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Conjugate pairs of categories and Quillen equivalent stable model categories of functors

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ABSTRACT

We describe the notion of a *conjugate pair* $(\mathcal{B}, \mathcal{A})$ of small categories, wherein maps in $\mathcal{B}$ admit a factorization by maps in the subcategory $\mathcal{A}$, much in the spirit of a "two-sided" calculus of fractions. When $(\mathcal{B}, \mathcal{A})$ is a conjugate pair, we prove that for any cofibrantly generated model category $\mathcal{C}$ there is an induced Quillen adjunction between the functor categories $[\mathcal{B}^{\mathrm{op}}, \mathcal{C}]$ and $[\mathcal{A}^{\mathrm{op}}, \mathcal{C}]$. When $\mathcal{C}$ is a left proper stable model category, this adjunction is a Quillen equivalence. Finally, we demonstrate that minor modifications of our arguments give the analogous result when $\mathcal{C}$ is instead assumed to be an **Ab**-category.

1. Introduction

1.1. Motivation and overview

Given a model category $\mathcal{C}$ and any small category $\mathcal{D}$, one can form the category $[\mathcal{D}, \mathcal{C}]$ of functors from $\mathcal{D}$ to $\mathcal{C}$. When $\mathcal{C}$ is cofibrantly generated, this category of diagrams inherits a model structure. A basic question is then to determine when two such model categories of diagrams in $\mathcal{C}$ are Quillen equivalent. In regarding the small category $\mathcal{D}$ as a "ring" and the functor category $[\mathcal{D}, \mathcal{C}]$ as a category of "left $\mathcal{D}$-modules," such problems are generalizations of classical Morita theory, and we will use this as inspiration for both our techniques and terminology.

In this article we develop an approach to this problem for the case when $\mathcal{C}$ is a stable model category. Precisely, we give conditions on pairs $(\mathcal{B}, \mathcal{A})$ of small categories such that there is an induced Quillen equivalence between the functor categories $[\mathcal{B}^{\mathrm{op}}, \mathcal{C}]$ and $[\mathcal{A}^{\mathrm{op}}, \mathcal{C}]$, where $\mathcal{C}$ may be any sufficiently nice stable model category. Roughly stated, our conditions amount to saying that $\mathcal{B}$ is constructed from $\mathcal{A}$ by a fairly intuitive calculus of "two-sided" fractions; we call such pairs of categories *conjugate pairs*. When $(\mathcal{B}, \mathcal{A})$ is a conjugate pair, we show that there is an associated "bimodule" $\mathcal{U}_+ : \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathrm{Sets}_*$ to the category of based sets, and by using formal Hom- and tensor-like constructions we obtain an adjoint pair

$$[\mathcal{B}^{\mathrm{op}}, \mathcal{C}] \underset{R}{\overset{L}{\rightleftarrows}} [\mathcal{A}^{\mathrm{op}}, \mathcal{C}].$$

Our main theorem then asserts the following:

Theorem 6.2. *Suppose that $\mathcal{C}$ is a left proper stable model category. If $(\mathcal{B}, \mathcal{A})$ is a conjugate pair of small categories, then the adjoint pair*

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$$[\mathcal{B}^{\mathrm{op}}, \mathcal{C}] \underset{R}{\overset{L}{\rightleftarrows}} [\mathcal{A}^{\mathrm{op}}, \mathcal{C}]$$

associated to the regular bimodule $\mathcal{U}_+ : \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathrm{Sets}_*$ *is a Quillen equivalence.*

The motivation for our approach comes from a single pair of categories that has appeared in both algebra and stable homotopy theory. Let $\mathcal{E}$ denote the category with objects the sets $\mathbf{n} = \{1, 2, \ldots, n\}$ and with surjective maps as the morphisms (where $\mathbf{0}$ is the empty set). Let Γ denote the category with objects the finite based sets $\mathbf{n}_+ = \{0, 1, 2, \ldots, n\}$ and morphisms the based maps, where 0 acts as the basepoint. It is often convenient to regard $\mathcal{E}$ as a subcategory of Γ by adding disjoint basepoints, in which case the maps in $\mathcal{E}$ send only the basepoint to the basepoint.

In [11] the author proves that the functor categories $[\Gamma^{\mathrm{op}}, \mathcal{C}]$ and $[\mathcal{E}^{\mathrm{op}}, \mathcal{C}]$ are equivalent when $\mathcal{C}$ is an abelian category (the covariant version is also obtained). The equivalence is given by the *cross effect* construction of Eilenberg and MacLane [6]. An additive generalization of Pirashvili's equivalence soon followed in [14]. There, the author gives conditions on pairs $(\mathcal{B}, \mathcal{A})$ of small categories enriched over R-modules so that there is an adjoint equivalence between the categories $[\mathcal{B}^{\mathrm{op}}, R\text{-mod}]$ and $[\mathcal{A}^{\mathrm{op}}, R\text{-mod}]$ of additive functors. Suitably enriched, the pair $(\Gamma, \mathcal{E})$ is an example of such a pair of "Morita equivalent" categories.

In roughly the same time period, similar equivalences were being noticed in stable homotopy theory, again arising from a (homotopical) cross effect. The categories $\mathcal{E}$ and Γ appear together in this way in several instances, such as in [1] and [2]. In his work on generalizations of the infinite symmetric product $\mathrm{SP}^\infty(-)$ to categories of S-algebras, Kuhn [9] explicitly mentions an apparent connection to Pirashvili's algebraic equivalence of categories. All of this suggests that the categories $[\Gamma^{\mathrm{op}}, \mathcal{C}]$ and $[\mathcal{E}^{\mathrm{op}}, \mathcal{C}]$ are Quillen equivalent when $\mathcal{C}$ is a stable model category. This is indeed the case, and follows immediately from our main result (Theorem 6.2 applied to Example 3.16).

Like Słomińska's work, we obtain our main result by abstracting certain features of the pair $(\Gamma, \mathcal{E})$ so that we may ask the same of any pair $(\mathcal{B}, \mathcal{A})$, where $\mathcal{A}$ is a subcategory of $\mathcal{B}$. Her strategy is to make use of the R-module structures on $\mathcal{B}$ and $\mathcal{A}$ to relate these categories via elaborate tensoring operations. However, our approach is more self-contained and "combinatorial" in nature, as we are not assuming that our categories are additive or enriched in any way.[1]

Fortunately, the main feature of the pair $(\Gamma, \mathcal{E})$ to be generalized is not hard to describe. In fact, it closely parallels the first isomorphism theorem in group theory. Precisely, every Γ-map $\gamma : \mathbf{m}_+ \to \mathbf{n}_+$ admits a uniquely determined three-fold factorization

$$\begin{tikzcd} \mathbf{m}_+ \arrow[r, "\gamma"] \arrow[d, "q"'] & \mathbf{n}_+ \\ \mathbf{r}_+ \arrow[r, "\alpha"] & \mathbf{s}_+ \arrow[u, "i"'] \end{tikzcd}$$

where

- q is the quotient map that collapses the "kernel" $\gamma^{-1}(0)$ of γ,
- i represents the inclusion of the image of γ into its codomain, and
- α is the unique epimorphism making the diagram commute.

Note that α has the additional property that only the basepoint is mapped to the basepoint. Thus we may forget the basepoints and view α as a morphism $\alpha : \mathbf{r} \to \mathbf{s}$ in $\mathcal{E}$. For a concrete example of this sort of factorization, see Example 3.17.

Thus we see that every Γ-map gives rise to a uniquely determined map in $\mathcal{E}$, and we can keep track of this assignment by recording the quotient map q and the inclusion map i. Furthermore, note that the category of inclusion maps is in some sense dual to the category of quotients. Abstracting all of this structure leads to our notion of a *conjugate pair* of small categories, of which $(\Gamma, \mathcal{E})$ is the prime example.

The added bonus of our approach is that we also obtain the additive analogue of Theorem 6.2. That is, we may replace the assumption that $\mathcal{C}$ is a stable model category with the assumption that $\mathcal{C}$ is only an additive category, and we then obtain a strict equivalence of functor categories; see Theorem 7.1 below. This requires only minor modifications of our proof for the stable case. Thus the notion of *conjugate pair* gives us both the algebraic and stable homotopy results essentially at once.

1.2. Organization of the paper

In Section 2.1 we lay the foundation for our version of Morita equivalence for categories of functors. Section 2.2 reviews the basic results and language of model categories and cofibrant generation (a standing assumption in this article). In Section 2.3 we provide the necessary background in stable model categories. This is all review; nothing here is new.

[1] Because we are working with non-additive categories, we necessarily lose a few of Słomińska's examples.

In Section 3.1 we begin the development of the small category side of the story, where we carefully define the notion of an *indexing category*. Such categories will play the role of the category of inclusion maps in the three-fold factorizations. This type of category will also index some fundamental (co)product decompositions, hence the name.

In Sections 3.2 and 3.3 we formalize three-fold factorization to the notion of a *conjugate pair* of small categories; plenty of examples follow in Section 3.4. The bulk of the hard work lies in Section 4, where we show that a conjugate pair allows for the creation of a special functor we call the *regular bimodule*. The regular bimodule conveniently encodes many naturality properties that are necessary for the proof of our main theorem, and these are detailed in Section 4.

Every bimodule creates an adjoint pair between functor categories. For a regular bimodule, the resulting right adjoint admits a nice product decomposition; this is the content of Section 5.1. In Section 5.2, we analyze the behavior of free functors and their pushouts under this adjunction. The entirety of Section 6 is devoted to finishing the proof of Theorem 6.2, our main result. Finally, the brief Section 7 shows how to modify our proof techniques to obtain the additive analogue of our main theorem.

2. Categorical preliminaries

2.1. The Morita theory context

For this discussion, let us fix a pointed category $\mathcal{C}$ having all limits and colimits. In the applications in this article, $\mathcal{C}$ will always be a pointed model category.

The constructions involved in our Morita theory arise naturally from products and coproducts indexed by based sets.[2] As an initial technicality we must be clear about how we will handle the basepoint. Given a based set S and an object C of $\mathcal{C}$, the product $\prod_S C$ will always stand for the ordinary product of copies of C, one copy for each non-basepoint element of S. That is, the factor corresponding to the basepoint of S will always be taken to be the zero object of $\mathcal{C}$. The same convention will apply to coproducts $\bigvee_S C$ indexed by based sets (as all of our categories will be pointed, we will be using the wedge symbol $\vee$ for the coproduct). It will be convenient to denote $\bigvee_S C$ by $C \otimes S$ on occasion. Recall that for a fixed object C, the assignment $S \mapsto \prod_S C$ is a contravariant functor of S, while forming coproducts $S \mapsto C \otimes S$ is covariant in S.

Let us now fix a small category $\mathcal{A}$. Given functors $G : \mathcal{A}^{\mathrm{op}} \to \mathcal{C}$ and $P : \mathcal{A}^{\mathrm{op}} \to \mathrm{Sets}_*$ we can form a new functor $\mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{C}$ by the assignment

$$(x, y) \mapsto \prod_{P(y)} G(x)$$

with the obvious action on morphisms. We define $\mathrm{End}_{\mathcal{A}}(P, G)$ to be the end of this functor (see [10] for a refresher on ends, if necessary). Thus there are "diagonal" structure maps $\Delta_a : \mathrm{End}_{\mathcal{A}}(P, G) \to \prod_{P(a)} G(a)$ which satisfy a universal mapping property with respect to all such diagonal transformations akin to a pullback.

It is entirely formal that this $\mathrm{End}_{\mathcal{A}}(-, -)$ construction has all of the expected properties of a Hom-like object: functoriality of the expected variances, correct behavior with (co)products, and a Yoneda lemma for representable (think: free) functors. Precisely, we have the following.

Proposition 2.1. *Let $\mathcal{C}$ be a complete pointed category.*

(a) $\mathrm{End}_{\mathcal{A}}(-, -)$ *defines a functor* $[\mathcal{A}^{\mathrm{op}}, \mathrm{Sets}_*]^{\mathrm{op}} \times [\mathcal{A}^{\mathrm{op}}, \mathcal{C}] \to \mathcal{C}$.
(b) *For functors $P, Q : \mathcal{A}^{\mathrm{op}} \to \mathrm{Sets}_*$ there is a natural isomorphism*

$$\mathrm{End}_{\mathcal{A}}(P \vee Q, G) \cong \mathrm{End}_{\mathcal{A}}(P, G) \times \mathrm{End}_{\mathcal{A}}(Q, G).$$

(c) *For functors $G, H : \mathcal{A}^{\mathrm{op}} \to \mathcal{C}$ there is a natural isomorphism*

$$\mathrm{End}_{\mathcal{A}}(P, G \times H) \cong \mathrm{End}_{\mathcal{A}}(P, G) \times \mathrm{End}_{\mathcal{A}}(P, H).$$

(d) *(Yoneda Lemma) If $P : \mathcal{A}^{\mathrm{op}} \to \mathrm{Sets}_*$ is the representable functor $P(-) = \mathcal{A}(-, a)_+$ then there is a natural isomorphism*

$$\mathrm{End}_{\mathcal{A}}(P, G) \cong G(a).$$

Likewise, we can use coends of coproducts to define a tensor product of functors when $\mathcal{C}$ is cocomplete. Fix a small category $\mathcal{B}$ and functors $F : \mathcal{B}^{\mathrm{op}} \to \mathcal{C}$ and $Q : \mathcal{B} \to \mathrm{Sets}_*$. From this we can form a functor $\mathcal{B}^{\mathrm{op}} \times \mathcal{B} \to \mathcal{C}$ via

$$(x, y) \mapsto F(x) \otimes Q(y),$$

[2] If we happen upon an unbased set X we may of course attach a disjoint basepoint, writing X_+ for this as is customary.

the coend of which we denote $F \otimes_{\mathcal{B}} Q$. Thus there are codiagonal structure maps $\nabla_b : \bigvee_{Q(b)} F(b) \to F \otimes_{\mathcal{B}} Q$ which satisfy a universal mapping property with respect to all such codiagonal transformations akin to a pushout. Again, this has all of the properties one would expect of a tensor product, so that the dual of Proposition 2.1 holds (with adjustments for variances).

The starting point in classical Morita theory is the simple fact that every bimodule gives an adjoint pair between categories of modules. In our situation, every functor of the form $P : \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathrm{Sets}_*$ will give rise to an adjoint pair between categories of contravariant functors; we will therefore call such functors *bimodules*.

To that end, fix a bimodule $P : \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathrm{Sets}_*$. Our $\mathrm{End}_{\mathcal{A}}(-,-)$ and $- \otimes_{\mathcal{B}} -$ constructions yield an adjoint pair

$$[\mathcal{B}^{\mathrm{op}}, \mathcal{C}] \underset{R}{\overset{L}{\rightleftarrows}} [\mathcal{A}^{\mathrm{op}}, \mathcal{C}]$$

described as follows. Given $F : \mathcal{B}^{\mathrm{op}} \to \mathcal{C}$ we define the functor $LF : \mathcal{A}^{\mathrm{op}} \to \mathcal{C}$ by

$$LF(a) = F \otimes_{\mathcal{B}} P(a,-)$$

with the obvious action on morphisms induced by P. That LF is a functor is due to the functoriality of our tensor product in the second slot, and that L is itself a functor is due to the functoriality in the first slot. Dually, given $G : \mathcal{A}^{\mathrm{op}} \to \mathcal{C}$ we define $RG : \mathcal{B}^{\mathrm{op}} \to \mathcal{C}$ by

$$RG(b) = \mathrm{End}_{\mathcal{A}}\big(P(-,b), G\big)$$

and both RG and R are appropriately functorial.

The proof that (L,R) is an adjoint pair is tedious yet completely formal. For the sake of being convincing, we will sketch the construction of the natural isomorphism

$$\mathrm{Hom}_{[\mathcal{A}^{\mathrm{op}},\mathcal{C}]}(LF, G) \longrightarrow \mathrm{Hom}_{[\mathcal{B}^{\mathrm{op}},\mathcal{C}]}(F, RG).$$

Suppose $\tau : LF \to G$ is a natural transformation; we are to construct a natural transformation $\tau^{\#} : F \to RG$ of functors $\mathcal{B}^{\mathrm{op}} \to \mathcal{C}$. Fix an object b of $\mathcal{B}$. Given an object x of $\mathcal{A}$ and an element $w \in P(x,b)$, we may form the composite

$$F(b) \xrightarrow{i_w} \bigvee_{P(x,b)} F(b) \xrightarrow{\nabla_b} LF(x) \xrightarrow{\tau_x} G(x).$$

Forming the product of such maps over $w \in P(x,b)$ yields a morphism

$$F(b) \xrightarrow{\delta_x} \prod_{P(x,b)} G(x).$$

One then checks that the system of maps $\{\delta_x : x \in \mathcal{A}\}$ is diagonally compatible, and so there is a unique induced map to the end

$$\tau_b^{\#} : F(b) \to RG(b).$$

This defines $\tau^{\#} : F \to RG$. The construction of $\eta^{\flat} : LF \to G$ from $\eta : F \to RG$ is completely dual.

2.2. Model categories

We will take as the definition of *model category* that which is set forth in Chapter 1 of Hovey's *Model Categories* [8]. In particular, we will assume that the functorial factorizations are fixed as part of the underlying model structure (others assume that such factorizations merely exist). Throughout, we use $\xrightarrow{\sim}$ to denote weak equivalences and $\xrightarrow{\cong}$ for isomorphisms. A map that is both a weak equivalence and a (co)fibration will be called an *acyclic (co)fibration*.

Results about model categories stated without proof are taken to be common knowledge. Proofs and complete details of such results may always be found in the contemporary standard references, such as [8], [7], or [5]. One such result, however, is worth emphasizing.

Suppose that $f : A \to B$ and $g : C \to D$ are two maps in a model category $\mathcal{C}$. The first projection map $A \times C \to A$ provides the composition

$$A \times C \xrightarrow{p_A} A \xrightarrow{f} B$$

and likewise for $g \circ p_C : A \times C \to D$. We now have maps with common domain $A \times C$, so we may form the map induced from the universal mapping property of $B \times D$, obtaining

$$(f \circ p_A) \times (g \circ p_C) : A \times C \to B \times D.$$

We will call this map (f,g) for short. The slogan "products of fibrations are fibrations" amounts to the fact that (f,g) is a fibration whenever both f and g are.

Proposition 2.2. *Let $\mathcal{C}$ be a model category, and suppose that $f : A \to B$ and $g : C \to D$ are maps in $\mathcal{C}$. If f and g are (acyclic) fibrations, then so is the map $(f, g) : A \times C \to B \times D$.*

The proof is straightforward. For the fibration case, one shows that (f, g) satisfies the right-lifting property with respect to all acyclic cofibrations, constructing the required lifts componentwise. The same argument (against "ordinary" cofibrations) establishes the acyclic fibration case. Of course, there is the dual result for coproducts of (acyclic) cofibrations, but we will not need this.

Beyond these observations, we will also be required to make some mild technical assumptions. For brevity's sake, we will bundle these assumptions into the term *model category*.

Convention. In this paper, the term *model category* shall always refer to a pointed, cofibrantly generated[3] model category.

All model categories in common practice are cofibrantly generated, so this is not a restrictive assumption. Moreover, we may always embed an unpointed model category $\mathcal{C}$ in a pointed one, as outlined in [8]. Precisely, let $\mathcal{C}_*$ denote the category of objects under the terminal object of $\mathcal{C}$. Forming coproducts against the terminal object provides a faithful embedding $\mathcal{C} \to \mathcal{C}_*$ that is left adjoint to the forgetful functor. Of course, $\mathcal{C}_*$ is pointed.

We close this section with a few remarks about cofibrant generation and categories of functors; see Chapter 11 of [7] for complete details. When $\mathcal{C}$ is a cofibrantly generated model category, the category $[\mathcal{D}^{\mathrm{op}}, \mathcal{C}]$ of functors $\mathcal{D}^{\mathrm{op}} \to \mathcal{C}$ inherits a model structure (here $\mathcal{D}$ can be any small category). This model structure is also cofibrantly generated, wherein weak equivalences and fibrations of diagrams are defined objectwise. We will need a complete description of the cofibrations.

Definition 2.3. Fix an object d of $\mathcal{D}$ and an object C of $\mathcal{C}$. The *free functor* generated by d and C is the functor $F_d^C : \mathcal{D}^{\mathrm{op}} \to \mathcal{C}$ given by

$$F_d^C(x) = C \otimes \mathcal{D}(x, d)_+ = \bigvee_{\mathcal{D}(x,d)_+} C.$$

Clearly every map $i : B \to C$ in $\mathcal{C}$ induces a natural transformation $i_* : F_d^B \to F_d^C$ of free functors. When $\mathcal{C}$ is cofibrantly generated, the cofibrations in $[\mathcal{D}^{\mathrm{op}}, \mathcal{C}]$ are generated by the maps of the form $i_* : F_d^B \to F_d^C$, where i is a generating cofibration in $\mathcal{C}$. Thus all cofibrations in the category of diagrams are obtained through transfinite compositions of pushouts of such maps (and retracts thereof).

In order to handle these transfinite compositions, we require one more technical tool before we proceed, namely a reasonable *sequential homotopy colimit* functor. A complete analysis of such "homotopy meaningful" sequential colimits may be found in Chapter I of [4]. Let $\mathcal{C}$ be a model category and fix an ordinal λ. By a *λ-sequence* in $\mathcal{C}$ we mean a functor $X : \lambda \to \mathcal{C}$, where λ is made into a category in the usual way. As is standard now, we will always assume that the natural induced map at any limit ordinal is an isomorphism (re-indexing at limit ordinals shows that this is no real restriction). Since $\mathcal{C}$ is cofibrantly generated, the category $[\lambda, \mathcal{C}]$ of all such sequences inherits a model structure with objectwise weak equivalences and fibrations.[4] In this model structure, a λ-sequence is cofibrant if it is objectwise cofibrant and each map in the sequence is a cofibration; see Example 4.3 of [4]. The following result on cofibrant diagrams appears (with proof) as Proposition 2.5 in [4] and as Proposition 17.9.1 in [7].

Proposition 2.4. *If $f : X \to Y$ is a weak equivalence between cofibrant λ-sequences in a model category, then the map* $\mathrm{colim}(f) : \mathrm{colim}(X) \to \mathrm{colim}(Y)$ *is a weak equivalence.*

It follows that the sequential colimit functor is homotopy meaningful on cofibrant objects, so that its total left derived functor exists. This is what we shall mean by the *homotopy colimit* $\mathrm{hocolim}(X)$ of a λ-sequence X. It is computed in the standard way, by taking the colimit of the cofibrant replacement $\widetilde{X}$ of X. Proposition 2.4 also implies that the natural map $\mathrm{colim}(\widetilde{X}) \to \mathrm{colim}(X)$ is a weak equivalence whenever X is a cofibrant λ-sequence. Thus, when X is cofibrant, $\mathrm{hocolim}(X) \cong \mathrm{colim}(X)$ in the homotopy category.

2.3. Stable model categories

A model category with zero object has much more structure than a model category without. Specifically, if $\mathcal{C}$ is a pointed model category there are naturally-defined loop and suspension functors $\Omega, \Sigma : \mathrm{ho}(\mathcal{C}) \to \mathrm{ho}(\mathcal{C})$. It is important to remember that these functors are well-defined only on the homotopy category, and not at the level of $\mathcal{C}$ itself.

[3] Either of the descriptions of cofibrant generation found in [8] or [7] suffice for our purposes.

[4] As outlined in [4], this is true even if $\mathcal{C}$ is not cofibrantly generated, though we will not need this fact.

The definition of ΣX goes as follows (and as expected). First, construct the cofibrant replacement $\widetilde{X}$ of X. Let $C(\widetilde{X})$ be the cylinder object of $\widetilde{X}$ arising from factoring the fold map $\nabla : \widetilde{X} \vee \widetilde{X} \to \widetilde{X}$ into a cofibration followed by an acyclic fibration. This provides the natural map

$$\widetilde{X} \vee \widetilde{X} \longrightarrow C(\widetilde{X})$$

into the two "ends" of the cylinder. The *suspension* ΣX of X is then defined as the pushout

$$\begin{tikzcd} \widetilde{X} \vee \widetilde{X} \arrow[r] \arrow[d] & C(\widetilde{X}) \arrow[d] \\ 0 \arrow[r] & \Sigma X. \end{tikzcd}$$

The definition of ΩX is completely dual. These constructions become functorial upon passage to the homotopy category.

So defined, Σ and Ω form an adjoint pair of endofunctors on $\mathrm{ho}(\mathcal{C})$. This structure even gives rise to fiber and cofiber sequences in the homotopy category (with the same warning that these don't make sense in the underlying model category). Modulo some fine details, a *cofiber sequence* in $\mathrm{ho}(\mathcal{C})$ is a diagram $X \to Y \to Z$ in the homotopy category isomorphic to a diagram $A \to B \to C$ originating from a pushout square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ 0 \arrow[r] & C \end{tikzcd}$$

in $\mathcal{C}$, where the map $A \to B$ is a cofibration of cofibrant objects. Consequently, C is also cofibrant. The map $B \to C$ is called the *cofiber* of $A \to B$, and similarly for $Y \to Z$.

We should point out that all of this was known to Quillen, appearing in his first treatise on model categories [12]. An updated (simplicial) approach appears in Chapter 6 of [8]. With these prerequisites out of the way, we may now give the definition of a *stable* model category.

Definition 2.5. A pointed model category $\mathcal{C}$ is *stable* if the functor Σ is an equivalence with inverse Ω.

It is immediate that a category of functors $[\mathcal{D}^{\mathrm{op}}, \mathcal{C}]$ is a cofibrantly generated stable model category whenever $\mathcal{C}$ is. The article [13] contains a multitude of examples of stable model categories: chain complexes, modules over a Frobenius ring, various species of spectra (equivariant and otherwise), presheaves of spectra, localized structures, and motivic examples.

The most important fact on stable model categories is that their homotopy categories are triangulated, the triangles arising from the cofiber sequences and the shifting being given by the suspension. Thus when $\mathcal{C}$ is stable, the homotopy category $\mathrm{ho}(\mathcal{C})$ is additive and fiber and cofiber sequences coincide up to sign, giving us access to 5-lemma type arguments. Proofs and precise statements of these and other deep results on stable model categories may be found in Chapter 7 of [8]. Beyond these, we will also need the following fact on "lower triangular" maps.

Proposition 2.6. *Let $\mathcal{C}$ be a stable model category. Suppose that $\{A_i\colon\ i \in I\}$ is a set of cofibrant objects and that $\{B_i\colon\ i \in I\}$ is a set of fibrant objects, where the index set I is a finite totally ordered set. Suppose that the map $f : \bigvee_{i\in I} A_i \to \prod_{i\in I} B_i$ has components $f_{ij} : A_i \to B_j$. If the diagonal components f_{ii} are weak equivalences and $f_{ij} = 0$ when $i < j$, then f is a weak equivalence.*

Proof. It suffices to prove this in the case of only two factors, so suppose without loss of generality that $I = \{1, 2\}$. Suppose that $f : A_1 \vee A_2 \to B_1 \times B_2$ is given by a matrix of the form

$$\begin{pmatrix} f_{11} & 0 \\ f_{21} & f_{22} \end{pmatrix}$$

where the diagonal entries are weak equivalences. One then checks that the diagram

$$\begin{tikzcd} A_1 \arrow[r, "i_1"] \arrow[d, "f_{11}"'] & A_1 \vee A_2 \arrow[r, "0\vee 1"] \arrow[d, "f"] & A_2 \arrow[d, "f_{22}"] \\ B_1 \arrow[r, "1\times 0"] & B_1 \times B_2 \arrow[r, "p_2"] & B_2 \end{tikzcd}$$

commutes precisely because $f_{12} = 0$. In the homotopy category of $\mathcal{C}$, the top row is a cofiber sequence and the bottom row is a fiber sequence. As $\mathcal{C}$ is a stable model category, fiber and cofiber sequences coincide in $\mathrm{ho}(\mathcal{C})$. We now have a map of two fiber sequences, f_{11} providing an isomorphism on the fiber, while f_{22} provides an isomorphism on the base. Thus f represents an isomorphism in $\mathrm{ho}(\mathcal{C})$, so f is a weak equivalence. □

3. Conjugate pairs of small categories

3.1. Indexing categories

EI-categories with extra structure will give us control over the indexing of intricate (co)product decompositions of the functors involved in our main theorem. In essence, these *indexing categories* will serve to parameterize the images and cokernels alluded to in the introduction.

For this we need some basic facts about EI-categories, all of which are easy to prove from the definitions. First, recall that an *EI-category* is one in which every endomorphism is an isomorphism. From this it follows that all retracts are isomorphisms. In any category with pullbacks, the condition that all retracts are isomorphisms is equivalent to every map being monic. Hence, in an EI-category with pullbacks, all maps are necessarily monic. An additional finiteness assumption is all that we require. We will denote by $\mathcal{I} \downarrow a$ the category of $\mathcal{I}$-objects over a, also known as the *comma category* over a.

Definition 3.1. A small EI-category $\mathcal{I}$ is an *indexing category* if all pullbacks exist and each category $\mathcal{I} \downarrow a$ has a skeleton with a finite number of objects.

The last condition simply asserts that for each object a there are only finitely many maps to a up to "covering" equivalence. Moreover, such covering equivalences are unique since all maps in an indexing category are monic.

Given an indexing category $\mathcal{I}$ and an object a, we will denote by $\mathrm{sk}(\mathcal{I} \downarrow a)$ a skeleton of the comma category $\mathcal{I} \downarrow a$. We may define a relation $\leqslant$ on $\mathrm{sk}(\mathcal{I} \downarrow a)$ as follows: given maps $i : x \to a$ and $j : y \to a$, we declare $i \leqslant j$ if there is a commutative triangle

$$\begin{tikzcd} x \arrow[rr, "k"] \arrow[dr, "i"'] & & y \arrow[dl, "j"] \\ & a & \end{tikzcd}$$

in $\mathcal{I}$.

Proposition 3.2. *Let $\mathcal{I}$ be an indexing category. For each object a, the category $\mathrm{sk}(\mathcal{I} \downarrow a)$ is a finite partially ordered set under the relation $\leqslant$ described above.*

Proof. This relation is plainly reflexive and transitive (even without taking the skeleton). To see antisymmetry, suppose that $i \leqslant j$ and $j \leqslant i$ as displayed in

$$\begin{tikzcd} x \arrow[r, "k"] \arrow[dr, "i"'] & y \arrow[r, "l"] \arrow[d, "j"] & x \arrow[dl, "i"] \\ & a. & \end{tikzcd}$$

Being an endomorphism, $k \circ l$ is an isomorphism, and this implies that k is a retract. But retracts are isomorphisms in an indexing category, so k is an isomorphism. Hence i and j are isomorphic objects in the comma category $\mathcal{I} \downarrow a$. As there is only one object from each isomorphism class in the skeleton, it follows that we have $i = j$. Thus, in $\mathrm{sk}(\mathcal{I} \downarrow a)$, the relation $\leqslant$ is antisymmetric. □

Example 3.3. Let P be a partially ordered set with greatest lower bounds for all pairs of elements (so that all pullbacks exist, as required). Suppose that the segment $\{x \in P \mid x \leqslant a\}$ is finite for each $a \in P$. Then the category formed from the poset P in the usual way is an indexing category. Hence any finite tree forms an indexing category (upon choosing a root), as does the subgroup lattice of a finite group.

Example 3.4. Given a subset A of the natural numbers (without 0), let A_+ denote $A \cup \{0\}$, where 0 will always play the role of the basepoint. Let $\mathcal{I}$ denote the category with objects the finite sets A_+ and morphisms the based injective functions. As pullbacks in $\mathcal{I}$ are given by intersections, it is easy to see that $\mathcal{I}$ is an indexing category.

Example 3.5. (After Example 11.2 of [15].) Fix a finite group G and a homogeneous G-set G/H. Denote by $\Gamma(G/H)$ the category with G/H as its only object and equivariant G-maps as the morphisms. Functors from this category into the category of R-modules give left $R\mathrm{Aut}(G/H)$-modules. It is well-known that all equivariant maps $G/H \to G/H$ are automorphisms, so $\Gamma(G/H)$ is an EI-category with pullbacks. The finiteness condition is obviously met, so $\Gamma(G/H)$ is an indexing category.

Example 3.6. Let G be a group and let $\mathcal{M}_G$ be the category of finite G-sets and equivariant monomorphisms. Pullbacks correspond to intersections and the finiteness condition on the skeleta is clearly satisfied. Thus $\mathcal{M}_G$ is an indexing category.

Example 3.7. Call a Γ-map $i : \mathbf{m}_+ \to \mathbf{n}_+$ *ordered* if $i(x) < i(y)$ whenever $x < y$. Letting $\mathcal{O}$ denote the subcategory of Γ consisting of the ordered maps, we see that $\mathcal{O}$ is an indexing category. It is clear that $\mathcal{O}(\mathbf{m}_+, \mathbf{n}_+)$ is in one-to-one correspondence with the subsets of $\mathbf{n} = \{1, 2, \ldots, n\}$ of order m. Under this correspondence, the pullback of two maps in $\mathcal{O}$ corresponds to the intersection of the subsets they represent. Moreover, we have $\mathrm{sk}(\mathcal{O} \downarrow \mathbf{n}_+) = \mathcal{O} \downarrow \mathbf{n}_+$ since $\mathcal{O}$ has no non-identity isomorphisms.

This last example has some additional structure that should be emphasized. Every ordered map $i : \mathbf{m}_+ \to \mathbf{n}_+$ has a natural dual $i^* : \mathbf{n}_+ \to \mathbf{m}_+$ that collapses the complement of the image of i to the basepoint and satisfies $i^* \circ i = 1$. These two properties in fact characterize the *collapse map* i^*. All such collapse maps give a subcategory of Γ.

It is clear that $(k \circ i)^* = i^* \circ k^*$, so that the category $\mathcal{O}^*$ of collapse maps is isomorphic to $\mathcal{O}^{\mathrm{op}}$. Moreover, a commutative square

$$\begin{tikzcd} \mathbf{m}_+ \arrow[r, "i"] \arrow[d, "j"'] & \mathbf{n}_+ \arrow[d, "k"] \\ \mathbf{r}_+ \arrow[r, "l"] & \mathbf{s}_+ \end{tikzcd}$$

in $\mathcal{O}$ is a pullback if and only if $j \circ i^* = l^* \circ k$ (recall that pullbacks are intersections here). This “interchange law” implies that all composites of the form $j \circ i^*$ $(i, j \in \mathcal{O})$ yield a subcategory of Γ: in order to compose two such maps, one must use the appropriate pullback to swap the two middle terms. Note that this new parent category contains both $\mathcal{O}$ and $\mathcal{O}^*$ as subcategories. In the next section we will see that every indexing category allows for a construction of this sort; see Example 3.15 below.

3.2. Axioms for conjugation

Let $\mathcal{U}$ be a category with subcategories $\mathcal{P}$ and $\mathcal{Q}$, all three having the same objects. We say that $\mathcal{U}$ *factors as* $\mathcal{Q} \circ \mathcal{P}$ if every morphism of $\mathcal{U}$ is expressible as a composition $q \circ p$ for some maps $q \in \mathcal{Q}$ and $p \in \mathcal{P}$. In this case we shall write $\mathcal{U} = \mathcal{Q} \circ \mathcal{P}$. For us, such factorizations will not necessarily be unique on the nose, but only up to the correct notion of equivalence via maps in an indexing category (see the second axiom below).

Definition 3.8. Suppose that $\mathcal{U}$ is a small category that factors as $\mathcal{U} = \mathcal{I} \circ \mathcal{A}$, where $\mathcal{I}$ is an indexing category and $\mathcal{A}$ contains all the isomorphisms from $\mathcal{I}$. We say that this factorization *admits conjugation* if the following three axioms hold:

- All hom-sets in $\mathcal{U}$ are finite.[5]
- The factorization $\mathcal{U} = \mathcal{I} \circ \mathcal{A}$ is unique up to lifting isomorphisms in $\mathcal{I}$. Precisely, for each commutative square

$$\begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d, "\beta"'] & b \arrow[d, "i"] \\ c \arrow[r, "j"] \arrow[ur, dashed] & d \end{tikzcd}$$

  with $\alpha, \beta \in \mathcal{A}$ and $i, j \in \mathcal{I}$, the indicated lift exists and is an isomorphism in $\mathcal{I}$.
- Given $\alpha : a \to b$ in $\mathcal{A}$ and $i : c \to b$ in $\mathcal{I}$, there is a pullback in $\mathcal{U}$ of the form

$$\begin{tikzcd} p \arrow[r, "i'"] \arrow[d, "\alpha'"'] & a \arrow[d, "\alpha"] \\ c \arrow[r, "i"] & b \end{tikzcd}$$

  with $\alpha' \in \mathcal{A}$ and $i' \in \mathcal{I}$.

[5] While this axiom will be needed only once, it is non-negotiable: see the proof of Lemma 5.5.

It is possible to rephrase various parts of this definition, obtaining an equivalent formulation of the version above. For instance, from our axioms one can prove that $\mathcal{A}$ must contain all the isomorphisms in $\mathcal{U}$, not just those of $\mathcal{I}$. Thus we could equally as well assume from the start that $\mathcal{A}$ contains all the isomorphisms in $\mathcal{U}$, but our version is equivalent and assumes less, so we won't bother. Likewise, one could instead assume that the hom-sets in $\mathcal{A}$ are finite as a replacement for the first axiom, but the version we've stated is more convenient for our purposes.

In essence, our axioms for conjugation serve to generalize the notions of image and inverse image, ensuring each is unique up to the "correct" kind of isomorphism. The motivation for this definition comes from the following important example.

Example 3.9. Let us say that a map γ in Γ is *regular* if $\gamma^{-1}(0) = \{0\}$. That is, γ is regular if it sends only the basepoint to the basepoint. Let $\mathcal{U}$ denote the subcategory of regular maps. By forgetting the basepoints, we see that $\mathcal{U}$ is equivalent to the category of finite unbased sets. Clearly the first axiom is met.

Take as our indexing category the category $\mathcal{O}$ of ordered maps in Γ (recall Example 3.7). By adding disjoint basepoints, we may take the category $\mathcal{E}$ of unbased epimorphisms to be a subcategory of Γ. Every map in $\mathcal{U}$ clearly factors as an epimorphism followed by the inclusion of the image into the original codomain. Once this image subset is uniquely represented by an $\mathcal{O}$-map, we see that $\mathcal{U}$ factors as $\mathcal{O} \circ \mathcal{E}$. It is immediate that the second axiom holds since such factorizations are unique on the nose (as $\mathcal{O}$ has no non-identity isomorphisms).

For the third axiom, it is instructive to check that the pullback of an $\mathcal{E}$-map $\alpha : \mathbf{m}_+ \to \mathbf{n}_+$ along an $\mathcal{O}$-map $i : \mathbf{k}_+ \to \mathbf{n}_+$ is the inverse image under α of the subset of $\mathbf{n}_+$ given by the image of i. It is then clear that parallel partners in the pullback square belong to the same subcategory, so the third axiom holds. Hence the factorization $\mathcal{U} = \mathcal{O} \circ \mathcal{E}$ admits conjugation.

The following observation will be needed only in the next section. The proof is a straightforward exercise in using our axioms for conjugation.

Proposition 3.10. *Suppose we have two pullback squares*

$$\begin{array}{ccc} p & \xrightarrow{i'} & a \\ {\scriptstyle\alpha'}\downarrow & & \downarrow{\scriptstyle\alpha} \\ c & \xrightarrow{i} & b \end{array} \qquad \begin{array}{ccc} p' & \xrightarrow{i''} & a \\ {\scriptstyle\alpha''}\downarrow & & \downarrow{\scriptstyle\alpha} \\ c & \xrightarrow{i} & b \end{array}$$

of the type addressed in the third axiom of Definition 3.8. Then the canonical morphism from p to p' is an isomorphism in $\mathcal{I}$.

3.3. Conjugate pairs

Throughout this discussion, suppose that $\mathcal{U} = \mathcal{I} \circ \mathcal{A}$ is a factorization admitting conjugation. We will construct a new category $\mathcal{B}$ containing all of the original data as subcategories, with $\mathcal{B}$ factoring as $\mathcal{B} = \mathcal{I} \circ \mathcal{A} \circ \mathcal{I}^{op}$. To foreshadow, we obtain $\mathcal{B}$ by attaching to maps in $\mathcal{U}$ formal pre-compositions by maps in $\mathcal{I}^{op}$, much in the spirit of the calculus of fractions. This results in the three-fold factorizations by cokernels and images alluded to in the introduction.

We will refer to the pair of categories $(\mathcal{B}, \mathcal{A})$ as a *conjugate pair*. This construction is a generalization of the *induction categories* of [15], or the category $\omega(G)$ of [16], both well-known to representation theorists. In fact, if one takes $\mathcal{A}$ to be Iso($\mathcal{I}$), the category of isomorphisms in $\mathcal{I}$, then our axioms for conjugation are trivially satisfied and the construction we give reduces to the others.

The category $\mathcal{B}$ will have the same objects as $\mathcal{U}$. A morphism $\beta : a \to b$ in $\mathcal{B}$ will be represented by a diagram

$$a \xleftarrow{i} a_1 \xrightarrow{\alpha} b_1 \xrightarrow{j} b$$

where $i, j \in \mathcal{I}$ and $\alpha \in \mathcal{A}$. Writing $i^* : a \to a_1$ for the formal opposite of i, we shall write $\beta = j \circ \alpha \circ i^*$. In order to get an honest category, we will have to identify some of these morphisms. We will declare the morphism above to be equivalent to

$$a \xleftarrow{k} a_2 \xrightarrow{\gamma} b_2 \xrightarrow{l} b$$

if there exist isomorphisms $\varphi, \psi \in \mathcal{I}$ making the entire diagram

$$\begin{tikzcd} & a_1 \arrow[dl, "i"'] \arrow[r, "\alpha"] \arrow[d, "\varphi"] & b_1 \arrow[dr, "j"] \arrow[d, "\psi"] & \\ a & a_2 \arrow[l, "k"] \arrow[r, "\gamma"] & b_2 \arrow[r, "l"] & b \end{tikzcd}$$

commute.

It is easy to check that this gives an equivalence relation, and one can take the morphisms in $\mathcal{B}$ to be equivalence classes of such diagrams. Alternatively, we may take the morphisms of $\mathcal{B}$ to be all formal composites of the form $\beta = j \circ \alpha \circ i^*$ as above, with the understanding that such representations are not unique. It is convenient to think of β as admitting a three-fold factorization

$$\begin{tikzcd} a \arrow[r, "\beta"] \arrow[d, "i^*"'] & b \\ a_1 \arrow[r, "\alpha"] & b_1 \arrow[u, "j"'] \end{tikzcd}$$

with such factorizations unique only up to adjustments by isomorphisms in $\mathcal{I}$. The latter point of view leads to simpler notation, so this is the approach we will take. We will refer to i^* (or even i) as the *cokernel* of β, and likewise we will call j its *image*.

Note. In any two three-fold factorizations of the given map β, the cokernel morphisms are isomorphic objects in the comma category $\mathcal{I} \downarrow a$. Likewise, the images are isomorphic in $\mathcal{I} \downarrow b$. Hence three-fold factorizations are unique if we require the $\mathcal{I}$-components to lie in a fixed skeleton of the relevant comma category.

Composition of such morphisms is defined in terms of pullbacks in the indexing category $\mathcal{I}$ and the axioms for conjugation. The composition of

$$a \xleftarrow{i} w \xrightarrow{\alpha} x \xrightarrow{j} b$$

with

$$b \xleftarrow{k} y \xrightarrow{\gamma} z \xrightarrow{l} c$$

is displayed in the following diagram:

$$\begin{tikzcd} & & p \arrow[r, "\alpha'"] \arrow[dl, "k''"'] & q \arrow[r, "\gamma'"] \arrow[dl, "k'"'] \arrow[dr, "j'"] & r \arrow[dr, "j''"] & & \\ & w \arrow[r, "\alpha"] \arrow[dl, "i"'] & x \arrow[dr, "j"] & & y \arrow[r, "\gamma"] \arrow[dl, "k"'] & z \arrow[dr, "l"] & \\ a & & & b & & & c. \end{tikzcd}$$

The middle diamond is the pullback of j along k, formed in $\mathcal{I}$. The upper-left square is the pullback of α along k' per the third axiom for conjugation; hence $\alpha' \in \mathcal{A}$ and $k'' \in \mathcal{I}$. Since $\mathcal{U} = \mathcal{I} \circ \mathcal{A}$, the map $\gamma \circ j'$ admits a factorization of the form $j'' \circ \gamma'$ where $j'' \in \mathcal{I}$ and $\gamma' \in \mathcal{A}$; this is the upper-right square. Hence the composition is given by $(lj'') \circ (\gamma'\alpha') \circ (ik'')^*$.

Of course, none of the steps in this composition are necessarily uniquely determined. However, it is easy to check that different choices would lead to equivalent morphisms, thanks to the axioms for conjugation and Proposition 3.10. Hence this composition is in fact well-defined, and this completes the description of the category $\mathcal{B} = \mathcal{I} \circ \mathcal{A} \circ \mathcal{I}^{op}$.

Definition 3.11. Suppose that $\mathcal{U} = \mathcal{I} \circ \mathcal{A}$ is a factorization admitting conjugation. If the category $\mathcal{B}$ is constructed as $\mathcal{I} \circ \mathcal{A} \circ \mathcal{I}^{op}$ as described above, we say that $(\mathcal{B}, \mathcal{A})$ is a *conjugate pair* of small categories.

Although not an explicit part of the notation, we will always assume that in every conjugate pair $(\mathcal{B}, \mathcal{A})$ we have fixed the underlying categories $\mathcal{U}$ and $\mathcal{I}$. Moreover, in some of the examples that follow, we may identify the resulting category $\mathcal{B} = \mathcal{I} \circ \mathcal{A} \circ \mathcal{I}^{op}$ with an isomorphic category having a more "down to earth" description, and do so without mention. Since isomorphic categories yield isomorphic categories of diagrams, this will have no impact on our results. That being said, in all of our proofs and arguments we will always work with the precise formulation of the category $\mathcal{B} = \mathcal{I} \circ \mathcal{A} \circ \mathcal{I}^{op}$ described above.

Proposition 3.12. *Suppose that $(\mathcal{B}, \mathcal{A})$ is a conjugate pair arising from the factorization $\mathcal{U} = \mathcal{I} \circ \mathcal{A}$.*

(a) *For maps $i, j \in \mathcal{I}$, we have $(j \circ i)^* = i^* \circ j^*$ whenever the composition is defined.*
(b) *For any map $i \in \mathcal{I}$, we have $i^* \circ i = 1$.*

Proof. These assertions follow immediately from the law of composition in $\mathcal{B}$. For the second, one needs only to recall that all maps in $\mathcal{I}$ are monic, hence the pullback (formed in $\mathcal{I}$) of i along itself may be given by completing the square with identity maps. $\square$

3.4. Examples

We start with the smallest and perhaps most instructive example.

Example 3.13. Let $\mathcal{I}$ be the category consisting of two objects and only one non-identity map, say

$$0 \xrightarrow{i} 1.$$

Taking $\mathcal{A}$ to be discrete, the resulting category $\mathcal{B} = \mathcal{I} \circ \mathcal{I}^{\mathrm{op}}$ has diagrammatic representation

$$\begin{tikzcd} 0 \arrow[r, shift left, "i"] & 1 \arrow[l, shift left, "i^*"] \end{tikzcd}$$

where $i^* \circ i = 1$. Note then that $i \circ i^*$ is an idempotent.

At this point it is instructive to recall what our main theorem would say here. It would assert that, for stable model categories $\mathcal{C}$, the functor category $[\mathcal{B}^{\mathrm{op}}, \mathcal{C}]$ is equivalent to $[\mathcal{A}^{\mathrm{op}}, \mathcal{C}]$. As $\mathcal{A}$ is discrete, the latter is simply $\mathcal{C} \times \mathcal{C}$. Hence the theorem says that to give a diagram

$$\begin{tikzcd} M \arrow[r, shift left, "i"] & N \arrow[l, shift left, "i^*"] \end{tikzcd}$$

in $\mathcal{C}$ with $i \circ i^*$ an idempotent is equivalent to giving two objects in $\mathcal{C}$ (namely, M and the "kernel" of i^*). Hence our main theorem is essentially a statement about the ability to split idempotents in $\mathcal{C}$; this is the elegant explanation of our work. (Of course, we are concerned with obtaining a Quillen equivalence and so we are splitting idempotents in the homotopy category, not $\mathcal{C}$ itself.)

Example 3.14. Let $\mathcal{I}$ be the category of a partially ordered set P as in Example 3.3. With $\mathcal{A}$ discrete, the category $\mathcal{B} = \mathcal{I} \circ \mathcal{I}^{\mathrm{op}}$ has the following description. A morphism $a : x \to y$ in $\mathcal{B}$ is just the statement that $a \leqslant x$ and $a \leqslant y$. The composition of $a : x \to y$ and $b : y \to z$ is the greatest lower bound of a and b. The category $\mathcal{I}$ appears in $\mathcal{B}$ as the maps $x : x \to y$ and similarly the maps $x : y \to x$ represent $\mathcal{I}^{\mathrm{op}}$ as a subcategory. We now see that each map $a : x \to y$ in $\mathcal{B}$ factors uniquely as

$$\begin{tikzcd} x \arrow[rr, "a"] \arrow[dr, "a"'] & & y \\ & a. \arrow[ur, "a"'] & \end{tikzcd}$$

With $\mathcal{A}$ the discrete category on the elements of P, we have that $(\mathcal{B}, \mathcal{A})$ is a conjugate pair.

Example 3.15. The two previous examples generalize: every indexing category gives rise to a conjugate pair. Given an indexing category $\mathcal{I}$, let $\mathcal{A}$ be the category $\mathrm{Iso}(\mathcal{I})$ of isomorphisms in $\mathcal{I}$. It is immediate that the axioms for conjugation are satisfied, and so we obtain a category $\mathcal{B}$ with $(\mathcal{B}, \mathrm{Iso}(\mathcal{I}))$ a conjugate pair. This is the non-additive version of the induction categories of [15].

Example 3.16. Here we obtain our motivating example, namely that $(\Gamma, \mathcal{E})$ is a conjugate pair. Recall the terminology and notation of Example 3.9, where we saw that $\mathcal{U} = \mathcal{O} \circ \mathcal{E}$ admits conjugation. We claim that the category $\mathcal{B}$ is equivalent to Γ. Recalling that $\mathcal{O}^{\mathrm{op}}$ is equivalent to the category $\mathcal{O}^*$ of collapse maps, we show that every map in Γ admits an internal three-fold factorization of the type $\mathcal{O} \circ \mathcal{E} \circ \mathcal{O}^*$.

Fix a Γ-map $\gamma : \mathbf{m}_+ \to \mathbf{n}_+$ and let $i : \mathbf{r}_+ \to \mathbf{m}_+$ be the $\mathcal{O}$-map representing the complement of the "kernel" $\gamma^{-1}(0)$ of γ. Upon using i^* to collapse the kernel to a point, γ induces a regular map $\overline{\gamma} : \mathbf{r}_+ \to \mathbf{n}_+$. This regular map $\overline{\gamma}$ then admits a factorization by an $\mathcal{E}$-map $\gamma' : \mathbf{r}_+ \to \mathbf{s}_+$ followed by the map $j : \mathbf{s}_+ \to \mathbf{n}_+$ representing the image of γ (which is also the image of $\overline{\gamma}$). Thus γ admits a three-fold factorization as $\gamma = j \circ \gamma' \circ i^*$:

$$\begin{tikzcd} \mathbf{m}_+ \arrow[r, "\gamma"] \arrow[d, "i^*"'] & \mathbf{n}_+ \\ \mathbf{r}_+ \arrow[r, "\gamma'"] & \mathbf{s}_+. \arrow[u, "j"'] \end{tikzcd}$$

Hence $(\Gamma, \mathcal{E})$ is a conjugate pair.

Example 3.17. We pause to give a concrete example of the previous factorization, since it has been the motivation for all of our work thus far. Let $\gamma : \mathbf{5}_+ \to \mathbf{3}_+$ be defined by

x	0	1	2	3	4	5
$\gamma(x)$	0	3	0	1	1	0

The three elements in $\{1, 3, 4\}$ do *not* map to the basepoint, so our collapse map $i^* : \mathbf{5}_+ \to \mathbf{3}_+$ in $\mathcal{O}^*$ is defined by

x	0	1	2	3	4	5
$i^*(x)$	0	1	0	2	3	0

Note that i^* and γ send the same elements to the basepoint. The image of γ is $\{0, 1, 3\}$ which is isomorphic to $\mathbf{2}_+$, so our image map $j : \mathbf{2}_+ \to \mathbf{3}_+$ in $\mathcal{O}$ is given by

x	0	1	2
$j(x)$	0	1	3

Finally, the map $\gamma' : \mathbf{3}_+ \to \mathbf{2}_+$ defined by

x	0	1	2	3
$\gamma'(x)$	0	2	1	1

is the unique map making the diagram

$$\begin{tikzcd} \mathbf{5}_+ \arrow[r, "\gamma"] \arrow[d, "i^*"'] & \mathbf{3}_+ \\ \mathbf{3}_+ \arrow[r, "\gamma'"] & \mathbf{2}_+ \arrow[u, "j"'] \end{tikzcd}$$

commute. Note that γ' is both regular and surjective, so that it belongs to $\mathcal{E}$.

Example 3.18. The $(\Gamma, \mathcal{E})$ example may be fattened up a bit. Let $\mathcal{B}$ denote the category with objects the finite based subsets A_+ of the natural numbers (with 0 acting as the basepoint) and morphisms the based maps. With $\mathcal{A}$ the subcategory of regular based surjections and $\mathcal{I}$ as in Example 3.4, we have that $(\mathcal{B}, \mathcal{A})$ forms a conjugate pair.

Example 3.19. Let $\mathcal{B}$ denote the category of Γ-maps β such that the inverse image $\beta^{-1}(x)$ of each nonzero point x is either empty or a singleton. That is, $\beta \in \mathcal{B}$ may send lots of elements to the basepoint, but modulo this, it is injective. With Σ denoting the category of regular permutations, $\mathcal{B}$ factors as $\mathcal{O} \circ \Sigma \circ \mathcal{O}^*$ and $(\mathcal{B}, \Sigma)$ is a conjugate pair.

4. Two natural decompositions

For the remainder of this paper, $(\mathcal{B}, \mathcal{A})$ will denote a fixed conjugate pair arising from a factorization $\mathcal{U} = \mathcal{I} \circ \mathcal{A}$. In this section we construct the bimodule $\mathcal{U}_+ : \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathrm{Sets}_*$ that will induce our Quillen equivalence. In Propositions 4.6 and 4.8 we show that this bimodule is "free" as a right $\mathcal{A}$-module and a "generator" in its left $\mathcal{B}$-module structure.

Definition 4.1. The maps in $\mathcal{B}$ lying in the subcategory $\mathcal{U} = \mathcal{I} \circ \mathcal{A}$ will be called the *regular maps*. If a map is not regular, we will say it is *singular*.

Thus, in our standard example Γ, a map γ is regular if and only if $\gamma^{-1}\{0\} = \{0\}$.

Note. It is easy to prove that a map $\beta = j \circ \alpha \circ i^*$ is regular if and only if i is an isomorphism in $\mathcal{I}$. Since $\mathcal{A}$ contains all the isomorphisms from $\mathcal{I}$, a regular map is always equivalent to one with the identity as its cokernel, as shown by the diagram

$$\begin{tikzcd} & a_1 \arrow[dl, "i"'] \arrow[r, "\alpha"] \arrow[d, "i"'] & b_1 \arrow[d, "1"] \arrow[dr, "j"] & \\ a & a \arrow[l, "1"] \arrow[r, "\alpha i^{-1}"] & b_2. \arrow[r, "j"'] & b \end{tikzcd}$$

Notation 4.2. Let $\mathcal{S}(a, b)$ denote the set of singular maps in $\mathcal{B}(a, b)$. We let $\mathcal{U}(a, b)_+$ denote the quotient set

$$\mathcal{U}(a, b)_+ = \mathcal{B}(a, b)/\mathcal{S}(a, b)$$

where we take the singular maps as a basepoint. As $\mathcal{B}(a, b)$ is the disjoint union of the regular and singular maps, this may also be regarded as the set of regular maps together with a disjoint basepoint (thus this notation is sensible).

Proposition 4.3. *In any conjugate pair* $(\mathcal{B}, \mathcal{A})$*, the singular maps have the following "ideal-like" properties*:

(a) *If* $\alpha \in \mathcal{A}(a, b)$ *and* $\gamma \in \mathcal{S}(b, c)$ *then* $\gamma \circ \alpha \in \mathcal{S}(a, c)$.
(b) *If* $\gamma \in \mathcal{S}(a, b)$ *and* $\beta \in \mathcal{B}(b, c)$ *then* $\beta \circ \gamma \in \mathcal{S}(a, c)$.

Proof. For the first statement, factor γ as $\gamma = j \circ \gamma' \circ i^*$ and suppose to the contrary that $\gamma \circ \alpha = (j \circ \gamma' \circ i^*) \circ \alpha$ is regular. The composition $i^* \circ \alpha$ is computed by a pullback diagram

$$\begin{tikzcd} p \arrow[r, "i'"] \arrow[d, "\alpha'"'] & a \arrow[d, "\alpha"] \\ c \arrow[r, "i"] & b \end{tikzcd}$$

as in the third axiom for conjugation (see Definition 3.8). Thus we determine that the three-fold factorization of $\gamma \circ \alpha$ is

$$\gamma \circ \alpha = \left(j \circ \gamma' \circ i^*\right) \circ \alpha = j \circ \left(\gamma' \circ \alpha'\right) \circ \left(i'\right)^*.$$

As $\gamma \circ \alpha$ is regular, i' must be an isomorphism. By our previous note we can arrange to take i' to be the identity. The pullback square above now tells us that $i \circ \alpha'$ is a valid factorization of the $\mathcal{A}$-map α, so i must be an isomorphism. Hence γ is regular, a contradiction.

The second assertion is a direct consequence of the following observation on indexing categories: if i and j are maps in $\mathcal{I}$ with $i \circ j$ an isomorphism, then both factors are isomorphisms as well. This implies that if i^* is not an isomorphism then neither is any composition $j^* \circ i^* = (i \circ j)^*$. Applying this fact with i^* as the cokernel of the singular map γ proves the second claim. □

Proposition 4.4. *Given a conjugate pair* $(\mathcal{B}, \mathcal{A})$*, the construction* $\mathcal{U}(-,-)_+$ *defines a functor* $\mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathrm{Sets}_*$.

Proof. Suppose we are given morphisms $\alpha : a \to b$ in $\mathcal{A}$ and $\beta : c \to d$ in $\mathcal{B}$. We then get an associated map $\mathcal{B}(b, c) \to \mathcal{B}(a, d)$ which sends a map $\gamma : b \to c$ to the composite $\beta \circ \gamma \circ \alpha$. By Proposition 4.3, this sends singular maps to singular maps. Hence this passes down to quotients, as desired. □

The functor $\mathcal{U}_+ : \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathrm{Sets}_*$ is the bimodule desired for our Morita equivalence. We will refer to this functor as the *regular bimodule*.

In the following, we will write $\mathrm{dom}(i)$ for the domain of a morphism i.

Lemma 4.5. *Suppose that* $(\mathcal{B}, \mathcal{A})$ *is a conjugate pair and fix an object a of* $\mathcal{A}$. *Every map* $\beta : b \to c$ *in* $\mathcal{B}$ *induces a map*

$$\beta_* : \bigvee_{i \in \mathrm{sk}(\mathcal{I} \downarrow b)} \mathcal{A}\left(a, \mathrm{dom}(i)\right)_+ \longrightarrow \bigvee_{j \in \mathrm{sk}(\mathcal{I} \downarrow c)} \mathcal{A}\left(a, \mathrm{dom}(j)\right)_+.$$

Furthermore, this assignment is functorial, so that wedge sums of the form

$$\bigvee_{i \in \mathrm{sk}(\mathcal{I} \downarrow b)} \mathcal{A}\left(a, \mathrm{dom}(i)\right)_+$$

give a functor $\mathcal{B} \to \mathrm{Sets}_*$ *in the b-variable.*

Verifying that this defines a functor is fairly straightforward, and at worst consists of checking a few special cases. The proof uses only three-fold factorizations and Proposition 4.3, therefore we will only describe the induced map β_*.

Suppose we are in the summand corresponding to $i : b' \to b$ in $\mathcal{I}$, and let $\alpha : a \to b'$ be a map in $\mathcal{A}$. We consider the composite

$$a \xrightarrow{\alpha} b' \xrightarrow{i} b \xrightarrow{\beta} c.$$

If this composite is singular, we define $\beta_*(\alpha)$ to be the basepoint. Otherwise it is regular, and thus admits a factorization

$$\begin{tikzcd} a \arrow[r, "\beta i \alpha"] \arrow[d, "1^*"'] & c \\ a \arrow[r, "\alpha'"] & c' \arrow[u, "j"'] \end{tikzcd}$$

where $\alpha' \in \mathcal{A}$ and $j \in \mathrm{sk}(\mathcal{I} \downarrow c)$. Note that when factored in this form, α' is uniquely determined. We then let $\beta_*(\alpha) = \alpha'$, corresponding to the summand indexed by j.

The next result states that the bimodule $\mathcal{U}_+ : \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathrm{Sets}_*$ is free as a right $\mathcal{A}$-module. Note that the naturality claim in the following is well-posed by the lemma.

Proposition 4.6. *Suppose that $(\mathcal{B}, \mathcal{A})$ is a conjugate pair. For each pair of objects a and b of $\mathcal{B}$, there is an isomorphism of based sets*

$$\mathcal{U}(a,b)_+ \cong \bigvee_{i \in \mathrm{sk}(\mathcal{I}\downarrow b)} \mathcal{A}\big(a, \mathrm{dom}(i)\big)_+$$

which is natural in both variables. In other words, for each object b of $\mathcal{B}$ there is a natural equivalence

$$\mathcal{U}(-,b)_+ \cong \bigvee_{i \in \mathrm{sk}(\mathcal{I}\downarrow b)} \mathcal{A}\big(-, \mathrm{dom}(i)\big)_+$$

of functors $\mathcal{A}^{\mathrm{op}} \to \mathrm{Sets}_*$ and these equivalences vary naturally with b.

The idea of the proof is simple: every regular map $\gamma : a \to b$ admits a factorization by a map $\alpha : a \to b'$ in $\mathcal{A}$ followed by the inclusion $i : b' \to b$ of the image of γ back into the codomain. Once a skeleton has been fixed, such a factorization is unique. Hence under our isomorphism, γ corresponds to $\alpha \in \mathcal{A}(a, b')_+$, landing in the summand indexed by i.

Next we carry out a similar analysis in the other variable.

Lemma 4.7. *Suppose that $(\mathcal{B}, \mathcal{A})$ is a conjugate pair and fix an object c of $\mathcal{B}$. Every map $\beta : a \to b$ in $\mathcal{B}$ induces a map*

$$\beta^* : \bigvee_{i \in \mathrm{sk}(\mathcal{I}\downarrow b)} \mathcal{U}\big(\mathrm{dom}(i), c\big)_+ \longrightarrow \bigvee_{j \in \mathrm{sk}(\mathcal{I}\downarrow a)} \mathcal{U}\big(\mathrm{dom}(j), c\big)_+.$$

Furthermore, this assignment is functorial, so that wedge sums of the form

$$\bigvee_{i \in \mathrm{sk}(\mathcal{I}\downarrow b)} \mathcal{U}\big(\mathrm{dom}(i), c\big)_+$$

give a functor $\mathcal{B}^{\mathrm{op}} \to \mathrm{Sets}_*$ in the b-variable.

Proposition 4.8. *Suppose that $(\mathcal{B}, \mathcal{A})$ is a conjugate pair. For each pair of objects b and c of $\mathcal{B}$, there is an isomorphism of based sets*

$$\mathcal{B}(b,c)_+ \cong \bigvee_{i \in \mathrm{sk}(\mathcal{I}\downarrow b)} \mathcal{U}\big(\mathrm{dom}(i), c\big)_+$$

which is natural in the first variable.

We remark that in general the isomorphisms of Proposition 4.8 are *not* natural in the second variable; counterexamples are easy to come by in $(\Gamma, \mathcal{E})$. Furthermore, by the finiteness condition on the indexing category $\mathcal{I}$, the wedge sums of Propositions 4.6 and 4.8 consist of only a finite number of summands.

As before, verifying the assorted claims is rather formal (yet very tedious) once the definition of the induced map β^* is made clear. To that end, suppose that $\beta : a \to b$ has three-fold factorization

$$\begin{tikzcd} a \arrow[r, "\beta"] \arrow[d, "i_1^*"'] & b \\ a_1 \arrow[r, "\beta_1"] & b_1 \arrow[u, "j_1"'] \end{tikzcd}$$

chosen with respect to the skeleta. We shall describe the map

$$\beta^* : \bigvee_{i \in \mathrm{sk}(\mathcal{I}\downarrow b)} \mathcal{U}\big(\mathrm{dom}(i), c\big)_+ \longrightarrow \bigvee_{j \in \mathrm{sk}(\mathcal{I}\downarrow a)} \mathcal{U}\big(\mathrm{dom}(j), c\big)_+$$

on the summand corresponding to a fixed map $i : b' \to b$ in $\mathcal{I}$. (This map will always send regular maps to regular maps, so the basepoint is of no concern here.) The composite

$$a_1 \xrightarrow{\beta_1} b_1 \xrightarrow{j_1} b \xrightarrow{i^*} b'$$

admits a factorization as a map $i_2^*: a_1 \to a_2$ followed by a regular map $\beta_2 : a_2 \to b'$ (so β_2 is simply the last two legs of the three-fold factorization). Given a regular map $\gamma : b' \to c$, we define

$$\beta^*(\gamma) = \gamma \circ \beta_2$$

landing in the summand corresponding to $j = i_1 \circ i_2 : a_2 \to a$.

5. The induced adjoint pair; free functors

5.1. The product decomposition

Thus far, from a conjugate pair $(\mathcal{B}, \mathcal{A})$ we obtain an associated bimodule $\mathcal{U}_+ : \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathrm{Sets}_*$. For a fixed model category $\mathcal{C}$, this in turn gives rise to an adjoint pair

$$[\mathcal{B}^{\mathrm{op}}, \mathcal{C}] \underset{R}{\overset{L}{\rightleftarrows}} [\mathcal{A}^{\mathrm{op}}, \mathcal{C}]$$

between model categories of functors, as described in Section 2.1 (where $P = \mathcal{U}_+$). Before we can show that this is a Quillen equivalence when $\mathcal{C}$ is stable, we must first establish that (L, R) is at least a Quillen pair. For this, the hypothesis of stability is not needed. The relevant fact here is that the bimodule $\mathcal{U}_+$ is free as a right $\mathcal{A}$-module.

Proposition 5.1. *Suppose that $\mathcal{C}$ is a model category and that $(\mathcal{B}, \mathcal{A})$ is a conjugate pair of small categories. For each functor $G : \mathcal{A}^{\mathrm{op}} \to \mathcal{C}$ and object b in $\mathcal{B}$, there is an isomorphism*

$$RG(b) \cong \prod_{i \in \mathrm{sk}(\mathcal{I}\downarrow b)} G\big(\mathrm{dom}(i)\big)$$

and this is natural in both G and b.

Proof. Recall that $RG(b) = \mathrm{End}_{\mathcal{A}}(\mathcal{U}(-, b)_+, G)$. Now substitute the natural isomorphism

$$\mathcal{U}(-, b)_+ \cong \bigvee_{i \in \mathrm{sk}(\mathcal{I}\downarrow b)} \mathcal{A}\big(-, \mathrm{dom}(i)\big)_+$$

of Proposition 4.6 and use the formal properties of $\mathrm{End}_{\mathcal{A}}(-, G)$ from Proposition 2.1 to obtain the desired isomorphism. □

This result has two important corollaries.

Corollary 5.2. *Suppose that $\mathcal{C}$ is a model category and that $(\mathcal{B}, \mathcal{A})$ is a conjugate pair. Then the adjoint pair (L, R) associated to the regular bimodule $\mathcal{U}_+$ is a Quillen pair.*

Proof. It is enough to check that R preserves fibrations and acyclic fibrations. Recall that the (acyclic) fibrations are defined objectwise in our categories of functors. Suppose that $\tau : G \to H$ is a fibration in $[\mathcal{A}^{\mathrm{op}}, \mathcal{C}]$. Upon evaluation at an object b of $\mathcal{B}$, Proposition 5.1 shows that the map $R\tau(b) : RG(b) \to RH(b)$ is of the type addressed in Proposition 2.2. Thus $R\tau(b)$ is a fibration as τ is objectwise, hence $R\tau$ is a fibration. The same argument shows that R preserves acyclic fibrations. □

Corollary 5.3. *Suppose that $\mathcal{C}$ is a model category and that $(\mathcal{B}, \mathcal{A})$ is a conjugate pair. Let $\tau : G \to H$ be a natural transformation of fibrant functors $G, H : \mathcal{A}^{\mathrm{op}} \to \mathcal{C}$. Then τ is a weak equivalence if and only if $R\tau$ is. Briefly, R detects and preserves weak equivalences between fibrant functors.*

Proof. Ken Brown's Lemma (see Lemma 1.1.12 of [8]) says that if R preserves acyclic fibrations between fibrant objects, then R preserves all weak equivalences between fibrant objects. Since our functor R preserves all acyclic fibrations, we see that R preserves all weak equivalences between fibrant functors.

The proof of the converse follows from the retract axiom in the model category $\mathcal{C}$. Supposing that $(f, g) : A \times C \to B \times D$ is a weak equivalence, it is clear that both f and g are retracts of the map (f, g). Since retracts preserve weak equivalences, both f and g are weak equivalences. Another appeal to Proposition 5.1 proves that if $R\tau$ is a weak equivalence, so is τ. □

Corollary 5.3 is one of two major ingredients in the proof of our main theorem; the missing ingredient is found below in Proposition 6.1. In the next section we build the necessary machinery to complete this final step.

5.2. Free functors and pushouts

We will write $\widehat{X}$ for the fibrant replacement of an object X in a model category. Recall that this comes equipped with a natural acyclic cofibration $i : X \to \widehat{X}$. Specialized to our model categories of diagrams, given $F : \mathcal{B}^{\mathrm{op}} \to \mathcal{C}$ its image LF in $[\mathcal{A}^{\mathrm{op}}, \mathcal{C}]$ has fibrant replacement $i : LF \to \widehat{LF}$. Our adjunction then provides a natural transformation $\eta'_F : F \to R(\widehat{LF})$ of functors $\mathcal{B}^{\mathrm{op}} \to \mathcal{C}$. This is clearly closely related to the honest unit map $\eta_F : F \to R(LF)$ of the adjunction, as displayed in the diagram

$$\begin{tikzcd} F \arrow[rr, "\eta'_F"] \arrow[dr, "\eta_F"'] & & R(\widehat{LF}) \\ & R(LF). \arrow[ur, "R(i)"'] & \end{tikzcd}$$

Our ultimate goal is to establish that this modified unit map $\eta'_F : F \to R(\widehat{LF})$ is a weak equivalence whenever F is cofibrant. We prove this by an induction argument using the cofibrant generation hypothesis, so as a first step we examine the free functors.

Lemma 5.4. *Suppose that $F : \mathcal{B}^{\mathrm{op}} \to \mathcal{C}$ is the free functor F_b^C (recall Definition 2.3). Then $LF : \mathcal{A}^{\mathrm{op}} \to \mathcal{C}$ may be computed by a natural isomorphism $LF(-) \cong C \otimes \mathcal{U}(-, b)_+$.*

Proof. This is just the associativity of our various tensor products, together with the Yoneda lemma. By hypothesis, $F(x) = C \otimes \mathcal{B}(x, b)_+$ so that

$$\begin{aligned} LF(a) &= F \otimes_{\mathcal{B}} \mathcal{U}(a, -)_+ \\ &= \big(C \otimes \mathcal{B}(-, b)_+\big) \otimes_{\mathcal{B}} \mathcal{U}(a, -)_+ \\ &\cong C \otimes \big(\mathcal{B}(-, b)_+ \otimes_{\mathcal{B}} \mathcal{U}(a, -)_+\big) \\ &\cong C \otimes \mathcal{U}(a, b)_+. \qquad \square \end{aligned}$$

From this point onward we will need the stability assumption on our model category $\mathcal{C}$. Note that up to this point it has not been used.

Lemma 5.5. *Let $\mathcal{C}$ be a stable model category. Suppose that $F : \mathcal{B}^{\mathrm{op}} \to \mathcal{C}$ is the free functor F_b^C with C a cofibrant object of $\mathcal{C}$. Then for each object y of $\mathcal{A}$ there is a weak equivalence*

$$\widehat{LF}(y) \xrightarrow{\sim} \prod_{\mathcal{U}(y,b)_+} \widehat{C}.$$

Proof. Consider the commutative square

$$\begin{tikzcd} LF(y) \arrow[r] \arrow[d] & \prod_{\mathcal{U}(y,b)_+} \widehat{C} \arrow[d] \\ \widehat{LF}(y) \arrow[r] \arrow[ur, dashed] & * \end{tikzcd}$$

where the top horizontal map is the natural one induced by the isomorphism

$$LF(y) \cong \bigvee_{\mathcal{U}(y,b)_+} C$$

of Lemma 5.4. The replacement map $C \to \widehat{C}$ is a weak equivalence with cofibrant domain and fibrant codomain, so the natural map

$$\bigvee_{\mathcal{U}(y,b)_+} C \longrightarrow \prod_{\mathcal{U}(y,b)_+} \widehat{C}$$

is a weak equivalence by Proposition 2.6 (here we need the finiteness of hom-sets in $\mathcal{U}$). Hence the upper horizontal map is a weak equivalence. Since $\widehat{C}$ is fibrant and $LF(y) \to \widehat{LF}(y)$ is an acyclic cofibration, the lifting axiom in the model category $\mathcal{C}$ shows that the dashed arrow above exists. The two-out-of-three axiom shows that this is a weak equivalence. $\square$

Proposition 5.6. *Let $\mathcal{C}$ be a stable model category. Suppose that $F : \mathcal{B}^{\mathrm{op}} \to \mathcal{C}$ is a free functor of the form F_b^C with C a cofibrant object of $\mathcal{C}$. Then the map $\eta'_F : F \to R(\widehat{LF})$ is a weak equivalence.*

Proof. We must show that for each object a, the natural map $\eta' : F(a) \to R(\widehat{LF})(a)$ is a weak equivalence. Recall that Proposition 4.8 supplies an isomorphism

$$\mathcal{B}(a,b)_+ \cong \bigvee_{i \in \mathrm{sk}(\mathcal{I}\downarrow a)} \mathcal{U}\big(\mathrm{dom}(i), b\big)_+$$

that is natural in the first variable. Using this in conjunction with Lemma 5.4 we obtain natural isomorphisms

$$\begin{aligned} F(a) &= C \otimes \mathcal{B}(a,b)_+ \\ &\cong C \otimes \bigvee_{i \in \mathrm{sk}(\mathcal{I}\downarrow a)} \mathcal{U}\big(\mathrm{dom}(i), b\big)_+ \\ &\cong \bigvee_{i \in \mathrm{sk}(\mathcal{I}\downarrow a)} C \otimes \mathcal{U}\big(\mathrm{dom}(i), b\big)_+ \\ &\cong \bigvee_{i \in \mathrm{sk}(\mathcal{I}\downarrow a)} LF\big(\mathrm{dom}(i)\big). \end{aligned}$$

Next we consider the composition

$$\bigvee_{i \in \mathrm{sk}(\mathcal{I}\downarrow a)} LF\big(\mathrm{dom}(i)\big) \cong F(a) \xrightarrow{\eta'_F} R(\widehat{LF})(a) \cong \prod_{j \in \mathrm{sk}(\mathcal{I}\downarrow a)} \widehat{LF}\big(\mathrm{dom}(j)\big). \tag{1}$$

Fix maps $i : x \to a$ and $j : y \to a$ in the indicated skeleton. Then i determines the inclusion of the summand $LF(x)$ into the above coproduct, while j determines the projection onto the factor $\widehat{LF}(y)$ out of the product. Let $f_{ij} : LF(x) \to \widehat{LF}(y)$ denote the corresponding composite through map (1) above, as displayed in the master diagram

$$\begin{tikzcd} C \otimes \mathcal{U}(x,b)_+ \cong LF(x) \arrow[r] \arrow[dddd, "f_{ij}"'] & \bigvee_{i \in \mathrm{sk}(\mathcal{I}\downarrow a)} LF(\mathrm{dom}(i)) \arrow[d, "\cong"] \\ & F(a) \arrow[d, "\eta'_F"] \\ & R(\widehat{LF})(a) \arrow[d, "\cong"] \\ & \\ \widehat{LF}(y) \arrow[d, "\sim"'] & \prod_{j \in \mathrm{sk}(\mathcal{I}\downarrow a)} \widehat{LF}(\mathrm{dom}(j)) \arrow[l] \\ \prod_{\mathcal{U}(y,b)_+} \widehat{C} & \end{tikzcd}$$

where we have appended the weak equivalence

$$\widehat{LF}(y) \xrightarrow{\sim} \prod_{\mathcal{U}(y,b)_+} \widehat{C}$$

of Lemma 5.5.

Recall that the index set $\mathrm{sk}(\mathcal{I}\downarrow a)$ is a finite poset. We claim that the "diagonal" maps f_{ii} are weak equivalences and that $f_{ij} = 0$ when either $i < j$ or when i and j are incomparable. Since the "middle" term $F(a)$ is $C \otimes \mathcal{B}(a,b)_+$, Lemma 5.4 shows that f_{ij} is induced by the composition

$$\mathcal{U}(x,b)_+ \longrightarrow \mathcal{B}(a,b)_+ \longrightarrow \mathcal{U}(y,b)_+ \tag{2}$$

after composing against the weak equivalence of Lemma 5.5. The composition in (2) sends a regular map $\gamma : x \to b$ to $\gamma \circ i^* \circ j$. Note that the behavior of this composition is completely dictated by $i^* \circ j$. When $i = j$ we have $i^* \circ j = i^* \circ i = 1$, hence f_{ii} is a weak equivalence, as claimed.

To prove that f_{ij} is the zero map in the other cases it suffices by Proposition 4.3 to show that $i^* \circ j$ is singular. In the case that $i < j$ there is a diagram

$$\begin{tikzcd} x \arrow[rr, "k"] \arrow[dr, "i"'] & & y \arrow[dl, "j"] \\ & a & \end{tikzcd}$$

in $\mathcal{I}$ where k is not an isomorphism (else $i = j$), and from $i = j \circ k$ we obtain $i^* \circ j = k^*$. As k is not an isomorphism, $i^* \circ j$ is singular, hence f_{ij} is the zero map.

The last case we must consider is when i and j are not comparable. If the pullback (formed in $\mathcal{I}$) of i and j is

$$\begin{tikzcd} z \arrow[r, "k"] \arrow[d, "l"'] & y \arrow[d, "j"] \\ x \arrow[r, "i"] & a \end{tikzcd}$$

then we have $l \circ k^* = i^* \circ j$. If k is an isomorphism we obtain $j \leqslant i$, a contradiction. Hence $i^* \circ j$ is singular, and f_{ij} is again the zero map.

We are almost in a position to apply Proposition 2.6 on the invertibility of lower triangular maps. We need only turn the finite partially ordered set $\mathrm{sk}(\mathcal{I} \downarrow a)$ into a totally ordered set, consistent with the original partial ordering (finiteness is crucial here). This is achieved by inserting new relations $<$ between the incomparable maps; see, for instance, Theorem 4.5.2 of [3] for details. There is some choice here of course, but this does not matter.

In such a linear ordering, $i < j$ now means one of two things: either $i < j$ in the original poset, or i and j were not comparable in the original partial ordering. In either event, f_{ij} is the zero map by our arguments above. Hence under this linear ordering the map (1) is lower triangular. Since C is cofibrant and $\widehat{C}$ is fibrant, Proposition 2.6 shows that (1) is a weak equivalence. Therefore η'_F is a weak equivalence, as desired. $\square$

Recall that a model category is *left proper* if the pushout of a weak equivalence along a cofibration is again a weak equivalence. If $\mathcal{C}$ is left proper, so is any category of functors taking values in $\mathcal{C}$. The following technical lemma is just a restatement of Proposition 13.5.6 of [7]. We will need it only in the case $\mathcal{M} = [\mathcal{B}^{\mathrm{op}}, \mathcal{C}]$.

Lemma 5.7. *Let $\mathcal{M}$ be a left proper model category and suppose we have a diagram in $\mathcal{M}$*

$$\begin{tikzcd} \widetilde{A} \arrow[r, "\sim"] \arrow[d, "\widetilde{f}"'] & A \arrow[r] \arrow[d, "f"] & X \arrow[d, "g"] \\ \widetilde{B} \arrow[r, "\sim"] & B \arrow[r] & Y \end{tikzcd}$$

in which the maps f and g are cofibrations, the right-hand square is a pushout, and the left-hand square exhibits $\widetilde{f}$ as a cofibrant replacement of f by a cofibration between cofibrant objects. Then there is a diagram

$$\begin{tikzcd} \widetilde{A} \arrow[r] \arrow[d, "\widetilde{f}"'] & \widetilde{X} \arrow[r, "\sim"] \arrow[d, "\widetilde{g}"] & X \arrow[d, "g"] \\ \widetilde{B} \arrow[r] & \widetilde{Y} \arrow[r, "\sim"] & Y \end{tikzcd}$$

in which the left-hand square is a pushout and the right-hand square exhibits $\widetilde{g}$ as a cofibrant replacement of g by a cofibration between cofibrant objects.

In short, this lemma says that when f is a cofibration, the pushout square of $B \xleftarrow{f} A \longrightarrow X$ may be sufficiently cofibrantly approximated by a pushout square in which all objects are cofibrant and the vertical maps are still cofibrations. Hence this new pushout square is a homotopy pushout, so the vertical maps have isomorphic cofibers in the homotopy category. In particular, with the zero object in the role of X, we may regard $A \longrightarrow B \longrightarrow B/A$ as a cofiber sequence in $\mathrm{ho}(\mathcal{M})$, as long as the map $A \longrightarrow B$ is a cofibration in $\mathcal{M}$ (from which it follows that B/A is necessarily cofibrant).

Proposition 5.8. *Let $\mathcal{C}$ be a left proper stable model category. Suppose that $F_b^A, F_b^B : \mathcal{B}^{op} \to \mathcal{C}$ are free functors and we have a pushout square*

$$\begin{tikzcd} F_b^A \arrow[r, "i_*"] \arrow[d] & F_b^B \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

in $[\mathcal{B}^{op}, \mathcal{C}]$ in which

(a) *the map $i : A \to B$ is a generating cofibration in the category $\mathcal{C}$ (so that $i_* : F_b^A \to F_b^B$ is a generating cofibration in the functor category), and*
(b) *the natural map $\eta'_X : X \to R(\widehat{LX})$ is a weak equivalence.*

Then the map $\eta'_Y : Y \to R(\widehat{LY})$ is a weak equivalence.

Proof. By the lemma and the remarks that follow (with $\mathcal{M} = [\mathcal{B}^{op}, \mathcal{C}]$ and $f = i_*$), we have cofiber sequences

$$F_b^A \longrightarrow F_b^B \longrightarrow F_b^{B/A}$$
$$X \longrightarrow Y \longrightarrow Y/X$$

in the homotopy category of $[\mathcal{B}^{op}, \mathcal{C}]$. Arising from parallel cofibrations in a pushout square, the cofibers must be isomorphic. Writing Z for the free functor $F_b^{B/A}$, we therefore have a cofiber sequence

$$X \longrightarrow Y \longrightarrow Z.$$

Since L is a left Quillen functor, $LX \longrightarrow LY \longrightarrow LZ$ is again a cofiber sequence, and this is also a fiber sequence as $[\mathcal{B}^{op}, \mathcal{C}]$ is a stable model category. Fibrant replacements are isomorphisms in the homotopy category, so $\widehat{LX} \longrightarrow \widehat{LY} \longrightarrow \widehat{LZ}$ is still a fiber sequence. Finally, R is right Quillen, so $R(\widehat{LX}) \longrightarrow R(\widehat{LY}) \longrightarrow R(\widehat{LZ})$ is a fiber sequence.

We now have a map of fiber sequences

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\eta'_X"'] & Y \arrow[r] \arrow[d, "\eta'_Y"] & Z \arrow[d, "\eta'_Z"] \\ R(\widehat{LX}) \arrow[r] & R(\widehat{LY}) \arrow[r] & R(\widehat{LZ}) \end{tikzcd}$$

in $\mathrm{ho}([\mathcal{B}^{op}, \mathcal{C}])$. Since B/A is the pushout of $0 \longleftarrow A \xrightarrow{i} B$ and i is a cofibration, we see that B/A is cofibrant. Proposition 5.6 now shows that η'_Z is an isomorphism in the homotopy category. By hypothesis, the same is true of η'_X. Since our map of fiber sequences is an isomorphism on the base and the fiber, the middle map must be an isomorphism as well. Hence η'_Y is a weak equivalence, as claimed. □

6. The main theorem

We are now in a position to provide the final missing ingredient.

Proposition 6.1. *Suppose that $\mathcal{C}$ is a left proper stable model category. For an arbitrary cofibrant functor $F : \mathcal{B}^{op} \to \mathcal{C}$, the modified unit map $\eta'_F : F \to R(\widehat{LF})$ is a weak equivalence.*

Proof. If F is a cofibrant diagram, it is either a cell complex or a retract thereof (we are using the language of Chapter 11 of [7] freely). The retract case follows easily from the cell complex case, so we assume that F is a cell complex. Thus there is an ordinal λ and a λ-sequence $X : \lambda \to [\mathcal{B}^{op}, \mathcal{C}]$ such that the transfinite composition of the sequence X is the map $0 \to F$. In short, the λ-sequence X satisfies:

(i) $X_0 = 0$,
(ii) $\operatorname{colim}_\alpha X_\alpha = F$, and
(iii) for each ordinal $\alpha < \lambda$, the map $X_\alpha \to X_{\alpha+1}$ fits into a pushout square

$$\begin{tikzcd} F_b^A \arrow[r, "i_*"] \arrow[d] & F_b^B \arrow[d] \\ X_\alpha \arrow[r] & X_{\alpha+1} \end{tikzcd}$$

in $[\mathcal{B}^{op}, \mathcal{C}]$ where $i_* : F_b^A \to F_b^B$ is a generating cofibration in the category of diagrams.

It follows that X is necessarily a cofibrant λ-sequence since F is cofibrant. As L is a left Quillen functor, LX is again cofibrant. Since fibrant replacement does not disturb any pre-existing cofibrancy, we see that $\widehat{LX}$ is both fibrant and cofibrant. Likewise, $\widehat{LF}$ is both fibrant and cofibrant.

The colimit structure maps give us a commutative square

$$\begin{tikzcd} X_\alpha \arrow[r, "\eta'_{X_\alpha}"] \arrow[d] & R(\widehat{LX}_\alpha) \arrow[d] \\ F \arrow[r, "\eta'_F"] & R(\widehat{LF}) \end{tikzcd}$$

for each ordinal $\alpha < \lambda$. Upon taking homotopy colimits we obtain a commutative square

$$\begin{tikzcd} \operatorname{hocolim}_\alpha X_\alpha \arrow[r, "\text{(C)}"] \arrow[d, "\text{(A)}"'] & \operatorname{hocolim}_\alpha R(\widehat{LX}_\alpha) \arrow[d, "\text{(B)}"] \\ F \arrow[r, "\gamma(\eta'_F)"] & R(\widehat{LF}) \end{tikzcd}$$

in the homotopy category, where γ is the canonical localization inverting weak equivalences. We claim that the maps labelled (A), (B), and (C) are isomorphisms. This would make $\gamma(\eta'_F)$ an isomorphism in the homotopy category, so that η'_F is then a weak equivalence, as desired. In fact, (A) is readily seen to be an isomorphism: since X is a cofibrant diagram, the natural map $\operatorname{hocolim}_\alpha X_\alpha \to \operatorname{colim}_\alpha X_\alpha = F$ is an isomorphism.

Next, we show that map (B) is an isomorphism in the homotopy category. The key observation is that $\widehat{LX}$ is again a cofibrant diagram so that we have

$$\operatorname*{hocolim}_\alpha \widehat{LX}_\alpha \cong \operatorname*{colim}_\alpha \widehat{LX}_\alpha$$

in the homotopy category. As sequential homotopy colimits and products commute in the stable case, upon evaluation at an object b of $\mathcal{B}$ we obtain a sequence of natural isomorphisms

$$\begin{aligned} \operatorname*{hocolim}_\alpha R(\widehat{LX}_\alpha)(b) &\cong \operatorname*{hocolim}_\alpha \prod_{i \in \mathrm{sk}(\mathcal{I}\downarrow b)} \widehat{LX}_\alpha\big(\mathrm{dom}(i)\big) \\ &\cong \prod_{i \in \mathrm{sk}(\mathcal{I}\downarrow b)} \operatorname*{hocolim}_\alpha \widehat{LX}_\alpha\big(\mathrm{dom}(i)\big) \\ &\cong \prod_{i \in \mathrm{sk}(\mathcal{I}\downarrow b)} \operatorname*{colim}_\alpha \widehat{LX}_\alpha\big(\mathrm{dom}(i)\big) \\ &\cong \prod_{i \in \mathrm{sk}(\mathcal{I}\downarrow b)} \widehat{LF}\big(\mathrm{dom}(i)\big) \\ &\cong R(\widehat{LF})(b). \end{aligned}$$

Hence map (B) is an isomorphism.

All that remains to be shown is that map (C) is an isomorphism. It suffices to prove that $X_\beta \to R(\widehat{LX}_\beta)$ is a weak equivalence for each ordinal $\beta < \lambda$, and for this we argue by transfinite induction. The base case ($\beta = 0$) is clear. Now fix an ordinal β and assume that $X_\alpha \to R(\widehat{LX}_\alpha)$ is a weak equivalence for each $\alpha < \beta$. There are two cases: either β has an immediate predecessor or it is a limit ordinal.

In the first case, $\beta = \alpha + 1$ for some α. By hypothesis, $X_\alpha \to R(\widehat{LX}_\alpha)$ is a weak equivalence. By examining statement (iii) above, we see that Proposition 5.8 implies that $X_\beta \to R(\widehat{LX}_\beta)$ is indeed a weak equivalence.

In the second case, β is a limit ordinal, so that $\operatorname{colim}_{\alpha<\beta} X_\alpha \to X_\beta$ is an isomorphism in $[\mathcal{B}^{op}, \mathcal{C}]$. An argument exactly like that for map (B) above shows that $\operatorname{hocolim}_{\alpha<\beta} R(\widehat{LX}_\alpha) \to R(\widehat{LX}_\beta)$ is an isomorphism in $\mathrm{ho}([\mathcal{B}^{op}, \mathcal{C}])$. Moreover, the inductive hypothesis implies that the map $\mathrm{hocolim}_{\alpha<\beta} X_\alpha \to \mathrm{hocolim}_{\alpha<\beta} R(\widehat{LX_\alpha})$ is an isomorphism as well. We then have a commutative square

$$\begin{tikzcd} \mathrm{hocolim}_{\alpha<\beta} X_\alpha \arrow[r, "\cong"] \arrow[d, "\cong"'] & \mathrm{hocolim}_{\alpha<\beta} R(\widehat{LX_\alpha}) \arrow[d, "\cong"] \\ X_\beta = \mathrm{colim}_{\alpha<\beta} X_\alpha \arrow[r] & R(\widehat{LX_\beta}) \end{tikzcd}$$

in the homotopy category with isomorphisms as indicated. It follows that $X_\beta \to R(\widehat{LX_\beta})$ is a weak equivalence. Transfinite induction now shows that map (C) is an isomorphism, as desired. □

Theorem 6.2. *Suppose that $\mathcal{C}$ is a left proper stable model category. If $(\mathcal{B}, \mathcal{A})$ is a conjugate pair of small categories, then the adjoint pair*

$$[\mathcal{B}^{\mathrm{op}}, \mathcal{C}] \underset{R}{\overset{L}{\rightleftarrows}} [\mathcal{A}^{\mathrm{op}}, \mathcal{C}]$$

associated to the regular bimodule $\mathcal{U}_+ : \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathrm{Sets}_*$ is a Quillen equivalence.

Proof. By the well-known criteria, it suffices to show that for each cofibrant functor $F : \mathcal{B}^{\mathrm{op}} \to \mathcal{C}$ and each fibrant functor $G : \mathcal{A}^{\mathrm{op}} \to \mathcal{C}$, a map $\tau : LF \to G$ is a weak equivalence if and only if its adjoint $\tau^\# : F \to RG$ is as well. Since the map $LF \to \widehat{LF}$ is an acyclic cofibration and G is fibrant, the lifting axiom shows that the map ϕ displayed in

$$\begin{tikzcd} LF \arrow[r, "\tau"] \arrow[d] & G \arrow[d] \\ \widehat{LF} \arrow[r] \arrow[ur, dashed, "\phi"] & * \end{tikzcd}$$

exists. Moreover, ϕ is a weak equivalence if and only if τ is. But $\phi : \widehat{LF} \to G$ is a map of fibrant functors, so Corollary 5.3 shows that ϕ is a weak equivalence if and only if $R\phi$ is. Putting this all together, we see that τ is a weak equivalence if and only if $R\phi$ is a weak equivalence.

The key observation is that the adjoint $\tau^\#$ is simply the composite

$$\tau^\# : F \xrightarrow{\eta_F} R(LF) \xrightarrow{R\tau} RG.$$

By our previous remarks this may in turn be factored as

$$\begin{tikzcd} F \arrow[r, "\eta_F"] \arrow[dr, "\eta'_F"'] & R(LF) \arrow[r, "R\tau"] \arrow[d] & RG \\ & R(\widehat{LF}). \arrow[ur, "R\phi"'] & \end{tikzcd}$$

Hence we have $\tau^\# = R\phi \circ \eta'_F$. As F is cofibrant, η'_F is a weak equivalence by Proposition 6.1. Thus $\tau^\#$ is a weak equivalence if and only if $R\phi$ is, which is true if and only if τ is a weak equivalence. It follows that (L, R) is a Quillen equivalence. □

Note. Neither the definition of a conjugate pair nor our assumptions on $\mathcal{C}$ are self-dual (there is no useful concept of a "fibrantly generated" model category). Hence one cannot formally dualize the proof of this theorem and obtain the covariant analogue.

7. The additive version

In this section we show how minor alterations to the previous arguments immediately grant us an analogue of Theorem 6.2 in the case that $\mathcal{C}$ is an **Ab**-category with all limits and colimits.[6] Precisely, we will outline the proof of the following:

[6] These assumptions imply that $\mathcal{C}$ is an additive category; equivalently, we could assume that $\mathcal{C}$ is additive with all limits and colimits.

Theorem 7.1. *Suppose that $\mathcal{C}$ is a complete and cocomplete* **Ab**-*category. If $(\mathcal{B}, \mathcal{A})$ is a conjugate pair of small categories, then the adjoint pair*

$$[\mathcal{B}^{\mathrm{op}}, \mathcal{C}] \underset{R}{\overset{L}{\rightleftarrows}} [\mathcal{A}^{\mathrm{op}}, \mathcal{C}]$$

associated to the regular bimodule $\mathcal{U}_+ : \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathrm{Sets}_*$ is an equivalence of categories.

It is possible to prove this from the ground up, first suitably enriching the categories $\mathcal{A}$ and $\mathcal{B}$ into additive categories by the standard trick and making all functors additive. As a matter of efficiency, we will simply recycle the model category results of the previous sections. These homotopical results will translate to the additive case essentially because $\mathrm{ho}(\mathcal{C})$ is naturally additive when $\mathcal{C}$ is stable.

Let $\mathcal{C}$ be a complete and cocomplete **Ab**-category. We give $\mathcal{C}$ a model structure by declaring the weak equivalences to be the isomorphisms, while every map will be both a fibration and a cofibration. Hence all objects are both fibrant and cofibrant and all homotopy adjectives become vacuous. This makes $\mathcal{C}$ a left proper model category, albeit not necessarily stable. Thus all the results from Section 5.1 carry over, as the stability assumption was never invoked there. The same is true of Lemma 5.4.

All that remains is to "repair" the results making use of the stable hypothesis so that they apply in this additive context. Examining its proof, we see that Lemma 5.5 still holds since the natural map

$$\bigvee_{\mathcal{U}(y,b)_+} C \longrightarrow \prod_{\mathcal{U}(y,b)_+} C$$

is an isomorphism in our additive category $\mathcal{C}$. Once again we are saved by the assumption that the hom-sets in $\mathcal{U}$ are finite. Similarly, Proposition 5.6 still holds as lower triangular maps with invertible diagonal entries remain invertible in any additive category.

Since our weak equivalences are now just isomorphisms, we see that Proposition 5.8 holds in the additive case. Similarly, homotopy colimits are now just colimits, and as finite products and coproducts coincide in $\mathcal{C}$ we see that (homotopy) colimits and products commute. Hence the proof of Proposition 6.1 goes through, so this result holds as well. With all of these prerequisites, the proof of Theorem 6.2 needs no alterations, and this provides a proof of Theorem 7.1.

Example 7.2. Let $\mathcal{C}$ be any complete and cocomplete abelian category. Applying the theorem to the conjugate pair $(\Gamma, \mathcal{E})$, we recover Pirashvili's first main result in [11].

In conclusion, we remark that this is not a complete Morita theory for stable model categories of diagrams. The best possible result would be a complete characterization of the pairs $(\mathcal{B}, \mathcal{A})$ yielding a Quillen equivalence, whereas we have found "only" a set of sufficient conditions. The difficulty in finding a set of necessary conditions lies in the fact that our domain categories $\mathcal{A}$ and $\mathcal{B}$ are not enriched in any way as in, say, the classical additive category/additive functor story. Without such enriched structure, it seems unlikely that information from $\mathcal{C}$ would pass down to something useful in both $\mathcal{A}$ and $\mathcal{B}$, but to be proven wrong about this would be pleasing.

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