.19), and that $E_{m,\infty}$ coincides with the description given in Section 1. We now apply Q_η. The dual of Theorem 5.2 now establishes that we obtain the top row with

$$E_{\infty,\infty} = \varinjlim_m E_{m,\infty} = \varinjlim_m \varprojlim_n E_{m,n},$$

provided only that we establish that

$$\begin{tikzcd} D_{n+1} \arrow[r, "\beta_{n+1,\infty}"] & E_{n+1,\infty} \\ D_n \arrow[u, two heads, "\sigma_n"] \arrow[r, "\beta_{n,\infty}"] & E_{n,\infty} \arrow[u, "\sigma_{n,\infty}"] \end{tikzcd} \tag{5.15}$$

is a push-out, for all n. It is plainly sufficient to show this for $n=0$, so we look at

$$\begin{tikzcd} D_1 \arrow[r, "\alpha_1"] & D_1 \arrow[r, "\beta_{1,\infty}"] & E_{1,\infty} \arrow[r, "\gamma_{1,\infty}"] & I \arrow[r, "\alpha''"] & I \\ D \arrow[u, two heads] \arrow[r, "\alpha"] & D \arrow[u, two heads, "\sigma"] \arrow[r, "\beta_{0,\infty}"] & E_{0,\infty} \arrow[u, "\sigma_{0,\infty}"] \arrow[r, "\gamma_{0,\infty}"] & I \arrow[u, equal] \arrow[r, "\alpha''"] & I \arrow[u, equal] \end{tikzcd} \tag{5.16}$$

Now the middle row of (5.14) is exact – the argument is exactly as for Theorem 4.5. Thus both the rows of (5.16) are exact and from this it readily follows that (5.15) (with $n=0$) is a push-out. (From this it also follows that $\sigma_{n,\infty}$ is an epimorphism, but this can be proved in many ways.) Since E_∞ was defined in Section 1 as $\varinjlim_m E_{m,\infty}$, we have established that $E_{\infty,\infty} = E_\infty$. Now since we could have executed $Q_\eta^\vee$ as $Q^\vee Q_\eta$ it follows immediately that

$$E_{\infty,\infty} = \varprojlim_n \varinjlim_m E_{m,n}$$

so that (5.11) is established.

The exactness of the top row of (5.14) follows exactly as in the proof of Theorem 4.5 and the determinations (4.22), (4.23), (4.26) of $E_{m,\infty}$, E_∞, $E_{\infty,n}$ respectively now follow from the appropriate exact sequences. $\square$

Remark. The reader should note that Theorem 5.3 as stated is valid in any abelian category in which the appropriate limits exist. There are no arguments essentially involving elements and diagram-chasing.

Of course, (4.22), (4.23) and (4.26) require modification in an arbitrary abelian category; the best description is then the statement of the appropriate exact sequence; thus

$$D_m \xrightarrow{\alpha_m} D_m \xrightarrow{\beta_{m,\infty}} E_{m,\infty} \xrightarrow{\gamma_{m,\infty}} I \xrightarrow{\alpha''} I\,, \tag{5.17}$$

$$U \xrightarrow{\alpha'} U \xrightarrow{\beta_\infty} E_\infty \xrightarrow{\gamma_\infty} I \xrightarrow{\alpha''} I\,, \tag{5.18}$$

$$U \xrightarrow{\alpha'} U \xrightarrow{\beta_{\infty,n}} E_{\infty,n} \xrightarrow{\gamma_{\infty,n}} D_n \xrightarrow{\alpha_n} D_n\,. \tag{5.19}$$

Of course the limit term could also be characterized by means of the Q_η^ν-process, but this would conceal the fact that it depends only on the spectral sequence and not on the exact couple.

The reader should also notice that the exact couple (1.1) ceases to be an exact couple "in the limit" but remains an exact sequence (5.18). We wish to stress that the exact couple disappears because we are carrying out both limiting and colimiting processes. It is thus a remarkable fact embedded in (5.11) that these two processes commute in our special case.

There would have to be some trivial modifications of detail in the case of a graded category as explained at the end of Section 4. It is unnecessary to enter into details.

Exercises:

5.1. Complete the details of the proof of Theorem 5.1.

5.2. Show that Theorem 5.2 is a special case of Theorem 5.1.

5.3. Show that the usual definition of the direct limit of a direct system of groups is a special case of the given definition of colimit.

5.4. Let $D(i,j)$ be a doubly-indexed family of non-zero abelian groups, $0 \leqq i < \infty$, $0 \leqq j < \infty$. Let $I^{\mathrm{opp}} = J$, where J is the ordered set of non-negative integers, and let $F: I \times J \to \mathfrak{Ab}$ be given by

$$F(i_0, j_0) = \bigoplus_{i \leqq i_0,\, j \leqq j_0} D(i,j)\,.$$

Complete the functor on $I \times J$ by the projections $F(i_0,j_0) \to F(i_1,j_0)$, if $i_1 \leqq i_0$, and the injections $F(i_0,j_0) \to F(i_0,j_1)$, if $j_0 \leqq j_1$. Show that

$$\varinjlim_J \varprojlim_I F \neq \varprojlim_I \varinjlim_J F\,.$$

5.5. Deduce that (5.15) is a push-out (for $n=0$) from the exactness of the rows of (5.16).

6. Rees Systems and Filtered Complexes

In Section 2 we studied filtered differential objects, and pointed out that, under certain conditions, the E_∞ term of the associated spectral sequence was obtained by applying the functor $Gr \circ H$ to the given filtered differential object. Explicit conditions in the case of a filtered chain complex were given in Section 3. Our main objective in this section is to examine the problem in complete generality, so as to be able to obtain necessary and sufficient conditions for

$$E_\infty \cong Gr \circ H(C), \quad C \in (\mathfrak{A}, d, f). \tag{6.1}$$

These conditions will then imply the relevant results of Section 3. Thus this section, and the next, can be omitted by the reader content with the situations covered by the finite convergence criteria of Section 3. Such a reader may also ignore Section 8, where we discuss, in greater generality than in Sections 2 and 3, the passage from $H(C)$ to $Gr \circ H(C)$.

We will generalize the framework of our theory in order to simplify the development. Given an abelian category $\mathfrak{A}$, consider triples (G, A, θ) consisting of a differential object G of $(\mathfrak{A}, d)$, a differential subobject A of G and an automorphism $\theta: G \xrightarrow{\sim} G$ such that $\theta A \subseteq A$. We thus obtain a category $\mathfrak{T}(\mathfrak{A}, d)$. If (see (2.1))

$$\cdots \subseteq C^{(p-1)} \subseteq C^{(p)} \subseteq \cdots \subseteq C, \qquad -\infty < p < \infty$$

is an object of $(\mathfrak{A}, d, f)$, we obtain a functor

$$F: (\mathfrak{A}, d, f) \to \mathfrak{T}(\mathfrak{A}^{\mathbb{Z}}, d) \tag{6.2}$$

by setting

$$G(C) = \bigoplus_{p \in \mathbb{Z}} C$$

$$A(C) = \bigoplus_{p \in \mathbb{Z}} C^{(p)}$$

(with the evident differential of p-degree 0), and defining $\theta: G(C) \to G(C)$ to be the morphism of degree $+1$ which is the identity on each component. Thus we will later use the functor F of (6.2) to apply our results on triples (G, A, θ) to filtered differential objects, by considering the triple $F(C)$, $C \in (\mathfrak{A}, d, f)$.

Given $(G, A, \theta) \in \mathfrak{T}(\mathfrak{A}, d)$, set $B = \theta A$. There are then exact sequences of differential objects (using the habitual notation of modules)

$$\begin{aligned} S_1 &: B \overset{i}{\rightarrowtail} A \overset{j}{\twoheadrightarrow} A/B \\ S_2 &: A/B \overset{\bar{i}}{\rightarrowtail} G/B \overset{\bar{j}}{\twoheadrightarrow} G/A \\ S_3 &: A \overset{i_A}{\rightarrowtail} G \overset{j_A}{\twoheadrightarrow} G/A \\ S_4 &: B \overset{i_B}{\rightarrowtail} G \overset{j_B}{\twoheadrightarrow} G/B \end{aligned} \tag{6.3}$$

If we write θ for the isomorphisms $G \cong G$, $A \cong B$, $G/A \cong G/B$, then these four sequences are connected by morphisms as follows

$$S_1 \xrightarrow{(1,\, i_A,\, \bar{i})} S_4 \underset{\theta}{\overset{(i,\, 1,\, \bar{j})}{\leftrightarrows}} S_3 \xrightarrow{(j,\, j_B,\, 1)} S_2 \tag{6.4}$$

where we use the symbol θ to represent any isomorphism induced by the isomorphism $\theta : (G, A) \xrightarrow{\sim} (G, B)$. Passing to homology, using the same symbols as in (6.3) for the induced homology morphisms and ω_i for the connecting homomorphism associated with the sequence S_i, $i = 1, 2, 3, 4$, we obtain the diagram

$$\begin{tikzcd}
 & H(B) \arrow[rr, "i"] \arrow[dl, "i_B"'] & & H(A) \arrow[dl, "j"'] \arrow[dr, "i_A"] & \\
H(G) \arrow[dr, "j_B"'] \arrow[r, phantom, "\textcircled{4}"] & {} & H(A/B) \arrow[ul, "\omega_1"'] \arrow[dl, "\bar{i}"'] \arrow[u, phantom, "\textcircled{1}"] \arrow[d, phantom, "\textcircled{2}"] & {} & H(G) \arrow[dl, "j_A"] \arrow[l, phantom, "\textcircled{3}"] \\
 & H(G/B) \arrow[rr, "\bar{j}"'] \arrow[uu, "\omega_4"'] & {} & H(G/A) \arrow[uu, "\omega_3"] \arrow[ul, "\omega_2"'] &
\end{tikzcd} \tag{6.5}$$

where the morphisms (6.4) imply the commutativity relations

$$\begin{aligned}
i_A i &= i_B, & \bar{j} j_B &= j_A, & \bar{i} j &= j_B i_A, \\
\omega_4 \bar{i} &= \omega_1, & j\omega_3 &= \omega_2, & i\omega_4 &= \omega_3 \bar{j}, \\
\theta i_A &= i_B \theta, & \theta j_A &= j_B \theta, & \theta\omega_3 &= \omega_4 \theta.
\end{aligned} \tag{6.6}$$

We use the isomorphism $\theta : S_3 \xrightarrow{\sim} S_4$ to bring triangle ④ into coincidence with triangle ③ in (6.5). That is, we write

$$D = H(A), \quad E = H(A/B), \quad \bar{D} = H(G/A), \quad \Gamma = H(G), \tag{6.7}$$

$$\begin{aligned}
\alpha &= i\theta, & \beta &= j, & \gamma &= \theta^{-1}\omega_1; \\
\bar{\alpha} &= \bar{j}\theta, & \bar{\beta} &= \omega_2, & \bar{\gamma} &= \theta^{-1}\bar{i}; \\
\xi &= \omega_3, & \varphi &= i_A, & \bar{\varphi} &= j_A,
\end{aligned} \tag{6.8}$$

and obtain the diagram

$$\begin{tikzcd}
 & D \arrow[rr, "\alpha"] \arrow[dl, "\varphi"'] & & D \arrow[dl, "\beta"'] \arrow[dr, "\varphi"] & \\
\Gamma \arrow[dr, "\bar{\varphi}"'] \arrow[r, phantom, "\textcircled{3}"] & {} & E \arrow[ul, "\gamma"'] \arrow[dl, "\bar{\gamma}"'] \arrow[u, phantom, "\textcircled{1}"] \arrow[d, phantom, "\textcircled{2}"] & {} & \Gamma \arrow[dl, "\bar{\varphi}"] \arrow[l, phantom, "\textcircled{3}"] \\
 & \bar{D} \arrow[rr, "\bar{\alpha}"'] \arrow[uu, "\xi"'] & {} & \bar{D} \arrow[uu, "\xi"] \arrow[ul, "\bar{\beta}"'] &
\end{tikzcd} \tag{6.9}$$

in which ① and ② are exact couples, ③ is an exact triangle, and

$$\alpha\xi = \xi\bar{\alpha}, \qquad \beta\xi = \bar{\beta}, \qquad \gamma = \xi\bar{\gamma}. \tag{6.10}$$

Moreover, there is an automorphism $\theta : \Gamma \xrightarrow{\sim} \Gamma$ such that

$$\theta\varphi = \varphi\alpha, \quad \bar{\varphi}\theta = \bar{\alpha}\bar{\varphi}, \quad \bar{\varphi}\theta^{-1}\varphi = \bar{\gamma}\beta. \tag{6.11}$$

We call (6.9), where the morphisms satisfy (6.10), a *Rees system* in $\mathfrak{A}$. If there is given an automorphism θ satisfying (6.11) we say that the Rees system is *special*. We thus get (with the evident definition of the morphisms) two categories $\mathfrak{R}(\mathfrak{A}, d)$, $\mathfrak{S}(\mathfrak{A}, d)$ and an underlying functor $U : \mathfrak{S} \to \mathfrak{R}$. In fact, in this book, all the Rees systems we meet will be special. However, we prefer to retain the notion of a Rees system, since not all our arguments require the existence of θ. We thus have described a functor $R : \mathfrak{T}(\mathfrak{A}, d) \to \mathfrak{S}(\mathfrak{A}, d)$. Notice that every exact couple, $\boldsymbol{EC}$, may be regarded as a Rees system by setting

$$\begin{aligned} ① &= \boldsymbol{EC}, \\ ② &= \boldsymbol{EC}, \\ \Gamma &= 0, \\ \xi &= 1, \end{aligned}$$

and this Rees system is trivially special. Thus we have a full embedding $E : \mathfrak{EC}(\mathfrak{A}) \to \mathfrak{S}(\mathfrak{A}, d)$. Notice also that, for the triple $F(C)$, where C is a filtered differential object, the exact couple (2.9) coincides with ① in (6.9). Thus, by extracting the exact couple ① from the Rees system (6.9) we get a functor $\bar{E} : \mathfrak{S}(\mathfrak{A}, d) \to \mathfrak{EC}(\mathfrak{A})$; and we have

$$\bar{E}E = 1,$$

and the following elementary proposition.

Proposition 6.1. *The diagram*

$$\begin{tikzcd} (\mathfrak{A}, d, f) \arrow[r, "\bar{H}"] \arrow[d, "F"] & \mathfrak{EC}(\mathfrak{A}^{\mathbb{Z}}) \arrow[d, shift left, "E"] \\ \mathfrak{T}(\mathfrak{A}^{\mathbb{Z}}, d) \arrow[r, "R"] & \mathfrak{S}(\mathfrak{A}^{\mathbb{Z}}, d) \arrow[u, shift left, "\bar{E}"] \end{tikzcd} \tag{6.12}$$

commutes. □

Theorem 6.2. *In the Rees system* (6.9) *the spectral sequences of the couples* ① *and* ② *coincide.*

Proof. The relations (6.10) assert that we have a morphism of $\mathfrak{EC}$,

$$(\xi, 1) : ② \to ①.$$

Applying the spectral sequence functor, we get

$$SS(\xi, 1) : SS② \to SS①.$$

But $SS(\xi, 1)$ is then a morphism of spectral sequences which is the identity at the E_0-level. It is, therefore, the identity. □

Thus we have a unique spectral sequence associated with any Rees system,

$$SS① = SS② . \tag{6.13}$$

In view of (6.13) it is natural to ask whether we may generalize the process, described in Section 1, for deriving an exact couple to obtain the *derived system* of a Rees system. We now present this generalization.

We base ourselves on the Rees system (6.9). We may plainly pass to the derived couples of the couples ① and ② – since, by (6.13), their spectral sequences coincide – and then $(\xi, 1)$ induces a morphism $(\xi_1, 1)$ of the derived couples, where $\xi_1 : D_1 \to D_1$. Now consider the diagram

$$\begin{tikzcd}
D_1 & & \bar{D}_1 \arrow[d, tail, "\bar{\varrho}"] \\
D \arrow[u, "\sigma"] \arrow[r, "\varphi"'] & \Gamma \arrow[r, "\bar{\varphi}"'] & \bar{D} .
\end{tikzcd}$$

Plainly $\bar{\varrho}$ is a factor of $\bar{\varphi}$; for $\bar{\varrho}$ is the kernel of $\bar{\beta}$ and $\bar{\beta}\bar{\varphi} = \beta\xi\bar{\varphi} = 0$. Similarly σ is a factor of φ, so that if we apply the $\theta_\sigma^{\bar{\varrho}}$-process to $(\varphi, \bar{\varphi})$ we get

$$D_1 \xrightarrow{\varphi_1} \Gamma \xrightarrow{\bar{\varphi}_1} \bar{D}_1, \quad \varphi_1 \sigma = \varphi, \quad \bar{\varrho}\bar{\varphi}_1 = \bar{\varphi} . \tag{6.14}$$

Moreover, the proof of Theorem 4.5 applies here to show that, since ③ is exact, namely,

$$\bar{D} \xrightarrow{\xi} D \xrightarrow{\varphi} \Gamma \xrightarrow{\bar{\varphi}} \bar{D} \xrightarrow{\xi} D ,$$

so is the derived triangle

$$\bar{D}_1 \xrightarrow{\xi_1} D_1 \xrightarrow{\varphi_1} \Gamma \xrightarrow{\bar{\varphi}_1} \bar{D}_1 \xrightarrow{\xi_1} D_1 . \tag{6.15}$$

Thus we have proved

Theorem 6.3. *The Rees system* (6.9) *induces a derived Rees system*

$$\begin{tikzcd}
 & D_1 \arrow[rr, "\alpha_1"] \arrow[dl, "\varphi_1"'] & & D_1 \arrow[dl, "\beta_1"'] \arrow[dr, "\varphi_1"] & \\
\Gamma \arrow[dr, "\bar{\varphi}_1"'] & & E_1 \arrow[ul, "\gamma_1"'] \arrow[dl, "\bar{\gamma}_1"] & & \Gamma \arrow[dl, "\bar{\varphi}_1"] \\
 & \bar{D}_1 \arrow[uu, "\xi_1"] \arrow[rr, "\bar{\alpha}"'] & & \bar{D}_1 \arrow[ul, "\bar{\beta}_1"] \arrow[uu, "\xi_1"'] &
\end{tikzcd} \tag{6.16}$$

□

Proposition 6.4. *Given a Rees system* (6.9) *and its derived system* (6.16), *we have* $\bar{\gamma}\beta = \bar{\varrho}\bar{\gamma}_1\beta_1\sigma$.

Proof. This follows immediately from the definitions of $\beta_1, \bar{\gamma}_1$ given in Section 1. A proof valid in any abelian category is given in [10; Prop. 7.16]. □

Proposition 6.5. *If* (6.9) *is special then* (6.16) *is special with the same* $\theta: \Gamma \cong \Gamma$.

Proof. Suppose given θ satisfying (6.11). Then

$$\theta\varphi_1\sigma = \theta\varphi = \varphi\alpha = \varphi_1\sigma\alpha = \varphi_1\alpha_1\sigma,$$

so that $\theta\varphi_1 = \varphi_1\alpha_1$. Similarly $\bar{\varphi}_1\theta = \bar{\alpha}_1\bar{\varphi}$. Finally

$$\bar{\varrho}\bar{\varphi}_1\theta^{-1}\varphi_1\sigma = \bar{\varphi}\theta^{-1}\varphi = \bar{\gamma}\beta = \bar{\varrho}\bar{\gamma}_1\beta_1\sigma,$$

so that $\bar{\varphi}_1\theta^{-1}\varphi_1 = \bar{\gamma}_1\beta_1$. □

Our principal interest lies, of course, in the Rees system associated with $F(C)$, where C is a filtered chain complex, and we use this application to motivate our next discussion. We are going to want to know when $E_\infty \cong Gr \circ H(C)$, so we look for $Gr \circ H(C)$ within the Rees system associated with $F(C)$. We may immediately prove

Proposition 6.6. *For the triple* $F(C)$, *we have*

$$Gr \circ H(C) = i_A H(A)/i_B H(B) = \ker j_A/\ker j_B.$$

Proof. Plainly, if $G = G(C)$, $A = A(C)$, then

$$i_A H(A) = \bigoplus_p \operatorname{im} H(C^{(p)}),$$
$$i_B H(B) = \bigoplus_p \operatorname{im} H(C^{(p-1)}).$$

Thus $i_A H(A)/i_B H(B) = \bigoplus_p \operatorname{im} H(C^{(p)})/\operatorname{im} H(C^{(p-1)}) = Gr \circ H(C)$. The second equality follows from exactness. □

It has been established that the couples ① and ② lead to identical spectral sequences such that $E = E_0 = H(A/B)$, so that we should look for conditions under which

$$E_\infty \cong i_A H(A)/i_B H(B). \tag{6.17}$$

Now, in the notation of the Rees system (6.9), obtained from (6.5), we have the relations

$$i_A H(A)/i_B H(B) = \varphi D/\theta\varphi D, \tag{6.18}$$
$$\ker j_A/\ker j_B = \ker\bar{\varphi}/\ker\bar{\varphi}\theta^{-1}.$$

We set, for *any* Rees system (6.9),

$$\Gamma^+ = \varphi D/\varphi\alpha D, \quad \Gamma^- = \ker\bar{\alpha}\bar{\varphi}/\ker\bar{\varphi}. \tag{6.19}$$

Then, in the Rees system associated with $F(C)$, we have

$$\Gamma^+ = Gr \circ H(C), \tag{6.20}$$

and, moreover,

Proposition 6.7. *In a special Rees system* (6.9) θ *induces an isomorphism*

$$\theta : \Gamma^- \xrightarrow{\sim} \Gamma^+ . \quad \square$$

Remark. There is an isomorphism $\Gamma^- \cong \Gamma^+$ in *any* Rees system, even if it is not special (see Theorem 7.25 of [10]).

Thus we are concerned, in studying the filtered chain complex C, to decide whether

$$E_\infty \cong \Gamma^+ . \tag{6.21}$$

We draw particular attention to the fact that the convergence criterion (6.21) is stated entirely within the Rees system, and we will give necessary and sufficient conditions for (6.21) to hold in the next section. The exact couple ① (or ②) of (6.9) plainly cannot contain the information to decide whether $E_\infty \cong Gr \circ H(C)$, since C does not appear in ①, only the filtering subcomplexes $C^{(p)}$. Thus it is preferable to replace the category $\mathfrak{E}\mathfrak{C}$ by the category $\mathfrak{S}$ in setting up the chain of functors leading from filtered chain complexes to spectral sequences. Specifically (see Proposition 6.1) we have the commutative diagram

$$\begin{tikzcd}
(\mathfrak{A}^{\mathbb{Z}}, d, f) \arrow[r, "R \circ F"] \arrow[d, equal] & \mathfrak{S}(\mathfrak{A}^{\mathbb{Z}\times\mathbb{Z}}, d) \arrow[r, "SS\ \bar{E}"] \arrow[d, "\bar{E}"] & \mathfrak{E}(\mathfrak{A}^{\mathbb{Z}\times\mathbb{Z}}) \arrow[d, equal] \\
(\mathfrak{A}^{\mathbb{Z}}, d, f) \arrow[r, "\bar{H}"] & \mathfrak{E}\mathfrak{C}(\mathfrak{A}^{\mathbb{Z}\times\mathbb{Z}}) \arrow[r, "SS"] & \mathfrak{E}(\mathfrak{A}^{\mathbb{Z}\times\mathbb{Z}}) ;
\end{tikzcd} \tag{6.22}$$

and the top row of (6.19) has the advantage over the bottom row that, in the Rees system $RF(C)$, we retain the information necessary for deciding whether $E_\infty \cong Gr \circ H(C)$, whereas in $\bar{H}(C)$ we can only decide *internal* questions relating to the convergence of the spectral sequence (e.g., whether it converges finitely). We wish to emphasize this point because many spectral sequences (for example, that which relates ordinary homology to a general homology theory in algebraic topology) do not arise from a filtered chain complex, but do lead naturally to a (special) Rees system.

We close this section by rendering explicit all the objects appearing in the Rees system (6.9) obtained from a filtered chain complex C and listing the bidegrees of the morphisms. Of course, the exact couple ① in (6.9) is just (2.9) and the exact couple ② in (6.9) is just (3.4).

Notice, first, that the term $\Gamma = H(C)$ is only graded, although we may, conventionally, bigrade it, as explained below. Then, referring to (6.9),

$$\left.\begin{aligned}
D &= \{D^{p,q}\}, \ D^{p,q} = H_q(C^{(p)}) ;\\
\bar{D} &= \{\bar{D}^{p,q}\}, \ \bar{D}^{p,q} = H_q(C/C^{(p-1)}) ;\\
E &= \{E^{p,q}\}, \ E^{p,q} = H_q(C^{(p)}/C^{(p-1)}) ;\\
\Gamma &= \{\Gamma^{p,q}\}, \ \Gamma^{p,q} = H_q(C) ;
\end{aligned}\right\} \tag{6.23}$$

$$\left.\begin{array}{l}\deg\alpha=(1,0),\ \deg\beta=(0,0),\ \deg\gamma=(-1,-1)\,;\\ \deg\bar{\alpha}=(1,0),\ \deg\bar{\beta}=(-1,-1),\ \deg\bar{\gamma}=(0,0)\,;\\ \deg\xi=(-1,-1),\ \deg\varphi=(0,0),\ \deg\bar{\varphi}=(1,0)\,;\\ \theta:\Gamma\to\Gamma \text{ is the identity morphism of degree } (1,0)\,.\end{array}\right\}\quad(6.24)$$

Here the p-degrees of $\varphi, \bar{\varphi}$ are, of course, purely conventional; what is important is that the p-degree of $\bar{\varphi}\varphi$ is 1.

Passing to the n^{th} derived Rees system of the Rees system (6.9) obtained from a filtered chain complex C, we obtain the bidegrees

$$\left.\begin{array}{l}\deg\alpha_n=(1,0),\ \deg\beta_n=(-n,0),\ \deg\gamma_n=(-1,-1)\,;\\ \deg\bar{\alpha}_n=(1,0),\ \deg\bar{\beta}_n=(-n-1,-1),\ \deg\bar{\gamma}_n=(0,0)\,;\\ \deg\xi_n=(-1,-1),\ \deg\varphi_n=(0,0),\ \deg\bar{\varphi}_n=(1,0)\,;\\ \deg d_n=(-n-1,-1)\,.\end{array}\right\}\quad(6.25)$$

We remark that the asymmetry between the degrees in the derived couples of ① and ② arises from our conventional insistence on regarding D_n and $\bar{D}_n$ as *sub*objects of D and $\bar{D}$, respectively. We would preserve symmetry by regarding D_n as a subobject and $\bar{D}_n$ as a quotient object.

We revert finally to the convention (3.3),

$$\bar{D}^{p,q}=H_q(C/C^{(p-1)})\,.$$

This convention was, as explained above, essential if we were to have symmetry between the degrees in the couples ① and ② of the Rees system $RF(C)$ – and so a chance of symmetry on the degrees of the derived couples. It is also consistent with the view that $Gr\circ H(C)$ is really a "self-dual" construction; one either considers the family of morphisms $H(C^{(p)})\xrightarrow{\varphi^p}H(C)$, passes to the induced epimorphisms of cokernels $\operatorname{coker}\varphi^{p-1}\twoheadrightarrow\operatorname{coker}\varphi^p$, and takes kernels – or one considers the family of morphisms $H(C)\xrightarrow{\bar{\varphi}^{p-1}}H(C/C^{(p-1)})$, passes to the induced monomorphisms of kernels $\ker\bar{\varphi}^{p-1}\rightarrowtail\ker\bar{\varphi}^p$, and takes cokernels.

Exercises:

6.1. Identify the morphisms of (6.9), including θ, for the Rees system of the triple $F(C)$, and establish the commutativity relations (6.10), (6.11).

6.2. Interpret the relations $\bar{\varphi}\theta^{-1}\varphi=\bar{\gamma}\beta$, $\bar{\varphi}_1\theta^{-1}\varphi_1=\bar{\gamma}_1\beta_1$ for the Rees system of the triple $F(C)$.

6.3. Establish the remark following Proposition 6.7.

6.4. Do the couples ① and ② of (6.9) together contain all information necessary to determine if $E_\infty\cong Gr\circ H(C)$?

6.5. Obtain a special Rees system for a *filtered cochain complex*, paying special attention to the degrees of the morphisms involved.

6.6. Formulate the ladder of a Rees system!

7. The Limit of a Rees System

In this section we introduce the *limit of a Rees system;* our particular interest is in obtaining necessary and sufficient conditions for the isomorphism (6.21) $E_\infty \cong \Gamma^+$ and to show how these conditions include those of Section 3.

We use the limiting processes introduced in Section 5, and obtain from (6.9), first, the diagram

$$\begin{tikzcd}
& I && U \arrow[dl, "\beta_\infty"'] \arrow[dr, "\varphi_U"] \\
\Gamma \arrow[dr, "\bar\varphi_{\bar I}"'] && E_\infty \arrow[ul, "\gamma_\infty"'] \arrow[dl, "\bar\gamma_\infty"] && \Gamma \\
& \bar I \arrow[uu, "\xi_I"] && \bar U \arrow[ul, "\bar\beta_\infty"] \arrow[uu, "\xi_U"']
\end{tikzcd} \tag{7.1}$$

The morphisms $\beta_\infty, \gamma_\infty, \bar\beta_\infty, \bar\gamma_\infty$ were defined in Section 5. The morphisms ξ_I, ξ_U are obtained by applying limit and colimit functors to the morphism $\xi:\bar D \to D$. The morphism φ_U is obtained by means of the morphisms $\varphi_n: D_n \to \Gamma$ of the successive derived Rees systems using the universal property of the colimit; and similarly for $\bar\varphi_{\bar I}$. We have the exact sequences

$$\begin{gathered}
U \xrightarrow{\alpha'} U \xrightarrow{\beta_\infty} E_\infty \xrightarrow{\gamma_\infty} I \xrightarrow{\alpha''} I\,, \\
\bar U \xrightarrow{\bar\alpha'} \bar U \xrightarrow{\bar\beta_\infty} E_\infty \xrightarrow{\bar\gamma_\infty} \bar I \xrightarrow{\bar\alpha''} \bar I\,.
\end{gathered} \tag{7.2}$$

Moreover, the commutativities

$$\beta_\infty \xi_U = \bar\beta_\infty\,, \qquad \gamma_\infty = \xi_I \bar\gamma_\infty\,, \qquad \xi_U \bar\alpha' = \alpha' \xi_U\,, \qquad \xi_I \bar\alpha'' = \alpha'' \xi_I \tag{7.3}$$

follow from the corresponding commutativities of the successive derived Rees systems. We claim

Theorem 7.1. *The sequences*

$$\begin{gathered}
\bar U \xrightarrow{\xi_U} U \xrightarrow{\varphi_U} \Gamma\,, \\
\Gamma \xrightarrow{\bar\varphi_I} \bar I \xrightarrow{\xi_I} I
\end{gathered}$$

are exact.

Proof. Since ξ_U, φ_U are induced by ξ, φ by passing to quotient objects, the exactness of $\bar U \xrightarrow{\xi_U} U \xrightarrow{\varphi_U} \Gamma$ follows immediately from that of $\bar D \xrightarrow{\xi} D \xrightarrow{\varphi} \Gamma$. Similarly for the second sequence. $\square$

Recall (6.19) that Γ^+ was defined as $\varphi D/\varphi\alpha D$ and Γ^- as $\ker\bar\alpha\bar\varphi/\ker\bar\varphi$. We immediately infer

Proposition 7.2.

$$\begin{aligned}
\Gamma^+ &= \varphi_U U/\varphi_U \alpha' U\,, \\
\Gamma^- &= \ker\bar\alpha''\bar\varphi_I/\ker\bar\varphi_I\,.
\end{aligned}$$

Proof. Obviously $\varphi_U U = \varphi D$, $\varphi_U \alpha' U = \varphi \alpha D$. As to the second expression, we may appeal to duality (the two expressions actually are dual, although their expressions disguise the fact!), or invoke the diagram

$$\begin{tikzcd} & \bar{I} \arrow[r, "\bar{\alpha}''"] \arrow[d, tail, "\bar{\nu}"] & \bar{I} \arrow[d, tail, "\bar{\nu}"] \\ \Gamma \arrow[ur, "\varphi_{\bar{I}}"] \arrow[r, "\bar{\varphi}"] & \bar{D} \arrow[r, "\bar{\alpha}"] & \bar{D} \end{tikzcd} \qquad \square$$

We also observe that, in a special Rees system with $\theta : \Gamma \xrightarrow{\sim} \Gamma$, the third of the identities (6.11) "goes to the limit", yielding

$$\bar{\varphi}_{\bar{I}} \theta^{-1} \varphi_U = \bar{\gamma}_\infty \beta_\infty . \tag{7.4}$$

We now put the facts together to yield the main theorem of this section.

Theorem 7.3. *The Rees system* (6.9) *gives rise to the limit diagram*

$$\begin{tikzcd} \operatorname{coker} \bar{\alpha}' \arrow[r, tail, "\xi'"] \arrow[d, equal] & \operatorname{coker} \alpha' \arrow[r, two heads, "\varphi^+"] \arrow[d, tail, "\beta_*"] & \Gamma^+ \arrow[d, tail, "\bar{\varphi}^+"] \\ \operatorname{coker} \bar{\alpha}' \arrow[r, tail, "\bar{\beta}_*"] & E_\infty \arrow[r, two heads, "\bar{\gamma}_*"] \arrow[d, two heads, "\gamma_*"] & \ker \bar{\alpha}'' \arrow[d, two heads, "\xi''"] \\ & \ker \alpha'' \arrow[r, equal] & \ker \alpha'' \end{tikzcd} \tag{7.5}$$

with exact rows and columns.

Proof. The exact sequence (5.18) yields the exact sequences involving β_*, γ_*; $\bar{\beta}_*, \bar{\gamma}_*$. The diagram

$$\begin{tikzcd} \bar{U} \arrow[r, "\xi_U"] \arrow[d, "\bar{\alpha}'"] & U \arrow[r, two heads, "\varphi_U"] \arrow[d, "\alpha'"] & \varphi_U U \arrow[d, dotted] \\ \bar{U} \arrow[r, "\xi_U"] & U \arrow[r, two heads, "\varphi_U"] & \varphi_U U \end{tikzcd}$$

immediately yields, by passing to cokernels, the exact sequence

$$\operatorname{coker} \bar{\alpha}' \xrightarrow{\xi'} \operatorname{coker} \alpha' \overset{\varphi^+}{\twoheadrightarrow} \Gamma^+ .$$

However, ξ' is monomorphic, in view of the diagram

$$\begin{tikzcd} \bar{U} \arrow[r, "\xi_U"] \arrow[d, "\bar{\alpha}'"] & U \arrow[d, "\alpha'"] \\ \bar{U} \arrow[r, "\xi_U"] \arrow[d, "\bar{\beta}_\infty"] & U \arrow[d, "\beta_\infty"] \\ E_\infty \arrow[r, equal] & E_\infty \end{tikzcd}$$

with exact columns. By duality we obtain the exact sequence

$$\Gamma^- \overset{\bar{\varphi}^-}{\rightarrowtail} \ker\bar{\alpha}'' \overset{\xi''}{\twoheadrightarrow} \ker\alpha'' .$$

Finally, we set $\bar{\varphi}^+ = \bar{\varphi}^- \theta^{-1}$ (Proposition 6.7) and establish the commutativity of the upper right hand square by appeal to (7.4). □

It is now obvious from Theorem 7.3 that $\Gamma^+ \cong E_\infty$ via a *natural* isomorphism if and only if $\operatorname{coker}\bar{\alpha}' = 0$ and $\ker\alpha'' = 0$. To make this statement quite precise, we need here the notion of a *homomorphic relation* between two objects X and Y of an abelian category $\mathfrak{A}$; this is simply a subobject of $X \oplus Y$. We also speak of a homomorphic relation *from* X *to* Y. Now it follows from Theorem 7.3 that we have an exact sequence

$$\operatorname{coker}\alpha' \overset{\{\beta_*, \varphi^+\}}{\rightarrowtail} E_\infty \oplus \Gamma^+ \overset{\langle\bar{\gamma}_*, -\bar{\varphi}^+\rangle}{\twoheadrightarrow} \ker\bar{\alpha}'' \tag{7.6}$$

so we obtain a homomorphic relation Θ from E_∞ to Γ^+ which is $\operatorname{im}\{\beta_*, \varphi^+\}$. Evidently Θ is natural.

Now a homomorphic relation can only be an isomorphism if it is a morphism (for the general theory of homomorphic relations see [22]). Thus we are led to the following important corollary, rendering precise the conclusion refered to above.

Corollary 7.4. *The homomorphic relation Θ from E_∞ to Γ^+ is an isomorphism if and only if*

$$\operatorname{coker}\bar{\alpha}' = 0, \qquad \ker\alpha'' = 0. \quad \square \tag{7.7}$$

Thus the conditions (7.7) are the necessary and sufficient conditions for the validity of (6.21); and hence of (6.17), $E_\infty \cong i_A H(A)/i_B H(B)$, for a Rees system arising from a triple (G, A, θ). We may apply this to the case of a differential filtered object

$$\cdots \subseteq C^{(p-1)} \subseteq C^{(p)} \subseteq \cdots \subseteq C, \qquad -\infty < p < \infty,$$

by means of the functor F (6.2). Then $\Gamma^+ = i_A H(A)/i_B H(B) = Gr \circ H(C)$. With a view to interpreting conditions (7.7) in this case, we define the subobject I_p of $H(C^{(p)})$ as

$$I_p = \bigcap_k \alpha^k H(C^{(p-k)}),$$

where $\alpha: H(C^{(p-1)}) \to H(C^{(p)})$ is induced by the inclusion; and we define the quotient object $\bar{U}_p$ of $H(C/C^{(p)})$ as

$$\bar{U}_p = H(C/C^{(p)}) / \bigcup_k \bar{\alpha}^{-k}(0),$$

where $\bar{\alpha}: H(C/C^{(p)}) \to H(C/C^{(p+1)})$ is induced by the inclusion of $C^{(p)}$ in $C^{(p+1)}$. We conclude

Theorem 7.5. *In the spectral sequence arising from a filtered differential object C, the homomorphic relation Θ from E_∞ to $Gr \circ H(C)$ is an* *isomorphism if and only if*

$$I_p \cap \alpha^{-1}(0) = 0\,, \qquad \bar{\alpha}' \bar{U}_p = \bar{U}_{p+1}\,, \quad \text{for all } p\,, \tag{7.8}$$

where $\bar{\alpha}'$ is induced by $\bar{\alpha}$. □

Let us finally observe how condition (3.9), that C be homologically finite, automatically – indeed, trivially – guarantees (7.8). For, in this case, we are dealing with a filtered graded differential object C and (3.9) (i) implies that $I_{p,q} = 0$ for all p, q so that $I_p = 0$ for all p, while (3.9) (ii) implies that $\bar{U}_{p,q} = 0$ for all p, q so that $\bar{U}_p = 0$ for all p.

More generally, we may paraphrase (7.8), in the case of a filtered (graded) differential group C almost precisely as follows. We say that an element of $H_q(C^{(p)})$ has *filtration* $-\infty$ if it belongs to $I_{p,q}$ that is, if it is in the image of $H_q(C^{(r)})$ for all $r \leqq p$; and we say that an element of $H_q(C^{(p)})$ is *stable* if it is non-zero in every $H_q(C^{(r)})$, $r \geqq p$. We apply similar terminology to $H_q(C/C^{(p)})$. Then (7.8) may be translated as saying: "elements of $H_q(C^{(p)})$ of filtration $-\infty$ are stable; stable elements of $H_q(C/C^{(p)})$ have filtration $-\infty$".

Exercises:

7.1. Specify the morphisms φ_U, $\varphi_{\bar{I}}$ of (7.1).
7.2. Prove (7.4).
7.3. Apply Theorem 7.5 to filtered cochain complexes.
7.4. Show that, in a category of modules, $\alpha' : U \to U$ is monomorphic. Give an example to show that $\alpha'' : I \to I$ need not be epimorphic.
7.5. Identify the sequence $\operatorname{coker} \alpha' \rightarrowtail E_\infty \twoheadrightarrow \ker \alpha''$ in the case of the spectral sequences associated with the couples of Exercises 1.5, 1.6. (These are called the *Bockstein* spectral sequences.) Consider both the case where C is of finite type and the general case.

8. Completions of Filtrations

Suppose given two filtered differential objects C and C' and a morphism $\varphi : C \to C'$. Thus we have

$$\begin{array}{ccccccccc}
\cdots \subseteqq & C^{(p-1)} & \subseteqq & C^{(p)} & \subseteqq & \cdots & \subseteqq & C \\
 & \downarrow{\scriptstyle \varphi^{(p-1)}} & & \downarrow{\scriptstyle \varphi^{(p)}} & & & & \downarrow{\scriptstyle \varphi} \\
\cdots \subseteqq & C'^{(p-1)} & \subseteqq & C'^{(p)} & \subseteqq & \cdots & \subseteqq & C'
\end{array} \tag{8.1}$$

Then φ induces a morphism of the associated spectral sequences, say,

$$\varphi_* : \boldsymbol{E} \to \boldsymbol{E}'\,.$$

Now it is easy to prove that the terms $E_{m,n}$ of Section 4 (see 4.17) depend naturally on the spectral sequence $\boldsymbol{E}$ (see Corollary 3.16 of [10]), and, in particular, the term $E_{m,n}$, $m, n \geqq k$, depends naturally on the part of the spectral sequence $\boldsymbol{E}$ beginning with E_k. We thus have immediately, in view of (5.11),

Proposition 8.1. *If* $\varphi_* : E_k \xrightarrow{\sim} E'_k$, *then* $\varphi_* : E_n \xrightarrow{\sim} E'_n$, $k \leqq n \leqq \infty$. $\square$

Theorem 7.5 now gives us conditions under which we may infer from $\varphi_* : E_\infty \xrightarrow{\sim} E'_\infty$ that

$$\varphi_* : Gr \circ H(C) \xrightarrow{\sim} Gr \circ H(C'). \tag{8.2}$$

Of course we really want to draw the inference that

$$\varphi_* : H(C) \xrightarrow{\sim} H(C') \tag{8.3}$$

and this section is mainly motivated by this problem: to give a reasonable set of conditions under which (8.2) implies (8.3). Certainly, the condition of homological finiteness for a filtered *graded* differential object immediately yields the proof of (8.3), given (8.2); for if C and C' satisfy this condition then the filtrations of $H_q(C)$, $H_q(C')$ are finite and a finite induction yields the desired conclusion. Thus this section may be omitted by those content to confine themselves to applications involving homologically finite filtered chain complexes.

Our aim, then, is to give conditions more general than those of homological finiteness which will still yield the conclusion (8.3) from (8.2). We introduce the notation

$$X^{p-1} \overset{\xi^p}{\rightarrowtail} X^p \overset{\nu^p}{\rightarrowtail} X \overset{\eta_p}{\twoheadrightarrow} X_p \overset{\xi_p}{\twoheadrightarrow} X_{p+1}, \tag{8.4}$$

where

$$\cdots \subseteqq X^{p-1} \subseteqq X^p \subseteqq \cdots \subseteqq X, \qquad -\infty < p < \infty \tag{8.5}$$

is a filtered object in the abelian category $\mathfrak{A}$, ξ^p, ν^p are the inclusions, η_p is the cokernel of ν^p, so that $X_p = X/X^p$, and $\xi_p \eta_p = \eta_{p+1}$. Thus X plays the role of $H(C)$ in the discussion. We may refer to

$$X \overset{\eta_p}{\twoheadrightarrow} X_p \overset{\xi_p}{\twoheadrightarrow} X_{p+1}$$

as the *cofiltration* associated with the filtration (8.5).

Definition. We say the filtration (8.5) is *left complete* if

$$(X; \nu^p) = \varinjlim (X^p, \xi^p);$$

we say the filtration is *right complete* if

$$(X; \eta_p) = \varprojlim (X_p, \xi_p),$$

we say the filtration (8.5) is *complete* if it is left complete and right complete.

Remarks. (i) A finite filtration is obviously complete. (ii) If $\mathfrak{A}$ is a category of modules then (8.5) is left complete if and only if $X = \bigcup_p X^p$. However the description of right completeness is even in this case more complicated. For we require two properties: (i) $\bigcap_p X^p = 0$ (dual to the property singled out as characterizing left completeness) and (ii) given a compatible set of elements $x_p \in X_p$ (i.e., $\xi_p(x_p) = x_{p+1}$), we require the *existence* of $x \in X$ with $\eta_p(x) = x_p$. We will see below just why this extra condition arises and we will give an example to show that it is essential.

Our aim is to show that, if the filtration of $H(C)$ is complete, then (8.3) follows from (8.2). To this end we consider the following situation. We suppose that, for all p, $v^p: X^p \rightarrowtail X$ factorizes as $X^p \overset{v'^p}{\rightarrowtail} Y \xrightarrow{\mu} X$, where μ is independent of p. Set $Y_p = Y/X^p$ and let $\eta'_p: Y \twoheadrightarrow Y_p$ be the projection. Then $\xi_p: X_p \twoheadrightarrow X_{p+1}$ induces $\xi'_p: Y_p \twoheadrightarrow Y_{p+1}$, and we have the commutative diagram

$$\begin{tikzcd} X^{p-1} \arrow[r, tail, "\xi^p"] \arrow[d, equal] & X^p \arrow[r, tail, "v'^p"] \arrow[d, equal] & Y \arrow[r, two heads, "\eta'_p"] \arrow[d, "\mu"] & Y_p \arrow[r, two heads, "\xi'_p"] \arrow[d, "\mu_p"] & Y_{p+1} \arrow[d, "\mu_{p+1}"] \\ X^{p-1} \arrow[r, tail, "\xi^p"] & X^p \arrow[r, tail, "v^p"] & X \arrow[r, two heads, "\eta_p"] & X_p \arrow[r, two heads, "\xi_p"] & X_{p+1} \end{tikzcd} \tag{8.6}$$

Proposition 8.2. *If* $(X; \eta_p) = \varprojlim (X_p, \xi_p)$, *then* $(Y; \eta'_p) = \varprojlim (Y_p, \xi'_p)$.

Proof. The right hand square of (8.6) is a pull-back since $\ker \xi'_p = \ker \xi_p$. It thus follows from Theorem 5.2 that

$$\begin{tikzcd} Y_{-\infty} \arrow[r, two heads, "\eta''_p"] \arrow[d, "\mu_{-\infty}"] & Y_p \arrow[d, "\mu_p"] \\ X \arrow[r, two heads, "\eta_p"] & X_p \end{tikzcd}$$

is a pull-back, where $(Y_{-\infty}; \eta''_p) = \varprojlim (Y_p, \xi'_p)$. But plainly

$$\begin{tikzcd} Y \arrow[r, two heads, "\eta'_p"] \arrow[d, "\mu"] & Y_p \arrow[d, "\mu_p"] \\ X \arrow[r, two heads, "\eta_p"] & X_p \end{tikzcd}$$

is also a pull-back, so that $(Y; \eta'_p) = (Y_{-\infty}; \eta''_p)$. $\square$

Now let us write X^p_q for X^p/X^q, $q \leqq p$. There is then a commutative square

$$\begin{tikzcd} X^p_q \arrow[r, tail, "\varrho^p_q"] \arrow[d, two heads, "\sigma^p_q"] & X^{p+1}_q \arrow[d, two heads, "\sigma^{p+1}_q"] \\ X^p_{q+1} \arrow[r, tail, "\varrho^p_{q+1}"] & X^{p+1}_{q+1} \end{tikzcd} \tag{8.7}$$

which is easily seen to be bicartesian. From Proposition 8.2 (and its dual) we infer

Proposition 8.3. (i) *If the filtration* (8.5) *is right complete, then*

$$\varprojlim_q (X_q^p, \sigma_q^p) = X^p$$

(ii) *If the filtration* (8.5) *is left complete, then*

$$\varinjlim_p (X_q^p, \varrho_q^p) = X_q . \quad \square$$

We may now prove our main theorem.

Theorem 8.4. *Let* $\psi: X \to X'$ *be a morphism of filtered objects in the abelian category* $\mathfrak{A}$. *Thus*

$$\begin{tikzcd} X^p \arrow[r, tail, "\nu^p"] \arrow[d, "\psi^p"] & X \arrow[r, two heads, "\eta_p"] \arrow[d, "\psi"] & X_p \arrow[d, "\psi_p"] \\ X'^p \arrow[r, tail, "\nu'^p"] & X' \arrow[r, two heads, "\eta'_p"] & X'_p \end{tikzcd} \tag{8.8}$$

Suppose that ψ *induces* $\psi_*: Gr(X) \xrightarrow{\sim} Gr(X')$. *If the filtrations of* X *and* X' *are left complete then* ψ_p *is an isomorphism. If the filtrations of* X *and* X' *are right complete then* ψ^p *is an isomorphism. If the filtrations of* X *and* X' *are complete then* ψ *is an isomorphism.*

Proof. We are given that ψ induces an isomorphism

$$\psi_*: X^p/X^{p-1} \xrightarrow{\sim} X'^p/X'^{p-1} .$$

It then follows by induction on $p-q$ that ψ induces an isomorphism $\psi_*: X_q^p \xrightarrow{\sim} X_q'^p$. For we have the commutative diagram

$$\begin{tikzcd} X_q^{p-1} \arrow[r, tail] \arrow[d, "\psi_*"] & X_q^p \arrow[r, two heads] \arrow[d, "\psi_*"] & X_{p-1}^p \arrow[d, "\psi_*"] \\ X_q'^{p-1} \arrow[r, tail] & X_q'^p \arrow[r, two heads] & X'^p_{p-1} . \end{tikzcd}$$

Thus ψ induces an isomorphism of the square (8.7) with the corresponding square for X'.

Now if the filtrations of X and X' are left complete it follows from Proposition 8.3 (ii) that $\psi_q: X_q \to X'_q$ is an isomorphism for all q. Similarly, if the filtrations of X and X' are right complete, $\psi^p: X^p \to X'^p$ is an isomorphism for all p. The final assertion of the theorem then follows immediately from (8.8). $\square$

We now take up the following question: suppose given a filtered object X in the abelian category $\mathfrak{A}$. Is it possible to associate with X, in a functorial manner, a filtered object Y such that (i) the filtration of Y is complete, and (ii) $Gr Y = Gr X$? We will show how this may be done. The process will be described as *completing* the filtration of X.

We return to (8.4) and construct $\varinjlim(X^p, \xi^p)$. Thus we obtain

$$\begin{tikzcd}
X^{p-1} \arrow[r, tail, "\xi^p"] \arrow[d, equal] & X^p \arrow[r, tail, "\nu^p"] \arrow[d, equal] & X \arrow[r, two heads, "\eta_p"] & X_p \arrow[r, two heads, "\xi_p"] & X_{p+1} \\
X^{p-1} \arrow[r, tail, "\xi^p"] & X^p \arrow[r, tail, "\nu^{\infty,p}"] & X^\infty \arrow[r, two heads, "\eta_p^\infty"] \arrow[u, "\lambda"] & X_p^\infty \arrow[r, two heads, "\xi_p^\infty"] \arrow[u, "\lambda_p"] & X_{p+1}^\infty \arrow[u, "\lambda_{p+1}"]
\end{tikzcd} \tag{8.9}$$

where $(X^\infty; \nu^{\infty,p}) = \varinjlim(X^p, \xi^p)$. Note that λ may be neither monomorphic nor epimorphic; but, if $\mathfrak{A}$ is a category of modules, λ is monomorphic. We now construct $\varprojlim(X_p^\infty, \xi_p^\infty)$. With an obvious notation we obtain

$$\begin{tikzcd}
X^{p-1} \arrow[r, tail, "\xi^p"] \arrow[d, equal] & X^p \arrow[r, tail, "\nu^p"] \arrow[d, equal] & X \arrow[r, two heads, "\eta_p"] & X_p \arrow[r, two heads, "\xi_p"] & X_{p-1} \\
X^{p-1} \arrow[r, tail, "\xi^p"] \arrow[d, "\kappa^{p-1}"] & X^p \arrow[r, tail, "\nu^{\infty,p}"] \arrow[d, "\kappa^p"] & X^\infty \arrow[r, two heads, "\eta_p^\infty"] \arrow[u, "\lambda"] \arrow[d, "\bar{\kappa}"] & X_p^\infty \arrow[r, two heads, "\xi_p^\infty"] \arrow[u, "\lambda_p"] \arrow[d, equal] & X_{p+1}^\infty \arrow[u, "\lambda_{p+1}"] \arrow[d, equal] \\
X^{p-1}_{-\infty} \arrow[r, tail, "\xi^p_{-\infty}"] & X^p_{-\infty} \arrow[r, tail, "\nu^{\infty,p}_{-\infty}"] & (X^\infty)_{-\infty} \arrow[r, two heads, "\eta^\infty_{-\infty}"] & X_p^\infty \arrow[r, two heads, "\xi_p^\infty"] & X_{p+1}^\infty
\end{tikzcd} \tag{8.10}$$

We call the bottom row of (8.10) the *completion* of the top row.

Theorem 8.5. *The completion is a complete filtration of* $Y = (X^\infty)_{-\infty}$ *and* $Gr\,Y = Gr\,X$.

Proof. By construction the filtration of Y is right complete. That it is left complete follows from the dual of Proposition 8.2.

Now, given (8.4), we obtain $Gr\,X$ either by

$$(Gr\,X)_p = \operatorname{coker} \xi^p$$

or by

$$(Gr\,X)_p = \ker \xi_{p-1}\,.$$

Since ξ^p is unchanged in passing from the first row of (8.10) to the second, and ξ_p^∞ is unchanged in passing from the second row to the third, it follows that $Gr\,Y = Gr\,X$. $\square$

Plainly the completion process as described is functorial. Moreover, it is self-dual in the following sense. Starting from (8.4) we may *first* construct $\varprojlim(X_p, \xi_p)$ and then construct the appropriate $\varinjlim$. We claim that if we do this we obtain (compare (8.10))

$$\begin{tikzcd}
X^{p-1} \arrow[r, tail, "\xi^p"] \arrow[d, "\kappa^{p-1}"] & X^p \arrow[r, tail, "\nu^p"] \arrow[d, "\kappa^p"] & X \arrow[r, two heads, "\eta_p"] \arrow[d, "\kappa"] & X_p \arrow[r, two heads, "\xi_p"] \arrow[d, equal] & X_{p+1} \arrow[d, equal] \\
X^{p-1}_{-\infty} \arrow[r, tail, "\xi^p_{-\infty}"] \arrow[d, equal] & X^p_{-\infty} \arrow[r, tail, "\nu^p_{-\infty}"] \arrow[d, equal] & X_{-\infty} \arrow[r, two heads, "\eta_{-\infty,p}"] & X_p \arrow[r, two heads, "\xi_p"] & X_{p+1} \\
X^{p-1}_{-\infty} \arrow[r, tail, "\xi^p_{-\infty}"] & X^p_{-\infty} \arrow[r, tail, "\nu^{\infty,p}_{-\infty}"] & (X_{-\infty})^\infty \arrow[r, two heads, "\eta^\infty_{-\infty,p}"] \arrow[u, "\bar{\lambda}"] & X_p^\infty \arrow[r, two heads, "\xi_p^\infty"] \arrow[u, "\lambda_p"] & X_{p+1}^\infty \arrow[u, "\lambda_{p+1}"]
\end{tikzcd} \tag{8.11}$$

with the same bottom row as in (8.10).

In particular,

$$(X^{\infty})_{-\infty} = (X_{-\infty})^{\infty}, \qquad \varprojlim_q \varinjlim_p (X_q^p; \varrho_q^p, \sigma_q^p) = \varinjlim_p \varprojlim_q (X_q^p; \varrho_q^p, \sigma_q^p). \tag{8.12}$$

The proof of these facts is similar to that of Proposition 8.2. The reader is advised to obtain proofs for himself as an exercise (see also [11]). It is also easy to prove, along the same lines, that the diagram

$$\begin{tikzcd} X^{\infty} \arrow[r, "\lambda"] \arrow[d, "\bar{\kappa}"] & X \arrow[d, "\kappa"] \\ X^{\infty}_{-\infty} \arrow[r, "\bar{\lambda}"] & X_{-\infty} \end{tikzcd} \tag{8.13}$$

is bicartesian.

Of course, the filtration (8.5) is left-complete if λ is an isomorphism (so then is $\bar{\lambda}$) and right-complete if κ is an isomorphism (so then is $\bar{\kappa}$). Our remark (ii) following the definition of completeness drew attention to the fact, that, in a category of modules, λ is monomorphic, so it is only necessary to check that λ is epimorphic. On the other hand, κ may fail to be epimorphic even for modules. As an example, let $X = \bigoplus_{n \geqq 0} \mathbb{Z}$, a countable direct sum of infinite cyclic groups, and let X^p be given by

$$\begin{aligned} X^p &= X, & p &\geqq 0 \\ &= \bigoplus_{n \geqq -p} \mathbb{Z}, & p &< 0. \end{aligned}$$

This yields a filtration of X

$$\cdots \subseteqq X^{p-1} \subseteqq X^p \subseteqq \cdots \subseteqq X, \tag{8.14}$$

which is certainly left complete! Passing to the associated cofiltration we obtain

$$X \xrightarrow{\eta_p} X_p \xrightarrow{\xi_p} X_{p+1},$$

where

$$\begin{aligned} X_p &= 0, & p &\geqq 0 \\ &= \bigoplus_{0 \leqq n \leqq -p-1} \mathbb{Z}, & p &< 0 \end{aligned}$$

and η_p, ξ_p are the obvious projections. However, in this case,

$$X_{-\infty} = \prod_{n \geqq 0} \mathbb{Z}$$

and $\kappa: X \to X_{-\infty}$ is the canonical injection $\bigoplus_{n \geqq 0} \mathbb{Z} \subseteqq \prod_{n \geqq 0} \mathbb{Z}$. Thus in this case $\bigcap_p X^p = 0$ (corresponding to the fact that κ is monomorphic), but the filtration (8.14) fails to be right complete.

Exercises:

8.1. Prove (8.12).
8.2. Prove that (8.13) is bicartesian.
8.3. Prove that, in (8.13), $0 \to \ker\lambda \to \ker\kappa\lambda \to \ker\kappa \to 0$ is a split short exact sequence. Prove the similar result for cokernels.
8.4. Give examples where, in (8.13), (i) λ is not epimorphic, (ii) κ is not monomorphic.
8.5. Check the facts stated for the filtration (8.14) and complete the filtration.
8.6. Give two examples from this chapter in which a limit commutes with a colimit.

9. The Grothendieck Spectral Sequence

Let $(\boldsymbol{B}, \partial', \partial'')$ be a double complex as defined in Chapter V, Section 1. Thus we have an *anti*commutative diagram

$$\begin{array}{ccc} B_{r,s} & \xrightarrow{\partial'} & B_{r-1,s} \\ \big\downarrow{\scriptstyle \partial''} & & \big\downarrow{\scriptstyle \partial''} \\ B_{r,s-1} & \xrightarrow{\partial'} & B_{r-1,s-1}\,, \end{array} \qquad \partial''\partial' + \partial'\partial'' = 0\,, \tag{9.1}$$

for each r, s. It will be convenient in this section to replace (9.1) by a commutative diagram; this we achieve by setting

$$\begin{aligned} d' &= \partial', \\ d'' &= (-1)^r \partial'' \quad \text{on} \quad B_{r,s}\,. \end{aligned} \tag{9.2}$$

Of course, we retain the same total differential $\partial = \partial' + \partial''$ in $\operatorname{Tot} \boldsymbol{B}$. We will regard the diagram

$$\begin{array}{ccc} B_{r,s} & \xrightarrow{d'} & B_{r-1,s} \\ \big\downarrow{\scriptstyle d''} & & \big\downarrow{\scriptstyle d''} \\ B_{r,s-1} & \xrightarrow{d'} & B_{r-1,s-1} \end{array} \tag{9.3}$$

as basic and refer to d', d'' as the *horizontal, vertical* differentials in B, respectively. We may also refer to ∂', ∂'' as horizontal, vertical differentials.

The complex $\operatorname{Tot} \boldsymbol{B}$ may now be filtered in the following two natural ways:

$$_1F^p(\operatorname{Tot} \boldsymbol{B})_n = \bigoplus_{\substack{r+s=n \\ r \leqq p}} B_{r,s}\,, \tag{9.4}$$

$$_2F^p(\operatorname{Tot} \boldsymbol{B})_n = \bigoplus_{\substack{r+s=n \\ s \leqq p}} B_{r,s}\,. \tag{9.5}$$

We shall refer to the filtration (9.4) as the *first* filtration of $\operatorname{Tot} \boldsymbol{B}$, and to the filtration (9.5) as the *second* filtration of $\operatorname{Tot} \boldsymbol{B}$. From these filtrations we obtain two spectral sequences.

Using the same notation as in (V. 1.2) and (V. 1.3) we have

Proposition 9.1. *For the (first) spectral sequence associated with the filtration* (9.4) *we have*

$$ {}_1E_0^{p,q} = H_{q-p}(B_{p,*}, \partial''), \qquad {}_1E_1^{p,q} = H_p(H_{q-p}(\boldsymbol{B}, \partial''), \partial'). \tag{9.6}$$

For the (second) spectral sequence associated with the filtration (9.5) *we have*

$$ {}_2E_0^{p,q} = H_{q-p}(B_{*,p}, \partial'), \qquad {}_2E_1^{p,q} = H_p(H_{q-p}(\boldsymbol{B}, \partial'), \partial''). \tag{9.7}$$

Proof. We prove (9.7) only, and so permit ourselves to write F^p for ${}_2F^p$.

Clearly, $F^p(\mathrm{Tot}\,\boldsymbol{B})_q/F^{p-1}(\mathrm{Tot}\,\boldsymbol{B})_q = B_{q-p,p}$. Moreover the differential $\partial = \partial' + \partial''$ on Tot $\boldsymbol{B}$ induces on this quotient the horizontal differential ∂'. This establishes the first assertion of (9.7).

Now d_0 in the spectral sequence is the composite

$$H_q(F^p/F^{p-1}) \xrightarrow{\gamma} H_{q-1}(F^{p-1}) \xrightarrow{\beta} H_{q-1}(F^{p-1}/F^{p-2}).$$

We choose a representative of $x \in H_q(F^p/F^{p-1})$ to be an element $b \in B_{q-p,p}$ such that $\partial' b = 0$. Then γx is the homology class of $\partial'' b$, and $\beta\gamma x$ is therefore just $\partial''_* x$, where ∂''_* is induced on $H(\boldsymbol{B}, \partial')$ by ∂''. □

Remark. We may, of course, write d', d'' for ∂', ∂'' in the statement of the proposition.

Definition. We say that the double complex $\boldsymbol{B}$ is *positive* if there exists n_0 such that

$$B_{r,s} = 0 \quad \text{if} \quad r < n_0 \quad \text{or} \quad s < n_0. \tag{9.8}$$

Proposition 9.2. *If $\boldsymbol{B}$ is positive, then both the first and the second spectral sequences* (9.6), (9.7) *converge finitely to the graded object associated with* $\{H_n(\mathrm{Tot}\,\boldsymbol{B})\}$, *suitably (finitely) filtered.*

Proof. By Theorem 3.5 we only have to verify that the filtrations (9.4), (9.5) are finite. But plainly, given (9.8),

$$ {}_1F^p(\mathrm{Tot}\,\boldsymbol{B})_n = 0 \quad \text{if} \quad p \leqq n_0 - 1,$$

$$ {}_1F^p(\mathrm{Tot}\,\boldsymbol{B})_n = (\mathrm{Tot}\,\boldsymbol{B})_n \quad \text{if} \quad p \geqq n - n_0;$$

and similarly for the second filtration. □

We are now ready to state and prove the existence and convergence theorem for the *Grothendieck spectral sequence.*

Suppose given three abelian categories $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ and additive functors $F: \mathfrak{A} \to \mathfrak{B}$, $G: \mathfrak{B} \to \mathfrak{C}$. Assume that $\mathfrak{A}$ and $\mathfrak{B}$ have enough injectives; this means, of course, that objects in $\mathfrak{A}$ and $\mathfrak{B}$ have injective resolutions. We thus can construct the right derived functors of F, G, and $G \circ F$. Theorem 9.3 will relate these derived functors by a spectral sequence, assuming an additional hypothesis. We shall say that an object B in $\mathfrak{B}$ is *(right) G-acyclic*, if

$$R^q G(B) = \begin{cases} G(B), & q = 0 \\ 0, & q \geqq 1. \end{cases} \tag{9.9}$$

Theorem 9.3 *(Grothendieck spectral sequence). Given $F : \mathfrak{A} \to \mathfrak{B}$, $G : \mathfrak{B} \to \mathfrak{C}$, assume that if I is an injective object of $\mathfrak{A}$, then $F(I)$ is G-acyclic. Then there is a spectral sequence $\{E_n(A)\}$ corresponding to each object A of $\mathfrak{A}$, such that*

$$E_1^{p,q} = (R^p G)(R^{q-p} F)(A) \Rightarrow R^q(GF)(A), \tag{9.10}$$

which converges finitely to the graded object associated with $\{R^q(GF)(A)\}$, suitably filtered.

Before starting the proof, we emphasize that there are other forms of the Grothendieck spectral sequence, involving left derived functors instead of right derived functors, or contravariant functors instead of covariant functors. These variations the reader will easily supply for himself, and will accept as proved once we have proved Theorem 9.3.

Proof. Take an injective resolution of A in $\mathfrak{A}$,

$$\boldsymbol{I} : I_0 \to I_1 \to I_2 \to \cdots. \tag{9.11}$$

Apply F to obtain the cochain complex in $\mathfrak{B}$,

$$FI_0 \to FI_1 \to FI_2 \to \cdots.$$

Suppose we have constructed a commutative diagram in $\mathfrak{B}$

$$\begin{tikzcd} FI_0 \arrow[r] \arrow[d, tail] & FI_1 \arrow[r] \arrow[d, tail] & FI_2 \arrow[r] \arrow[d, tail] & \cdots \\ J_{0,0} \arrow[r] \arrow[d] & J_{1,0} \arrow[r] \arrow[d] & J_{2,0} \arrow[r] \arrow[d] & \cdots \\ J_{0,1} \arrow[r] \arrow[d] & J_{1,1} \arrow[r] \arrow[d] & J_{2,1} \arrow[r] \arrow[d] & \cdots \\ \vdots & \vdots & \vdots & \end{tikzcd} \tag{9.12}$$

such that each row is a cochain complex and the r^{th} column is an (augmented) injective resolution of FI_r, $r = 0, 1, 2, \ldots$

Apply G to obtain the double (cochain) complex $\boldsymbol{B}$,

$$\begin{tikzcd}
GJ_{0,0} \arrow[r, "d'"] \arrow[d, "d''"] & GJ_{1,0} \arrow[r, "d'"] \arrow[d, "d''"] & GJ_{2,0} \arrow[r] \arrow[d, "d''"] & \cdots \\
GJ_{0,1} \arrow[r, "d'"] \arrow[d, "d''"] & GJ_{1,1} \arrow[r, "d'"] \arrow[d, "d''"] & GJ_{2,1} \arrow[r] \arrow[d, "d''"] & \cdots \\
GJ_{0,2} \arrow[r, "d'"] \arrow[d] & GJ_{1,2} \arrow[r, "d'"] \arrow[d] & GJ_{2,2} \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd} \tag{9.13}$$

First we study the spectral sequence based on the first filtration (9.4) of $\operatorname{Tot}\boldsymbol{B}$. Thus ${}_1E^{p,q}$ is computed by applying the vertical differential so that, since $F(I)$ is G-acyclic,

$$\begin{aligned} {}_1E^{p,q} &= GFI_q, \quad p=q, \\ &= 0, \quad p \neq q. \end{aligned}$$

Computing ${}_1E_1$, we find

$$\begin{aligned} {}_1E_1^{p,q} &= R^q(GF)(A), \quad p=q, \\ &= 0, \quad p \neq q. \end{aligned} \tag{9.14}$$

Now, by the dual of (2.11), $\deg d_r = (r+1, 1)$ for the r^{th} differential d_r of the spectral sequence. Thus (9.14) implies that

$$d_r = 0, \quad r \geqq 1,$$

so that, for all $r \geqq 1$,

$$\begin{aligned} {}_1E_r^{p,q} &= R^q(GF)(A), \quad p=q, \\ &= 0, \quad p \neq q, \end{aligned} \tag{9.15}$$

and consequently

$$\begin{aligned} {}_1E_\infty^{p,q} &= R^q(GF)(A), \quad p=q, \\ &= 0, \quad p \neq q. \end{aligned} \tag{9.16}$$

Then Proposition 9.2 ensures that $H^q(\operatorname{Tot}\boldsymbol{B})$ is (finitely) filtered by subobjects whose successive quotients are ${}_1E_\infty^{p,q}$. Since, for fixed q, only one ${}_1E_\infty^{p,q}$ can be non-zero, we conclude

$$H^q(\operatorname{Tot}\boldsymbol{B}) = R^q(GF)(A). \tag{9.17}$$

This exhausts the utility of the first spectral sequence. We now turn to the second spectral sequence; we will permit ourselves to write E instead of ${}_2E$ in discussing this spectral sequence. We will find that it is necessary to construct (9.12) in a very specific way in order to obtain a valuable result from the second spectral sequence. In fact, we construct an injective resolution of FI in the category of cochain complexes in $\mathfrak{B}$, relative to monomorphisms which induce cohomology monomorphisms (see IX.1).

We write F_r for FI_r and display the cocycles and coboundaries of the cochain-complex

$$F_0 \to F_1 \to F_2 \to \cdots$$

as

$$Z_0 \rightarrowtail F_0 \twoheadrightarrow B_1 \rightarrowtail Z_1 \rightarrowtail F_1 \twoheadrightarrow B_2 \rightarrowtail Z_2 \rightarrowtail F_2 \twoheadrightarrow \cdots. \tag{9.18}$$

We prove

Lemma 9.4. *We may resolve* (9.18) *as*

$$\begin{tikzcd}
Z_0 \arrow[r, tail] \arrow[d, tail] & F_0 \arrow[r, two heads] \arrow[d, tail] & B_1 \arrow[r, tail] \arrow[d, tail] & Z_1 \arrow[r, tail] \arrow[d, tail] & F_1 \arrow[r, two heads] \arrow[d, tail] & B_2 \arrow[r, tail] \arrow[d, tail] & \cdots \\
K_{0,0} \arrow[r, tail] \arrow[d] & J_{0,0} \arrow[r, two heads] \arrow[d] & L_{1,0} \arrow[r, tail] \arrow[d] & K_{1,0} \arrow[r, tail] \arrow[d] & J_{1,0} \arrow[r, two heads] \arrow[d] & L_{2,0} \arrow[r, tail] \arrow[d] & \cdots \\
K_{0,1} \arrow[r, tail] \arrow[d] & J_{0,1} \arrow[r, two heads] \arrow[d] & L_{1,1} \arrow[r, tail] \arrow[d] & K_{1,1} \arrow[r, tail] \arrow[d] & J_{1,1} \arrow[r, two heads] \arrow[d] & L_{2,1} \arrow[r, tail] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd} \tag{9.19}$$

where each column is an (augmented) injective resolution of the object appearing at its head, and

$$K_{r,s} \rightarrowtail J_{r,s} \twoheadrightarrow L_{r+1,s}$$

is exact.

Proof. We already know (Lemma III.5.4; see also the proof of Theorem IV.6.1) how to resolve $Z_0 \rightarrowtail F_0 \twoheadrightarrow B_1$. Given the resolution of B_1, we choose an arbitrary resolution of Z_1/B_1 and resolve $B_1 \rightarrowtail Z_1 \twoheadrightarrow Z_1/B_1$. We thus obtain a resolution of Z_1. We then use an arbitrary resolution of B_2 to yield a resolution of $Z_1 \rightarrowtail F_1 \twoheadrightarrow B_2$, and so we step steadily to the right along (9.18). □

When diagram (9.12) is constructed according to the prescription of Lemma 9.4, we will speak of (9.12) as a *resolution* of FI.

We note that by construction of (9.19) the sequence

$$Z_r/B_r \rightarrowtail K_{r,0}/L_{r,0} \to K_{r,1}/L_{r,1} \to K_{r,2}/L_{r,2} \to \cdots, \qquad r = 0, 1, 2, \ldots \tag{9.20}$$

is an injective resolution of Z_r/B_r.

Now since all the objects in the resolution of (9.18) are injective, all monomorphisms split. Thus when we apply the additive functor G to the resolution we maintain all exactness relations.

In particular we note that

$$G(K_{r,s}/L_{r,s}) = GK_{r,s}/GL_{r,s}, \tag{9.21}$$

since $L_{r,s} \rightarrowtail K_{r,s}$ splits. Finally we recall that

$$Z_r/B_r = R^r F(A). \tag{9.22}$$

We complete the proof of Theorem 9.3 by supposing (9.12) constructed, as in Lemma 9.4, to be a resolution of FI. We now study the spectral sequence $E = {}_2E$, based on the filtration (9.5). Thus $E^{p,q}$ is computed by applying the horizontal differential to (9.13), so that, by (9.21),

$$\begin{aligned} E^{p,q} &= H^{q-p}(GJ_{*,p}, d') \\ &= GK_{q-p,p}/GL_{q-p,p} \\ &= G(K_{q-p,p}/L_{q-p,p}). \end{aligned}$$

$E_1^{p,q}$ is now computed by applying the vertical differential. In view of (9.20) and (9.22), we have

$$\begin{aligned} E_1^{p,q} &= R^p G(Z_{q-p}/B_{q-p}) \\ &= (R^p G)\ (R^{q-p} F)(A). \end{aligned}$$

Since (9.13) is positive, Proposition 9.2 guarantees good convergence and the theorem follows from (9.17) and Proposition 9.2. $\square$

Remark. We will show below that it is essential to construct the diagram (9.12) as in Lemma 9.4 to obtain the desired result. (See Remark (i) following the proof of Theorem 9.5.)

We will apply Theorem 9.3 to obtain a spectral sequence, due to Lyndon and Hochschild-Serre, in the cohomology of groups. We will defer other applications of Theorem 9.3 to the exercises.

Thus we now consider a short exact sequence of groups

$$N \overset{i}{\rightarrowtail} K \overset{p}{\twoheadrightarrow} Q \tag{9.23}$$

in other words, N is a normal subgroup of K with quotient group Q. Let $\mathfrak{A}$ be the category of (left) K-modules; let $\mathfrak{B}$ be the category of (left) Q-modules, and let $\mathfrak{C}$ be the category of abelian groups. Further, consider the functors

$$\mathfrak{A} \xrightarrow{F} \mathfrak{B} \xrightarrow{G} \mathfrak{C} \tag{9.24}$$

where $F(A) = \mathrm{Hom}_N(\mathbb{Z}, A) = A^N$, the subgroup of A consisting of elements fixed under N; and $G(B) = \mathrm{Hom}_Q(\mathbb{Z}, B) = B^Q$. It is then plain that A^N acquires the structure of a Q-module by means of the action

$$(px) \circ a = xa, \quad x \in K, \quad a \in A^N,$$

in such a way that F is indeed an additive functor from $\mathfrak{A}$ to $\mathfrak{B}$; G is evidently an additive functor from $\mathfrak{B}$ to $\mathfrak{C}$, and

$$GF(A) = \mathrm{Hom}_K(\mathbb{Z}, A) = A^K. \tag{9.25}$$

We are now ready to prove

Theorem 9.5 (Lyndon-Hochschild-Serre). *Given the short exact sequence of groups*

$$N \overset{i}{\rightarrowtail} K \overset{p}{\twoheadrightarrow} Q$$

and a K-module A, there is a natural action of Q on the cohomology groups $H^m(N, A)$. Moreover, there is a spectral sequence $\{E_n(A)\}$ such that

$$E_1^{p,q} = H^p(Q, H^{q-p}(N, A)) \Rightarrow H^q(K, A),$$

which converges finitely to the graded group associated with $\{H^q(K, A)\}$, suitably filtered.

Proof. We must first verify the hypotheses of Theorem 9.3 for the functors F and G of (9.24). We have already remarked that F and G are additive, so it remains to show that, if I is an injective K-module, then I^N is G-acyclic; in fact, we show that I^N is an injective Q-module. For the functor F is right adjoint to $\bar{F} : \mathfrak{B} \to \mathfrak{A}$, where $\bar{F}(B)$ is the abelian group B with the K-module structure given by $xb = (px) \circ b$. Since $\bar{F}$ plainly preserves monomorphisms, F preserves injectives (Theorem IV. 12.1).

Thus we may apply Theorem 9.3, and it is merely a question of identifying the (right) derived functors involved. Since $\mathbb{Z}K$ is a *free* $\mathbb{Z}N$-module, it follows that a K-injective resolution of A is also an N-injective resolution. Moreover, given any such K-injective resolution of A,

$$I_0 \to I_1 \to I_2 \to \cdots$$

the complex

$$\mathrm{Hom}_N(\mathbb{Z}, A) \to \mathrm{Hom}_N(\mathbb{Z}, I_0) \to \mathrm{Hom}_N(\mathbb{Z}, I_1) \to \cdots$$

acquires the structure of a Q-complex. Thus the cohomology groups $H^m(N, A)$ also acquire the structure of Q-modules and

$$R^m F(A) = H^m(N, A) \tag{9.26}$$

as Q-modules. Since, plainly,

$$R^m G(B) = H^m(Q, B),$$

$$R^m(GF)(A) = H^m(K, A),$$

the theorem follows by quoting Theorem 9.3. □

Remarks. (i) As we have indicated, Theorem 9.5 makes it plain that the diagram (9.12) must be constructed as in Lemma 9.4 in order to achieve the required result. For, since, in this case, the functor $F : \mathfrak{A} \to \mathfrak{B}$ maps injectives to injectives, the identity map of the cochain complex $FI_0 \to FI_1 \to FI_2 \to \cdots$ could be regarded as an example of (9.12). But, for this diagram, we plainly have

$$E^{p,q} = \begin{cases} R^q(GF)(A), & p=0, \\ 0, & \text{otherwise}. \end{cases}$$

so that we achieve nothing. Thus, although it may not be absolutely essential to choose (9.12) to be a *resolution* of $F\boldsymbol{I}$, in the sense of Lemma 9.4, we must certainly avoid arbitrary choice. Moreover, we see that we do not gain in simplicity of demonstration of Theorem 9.3 by replacing the hypothesis that $F(I)$ is G-acyclic by the more restrictive hypothesis that $F(I)$ is injective.

(ii) The form of the Grothendieck spectral sequence, involving left derived functors instead of right derived functors, to which we have already drawn the reader's attention, readily implies a spectral sequence analogous to that of Theorem 9.5, but stated in terms of homology instead of cohomology. Given (9.23), we choose our categories $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ as in the proof of Theorem 9.5 but now $F: \mathfrak{A} \to \mathfrak{B}$ is given by

$$F(A) = \mathbb{Z} \otimes_N A,$$

and $G: \mathfrak{B} \to \mathfrak{C}$ is given by

$$G(B) = \mathbb{Z} \otimes_Q B,$$

so that

$$GF(A) = \mathbb{Z} \otimes_K A.$$

One reasons that F preserves projectives, since F is left adjoint to $\bar{F}: \mathfrak{B} \to \mathfrak{A}$, which is the same $\bar{F}$ as in the proof of Theorem 9.5, and which preserves epimorphisms. The rest of the argument may certainly be left to the reader. We give below some exercises which exploit the homology form of the Lyndon-Hochschild-Serre spectral sequence.

The question also arises of the functoriality of the Grothendieck spectral sequence with respect to the object A. The conclusion – as in so many applications of spectral sequence theory – is that the spectral sequence $\{E_n(A)\}$ of Theorem 9.3 is functorial from $n=1$ onwards; indeed, the determination of E_0 in the proof of the theorem shows that this is as much as could be hoped for. The proof of this fact, involving the notion of *homotopy* of morphisms of double complexes, is deferred to the exercises (see Exercise 9.7).

Exercises:

9.1. Confirm that, in Lemma 9.4, we have constructed an injective resolution of $F\boldsymbol{I}$ relative to monomorphisms which induce cohomology monomorphisms.

9.2. In the spectral sequence of Proposition 9.1 show that there is an exact sequence

$$H_2(\operatorname{Tot} \boldsymbol{B}) \to E_1^{2,2} \xrightarrow{d_1} E_1^{0,1} \to H_1(\operatorname{Tot} \boldsymbol{B}) \to E_1^{1,1} \to 0.$$

Identify the terms in the special case of the Grothendieck spectral sequence and in the Lyndon-Hochschild-Serre spectral sequence. Compare with the sequences (VI. 8.2).

9.3. Apply the Grothendieck spectral sequence to the following situation. Let $\mathfrak{n} \rightarrowtail \mathfrak{g} \twoheadrightarrow \mathfrak{h}$ be a short exact sequence of Lie algebras over K, and let A be a $\mathfrak{g}$-module. Consider the functors $F: \mathfrak{M}_{\mathfrak{g}} \to \mathfrak{M}_{\mathfrak{h}}$, $F(A) = \mathrm{Hom}_{\mathfrak{n}}(K, A)$, and $G: \mathfrak{M}_{\mathfrak{h}} \to \mathfrak{Ab}$, $G(B) = \mathrm{Hom}_{\mathfrak{h}}(K, B)$. Deduce a spectral sequence (Hochschild-Serre) with $E_1^{p,q} = H^p(\mathfrak{h}, H^{q-p}(\mathfrak{n}, A))$, converging to the graded vector space $\{H^q(\mathfrak{g}, A)\}$, suitably filtered. Identify the sequence of Exercise 9.2 in this case.

9.4. Carry out the program outlined in Remark (ii) at the end of the section.

9.5. Let $N \rightarrowtail K \twoheadrightarrow Q$ be exact with N central in K, and let A be a trivial K-module. Show that $H_m(N, A)$, $H^m(N, A)$ are trivial Q-modules.

9.6. Let G be a finite p-group. Show that if $|G| = p^n$, then $|H_2(G, \mathbb{Z})| \leq p^{n(n-1)/2}$ and that this inequality is best possible. $\{|G|$ is the order of the group $G\}$.

9.7. Let $\varphi, \psi: \boldsymbol{B} \to \tilde{\boldsymbol{B}}$ be morphisms of double complexes. We say that φ, ψ are *homotopic*, and write $\varphi \simeq \psi$, if there exist families of morphisms

$$\Sigma'_{r,s}: B_{r,s} \to \tilde{B}_{r+1,s}, \qquad \Sigma''_{r,s}: B_{r,s} \to \tilde{B}_{r,s+1},$$

such that $\boldsymbol{d}''\boldsymbol{\Sigma}' = \boldsymbol{\Sigma}'\boldsymbol{d}''$, $\boldsymbol{d}'\boldsymbol{\Sigma}'' = \boldsymbol{\Sigma}''\boldsymbol{d}'$, and

$$\psi - \varphi = \boldsymbol{d}'\boldsymbol{\Sigma}' + \boldsymbol{\Sigma}'\boldsymbol{d}' + \boldsymbol{d}''\boldsymbol{\Sigma}'' + \boldsymbol{\Sigma}''\boldsymbol{d}''.$$

Show that $\varphi \simeq \psi$ is an equivalence relation, and that, if $\varphi \simeq \psi$, then

$$\mathrm{Tot}\,\varphi \simeq \mathrm{Tot}\,\psi, \qquad E_1(\mathrm{Tot}\,\varphi) = E_1(\mathrm{Tot}\,\psi),$$

where E_1 refers to either spectral sequence of Proposition 9.1. Deduce that the Grothendieck spectral sequence is functorial in A, from E_1 onwards, including the identification of $E_\infty^{*,q}$ with the associated graded object of $R^q(GF)(A)$, suitably filtered.

9.8. Use the spectral sequences associated with a double complex to show the balance of Ext_Λ. (Hint: Let $\boldsymbol{I}$ be an injective resolution of B and let $\boldsymbol{P}$ be a projective resolution of A. Form the double (cochain) complex $\mathrm{Hom}_\Lambda(\boldsymbol{P}, \boldsymbol{I})$ and consider its associated spectral sequences (Proposition 9.1).) Find a similar proof for the balance of Tor^Λ.

9.9. Let the group G be given by the presentation $(x, y; x^m = y^m = x^{-1}y^{-1}xy)$, where m is an odd prime. Show that x generates a normal subgroup of order m^2, with quotient of order m. Thus we get a group extension

$$C_{m^2} \rightarrowtail G \twoheadrightarrow C_m \qquad (*)$$

with natural generators x of C_{m^2} and $\bar{y}$ of C_m, where $\bar{y}$ is the image of y. Show that the action of C_m on C_{m^2} is given by $\bar{y} \circ x = x^{m+1}$. Use Exercise VI. 7.6 to compute the E_1 term of the Lyndon-Hochschild-Serre spectral sequence in homology for the extension $(*)$. Conclude that $H_n(G) = 0$ for $0 < n < 2m-1$, n even, and that, for $0 < n < 2m-1$, n odd, there is an exact sequence $\mathbb{Z}_m \rightarrowtail H_n(G) \twoheadrightarrow \mathbb{Z}_m$. Show that for $n = 2m-1$ this latter result is not true.

IX. Satellites and Homology

Introduction

In Chapters VI and VII we gave "concrete" applications of the theory of derived functors established in Chapter IV, namely to the category of groups and the category of Lie algebras over a field K. In this chapter our first purpose is to broaden the setting in which a theory of derived functors may be developed. This more general theory is called *relative homological algebra*, the relativization consisting of replacing the class of *all* epimorphisms (monomorphisms) by a suitable subclass in defining the notion of *projective (injective)* object. An important example of such a relativization, which we discuss explicitly, consists in taking, as our *projective class* of epimorphisms in the category $\mathfrak{M}_\Lambda$ of Λ-modules, those epimorphisms which split as abelian group homomorphisms.

The theory of (left) *satellites* of a given additive functor $H:\mathfrak{A}\to\mathfrak{B}$ between abelian categories, with respect to a projective class $\mathscr{E}$ of epimorphisms in $\mathfrak{A}$, is developed in Section 3, and it is shown that if H is right $\mathscr{E}$-exact, then these satellites coincide with the *left derived functors* of H, again taken relative to the class $\mathscr{E}$, as defined in Section 2. Examples are given in Section 4.

In the second half of the chapter we embark on a further, and more ambitious, generalization of the theory. We associate with functors $T:\mathfrak{U}\to\mathfrak{A}$, $J:\mathfrak{U}\to\mathfrak{B}$, where $\mathfrak{U}$, $\mathfrak{B}$ are small categories and $\mathfrak{A}$ is abelian, objects $H_n(J,T)$ of the functor category $[\mathfrak{B},\mathfrak{A}]$ which deserve to be called the (absolute) homology of J with coefficients in T. This is achieved by taking satellites of the Kan extension $\tilde{J}$ evaluated at T, so that some category-theoretical preparation is needed in order to develop these ideas. Relative J-homology may also be defined by prescribing projective classes of epimorphisms in the functor category $[\mathfrak{U},\mathfrak{A}]$. Examples are given in the final section to show how this notion of J-homology generalizes the examples of homology theories already discussed in this book; moreover, the Grothendieck spectral sequence, described in Chapter VIII, is applied to this very general situation to yield, by further specialization, the Lyndon-Hochschild-Serre spectral sequence.

The chapter closes with indications of further applications of the idea of J-homology. We mention, for example, the homology theory of commutative K-algebras, which we regard as an example of this type of homology theory. However, we do not enter into the set-theoretical questions which arise if, as in this case, the categories $\mathfrak{U}$, $\mathfrak{B}$ can no longer be assumed to be small. The exercises at the end of the final section are, in the main, concerned with further applications of the theory and should be regarded as suggesting directions for further reading beyond the scope of this book.

1. Projective Classes of Epimorphisms

Let $\mathfrak{A}$ be an abelian category and let $\mathscr{E}$ be a class of epimorphisms in $\mathfrak{A}$.

Definition. The object P in $\mathfrak{A}$ is called *projective* rel ε, where $\varepsilon: B \to C$ is an epimorphism in $\mathfrak{A}$, if $\varepsilon_*: \mathfrak{A}(P, B) \to \mathfrak{A}(P, C)$ is surjective. P is called *$\mathscr{E}$-projective* if it is projective rel ε for every ε in $\mathscr{E}$.

It is clear that $P_1 \oplus P_2$ is projective rel ε if, and only if, both P_1 and P_2 are projective rel ε.

Definition. The *closure*, $C(\mathscr{E})$, of $\mathscr{E}$, consists of those epimorphisms ε in $\mathfrak{A}$ such that every $\mathscr{E}$-projective object of $\mathfrak{A}$ is also projective rel ε. Plainly $\mathscr{E} \subseteqq C(\mathscr{E})$ and $C(C(\mathscr{E})) = C(\mathscr{E})$. The class $\mathscr{E}$ is *closed* if $\mathscr{E} = C(\mathscr{E})$.

We will henceforth be mainly concerned with closed classes of epimorphisms (though we will often have to prove that our classes are closed). We note the following elementary results.

Proposition 1.1. *A closed class of epimorphisms is closed under composition and direct sums.* $\square$

Proposition 1.2. *A closed class of epimorphisms contains every projection $\pi: A \oplus B \twoheadrightarrow A$.*

Proof. Every object is projective rel π. $\square$

Of course, Proposition 1.2 includes the fact that a closed class contains all isomorphisms and the maps $B \to 0$.

The following are important examples of classes of epimorphisms in the category $\mathfrak{M}_\Lambda$ of (left) Λ-modules.

(a) $\mathscr{E}_0$, *the class of all split epimorphisms.* This is obviously a closed class; for every object is $\mathscr{E}_0$-projective, and if $\varepsilon: B \to C$ is not split, then C is plainly not projective rel ε.

(b) $\mathscr{E}_1$, *the class of all epimorphisms in* $\mathfrak{M}_\Lambda$. This is, even more obviously, a closed class; and the $\mathscr{E}_1$-projectives are precisely the projectives.

(c) $\mathscr{E}_2$, *the class of all epimorphisms in* $\mathfrak{M}_\Lambda$ *which split as epimorphisms of abelian groups.* This is, much less obviously, a closed class. We leave the proof to the reader, with the hint that, for any abelian group G, the Λ-module $\Lambda \otimes G$ is $\mathscr{E}_2$-projective (see also example (b) in Section 4).

(d) $\mathscr{E}_3$, *the class of all pure epimorphisms of abelian groups.* Recall that an epimorphism $\varepsilon: B \to C$ is pure if, and only if, given any n, then to every $c \in C$ with $nc = 0$ there exists $b \in B$ with $\varepsilon b = c$ and $nb = 0$ (see Exercise I.1.7). We leave it to the reader to show that P is $\mathscr{E}_3$-projective if, and only if, P is a direct sum of cyclic groups, and that the class $\mathscr{E}_3$ is closed. (For the "only if" part in that statement, one needs the deep result that a subgroup of a direct sum of cyclic groups is again a direct sum of cyclic groups [19].)

Definition. Let $\mathscr{E}$ be a closed class of epimorphisms in $\mathfrak{A}$. A morphism φ in $\mathfrak{A}$ is *$\mathscr{E}$-admissible* if, in the canonical splitting $\varphi = \mu\varepsilon$, μ monic, ε epic, we have $\varepsilon \in \mathscr{E}$. An exact sequence in $\mathfrak{A}$ is *$\mathscr{E}$-exact* if all its morphisms are $\mathscr{E}$-admissible. A complex in $\mathfrak{A}$,

$$\boldsymbol{K}: \cdots \to K_n \to K_{n-1} \to \cdots \to K_0$$

is called *$\mathscr{E}$-projective* if each K_n is $\mathscr{E}$-projective; $\boldsymbol{K}$ is called *$\mathscr{E}$-acyclic* if the augmented complex

$$\cdots \to K_n \to K_{n-1} \to \cdots \to K_0 \to H_0(\boldsymbol{K}) \to 0$$

is $\mathscr{E}$-exact. $\boldsymbol{K}$ is an *$\mathscr{E}$-projective resolution* of A if it is $\mathscr{E}$-projective, $\mathscr{E}$-acyclic, and $H_0(\boldsymbol{K}) \cong A$.

The following comparison theorem is an obvious generalization of Theorem IV.4.1; we omit the proof for this reason.

Theorem 1.3. *Let* $\boldsymbol{K}: \cdots \to K_n \to K_{n-1} \to \cdots \to K_0$, *and*

$$\boldsymbol{L}: \cdots \to L_n \to L_{n-1} \to \cdots \to L_0$$

be two complexes in $\mathfrak{A}$. *If* $\boldsymbol{K}$ *is $\mathscr{E}$-projective and* $\boldsymbol{L}$ *is $\mathscr{E}$-acyclic, then every morphism* $\varphi: H_0(\boldsymbol{K}) \to H_0(\boldsymbol{L})$ *lifts to a morphism of complexes* $\boldsymbol{\varphi}: \boldsymbol{K} \to \boldsymbol{L}$ *whose homotopy class is uniquely determined.* □

Definition. A closed class $\mathscr{E}$ of epimorphisms in $\mathfrak{A}$ is said to be *projective* if, to each object A of $\mathfrak{A}$ there is an epimorphism $\varepsilon: P \to A$ in $\mathscr{E}$ with P $\mathscr{E}$-projective. If $K \overset{\mu}{\rightarrowtail} P$ is the kernel of ε, we call $K \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ an *$\mathscr{E}$-projective presentation* of A.

Obviously, if $\mathscr{E}$ is a projective class, every object admits an $\mathscr{E}$-projective resolution.

All the notions of this section may plainly be dualized to a consideration of classes $\mathscr{M}$ of *monomorphisms* in $\mathfrak{A}$ leading finally to the notion of *injective* classes of monomorphisms. We leave the explicit formulations to the reader.

Finally we remark that a class $\mathscr{E}$ of epimorphisms gives rise in a natural way to a class $\mathscr{M}$ of monomorphisms, namely the class consisting of all kernels of epimorphisms in $\mathscr{E}$.

Exercises:

1.1. Prove that the class $\mathscr{E}_2$ in $\mathfrak{M}_\Lambda$ is closed.
1.2. Prove Theorem 1.3.
1.3. Suppose that $\mathscr{E}$ is a projective class of epimorphisms. By analogy with Theorem I. 4.7, give different characterizations for P to be $\mathscr{E}$-projective.
1.4. Prove the facts claimed in example (d). Dualize.
1.5. Let $\gamma: \Lambda \to \Lambda'$ be a ring homomorphism. Let $U^\gamma: \mathfrak{M}_{\Lambda'} \to \mathfrak{M}_\Lambda$ be the change-of-rings functor (see Section IV.12). Define a class $\mathscr{E}'$ of epimorphisms in $\mathfrak{M}_{\Lambda'}$ as follows: The epimorphism $\varepsilon': B' \to C'$ is in $\mathscr{E}'$ if, and only if, the homomorphism $U^\gamma \varepsilon': U^\gamma B' \to U^\gamma C'$ in $\mathfrak{M}_\Lambda$ splits. Is the class $\mathscr{E}'$ projective?
1.6. Interpret the $\mathscr{E}$-projectives, for $\mathscr{E}$ a projective class, as ordinary projectives in a suitable category.
1.7. Identify relative projective G-modules (see Section VI. 11) as $\mathscr{E}$-projectives for a suitable class $\mathscr{E}$ of epimorphisms. Do a similar exercise for relative injective G-modules.

2. $\mathscr{E}$-Derived Functors

We now imitate the development in Chapter IV. Let $\mathfrak{A}$, $\mathfrak{B}$ be abelian categories, let $T: \mathfrak{A} \to \mathfrak{B}$ be an additive functor and let $\mathscr{E}$ be a projective class in $\mathfrak{A}$. Given an object A in $\mathfrak{A}$, let $\boldsymbol{K}_A$ be an $\mathscr{E}$-projective resolution of A. Then Theorem 1.3 enables us to infer that the object $H_n(T\boldsymbol{K}_A)$ depends only on A and yields an additive functor $\mathfrak{A} \to \mathfrak{B}$ which we call the n^{th} *left $\mathscr{E}$-derived functor of T*, and write $L_n^{\mathscr{E}} T$, or merely $L_n T$ if the context ensures there is no ambiguity.

The development now proceeds just as in the "absolute" case ($\mathscr{E} = \mathscr{E}_1$); we will only be explicit when the relativization introduces a complication into the argument. This occurs in obtaining the first of the two basic exact sequences.

Theorem 2.1. *Let $0 \to A' \to A \to A'' \to 0$ be a short $\mathscr{E}$-exact sequence in $\mathfrak{A}$. Then, for any additive functor $T: \mathfrak{A} \to \mathfrak{B}$ there exist connecting homomorphisms $\omega_n: L_n^{\mathscr{E}} T(A'') \to L_{n-1}^{\mathscr{E}} T(A')$ such that the sequence*

$$\cdots \to L_n T(A') \to L_n T(A) \to L_n T(A'') \xrightarrow{\omega_n} L_{n-1} T(A') \to \cdots$$
$$\cdots \to L_0 T(A') \to L_0 T(A) \to L_0 T(A'') \to 0$$

is exact.

Proof. As in the absolute case, the proof hinges on the following key lemma.

Lemma 2.2. *The short $\mathscr{E}$-exact sequence* $0 \to A' \xrightarrow{\alpha'} A \xrightarrow{\alpha''} A'' \to 0$ *may be embedded in a diagram*

$$\begin{tikzcd} K' \arrow[r, tail, "\kappa'"] \arrow[d, tail, "\mu'"] & K \arrow[r, two heads, "\kappa''"] \arrow[d, tail, "\mu"] & K'' \arrow[d, tail, "\mu''"] \\ P' \arrow[r, tail, "\lambda'"] \arrow[d, "\varepsilon'"] & P \arrow[r, two heads, "\lambda''"] \arrow[d, "\varepsilon"] & P'' \arrow[d, "\varepsilon''"] \\ A' \arrow[r, tail, "\alpha'"] & A \arrow[r, two heads, "\alpha''"] & A'' \end{tikzcd} \tag{2.1}$$

with all rows and columns $\mathscr{E}$-exact and P', P, P'' $\mathscr{E}$-projective.

Proof of Lemma. The construction of (2.1) is exactly as in the absolute case. Thus we take $\mathscr{E}$-projective presentations of A' and A''; set $P = P' \oplus P''$, λ' being the injection and λ'' the projection; define ε as $\langle \alpha'\varepsilon', \theta \rangle$, where $\theta : P'' \to A$ lifts ε'', so that $\alpha''\theta = \varepsilon''$; and μ is the kernel of ε. The extra points requiring verification are (i) $\varepsilon \in \mathscr{E}$, (ii) $\kappa'' \in \mathscr{E}$; it is, of course, trivial (see Proposition 1.2) that $\lambda'' \in \mathscr{E}$.

In proving (i) and (ii) we must, of course, use the fact that $\mathscr{E}$ is closed. Thus we suppose Q $\mathscr{E}$-projective and seek to lift an arbitrary morphism $\varphi : Q \to A$ into P,

$$\begin{tikzcd} & Q \arrow[dl, dotted, "\psi"'] \arrow[d, "\varphi"] \\ P \arrow[r, two heads, "\varepsilon"] & A\,. \end{tikzcd}$$

Equivalently, we seek $\psi' : Q \to P'$, $\psi'' : Q \to P''$, such that $\alpha'\varepsilon'\psi' + \theta\psi'' = \varphi$. Now since ε'' is in $\mathscr{E}$, we may lift $\alpha''\varphi : Q \to A''$ into P'', that is, we find $\psi'' : Q \to P''$ with $\varepsilon''\psi'' = \alpha''\varphi$. Let ι embed P'' in P. Then $\lambda''\iota = 1$, $\varepsilon\iota = \theta$, so $\alpha''\varphi = \varepsilon''\psi'' = \varepsilon''\lambda''\iota\psi'' = \alpha''\varepsilon\iota\psi'' = \alpha''\theta\psi''$. Thus $\varphi = \alpha'\varrho + \theta\psi''$, $\varrho : Q \to A'$. Since ε' is in $\mathscr{E}$, we may lift ϱ into P', that is, we find $\psi' : Q \to P'$ with $\varepsilon'\psi' = \varrho$. Then $\varphi = \alpha'\varepsilon'\psi' + \theta\psi''$ and (i) is proved.

To prove (ii), we again suppose Q $\mathscr{E}$-projective and consider the lifting problem

$$\begin{tikzcd} & Q \arrow[dl, dotted, "\psi"'] \arrow[d, "\varphi"] \\ K \arrow[r, two heads, "\kappa''"] & K''\,. \end{tikzcd}$$

We first lift $\mu''\varphi$ into P, that is, we find $\sigma : Q \to P$ with $\lambda''\sigma = \mu''\varphi$. Then $\alpha''\varepsilon\sigma = \varepsilon''\lambda''\sigma = \varepsilon''\mu''\varphi = 0$, so that $\varepsilon\sigma = \alpha'\tau$, $\tau : Q \to A'$. Since ε' is in $\mathscr{E}$, we lift τ to $\tau' : Q \to P'$ with $\varepsilon'\tau' = \tau$. Set $\bar\sigma = \sigma - \lambda'\tau'$. Then $\lambda''\bar\sigma = \mu''\varphi$ and $\varepsilon\bar\sigma = \varepsilon\sigma - \varepsilon\lambda'\tau' = \varepsilon\sigma - \alpha'\varepsilon'\tau' = \alpha'\tau - \alpha'\tau = 0$. Thus $\bar\sigma = \mu\psi$ with $\psi : Q \to K$. Finally, $\mu''\kappa''\psi = \lambda''\mu\psi = \lambda''\bar\sigma = \mu''\varphi$, so $\kappa''\psi = \varphi$ and (ii) is proved. □

The reader should now have no difficulty in deriving Theorem 2.1 from Lemma 2.2. We state the other basic exactness theorem without proof.

Theorem 2.3. *Let* $0 \to T' \to T \to T'' \to 0$ *be a sequence of additive functors* $\mathfrak{A} \to \mathfrak{B}$ *which is* $\mathscr{E}$*-exact on* $\mathscr{E}$*-projectives. Then, for any object* A *in* $\mathfrak{A}$*, there exist connecting homomorphisms* $\omega_n : L_n^{\mathscr{E}} T''(A) \to L_{n-1}^{\mathscr{E}} T'(A)$ *such that the sequence*

$$\cdots \to L_n T'(A) \to L_n T(A) \to L_n T''(A) \xrightarrow{\omega_n} L_{n-1} T'(A) \to \cdots$$

$$\cdots \to L_0 T'(A) \to L_0 T(A) \to L_0 T''(A) \to 0$$

is exact. □

Definition. A functor $T : \mathfrak{A} \to \mathfrak{B}$ is called *right* $\mathscr{E}$*-exact* if, for every $\mathscr{E}$-exact sequence

$$A' \to A \to A'' \to 0$$

the sequence $TA' \to TA \to TA'' \to 0$ is exact.

Proposition 2.4. *A right* $\mathscr{E}$*-exact functor is additive.*

Proof. Since zero objects of $\mathfrak{A}$ are precisely those A such that $A \xrightarrow{1} A \xrightarrow{1} A \to 0$ is exact (and hence $\mathscr{E}$-exact), it follows that if T is right $\mathscr{E}$-exact then $T(0) = 0$. The proof is now easily completed as in the absolute case by considering the $\mathscr{E}$-exact sequence

$$0 \to A \to A \oplus B \to B \to 0. \quad \square$$

Proposition 2.5. *T is right $\mathscr{E}$-exact if, for every short $\mathscr{E}$-exact sequence* $0 \to A' \to A \to A'' \to 0$*, the sequence* $TA' \to TA \to TA'' \to 0$ *is exact.* □

Proposition 2.6. *For any additive functor T, $L_0 T$ is right $\mathscr{E}$-exact.*

Proof. Apply Proposition 2.5 and Theorem 2.1. □

Theorem 2.7. *For any additive functor* $T : \mathfrak{A} \to \mathfrak{B}$ *there is a natural transformation* $\tau : L_0 T \to T$ *which is an equivalence if, and only if, T is right $\mathscr{E}$-exact.*

Proof. Let $\cdots \to P_1 \to P_0$ be an $\mathscr{E}$-projective resolution of A. Then

$$TP_1 \to TP_0 \to L_0 T(A) \to 0$$

is exact, by definition; and

$$TP_1 \to TP_0 \to TA$$

is differential. This yields $\tau_A : L_0 T(A) \to TA$. The standard argument now yields the independence of τ_A of the choice of resolution and the fact that τ is natural. If T is right $\mathscr{E}$-exact, then $TP_1 \to TP_0 \to TA \to 0$ is exact, so that τ is an equivalence. The converse follows immediately from Proposition 2.6, so the theorem is proved. $\square$

We will content ourselves here with just one example, but will give many more examples in Section 4. Consider the projective class $\mathscr{E}_3$ of pure epimorphisms in the category of abelian groups (see example (d) in Section 2). It may then be shown that the left $\mathscr{E}_3$-derived functors $L_n(A\otimes -)$ are trivial for $n\geqq 1$. Taking $T=\mathrm{Hom}(-,B)$ as base functor, we can construct the right $\mathscr{E}_3$-derived functors $R_n T$. It turns out that $R_n^{\mathscr{E}_3} T$ is trivial for $n\geqq 2$. Define $\mathrm{Pext}(-,B)=R_1^{\mathscr{E}_3} T(-)$. Then, if $0\to A'\to A\to A''\to 0$ is a pure exact sequence, we have exact sequences $0\to \mathrm{Hom}(A'',B)\to \mathrm{Hom}(A,B)\to \mathrm{Hom}(A',B)\to \mathrm{Pext}(A'',B)\to \mathrm{Pext}(A,B)\to \mathrm{Pext}(A',B)\to 0$; $0\to \mathrm{Hom}(B,A')\to \mathrm{Hom}(B,A)\to \mathrm{Hom}(B,A'')\to \mathrm{Pext}(B,A')\to \mathrm{Pext}(B,A)\to \mathrm{Pext}(B,A'')\to 0$.

We remark again that everything we have done here is readily dualizable to *right* $\mathscr{M}$-derived functors, based on an *injective* class $\mathscr{M}$ of monomorphisms. The reader should certainly formulate the theorems dual to Theorems 2.1, 2.3.

Exercises:

2.1. Prove Theorems 2.1, 2.3.
2.2. Evaluate $L_n^{\mathscr{E}} TP$ where T is additive and P is $\mathscr{E}$-projective.
2.3. What is $L_m^{\mathscr{E}} L_n^{\mathscr{E}} T$?
2.4. Show that $L_m^{\mathscr{E}} T$ is additive.
2.5. Prove the analog of Proposition IV. 5.5.
2.6. Compute for the class $\mathscr{E}_0$ in $\mathfrak{M}_\Lambda$ the functors $R^n \mathrm{Hom}_\Lambda(-,B)$.
2.7. Prove the assertions made in discussing the example relating to $\mathscr{E}_3$ at the end of the section.
2.8. Prove, along the lines of Theorem III. 2.4, that $\mathrm{Pext}(A,B)$ classifies pure extensions.

3. $\mathscr{E}$-Satellites

Let $\mathscr{E}$ again be a projective class of epimorphisms in the abelian category $\mathfrak{A}$ and let $\mathfrak{B}$ be an abelian category.

Definition. An *$\mathscr{E}$-connected sequence of functors* $\boldsymbol{T}=\{T_j\}$ from $\mathfrak{A}$ to $\mathfrak{B}$ consists of

(i) additive functors $T_j:\mathfrak{A}\to\mathfrak{B}$, $j=\cdots,-1,0,1,\ldots$,

(ii) connecting morphisms $\omega_j: T_j(A'')\to T_{j-1}(A')$, $j=\cdots,-1,0,1,\ldots$, corresponding to each short $\mathscr{E}$-exact sequence $0\to A'\to A\to A''\to 0$, which are natural in the obvious sense (i.e., the ω_j are natural transformations of functors from the category of short $\mathscr{E}$-exact sequences in $\mathfrak{A}$ to the category of morphisms in $\mathfrak{B}$).

Proposition 3.1. *If* $\boldsymbol{T}$ *is an* $\mathscr{E}$*-connected sequence of functors, the sequence*

$$\cdots \to T_j(A') \to T_j(A) \to T_j(A'') \xrightarrow{\omega_j} T_{j-1}(A') \to T_{j-1}(A) \to \cdots$$

is differential for each short $\mathscr{E}$*-exact sequence* $0 \to A' \to A \to A'' \to 0$ *in* $\mathfrak{A}$.

Proof. The sequence is certainly differential at $T_j(A)$. That it is differential at $T_j(A'')$ follows by naturality from the diagram

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] & A \arrow[r] & A'' \arrow[r] & 0\,; \end{tikzcd}$$

that it is differential at $T_{j-1}(A')$ follows by naturality from the diagram

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & A'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] & A \arrow[r] & 0 \arrow[r] & 0\,. \end{tikzcd} \qquad \square$$

An example of an $\mathscr{E}$-connected sequence of functors is afforded by the left $\mathscr{E}$-derived functors of an additive functor S (we set $L_j S = 0, j < 0$).

It is clear what we should understand by a *morphism* of $\mathscr{E}$-connected sequences of functors $\boldsymbol{\varphi}: \boldsymbol{T} \to \boldsymbol{T}'$; it consists of natural transformations $\varphi_j : T_j \to T_j'$, $j = \cdots, -1, 0, 1, \ldots$ such that, for every $\mathscr{E}$-exact sequence $0 \to A' \to A \to A'' \to 0$, the square

$$\begin{tikzcd} T_j(A'') \arrow[r, "\omega_j"] \arrow[d, "\varphi_j"] & T_{j-1}(A') \arrow[d, "\varphi_{j-1}"] \\ T_j'(A'') \arrow[r, "\omega_j'"] & T_{j-1}'(A') \end{tikzcd}$$

commutes for all j. We are now ready for our main definition of this section.

Definition. Let $H: \mathfrak{A} \to \mathfrak{B}$ be an additive functor. An $\mathscr{E}$-connected sequence of functors $\boldsymbol{S} = \{S_j\}$, with $S_0 = H$, is called the *left* $\mathscr{E}$*-satellite* of H if it satisfies the following universal property:

To every $\mathscr{E}$-connected sequence of functors $\boldsymbol{T}$ and every natural transformation $\varphi: T_0 \to S_0$ there exists a unique morphism $\boldsymbol{\varphi}: \boldsymbol{T} \to \boldsymbol{S}$ with $\varphi_0 = \varphi$.

We immediately remark that, since the left $\mathscr{E}$-satellite is defined by a universal property, it is unique up to canonical isomorphism. We may thus write $S_j = S_j^{\mathscr{E}} H$ if the satellite exists (we may suppress $\mathscr{E}$ if the context permits). We also remark that it follows from the definition of a left satellite that $S_j = 0$ for $j < 0$. For, given a left $\mathscr{E}$-satellite $\boldsymbol{S}$, we define a connected sequence of functors $\boldsymbol{S}' = \{S'_j\}$ by

$$S'_j = \begin{cases} S_j, & j \geqq 0 \\ 0, & j < 0 \end{cases} \qquad \omega'_j = \begin{cases} \omega_j, & j > 0 \\ 0, & j \leqq 0. \end{cases}$$

It is then plain that $\boldsymbol{S}'$ also satisfies the universal property of a left satellite. Hence, by uniqueness $\boldsymbol{S} = \boldsymbol{S}'$ and $S_j = 0$, $j < 0$. We next take up the question of existence of left satellites.

Theorem 3.2. *If $H: \mathfrak{A} \to \mathfrak{B}$ is a right $\mathscr{E}$-exact functor, where $\mathscr{E}$ is a projective class of epimorphisms in $\mathfrak{A}$, then the $\mathscr{E}$-connected sequence of functors $\{L_j^{\mathscr{E}} H\}$ is the left $\mathscr{E}$-satellite of H, $L_j^{\mathscr{E}} H = S_j^{\mathscr{E}} H$ ($L_j^{\mathscr{E}} H = 0, j < 0$).*

Proof. Since H is right $\mathscr{E}$-exact, $H = L_0 H$. Thus we suppose given an $\mathscr{E}$-connected sequence of functors $\boldsymbol{T}$ and a natural transformation $\varphi: T_0 \to L_0 H$ and have to show that there exist unique natural transformations $\varphi_j: T_j \to L_j H$ with $\varphi_0 = \varphi$, such that the diagram

$$\begin{array}{ccc} T_j(A'') & \xrightarrow{\omega_j} & T_{j-1}(A') \\ \downarrow{\scriptstyle \varphi_j} & & \downarrow{\scriptstyle \varphi_{j-1}} \\ L_j H(A'') & \xrightarrow{\omega_j} & L_{j-1} H(A') \end{array} \tag{3.1}$$

commutes for all short $\mathscr{E}$-exact sequences $0 \to A' \to A \to A'' \to 0$.

We first remark that for $j < 0$, φ_j is the trivial map. For $j = 0$ we have $\varphi_0 = \varphi$, and, for $j > 0$, we define φ_j inductively. Thus we suppose φ_k defined for $k \leqq j$, $j \geqq 0$, to commute with ω_k as in (3.1), and we proceed to define φ_{j+1}. Let $0 \to K \to P \to A \to 0$ be an $\mathscr{E}$-projective presentation of A. Then we have a commutative diagram

$$\begin{array}{ccccccccc} \cdots \to T_{j+1} P & \longrightarrow & T_{j+1} A & \xrightarrow{\omega_{j+1}} & T_j K & \longrightarrow & T_j P & \to \cdots \\ & & \vdots & & \downarrow{\scriptstyle \varphi_j} & & \downarrow{\scriptstyle \varphi_j} & \\ 0 & \longrightarrow & L_{j+1} H A & \xrightarrow{\omega_{j+1}} & L_j H K & \longrightarrow & L_j H P & \to \cdots, \end{array}$$

with the bottom row exact. This yields a unique candidate for $\varphi_{j+1}: T_{j+1} A \to L_{j+1} H A$. We prove that φ_{j+1} is a natural transformation, independent of the choice of presentation, in the following lemma.

Lemma 3.3. *Let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & K' & \longrightarrow & P' & \longrightarrow & A' & \longrightarrow & 0 \end{array}$$

be a morphism of $\mathscr{E}$-projective presentations. Let $\varphi_{j+1}: T_{j+1} A \to L_{j+1} H A$ be defined by means of the top row, $\varphi'_{j+1}: T_{j+1} A' \to L_{j+1} H A'$ by means *of the bottom row. Then the diagram*

$$\begin{tikzcd} T_{j+1}A \arrow[r, "T_{j+1}\alpha"] \arrow[d, "\varphi_{j+1}"] & T_{j+1}A' \arrow[d, "\varphi'_{j+1}"] \\ L_{j+1}HA \arrow[r, "L_{j+1}H\alpha"] & L_{j+1}HA' \end{tikzcd} \tag{3.2}$$

commutes.

Proof of Lemma. Embed (3.2) in the cube

$$\begin{tikzcd} T_{j+1}A \arrow[rr] \arrow[dr, "T_{j+1}\alpha"] \arrow[dd, "\varphi_{j+1}"] & & T_jK \arrow[dr] \arrow[dd] & \\ & T_{j+1}A' \arrow[rr] \arrow[dd, "\varphi'_{j+1}"] & & T_jK' \arrow[dd] \\ L_{j+1}HA \arrow[rr] \arrow[dr, "L_{j+1}H\alpha"] & & L_jHK \arrow[dr] & \\ & L_{j+1}HA' \arrow[rr, "\omega_{j+1}"] & & L_jHK' \end{tikzcd}$$

All remaining faces of the cube commute and $\omega_{j+1}: L_{j+1}HA' \to L_jHK'$ is monomorphic. Thus the face (3.2) also commutes. $\square$

It remains to establish that the definition of φ_{j+1} yields commutativity in the square

$$\begin{tikzcd} T_{j+1}A'' \arrow[r, "\omega_{j+1}"] \arrow[d, "\varphi_{j+1}"] & T_jA' \arrow[d, "\varphi_j"] \\ L_{j+1}HA'' \arrow[r, "\omega_{j+1}"] & L_jHA' \end{tikzcd} \tag{3.3}$$

corresponding to the short $\mathscr{E}$-exact sequence $0 \to A' \to A \to A'' \to 0$. Let

$$\begin{tikzcd} 0 \arrow[r] & K'' \arrow[r] \arrow[d] & P'' \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A' \arrow[r] & A \arrow[r] & A'' \arrow[r] & 0 \end{tikzcd} \tag{3.4}$$

be a morphism from an $\mathscr{E}$-projective presentation of A'' to the given $\mathscr{E}$-exact sequence, and consider the squares

$$\begin{tikzcd} T_{j+1}A'' \arrow[r] \arrow[d, "\varphi_{j+1}"] & T_jK'' \arrow[d, "\varphi_j"] \\ L_{j+1}HA'' \arrow[r] & L_jHK'' \end{tikzcd} \qquad \begin{tikzcd} T_jK'' \arrow[r] \arrow[d, "\varphi_j"] & T_jA' \arrow[d, "\varphi_j"] \\ L_jHK'' \arrow[r] & L_jHA' \end{tikzcd} \tag{3.5}$$

The first commutes by definition of φ_{j+1}, the second by the naturality of φ_j. Moreover, the naturality of ω_{j+1} in the definition of an $\mathscr{E}$-connected sequence of functors, applied to (3.4), ensures that the composite of the two squares (3.5) is just (3.3). This completes the proof of the theorem. □

Corollary 3.4. *If $H:\mathfrak{A}\to\mathfrak{B}$ is an additive functor, then*

$$S_j^{\mathscr{E}}(L_0 H)=L_j^{\mathscr{E}}H\,,\quad j\geqq 0\,.$$

Proof. It is sufficient to observe that $L_0 H$ is right $\mathscr{E}$-exact and $L_j L_0 H = L_j H$, since $L_0 HP = HP$ for any $\mathscr{E}$-projective P. □

We remark that we have not established the existence of $\mathscr{E}$-satellites of arbitrary additive functors, nor have we established the existence of $\mathscr{E}$-satellites of right $\mathscr{E}$-exact functors if $\mathscr{E}$ is merely supposed to be a *closed* class of epimorphisms in $\mathfrak{A}$ (the *definitions* of this section make perfectly good sense without supposing $\mathscr{E}$ to be a projective class). This second question is reminiscent of the discussion in Chapter IV of characterizations of derived functors without the use of projectives. A discussion of the question may be found in Buchsbaum [5]; see Exercise 3.3.

Again, we may dualize. Here we should be somewhat explicit as the notational convention relating to $\mathscr{M}$-connected sequences of functors has the connecting morphisms ω_j from the domain $T_j A''$ to the codomain $T_{j+1}A'$ (instead of $T_{j-1}A'$). The dual of Theorem 3.2 then reads

Theorem 3.5. *If $H:\mathfrak{A}\to\mathfrak{B}$ is a left $\mathscr{M}$-exact functor, where $\mathscr{M}$ is an injective class of monomorphisms in $\mathfrak{A}$, then the $\mathscr{M}$-connected sequence of functors $\{R_j^{\mathscr{M}}H\}$ is the right $\mathscr{M}$-satellite of H, $(R_j^{\mathscr{M}}H=0, j<0)$. That is,*

$$R_j^{\mathscr{M}}H=S_j^{\mathscr{M}}H\,.\quad □$$

We will also have need of a contravariant form. Obviously a projective class $\mathscr{E}$ in $\mathfrak{A}$ gives rise to an injective class $\mathscr{E}^*$ in $\mathfrak{A}^{\mathrm{opp}}$. If $H:\mathfrak{A}\to\mathfrak{B}$ is *contravariant*, then we regard H as a functor $\mathfrak{A}^{\mathrm{opp}}\to\mathfrak{B}$ and if H is left $\mathscr{E}^*$-exact, we infer that $\{R_j^{\mathscr{E}^*}H\}$ is the right $\mathscr{E}^*$-satellite of H. Note that $R_j H$ is defined by means of an $\mathscr{E}^*$-injective resolution in $\mathfrak{A}^{\mathrm{opp}}$, that is, by means of an $\mathscr{E}$-projective resolution in $\mathfrak{A}$.

Exercises:

3.1. Using the projective class $\mathscr{E}=\mathscr{E}_1$ in $\mathfrak{M}_\Lambda$, show that $\mathrm{Ext}_\Lambda^n(-,B)$ is the right $\mathscr{E}^*$-satellite of $\mathrm{Hom}_\Lambda(-,B)$. Using the injective class $\mathscr{M}=\mathscr{M}_1$ of all monomorphisms in $\mathfrak{M}_\Lambda$, show that $\overline{\mathrm{Ext}}_\Lambda^n(A,-)$ is the right $\mathscr{M}$-satellite of $\mathrm{Hom}_\Lambda(A,-)$. Using the fact that $\overline{\mathrm{Ext}}_\Lambda^n(A,-)$ gives rise to a connected sequence also in the *first* variable, show that the universal property of $\mathrm{Ext}_\Lambda^n(-,B)$ yields a natural transformation

$$\eta:\overline{\mathrm{Ext}}_\Lambda^n(A,B)\to\mathrm{Ext}_\Lambda^n(A,B)\,.$$

Using the fact that $\overline{\mathrm{Ext}}^n_\Lambda(P, B)=0$ for all $\mathscr{E}$-projectives P, show that η is a natural equivalence.

3.2. Do a similar exercise to 3.1 for the bifunctor $-\otimes_\Lambda -$ instead of $\mathrm{Hom}_\Lambda(-,-)$.

3.3. (Definition of satellites following Buchsbaum [5].) Let $\mathfrak{A}, \mathfrak{B}$ be abelian categories, and let $H: \mathfrak{A} \to \mathfrak{B}$ be a (covariant) additive functor. Suppose that $\mathfrak{B}$ has limits. For A in $\mathfrak{A}$ consider the totality of all short exact sequences

$$E: 0 \to B \to E \to A \to 0 .$$

Define a partial ordering as follows: $E' < E$ if there exists $\varphi: E \to E'$ and $\psi: B \to B'$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, "\psi"] & E \arrow[r] \arrow[d, "\varphi"] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B' \arrow[r] & E' \arrow[r] & A \arrow[r] & 0 \end{tikzcd} \tag{$*$}$$

is commutative. Consider then $H_E = \ker(HB \to HE)$. Show that for $E' < E$ the map $H\psi: HB \to HB'$ induces a map $\theta^E_{E'}: H_E \to H_{E'}$, which is independent of the choice of φ and ψ in $(*)$. Define $S_1 A = \varprojlim (H_E, \theta^E_{E'})$. (Note that there is a set-theoretical difficulty, for the totality of sequences E need not be a set. Although this difficulty is not trivial, we do not want the reader to concern himself with it at this stage.) Show that S_1 is made into a covariant functor by the following procedure. For $\alpha: A \to \bar{A}$ and for $\bar{E}: 0 \to B \to \bar{E} \to \bar{A} \to 0$, show that there exists $E: 0 \to B \to E \to A \to 0$ such that there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\alpha"] & 0 \\ 0 \arrow[r] & B \arrow[r] & \bar{E} \arrow[r] & \bar{A} \arrow[r] & 0 . \end{tikzcd} \tag{$**$}$$

Using $(**)$ define maps $H_E \to H_{\bar{E}}$, and, passing to the limit, define a morphism $\alpha_*: S_1 A \to S_1 \bar{A}$. Show that with this definition S_1 becomes a functor. Starting with H, we thus have defined a functor $S_1 = S_1(H)$. Show that S_1 is additive. Setting $S_0 = H$, define $S_n(H) = S_1(S_{n-1})$, $n = 1, 2, \ldots$.

Given a short exact sequence

$$0 \to A' \to A \to A'' \to 0 ,$$

show that the definition of $S_1 A''$ yields a morphism $\omega = \omega_1: S_1 A'' \to HA'$. By induction define morphisms $\omega_n: S_n A'' \to S_{n-1} A'$, $n = 1, 2, \ldots$. Show that $S = (S_j, \omega_j)$ is an $\mathscr{E}$-connected sequence of functors, where $\mathscr{E} = \mathscr{E}_1$ is the class of all epimorphisms in $\mathfrak{A}$. Finally show that S has the universal property required of the left $\mathscr{E}$-satellite of H.

Dualize.

Consider the case of a contravariant functor.

Replace $\mathscr{E} = \mathscr{E}_1$ by other classes of epimorphisms in $\mathfrak{A}$.

3.4. Show that for H *not* right exact, the left satellite of H is *not* given by the left derived functor of H.

3.5. Give a form of the Grothendieck spectral sequence (Theorem VIII. 9.3) valid for $\mathscr{E}$-derived functors.

4. The Adjoint Theorem and Examples

For the definition of satellites and derived functors of functors $\mathfrak{A} \to \mathfrak{B}$ we had to specify a class of epimorphisms $\mathscr{E}$ in $\mathfrak{A}$. For the construction of derived functors it was essential that the class $\mathscr{E}$ be projective. We now discuss how the theory of adjoint functors may be used to transfer projective classes of epimorphisms from one category to another.

Let $\mathscr{E}$ be a projective class in the abelian category $\mathfrak{A}$ and let $F : \mathfrak{A} \to \mathfrak{A}'$, $U : \mathfrak{A}' \to \mathfrak{A}$, where $\mathfrak{A}'$ is also abelian, be a pair of adjoint functors. We will suppose U *faithful*, so that if $U\varepsilon$ is an epimorphism in $\mathfrak{A}$ then ε is an epimorphism in $\mathfrak{A}'$. In particular $\mathscr{E}' = U^{-1}\mathscr{E}$ will be a class of epimorphisms. We now prove the theorem which gives effect to the objective described in the first paragraph; we will then give various examples.

Theorem 4.1. *Under the hypotheses above, $\mathscr{E}' = U^{-1}\mathscr{E}$ is a projective class of epimorphisms in $\mathfrak{A}'$. The objects FP where P is $\mathscr{E}$-projective in $\mathfrak{A}$ are $\mathscr{E}'$-projective and are sufficient for $\mathscr{E}'$-presenting objects of $\mathfrak{A}'$, so that the $\mathscr{E}'$-projectives are precisely the direct summands of objects FP.*

Proof. First observe that F sends $\mathscr{E}$-projectives to $\mathscr{E}'$-projectives. For if P is $\mathscr{E}$-projective and $\varepsilon' : A_1' \twoheadrightarrow A_2'$ is in $\mathscr{E}'$, then $U\varepsilon' \in \mathscr{E}$, so that $(U\varepsilon')_* : \mathfrak{A}(P, UA_1') \twoheadrightarrow \mathfrak{A}(P, UA_2')$. But this means that

$$\varepsilon'_* : \mathfrak{A}'(FP, A_1') \twoheadrightarrow \mathfrak{A}'(FP, A_2')$$

so that FP is $\mathscr{E}'$-projective.

Next we prove that $\mathscr{E}'$ is closed. Thus we suppose given $\alpha' : A_1' \twoheadrightarrow A_2'$ such that, for any $\mathscr{E}'$-projective P', the map $\alpha'_* : \mathfrak{A}'(P', A_1') \twoheadrightarrow \mathfrak{A}'(P', A_2')$ is surjective. Take, in particular, $P' = FP$, where P is $\mathscr{E}$-projective. Then it follows that $(U\alpha')_* : \mathfrak{A}(P, UA_1') \twoheadrightarrow \mathfrak{A}(P, UA_2')$ for all $\mathscr{E}$-projectives P. Since $\mathscr{E}$ is a projective class it follows first that $U\alpha'$ is epimorphic and then that $U\alpha' \in \mathscr{E}$. Thus $\alpha' \in \mathscr{E}'$, so $\mathscr{E}'$ is closed.

Finally we prove that every A' may be $\mathscr{E}'$-presented. First we $\mathscr{E}$-present UA' by $P \overset{\varepsilon}{\twoheadrightarrow} UA'$, $\varepsilon \in \mathscr{E}$. Let $\varepsilon' : FP \to A'$ be adjoint to ε; it remains to show that $\varepsilon' \in \mathscr{E}'$. We have the diagram

$$P \xrightarrow{\delta} UFP \xrightarrow{U\varepsilon'} UA', \qquad U\varepsilon' \circ \delta = \varepsilon$$

where δ is the unit of the adjunction. Thus $U\varepsilon'$, and hence ε', is epimorphic. Also if $\varphi : Q \to UA'$ is a morphism of the $\mathscr{E}$-projective Q to UA', then φ may be lifted back to P and hence, *a fortiori*, to UFP. Thus, since $\mathscr{E}$ is closed, $U\varepsilon' \in \mathscr{E}$ so that $\varepsilon' \in \mathscr{E}'$. Finally we see that if A' is $\mathscr{E}'$-projective it is a direct summand of FP, i.e., there is $\iota : A' \to FP$ with $\varepsilon'\iota = 1$. $\square$

We may also appeal to the dual of Theorem 4.1. We now discuss examples. Our examples are related to the change of rings functor

$U = U^\gamma : \mathfrak{M}_{\Lambda'} \to \mathfrak{M}_\Lambda$ associated with a ring homomorphism $\gamma : \Lambda \to \Lambda'$ (see Section IV. 12). Thus, henceforth in this section, $\mathfrak{A} = \mathfrak{M}_\Lambda$, $\mathfrak{A}' = \mathfrak{M}_{\Lambda'}$. Recall that, if A' is a (left) Λ'-module, then UA' has the same underlying abelian group as A', the Λ-module structure being given by

$$\lambda a' = \gamma(\lambda) a' .$$

As a special case, $\Lambda = \mathbb{Z}$ and γ is given by $\gamma(1) = 1_{\Lambda'}$ (γ is called the *unit*). Then UA' simply forgets the Λ'-module structure of A' and retains the abelian group structure; we refer to this as the *forgetful case*.

In general, as we know, U has a left adjoint $F^l : \mathfrak{M}_\Lambda \to \mathfrak{M}_{\Lambda'}$, given by $F^l(A) = \Lambda' \otimes_\Lambda A$, and a right adjoint $F^r : \mathfrak{M}_\Lambda \to \mathfrak{M}_{\Lambda'}$, given by $F^r(A) = \mathrm{Hom}_\Lambda(\Lambda', A)$, $A \in \mathfrak{M}_\Lambda$. Thus U preserves monomorphisms and epimorphisms (obvious anyway); it is plain that U is faithful.

(a) Let $\mathscr{E} = \mathscr{E}_1$ be the class of all epimorphisms in $\mathfrak{M}_\Lambda$. Then $\mathscr{E}'_1$ is the class of all epimorphisms in $\mathfrak{M}_{\Lambda'}$ since U preserves epimorphisms. We observe that, by the argument of Theorem 4.1, we can present every Λ'-module by means of a module of the form $\Lambda' \otimes_\Lambda P$, where P is a projective Λ-module. If we take the functor $C \otimes_{\Lambda'} - : \mathfrak{M}_{\Lambda'} \to \mathfrak{Ab}$, the left $\mathscr{E}'_1$-satellite may be seen by Theorem 3.2 to be the connected sequence of functors $\mathrm{Tor}_n^{\Lambda'}(C, -)$.

(b) We get a more genuinely relative theory by taking $\mathscr{E} = \mathscr{E}_0$, the class of all split epimorphisms in $\mathfrak{M}_\Lambda$. Then $\mathscr{E}'_0$ consists of those epimorphisms in $\mathfrak{M}_{\Lambda'}$ which split as epimorphisms of Λ-modules. Thus, in the forgetful case, $\mathscr{E}'_0$ is the class $\mathscr{E}_2$ of Section 1. Of course, every Λ-module is $\mathscr{E}_0$-projective, so we may use the Λ-modules $\Lambda' \otimes_\Lambda B$ for $\mathscr{E}'_0$-projective presentations in $\mathfrak{M}_{\Lambda'}$. If we again take the functor $C \otimes_{\Lambda'} - : \mathfrak{M}_{\Lambda'} \to \mathfrak{Ab}$, the left $\mathscr{E}'_0$-satellite is computed by means of left $\mathscr{E}'_0$-derived functors and it is customary to denote these derived functors by $\mathrm{Tor}_n^\gamma(C, -)$ or, if $\gamma : \Lambda \to \Lambda'$ is an embedding, also by $\mathrm{Tor}_n^{(\Lambda', \Lambda)}(C, -)$. We obtain results for this relative Tor (exact sequences, balance between left and right), just as for the absolute Tor.

(c) Let $\mathscr{M} = \mathscr{M}_1$ be the class of all monomorphisms in $\mathfrak{M}_\Lambda$. This class is injective, and, since U preserves monomorphisms the class $\mathscr{M}'_1 = U^{-1}\mathscr{M}_1$ consists of all monomorphisms in $\mathfrak{M}_{\Lambda'}$. Thus, by the dual of Theorem 4.1 $\mathscr{M}'_1$ is injective. Note that, in the forgetful case, this implies that the Λ'-modules $\mathrm{Hom}(\Lambda', D)$, where D is a divisible abelian group, are injective, and also provide enough injectives in $\mathfrak{M}_{\Lambda'}$ (compare Theorem I. 8.2). Now consider the left $\mathscr{M}'_1$-exact functor

$$S = \mathrm{Hom}_{\Lambda'}(C, -) : \mathfrak{M}_{\Lambda'} \to \mathfrak{Ab} .$$

Then we may apply Theorem 3.5 to infer that the right $\mathscr{M}'_1$-satellite of $\mathrm{Hom}_{\Lambda'}(C, -)$ consists of the connected sequence of right-derived functors $\mathrm{Ext}_{\Lambda'}^n(C, -)$.

(d) Now let $\mathscr{M} = \mathscr{M}_0$, the class of all split monomorphisms in $\mathfrak{M}_\Lambda$. This is an injective class, and $\mathscr{M}'_0 = U^{-1}\mathscr{M}_0$ is the class of those monomorphisms in $\mathfrak{M}_{\Lambda'}$ which split as monomorphisms in $\mathfrak{M}_\Lambda$. Again, this is an injective class and we have enough $\mathscr{M}'_0$-injectives consisting of the Λ'-modules $\operatorname{Hom}_\Lambda(\Lambda', A)$, where A is an arbitrary Λ-module. We refer again to the functor $\operatorname{Hom}_{\Lambda'}(C, -): \mathfrak{M}_{\Lambda'} \to \mathfrak{Ab}$, which is, of course, left $\mathscr{M}'_0$-exact. Theorem 3.5 ensures that the right $\mathscr{M}'_0$-satellite of $\operatorname{Hom}_{\Lambda'}(C, -)$ consists of the connected sequences of right $\mathscr{M}'_0$-derived functors of $\operatorname{Hom}_{\Lambda'}(C, -)$, which we write $\operatorname{Ext}^n_\gamma(C, -)$ or, if $\gamma: \Lambda \to \Lambda'$ is an embedding, also by $\operatorname{Ext}^n_{(\Lambda', \Lambda)}(C, -)$. Again, the reader should check that these relative Ext groups have the usual properties; see also Example (e) below.

(e) Here we exploit the contravariant form of Theorem 3.2. We revert to the projective classes $\mathscr{E}_i$, $i = 0, 1$, of (a), (b) and now regard the projective class $\mathscr{E}'_i$ in $\mathfrak{M}_{\Lambda'}$ as an injective class $\mathscr{E}'^*_i$ in $\mathfrak{M}^{\text{opp}}_{\Lambda'}$. The contravariant functor $\operatorname{Hom}_{\Lambda'}(-, C): \mathfrak{M}_{\Lambda'} \to \mathfrak{Ab}$ is left exact. We may thus describe the right $\mathscr{E}'^*_i$-satellite of $\operatorname{Hom}_{\Lambda'}(-, C)$ in terms of the right $\mathscr{E}'^*_i$-derived functors of $\operatorname{Hom}_{\Lambda'}(-, C)$. If $i = 1$, we obtain the usual Ext functors, $\operatorname{Ext}^n_{\Lambda'}(-, C)$; if $i = 0$, we obtain the relative Ext functors denoted by $\operatorname{Ext}^n_\gamma(-, C)$ or, if γ is an embedding, by $\operatorname{Ext}^n_{(\Lambda', \Lambda)}(-, C)$.

Exercises:

4.1. In analogy with Exercise 3.1, prove a balance theorem for $\operatorname{Tor}^{(\Lambda, \Lambda')}_n(-, -)$ and $\operatorname{Ext}^n_{(\Lambda, \Lambda')}(-, -)$.

4.2. Attach a reasonable meaning to the symbols $\operatorname{Ext}^n_{\mathscr{E}_0}(-, B)$, $\operatorname{Tor}^{\mathscr{E}_0}_n(-, B)$.

4.3. Let $\mathscr{E}' \subseteqq \mathscr{E}$ be two projective classes. If $(L^{\mathscr{E}}_n T)\ P' = 0$ for $n \geqq 1$ and all $\mathscr{E}'$-projectives P' (P' is then called *$\mathscr{E}$-acyclic* for T), prove that $L^{\mathscr{E}}_n T = L^{\mathscr{E}'}_n T$ for all $n \geqq 1$. (Hint: Proceed by induction, using $\mathscr{E}'$-projective presentations.)

4.4. Use Exercise 4.3 to show that $\operatorname{Ext}^n_{\mathscr{M}_1}(\mathbb{Z}, -) = \operatorname{Ext}^n_{\mathscr{M}_2}(\mathbb{Z}, -)$ where $\mathscr{M}_1$ denotes the injective class of all monomorphisms in $\mathfrak{M}_G = \mathfrak{M}_{\mathbb{Z}G}$, and $\mathscr{M}_2$ denotes the injective class of all monomorphisms in $\mathfrak{M}_G$ which split as homomorphisms of abelian groups. Obtain a corresponding result with Ext replaced by Tor.

4.5. Generalize the argument of Exercise 4.4 by replacing the ring $\mathbb{Z}G$ by a suitable ring Λ. (One cannot generalize to arbitrary rings Λ!)

5. Kan Extensions and Homology

In this section we describe a very general procedure for obtaining homology theories; we will first give the abstract development and then illustrate with examples.

Let $\mathfrak{U}$, $\mathfrak{B}$ be two small categories and let $J: \mathfrak{U} \to \mathfrak{B}$ be a functor. Let $\mathfrak{A}$ be a category admitting colimits (for example $\mathfrak{Ab}, \mathfrak{M}_\Lambda$). Now given a functor $S: \mathfrak{B} \to \mathfrak{A}$, $S \circ J: \mathfrak{U} \to \mathfrak{A}$ is a functor so that J induces a functor between functor categories.

$$J^* : [\mathfrak{B}, \mathfrak{A}] \to [\mathfrak{U}, \mathfrak{A}], \tag{5.1}$$

by the rule $J^*(S) = S \circ J$. We prove

Theorem 5.1. *If $\mathfrak{A}$ admits colimits, then $J^* : [\mathfrak{B}, \mathfrak{A}] \to [\mathfrak{U}, \mathfrak{A}]$ has a left adjoint.*

Proof. For any object V in $\mathfrak{B}$, form the category $\mathfrak{J}_V$ of *J-objects over V* as follows. An object of $\mathfrak{J}_V$ is a pair (U, ψ) consisting of an object U of $\mathfrak{U}$ and a morphism $\psi : JU \to V$. A morphism $\varphi : (U, \psi) \to (U', \psi')$ is a morphism $\varphi : U \to U'$ in $\mathfrak{U}$ such that the diagram

$$\begin{tikzcd} JU \arrow[dr, "\psi"] \arrow[dd, "J\varphi"'] & \\ & V \\ JU' \arrow[ur, "\psi'"'] & \end{tikzcd}$$

commutes. With the evident law of composition, $\mathfrak{J}_V$ is a category. Given a functor $T : \mathfrak{U} \to \mathfrak{A}$, we define a functor $T_V : \mathfrak{J}_V \to \mathfrak{A}$ by the rule

$$T_V(U, \psi) = T(U), \qquad T_V(\varphi) = T(\varphi). \tag{5.2}$$

We now set

$$\tilde{J}T(V) = \varinjlim T_V. \tag{5.3}$$

This makes sense since $\mathfrak{A}$ admits colimits. Notice that $\tilde{J}T(V)$ is a certain object A_V of $\mathfrak{A}$, furnished with morphisms $\varrho_V(U, \psi) : T_V(U, \psi) \to A_V$, such that

$$\varrho_V(U', \psi') \circ T_V(\varphi) = \varrho_V(U, \psi) \tag{5.4}$$

for $\varphi : (U, \psi) \to (U', \psi')$ in $\mathfrak{J}_V$; and satisfying the usual universal property.

Now let $\theta : V_1 \to V$ in $\mathfrak{B}$. It is then easy to see that θ induces a morphism $\tilde{\theta} : A_{V_1} \to A_V$, determined by the equations

$$\tilde{\theta} \circ \varrho_{V_1}(U_1, \psi_1) = \varrho_V(U_1, \theta\psi_1), \tag{5.5}$$

for $\psi_1 : JU_1 \to V_1$. Moreover, the rule

$$\tilde{J}T(V) = A_V, \qquad \tilde{J}T(\theta) = \tilde{\theta} \tag{5.6}$$

plainly yields a functor $\tilde{J}T : \mathfrak{B} \to \mathfrak{A}$.

We next show that $\tilde{J}$ is a functor, $\tilde{J} : [\mathfrak{U}, \mathfrak{A}] \to [\mathfrak{B}, \mathfrak{A}]$. Let $S, T : \mathfrak{U} \to \mathfrak{A}$ be two functors and let $\lambda : T \to S$ be a natural transformation of functors. Then we define a natural transformation $\lambda_V : T_V \to S_V$ of functors $\mathfrak{J}_V \to \mathfrak{A}$ by setting $\lambda_V(U, \psi) = \lambda(U) : T_V(U, \psi) \to S_V(U, \psi)$. Let $\varinjlim S_V$ consist of the object B_V together with morphisms $\sigma_V(U, \psi) : S_V(U, \psi) \to B_V$. We then determine a natural transformation $\tilde{\lambda}: \tilde{J}T \to \tilde{J}S$ of functors $\mathfrak{U} \to \mathfrak{A}$ by the rule

$$\tilde{\lambda}(V) \circ \varrho_V(U, \psi) = \sigma_V(U, \psi) \circ \lambda_V(U, \psi). \tag{5.7}$$

Plainly, $\tilde{\lambda}$ is a natural transformation; plainly, too, if we set $\tilde{J}(\lambda) = \tilde{\lambda}$, then $\tilde{J}$ is a functor $[\mathfrak{U}, \mathfrak{A}] \to [\mathfrak{B}, \mathfrak{A}]$. It remains to show that $\tilde{J}$ is left adjoint to J^*. We now suppose given functors $T: \mathfrak{U} \to \mathfrak{A}$, $S: \mathfrak{B} \to \mathfrak{A}$ and a natural transformation $\tau: \tilde{J}T \to S$. We define a natural transformation $\tau' = \eta(\tau): T \to SJ$ by

$$\tau'(U) = \tau(JU) \circ \varrho_{JU}(U, 1), \quad U \text{ in } \mathfrak{U}, \tag{5.8}$$

$$TU \xrightarrow{\varrho_{JU}(U,1)} \tilde{J}TJU \xrightarrow{\tau(JU)} SJU.$$

Also given a natural transformation $\sigma: T \to SJ$, we define a natural transformation $\bar{\sigma} = \bar{\eta}(\sigma): \tilde{J}T \to S$ by

$$\bar{\sigma}(V) \circ \varrho_V(U, \psi) = S\psi \circ \sigma(U), \tag{5.9}$$

$$V \text{ in } \mathfrak{B}, \quad U \text{ in } \mathfrak{U}, \quad \psi: JU \to V \text{ in } \mathfrak{B}.$$

It is easy to verify that τ', η, $\bar{\sigma}$, $\bar{\eta}$ are natural; we conclude by showing that η and $\bar{\eta}$ are mutual inverses. First, if $\tau: \tilde{J}T \to S$, then

$$\overline{\tau'}(V) \circ \varrho_V(U, \psi) = S\psi \circ \tau'(U) = S\psi \circ \tau(JU) \circ \varrho_{JU}(U, 1).$$

Now consider the diagram

$$\begin{tikzcd}
TU \arrow[r, "\varrho_{JU}(U,1)"] \arrow[dr, "\varrho_V(U,\psi)"'] & \tilde{J}TJU \arrow[r, "\tau(JU)"] \arrow[d, "\tilde{J}T\psi"] & SJU \arrow[d, "S\psi"] \\
& \tilde{J}TV \arrow[r, "\tau(V)"] & SV.
\end{tikzcd}$$

The triangle commutes by the definition of $\tilde{J}T\psi$ (5.5), and the square commutes by the naturality of τ. Thus

$$\overline{\tau'}(V) \circ \varrho_V(U, \psi) = S\psi \circ \tau(JU) \circ \varrho_{JU}(U, 1) = \tau(V) \circ \varrho_V(U, \psi),$$

so that $\overline{\tau'} = \tau$, or $\bar{\eta}\eta = 1$.

Next, $\bar{\sigma}'(U) = \bar{\sigma}(JU) \circ \varrho_{JU}(U, 1) = \sigma(U)$, by (5.9), so that $\bar{\sigma}' = \sigma$, or $\eta\bar{\eta} = 1$. This completes the proof of the theorem. □

Note that if, for some $V \in \mathfrak{B}$, $\mathfrak{B}(JU, V)$ is empty for every $U \in \mathfrak{U}$, then $\tilde{J}T(V)$ is just the initial object in $\mathfrak{A}$, so that this case need not be regarded as exceptional.

Definition. The functor $\tilde{J}: [\mathfrak{U}, \mathfrak{A}] \to [\mathfrak{B}, \mathfrak{A}]$ is called the *(left) Kan extension.*

The term "extension" is justified by the following proposition.

Proposition 5.2. *If $J: \mathfrak{U} \to \mathfrak{B}$ is a full embedding, then, for any $T: \mathfrak{U} \to \mathfrak{A}$, $\tilde{J}T$ does extend T in the sense that $(\tilde{J}T)J = T$.*

Proof. Let $U \in \mathfrak{U}$ and consider the category $\mathfrak{J}_{JU}$. There is a subcategory of $\mathfrak{J}_{JU}$ consisting of just the object $(U, 1)$ and its identity morphism. Now, given any object (U_1, ψ_1) of $\mathfrak{J}_{JU}$, there is a unique morphism $\varphi: (U_1, \psi_1) \to (U, 1)$ with $J\varphi = \psi_1$, since J is a full embedding. It is then obvious that $\varinjlim T_{JU}$ is just TU, with $\varrho_{JU}(U_1, \psi_1) = T(\varphi)$. This proves the proposition. □

Remark. We have, in this proof, a very special case of a *cofinal* functor, namely the embedding of the object $(U, 1)$ in $\mathfrak{J}_{JU}$; it is a general fact that *colimits are invariant under cofinal functors* in the sense that $\varinjlim T = \varinjlim TK$, where $K: \mathfrak{C} \to \mathfrak{D}$ is a cofinal functor of small categories and $T: \mathfrak{D} \to \mathfrak{A}$. For the definition of a cofinal functor, generalizing the notion of a cofinal subset of a directed set, see Exercise 5.4.

We now construct the Kan extension in a very special situation. Let $\mathfrak{U} = \mathbf{1}$ be the category with one object 1 and one morphism. Then clearly $[\mathbf{1}, \mathfrak{A}]$ may be identified with $\mathfrak{A}$. Let $V \in \mathfrak{B}$, and let $J = K_V: \mathbf{1} \to \mathfrak{B}$ be the functor $K_V(1) = V$; then the functor $K_V^*: [\mathfrak{B}, \mathfrak{A}] \to [\mathbf{1}, \mathfrak{A}] = \mathfrak{A}$ is just evaluation at V, i.e., for $T: \mathfrak{B} \to \mathfrak{A}$ we have $K_V^* T = TV$.

Proposition 5.3. *The Kan extension $\tilde{K}_V: \mathfrak{A} \to [\mathfrak{B}, \mathfrak{A}]$ is given by*

$$(\tilde{K}_V A) V' = \coprod_{v \in \mathfrak{B}(V, V')} A_v, \quad A_v = A, \tag{5.10}$$

with the obvious values on morphisms.

Proof. Of course it is possible to prove the implied adjointness relation directly. However, we shall apply the general construction of Theorem 5.1. So let $J = K_V$, then $\mathfrak{J}_V$ is the category with objects

$$(\mathbf{1}, v) = v: J(\mathbf{1}) = V \to V'$$

and identity morphisms only. For the functor $T: \mathbf{1} \to \mathfrak{A}$ with $T(\mathbf{1}) = A$ the functor $T_{V'}: \mathfrak{J}_{V'} \to \mathfrak{A}$ is given by $T_{V'}(\mathbf{1}, v) = T(\mathbf{1}) = A$ (see (5.2)). Hence, by (5.3), the Kan extension of T evaluated at V' is just the coproduct $\coprod_{v \in \mathfrak{B}(V, V')} A_v$, where $A_v = A$. □

Next we discuss the Kan extension in a slightly more general situation than that covered by Proposition 5.3. Let $\mathfrak{U} = \mathfrak{B}_d$ be the discrete subcategory of $\mathfrak{B}$, and let $I: \mathfrak{B}_d \to \mathfrak{B}$ be the embedding. Note that $[\mathfrak{B}_d, \mathfrak{A}]$ may be interpreted as the product category $\prod_{V \in \mathfrak{B}} [\mathbf{1}_V, \mathfrak{A}] = \prod_{V \in \mathfrak{B}} \mathfrak{A}_V$, where $\mathfrak{A}_V$ is just a copy of $\mathfrak{A}$. We denote objects in $[\mathfrak{B}_d, \mathfrak{A}]$ therefore by $\{A_V\}$. Note also that $I^*: [\mathfrak{B}, \mathfrak{A}] \to [\mathfrak{B}_d, \mathfrak{A}]$ is the functor given by

$$(I^* T)_V = (I^* T) V = T \circ I(V) = TV$$

where $T: \mathfrak{B} \to \mathfrak{A}$; in other words,

$$I^* T = \{TV\}.$$

Corollary 5.4. *The Kan extension* $\tilde{I}: [\mathfrak{B}_d, \mathfrak{A}] \to [\mathfrak{B}, \mathfrak{A}]$ *is given by*

$$(\tilde{I}\{A_V\})V' = \coprod_{V \in \mathfrak{B}} (\tilde{K}_V A_V)V' = \coprod_{V \in \mathfrak{B}} \coprod_{v \in \mathfrak{B}(V,V')} (A_V)_v,$$

where $(A_V)_v = A_V$, *with the obvious values on morphisms.*

This follows easily from the following lemma.

Lemma 5.5. *Let* $F_i: \mathfrak{C}_i \to \mathfrak{D}$ *be a left adjoint of* $G_i: \mathfrak{D} \to \mathfrak{C}_i$, *and suppose that* $\mathfrak{D}$ *has coproducts. Define* $G: \mathfrak{D} \to \prod_i \mathfrak{C}_i$ *by* $GD = \{G_i D\}$. *Then* $F: \prod_i \mathfrak{C}_i \to \mathfrak{D}$, *defined by* $F\{C_i\} = \coprod_i F_i C_i$, *is a left adjoint to* G.

Proof. $\mathfrak{D}(F\{C_i\}, D) = \mathfrak{D}\left(\coprod_i F_i C_i, D\right) = \prod_i \mathfrak{D}(F_i C_i, D) \cong \prod_i \mathfrak{C}_i(C_i, G_i D)$ $= \left(\prod_i \mathfrak{C}_i\right)(\{C_i\}, \{G_i D\}) = \left(\prod_i \mathfrak{C}_i\right)(\{C_i\}, GD)$. $\square$

Plainly, Corollary 5.4 follows immediately from Lemma 5.5 and Proposition 5.3. $\square$

Going back to the general case, let $J: \mathfrak{U} \to \mathfrak{V}$ be a functor and let $\mathfrak{A}$ be an *abelian* category with colimits. Then (see Exercise II. 9.6) $[\mathfrak{U}, \mathfrak{A}]$ and $[\mathfrak{V}, \mathfrak{A}]$ are abelian categories and, moreover, the Kan extension $\tilde{J}: [\mathfrak{U}, \mathfrak{A}] \to [\mathfrak{V}, \mathfrak{A}]$ exists. Since $\tilde{J}$ is defined as a left adjoint (to J^*) it preserves epimorphisms, cokernels and coproducts; in particular, $\tilde{J}$ is right exact. Denoting by $\mathscr{E}'_1$ the class of all epimorphisms in $[\mathfrak{U}, \mathfrak{A}]$, we make the following definition.

Definition. Let $T: \mathfrak{U} \to \mathfrak{A}$ be a functor. We define the *(absolute) homology* $H_*(J, T)$ *of* J *with coefficients in* T as the left $\mathscr{E}'_1$-satellite of the Kan extension $\tilde{J}$ evaluated at T

$$H_n(J, T) = (S_n \tilde{J})T, \quad n = 0, 1, \ldots . \tag{5.11}$$

We may also, for convenience, refer to this type of homology as *J-homology*. By definition $H_n(J, T)$ is a functor from $\mathfrak{V}$ into $\mathfrak{A}$, and $H_0(J, T) = \tilde{J}T$. Next we take up the question of the existence of J-homology. We shall apply Theorem 4.1 to show that, if $\mathfrak{A}$ has enough projectives, then so does $[\mathfrak{U}, \mathfrak{A}]$; that is to say, the class $\mathscr{E}'_1$ of all epimorphisms in $[\mathfrak{U}, \mathfrak{A}]$ is projective. By Theorem 3.2 the satellite of $\tilde{J}$ may then be computed via $\mathscr{E}'_1$-projective resolutions in $[\mathfrak{U}, \mathfrak{A}]$. To this end consider $I: \mathfrak{U}_d \to \mathfrak{U}$ and the Kan extension $\tilde{I}: [\mathfrak{U}_d, \mathfrak{A}] \to [\mathfrak{U}, \mathfrak{A}]$. By Theorem 5.1 $\tilde{I}$ exists, and its form is given by Corollary 5.4. Since $[\mathfrak{U}_d, \mathfrak{A}]$ may be identified with the category $\prod_{U \in \mathfrak{U}} \mathfrak{A}_U$, where $\mathfrak{A}_U$ is a copy of $\mathfrak{A}$, it is clear that $I^*: [\mathfrak{U}, \mathfrak{A}] \to [\mathfrak{U}_d, \mathfrak{A}]$ is faithful. By Theorem 4.1 the adjoint pair $\tilde{I} \dashv I^*$ may then be used to transfer projective classes from $[\mathfrak{U}_d, \mathfrak{A}]$ to $[\mathfrak{U}, \mathfrak{A}]$. Clearly, if $\mathscr{E}_1$ denotes the class of all epimorphisms in $[\mathfrak{U}_d, \mathfrak{A}]$, $(I^*)^{-1}(\mathscr{E}_1)$ is the class $\mathscr{E}'_1$ of all epimorphisms in $[\mathfrak{U}, \mathfrak{A}]$. Now since $\mathfrak{A}$ has enough projectives, the category $[\mathfrak{U}_d, \mathfrak{A}]$ has enough projectives, and $\mathscr{E}_1$ is a projective class. By Theorem 4.1 it follows that the class $\mathscr{E}_1'$ is projective. We therefore have

Theorem 5.6. *If $\mathfrak{A}$ has enough projectives then the J-homology*

$$H_*(J, -): [\mathfrak{U}, \mathfrak{A}] \to [\mathfrak{B}, \mathfrak{A}]$$

exists. It may be computed as the left $\mathscr{E}_1'$-derived functor of the Kan extension,

$$H_n(J, T) = (L_n^{\mathscr{E}_1'} \tilde{J})\, T, \qquad n = 0, 1, \ldots . \quad \square \tag{5.12}$$

We remark that Theorem 4.1 and Corollary 5.4 yield the form of the projectives. A functor $S: \mathfrak{U} \to \mathfrak{A}$ is, by the last part of Theorem 4.1, $\mathscr{E}_1'$-projective if and only if it is a direct summand of a functor $\tilde{I}\{P_U\}$ for $\{P_U\}$ a projective in $[\mathfrak{U}_d, \mathfrak{A}]$. But this, of course, simply means that each P_U is projective in $\mathfrak{A}$. Thus, by Corollary 5.4, the functor S is $\mathscr{E}_1'$-projective if and only if it is a direct summand of a functor $\bar{S}: \mathfrak{U} \to \mathfrak{A}$ of the form

$$\bar{S}(U') = \coprod_{U \in \mathfrak{U}} \coprod_{v \in \mathfrak{U}(U, U')} (P_U)_v,$$

where $(P_U)_v = P_U$ is a projective object in $\mathfrak{A}$.

Corollary 5.7. *Let $0 \to T' \to T \to T'' \to 0$ be a sequence of functors in $[\mathfrak{U}, \mathfrak{A}]$ which is $\mathscr{E}_1'$-exact. Then there is a long exact sequence of functors in $[\mathfrak{B}, \mathfrak{A}]$,*

$$\cdots \to H_n(J, T') \to H_n(J, T) \to H_n(J, T'') \to H_{n-1}(J, T') \to \cdots . \quad \square \tag{5.13}$$

We remark that the above definitions and development may be dualized by replacing $\mathfrak{A}$ by $\mathfrak{A}^{\mathrm{opp}}$ to yield *co*homology. The reader should conscientiously carry out at least part of this dualization process, since it differs from that employed in Chapter IV in describing derived functors of covariant and contravariant functors in that, here, it is the *codomain* category $\mathfrak{A}$ of our functor T which is replaced by its opposite $\mathfrak{A}^{\mathrm{opp}}$.

Our approach has used the existence of enough projectives in the category $\mathfrak{A}$. However, instead of defining the homology using the class $\mathscr{E}_1'$ in $[\mathfrak{U}, \mathfrak{A}]$ it is possible to define a (relative) homology as the left satellite with respect to the class $\mathscr{E}_0'$ of all epimorphisms in $[\mathfrak{U}, \mathfrak{A}]$ which are *objectwise split*, meaning that the evaluation at any U in $\mathfrak{U}$ is a split epimorphism in $\mathfrak{A}$. It is then plain that $\mathscr{E}_0'$ is just $(I^*)^{-1}(\mathscr{E}_0)$ where $\mathscr{E}_0$ denotes the class of all epimorphisms in $[\mathfrak{U}_d, \mathfrak{A}]$ which are objectwise split. We can then define a *relative* homology,

$$\hat{H}_n(J, T) = (S_n^{\mathscr{E}_0'} \tilde{J})\, T, \qquad n = 0, 1, \ldots \tag{5.14}$$

as the left satellite of the Kan extension with respect to the class $\mathscr{E}_0'$. Since the class $\mathscr{E}_0$ is clearly projective in $[\mathfrak{U}_d, \mathfrak{A}]$, we may compute the relative homology as the left $\mathscr{E}_0'$-derived functor of the Kan extension.

This definition clearly works even if $\mathfrak{A}$ lacks enough projectives. Moreover it follows from Proposition 5.8 below that if $\mathfrak{A}$ has enough projectives and *exact coproducts* then the relative and the absolute homology coincide. An abelian category is said to have *exact coproducts* if coproducts of short exact sequences are short exact – equivalently, if coproducts of monomorphisms are monomorphisms.

Proposition 5.8. *Let $\mathfrak{A}$ have enough projectives and exact coproducts. If $R \in [\mathfrak{U}, \mathfrak{A}]$ is an $\mathscr{E}_0'$-projective functor, then $(L_n^{\mathscr{E}_1'} \tilde{J}) R = 0$ for $n \geqq 1$.*

Proof. Clearly every functor $\mathfrak{U}_d \to \mathfrak{A}$ is $\mathscr{E}_0$-projective. Thus, since $(L_n^{\mathscr{E}_1'} \tilde{J})$ is additive and $\mathfrak{A}$ has exact coproducts, it is enough, by Theorem 4.1 and Corollary 5.4, to prove the assertion for $R = \tilde{K}_U A_U : \mathfrak{U} \to \mathfrak{A}$ where $A = A_U$ is an arbitrary object in $\mathfrak{A}$. Now choose a projective resolution

$$\boldsymbol{P} : \cdots \to P_n \to P_{n-1} \to \cdots \to P_0$$

of A in $\mathfrak{A}$. Then, since coproducts are exact in $\mathfrak{A}$,

$$\tilde{K}_U \boldsymbol{P} : \cdots \to \tilde{K}_U P_n \to \tilde{K}_U P_{n-1} \to \cdots \to \tilde{K}_U P_0$$

is an $\mathscr{E}_1'$-projective resolution of R. Since trivially

$$\tilde{J}(\tilde{K}_U P_n) = \tilde{K}_{JU} P_n \tag{5.14}$$

the complex $\tilde{J}(\tilde{K}_U \boldsymbol{P})$ is again acyclic, whence the assertion follows. □

We have the immediate collorary (see Exercise 4.3).

Theorem 5.9. *Let $\mathfrak{A}$ be an abelian category with enough projectives and exact coproducts. Let $\mathfrak{U}$ and $\mathfrak{B}$ be small categories and let $J : \mathfrak{U} \to \mathfrak{B}$, $T : \mathfrak{U} \to \mathfrak{A}$ be functors. Then*

$$H_n(J, T) \cong \hat{H}_n(J, T) . \quad \square$$

Exercises:

5.1. Justify the statement that if $\mathfrak{B}(JU, V)$ is empty for some V and all U, then $\tilde{J}T(V)$ is just the initial object in $\mathfrak{A}$.

5.2. Formulate the concept of the *right* Kan extension.

5.3. Give an example where $J : \mathfrak{U} \to \mathfrak{B}$ is an embedding but $\tilde{J}T$ does not extend T.

5.4. A category $\mathfrak{C}$ is said to be *cofiltering* if it is small and connected and if it enjoys the following two properties:

(i) given A, B in $\mathfrak{C}$, there exists C in $\mathfrak{C}$ and morphisms $\alpha : A \to C$, $\beta : B \to C$ in $\mathfrak{C}$;

(ii) given $X \underset{\psi}{\overset{\varphi}{\rightrightarrows}} Y$ in $\mathfrak{C}$, there exists $\theta : Y \to Z$ in $\mathfrak{C}$ with $\theta\varphi = \theta\psi$.

A functor $K : \mathfrak{C} \to \mathfrak{D}$ from the cofiltering category $\mathfrak{C}$ to the cofiltering category $\mathfrak{D}$ is said to be *cofinal* if it enjoys the following two properties:

(i) given B in $\mathfrak{D}$, there exists A in $\mathfrak{C}$ and $\psi : B \to KA$ in $\mathfrak{D}$;

(ii) given $B \underset{\psi}{\overset{\varphi}{\rightrightarrows}} KA$ in $\mathfrak{D}$, there exists $\theta : A \to A_1$ in $\mathfrak{C}$ with $(K\theta)\varphi = (K\theta)\psi$.

Prove that, if $T: \mathfrak{D} \to \mathfrak{A}$ is a functor from the cofiltering category $\mathfrak{D}$ to the category $\mathfrak{A}$ with colimits and if $K: \mathfrak{C} \to \mathfrak{D}$ is a cofinal functor from the cofiltering category $\mathfrak{C}$ to $\mathfrak{D}$, then $\varinjlim T = \varinjlim TK$. (You should make the nature of this equality quite precise.)

5.5. Prove Proposition 5.3 directly.

5.6. Prove that, under the hypotheses of Proposition 5.8, the connected sequences of functors $\{L_n^{\mathscr{E}'_1} \tilde{J}\}$ and $\{L_n^{\mathscr{E}'_0} \tilde{J}\}$ are equivalent.

[Further exercises on the material of this section are incorporated into the exercises at the end of Section 6.]

6. Applications: Homology of Small Categories, Spectral Sequences

We now specialize the situation described in the previous section. Let $\mathfrak{B} = \mathbf{1}$, the category with one object and only one morphism, and let $J: \mathfrak{U} \to \mathfrak{B}$ be the obvious functor. Thus, for $T: \mathfrak{U} \to \mathfrak{A}$, we define $H_n(\mathfrak{U}, T)$ by

$$H_n(\mathfrak{U}, T) = H_n(J, T), \qquad n \geqq 0, \tag{6.1}$$

and call it the *homology of the small category* $\mathfrak{U}$ *with coefficients in* T.

We will immediately give an example. Let $\mathfrak{U} = G$ where G is a group regarded as a category with one object, let $\mathfrak{B} = \mathfrak{U}_d = \mathbf{1}$, and let J be the obvious functor. Take $\mathfrak{A} = \mathfrak{Ab}$ the category of abelian groups. The functor $T: \mathfrak{U} \to \mathfrak{A}$ may then be identified with the G-module $A = T(\mathbf{1})$, so that $[\mathfrak{U}, \mathfrak{A}] = \mathfrak{M}_G$. The category $[\mathfrak{B}, \mathfrak{A}] = \mathfrak{A}$ is just the category of abelian groups. The functor $J^*: [\mathfrak{B}, \mathfrak{A}] \to [\mathfrak{U}, \mathfrak{A}]$ associates with an abelian group A the trivial G-module A. The Kan extension $\tilde{J}$ is left adjoint to J^*, hence it is the functor $-_G: [\mathfrak{U}, \mathfrak{A}] \to [\mathfrak{B}, \mathfrak{A}]$ associating with a G-module M the abelian group M_G. Since the class $\mathscr{E}'_1$ in $[\mathfrak{U}, \mathfrak{A}]$ is just the class of all epimorphisms in $\mathfrak{M}_G$, we have

$$H_n(J, T) = H_n(G, A), \qquad n \geqq 0, \tag{6.2}$$

where $A = T(\mathbf{1})$, so that group homology is exhibited as a special case of the homology of small categories. Moreover the long exact sequence (5.13) is transformed under the identification (6.2) into the exact coefficient sequence in the homology of groups.

We next consider the situation

$$\mathfrak{U} \xrightarrow{J} \mathfrak{B} \xrightarrow{I} \mathfrak{W}$$

where J, I are two functors between small categories. The Grothendieck spectral sequence may then be applied to yield (see Theorem VIII. 9.3).

Theorem 6.1. *Let* $J: \mathfrak{U} \to \mathfrak{B}$, $I: \mathfrak{B} \to \mathfrak{W}$ *be two functors between small categories, and let* $\mathfrak{A}$ *be an abelian category with colimits and enough* *projectives. Then there is a spectral sequence*

$$E_1^{p,q} = H_p(I, H_{q-p}(J, -)) \Rightarrow H_q(IJ, -). \tag{6.3}$$

Proof. We only have to show that projectives in $[\mathfrak{U}, \mathfrak{A}]$ are transformed by $\tilde{J}$ into $\tilde{I}$-acyclic objects in $[\mathfrak{B}, \mathfrak{A}]$. Since $\tilde{J}$ is additive it is enough to check this claim on functors $R = \tilde{K}_U P_U : \mathfrak{U} \to \mathfrak{A}$. But then $\tilde{J}(\tilde{K}_U P_U) = \tilde{K}_{JU} P_U$ (by 5.14) which is not only $\tilde{I}$-acyclic in $[\mathfrak{B}, \mathfrak{A}]$, but even projective. □

We give the following application of Theorem 6.1. Let $\mathfrak{U} = G$, where G is a group regarded as a category with just one object, let $\mathfrak{B} = Q$ be a quotient group of G, and let $\mathfrak{W} = \mathbf{1}$. I, J are the obvious functors. Let $\mathfrak{A}$ be the category of abelian groups. Theorem 6.1 then yields the spectral sequence

$$E_1^{p,q} = H_p(Q, H_{q-p}(J, -)) \Rightarrow H_q(G, -). \tag{6.4}$$

In order to discuss $H_*(J, -)$ in this special case, we note that $[\mathfrak{B}, \mathfrak{A}]$ may be identified with the category $\mathfrak{M}_Q$ of Q-modules. If M is in $\mathfrak{M}_Q$, J^*M is M regarded as a G-module. It then follows that for M' in $\mathfrak{M}_G$, $\tilde{J}M' = \mathbb{Z}Q \otimes_G M'$, since $\tilde{J}$ is left adjoint to J^*. We thus obtain

$$H_r(J, -) = \mathrm{Tor}_r^G(\mathbb{Z}Q, -) \cong H_r(N, -), \qquad r \geqq 0,$$

as functors to $\mathfrak{M}_Q$, where N is the normal subgroup of G with $G/N = Q$. The spectral sequence (6.4) is thus just the Lyndon-Hochschild-Serre spectral sequence for the homology of groups.

We would like to remark that the procedures described in this section are really much more general than our limited tools allow us to show. Since we did not want to get involved in set-theoretical questions, we have had to suppose that both $\mathfrak{U}$ and $\mathfrak{B}$ are small categories. However, one can show that under certain hypotheses the theory still makes sense when $\mathfrak{U}$ and $\mathfrak{B}$ are not small. We mention some examples of this kind.

(a) Let $\mathfrak{U}$ be the full subcategory of $\mathfrak{M}_\Lambda$ consisting of *free* Λ-modules. Let $\mathfrak{B} = \mathfrak{M}_\Lambda$, and let J be the obvious functor. Thus $J^* : [\mathfrak{B}, \mathfrak{A}] \to [\mathfrak{U}, \mathfrak{A}]$ is just the restriction. It may be shown that for every additive functor $T : \mathfrak{U} \to \mathfrak{A}$

$$H_n(J, T) = L_n T, \qquad n \geqq 0,$$

where $L_n T$ denotes the usual n^{th} left derived functor of $T : \mathfrak{U} \to \mathfrak{A}$.

(b) Let $\mathfrak{U}$ be the full subcategory of $\mathfrak{G}$, the category of groups, consisting of all *free* groups. Let $\mathfrak{B} = \mathfrak{G}$, and let $\mathfrak{A}$ be the category of abelian groups. Again, $J : \mathfrak{U} \to \mathfrak{B}$ is the obvious functor. Let $R_A : \mathfrak{U} \to \mathfrak{A}$ be the functor which assigns, to the free group F, the abelian group

$$IF \otimes_F A = F_{ab} \otimes A,$$

for A a fixed abelian group. It may then be show that

$$H_n(J, R_A)\, G = H_{n+1}(G, A)\,, \qquad n \geqq 1\,,$$
$$H_0(J, R_A)\, G = G_{ab} \otimes A\,.$$

Thus we obtain, essentially, the homology of G with trivial coefficients. However, more generally, we may obtain the homology of G with coefficients in an arbitrary G-module A, by taking for $\mathfrak{U}$ the category of free groups *over* G, for $\mathfrak{V}$ the category of all groups over G, and for $J : \mathfrak{U} \rightarrow \mathfrak{V}$ the functor induced by the imbedding. Then we may define a functor $T_A : \mathfrak{U} \rightarrow \mathfrak{A}$ by

$$T_A(F \xrightarrow{f} G) = IF \otimes_F A$$

where A is regarded as an F-module via f. One obtains

$$H_n(J, T_A)\, 1_G = H_{n+1}(G, A)\,, \qquad n \geqq 1\,,$$
$$H_0(J, T_A)\, 1_G = IG \otimes_G A\,.$$

Proceeding analogously, it is now possible to define homology theories in any category $\mathfrak{V}$ once a subcategory $\mathfrak{U}$ (called the *category of models*) and a base functor are specified. This is of significant value in categories where it is not possible (as it is for groups and Lie algebras over a field) to define an appropriate homology theory as an ordinary derived functor. As an example, we mention finally the case of *commutative K-algebras*, where K is a field.

(c) Let $\mathfrak{V}'$ be the category of commutative K-algebras. Consider the category $\mathfrak{V} = \mathfrak{V}'/V$ of commutative K-algebras over the K-algebra V. Let $\mathfrak{U}$ be the full subcategory of *free* commutative (i.e., polynomial) K-algebras over V, and let $J : \mathfrak{U} \rightarrow \mathfrak{V}$ be the obvious embedding. Then

$$H_n(J, \operatorname{Diff}(-, A))$$

defines a good homology theory for commutative K-algebras. Here A is a V-module and the abelian group $\operatorname{Diff}(U \xrightarrow{f} V, A)$ is defined as follows. Let M be the kernel of the map $m : U \otimes_K U \rightarrow U$ induced by the multiplication in U. Then $\operatorname{Diff}(U \xrightarrow{f} V, A) = M/M^2 \otimes_U A$ where A is regarded as a U-module via f.

Exercises:

6.1. State a "Lyndon-Hochschild-Serre" spectral sequence for the homology of small categories.

6.2. Let $\mathfrak{A}$ be an abelian category and let $\mathfrak{U}, \mathfrak{V}$ be small additive categories. Denote by $[\mathfrak{U}, \mathfrak{A}]_+$ the full subcategory of $[\mathfrak{U}, \mathfrak{A}]$ consisting of all *additive* functors. Given an *additive* functor $J : \mathfrak{U} \rightarrow \mathfrak{V}$, define the *additive Kan extension* $\tilde{J}^+$ as a left adjoint to

$$J^* : [\mathfrak{V}, \mathfrak{A}]_+ \rightarrow [\mathfrak{U}, \mathfrak{A}]_+\,.$$

Along the lines of the proof of Theorem 5.1, prove the existence of $\tilde{J}^+$ in case $\mathfrak{A}$ has colimits. Prove an analog of Proposition 5.2.

6.3. In the setting of Exercise 6.2 define an additive J-homology by

$$H_n^+(J, -) = (S_n^{\mathscr{E}_1} \tilde{J}^+) : \mathfrak{B} \to \mathfrak{A}, \qquad n \geqq 0.$$

Show the existence of this homology if $\mathfrak{A}$ has enough projectives.

6.4. Let $\mathfrak{U} = \Lambda$ be an augmented algebra over the commutative ring K regarded as an additive category with a single object. Set $\mathfrak{B} = K$, and let $J : \Lambda \to K$ be the augmentation. What is $H_n^+(J, T)$ for $T : \Lambda \to \mathfrak{Ab}$ an additive functor, i.e., a Λ-module? ($H_n^+(J, T)$ is then called the n^{th} *homology group* of Λ with coefficients in the Λ-module T.) What is $H_n^+(J, T)$ when (a) $\mathfrak{U} = \mathbb{Z}G$, the groupring of G, $K = \mathbb{Z}$; (b) K is a field and $\mathfrak{U} = U\mathfrak{g}$, the universal envelope of the K-Lie algebra $\mathfrak{g}$? Dualize.

6.5. State a spectral sequence theorem for the homology of augmented algebras. Identify the spectral sequence in the special cases referred to in Exercise 6.4.

Note to the third corrected printing (see page 107).

Shelah has proved that, if the rank of A is uncountable, then the truth of the conjecture

$$\operatorname{Ext}(A, \mathbb{Z}) = 0 \text{ implies } A \text{ free}$$

—known as the *Whitehead conjecture* after J. H. C. Whitehead who first formulated it—depends on the set theory adopted. The following references may be useful to the interested reader:

S. Shelah: Infinite abelian groups, Whitehead problem and some constructions. Israel J. Math. 18 (1974), 243–256.

H. Hiller, M. Huber and S. Shelah: The structure of $\operatorname{Ext}(A, \mathbb{Z})$ and $V = L$. Math. Z. 162 (1978), 39–50.

P. Eklof: Set-theoretic methods in homological algebra and abelian groups. Séminaire de Mathématiques Supérieures 69, University of Montreal Press (1980), 117 pp.

X. Some Applications and Recent Developments

The first section of this chapter describes how homological algebra arose by abstraction from algebraic topology and how it has contributed to the knowledge of topology. The other four sections describe applications of the methods and results of homological algebra to other parts of algebra. Most of the material presented in these four sections was not available when this text was first published. Since then homological algebra has indeed found a large number of applications in many different fields, ranging from finite and infinite group theory to representation theory, number theory, algebraic topology, and sheaf theory. Today it is a truly indispensable tool in all these fields. For the purpose of illustrating to the reader the range and depth of these developments, we have selected a number of different topics and describe some of the main applications and results. Naturally, the treatments are somewhat cursory, the intention being to give the flavor of the homological methods rather than the details of the arguments and results.

1. Homological Algebra and Algebraic Topology

Homological algebra originated, as we have said, as an abstraction from algebraic topology (see the Introduction to our text and the Introduction to Chapter VI). Here we would like to go into somewhat greater detail in forging that connection, since many students today study homological algebra without the benefit of a prior familiarity with the related concepts of algebraic topology.

The primary link with algebraic topology is via the homology theory of topological spaces. Let us consider the simplest case, that of a polyhedron triangulated as a *simplicial complex* K. Thus K is a union of simplexes s^n of various dimensions $n \geq 0$. A *simplex* s^n is just the convex hull of $n + 1$ independent points, the *vertices* of s^n, in some Euclidean space. If s and t are two simplexes of K, then either they are disjoint or they intersect in a common *face*. We may *orient* the simplexes of K; then the oriented n-simplexes become a *basis* for a free abelian group $C_n(K)$, the nth *chain group* of K (with integer coefficients). The *boundary* of an oriented n-simplex is an $(n-1)$-chain according to the formula

$$\partial(a_0 a_1 \cdots a_n) = \sum_{i=0}^{n} (-1)^i (a_0 a_1 \cdots \hat{a}_i \cdots a_n),$$

where $a_0, a_1, \ldots, a_n$ are the vertices of the oriented n-simplex, and $\hat{a}_i$ indicates that the vertex a_i is to be *deleted* (compare formula (13.3) of Chapter VI). Then $\partial\partial = 0$, so that $(\mathbf{C}(K), \partial)$ is a free abelian chain complex. We may then carry out the various processes described in the text. Thus we may pass to homology, obtaining the (integral) homology groups $H_*(K)$; we may tensor with an abelian coefficient group G, obtaining the homology groups $H_*(K; G)$ of K with coefficients in G; we may form the *cochain complex* $\operatorname{Hom}(\mathbf{C}(K), G)$, and obtain the cohomology groups $H^*(K; G)$ of K with coefficients in G; and we have *universal coefficient theorems* (see Theorems 2.5 and 3.3 of Chapter V) relating these homology and cohomology groups with coefficients in G to the integral homology groups $H_*(K)$. Moreover, we may introduce *chain maps* and *chain homotopies* as prototypes of the general concepts described in our text. Thus a simplicial map $f: K \to L$, that is, a function from the vertices of K to the vertices of L which sends vertices of a common simplex to vertices of a common simplex, induces in an obvious way a chain-map $\varphi: \mathbf{C}(K) \to \mathbf{C}(L)$. A continuous map of the underlying polyhedra induces (but not uniquely) a simplicial map of a sufficiently fine subdivision of K into L; and homotopic continuous maps lead to chain-homotopic chain-maps. There is a vitally important chain-map $\omega: \mathbf{C}(K) \to \mathbf{C}(K')$ from the chain-complex of K to the chain-complex of K', the (first) barycentric subdivision of K, which induces an isomorphism of homology groups and serves to explain (and to prove) the topological invariance – indeed the homotopy invariance – of the homology groups.

The developments described thus far may be generalized to a broader class of topological spaces beyond the polyhedra. Thus one may consider Whitehead *cell-complexes* (or CW-complexes) instead of simplicial complexes; or one may use the so-called *singular theory*, building a simplicial complex from the *singular simplexes* of an arbitrary topological space (see, for example, [D]).

Let us now show how to apply the fundamental Theorem 2.1 of Chapter IV, proving that a short exact sequence of chain-complexes induces a long exact homology sequence. The first application is obvious and immediate: if $B' \rightarrowtail B \twoheadrightarrow B''$ is a short exact sequence of abelian groups then, since $\mathbf{C}(K)$ is free abelian, $\mathbf{C}(K) \otimes B' \rightarrowtail \mathbf{C}(K) \otimes B \twoheadrightarrow \mathbf{C}(K) \otimes B''$ is a short exact sequence of chain-complexes, yielding the *exact coefficient sequence in homology*

$$\cdots \to H_n(K; B') \to H_n(K; B) \to H_n(K; B'') \to H_{n-1}(K; B') \to \cdots.$$

Likewise $A' \rightarrowtail A \twoheadrightarrow A''$ induces an *exact coefficient sequence in cohomology*

$$\cdots \to H^n(K; A') \to H^n(K; A) \to H^n(K; A'') \to H^{n+1}(K; A') \to \cdots.$$

More subtle is the following. If L is a subcomplex of the simplicial complex K, then, for each n, $C_n(L)$ is a direct summand in $C_n(K)$, since it is a free abelian group with basis a subset of the basis of $C_n(K)$. Moreover, the boundary operator in $\mathbf{C}(K)$ restricts to the boundary operator in $\mathbf{C}(L)$. Thus $\mathbf{C}(L)$ is a subcomplex of the chain-complex $\mathbf{C}(K)$ such that, for each n, $C_n(L)$ is a direct summand in $C_n(K)$. It follows that we can form the quotient complex $\mathbf{C}(K, L) = \mathbf{C}(K)/\mathbf{C}(L)$ and $C_n(K, L)$ is again free abelian. The short exact sequence $\mathbf{C}(L) \rightarrowtail \mathbf{C}(K) \twoheadrightarrow \mathbf{C}(K, L)$ splits in each dimension. It thus remains short exact when we tensor with an abelian group B,

$$\mathbf{C}(L) \otimes B \rightarrowtail \mathbf{C}(K) \otimes B \twoheadrightarrow \mathbf{C}(K, L) \otimes B,$$

yielding the *exact homology sequence of the pair* (K, L),

$$\cdots \to H_n(L; B) \to H_n(K; B) \to H_n(K, L; B) \to H_{n-1}(L; B) \to \cdots.$$

Similarly, there is an *exact cohomology sequence of the pair* (K, L)

$$\cdots \to H^n(K, L; A) \to H^n(K; A) \to H^n(L; A) \to H^{n+1}(K, L; B) \to \cdots.$$

The groups $H_n(K, L; B)$, $H^n(K, L; A)$ are homotopy invariants of the *polyhedral pair* underlying the *simplicial pair* of complexes K, L.

Now let R be a (coefficient) ring. Let $u: C_m(K) \to R$, $v: C_n(K) \to R$ be an m-cochain of K and an n-cochain of K with coefficients in R. We define an $(m + n)$-cochain of K with coefficients in R by the rule

$$uv(a_0 a_1 \cdots a_{m+n}) = u(a_0 a_1 \cdots a_m) v(a_m a_{m+1} \cdots a_{m+n}).$$

It turns out that this rule defines a graded ring structure in the cohomology group $H^*(K; R)$, which is commutative in the graded sense, i.e.,

$$[u][v] = (-1)^{mn}[v][u], \qquad [u] \in H^m, \quad [v] \in H^n,$$

if R is itself commutative.

There is a version of the Künneth Theorem (see Chapter V) for topological spaces, but it has an extra feature to it compared with the purely algebraic version. If X, Y are two spaces and we wish to compute the homology of the topological product $X \times Y$, then we form free abelian chain-complexes $\mathbf{C}(X)$, $\mathbf{C}(Y)$ yielding the integral homology of X, Y, respectively (for instance, we use simplicial or singular chains) and there is then a natural chain-map

$$\tau: \mathbf{C}(X) \otimes \mathbf{C}(Y) \to \mathbf{C}(X \times Y)$$

yielding an isomorphism in homology; in fact, τ is a chain-homotopy equivalence.

Then, by using τ, we reduce the problem of calculating the homology groups of $X \times Y$ to that of calculating the homology groups of $\mathbf{C}(X) \otimes \mathbf{C}(Y)$; and that is achieved by means of the Künneth formula of Chapter V.

The existence and properties of τ are established by showing what is known as the Eilenberg–Zilber Theorem; they may be established using a standard technique of algebraic topology known as the method of *acyclic models.*

As indicated in the Introduction to Chapter VI, there is a very close and important connection between the cohomology of groups and algebraic topology. Let $\tilde{K}$ be the universal cover of K, which is assumed connected. Thus there is a covering (simplicial) map $p:\tilde{K} \to K$ which maps $\tilde{K}$ onto K and is a local homeomorphism, and $\tilde{K}$ is simply-connected. Then the chain-group $C_n(\tilde{K})$ is a free $\pi_1(K)$-module, and $\pi_1(K)$ freely permutes the n-simplexes of $\tilde{K}$ lying over a given n-simplex of K. If B is a $\pi_1(K)$-module, then we may form $B \otimes_{\pi_1(K)} \mathbf{C}(\tilde{K})$ and the homology groups of this chain-complex are just the homology groups of K with *local coefficients* B, usually written $H_*(K; \{B\})$. In the particular case where B is an abelian group (trivial $\pi_1(K)$-module), this reduces to just $H_*(K; B)$. Similarly, if A is a $\pi_1(K)$-module we form the cochain-complex $\mathrm{Hom}_{\pi_1(K)}(\mathbf{C}(\tilde{K}), A)$, whose cohomology groups $H^*(K; \{A\})$ are the cohomology groups of K with *local coefficients* A, reducing to the usual cohomology groups $H^*(K; A)$ if A is an abelian group.

Now let $K(\pi, 1)$ be a cell-complex with fundamental group π whose universal cover is contractible. For example, the circle S^1 is a $K(C, 1)$, where C is the infinite cyclic group, since its universal cover is the line $\mathbb{R}$. Such a space $K(\pi, 1)$ is often called an Eilenberg–MacLane complex. Hurewicz observed that the homology groups of $K(\pi, 1)$ are uniquely determined by π; in fact, it is not difficult to see that the homotopy type of $K(\pi, 1)$ is determined by π. Even more precisely, a homomorphism $\varphi:\pi \to \pi'$ induces a well-defined homotopy class of (pointed) maps $f:K(\pi, 1) \to K(\pi', 1)$ such that the homomorphism of fundamental groups induced by f is precisely φ. Moreover, the functor $K(-, 1)$ from the category of groups to the category of homotopy classes of (pointed) cell-complexes is right adjoint (see Section 7 of Chapter II) to the fundamental group functor, so that there is a natural equivalence

$$[X, K(\pi, 1)] \simeq \mathrm{Hom}(\pi_1(X), \pi).$$

Now let $\tilde{K}$ be the universal cover of $K(\pi, 1)$. Then $\mathbf{C}(\tilde{K})$ is a free π-complex with vanishing homology in positive dimensions and such that $H_0 = \mathbb{Z}$. Thus if $\varepsilon: C_0(\tilde{K}) \twoheadrightarrow \mathbb{Z}$ is the augmentation, sending each vertex to 1, $\mathbf{C}(\tilde{K})$ provides a free resolution of $\mathbb{Z}$ as (trivial) π-module, so that

$$H_n(K(\pi, 1); \{B\}) = H_n(\pi, B), \qquad H^n(K(\pi, 1); \{A\}) = H^n(\pi, A).$$

Thus our theory in Chapter VI precisely provides a description of the way in which the fundamental group π of the Eilenberg–MacLane space $K(\pi, 1)$ determines its homology and cohomology groups.

We have already described how the cohomology groups of a simplicial complex (and hence of any polyhedron) with coefficients in a ring R may be turned into a graded ring. So may the cohomology groups of π (see Exercise 13.6 of Chapter VI), and then

$$H^*(K(\pi, 1); R) \simeq H^*(\pi, R)$$

as rings.

There are now many ways in which the algebra and topology interplay between π and $K(\pi, 1)$. Let us be content with just one example.

Theorem 1.1. *If π has an element of finite order, then there can be no finite-dimensional model for $K(\pi, 1)$.*

Proof. First consider $K(C_m, 1)$, where C_m is a cyclic group of order $m \geq 2$. Since $\mathbf{C}(\tilde{K}(C_m, 1))$ is a free C_m-resolution of $\mathbb{Z}$, and since C_m has nonzero integral homology groups in all odd dimensions (see Section 7 of Chapter VI), $\tilde{K}(C_m, 1)$, and hence $K(C_m, 1)$, must have cells in arbitrarily high dimensions, i.e., it must be infinite dimensional.

Now let π have an element of order m. Then $\mathbf{C}(\tilde{K}(\pi, 1))$ is a free C_m-resolution, since C_m is a subgroup of π (Lemma 1.3 of Chapter VI). Thus $\tilde{K}(\pi, 1)$, and hence $K(\pi, 1)$, has cells in arbitrarily high dimensions, and so must be infinite dimensional. □

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2. Nilpotent Groups

Homological algebra, in particular the (co)homology theory of groups has seen many applications to the structure theory of groups, both finite and infinite. Here we shall describe one of these applications, the localization theory of nilpotent groups. This theory was developed extensively between 1960 and 1980 in order to facilitate and deepen the homotopy-theoretical study of *nilpotent spaces* (for a definition see below), but it also has considerable purely algebraic interest.

For a group G the series of subgroups $\gamma^i G$, the *lower central series* of G, is defined by

$$\gamma^1(G) = G, \qquad \gamma^{i+1}(G) = [G, \gamma^i(G)], \qquad i \geq 1,$$

where for two subgroups U, V of G the symbol $[U, V]$ denotes the subgroup of G generated by all elements $u^{-1}v^{-1}uv$ with $u \in U$ and $v \in V$. It is easy to see that $\gamma^i(G)$ is a normal subgroup of G and that $\gamma^i(G)/\gamma^{i+1}(G)$ is a central, and hence also abelian, subgroup of $G/\gamma^{i+1}(G)$. A group G is called *nilpotent* of class $\leq c$ if $\gamma^{c+1}(G) = 1$; the smallest such c is called the *nilpotency class* of G, nil G. From this definition it is clear that a nilpotent group can be obtained by a finite number of successive central extensions, starting with an abelian group. This is the key to the method of obtaining detailed results about the homology and the localization of nilpotent groups.

We consider a possibly empty family P of primes. As usual, we call the integer n a *P-number* if the prime factors of n lie in P, and a *P'-number* if they lie outside P. A group N is called *P-local* if the function $x \mapsto x^q$, $x \in N$, is bijective for all P'-numbers q. A *commutative* group A is thus P-local if and only if it is uniquely divisible by P'-numbers, and this is equivalent to saying that A is a $\mathbb{Z}_P$-module, where $\mathbb{Z}_P$ is the ring of integers *localized* at P. For abelian groups the tensor product with $\mathbb{Z}_P$ defines a functor that associates with any abelian group A a P-local abelian group $A_P = \mathbb{Z}_P \otimes A$. Attached to this P-localization functor is a natural transformation given by the obvious map $l: A \to A_P = \mathbb{Z}_P \otimes A$.

This P-localization functor, defined on the category of abelian groups, can be extended to a P-localization functor on the category of nilpotent groups. It associates with any nilpotent group N a P-local group N_P, and there is a natural transformation $l: N \to N_P$ with the following universal property: for all P-local nilpotent groups M and all homomorphisms $\varphi: N \to M$, there exists a unique homomorphism $\psi: N_P \to M$ with $\psi l = \varphi$. In that case, one says that the map $l: N \to N_P$ *P-localizes.*

The existence of the P-localizing functor on the category of nilpotent groups can be proved using homological algebra, in particular, the (co)homology theory of groups, and induction on the nilpotency class. The general result is as follows:

Proposition 2.1. *Let N be a nilpotent group. Then there is a P-local group N_P and a map $l: N \to N_P$ which P-localizes; and* nil $N_P \leq$ nil N. *Moreover, a homomorphism $\tilde{l}: N \to M$ between nilpotent groups P-localizes if and only if the induced map $l_n: H_n(N) \to H_n(M)$ P-localizes for all $n \geq 1$. In particular, a nilpotent group is P-local if and only if its homology groups in positive dimensions are P-local.*

The group N_P, as well as the map l, may be constructed, as we have said, by induction on the nilpotency class of the group N. To start the induction, one has to show that, for an abelian group A, the map $l: A \to \mathbb{Z}_P \otimes A$ induces localizing maps $l_n: H_n(A) \to H_n(\mathbb{Z}_P \otimes A)$ for all

$n \geq 1$. This is done by an explicit calculation. For the inductive step one considers the central extension

$$\gamma^c N \rightarrowtail N \twoheadrightarrow N/\gamma^c N, \tag{2.1}$$

where c is the nilpotency class of N. One then proves that the natural map

$$H^2((N/\gamma^c N)_P, (\gamma^c N)_P) \to H^2(N/\gamma^c N, (\gamma^c N)_P)$$

is an isomorphism, and uses this to construct the group N_P as a central extension

$$(\gamma^c N)_P \rightarrowtail N_P \twoheadrightarrow (N/\gamma^c N)_P, \tag{2.2}$$

together with a map l from the central extension (2.1) to the central extension (2.2). The homological property of the map l of the proposition finally is deduced from the Lyndon–Hochschild–Serre spectral sequence (see Chapter VIII, Section 9) associated with the two group extensions.

This construction procedure may also be used to establish further key properties of the localizing functor. We mention the following:

(i) If $N' \rightarrowtail N \twoheadrightarrow N''$ is a short exact sequence of nilpotent groups, then the sequence $N'_P \rightarrowtail N_P \twoheadrightarrow N''_P$ is exact.

(ii) A homomorphism $\theta: N \to K$ of nilpotent groups P-localizes if and only if K is P-local and θ is P-bijective. Here we say that θ is *P-injective* if $\ker \theta$ consists exclusively of P-torsion elements; θ is *P-surjective* if, given any $y \in K$, there exists a P'-number q such that $y^q \in \operatorname{im} \theta$; and θ is *P-bijective* if it is P-injective and P-surjective.

Property (ii) often makes it possible to detect the P-localization where it would be hard (if not impossible) to establish the universal property directly.

In a certain sense the homology theory of groups is easy to handle when we confine ourselves to nilpotent groups. It is convenient to introduce the idea of a Serre class of nilpotent groups generalizing the classical notion of a Serre class of abelian groups (see [HR] and [S]). Thus, a Serre class $\mathscr{C}$ of abelian groups satisfies the following axioms:

(I) Given a short exact sequence of abelian groups $A' \rightarrowtail A \twoheadrightarrow A''$, then $A', A'' \in \mathscr{C}$ if and only if $A \in \mathscr{C}$;

(II) if $A, B \in \mathscr{C}$, then $A \otimes B$ and $\operatorname{Tor}(A, B) \in \mathscr{C}$; and

(III) if $A \in \mathscr{C}$, then $H_n A \in \mathscr{C}$ for $n \geq 1$.

(There are possible variants, but these are the axioms we will assume.) Now let $\mathscr{C}$ be any Serre class of abelian groups. We say that the nilpotent group N belongs to the class $\mathscr{C}'$ if and only if $N_{\mathrm{ab}} = N/\gamma^2 N$ is in $\mathscr{C}$. The class $\mathscr{C}'$ may rightly be called a Serre class of nilpotent groups, since it satisfies properties analogous to (I) and (III). To see this, one first shows that for $N \in \mathscr{C}'$ the successive quotients $\gamma^i N/\gamma^{i+1}N$ belong to $\mathscr{C}$. This allows one to prove that the nilpotent group N belongs to $\mathscr{C}'$ if (and only if) the integral homology groups $H_n N$, $n \geq 1$, all belong to $\mathscr{C}$. Using this, it is not hard to see that, given an extension of nilpotent groups $N' \rightarrowtail N \twoheadrightarrow N''$, one has $N', N'' \in \mathscr{C}'$ if and only if $N \in \mathscr{C}'$.

It is obvious from the above that if $\mathscr{C}$ is the (non-Serre) class of P-local abelian groups, then the class $\mathscr{C}'$ is the class of P-local nilpotent groups. There are various interesting classes of nilpotent groups that may be defined in a similar way. We mention the following example. Consider the Serre class $\mathscr{C}$ of abelian groups whose P-localization is a finitely generated $\mathbb{Z}_P$-module. Then the associated class $\mathscr{C}'$ of nilpotent groups consists of P-local groups N_P which have the property that there is a finite subset S which generates N_P as a P-local group in the sense that N_P is the smallest P-local subgroup containing S. Of particular interest in applications is the Serre class $\tilde{\mathscr{C}}$ consisting of the abelian groups whose p-localization is finitely generated as a module over $\mathbb{Z}_p$ for all primes p (the set P consists of just one prime p), and its associated class $\tilde{\mathscr{C}}'$ of nilpotent groups. It is of course substantially larger than the class of finitely generated nilpotent groups.

There is a *relative* version of the localization of nilpotent groups. Let the group Q act on the group N. We construct a *lower central series* of N as a Q-group by the rule

$$\gamma_Q^1 N = N, \qquad \gamma_Q^{i+1} N = \mathrm{gp}\langle a(xb)a^{-1}b^{-1}\rangle, \qquad i \geq 1,$$

where $a \in N$, $b \in \gamma_Q^i N$, $x \in Q$. Then N is called *Q-nilpotent of class $\leq c$* if $\gamma_Q^{c+1} N = 1$. It is not hard to see that relative nilpotency naturally arises in the study of extensions of nilpotent groups. For in the extension $N \rightarrowtail G \twoheadrightarrow Q$ the group G is nilpotent if and only if Q is nilpotent and N is G-nilpotent. Moreover, one can show by arguments similar to those sketched in connection with Serre classes that a nilpotent Q-group N is Q-nilpotent if and only if N_{ab} is Q-nilpotent. If N is Q-nilpotent, then Q acts on N_P so that $\gamma_Q^i N_P = (\gamma_Q^i N)_P$. If Q is itself nilpotent and acts nilpotently on N there is an induced nilpotent action of Q_P on N_P such that $\gamma_Q^i N_P = \gamma_{Q_P}^i N_P$.

From the point of view of application to homotopy theory, the case when N is abelian is of especial importance. For a path-connected space X is called *nilpotent* if its fundamental group $\pi_1(X)$ is nilpotent and acts nilpotently on all higher homotopy groups $\pi_n(X)$, $n \geq 2$. (Note that these groups are abelian.) Then localization theory may be extended in a natural way to nilpotent spaces. A nilpotent space Y is called *P-local* if its homotopy groups $\pi_i(Y)$, $i \geq 1$, are P-local. There is a procedure to associate with any nilpotent space X a P-local nilpotent space X_P, and a natural transformation $l_P : X \to X_P$ which P-localizes, i.e., it satisfies the following universal property: for all P-local nilpotent spaces Y and all continuous maps $f : X \to Y$, there exists a continuous map, unique up to homotopy, $g : X_P \to Y$ with $gl \simeq f$. It turns out that a continuous map $\tilde{l} : X \to Z$ P-localizes if and only if the induced homomorphisms from $\pi_n(X)$ to $\pi_n(Z)$, $n \geq 1$, P-localize. We finally mention that the full notion of Q-nilpotency is needed to define a nilpotent fiber space and its P-localization.

It is also worth remarking that the isomorphism

$$l^* : H^n(Q_P, A_P) \simeq H^n(Q, A_P),$$

exploited above when Q acts trivially on the $\mathbb{Z}_P$-module A_P, remains valid when Q acts nilpotently on A_P.

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3. Finiteness Conditions on Groups

As explained in Section 1, there is an intimate relationship between the cohomology of groups and algebraic topology. We shall say a few more things about it here, concentrating on various topological and cohomological finiteness conditions on groups. This area has seen a rapid development in the past years, and many important connections with number theory and the theory of algebraic groups have surfaced.

Let us first recall that the cohomology of a group G with coefficients in the $\mathbb{Z}G$-module A can be obtained in a topological framework via an Eilenberg–MacLane space $K(G, 1)$. The cohomology of $K(G, 1)$ with local coefficients $\{A\}$ is the cohomology of the group G with coefficients in A. In fact, it is explained in Section 1 that if $X = K(G, 1)$ is realized as a CW-complex, the chain complex of its universal cover $\tilde{X}$ provides a $\mathbb{Z}G$-free resolution of $\mathbb{Z}$.

From a topological point of view there is thus considerable interest in groups G that admit a $K(G, 1)$ with certain additional properties, properties which have already proved to be important in algebraic topology. The first class of groups defined using such a scheme is as follows:

Definition. A group G is said to be of type F if and only if it admits a $K(G, 1)$ which is a *finite* CW-complex.

If G is a group of type F it admits a $K(G, 1)$ which is a finite and hence also finite-dimensional CW-complex. The chain complex of its universal cover therefore provides a $\mathbb{Z}G$-free resolution of $\mathbb{Z}$, i.e., an exact sequence

$$0 \to L_m \to L_{m-1} \to \cdots \to L_1 \to L_0 \to \mathbb{Z} \to 0,$$

with L_i a free $\mathbb{Z}G$-module of finite rank for $0 \le i \le m$.

In particular, it follows that a group of type F has cohomological dimension $\le m$, for some m. We say that a group G has *cohomological dimension* $\le m$, cd $G \le m$ if $H^q(G, A) = 0$ for all $\mathbb{Z}G$-modules A and all $q > m$. It is not hard to show that cd $G \le m$ is equivalent to the following (see Chapter VI, Exercise 15.6):

For every $\mathbb{Z}G$-projective resolution of $\mathbb{Z}$

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

the image of $P_m \to P_{m-1}$ is projective.

It is apparent from the above that the class of groups G with cd $G < \infty$ contains the class F as well as the class of groups that admit a *finite-dimensional* (but not necessarily finite) $K(G, 1)$. In Section 1 we have seen that a group G which contains torsion cannot have finite-cohomological dimension, and it therefore cannot be of type F. Examples show that torsion-free groups too, in general, have infinite cohomological dimension.

There are refinements of the concepts introduced above in several directions. Of the many possibilities we mention those which have turned out to be the most important ones.

Definition. (i) A group G is said to be of type F_m if it admits a $K(G, 1)$ whose m-skeleton consists of only finitely many cells.

(ii) A group G is said to be of type F_∞ if it admits a $K(G, 1)$ such that its m-skeleton for all $m \ge 0$ consists of only finitely many cells.

(iii) A group G is said to be of type FP_m if the trivial module $\mathbb{Z}$ admits a $\mathbb{Z}G$-projective resolution

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0,$$

such that P_i is finitely generated for $0 \le i \le m$.

(iv) A group G is said to be of type FP_∞ if the trivial module $\mathbb{Z}$ admits a $\mathbb{Z}G$-projective resolution consisting entirely of finitely generated projective modules.

Clearly a group of type F_m is also of type FP_m. It is a little exercise (in homological algebra) to prove that FP_∞ is equivalent to FP_m for all m. It is a somewhat bigger exercise (in algebraic topology) to show that F_∞ is equivalent to F_m for all m. Also, it can easily be shown that if G is a group of type FP_m, then the trivial module $\mathbb{Z}$ even admits a *free* resolution consisting of finitely generated *free* modules up to dimension m.

Finite groups plainly are both of type F_m and of type FP_m for all $m \geq 0$. Also, it is not hard to see that a group is of type F_1 (or of type FP_1) if and only if it is finitely generated. Moreover, a standard procedure in algebraic topology yields the following: If G is a *finitely presentable group* (for a definition see Chapter VI, Exercise 9.2), then there is a $K(G, 1)$ whose 2-skeleton is finite: one starts with a bouquet of 1-cells, one cell for each generator, and attaches 2-cells according to the (finitely many) relators. To get an aspherical space one then successively adds cells of higher dimension to "kill" the homotopy groups. A finitely presentable group G is thus of type* F_2 and hence also of type FP_2. The question whether the converse is true, i.e., whether a group of type FP_2 is also of type F_2 had been a long-standing open question. It has been settled only recently in the negative by M. Bestvina and N. Brady [BB]. They show that for all $m \geq 2$ the classes F_m and FP_m are in fact different.

It is obvious that a group of type F_m or FP_m has the property that the integral homology and cohomology groups $H_i(G)$, $H^i(G)$ are finitely generated for $0 \leq i \leq m$. An early example of Stallings [S], already mentioned in Chapter VI, Exercise 15.2, shows that there are finitely presentable groups with the property that $H_3(G)$ is not finitely generated. Stallings' group thus is in F_2 but not in F_3 (nor in FP_3). Similar examples may be constructed to show that for any $m \geq 1$ the class F_{m+1} is strictly smaller than the class F_m.

The relevance of these concepts stems from the fact that a great many of the groups that play a central role in number theory, algebraic group theory, or Lie theory are indeed of type F_m or FP_m for some m. We mention as examples the group $SL(2, \mathbb{Z})$ and, more generally, the arithmetic groups, the discrete subgroups of Lie groups, and the automorphism groups of finitely generated free groups.

Many of the groups of type F_m or FP_m for some m which arise in other fields have the additional feature that they contain a subgroup of finite index which has finite cohomological dimension. We say that a group G is of *virtual cohomological dimension* $\leq m$, vcd $G \leq m$, if there is a subgroup U of finite index in G with** cd $U \leq m$. Note that U is necessarily torsion free; moreover, it can be shown that U is of type F_m (or FP_m) if and only if G is of type F_m (or FP_m). This shows once more the relevance of the concept of finite cohomological dimension.

There is no apparent reason why there should be a connection between finite virtual cohomological dimension and type FP_∞. In fact, there are torsion-free groups which are of type FP_∞, but of infinite cohomological dimension: K. Brown and R. Geoghegan [BG] have

* The converse also holds: if G is of type F_m, $m \geq 2$, then G is finitely presentable.

** Remarkably, cd U is independent of the choice of U.

shown that Thompson's group T has this property; the group T is defined by the presentation $T = (x_0, x_1, x_2, \ldots; x_i^{-1}x_nx_ix_{n+1}^{-1}$ for $i < n)$. This example contrasts with a recently obtained, and somewhat surprising, result of P.H. Kropholler (partly in collaboration with J. Cornick): if G is a group of type FP_∞ which is linear or soluble (in fact, a much wider class is treated in that paper), then G is also of finite virtual cohomological dimension [K]. We note in addition that, for both linear groups and soluble groups, the fact that the cohomological dimension is finite has a rather well-understood group-theoretic meaning [AS], [St]. For the special case of nilpotent groups we shall come back to this point below.

Of course, other properties of topological spaces may be used to give, or better to suggest, definitions for further classes of groups. For example, we may consider groups that admit a $K(G, 1)$ which is a closed manifold of dimension m. Clearly, such a group is of type F_m and also of cohomological dimension m. But of course more is true. The cohomology of the group G in this case satisfies a duality modeled on the Poincaré duality in the cohomology and homology of a manifold. This has led to the definition of so-called Poincaré duality groups (see [JW] and [B]). A group G is called a *Poincaré duality group* (of dimension m) if its cohomology and homology groups are related by a duality mimicking the Poincaré duality of a closed orientable manifold (of dimension m), i.e., there is a natural equivalence of functors

$$H^i(G, -) \simeq H_{m-i}(G, -), \qquad i = 0, 1, \ldots, m.$$

Of course, the fundamental group of a closed orientable surface of genus ≥ 1 is a Poincaré duality group of dimension* 2. There are many other examples. In particular, it is possible to show (using induction and the Lyndon–Hochschild–Serre spectral sequence) that any finitely generated torsion-free nilpotent group N is a Poincaré duality group. Its dimension m is given by the so-called Hirsch number $h(N)$ of N. The Hirsch number $h(N)$ is defined as the sum of the torsion-free ranks of the finitely many nontrivial successive quotients $\gamma_i N/\gamma_{i+1}N$ of the lower central series of N (see Section 2).

Taking as a model, instead of a closed manifold, a manifold (of dimension m) with boundary leads to the concept of a duality group (of dimension m) (see [BE]). Again the cohomology and homology of the group are related in a specific way to each other, but the coefficients change according to the presence of the boundary. More precisely, a group G is said to be a *duality group* of dimension m if there is a $\mathbb{Z}G$-module C and a natural equivalence of functors

* The converse is also true (see [E]).

$$H^i(G, -) \simeq H_{m-i}(G, C \otimes -), \qquad i = 0, 1, \dots, m.$$

Clearly, Poincaré duality groups are special cases of duality groups, but there are others. Any noncyclic knot group, i.e., the fundamental group of the space of a knot, is an example of a duality group of dimension 2 which is not a Poincaré duality group. It is a deep theorem of Borel and Serre [BS] that any torsion-free arithmetic group is a duality group.

We finally note that, although it is not part of the definition of duality and Poincaré duality groups of dimension m, these groups necessarily turn out to be of type FP_m and of cohomological dimension m.

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4. Modular Representation Theory

One of the most active fields of application of homological algebra over the past 10 or 15 years is the representation theory of groups. This concerns both the modular representation theory of finite groups and the representation theory of algebraic groups. In this short survey section we shall only be able to describe some parts of the development in the modular representation theory of finite groups.

Let G be a finite group and let k be a (big enough) field of characteristic p. The linear representations of G over k correspond to the finite-dimensional kG-modules. The theorem of Maschke (see Theorem 16.6 in Chapter VI) says that the kG-modules are semisimple if p does not divide the group order $|G|$. The converse can easily be shown to be true too. This result can be expressed in homological terms as follows:

Theorem 4.1 (Maschke). *There exist kG-modules A, B with $\mathrm{Ext}^1_{kG}(A, B) \neq 0$ if and only if p divides the group order $|G|$.*

Modular representation theory concerns itself with the case where p divides the group order $|G|$. To record the information about the nontrivial extensions a graph $\Gamma(G)$ may be defined. The vertices of $\Gamma(G)$ correspond bijectively to the simple modules $S_0 = k, S_1, \ldots, S_l$. The graph $\Gamma(G)$ has an edge connecting S_i and S_j if and only if $\mathrm{Ext}^1_{kG}(S_i, S_j) \neq 0$. Two simple modules S_i and S_j are said to be in the same *block* $\mathscr{B}$ if the vertices S_i and S_j lie in the same connected component of $\Gamma(G)$. The block containing $S_0 = k$ is called *principal*. The notion of blocks was introduced by R. Brauer in the 1940s using characters. Over the years it has been realized that the notion is basically of a homological nature. The graph $\Gamma(G)$ described above yields a simple and transparent homological definition.

Much of the role simple modules play in complex representation theory is played in modular representation theory by the indecomposable modules. These are the modules which cannot be written in a nontrivial way as a direct sum of submodules. It is not hard to show, by induction on the composition length and using the long exact Ext-sequence, that the composition factors of an indecomposable module all belong to the same block. For finite groups the number of (isomorphism classes of) simple modules is finite; however, the number of (isomorphism classes) of indecomposable modules is in most cases infinite. There are finitely many indecomposable kG-modules, if and only if the p-Sylow subgroups of G are cyclic. In this case, one says that the *representation type* of kG is finite. The classical modular representation theory of Brauer, the theory of cyclic defect, is able to handle that case completely and to give a complete classification of the indecomposable modules. For infinite representation type this cannot be achieved: it can be shown that the representation theory of kG in most of these cases contains the representation theory of the polynomial algebra in two noncommuting indeterminates, which is known to be too complicated to deal with effectively.

In order to make progress in the general case the class of indecomposable modules must be divided into subclasses with special properties. The techniques of subdivision in many of these cases involve homological notions. In the sequel we shall attempt to describe some of the techniques used.

One first possibility, of course, is to sort out the indecomposable modules which are *projective* over kG. (We add here that, over kG, the notions projective and injective coincide.) It turns out that these modules are precisely the indecomposable direct summands of the module kG. They are called the *principal indecomposable modules*. It is not hard to see that they correspond bijectively to the simple modules. Given their nature, it is not surprising that the structure of these modules is of great interest in modular representation theory. Much work has been done in that direction; many results, either in the form of general theorems or in the form of explicit results about specified finite groups, are known (see, for example, [St] and [MSS]).

A further technique to deal with indecomposable modules is given by Green's theory (see [G] and [A]). The approach uses the homological notion of relative projectivity (see Chapter IX, Sections 1, 2, 3), and thus in fact can be regarded as sorting out classes of indecomposable modules generalizing the class of projective modules.

We recall that a kG-module M is called *projective relative to a subgroup* U *of* G if and only if every short exact sequence of kG-modules

$$0 \to A \to B \to M \to 0$$

whose restriction to kU splits, also splits over kG. It is not hard to see that if S is a p-Sylow subgroup of G then every kG-module is projective relative to S. Let M now be an indecomposable module. Consider the class of subgroups U with the property that M is projective relative to U, and let V be a minimal element in that class. By the above remark we know that V is a p-subgroup. It is a remarkable fact which lies at the basis of Green's theory that V is uniquely determined up to conjugation in G. It is called the *vertex* of M.

If the module M is projective relative to the subgroup U, then the short exact sequence

$$0 \to K \to kG \otimes_{kU} M \to M \to 0,$$

which obviously splits over kU, also splits over kG. In other words, M is a direct summand of $kG \otimes_{kU} M$. If now M is indecomposable, then it follows from this that there is an indecomposable kU-module N such that M is a direct summand of $kG \otimes_{kU} N$. If U is taken to be the vertex

V of M, it turns out that the kV-module N is essentially uniquely determined. It is called the *source* of M.

The theory of *vertices and sources* due to J.A. Green has proved to be extremely important in modular representation theory. It can be used, for example, to set up the so-called *Green correspondence*. The full theory is too involved to be described here, but it is possible to describe a special case. Let Q be a p-subgroup of G and let L be the normalizer $N_G(Q)$. Then there is a correspondence between the (isomorphism classes of) indecomposable kG-modules with vertex Q and the (isomorphism classes of) indecomposable kL-modules with vertex Q. Moreover, the correspondence is given in a conceptually simple way: If M is an indecomposable kG-module with vertex Q, then its *Green correspondent* N is the (uniquely determined) indecomposable direct summand of the restriction M_L with vertex Q. The converse map is given in a similar way: If N is an indecomposable kL-module with vertex Q, then its *Green correspondent* M is the (uniquely determined) indecomposable direct summand of the kG-module $kG \otimes_{kL} N$ with vertex Q.

We may add here that the Green correspondence can be regarded as an extension of the so-called Brauer correspondence, which was introduced by R. Brauer in the framework of character theory in the early 1940s. In fact, Brauer's defect groups of G can be interpreted as vertex groups of certain well-defined kG-modules (see [G]).

The control of homomorphisms under the Green correspondence (or under similar correspondences) is usually a complicated issue. It may require us to pass from the module category to more complicated categories, like the stable or derived category. These homological constructions have been used not only in this context but also in a great number of other applications, and we shall therefore devote a special survey section to them (Section 5).

Apart from the possibilities already described of sorting out subclasses of indecomposable modules there are other possibilities. One of them is connected with the notion of *complexity*. We shall go into this in somewhat more detail now, since it is connected with other important developments in the cohomology theory of groups.

Let us first consider the cohomology group $H^*(G, k)$ of G with the field k as coefficients. Under the cup-product $H^*(G, k)$ becomes a graded algebra over k, which is commutative in the graded sense (see Exercise 13.6 of Chapter VI). Restricting to even degrees, one obtains a commutative algebra over k. It has been shown that for G finite the algebra $H^*(G, k)$ is finitely generated (see Evens [E2] and Venkov [V]). Properties of the group G can now be linked to properties of the graded commutative ring $H^*(G, k)$.

One such property which has been looked at in detail is the so-called *Krull dimension*. For our purpose it is convenient to measure the

Krull dimension of $H^*(G, k)$ by the rate of growth of the sequence $d_n = \dim_k H^n(G, k)$, $n = 0, 1, 2, \ldots$. The *growth* of a sequence $\{s_n\}$ of integers is defined to be the least nonnegative integer c such that there is a constant C with

$$s_n \le Cn^{c-1}$$

for all $n \ge 0$. The growth $c = c(G)$ of the sequence $\{d_n\}$ may be taken as the definition of the Krull dimension of the algebra $H^*(G, k)$. Obviously one has $c(G) = 1$ if and only if the dimensions of the homology groups $H^n(G, k)$ are uniformly bounded. It is not hard to see, using the restriction map, that this is the case if and only if the p-Sylow subgroup of G is cyclic or, if $p = 2$, the 2-Sylow subgroup is cyclic or generalized quaternion.

It was a major breakthrough in the theory when Quillen [Q] proved the following result (previously conjectured by Atiyah and Swan):

Theorem 4.2 (Quillen). *The Krull dimension $c(g)$ of $H^*(G, k)$ equals the maximum of the Krull dimensions $c(E)$ of $H^*(E, k)$ where E runs through the elementary abelian p-subgroups of G.*

The Krull dimension $c(E)$ of the elementary abelian p-group E of order p^n is seen to be n. Thus Quillen's result also says that the Krull dimension of $H^*(G, k)$ equals the p-rank of the group G, i.e., the maximal rank of the elementary abelian p-subgroups E of G.

In developing the consequences of Quillen's result it turned out that it is possible to associate with an indecomposable kG-module M a numerical invariant, called the *complexity* $c_G(M)$, such that $c_G(k)$ is the Krull dimension of $H^*(G, k)$. For the purpose of this short essay it is easiest to define $c_G(M)$ via projective resolutions of M. Let

$$\mathbf{P}: \quad \cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

be a kG-projective resolution of M. Set $p_n = \dim_k P_n$, and let $c(\mathbf{P})$ be the growth rate of the sequence $\{p_n\}$. Then the complexity $c_G(M)$ of the module M is defined as the minimum of $c(\mathbf{P})$ taken over all kG-projective resolutions of M. It can be shown that $c_G(M)$ is realized by the *minimal* projective resolution. This is the resolution where, in the stepwise construction, the dimension of P_n is minimal, i.e., where P_n is taken to be the projective cover (see Chapter I, Section 9) of $\ker(P_{n-1} \to P_{n-2})$. Note that projective covers do exist in this context! It turns out that for $M = k$ one has $c_G(k) = c(G)$, so that the notion of complexity of a module in fact generalizes the notion of Krull dimension of the algebra $H^*(G, k)$. It is a remarkable fact that $c_G(M)$ only depends on the restrictions M_E of M to the elementary abelian p-subgroups E of G. Much as in Quillen's Theorem one has (see [AE1] and [C2]):

Theorem 4.3 (Alperin, Evens; Carlson). *The complexity $c_G(M)$ of a module M is the maximum of the complexities $c_E(M_E)$ of the restrictions M_E of M to the elementary abelian p-subgroups E of G.*

It follows immediately from the definition of the complexity that $c_G(M) = 0$ if and only if the module M is kG-projective. The above theorem thus implies as corollary the following surprising result, which had actually been obtained earlier:

Corollary 4.4 (Chouinard). *The module M is kG-projective if and only if, for every elementary abelian p-subgroup of G, the restriction M_E is kE-projective.*

It is obvious now that the class of indecomposable modules can be subdivided according to complexity. In this direction much work has already been done, and it remains an area of intensive research.

In the framework of commutative algebra the Krull dimension of $H^*(G, k)$ can also be interpreted as the dimension of the maximal ideal spectrum X_G of $H^*(G, k)$. (To be precise one has to restrict attention to the even-dimensional part $H^{2*}(G, k)$ of the cohomology algebra, if $p \neq 2$.) This approach too can be generalized to modules. Given the module M a certain subvariety $X_G(M)$ of $X_G = X_G(k)$ is defined in such a way that its dimension is $c_G(M)$. We cannot go into the details of the definition here, but would like to remark that the theory of these varieties has proved to be of great value both in the cohomology theory of finite groups and in modular representation theory (see [AE2], [C2], [B], and [C1]).

It is perhaps worth saying a few words about the techniques of proof used in the whole area around the notion of complexity. A very crucial role is played by the Lyndon–Hochschild–Serre spectral sequence associated with a group extension (see Theorem 9.5 in Chapter VIII). The proofs in most cases, however, use much more than the mere existence of that spectral sequence; properties of naturality and rather delicate properties connected with the products in the cohomology of groups play an important role. The part of the theory described above which is connected with elementary abelian subgroups depends in an apparently essential way on a celebrated lemma of Serre [S] which characterizes elementary abelian p-groups by properties of their cohomology algebra. The proof of this crucial lemma in turn seems to require Steenrod operations or a related tool, like Evens' norm map (see Benson [B]).

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5. Stable and Derived Categories

Often the main result of a mathematical theory consists in a statement about the equivalence of two categories. In homological algebra the categories involved are usually abelian or at least additive, and we shall of course concentrate here on this case. Most of our examples will be taken from representation theory, and the categories in question will be module categories over some group algebra. An equivalence of categories, in principle, gives a lot of information about the situation, although one should be aware of the fact that it may be hard in general to extract that information from the equivalence. To illustrate this point we briefly mention the case where the categories $\mathfrak{M}_R$ and $\mathfrak{M}_S$ for the two rings R and S are equivalent. This clearly says something important about the rings R and S. The precise statement, however, is not at all obvious. The problem is studied by the so-called Morita theory (see, for example, [CR]). The result is that the ring S is isomorphic to an endomorphism ring of a projective generator of the category $\mathfrak{M}_R$ and vice versa. (A projective generator in an abelian category $\mathfrak{A}$ is an object G which is projective and has the property that for each object A of $\mathfrak{A}$ there is a nonzero morphism $\varphi : G \to A$. See Chapter I, Sections 7 and 8, where the dual of this notion has been used informally.) It is shown in Morita theory that a functor establishing the equivalence of the two categories can be realized as a tensor product with a certain left-S-right-R-module (respectively, a left-R-right-S-module).

A specific example of a Morita equivalence between two module categories is given as follows. Let K be a commutative ring, and let R be the $n \times n$-matrix ring $M_n(K)$ over K. Then the module categories $\mathfrak{M}_K$ and $\mathfrak{M}_{M_n(K)}$ are equivalent. The equivalence is given by the functor $F = P \otimes - : \mathfrak{M}_{M_n(K)} \to \mathfrak{M}_K$, $F(Y) = P \otimes_{M_n(K)} Y$ where P is the free (left)-K-module on n elements, which is also regarded as a (right)-$M_n(K)$-module in the obvious way. Note that $M_n(K)$ can be identified with the K-endomorphism ring of the (right)-K-module $\operatorname{Hom}(P, K)$.

Morita equivalences occur frequently in modular representation theory. A simple specific example is as follows. Let k be a field of characteristic p, let G be a finite group, and let N be a nontrivial normal p'-subgroup. Let M be a simple kN-module. For $x \in G$ we define the conjugate module xM: As k-vector space xM is M; the N operation is given by $n \circ (xm) = x((x^{-1}nx)m)$. The elements $x \in G$ with $xM \simeq M$ as kN-modules form a subgroup T of G, the so-called *inertial subgroup* of M. Let $\mathfrak{T}$ denote the category of kT-modules which, upon restriction to N, become a direct sum of copies of M. Finally, let $\mathfrak{G}$ be the subcategory of $\mathfrak{M}_{kG}$ whose indecomposable modules, restricted to N, have a direct summand isomorphic to M. It is part of the so-called Clifford theory that the categories $\mathfrak{T}$ and $\mathfrak{G}$ are equivalent, the equivalence being given by induction from T to G, i.e., we associate with A in $\mathfrak{T}$ the module $kG \otimes_{kT} A$.

It is clear that equivalence in many cases is too strong a property to describe the relationship between categories that are given a priori; the relationship is usually of a much weaker form. Nevertheless, it may happen that it can be expressed by an assertion saying that two categories *associated* in some way with the given situation are equivalent. Instances of the phenomenon just described are plentiful; the representation theory of algebraic or of finite groups provides many important examples.

The relationship may take many different forms, but we shall restrict ourselves to two cases which occur particularly often in this context: the *stable* and *derived* equivalence. Also, it has turned out that the procedure leading from the given abelian category to the associated stable or derived category is, despite its abstractness, extremely suitable to express certain homological properties of the categories involved. We shall therefore describe these important constructions in some detail. Although it is a fact that the constructions yield their greatest benefits in a general context, we shall restrict ourselves here mostly to the concrete situation of a module category or a full subcategory of a module category. We do that in order to keep the description as transparent and nontechnical as possible.

I. We shall first concern ourselves with the *stable category* associated with a module category. Thus let R be a ring and let $\mathfrak{M}_R$ be the category of left modules over R. We call a homomorphism $f: A \to B$ *projective* if it factors through a projective module, i.e., if and only if there exists a projective module P and homomorphisms $g: A \to P$ and $h: P \to B$ with $f = h \circ g$. It is easy to see (compare Chapter IV, Exercise 5.7) that the projective homomorphisms from A to B form a subgroup $\mathrm{Phom}_R(A, B)$ of $\mathrm{Hom}_R(A, B)$. We now set

$$\underline{\mathrm{Hom}}_R(A, B) = \mathrm{Hom}_R(A, B)/\mathrm{Phom}_R(A, B).$$

(Note that in Chapter IV, Exercise 5.7, the (old-fashioned) notation $\Pi P(A, B)$ is used to denote this group.) The *stable category* $\mathfrak{S}\mathfrak{t}\mathfrak{M}_R$ of $\mathfrak{M}_R$ is then defined as follows: The objects are the same as those in $\mathfrak{M}_R$. The morphisms are the equivalence classes of morphisms in $\mathfrak{M}_R$ modulo the projective homomorphisms; the set of morphisms from A to B in $\mathfrak{S}\mathfrak{t}\mathfrak{M}_R$ is thus given by $\underline{\mathrm{Hom}}_R(A, B)$.

It follows immediately that the projective modules, regarded as objects in $\mathfrak{S}\mathfrak{t}\mathfrak{M}_R$, become isomorphic to 0, the null module. Moreover, two modules A and B are isomorphic in $\mathfrak{S}\mathfrak{t}\mathfrak{M}_R$ if and only if there exist projective modules P and Q with $A \oplus P \simeq B \oplus Q$.

We note in passing that the notion of a projective homomorphism (and its dual, the notion of an injective homomorphism) is formally related to the homotopy theory of topological spaces. The notion, with different terminology, was introduced by B. Eckmann and P. Hilton (see [E] and [H]). In these papers a homomorphism $f: A \to B$ is termed *p-nullhomotopic* if it can be factored through every module $\bar{B}$ of which B is a quotient. It is easy to see that a homomorphism is *p-nullhomotopic* if and only if it is projective. This notion has given rise to a "homotopy theory" of modules; in today's terminology this is nothing else but the theory of the stable category.

The stable category is the appropriate category to study certain notions of homological algebra. We illustrate this by the following simple example. Let A be in $\mathfrak{M}_R$ and let

$$0 \to K \to P \to A \to 0$$

be a projective presentation of A (see Chapter III, Section 2). In many basic propositions of homological algebra (construction of projective resolutions, dimension shifting, etc.), the *kernel* K plays a crucial role, and it is tempting to use a notation for K that indicates in some way the dependence of K on A, for example, $\Omega(A)$, to denote K. This may, however, be somewhat misleading to the reader, since $\Omega(-)$ is *not* a functor on $\mathfrak{M}_R$. However, in the category $\mathfrak{St}\mathfrak{M}_R$, the passage from (the object corresponding to) A to (the object corresponding to) $\Omega(A)$ becomes a functor (see [H]): up to isomorphism in $\mathfrak{St}\mathfrak{M}_R$ the kernel K does not depend on the chosen projective presentation of A, and, given the map $\varphi: A \to A'$, the lifting of φ produces a well-defined element $\Omega(\varphi)$ in $\underline{\mathrm{Hom}}_R(A, A')$. The proof of this fact is a little exercise; it can also be read off from the material in Chapter III, Section 2. It is obvious that the endofunctor Ω of the category $\mathfrak{St}\mathfrak{M}_R$ will play an important role in many circumstances. It should be noted that the passage from $\mathfrak{M}_R$ to $\mathfrak{St}\mathfrak{M}_R$ is not without problems: the stable category is not in general abelian.

In some cases the categories $\mathfrak{St}\mathfrak{M}_R$ and $\mathfrak{St}\mathfrak{M}_S$ are equivalent even if $\mathfrak{M}_R$ and $\mathfrak{M}_S$ are not equivalent. One then speaks of a *stable equivalence* between these two categories of modules. A simple example of stable (self-)equivalence is given by the functor Ω in the category $\mathfrak{St}\mathfrak{M}_{kG}$ where G is a finite group and k is a field of characteristic p dividing the group order. For in this case the functor Ω is invertible, and thus establishes a nontrivial self-equivalence.

Often a stable equivalence, i.e., an equivalence between the stable categories $\mathfrak{St}\mathfrak{M}_R$ and $\mathfrak{St}\mathfrak{M}_S$, is induced from a functor $F: \mathfrak{M}_R \to \mathfrak{M}_S$. There are many examples of this kind in modular representation theory. We mention the following: Let k be a field of characteristic p, and let G be a finite group with the property that the intersection of any two p-Sylow-subgroups S and $S' = xSx^{-1}$ is either S or $\{e\}$. (This property is usually called the *trivial intersection property*.) If H denotes the normalizer of S in G, then there is a stable equivalence F between the principal block of kG and the principal block of kH. (In modular representation theory the principal block is a certain well-defined full subcategory of the category of modules over the group, as explained in Section 4.) The stable equivalence F is given on the level of module categories by induction, i.e., one associates with the kH-module A the module $kG \otimes_{kH} A$.

II. As far as homological algebra is concerned the *derived category* is the ultimate tool (for a complete treatment of the theory see, for example, [G] and [W]). Starting from the category $\mathfrak{C}$ of cochain[1] complexes $\mathbf{C}$ in, say, a (fixed) abelian category $\mathfrak{A}$, one first passes to the associated

[1] The theory of derived categories has emerged in sheaf theory, and in this situation it is natural to consider *cochain* complexes and not chain complexes. The terminology was chosen accordingly when the theory developed. As a consequence of this, all the subsequent literature presents the theory in the framework of cochain complexes. Obviously, we do not want to depart from this convention in this short survey section.

homotopy category (see Chapter IV, Section 3), which we now denote by $\mathfrak{K}$. We recall that $\mathfrak{K}$ has the same object as $\mathfrak{C}$, and the morphisms are the homotopy classes of cochain maps. We also recall that the homology functor on $\mathfrak{C}$ factors through $\mathfrak{K}$. In particular, two cochain complexes $\mathbf{C}$, $\mathbf{C}'$, which become isomorphic in $\mathfrak{K}$, i.e., so-called homotopy equivalent cochain complexes, have the same homology, $H(\mathbf{C}) \simeq H(\mathbf{C}')$. In Chapter IV, Section 3, we have given an example where the converse is not true. Here then is the starting point for the second step in the construction of the derived category $\mathfrak{D}$. But before we proceed we mention that the category $\mathfrak{K}$ does not have as much structure as one would hope for: although it is additive, it is not in general abelian. It is the achievement of J.L. Verdier to have discovered the importance of the structure of a *triangulated category*. The category $\mathfrak{K}$ has that structure and this makes the subsequent steps possible. Essential parts of the structure of a triangulated category are an additive invertible endofunctor T, and a distinguished class of *triangles* (f, g, h), i.e., sequences of the form

$$\mathbf{C} \xrightarrow{f} \mathbf{D} \xrightarrow{g} \mathbf{E} \xrightarrow{h} T\mathbf{C}.$$

There is a rather long list of axioms which must be satisfied, but which we cannot describe here. In our context the functor T is given by "translating" by one step: If $\mathbf{C} = \{C^n\}$ is a cochain complex, one defines $T(\mathbf{C})$ by $\big(T(\mathbf{C})\big)^n = C^{n+1}$. To describe the distinguished class of triangles one needs the construction of the (mapping) *cone* of a cochain map (see Chapter IV, Exercise 1.2, where the mapping cone is defined for a map of chain complexes, and Chapter IV, Exercise 2.1). Given a cochain map $f:\mathbf{C} \to \mathbf{D}$ the cone $\mathbf{E} = \mathbf{E}_f$ of f is the cochain complex with $\mathbf{E}_f^n = C^{n+1} \oplus D^n$ and with differential $\delta_{\mathbf{E}}(c_{n+1}, d_n) = \big(-\delta_{\mathbf{C}}(c_{n+1}), f(c_{n+1}) + \delta_{\mathbf{D}}(d_n)\big)$. By definition, a triangle in $\mathfrak{K}$ belongs to the distinguished class if it is of the following form (up to isomorphism in $\mathfrak{K}$): The map $f:\mathbf{C} \to \mathbf{D}$ is any cochain map, $\mathbf{E}$ is its cone, the cochain map $g:\mathbf{D} \to \mathbf{E}$ is given by $g(d_n) = (0, d_n)$, and the cochain map $h:\mathbf{E} \to T\mathbf{C}$ is given by $h(c_{n+1}, d_n) = c_{n+1}$.

From the triangulated category $\mathfrak{K}$ the derived category $\mathfrak{D}$ is then obtained by a process of "localization". As mentioned above the category $\mathfrak{K}$ may contain so-called *homology isomorphisms*, i.e., morphisms in $\mathfrak{K}$ that induce isomorphisms in homology. These homology isomorphisms in $\mathfrak{K}$ are inverted to yield the derived category $\mathfrak{D}$. The process of inverting a class of morphisms in a category is much like the process of localization in commutative algebra, but – as the reader may imagine – there are quite a number of technical problems that must be overcome (see P. Gabriel and M. Zisman [GZ]). The derived category $\mathfrak{D}$ turns out to be additive and triangulated. Under the canonical functor $P:\mathfrak{K} \to \mathfrak{D}$ homology isomorphisms become invertible, and the functor P is universal with regard to this property.

It is an important property that the derived category $\mathfrak{D}$ admits a functor $J : \mathfrak{A} \to \mathfrak{D}$: for A in $\mathfrak{A}$, the object $J(A)$ is represented by the cochain complex concentrated in degree zero with $(J(A))^\circ = A$. It turns out that the functor J is full and faithful.

There are variants of the derived category $\mathfrak{D}$. Thus, starting with the category $\mathfrak{C}^+$ of cochain-complexes which are bounded below, and proceeding in the same way, one ends up with the derived category $\mathfrak{D}^+$. The categories $\mathfrak{C}^-$, or $\mathfrak{C}^b$, consisting of complexes which are bounded above or bounded below *and* above, in a similar way yield the derived categories $\mathfrak{D}^-$ and $\mathfrak{D}^b$, respectively. It turns out that the categories $\mathfrak{D}^+$, $\mathfrak{D}^-$, and $\mathfrak{D}^b$ are all full subcategories of $\mathfrak{D}$, containing the image under the functor J.

In order to explain the relevance of the derived category we shall now assume that the category $\mathfrak{A}$ admits enough injectives. In that case, we may relate the concept of derived categories to the classical tool of injective resolutions. For this we introduce the category $\mathfrak{C}(I)^+$, as the full subcategory of $\mathfrak{C}^+$ of cochain complexes consisting of injective objects. Clearly injective resolutions are objects in $\mathfrak{C}(I)^+$. For many facts about injective resolutions which are basic in the classical development of homological algebra the natural framework is the homotopy category $\mathfrak{K}(I)^+$. The following key result then explains why derived categories form a framework which is a vast generalization of classical homological algebra.

Theorem 5.1. *If the category $\mathfrak{A}$ admits enough injectives, then the categories $\mathfrak{K}(I)^+$ and $\mathfrak{D}^+$ are equivalent.*

Dually of course one has a statement about projective objects and the category $\mathfrak{D}^-$. (The reader should compare this with the assertions in Chapter IV, Exercises 4.1 and 4.2). It is clear from this result that the derived category may be used to introduce derived functors, and other tools of homological algebra, like spectral sequences, even in the absence of enough injectives and/or projectives. This becomes extremely important, for example, in sheaf theory, where these techniques play a crucial role. In fact, it was sheaf theory that prompted the development of the theory of derived categories.

To explain the definition of derived functors we consider two abelian categories $\mathfrak{A}$ and $\mathfrak{B}$, and an additive functor $F : \mathfrak{K}(\mathfrak{A}) \to \mathfrak{K}(\mathfrak{B})$ which transforms distinguished triangles in $\mathfrak{K}(\mathfrak{A})$ into distinguished triangles in $\mathfrak{K}(\mathfrak{B})$.

The right derived functor of F then consists of a functor $RF : \mathfrak{D}^+(\mathfrak{A}) \to \mathfrak{D}(\mathfrak{B})$ and a natural transformation $\xi_F : P_{\mathfrak{B}} \circ F \to RF \circ P_{\mathfrak{A}}$ such that the following universal property is satisfied: For every functor $G : \mathfrak{D}^+(\mathfrak{A}) \to \mathfrak{D}(\mathfrak{B})$ and for every natural transformation $\eta : P_{\mathfrak{B}} \circ F \to G \circ P_{\mathfrak{A}}$ there is a unique natural transformation $\zeta : RF \to G$ with $\eta = (\zeta \circ P_{\mathfrak{A}}) \circ \xi_F$. (Compare Chapter IX, Section 3, where the dual universal property is stated to define the (left!)-$\mathscr{E}$-satellite of a functor.)

The above definition generalizes the usual definition of the right derived functor. If F is induced from a left exact additive functor $F' : \mathfrak{A} \to \mathfrak{B}$ and if the category $\mathfrak{A}$ admits enough injectives then the usual nth right derived functor (as defined in Chapter IV, Section 5) turns out to coincide with the functor $H^n(RF)$, where H^n denotes the process of taking homology in degree n. (Note that, if F is not supposed to be *left exact*, the functor $H^n(RF)$ corresponds to the nth right satellite of F in the sense of Chapter IX, Section 3.)

Given two abelian categories $\mathfrak{A}$ and $\mathfrak{B}$, it may happen that their derived categories become equivalent. One then speaks of a derived equivalence between $\mathfrak{A}$ and $\mathfrak{B}$. An example of this kind is provided by the following situation, again in modular representation theory. Let k be a field of characteristic p, let G be a finite group with cyclic p-Sylow-subgroup P, and let H be the normalizer of P in G. Then the derived categories $\mathfrak{D}^b(\mathfrak{A})$ and $\mathfrak{D}^b(\mathfrak{B})$ are equivalent where $\mathfrak{A}$ and $\mathfrak{B}$ are the principal blocks of kG and kH, respectively.

As already mentioned, the tool of derived categories was introduced to handle certain problems in sheaf theory. It is in this area where derived categories have been applied most successfully. We may mention that many of these results have the form we have talked about here, namely, that two derived categories of sheaves are equivalent. The interested reader may consult the appropriate literature; we mention [I], [Ha], and [B].

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