Number 383

Alex Heller

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ABSTRACT

If, for a small category **C**, $\underline{\Pi}$ **C** is the homotopy category of the functor category $K^{\mathbf{C}}$, where K is the category of simplicial sets, then $\underline{\Pi}$ is an example of a hyperfunctor, taking small categories, functors and natural transformations to categories, functors and natural transformations. We show that $\underline{\Pi}$ has a number of additional properties, which we take as the axioms of a "homotopy theory." From them we recreate much of the familiar structure of homotopy theory, including standard theorems on homotopy limits and localization, and give a description of algebras-up-to-homotopy designed to illuminate the theory of loop-spaces.

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INTRODUCTION

1. In the forty-odd years since Eilenberg and Steenrod wrote down the axioms for homology theory ([6]) there have been repeated attempts to "do the same thing" for homotopy theory: I am myself guilty of one of these ([9]). The most fruitful and influential axiomatization is that of Quillen ([21]). D. W. Anderson ([2]) discusses at some length why he believes all these efforts to be less than adequate. I concur with his remarks and see no need to repeat them here. He goes on to propose another, which seems to me to be the most promising one yet suggested. Unfortunately this proposal has not been fully implemented. Most recently Grothendieck ([8]) has considered a number of ways in which homotopy theory might arise, but his interests, if I understand them, are rather at cross purposes with those of homotopy theorists.

If I venture to put forward yet another axiomatization it is because I believe that Anderson's axioms stop just short of a final desideratum, viz. that of characterizing "homotopy theory" in a homotopy-invariant way. Almost all descriptions thus far proposed undertake to define the homotopy theory of something, with the aim of making that thing sufficiently general to encompass all conceivable examples. Thus, for example, classical homotopy theory is the homotopy theory of the category of topological spaces, while in [9] I attempted to discuss the homotopy theory of cell-complexes, so generalized as to include the stable category of spectra. Quillen develops the homotopy theory of model categories, Anderson that of categories provided with a fraction-functor.

In contrast the notion here set forth is absolute rather than prepositional in character: what is considered is "homotopy theory" as opposed to "the homotopy theory of ... ." To the best of my knowledge a similar procedure is followed only by Puppe ([20]), in defining "stable categories" (cf. also Verdier [27]) which however are not intended to express all the structure of homotopy theory.

The reason that expositions of homotopy theory have had this character is not far to seek. The practice of homotopy theory involves essentially the making of such geometrical constructions as mapping cones and other adjunction-spaces, fibres and other such pullbacks; more general limits and colimits in the category of, say, topological spaces equally well occur. But homotopy categories typically lack such limits and colimits. It has thus

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appeared to be necessary to have some precursor-category in which the constructions can be made, the homotopy theory then appearing as the homotopy theory _of_ that precursor.

To be sure, when homotopy theory meant only the homotopy theory of topological spaces it might well have been felt that the precursor was indeed the primary object of study, and that its presence was therefore not obtrusive. The later development of the subject, beginning with simplicial sets, might already cast some doubt on the justice of this feeling. The fact, now familiar, that different precursors may give rise to what is palpably the same homotopy theory (e.g. spaces, simplicial sets, small categories [15], topogenic groupoids [11] etc.) suggests that it might be desirable to study that homotopy theory divorced from the accidents of its provenance. To venture an analogy, it would be uncomfortable to have to study groups only in terms of generators and relations, without an independent definition of the notion of a group.

2. The keys to an invariant definition of homotopy theory are the notions of homotopy limit and colimit (cf. [4], [21], [28]). These permit us to make the constructions of homotopy theory in a homotopy-invariant way. It was Anderson who pointed out (in [27]) the centrality of these notions and who supplied an essential step in understanding their role by generalizing them to homotopy Kan extensions.

The plan adopted here is to regard the homotopy Kan extensions, and in particular homotopy limits and colimits, as being provided by an additional structure on the underlying homotopy category. To see how this works we may look briefly at the ordinary theory of Kan extensions in a complete category. Associated to such a category we have, for each "diagram-scheme" (i.e. small category) a category of diagrams; functors between diagram-schemes give rise by composition to functors between diagram categories; Kan extensions are adjoints to these functors.

Since all of this is uniquely determined by the original complete category we do not usually think of it as additional structure. In the homotopy theory of topological spaces (to consider one example) we deal however not with diagram-categories in the homotopy category of spaces but rather with homotopy categories of the diagram-categories of spaces. It is these which constitute the additional structure in question in the example. More generally this additional structure will be so characterized that the ordinary theory of Kan extensions is a special case or, to put it a little differently, that homotopy theory appears as a generalization of the theory of limits and colimits in categories.

The precise description of that structure on a category which is needed to characterize a homotopy theory requires certain ideas from category theory which are not fully covered in such standard sources as [18]. We include therefore in Chapter I brief descriptions of localization and of fibrations of categories before passing to the definition of the central notion of hyperfunctors. The chapter concludes with a modest account of the fragment of 2-category theory needed for the elucidation of their relevant properties.

In Chapter II we formulate the definition of a homotopy theory or, more properly, a left or a right homotopy theory (corresponding to a left-complete or a right-complete category) as a hyperfunctor enjoying a list of properties which thus serve as the axioms for homotopy theory, and go on to prove that the standard homotopy theory, modeled for convenience on simplicial sets, constitutes such a homotopy theory. This requires a substantial argument since it subsumes most of the elementary facts about homotopy limits and colimits which are reproved here using a technique first introduced by Steenrod in his exposition of the Milgram bar construction ([25], cf. also [10], [13]), and which recurs in the invariant treatment below. It is necessary to have an alternative description of this standard homotopy theory in terms of the category of small categories, which is adduced here as well.

The remaining chapters examine consequences of the axioms set forth in Chapter II. In Chapter III the basic techniques for computing with homotopy Kan extensions are developed and a density theorem is proved. This theorem, itself an essential tool in the further development, is of interest as showing that the hyperfunctor which constitutes the homotopy theory on an underlying category is in a suitable weak sense generated by that category.

The standard homotopy theory plays a central role in homotopy theory in somewhat the same way that the category of sets does in ordinary category theory. Thus any homotopy theory admits the standard theory as "operators" both convariantly and contravariantly or, in the jargon of category theory, is tensored and cotensored over the standard theory. This structure is the subject matter of Chapter IV and provides the context for the discussion there of homotopical finality of functors between small categories and its effect on homotopy limits.

Chapters V and VI are devoted to the development within our context of two major themes of homotopy theory, *viz*. the theory of localization and that of what we have chosen to call here homotopical algebra, that is to say the abstract version of the homotopy theory of topological algebra.

We propose a systematic definition of localization and colocalization within a homotopy theory, covering not only the cases discussed by Sullivan ([26]), Adams ([1]) and Bousfield ([3]), but others (e.g. Postnikov systems) as well. A principal result is that localizations and colocalizations of homotopy theories are, once more, homotopy theories.

The use made here of the expression "homotopical algebra" occasions an apology to Quillen, whose seminal work on model categories [21] bears that title. It seems to me, nevertheless, that it is underutilized and I have ventured to steal it from him in order to indicate a homotopical version, due in spirit to Segal ([23]), of the description of universal algebra introduced by Lawvere ([16]). An appropriate localization theorem shows that in a wide class of homotopy theories the homotopical algebras of a given type form a localization and thus constitute a homotopy theory.

The loop-space functor, which may be defined in an arbitrary pointed homotopy theory via its cotensor structure, always takes its values in the theory of homotopical groups. The theorem of Segal ([23]), in the tradition of Stasheff ([24]), shows in fact that, in the classical case, the loop-space is an equivalence of the homotopy theory of pointed connected spaces with that of homotopical groups. In the same spirit it is possible to define the notion of a homotopical multialgebra -- not, as in the standard case, reducible to that of an algebra -- and to use it to characterize iterated loop-spaces.

3. Much remains to be worked out. I shall mention some outstanding problems without attempting to assign them priorities or suggesting that others may not be equally interesting.

There seems to be little doubt that homological algebra in a complete abelian category constitutes, in some way, a homotopy theory. The problem here is to determine the most useful way -- or ways -- of saying this.

Stable homotopy theory resists immediate subsumption, for what appear to be inessential technical reasons. The recent work of Elmendorf ([7]) may perhaps provide the necessary information, though at a considerable cost in complexity and with an unfortunate restriction to the classical case. An interesting possibility is that of inventing, for homotopy theory, an analogue of finite completeness and developing a theory of homotopical completion. It might be hoped that the homotopy theory of finite spectra would be easily seen to be such a finitary homotopy theory and that the full stable homotopy theory might be constructed as its homotopical completion. An unstable version of this, constructing the standard homotopy theory from a putative finitary one, would in itself be of interest.

A technical problem which seems to be of some importance is that of lifting diagrams in homotopy categories to objects in homotopy categories of diagrams. A start is made in III.3 below, but it is clear that this only begins to touch the problem, whose resolution ought to lead to improved representability and localization theorems.

Finally, Segal's result on loop spaces, mentioned above, still needs to be formulated within the context of general homotopy theory. Examples show that it is not always true; the correct hypothesis is not clear. Segal's result on infinite loop-spaces might have been treated here in an *ad hoc* fashion, but I have thought it better to omit it until a better understanding of its status in axiomatic homotopy theory becomes available.

CHAPTER I
CATEGORICAL PRELIMINARIES

We presuppose a general familiarity with the basic notions of categories theory. The purpose of this chapter is to fix terminology and notation, which sometimes remains labile in the literature, and to reintroduce, or perhaps in one case introduce, some slightly more esoteric ones.

1. BASIC NOTIONS AND LOCALIZATION

A graph Γ consists of a class Γ_0 of nodes, a class Γ_1 of arrows and maps $d_0, d_1 \colon \Gamma_1 \to \Gamma_0$, the source and target, such that for any $x,y \in \Gamma_0$

$$(x,y) = \{a \in \Gamma_1 \mid d_0 a = x,\ d_1 a = y\}$$

is a set. We define also $\Gamma_2 = \{(b,a) \mid a,b \in \Gamma_1,\ d_0 b = d_1 a\}$. The graph Γ is small if Γ_0 is a set; it follows that Γ_1 is then also a set.

A category C is thus a graph provided with certain additional structures, viz. a unit map id: $C_0 \to C_1$ and a composition map $C_2 \to C_1$, denoted by $(g,f) \mapsto gf$, satisfying the usual conditions. We adopt the usual course of referring to elements of C_0 and C_1 as objects and morphisms of C and d_0, d_1 as the domain and codomain maps. C is small if its underlying graph is small. We shall often denote small categories by bold-face capitals: $\mathbf{C}$.

The small graphs are the objects of the category Gph, with the evident morphisms. The small categories are the objects of the category Cat, whose morphisms are functors. Cat is cartesian-closed, the "internal hom" functor being $\mathbf{D}^{\mathbf{C}}$, the category of functors and natural transformations.

The forgetful functor Cat $\to$ Gph has a left adjoint. By a free category we mean one isomorphic to a value of this left adjoint. Such a category is freely generated by the graph whose nodes are the objects of the category and whose arrows are those morphisms which are neither identities nor compositions of non-identity morphisms.

A category C is finite if C_1 (and hence C_0) is finite. A moment's thought will show that a finite free category contains no proper endomorphisms: if $d_0 f = d_1 f$ then $f = \mathrm{id}(d_0 f)$. Examples of finite free categories are the discrete categories $n = \{0,1,\dots,n-1\}$ and the ordered categories $\mathbf{n} = \{0 \to 1 \to \cdots \to n-1\}$, for $n = 0,1,2,\dots$, as well as the subcategories

$$\underline{\Lambda} = \begin{pmatrix} & 0,0 & \\ \swarrow & & \searrow \\ 1,0 & & 0,1 \end{pmatrix}, \qquad \underline{\Lambda}^{op} = \begin{pmatrix} 1,0 & & 0,1 \\ & \searrow \ \swarrow & \\ & 1,1 & \end{pmatrix}$$

of 2×2, which is not itself free. Another useful example is $P = (0 \rightrightarrows 1)$.

A <u>congruence</u> in a category C is an equivalence relation $\sim$ in C_1 such that $f \sim g$ implies $d_i f = d_i g$, $i = 0,1$, which is furthermore preserved by composition on either side. Thus a functor $F: C \to \mathcal{D}$ determines a congruence $\sim_F$ be setting $f \sim_F g$ when $d_i f = d_i g$, $i = 0,1$, and $Ff = Fg$. A functor $F: C \to \mathcal{D}$ is a <u>quotient functor</u> if any functor $G: C \to E$ such that $\sim_G \supset \sim_F$ factors as HF for a unique $H: \mathcal{D} \to E$. Any congruence is a $\sim_F$; for a congruence in C, $F: C \to \mathcal{D} = C/\sim$, where $\mathcal{D}_0 = C_0$ and $\mathcal{D}_1$ consists of the equivalence classes in C_1.

A functor $F: C \to \mathcal{D}$ is a <u>weak quotient functor</u> if any $G: C \to E$ such that $\sim_G \supset \sim_F$ is naturally isomorphic to HF for some $H: \mathcal{D} \to E$ which is itself unique up to natural isomorphism. It is easy to see that F is a weak quotient functor is and only if it is <u>full</u>, i.e. all $C(X,Y) \to \mathcal{D}(FX,FY)$ are surjective, and <u>replete</u>, i.e. any $W \in \mathcal{D}_0$ is isomorphic to some FX. It is then a quotient functor if and only if $F_0: C_0 \approx \mathcal{D}_0$.

For any $F: C \to \mathcal{D}$ let Ker F be the subcategory of C with $(\text{Ker } F)_0 = C_0$ and $(\text{Ker } F)_1$ containing those morphisms which are <u>inverted</u> by F, i.e. those f such that Ff is an isomorphism in $\mathcal{D}$. We shall say that F is a <u>fraction functor</u> if any $G: C \to E$ such that Ker $G \supset$ Ker F factors as HF for a unique $H: \mathcal{D} \to E$. F is a <u>weak fraction functor</u> if any such G is naturally isomorphic to HF for some H, itself unique up to natural isomorphism. A weak fraction functor $F: C \to \mathcal{D}$ is a fraction functor if and only if $F: C_0 \approx \mathcal{D}_0$.

A relation in the class C_1 of morphisms of a category C which refines $d_i f = d_i g$, $i = 0,1$ is contained in a smallest congruence, which it is said to <u>generate</u>. A subset S of the morphisms C_1 of a small category C is contained in a smallest Ker F for $F: C \to \mathcal{D}$ a fraction functor. The corresponding assertion for categories which are not small is in general false. If such an F exists, we write $\mathcal{D} = C[S^{-1}]$ and say that it is the <u>category of fractions with respect to</u> S. A <u>weak category of fractions with respect to</u> S is defined analogously. It is easy to see that the existence of the latter guarantees the existence of the former.

If S is a full subcategory of C^2 -- which is another way more suited to our present purposes of saying a subclass of C_1 -- an object X of C is <u>injective</u> <u>(local)</u> with respect to S if, for all f in S, $C(f,X)$ is surjective (bijective). We write inj S (loc S) for the full subcategory of C containing these injective (local) objects. It is evident that inj S, loc S are replete subcategories of C as well.

Conversely, if $\mathcal{D}$ is a full subcategory of $\mathcal{C}$, we say that an f in $\mathcal{C}^2$ <u>injectivizes</u> (<u>localizes</u>) $\mathcal{D}$ if, for all $X \in \mathcal{D}_0$, $\mathcal{C}(f,X)$ is surjective (bijective). The corresponding full subcategories of $\mathcal{C}^2$ are inj*$\mathcal{D}$, loc*$\mathcal{D}$; they are of course replete as well.

Evidently inj, inj* and loc,loc* give Galois correspondences between full subcategories of $\mathcal{C}^2$ and $\mathcal{C}$, the Galois-closed subcategories being loc loc* loc S = loc S and so forth.

We shall say that $\mathcal{C}$ <u>admits a localization with respect to</u> $S \subset \mathcal{C}^2$ if the inclusion J: loc $S \subset \mathcal{C}$ has a left adjoint Loc, which is then called the <u>localizing functor relative to</u> S. If η is the unit of the adjunction Loc $\dashv$ J and X is in $\mathcal{C}$, the morphism η_X: X $\to$ Loc X is called the <u>localization</u> of X. It is easy to see that Loc is a weak fraction functor with respect to S (regarded now as a subclass of $\mathcal{C}_1$) so that $\mathcal{C}$ admits a category of fractions F: $\mathcal{C} \to \mathcal{C}[S^{-1}]$, and that J Loc then factors as GF with F $\dashv$ G.

Dualizing, we may introduce the notions of <u>projectivity</u> and <u>colocality</u> with respect to a full $S \subset \mathcal{C}^2$, with analogous properties.

2. FIBRATIONS

We recall that if F: $\mathcal{A} \to \mathcal{C}$, G: $\mathcal{B} \to \mathcal{C}$ are functors, the comma-category $(F \downarrow G)$ has objects (A,B,f) where $A \in \mathcal{A}_0$, $B \in \mathcal{B}_0$, f: FA $\to$ GB and morphisms (a,b) where

$$\begin{array}{ccccccc} A & \longmapsto & FA & \xrightarrow{f} & GB & \longleftarrow\!\!\shortmid & B \\ {\scriptstyle a}\downarrow & & {\scriptstyle Fa}\downarrow & & \downarrow{\scriptstyle Fb} & & \downarrow{\scriptstyle b} \\ A' & \longmapsto & FA' & \xrightarrow[f']{} & GB' & \longleftarrow\!\!\shortmid & B' \end{array}$$

f'(Fa) = (Fb)f. If we regard an object C of $\mathcal{C}$ as a morphism C: $\mathbf{1} \to \mathcal{C}$ and let $\mathcal{C}$ stand for its own identity functor, then such expressions as $(C \downarrow \mathcal{C})$, $(\mathcal{C} \downarrow C)$ are special cases. We shall prefer them to, e.g., $\mathcal{C}/C$ or $C \setminus \mathcal{C}$.

If P: $\mathcal{E} \to \mathcal{B}$ and $B \in \mathcal{B}_0$, we denote by $\mathcal{E}_B$ the subcategory of $\mathcal{E}$ having objects E such that PE = B and morphisms f such that Pf = id_B. If $\mathcal{E}$ and $\mathcal{B}$ are small, then

$$\begin{array}{ccc} \mathcal{E}_B & \xrightarrow{U} & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathbf{1} & \xrightarrow{B} & \mathcal{B} \end{array}$$

is a pullback in Cat. $\mathcal{E}_B$ is called the fibre of $\mathcal{E}$ (or of P) over B.

If $(B \downarrow P)$, $(P \downarrow B)$ are the comma-categories and J, J' the forgetful functors, then there are functors I, I' such that

(2.1)
$$\begin{array}{ccc} \mathcal{E}_B & \xrightarrow{\ I\ } & (B \downarrow P) \\ {\scriptstyle I'}\downarrow & \overset{U}{\searrow} & \downarrow{\scriptstyle J} \\ (P \downarrow B) & \xrightarrow[\ J'\ ]{} & \mathcal{E} \end{array}$$

commutes, viz. $IE = (B \xrightarrow{id} B \leftarrow\!\shortmid E)$, $I'E = (E \mapsto B \xrightarrow{id} B)$.

The functor P is a <u>fibration</u> if, for all $B \in \mathcal{B}_0$ the functor I in 2.1 has a right adjoint. It is an <u>opfibration</u> if, for each B, the functor I' has a left adjoint or, equivalently, if and only if $P^{op}: \mathcal{E}^{op} \to \mathcal{B}^{op}$ is a fibration.

A fibration or opfibration P is <u>locally small</u> if each $\mathcal{E}_B$ is small. If $\mathcal{E}$ is small then P is certainly locally small; if P is locally small and $\mathcal{B}$ is small then $\mathcal{E}$ is small.

P is a <u>discrete fibration</u> (<u>opfibration</u>) if each $\mathcal{E}_B$ is a discrete category. These are characterized by the unique path-lifting properties:

$$\begin{array}{ccc} \mathcal{E}_1 & \xrightarrow{\ d_i\ } & \mathcal{E}_0 \\ {\scriptstyle P_1}\downarrow & & \downarrow{\scriptstyle P_0} \\ \mathcal{B}_1 & \xrightarrow{\ d_i\ } & \mathcal{B}_0 \end{array}$$

is a pullback precisely when P is a discrete fibration (i = 1) or opfibration (i = 0). For example, for any $F: \mathcal{C} \to \mathcal{D}$ and any $D \in \mathcal{D}_0$ the forgetful functors $(D \downarrow F) \to \mathcal{C}$, $(F \downarrow D) \to \mathcal{C}$ are, respectively, a discrete fibration and a discrete opfibration.

If we relax our usage of the word "pullback" so as to be able to speak of a pullback of (not necessarily small) categories, so that for example $\mathcal{E}_B$ is always a pullback as described above, we may record the following phenomenon.

(2.2) A pullback of a fibration (opfibration) of categories is again a fibration (opfibration).

If $F: \mathcal{B} \to$ Cat the category $\mathcal{B} \ltimes F$ which we shall, because of the analogy with the group-theoretical construction it generalizes, call the <u>left semi-direct product</u> is defined by setting

(2.3)
$$(\mathcal{B} \propto F)_0 = \{(B,X) \mid B \in \mathcal{B}_0,\ X \in (FB)_0\} = \coprod_B (FB)_0$$
$$(\mathcal{B} \propto F)((B,X),(B',X')) = \coprod_{f:\ B \to B'} (FB')((Ff)X,X')$$

with composition given by

$$(f',\phi')(f,\phi) = (f'f,\phi'(Ff)\phi).$$

The projection P, the functor taking (B,X) to B and (f,ϕ) to f is then a locally small opfibration. For $(\mathcal{B} \ltimes F)_B = FB$. The left adjoint to $I'\colon FB \to (P \downarrow B)$ is given by

$$(P(B',X') \xrightarrow{f} B) \longmapsto ((Ff)X').$$

Dually, for $F\colon \mathcal{B}^{op} \to \mathrm{Cat}$ we set $F \rtimes \mathcal{B} = (\mathcal{B}^{op} \ltimes F)^{op}$. If P is the projection of the left semidirect product $\mathcal{B}^{op} \ltimes F$, then P^{op}, the projection of $F \rtimes \mathcal{B}$ is a locally small fibration. To spell out this definition,

$$(F \rtimes \mathcal{B})_0 = \coprod_B (FB)_0$$

$$(F \rtimes \mathcal{B})((B,X),(B',X')) = \coprod_{f:B \to B'} (FB)(X,(Ff)X')$$

$$(f',\phi')(f,\phi) = (f'f,((Ff)\phi)\phi').$$

The category $F \rtimes \mathcal{B}$ is of course the right semidirect product.

Identifying sets with discrete small categories functors $F\colon \mathcal{B}^{op} \to \mathrm{Sets}$, $F\colon \mathcal{B} \to \mathrm{Sets}$ give rise to locally small discrete fibrations and opfibrations. Indeed this process gives rise to an equivalence. For if $P\colon \mathcal{E} \to \mathcal{B}$ is a locally small discrete fibration, then $FB = \mathcal{E}_B$ defines a functor $\mathcal{B}^{op} \to \mathrm{Sets}$, the effect on morphisms being determined by the unique path-lifting property, and $F \rtimes \mathcal{B} \approx \mathcal{E}$. An analogous treatment of nondiscrete fibrations is also available, but takes us further into the domain of 2-category theory than we have occasion to penetrate.

The semidirect products have more-or-less evident functorial properties. If $F,G\colon \mathcal{B} \to \mathrm{Cat}$ and $\phi\colon F \to G$ is a natural transformation, then $\mathcal{B} \ltimes \phi\colon \mathcal{B} \ltimes F \to \mathcal{B} \ltimes G$ is defined in the obvious way, as also, for $F,G\colon \mathcal{B}^{op} \to \mathrm{Cat}$, $\phi\colon F \to G$, is $\phi \rtimes \mathcal{B}\colon F \rtimes \mathcal{B} \to G \rtimes \mathcal{B}$. If $\mathcal{B}$ is a small category, then $\mathcal{B}\ltimes -\colon \mathrm{Cat}^{\mathcal{B}} \to \mathrm{Cat}$ and $- \rtimes \mathcal{B}\colon \mathrm{Cat}^{\mathcal{B}^{op}} \to \mathrm{Cat}$ are functors.

Finally, if $W\colon \mathcal{B}^{op} \times \mathcal{C} \to \mathrm{Cat}$, we may construct $\mathcal{C} \ltimes W\colon \mathcal{B}^{op} \to \mathrm{Cat}$ and $W \rtimes \mathcal{B}\colon \mathcal{C} \to \mathrm{Cat}$ and thus $\mathcal{C} \ltimes (W \rtimes \mathcal{B})$, $(\mathcal{C} \ltimes W) \rtimes \mathcal{B}$. These are easily seen to be isomorphic and we are led to write $\mathcal{C} \ltimes W \rtimes \mathcal{B}$ for both. The projections

(2.4) $$\mathcal{C} \xleftarrow{P'} \mathcal{C} \ltimes W \rtimes \mathcal{B} \xrightarrow{P''} \mathcal{B}$$

are, respectively, an opfibration and a fibration.

3. HYPERFUNCTORS

The fundamental notion with which we shall be occupied below is that of a hyperfunctor. Indeed, a homotopy theory will be defined as a hyperfunctor satisfying a number of conditions to be set forth in II§1. Similar objects have been introduced before, most notably by Lawvere [17], who has used the name "hyperdoctrine" for his variant. The term "hyperfunctor" has been adopted here in deference to that usage.

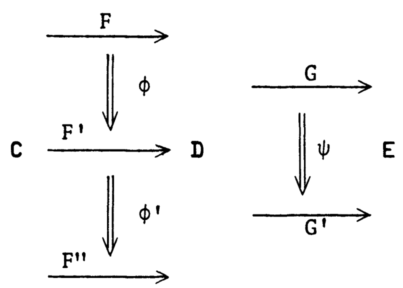
Our modest assumptions about set theory prevent us from adopting the compendious definition of a hyperfunctor as a strict 2-functor from Cat^{op} to some category of "large" categories. We adopt the more conservative course of saying that a <u>hyperfunctor</u> is a function T which assigns

(i) to each small category $\mathbf{C}$ a category $T\mathbf{C}$,

(ii) to each functor $F\colon \mathbf{C} \to \mathbf{D}$ a functor $TF\colon T\mathbf{D} \to T\mathbf{C}$,

(iii) to each natural transformation $\phi\colon F \to G$ a natural transformation $T\phi\colon TF \to TG$,

in such a way that identity functors are taken to identity functors, identity natural transformations are taken to identity natural transformations, and all compositions are preserved. Thus if



is a diagram of functors and natural transformations, then

$$
(3.1)\qquad
\begin{aligned}
T(GF) &= (TF)(TG)\colon T\mathbf{D} \to T\mathbf{C}\\
T(\phi'\phi) &= (T\phi')(T\phi)\colon TF \to TF''\\
T(\psi F) &= (TF)(T\psi)\colon T(GF) \to T(G'F)\\
T(G\phi) &= (T\phi)(TG)\colon T(GF) \to T(GF').
\end{aligned}
$$

The most obvious example is that of a <u>representable</u> hyperfunctor. For any category C, $\mathbf{C} \longmapsto C^{\mathbf{C}}$ defines a hyperfunctor with, for functors F and natural transformations ϕ, TF and $T\phi$ defined by composition. To forestall misunderstanding let us say right now that interesting homotopy theories are not for the most part such representable hyperfunctors.

If T is a hyperfunctor and $\mathbf{C}$ is a small category, then $\mathbf{D} \longmapsto T(\mathbf{C} \times \mathbf{D})$ is once more a hyperfunctor which we denote by $T[\mathbf{C}]$, so that $T[\mathbf{C}]\mathbf{D} = T(\mathbf{C} \times \mathbf{D})$, $T[\mathbf{C}]F = T(\mathbf{C} \times F)$ and so forth. Here $\mathbf{C}$ plays the role of a parameter.

We shall not have to consider the analogous notions with the variance in functors and/or natural transformations reversed. Note, however, that if T is a hyperfunctor, then

$$\mathbf{C} \to (T(\mathbf{C}^{op}))^{op} = T^*\mathbf{C} \tag{3.2}$$

gives another hyperfunctor, the <u>bidual</u> of T.

Let us adduce here some elementary properties of hyperfunctors.

PROPOSITION 3.3:

(i) If $F: \mathbf{C} \to \mathbf{D}$, $G: \mathbf{D} \to \mathbf{C}$ and η, ε are the unit and counit of an adjunction $F \dashv G$, then for any hyperfunctor T, $T\varepsilon$ and $T\eta$ are the unit and counit of an adjunction $TG \dashv TF$.

(ii) If $\bot: \mathbf{1} \to \mathbf{C}$ is initial, then, for any $\mathbf{D}$, $Tpr_{\mathbf{D}} \dashv T(\mathbf{D} \times \bot)$. Dually, if $\top: \mathbf{1} \to \mathbf{C}$ is terminal, then $T(\mathbf{D} \times \top) \dashv Tpr_{\mathbf{D}}$.

This is an immediate consequence of 4.1.

For any hyperfunctor T evaluation defines, for each $\mathbf{C}$, $\mathbf{D}$, a functor $\mathbf{D}^{\mathbf{C}} \times T\mathbf{D} \to T\mathbf{C}$ with naturality properties clear from 4.1. Taking $\mathbf{C} = \mathbf{1}$ and transposing we get $dgm_{\mathbf{D}} : T\mathbf{D} \to (T\mathbf{1})^{\mathbf{D}}$. The naturality properties of these functors are again evident: if $F: \mathbf{C} \to \mathbf{D}$, then

$$\begin{array}{ccc} T\mathbf{D} & \xrightarrow{TF} & T\mathbf{C} \\ {\scriptstyle dgm_{\mathbf{D}}}\downarrow & & \downarrow{\scriptstyle dgm_{\mathbf{C}}} \\ (T\mathbf{D})^{\mathbf{D}} & \xrightarrow{F^*} & (T\mathbf{1})^{\mathbf{C}} \end{array}$$

where $F^* = (T\mathbf{1})^F$ commutes.

If X is in $T\mathbf{D}$ we shall call $dgm_{\mathbf{D}} X$ the <u>underlying diagram</u> of X and shall allow ourselves to write X_d instead of $(dgm_{\mathbf{D}} X)_d$ for d in $\mathbf{D}$.

Writing $t: \mathbf{D} \to \mathbf{1}$ for the unique functor, Tt is denoted by $T\text{-const}_{\mathbf{D}}$. The diagram of $T\text{-const}_{\mathbf{D}} X$ is the constant diagram $\text{const}_{\mathbf{D}} X$ with value X.

It is perhaps worthwhile to record an alternate description of dgm X. If $d: \mathbf{1} \to \mathbf{D}$ is an object of $\mathbf{D}$, then $(dgm\ X)_d = (Td)X$. If $\phi: c \to d$ is a morphism of $\mathbf{D}$, i.e. a natural transformation

$$\mathbf{1} \underset{d}{\overset{c}{\Downarrow \phi}} \mathbf{D} \quad ,$$

then $X_\phi = T\phi: X_c \to X_d$.

If T and T' are hyperfunctors, a strict hypernatural transformation Φ: $T \to T'$ is a function which assigns to each small category $\mathbf{C}$ a functor $\Phi_{\mathbf{C}}$: $T\mathbf{C} \to T'\mathbf{C}$ in such a way that for any F: $\mathbf{C} \to \mathbf{D}$ the diagram

$$\begin{array}{ccc} T\mathbf{D} & \xrightarrow{\Phi_{\mathbf{D}}} & T'\mathbf{D} \\ {\scriptstyle TF}\downarrow & & \downarrow{\scriptstyle T'F} \\ T\mathbf{C} & \xrightarrow{\Phi_{\mathbf{C}}} & T'\mathbf{C} \end{array}$$

commutes. These will not exhaust the possibilities for "morphisms" between hyperfunctors, but they are exemplified by the inclusions of subhyperfunctors, S being a subhyperfunctor of T if, indeed, each $S\mathbf{C}$ is a subcategory of $T\mathbf{C}$, the inclusion being a strict hypernatural transformation.

The notions of full or replete subhyperfunctors are self-defining. We shall say that a subhyperfunctor S of T is maximal if it is maximal among those $S' \subset T$ for which $S'\mathbf{1} = S\mathbf{1}$, that is to say X (or f) is in $S\mathbf{C}$ if and only if, for each d: $\mathbf{1} \to \mathbf{D}$, X_d (or f_d) is in $S\mathbf{1}$. Thus the maximal (full, replete) subhyperfunctors of T correspond bijectively to the (full, replete) subcategories of $T\mathbf{1}$.

We shall also have occasion to speak of hyperfunctors of several variables. A hyperfunctor T of two variables, for example, assigns to each pair $\mathbf{C}$, $\mathbf{D}$ of small categories, a category $T(\mathbf{C}, \mathbf{D})$, to each pair of functors F: $\mathbf{C}' \to \mathbf{C}$, G: $\mathbf{D}' \to \mathbf{D}$ a functor $T(F,G)$: $T(\mathbf{C}', \mathbf{D}') \to T(\mathbf{C},\mathbf{D})$ and to each pair of natural transformations ϕ: $F \to F'$, ψ: $G \to G'$ a natural transformation $T(\phi,\psi)$: $T(F,G) \to T(F',G')$, respecting all identities and compositions in Cat $\times$ Cat. For example, if T is a hyperfunctor of one variable, then $\mathbf{C}, \mathbf{D} \to T(\mathbf{C} \times \mathbf{D})$ and $\mathbf{C},\mathbf{D} \longmapsto T\mathbf{C} \times T\mathbf{D}$ are both hyperfunctors of two variables.

We could have subsumed this notion under that of hyperfunctors of one variable by generalizing to hyperfunctors defined on $\mathbf{2}$-categories, but we shall not need this generalization.

4. WEAK COMMUTATIVITY

In order to state precisely many of the results below, we are forced to immerse ourselves in the icy waters of $\mathbf{2}$-category theory. We shall make the plunge as shallow as possible.

A commutative square

$$\begin{array}{ccc} X & \xrightarrow{x} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{y} & Y' \end{array}$$

in a category C may usefully be thought of as a morphism $(x,y)\colon f \to f'$ in C^2 or, as we shall say, a ^{2}morphism in C. It may equally well be thought of as a ^{2}morphism $(f,f')\colon x \to y$. Analogy would then give us "^{2}functors" for similar diagrams of categories and functors. But in this case commutativity may well be an excessive demand. If we are given a square

$$(4.1)\qquad \begin{array}{ccc} A & \xrightarrow{A} & A' \\ {\scriptstyle F}\downarrow & \phi \nearrow\!\!\!\nearrow & \downarrow{\scriptstyle F'} \\ B & \xrightarrow[B]{} & B' \end{array}$$

where $\phi\colon BF \to F'A$ is a natural transformation we shall say that the square is weakly commutative and that (A,B,ϕ) is a weak left ^{2}functor $(A,B,\phi)\colon F \xrightarrow[L]{} F'$ or equivalently that (F,F',ϕ) is a weak right ^{2}functor $(F,F',\phi)\colon A \xrightarrow[R]{} B$. If ϕ is a natural isomorphism, we say that the square is strongly commutative and that (A,B,ϕ), (F,F',ϕ) are strong left and right ^{2}functors. Strict commutativity and strict ^{2}functors constitute the special case $F'A = BF$, $\phi = id_{F'A}$.

If also $(A',B',\phi')\colon F' \xrightarrow[L]{} F''$, the composition $(A',B',\phi') \circ (A,B,\phi)\colon F \xrightarrow[L]{} F''$ is defined to be $(A'A, B'B, (\phi'A)(B'\phi))$. Composition of right weak ^{2}functors is defined analogously. Both compositions are associative with evident units. Compositions of strong ^{2}functors are again strong.

If (A_0,B_0,ϕ_0), $(A_1,B_1,\phi_1)\colon F \xrightarrow[L]{} F'$ a ^{2}natural transformation, $(\alpha,\beta)\colon (A_0,B_0,\phi_0) \to (A_1,B_1,\phi_1)$ is a pair of natural transformations $\alpha\colon A_0 \to A_1$, $\beta\colon B_0 \to B_1$ such that $(F'\alpha)\phi_0 = \phi_1(\beta F)$. These, of course, compose in the obvious way.

Now suppose that $(A,B,\phi)\colon F \xrightarrow[L]{} F'$ and that F and F' are provided with left adjoints $F_* \dashv F$, $F'_* \dashv F'$, the adjunction being specified by units η, η' and counits ε, ε'. Let ϕ_* be the composition

$$(4.2)\qquad F'_*B \xrightarrow{F'_*B\eta} F'_*BFF_* \xrightarrow{F'_*\phi F_*} F'_*F'AF_* \xrightarrow{\varepsilon' AF_*} AF_*.$$

We then have the weakly commutative square

$$\begin{array}{ccc} B & \xrightarrow{B} & B' \\ {\scriptstyle F_*}\downarrow & \phi_* \swarrow\!\!\!\swarrow & \downarrow{\scriptstyle F'_*} \\ A & \xrightarrow[A]{} & A' \end{array}$$

so that $(B,A,\phi_*)\colon F_* \xrightarrow[R]{} F'_*$. We write $(B,A,\phi_*) = (A,B,\phi)_*$ and call it the left adjoint of (A,B,ϕ) with respect to the given adjunctions.

Similarly, if $(A,B,\phi)\colon F \xrightarrow[R]{} F'$ and F, F' are supplied with right adjoints F^*, F'^* we construct $\psi^*\colon AF^* \to F'^*B$ and $(A,B,\phi)^* = (B,A,\phi^*)\colon F^* \xrightarrow[L]{} F'^*$.

Of course $((A,B,\phi)_*)^* = (A,B,\phi)$.

PROPOSITION 4.3. If $(A,B,\phi)\colon F \xrightarrow[L]{} F'$, $(A',B',\phi')\colon F' \xrightarrow[L]{} F''$ and F, F', F'' are provided with left adjoints, then

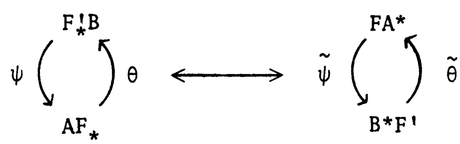
$$((A',B',\phi') \circ (A,B,\phi))_* = (A',B',\phi')_* \circ (A,B,\phi)_*.$$

This results from a tedious but entirely straightforward computation, as does the next statement.

PROPOSITION 4.4. If $(A_i,B_i,\phi_i)\colon F \xrightarrow[L]{} F'$, $i = 0,1$ and $(\alpha,\beta)\colon (A_0,B_0,\phi_0) \to (A_1,B_1,\phi_1)$, and if F and F' are supplied with left adjoints, then $(\beta,\alpha)\colon (A_0,B_0,\phi_0)_* \to (A_1,B_1,\phi_1)_*$.

Now suppose that $(A,B,\phi)\colon F \xrightarrow[L]{} F'$ and that F, F' are supplied with two sets of left adjoints, F_*, $F_*^!$ and $F_\#$, $F_\#^!$, the latter with units and counits, let us say $\eta_\#$, $\eta_\#^!$, $\varepsilon_\#$, $\varepsilon_\#^!$. Corresponding to the latter we also have $(B,A,\phi_\#)\colon F_\# \xrightarrow[R]{} F_\#^!$ with $\phi_\#$ constructed as in 4.2 or, equivalently, $(F_\#,F_\#^!,\phi_\#)\colon A \xrightarrow[L]{} B$. But $(\varepsilon F_*)(F_\#\eta)\colon F_\# \to F_*$ and $(\varepsilon' F_*^!)(F_\#^!\eta')\colon F_\#^! \to F_*^!$ are isomorphisms (this expresses the uniqueness of adjoints), giving a natural isomorphism $(F_\#,F_\#^!,\phi_\#) \to (F_*,F_*^!,\phi_*)$. Thus if ϕ_* is an isomorphism so also is $\phi_\#$. In other words, the property, that ϕ_* be an isomorphism, is independent of the choice of adjoints. We shall, accordingly, say that the square 4.1 has the <u>Beck-Chevalley property</u>, or is a <u>B - C square</u>, when ϕ_* is an isomorphism.

It might seem that we ought to distinguish between left and right B - C squares, 4.1 being a right B - C square if A and B have right adjoints and ϕ^* is an isomorphism. In fact this distinction is unnecessary. For suppose we are given adjunctions $F_* \dashv F$, $F_*^! \dashv F'$, $A \dashv A^*$, $B \dashv B^*$. These will give rise to adjunctions $F_*^!B \dashv B^*F'$, $AF_* \dashv FA^*$ and thus to a bijective "transposition"



PROPOSITION 4.5. If $(A,B,\phi)\colon F \xrightarrow[L]{} F'$ and A, B, F, F' have adjoints as indicated above, then $\phi^* = \tilde{\phi}_*$.

This is, once again, the result of a straightforward computation. It implies, of course, that such a square has the Beck-Chevalley property in either sense if and only if it has it in the other.

It was indicated above that the notion of strict hypernatural transformation between hyperfunctors was not adequate to the applications to follow. We now have the apparatus with which to generalize it. If T and T' are hyperfunctors, a <u>left weak hypernatural transformation</u> $\Phi: T \xrightarrow[L]{} T'$ is a function which associates to each small category a functor $\Phi_{\mathbf{C}}: T_{\mathbf{C}} \to T'_{\mathbf{C}}$ and to each functor $F: \mathbf{C} \to \mathbf{D}$ a natural transformation $\Phi_F: (\Phi_{\mathbf{C}})TF) \to (T'F)(\Phi_{\mathbf{D}})$, so that $(\Phi_D,\Phi_C,\Phi_F): TF \xrightarrow[L]{} T'F$, in such a way as to preserve composition, so that if also $G: \mathbf{D} \to \mathbf{W}$ then

$$(T(GF),T'(GF),\Phi_{GF}) = (TG,T'G,\Phi_G) \circ (TF,T'F,\Phi_F).$$

<u>Right weak hypernatural transformations</u>, and <u>left and right strong hypernatural transformations</u> are defined by analogy.

If $\Phi: T \xrightarrow[L]{} T'$, $F: \mathbf{C} \to \mathbf{D}$ and both TF and T'F have left adjoints, we may say that Φ <u>preserves the left adjoint</u> at F if (Φ_D,Φ_C,Φ_F) has the Beck-Chevalley property. The notion that a $\Psi: T \xrightarrow[R]{} T'$ preserve right adjoints is of course dual. Properly speaking it does not make sense to assert of a Φ, as above, that it preserves right adjoints. But if Φ is a strong left hypernatural transformation, then, using the Φ_F^{-1}, we may in the obvious way define a right hypernatural transformation of which the assertion can be made.

CHAPTER II
HOMOTOPY THEORIES

In this chapter we shall define the notion of a homotopy theory and show that the standard homotopy theory is a homotopy theory in this sense. We shall for this purpose describe the standard theory by means of simplicial sets. This is a matter of convenience; topological spaces would have done as well since they give rise to the same homotopy theory. Indeed the same may be said of small categories, and we shall, in the interest of later applications, see just how the homotopy theory of Cat is the same as that of simplicial sets.

1. LEFT AND RIGHT HOMOTOPY THEORIES

Homotopy theory is a sort of generalization of the theory of complete categories and like that theory comes in two dual forms. Thus we shall define dual notions of left and right homotopy theories, a homotopy theory being both a left and a right homotopy theory.

Let us begin by listing a number of conditions on a hyperfunctor T.

(H0) For any family $\{\mathbf{C}_\alpha\}$ of small categories, the canonical functor $T(\coprod_\alpha \mathbf{C}_\alpha) \to \prod_\alpha T\mathbf{C}_\alpha$ is an isomorphism.

(H1) For any $\mathbf{C}$ the functor $dgm_{\mathbf{C}}\colon T\mathbf{C} \to (T1)^{\mathbf{C}}$ reflects isomorphisms, i.e. if $dgm_{\mathbf{C}} f$ is an isomorphism then so also is f.

(H2) If $\mathbf{F}$ is a finite free category, then, for any $\mathbf{C}$, $dgm_{\mathbf{F}}\colon T(\mathbf{C} \times \mathbf{F}) \to (T\mathbf{C})^{\mathbf{F}}$ is a weak quotient functor.

So far, these conditions are self-dual. The remaining ones come in dual pairs.

(H3L) For any $F\colon \mathbf{C} \to \mathbf{D}$, TF has a left adjoint $L_T F$.

(H4L) If $P\colon \mathbf{E} \to \mathbf{B}$ is a discrete fibration, then TP has a right adjoint $R_T P$, $P^*\colon (T1)^{\mathbf{B}} \to (T1)^{\mathbf{E}}$ has a right adjoint (conventionally called Ran_P) and the strictly commutative square

$$\begin{array}{ccc} T\mathbf{B} & \xrightarrow{\ \ TP\ \ } & T\mathbf{E} \\ {\scriptstyle dgm_{\mathbf{B}}}\downarrow & \overset{id}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle dgm_{\mathbf{E}}} \\ (T1)^{\mathbf{B}} & \xrightarrow{\ \ P^*\ \ } & (T1)^{\mathbf{E}} \end{array}$$

has the Beck-Chevalley property.

Conditions H3R and H4R are the strict duals of H3L and H4L. Thus:

(H4R) If $P: \mathbf{E} \to \mathbf{B}$ is a discrete opfibration, then TP has a left adjoint $L_T P$, P^* has a left adjoint (Lan_P) and the square above, with the identity natural transformation reversed, has the Beck-Chevalley property.

We shall say that a hyperfunctor T is a left homotopy theory if it satisfies the conditions H0,1,2,3L,4L and a right homotopy theory if it satisfies H0,1,2,3R,4R. If it is both a left and a right homotopy theory, it is a homotopy theory.

The left adjoints $L_T F$ and right adjoints $R_T F$, when they exist, will be referred to as homotopy Kan extensions and the subscripts will often be dropped when they can be inferred from the context. Like ordinary Kan extensions these are of course determined only up to isomorphism. This is the reason for the material of I§4. The homotopy Kan extensions along the unique functors $\mathbf{C} \to \mathbf{1}$, when they exist, are called homotopy colimit and homotopy limit and are denoted by $T\text{-colim}_{\mathbf{D}}$ and $T\text{-lim}_{\mathbf{D}}$.

We see immediately that the ordinary theory of completeness in categories is subsumed under homotopy theory.

PROPOSITION 1.1. The representable hyperfunctor $\mathbf{D} \mapsto C^{\mathbf{D}}$ is a left (right) homotopy theory if and only if C is closed under colimits and products (limits and coproducts).

In this case each $\mathrm{dgm}_{\mathbf{C}}$ is the identity, $LF = \mathrm{Lan}_F$, $RF = \mathrm{Ran}_F$. The necessity is obvious. For the sufficiency, it is enough to observe that if $P: \mathbf{E} \to \mathbf{B}$ is a discrete fibration and $X: \mathbf{E} \to C$, then $(\mathrm{Ran}_P X)_b = \prod_{Pc=b} X_c$.

To this we may add the following immediate consequence of the definition.

PROPOSITION 1.2. If T is a left homotopy theory, then so also, for each $\mathbf{C}$, is $T[\mathbf{C}]$, while the bidual T^* of T is a right homotopy theory.

2. THE STANDARD EXAMPLE

In order no conform with the traditional notation, we write $\mathbf{n} = [n - 1]$ for $n = 1,2,\ldots$. The full subcategory of Cat containing these is $\underline{\Delta}$, the simplicial index category. The functor category $K = \mathrm{Sets}^{\underline{\Delta}^{op}}$ is the category of simplicial sets. A morphism f in K is called a weak equivalence if its geometric realization $|f|$ is a homotopy equivalence of topological spaces.

More generally, for any small category $\mathbf{C}$ a morphism f of $K^{\mathbf{C}}$ is a weak equivalence if, for each $c \in \mathbf{C}_0$, f_c is a weak equivalence of K.

PROPOSITION 2.1. For each small category $\mathbf{C}$, $K^{\mathbf{C}}$ admits a category of fractions $\mathrm{Ho}(K^{\mathbf{C}})$ with respect to its weak equivalences.

This is a standard theorem (cf. for example Kan-Bousfield [4]). It is also a corollary of Theorem 4.5 below.

If F: $\mathbf{C} \to \mathbf{D}$ then F*: $K^{\mathbf{D}} \to K^{\mathbf{C}}$ preserves weak equivalences and thus induces a functor HoF*: Ho($K^{\mathbf{D}}$) $\to$ Ho($K^{\mathbf{C}}$). Making the evident observation about natural transformations, we see that $\mathbf{C} \longmapsto$ Ho($K^{\mathbf{C}}$) is a hyperfunctor which we denote henceforth by $\underline{\Pi}$.

THEOREM 2.2. $\underline{\Pi}$ is a homotopy theory.

The proof of this theorem will occupy most of Chapter II. We shall refer to $\underline{\Pi}$ as the <u>standard</u> homotopy theory.

There are a number of reasons for dignifying $\underline{\Pi}$ with this adjective. Most immediately, $\underline{\Pi}\mathbf{1}$ = HoK is indeed the homotopy category which still occupies the attention of homotopy theorists, while $\underline{\Pi}$ itself should be thought of as containing the generalized theory of limits in HoK.

To this must be added its surprising ubiquity. Suppose C is a category provided with a class of morphisms to be referred to as "weak equivalences". If it satisfies the conclusion of 2.1, it gives rise to a hyperfunctor $\mathbf{C} \longmapsto$ Ho($C^{\mathbf{C}}$) which may be fortunate enough to be a homotopy theory; this is, approximately, the notion of homotopy theory proposed by D. W. Anderson [2]. What is surprising is how often this "new" homotopy theory turns out to be equivalent to $\underline{\Pi}$.

The prime example is of course the original one. If Top is the category of topological spaces and S: Top $\to K$ is the singular-complex functor, so that $| \; | \dashv$ S, a weak (homotopy) equivalence in Top is a map f such that $|Sf|$ is a homotopy equivalence. The unit and the counit of the adjunction are weak equivalences, so that, for any $\mathbf{C}$, S and $| \; |$ induce inverse equivalences of the hyperfunctor $\mathbf{C} \longmapsto$ Ho(Top$^{\mathbf{C}}$) with $\underline{\Pi}$.

A variant of this is furnished by the original notion of "semi-simplicial complexes". If J: $\underline{\Delta}^{\#} \subset \underline{\Delta}$ is the inclusion of the subcategory containing the injective maps, then the unit and counit of the adjunction $\mathrm{Lan}_J \dashv J^*$: $K \to \mathrm{Sets}^{\underline{\Delta}^{\#op}}$ are both weak equivalences and the hyperfunctor $\mathbf{C} \mapsto$ Ho($\mathrm{Sets}^{\underline{\Delta}^{\#op} \times \mathbf{C}}$) is again equivalent to $\underline{\Pi}$.

A number of interesting examples might be adduced (cf. for example [11]), but are beyond the scope of our present interests, except for the one provided by the category of small categories itself. The functor "nerve", N: Cat $\to K$ may be used to define a notion of weak equivalence in Cat, <u>viz</u>. F: $\mathbf{C} \to \mathbf{D}$ is a weak equivalence if NF is one in K, and thus in each of the categories Cat$^{\mathbf{C}}$. It was apparently first observed by D. Latch [15] that Cat has a category of fractions HoCat respect to these and that HoN: HoCat $\to$ HoK

is an equivalence of categories. We shall see in §6 below that this statement is true for all $\mathrm{Ho}(\mathrm{Cat}^{\mathbf{C}}) \to \mathrm{Ho}(K^{\mathbf{C}})$ so that $\mathbf{C} \to \mathrm{Ho}(\mathrm{Cat}^{\mathbf{C}})$ is again a homotopy theory equivalent to $\underline{\Pi}$.

A final reason for calling $\underline{\Pi}$ "standard" is investigated in Chapter IV below. It appears that $\underline{\Pi}$ "acts on" homotopy theories, on the left for left homotopy theories, on the right for right homotopy theories, the two actions on a homotopy theory being adjoint; we shall adopt the terminology of category theory by saying that a (left, right) homotopy theory T is (tensored, cotensored) over $\underline{\Pi}$. In the ordinary theory of complete categories, the category of sets plays the analogous role. The suggestion is that $\underline{\Pi}$ occupies the same central position in homotopy theory that Sets does in category theory.

3. CLOSED QUILLEN MODEL CATEGORIES

We review here the notion of a closed model structure in the sense of Quillen [21,22] and see how it is exemplified in the category of simplicial sets.

In any category we shall say that morphisms $u: A \to X$, $p: E \to B$ are <u>transverse</u> and write $u \vdash\!\vdash p$ if for any $f: A \to E$, $g: X \to B$ such that $pf = gu$ there is an $h: X \to E$ with $hu = f$, $ph = g$. We extend the notion to transversality of classes of morphisms, $\mathcal{A} \vdash\!\vdash \mathcal{B}$ having the obvious interpretation, and write also $\mathcal{A}^{\vdash\vdash} = \{p \mid \mathcal{A} \vdash\!\vdash p\}$ and ${}^{\vdash\vdash}\mathcal{B} = \{u \mid u \vdash\!\vdash \mathcal{B}\}$. Thus $\mathcal{A} \vdash\!\vdash \mathcal{B}$, $\mathcal{A} \subset {}^{\vdash\vdash}\mathcal{B}$, $\mathcal{A}^{\vdash\vdash} \supset \mathcal{B}$ are equivalent. The classes $\mathcal{A}^{\vdash\vdash}$, ${}^{\vdash\vdash}\mathcal{B}$ contain all isomorphisms, are closed under composition and retraction (in the morphism category) and under various limits and colimits. For example $\mathcal{A}^{\vdash\vdash}$ is closed under pullback and products, ${}^{\vdash\vdash}\mathcal{B}$ under pushouts and coproducts.

If C is a finitely complete and cocomplete category, by a <u>class of weak equivalences</u> in C, we mean a class $\mathcal{E}$ of morphisms closed under retraction (in C^2) and under composition and cancellation, i.e. such that any two of f, g, fg in $\mathcal{E}$ implies that the third is as well. By a <u>closed Quillen model structure in</u> C, <u>relative to a class</u> $\mathcal{E}$ <u>of weak equivalences</u>, we mean a pair of classes of morphisms, Cof, Fib, whose members are called, respectively, <u>cofibrations</u> and <u>fibrations</u>, each closed under retraction in C^2, and such that

(Q1) $\mathrm{Cof} \vdash\!\vdash \mathrm{Fib} \cap \mathcal{E}$, $\mathrm{Cof} \cap \mathcal{E} \vdash\!\vdash \mathrm{Fib}$.

(Q2) Any morphism in C has a <u>left factorization</u> $f''f'$ with $f'' \in \mathrm{Fib}$, $f' \in \mathrm{Cof} \cap \mathcal{E}$ and a <u>right factorization</u> $g''g'$ with $g'' \in \mathrm{Fib} \cap \mathcal{E}$, $g' \in \mathrm{Cof}$.

A category C supplied with such a structure is a <u>closed Quillen model category</u> or CQMC. If further the factorizations of Q2 are given by functors $L,R: C^2 \to C^3$, say

$$(f\colon X \to Y) \longmapsto \begin{cases} Lf = (X \xrightarrow{L'f} \hat{L}f \xrightarrow{L''f} Y) \\ Rf = (X \xrightarrow{R'f} \hat{R}f \xrightarrow{R''f} Y), \end{cases}$$

then $\mathcal{C}$ has the structure of a functorial CQMC.

PROPOSITION 3.1. In a CQMC

$$\mathrm{Fib} = (\mathrm{Cof} \cap \mathcal{E})^{\vdash\vdash} \qquad {}^{\vdash\vdash}\mathrm{Fib} = \mathrm{Cof} \cap \mathcal{E}$$

$$\mathrm{Cof} = {}^{\vdash\vdash}(\mathrm{Fib} \cap \mathcal{E}) \qquad \mathrm{Cof}^{\vdash\vdash} = \mathrm{Fib} \cap \mathcal{E}.$$

A proof may be found in [21]; in fact it is an easy exercise.

PROPOSITION 3.2 ([21]): If $\mathcal{C}$ is a CQMC it possesses a category of fractions $\mathrm{Ho}\mathcal{C} = \mathcal{C}[\mathcal{E}^{-1}]$ with respect to the weak equivalences.

We shall not repeat the proof here, but rather recall its ingredients. If $\emptyset$, $*$ are the initial and terminal objects of $\mathcal{C}$ and object X is fibrant (cofibrant, bifibrant) if $X \to *$ is a fibration ($\emptyset \to X$ is a cofibration, both). A congruence, ~, called homotopy, may be introduced in the full subcategory of bifibrant objects of $\mathcal{C}$, in analogy with the familiar construction in topology. The corresponding quotient category is then a weak fraction category, whose existence implies that of a fraction category.

In the category $\mathcal{K} = \mathrm{Sets}^{\underline{\Delta}^{op}}$ of simplicial sets, with $\mathcal{E}$ the class of simplicial maps f whose geometric realizations $|f|$ are homotopy equivalences in the category of topological spaces, one takes for Cof the class of injective simplicial maps and sets $\mathrm{Fib} = (\mathrm{Cof} \cap \mathcal{E})^{\vdash\vdash}$.

THEOREM 3.3 (Quillen [21]): $\mathcal{K}$ has the structure of a functorial CQMC.

A left factorization is given by Kan's Ex^{∞} functor and a right factorization is deduced from this, using the classical mapping cylinders. A central point in the argument is the fact that $\mathrm{Fib} = H^{\vdash\vdash}$ where H is the (countable) set of inclusions $\Lambda^n_q \subset \underline{\Delta}(-,[n])$, the "horn" inclusions.

This fact is of course implicated in several of the following properties of $\mathcal{K}$. These come in dual pairs.

PROPOSITION 3.4. In $\mathcal{K}$,

(o) If for any family $\{f_i\colon X_i \to Y_i\}$ of weak equivalences each X_i and Y_i is cofibrant, then $\coprod f_i\colon \coprod X_i \to \coprod Y_i$ is a weak equivalence.

(i) If $f_i = X_i \to Y_i$ and each $f_i \in \mathrm{Cof} \cap \mathcal{E}$, then $\coprod f_i \in \mathrm{Cof} \cap \mathcal{E}$.

(ii) If $X, X'\colon \mathbb{N} \to \mathcal{K}$ are sequences of cofibrations, $f\colon X \to X'$ and each $f_i\colon X_i \to X'_i$ is a weak equivalence, then $\mathrm{colim}\, f\colon \mathrm{colim}\, X \to \mathrm{colim}\, X'$ is a weak equivalence.

(iii) If $X\colon N^{op} \to K$, $A \to X$ is a limit cone and each $A \to X_i$ is a cofibration, then so also is $A \to \lim X$.

Dually,

(o*) If $\{f_i\colon X_i \to T_i\}$ is a family of weak equivalences and each X_i, Y_i is fibrant, then $\prod f_i\colon \prod X_i \to \prod Y_i$ is a weak equivalence.

(i*) If each f_i is in $\mathrm{Fib} \cap E$, then $\prod f_i \in \mathrm{Cof} \cap E$.

(ii*) If $X, X'\colon N^{op} \to K$ are sequences of fibrations, $f\colon X \to X'$ and each f_i is a weak equivalence, then so also is $\lim f\colon \lim X \to \lim X'$.

(iii*) If $X\colon N \to K$, $X \to B$ is a colimit cone and each $X_i \to B$ is a fibration, then so also is $\operatorname{colim} X \to B$.

(In some instances, e.g. (o), we have added redundant hypotheses to emphasize the duality.)

4. MODEL STRUCTURE IN $K^{\mathbf{C}}$

If $\mathbf{C}$ is a small category, then, as indicated in §2, we define the class $E_{\mathbf{C}}$ of weak equivalences in $K^{\mathbf{C}}$ to consist of those f such that each f_c is a weak equivalence in K. The classes Fib_w of <u>weak fibrations</u> and Cof_w of <u>weak cofibrations</u> consist respectively of those f which for each c have $f_c \in \mathrm{Fib}$ (resp. $f_c \in \mathrm{Cof}$). The left and right factorizations L and R in K give rise by composition to factorizations in $K^{\mathbf{C}}$ which we again denote by L and R. Thus for f in $K^{\mathbf{C}}$, $L'f \in \mathrm{Cof}_w \cap E_{\mathbf{C}}$, $L''f \in \mathrm{Fib}_w$ with R having the dual properties.

If $\mathbf{C}$ is discrete then Cof_w, Fib_w, L, R give to $K^{\mathbf{C}}$ the structure of a functorial CQMC and we drop the subscript w. In general, we go on to define the classes Fib_s of <u>strong fibrations</u> and Cof_s of <u>strong cofibrations</u> by

$$\mathrm{Fib}_s = (\mathrm{Cof}_w \cap E_{\mathbf{C}})^{\vdash\!\vdash}, \qquad \mathrm{Cof}_s = {}^{\vdash\!\vdash}(\mathrm{Fib}_w \cap E_{\mathbf{C}}).$$

We shall see that each of the pairs $(\mathrm{Cof}_s, \mathrm{Fib}_w)$ and $(\mathrm{Cof}_w, \mathrm{Fib}_s)$ belong to functorial CQMC structures relative to $E_{\mathbf{C}}$, the <u>left</u> and <u>right model structures</u> in $K^{\mathbf{C}}$. These structures will require factorizations, ${}^{L}L$ and ${}^{L}R$ for the left, and ${}^{R}L$, ${}^{R}R$ for the right structure, which we now proceed to construct. The idea behind their construction goes back to Steenrod [25]; we shall accordingly refer to them as the <u>Steenrod factorizations</u>. They will also occur in the proof in §5 of Theorem 2.2.

We begin by constructing ${}^{L}R$ and from it ${}^{L}L$. The remaining factorizations, ${}^{R}L$ and ${}^{R}R$ are constructed dually, so that an explicit description will not be necessary.

LEMMA 4.1. If $c\colon \mathbf{1} \to \mathbf{C}$ is an object of $\mathbf{C}$, then $\mathrm{Lan}_c \mathrm{Cof} \subset \mathrm{Cof}_s$, $\mathrm{Lan}_c(\mathrm{Cof} \cap E) \vdash\!\vdash \mathrm{Fib}_w$, $\mathrm{Lan}_c(\mathrm{Cof} \cap E) \subset E_{\mathbf{C}}$.

The first two assertions are direct consequences of the adjunction $Lan_c \dashv c^*$, c^* being of course evaluation at c. The third follows from 3.4(o) and the observation that $(Lan_c X)_d$ is $\mathbf{C}(c,d) \times X$, the $\mathbf{C}(c,d)$ copower of X.

Let $J\colon \mathbf{C}_0 \to \mathbf{C}$ be the inclusion and write η, ε for the unit and counit of $Lan_J \dashv J^*$.

LEMMA 4.2. Suppose that X is in $K^{\mathbf{C}}$, that $f\colon J^*X \to A$ is a cofibration and that

$$\begin{array}{ccc} Lan_J J^*X & \xrightarrow{\;Lan_J f\;} & Lan_J A \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle v} \\ X & \longrightarrow & X' \end{array}$$

is a pushout in $K^{\mathbf{C}}$. Then $(J^*v)\eta_A\colon A \to J^*X'$ is a cofibration.

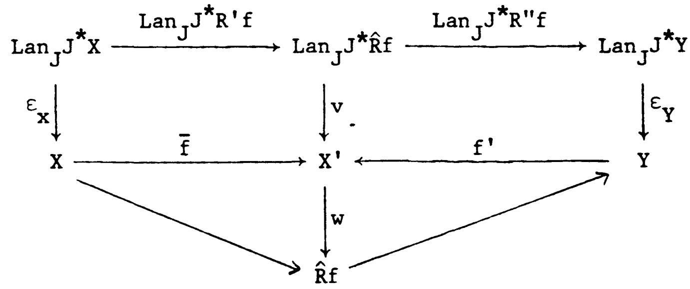
It will be sufficient to evaluate at an arbitrary $c \in \mathbf{C}_0$. The map $(Lan_J f)_c$ factorizes as in the second row of the diagram

$$\begin{array}{ccccc} & & A_c & \xrightarrow{\;1\;} & A_c \\ & & \downarrow & & \downarrow{\scriptstyle (\eta_A)_c} \\ \coprod\limits_{\substack{\phi:\, d\to c\\ \phi\neq 1_c}} X_d \sqcup X_c & \xrightarrow{\;1\sqcup f_c\;} & \Big(\coprod\limits_{\substack{\phi:\, d\to c\\ \phi\neq 1_c}} X_d\Big) \sqcup A_c & \xrightarrow{\;(\coprod f_d)\sqcup 1\;} & \coprod\limits_{\substack{\phi:\, d\to c\\ \phi\neq 1_c}} A_d \sqcup A_c \\ \downarrow{\scriptstyle (\psi\;\; 1)} & & \downarrow{\scriptstyle (\psi'\;\; 1)} & & \downarrow{\scriptstyle v_c} \\ X_c & \xrightarrow{\;f_c\;} & A_c & \xrightarrow{\;v_c(\eta_A)_c\;} & X'_c \end{array}$$

in which the maps from the first row to the second are right injections, ψ and ψ' are, respectively, given by $\psi\, inj_\phi = \phi$, $\psi'\, inj_\phi = f_c\phi$ and both squares at the bottom are pushouts. The morphisms $(\coprod f_d) \sqcup 1$, as a coproduct of cofibrations, is a cofibration, hence also $v_c(\eta_A)_c = ((J^*v)\eta_A)_c$.

We have eschewed here a shorter argument in favor of one that dualizes.

The factorization ${}^L R$ arises by iteration of the following basic construction. If $f\colon X \to Y$ in $K^{\mathbf{C}}$ the following commutative diagram is determined by demanding that the left-hand square be a pushout and that $wv = \varepsilon_{\hat{R}f}$.

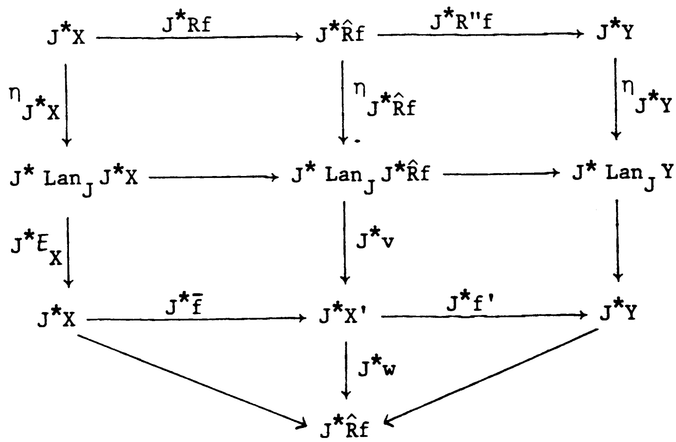


Iterating this construction we arrive at a sequence $X = X_0 \xrightarrow{x_0} X_1 \xrightarrow{x_2} X_2 \to \cdots$ together with morphisms $f = f_0\colon X_0 \to Y$, $f_1\colon X_1 \to Y$, $f_2\colon X_2 \to Y$, giving it the structure of a colimit cone, by setting $X_{n+1} = X_n'$, $f_{n+1} = f_n'$, $x_n = \bar{f}_n$. We set ${}^L\hat{R}f = \mathrm{colim}(X_0 \to X_1 \to \cdots)$ with ${}^LR'f\colon X \to {}^L\hat{R}f$ the injection and ${}^LR''f\colon {}^L\hat{R}f \to Y$ the colimit of $\{f_n\}$, giving a factorization ${}^LR$.

LEMMA 4.4. All x_n are in Cof_s; also ${}^LR'f \in \mathrm{Cof}_s$. ${}^LR''f \in \mathrm{Fib}_w \cap \mathcal{E}_C$. If f is a weak equivalence, then also ${}^LR'f \in \mathcal{E}_C$ and ${}^LR'f \mapstochar\!\!\!-\!\!\!-\, \mathrm{Fib}_w$.

That $x_n \in \mathrm{Cof}_s$ follows from 4.1; the closure properties of Cof_s imply that ${}^LR'f \in \mathrm{Cof}_s$. The same lemma implies the last assertion as well.

To see that ${}^LR''f \in \mathrm{Fib}_w \cap \mathcal{E}_C$ we apply J* to the basic construction 4.3 and add the units of the adjunction to get



Using this we may interpolate terms in the result of applying J* to the colimit cone defining ${}^LR$ to get in K^{C_0} the colimit cone below in which each $J^*R''f_n$ is both a fibration and a weak equivalence. By Lemma 4.2 all the morphisms in the top row are cofibrations. Thus by 3.4(ii), (iii*), $J^{*\,L}R''f$ is both a fibration and a weak equivalence, which implies our assertion.

$$\begin{array}{ccccccccc}
J^*X_0 & \xrightarrow{J^*R'f_0} & J^*\hat{R}f_0 & \xrightarrow{(J_v^*)\eta J^*\hat{R}f_0} & J^*X_1 & \xrightarrow{J^*R'f_1} & J^*\hat{R}f_1 & \to & \cdots \\
{\scriptstyle J^*f_0}\downarrow & & \downarrow{\scriptstyle J^*R''f_0} & & \downarrow{\scriptstyle J^*f_1} & & \downarrow{\scriptstyle J^*R''f_1} & & \\
J^*Y & \xrightarrow{1} & J^*Y & \longrightarrow & J^*Y & \longrightarrow & J^*Y & \to & \cdots
\end{array}$$

In particular, ${}^LR$ is a right factorization for $(\mathrm{Cof}_s,\mathrm{Fib}_w)$. A left factorization ${}^LL$ may be obtained by taking, for $f\colon X \to Y$,

$$ {}^LLf = \left(X \xrightarrow{{}^LR'L'f} {}^L\hat{R}L'f \xrightarrow{(L''f)({}^LR''L'f)} Y\right). $$

Finally, if u is a strong cofibration in $\mathcal{K}^{\mathbf{C}}$, then u is a retract of ${}^LR'u$. From 4.4, if u is also a weak equivalence, then ${}^LR'u \mapsto\!\!\!\!+ \mathrm{Fib}_w$, hence also $u \mapsto\!\!\!\!+ \mathrm{Fib}_w$.

THEOREM 4.5. ${}^LL$, ${}^LR$ give to $(\mathrm{Cof}_s,\mathrm{Fib}_w)$ the structure of a functorial CQMC relative to $\mathcal{E}_{\mathbf{C}}$. Dually, ${}^RL$, ${}^RR$ give to $(\mathrm{Cof}_w,\mathrm{Fib}_s)$ the structure of a functorial CQMC, again relative to $\mathcal{E}_{\mathbf{C}}$.

5. $\underline{\Pi}$ AS A HOMOTOPY THEORY

We proceed, finally, to the proof of Theorem 2.2. To begin with, Proposition 2.1 follows from 3.2 and Theorem 4.5, so that $\underline{\Pi}\mathbf{C} = \mathrm{Ho}(\mathcal{K}^{\mathbf{C}})$ is well defined. We shall show that $\underline{\Pi}$ is a left homotopy theory; since all the arguments dualize, we conclude that it is a right homotopy theory as well.

Axioms H0 and H1 are immediate consequences of the definition. For H2 it is convenient to introduce the following convention. Suppose $\mathbf{F}$ is a finite free category whose generating graph is Γ. For a functor $X\colon \mathbf{F} \to \mathcal{K}$ we consider the condition

(*) For any $c \in \mathbf{C}_0$

$$\coprod_{\gamma\in\Gamma(d,c)} X_d \xrightarrow{\ \xi\ } X_c$$

is a cofibration, where $\xi\,\mathrm{inj}_\gamma = X_\gamma$.

PROPOSITION 5.1. If $\mathbf{F}$ is a finite free category, then

(i) If X has the property (*), then X is strongly cofibrant (i.e. $\emptyset \to X$ is in Cof_s).

(ii) For any Y in $\mathcal{K}^{\mathbf{F}}$ there is an $X \to Y$ in $\mathrm{Fib}_w \cap \mathcal{E}_{\mathbf{F}}$ such that X has the property (*).

(iii) Thus X is strongly cofibrant if and only if it has the property (*).

(iv) For any $W\colon \mathbf{F} \to \mathrm{Ho}K$ there is a cofibrant X with dgm $X \approx W$.

(v) If X is strongly cofibrant and Y is weakly fibrant, then

$$K^{\mathbf{C}}(X,Y) \to (\mathrm{Ho}K)^{\mathbf{F}}(\mathrm{dgm}\ X,\ \mathrm{dgm}\ Y) \text{ is surjective.}$$

With the exception of (iii), which follows immediately from (i) and (ii), all these assertions may be proved by induction on the number of objects of $\mathbf{F}$. If there is only one, then $\mathbf{F} \approx \mathbf{1}$ and there is nothing to prove. In general $\mathbf{F}_0$ is ordered by setting $d \leq c$ when $\mathbf{F}(d,c) \neq \emptyset$. Choosing some maximal c, let $\mathbf{F}'$ be the full subcategory containing the remaining objects and, in general, denote by $(\ )'$ the restriction to $\mathbf{F}'$.

For (i) the inductive step goes as follows. Suppose $p\colon \mathbf{E} \to \mathbf{B}$ is in $\mathrm{Fib}_w \cap E_{\mathbf{F}}$ and $f\colon X \to B$. We must show that there is a $g\colon X \to E$ with $pg = f$. By the hypothesis of induction, there is an $h\colon X' \to E'$ with $p'h = f'$. In K we have, then, the commutative square

$$\begin{array}{ccc} \coprod_{\gamma\colon d \to c} X_d & \xrightarrow{\ \phi\ } & E_c \\ \xi \downarrow & & \downarrow p_c \\ X_c & \xrightarrow{\ f_c\ } & B_c \end{array}$$

where $\phi\ \mathrm{inj}_\gamma = E_\gamma h_d$. But then there is a $g_c\colon X_c \to E_c$ with $g_c\xi = \phi$, $p_c g_c = f_c$, completing the construction of g.

The inductive steps in (ii), (iv) and (v) are all of this general pattern and need not be detailed here. Axiom H2, of course, follows immediately.

Now suppose $P\colon \mathbf{E} \to \mathbf{B}$ is a discrete fibration. Then $\mathrm{Ran}_P\colon K^{\mathbf{E}} \to K^{\mathbf{B}}$ is given by $(\mathrm{Ran}_P X)_b = \prod_{P\ell=b} X$. Let $(K^{\mathbf{E}})^\#$ and $(K^{\mathbf{B}})^\#$ be the full subcategories of $K^{\mathbf{E}}$ and $K^{\mathbf{B}}$ containing the weakly fibrant objects. Then $(K^{\mathbf{E}})^\# \to \underline{\Pi}\mathbf{E}$ and $(K^{\mathbf{B}})^\# \to \underline{\Pi}\mathbf{B}$ are weak fraction categories. Both P^* and Ran_P restrict to these categories and, since they preserve weak equivalences, define functors on the fraction categories, *viz.* $\underline{\Pi}P\colon \underline{\Pi}\mathbf{B} \to \underline{\Pi}\mathbf{E}$ and, say, $RP\colon \underline{\Pi}\mathbf{E} \to \underline{\Pi}\mathbf{B}$. The unit and counit of $P^* \dashv \mathrm{Ran}_P$, similarly restricted, then define the unit and counit of $\underline{\Pi}P \dashv RP$, thus proving H4L.

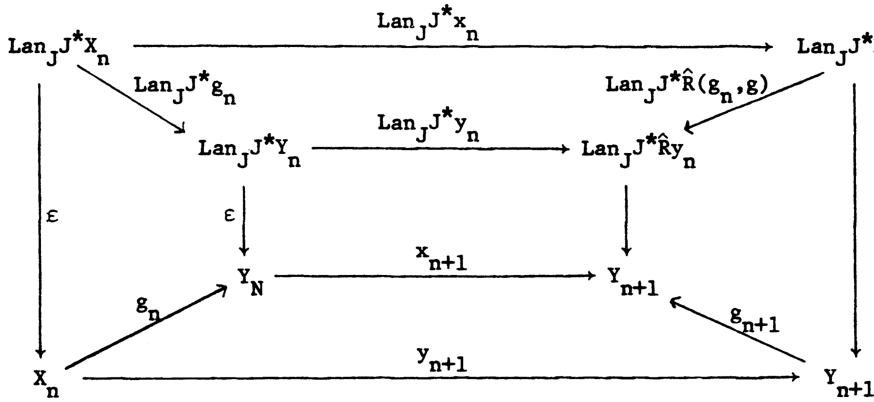
We turn finally to the proof of H3L. For any $\mathbf{C}$ we may define $T\colon K^{\mathbf{C}} \to K^{\mathbf{C}}$ by $TX = {}^L\hat{R}(\emptyset \to X)$. Then $\tau_X = {}^LR''(\emptyset \to X)\colon TX \to X$ is a weak equivalence.

LEMMA 5.2. If $F\colon \mathbf{C} \to \mathbf{D}$ then $\mathrm{Lan}_F T\colon K^{\mathbf{C}} \to K^{\mathbf{D}}$ preserves weak equivalences.

We refer to the construction of ${}^LR$ in §4. If $g\colon X \to Y$ is a weak equivalence in $K^{\mathbf{C}}$, then Tg is the colimit of a diagram in which the successive g_n

$$\begin{array}{ccccccccc}
\emptyset & = & X_0 & \xrightarrow{x_0} & X_1 & \xrightarrow{x_1} & X_2 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & & \\
\emptyset & = & Y_0 & \xrightarrow{y_0} & T_1 & \xrightarrow{y_1} & T_2 & \longrightarrow & \cdots
\end{array}$$

are determined by commutativity in the diagrams



in which the horizontal arrows are cofibrations and the two squares are push-outs. Inductively, g_n and hence $Lan_J J^* g_n$ are weak homotopy equivalences. Since g is a weak equivalence, so also is $\hat{R}(g_n, g)$, and thus $Lan_J J^* \hat{R}(g_n, g)$ as well. It follows that g_{n+1} is a weak equivalence for each n.

If we write $K\colon \mathbf{D}_0 \to \mathbf{D}$ for the inclusion, then $KF_0 = FJ$ so that $J^*F^* = F_0^*K^*$ and $Lan_F Lan_J = Lan_K Lan_{F_0}$. Since $(Lan_{F_0} X)_d = \coprod_{F_0 c = d} X_c$, Lan_{F_0} preserves weak equivalences, hence also $Lan_F Lan_J$. Thus, applying Lan_F to 5.3 we see that $Lan_F g_n$ is a weak equivalence for all n, hence also (by 3.3 ii) $Lan_F Tg$.

We should remark parenthetically here that the statement "Lan_{F_0} preserves weak equivalences" dualizes to "Ran_{F_0} preserves weak equivalences between weakly fibrant objects", which is sufficient to prove the dual of 5.2.

Thus $Lan_F T$ induces a functor $LF\colon Ho(K^{\mathbf{C}}) \to Ho(K^{\mathbf{D}})$. We assert that it is left adjoint to $\underline{\Pi}F = HoF^*$. For the composition

$$Lan_F TF^* \xrightarrow{Lan_F \tau F^*} Lan_F F^* \xrightarrow{\varepsilon} id_{K^{\mathbf{D}}}$$

induces a natural transformation $\bar{\varepsilon}\colon (LF)(\underline{\Pi}F) \to id_{Ho(K^{\mathbf{D}})}$, ηT a natural transformation $\tilde{\eta}\colon HoT \to (\underline{\Pi}F)(LF)$ and τ a natural isomorphism $\tilde{\tau}\colon HoT \to 1_{Ho(K^{\mathbf{C}})}$. If $\bar{\eta} = \tilde{\eta}\tilde{\tau}^{-1}$, then $\bar{\eta}$ and $\bar{\varepsilon}$ are the unit and counit of $LF \dashv \underline{\Pi}F$.

6. THE STANDARD THEORY VIA SMALL CATEGORIES

We recall that $\underline{\Delta}$ is a full subcategory of Cat. We may thus define N: Cat $\to$ K, the <u>nerve</u>, by $(N\mathbf{C})_n = \mathrm{Cat}([n],\mathbf{C})$. An F: $\mathbf{C} \to \mathbf{D}$ in Cat is a <u>weak equivalence</u> if NF is a weak equivalence in K. More generally, we shall say that a morphism F in $\mathrm{Cat}^{\mathbf{C}}$ is a <u>weak equivalence</u> if F_c is a weak equivalence in Cat for all $c \in \mathbf{C}$.

THEOREM 6.1.

(i) For each $\mathbf{C}$, $\mathrm{Cat}^{\mathbf{C}}$ admits a category of fractions $\mathrm{Ho}(\mathrm{Cat}^{\mathbf{C}})$ with respect to its weak equivalences, so that $\mathbf{C} \mapsto \mathrm{Ho}(\mathrm{Cat}^{\mathbf{C}})$ is a hyperfunctor.

(ii) $\mathrm{Ho}(N^{\mathbf{C}})\colon \mathrm{Ho}(\mathrm{Cat}^{\mathbf{C}}) \to \mathrm{Ho}(K^{\mathbf{C}}) = \underline{\Pi}\mathbf{C}$ is a strict hypernatural transformation which for each $\mathbf{C}$ is an equivalence of categories.

We may paraphrase this by saying that $\mathbf{C} \mapsto \mathrm{Ho}(\mathrm{Cat}^{\mathbf{C}})$ is a homotopy theory equivalent to $\underline{\Pi}$.

The proof will begin with the definition of $\Gamma\colon K \to \mathrm{Cat}$ by $\Gamma X = \underline{\Delta}^{op} \propto X$ (cf. I§3. We might note that Γ is not the left adjoint of N, which loses too much information for our purposes.) Then

$$(N\Gamma X)_n = \coprod_{u\colon [n] \to \underline{\Delta}^{op}} X_{u0} \tag{6.2}$$

so that an element $\mathrm{inj}_u x$ of $(N\Gamma X)_n$ gives rise to the data

$$\begin{array}{cccccccccc} & u0 & \longleftarrow & u1 & \longleftarrow & \cdots & \longleftarrow & un & & \text{in } \underline{\Delta} \\ & X_{u0} & \longrightarrow & X_{u1} & \longrightarrow & \cdots & \longrightarrow & X_{un} & & \text{in Sets} \\ x = & x_0 & \longmapsto & x_1 & \longmapsto & \cdots & \longmapsto & x_n & . & \end{array}$$

If $\omega\colon [m] \to [n]$ in $\underline{\Delta}$ then $(N\Gamma X)_\omega \, \mathrm{inj}_u x = \mathrm{inj}_{u\omega} x_{\omega 0}$.

Associated with any $u\colon [n] \to \underline{\Delta}^{op}$ in Cat is a morphism $\bar{u}\colon [n] \to u0$ in $\underline{\Delta}$, defined by

$$\begin{array}{ccccc} j & \longmapsto & 0 & \longmapsto & \bar{u}j \\ \cap & & \cap & & \cap \\ [n] & & uj & & u0 \end{array} \quad . \tag{6.3}$$

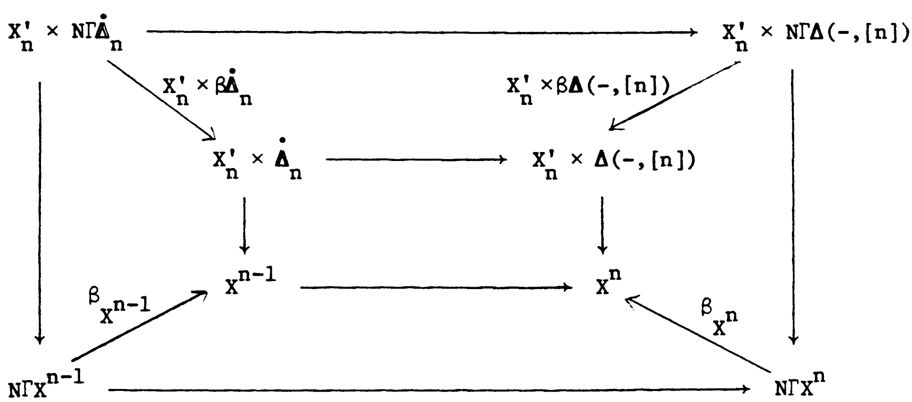
It is easy to see that for $\omega\colon [m] \to [n]$ the diagram

$$\begin{array}{ccc} [m] & \xrightarrow{\overline{u\omega}} & u\omega 0 \\ \omega\downarrow & & \downarrow \\ [n] & \xrightarrow{\bar{u}} & u0 \end{array}$$

in $\underline{\Delta}$ commutes. Thus we may define β_X: $N\Gamma X \to X$ by $\beta_X \mathrm{inj}_u x = X_{\bar{u}} x$, yielding a natural transformation β: $N\Gamma \to \mathrm{id}_K$.

LEMMA 6.4. For any X in K, β_X is a weak equivalence.

In fact the functor $N\Gamma$ is the "barycentric subdivision". We need only make this explicit. Now $\underline{\Delta}(-,[n])$ is the simplicial set corresponding to the ordered n-simplex and $N\Gamma\underline{\Delta}(-,[n])$ the simplicial set corresponding to its barycentric subdivision. Since both are weakly contractible, $\beta_{\underline{\Delta}(-,[n])}$ is certainly a weak equivalence. From 6.2 we see that $N\Gamma$ preserves cofibrations, i.e. injective simplicial maps, and coproducts. Thus, writing X^n for the n-skeleton of X, $\dot{\underline{\Delta}}_n$ for the n-1 skeleton of $\underline{\Delta}(-,[n])$ and X'_n for the set of non-degenerate n-simplices of X, we may construct in K the commutative diagrams



in which both squares are pushouts and the horizontal arrows are cofibrations. Inductively, the other instances of β being weak equivalences, so also is β_{X^n} and thus, finally, β_X as well.

On the other hand, $(\Gamma\mathbf{C})_0 = \{v\colon [n] \to \mathbf{C}\}$ the morphisms being all of the form $v \to v\omega$ for $\omega\colon [m] \to [n]$ in $\underline{\Delta}$. Thus we may define $\gamma_{\mathbf{C}}$: $\Gamma N\mathbf{C} \to \mathbf{C}$ by $\gamma v = v0$, where $\gamma(v \to v\omega)\colon v0 \to v\omega 0$ is extracted from $v0 \to v1 \to \cdots \to vn$, yielding a natural transformation γ: $\Gamma N \to \mathrm{id}_{\mathrm{Cat}}$.

LEMMA 6.5. $N\gamma = \beta N$. Thus, for all $\mathbf{C}$, $\gamma_{\mathbf{C}}$ is a weak equivalence.

For $(N\Gamma N\mathbf{C})_n = \coprod_{u\colon [n] \to \underline{\Delta}^{op}} \{v\colon u0 \to \mathbf{C}\}$ so that an element $\mathrm{inj}_u v$ is a diagram

$$\mathbf{C} \xleftarrow{\;v\;} u0 \longleftarrow u1 \longleftarrow \cdots \longleftarrow un$$

in Cat, whence $\beta\ \mathrm{inj}_u v = v\bar{u}$. But $(N\gamma)\mathrm{inj}_u v$ is obtained by evaluating each of the compositions $uj \to u0 \to \mathbf{C}$ at 0 so that, referring to 6.3, it is also given by $v\bar{u}$.

Theorem 6.1 now follows immediately.

We may recall at this point the familiar fact that, since N preserves products and $N2 = \underline{\Delta}(-,[1])$, natural transformations in Cat become homotopies in K. In particular, if **C** has an initial or a terminal object then N**C** is weakly contractible.

An alternate description of the nerve functor at the homotopy level shows strikingly how homotopy theory can differ from the ordinary theory of limits.

PROPOSITION 6.6. For any small category **C**, $N\mathbf{C} \approx \underline{\Pi}$ - $\mathrm{colim}_{\mathbf{C}}*$.

By * we mean here the constant functor with value $\underline{\Delta}(-,[0])$, the terminal object of $\underline{\Pi}\mathbf{C}$.

Let us consider the functor $N^{\#}\mathbf{C} = N(\mathbf{C}{\downarrow}-)\colon \mathbf{C} \to K$. Since id_c is terminal in $(\mathbf{C}{\downarrow}c)$ all $(N^{\#}\mathbf{C})_c$ are weakly contractible and $N^{\#}\mathbf{C} \to *$ is a weak equivalence. But $N^{\#}\mathbf{C}$ is strongly cofibrant in $K^{\mathbf{C}}$. For $((N^{\#}\mathbf{C})_c)_n$ consists of diagrams $\gamma = (c_0 \to c_1 \to \cdots \to c_n \xrightarrow{u} c)$ in **C**, and if $v\colon c \to c'$ then $(N^{\#}\mathbf{C})_v\gamma = (c_0 \to c_1 \to \cdots \to c_n \xrightarrow{vu} c')$. Thus every γ has a unique representation in the form $\gamma = (N^{\#}\mathbf{C})_u\bar{\gamma}$ where $\bar{\gamma}$ is of the special type $(c_0 \to c_1 \to \cdots \to c_n \xrightarrow{1} c_n)$.

But this readily translates into the assertion that for each $n > 0$

$$\begin{array}{ccc} \mathrm{Lan}_J J^*(N^{\#}\mathbf{C})^{n-1} & \longrightarrow & \mathrm{Lan}_J J^*(N^{\#}\mathbf{C})^{n} \\ \downarrow & & \downarrow \\ (N^{\#}\mathbf{C})^{n-1} & \longrightarrow & (N^{\#}\mathbf{C})^{n} \end{array}$$

is a pushout, where $J\colon \mathbf{C}_0 \to \mathbf{C}$ is the inclusion, and the cofibrancy of $(N^{\#}\mathbf{C})^n$ follows inductively using 4.2, hence that of $N^{\#}\mathbf{C}$ as well.

Thus $\underline{\Pi}$ - $\mathrm{colim}_{\mathbf{C}}* \approx \mathrm{colim}_{\mathbf{C}}N^{\#}\mathbf{C}$. But this colimit is just N**C**, the injections $(N^{\#}\mathbf{C})_c \to N\mathbf{C}$ being given by $(c_0 \to \cdots \to c_n \to c) \mapsto (c_0 \to \cdots \to c_n)$.

Let us conclude by adducing some homotopical relations between the several symmetries in our categories. In Cat, $\mathbf{C} \mapsto \mathbf{C}^{op}$ is such a symmetry, that is to say an automorphism of period 2. If $\iota_n\colon [n] \to [n]$ is the order reversing map, then a symmetry $\iota\colon \underline{\Delta} \to \underline{\Delta}$ is defined by $[n] \mapsto [n]$, $\omega \mapsto \iota_n\omega\iota_m$ for $\omega\colon [m] \to [n]$, giving rise, by composition, to a symmetry in K which we write $X \mapsto X^{\iota}$.

The following relations between these are immediate

$$N(\mathbf{C}^{op}) = (N\mathbf{C})^{\iota}$$

(6.6)

$${}^{op}\Gamma X = (\Gamma X)^{op}, \text{ where } {}^{op}\Gamma X = X \rtimes \underline{\Delta} \quad (\text{recall } \Gamma X = \underline{\Delta}^{op} \ltimes X).$$

To these we may add

(6.7) $$\Gamma(X^{\iota}) \approx \Gamma X, \quad \text{whence} \quad \Gamma N\mathbf{C}^{op} \approx \Gamma N\mathbf{C},$$

the isomorphisms being given by the identity on objects and $\omega \mapsto \iota\omega$ on morphisms.

Together with 6.4.5 they lead to the following conclusions

PROPOSITION 6.8.

(i) $\mathbf{C}$ and $\mathbf{C}^{op}$ are naturally isomorphic in HoCat.

(ii) X and X^{ι} are naturally isomorphic in HoK.

(iii) HoΓ and Ho$({}^{op}\Gamma)$: Ho$K \to$ HoCat are naturally isomorphic.

CHAPTER III

PROPERTIES OF HOMOTOPY THEORIES

We begin here to explore the consequences of our axioms for homotopy theory, viz. conditions H0 - H4L,R of II§1. The subject matter is accordingly rather technical than conceptual. But the density theorem, 4.2 below, while it will have important applications, also is of interest as supporting the intuition that a homotopy theory T ought to be thought of as consisting of extra structure on its underlying homotopy category T**1**. It asserts in effect that the objects of T**1**, in a suitable homotopical sense, generate all of the categories T**C**.

1. LIMITS AND WEAK LIMITS

If T is a left homotopy theory and **C** is a discrete category, then T-const$_{\mathbf{C}}$: T**1** → T**C** has a left adjoint by H3L and since **C** → **1** is a discrete fibration, a right adjoint by H4L. In view of H0 this implies that T**1** is supplied with both coproducts and products. Since T[**D**] is again a left homotopy theory, the same is true of T**D**. The conclusion being self-dual, we conclude that it holds for right homotopy theories as well. We shall adopt the convention of writing ∅, * for the initial and terminal objects of the categories T**C**.

Other limits and colimits are however often absent. We may at the risk of pedantry, observe that in $\underline{\Pi}\mathbf{1} \approx$ Ho(Top) the diagram $* \longleftarrow S^2 \xrightarrow{f} S^2$ where f is of degree 2 lacks a pushout. For if P were a pushout then, cohomology being representable, the cohomology of P with coefficients Z and Z/2 would violate the universal coefficient theorem.

On the other hand, if T is a left homotopy theory and **F** is a finite free category, then for any diagram W: **F** → T**1** there is an X in T**F** with dgm X ≈ W.

PROPOSITION 1.1. If **F** is a finite free category and X is in T**F**, then T - colim$_{\mathbf{F}}$X is a weak colimit of dgm$_{\mathbf{F}}$X.

This is implied by the fact that for any A in T**1**

$$T\mathbf{1}(T\text{-colim}_{\mathbf{F}}X,A) \approx T\mathbf{F}(X,T\text{-const}_{\mathbf{F}}A) \xrightarrow{\text{dgm}_{\mathbf{F}}} (T\mathbf{1})^{\mathbf{F}}(\text{dgm}_{\mathbf{F}}X,\ \text{const}_{\mathbf{F}}A)$$

is surjective.

This supplies a weak colimit for W. It is not of course functorial in W. But if also dgm Y ≈ W, then there is an f: X → Y such that dgm f is an isomorphism and $T\text{-colim}_{\mathbf{F}}Y \approx T\text{-colim}_{\mathbf{F}}X$. In other words the weak colimits provided by this construction are unique up to non-canonical isomorphism. We shall refer to them as <u>privileged weak colimits</u>.

In particular, taking $\mathbf{F} = \underline{\Lambda}$, we see that each $\mathbf{TC}$ has privileged weak pushouts or, as we shall sometimes say, <u>homotopy pushouts</u>. We have already seen that it has coproducts.

PROPOSITION 1.2. If T is a left homotopy theory, then any diagram in any $\mathbf{TC}$ has a weak colimit.

Dual results hold of course for right homotopy theories.

Let us take the time to insist on the special character of privileged weak colimits in the case $\mathbf{F} = \underline{\Lambda}$. Writing $J\colon \underline{\Lambda} \to \mathbf{2}^2$ for the inclusion, a privileged weak pushout or homotopy pushout is an X in $(\mathbf{T1})^{\mathbf{2}\times\mathbf{2}}$ isomorphic to some $\mathrm{dgm}(LJ)\overline{X}$.

LEMMA 1.3.

(i) If X and Y are homotopy pushouts and $f\colon J^*X \approx J^*Y$, then $f = J^*\hat{f}$ for some $\hat{f}\colon X \approx Y$.

(ii) If in the commutative diagram

$$\begin{array}{ccccc} X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 \\ \downarrow & & \downarrow & & \downarrow \\ Y_0 & \longrightarrow & Y_1 & \longrightarrow & Y_2 \end{array}$$

in $\mathbf{T1}$ both squares are homotopy pushouts, then so also is the rectangle. If the rectangle and the left-hand square are homotopy pushouts, then so is the right-hand square.

Both (i) and (ii) are familiar properties of genuine pushouts, but are in general false for weak pushouts.

The next proposition will often be used below. Here it will serve to give us some insight into the existence of genuine limits and colimits in a homotopy theory.

PROPOSITION 1.4. If T is a left homotopy theory and $d\colon \mathbf{1} \to \mathbf{D}$, then $Td\colon \mathbf{TD} \to \mathbf{T1}$ has a right adjoint Rd and, for X in $\mathbf{T1}$, $\mathrm{dgm}_{\mathbf{D}}(Rd)X \approx X^{\mathbf{D}(-,d)}$.

For we may factor d as $\mathbf{1} \to (\mathbf{D}{\downarrow}d) \xrightarrow{J} \mathbf{D}$ where $\top = \mathrm{id}_d$ is terminal and J, the forgetful functor, is a discrete fibration. Then by (I.3.3, ii) $T_\top$ has a right adjoint, <u>viz</u>. $T\text{-const}_{(\mathbf{D}{\downarrow}d)}$, while RJ exists by H4L. The computation of $\mathrm{dgm}_{\mathbf{D}}(Rd)X$ now follows from the fact that $\{w \in (\mathbf{D}{\downarrow}d)_0 \mid Jw = d'\} = \mathbf{D}(d',d)$.

Thus Td, having adjoints on both sides, preserves whatever limits and colimits may exist in $\mathbf{TD}$. Since $Td = d^*\mathrm{dgm}_{\mathbf{D}}$ and limits and colimits in $(\mathbf{T1})^{\mathbf{D}}$

are computed "pointwise", we conclude that $dgm_{\mathbf{D}}$ preserves limits and colimits too. Axiom H1 asserts moreover that it reflects isomorphism.

PROPOSITION 1.5. If T is a left homotopy theory and $T\mathbf{C}$ is closed either under limits or under colimits, then $dgm_{\mathbf{C}}: T\mathbf{C} \to (T\mathbf{1})^{\mathbf{C}}$ is faithful.

For in this case two morphisms are equal if and only if their equalizer or coequalizer is an isomorphism.

Thus a left or a right homotopy theory T such that each $T\mathbf{C}$ is either complete or cocomplete is imbedded in the representable hyperfunctor $\mathbf{C} \mapsto (T\mathbf{1})^{\mathbf{C}}$. When $\mathbf{C}$ is finite and free, the imbedding is of course an equivalence.

2. COMPUTING WITH HOMOTOPY KAN EXTENSIONS

If $F: \mathbf{C} \to \mathbf{D}$, $d: \mathbf{1} \to \mathbf{D}$ we may construct, as in I§5, the weakly commutative square

$$\begin{array}{ccc} (F\downarrow d) & \xrightarrow{\ t\ } & \mathbf{1} \\ {\scriptstyle J}\downarrow & \overset{\phi}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle d} \\ \mathbf{C} & \xrightarrow{\ F\ } & \mathbf{D} \end{array}$$

with $\phi_{c,u:\ Fc\to d} = u$. If T is a left homotopy theory, we may apply it to the diagram to get

$$(2.1)\qquad \begin{array}{ccc} T\mathbf{D} & \xrightarrow{\ Td\ } & \mathbf{1} \\ {\scriptstyle TF}\downarrow & \overset{T\phi}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle Tt\ =\ T\text{-}const_{(F\downarrow d)}} \\ T\mathbf{C} & \xrightarrow{\ TJ\ } & T(F\downarrow d) \end{array} \quad .$$

PROPOSITION 2.2. The square 2.1 has the Beck-Chevalley property.

This observation is central to the understanding of the homotopy Kan extension, yielding, for example, the analogue of the computation of the ordinary Kan extension in terms of limits.

What we are primarily interested in is the statement that $(T\phi)_*$ is an isomorphism. But Td also has a right adjoint by 1.4, and TJ has a right adjoint since J is a discrete fibration. Thus by I.4.5 it is sufficient to show that $(T\phi)^*$ is an isomorphism. We recall that $(T\phi)^*$ is the composition

$$(2.3)\quad (TF)(Rd) \xrightarrow{\eta(TF)(Rd)} (RJ)(TJ)(TF)(Rd) \xrightarrow{(RJ)\phi(Rd)} (RJ)(Tt)(Td)(Rd) \xrightarrow{(RJ)(Tt)\varepsilon} (RJ)(Tt)$$

where η and ε are the appropriate unit and counit.

Let us choose an A in $T\mathbf{1}$ and a $c \in \mathbf{C}_0$. Then, using the computations of right adjoints given by 1.4 and H4L, we may apply the sequence (2.3) of functors and natural transformations to A and evaluate at c, getting

$$ {}_{A}\mathbf{D}(Fc,d) \longrightarrow {}_{A}\mathbf{D}(Fc,d) \times \mathbf{D}(Fc,d) \longrightarrow {}_{A}\mathbf{D}(d,d) \times \mathbf{D}(Fc,d) \longrightarrow {}_{A}\mathbf{D}(Fc,d) $$

with morphisms given by

$$ \begin{array}{ccccccc} u' & \longleftarrow\!\shortmid & (u',u) & & (1,u) & \longleftarrow\!\shortmid & u \\ \mathbf{D}(Fc,d) & \longleftarrow & \mathbf{D}(Fc,d)\times\mathbf{D}(Fc,d) & \longleftarrow & \mathbf{D}(d,d)\times\mathbf{D}(Fc,d) & \longleftarrow & \mathbf{D}(Fc,d) \\ & & (vu,u) & \longleftarrow\!\shortmid & (v,u). & & \end{array} $$

But the composition of these is the identity.

COROLLARY 2.4. If X is in $T\mathbf{C}$ then $((LF)X)_d \approx (T\text{-colim}_{(F\downarrow d)})(TJ)X$.

In other words we can express not (LF)X, but at least its underlying diagram, in terms of homotopy colimits.

COROLLARY 2.5. A left homotopy theory satisfies H4R.

The existence of the left adjoint follows of course from H3L, while 2.4 tells us how to compute it. For if $P\colon \mathbf{E} \to \mathbf{B}$ is a discrete opfibration and, for $b \in \mathbf{B}_0$, $I\colon \mathbf{E}_b \to (P\downarrow b)$ is the canonical injection, then $T\text{-colim}_{P\downarrow b} \approx (T\text{-colim}_{\mathbf{E}_b})(TI)$. But $\mathbf{E}_b$ is discrete, so that $T\text{-colim}_{\mathbf{E}_b}$ is just the coproduct.

Kan extensions were so named because, along the inclusion of a full subcategory, they yield genuine extensions. The same conclusion is true for homotopy Kan extensions.

PROPOSITION 2.6. If T is a left homotopy theory and $U\colon \mathbf{C} \to \mathbf{D}$ is a full inclusion, then the unit $\eta\colon 1_{T\mathbf{C}} \to (TU)(LU)$ is an isomorphism.

For if $c \in \mathbf{C}_0$ then $U\downarrow c$ has terminal object 1_c. Thus, by 2.4, $((LU)X)_c \approx X_c$ so that the conclusion follows from H1.

PROPOSITION 2.7. Let

$$ \begin{array}{ccc} \mathbf{E}' & \xrightarrow{G} & \mathbf{E} \\ {\scriptstyle P'}\downarrow & & \downarrow{\scriptstyle P} \\ \mathbf{B}' & \xrightarrow{F} & \mathbf{B} \end{array} $$

be a pullback with P an opfibration. If T is a left homotopy theory, then

$$ \begin{array}{ccc} T\mathbf{B} & \xrightarrow{TF} & T\mathbf{B}' \\ {\scriptstyle TP}\downarrow & {\scriptstyle 1}\nearrow\!\!\!\nearrow & \downarrow{\scriptstyle TP'} \\ T\mathbf{E} & \xrightarrow[TG]{} & T\mathbf{E}' \end{array} $$

has the Beck-Chevalley property.

(Compare Anderson [2], where the same conclusion is asserted under different hypothesis.)

Let us treat first the special case $\mathbf{B}' = \mathbf{1} \xrightarrow{b} \mathbf{B}$, $\mathbf{E}' = \mathbf{E}_b$. Here $T\text{-colim}_{P\downarrow b} = (T\text{-colim}_{\mathbf{E}_b})(TI)$ where $I\colon \mathbf{E}_b \to (P\downarrow b)$ is the canonical injection. By 2.4, $(Tb)(LP) \approx (T\text{-colim}_{(P\downarrow b)})(TJ) \approx (T\text{-colim}_{\mathbf{E}_b})T(JI)$, while $JI\colon \mathbf{E}_b \to \mathbf{E}$ is of course the inclusion, i.e. the pullback of b along P.

For the general case we consider the diagram

$$\begin{array}{ccccc} \mathbf{E}_{Fb} = \mathbf{E}'_b & \longrightarrow & \mathbf{E}' & \xrightarrow{G} & \mathbf{E} \\ \downarrow & & \downarrow{\scriptstyle P'} & & \downarrow{\scriptstyle P} \\ \mathbf{1} & \xrightarrow{b} & \mathbf{B}' & \xrightarrow{F} & \mathbf{B} \end{array}$$

leading to

$$\begin{array}{ccccc} T\mathbf{B} & \xrightarrow{TF} & T\mathbf{B}' & \longrightarrow & T\mathbf{1} \\ {\scriptstyle TP}\downarrow & {\scriptstyle 1}\nearrow\!\!\!\nearrow & \downarrow{\scriptstyle TP'} & {\scriptstyle 1}\nearrow\!\!\!\nearrow & \downarrow{\scriptstyle \mathrm{const}_{\mathbf{E}'_b}} \\ T\mathbf{E} & \longrightarrow & T\mathbf{E}' & \longrightarrow & T\mathbf{E}'_b \end{array}$$

in which both the rectangle and the right-hand square, for all $b \in \mathcal{B}_0$, have the Beck-Chevalley property. It follows from H1 that the left-hand square has it as well.

PROPOSITION 2.8. If $F\colon \mathbf{C} \to \mathbf{D}$ and $E'\colon \mathbf{C}' \to \mathbf{D}'$, and T is a left homotopy theory, then

$$\begin{array}{ccc} T(\mathbf{D}\times\mathbf{D}') & \xrightarrow{T(\mathbf{D}\times F')} & T(\mathbf{D}\times\mathbf{C}') \\ {\scriptstyle T(F\times\mathbf{D}')}\downarrow & {\scriptstyle 1}\nearrow\!\!\!\nearrow & \downarrow{\scriptstyle T(F\times\mathbf{C}')} \\ T(\mathbf{C}\times\mathbf{D}') & \xrightarrow[T(\mathbf{C}\times F')]{} & T(\mathbf{C}\times\mathbf{C}') \end{array}$$

is a Beck-Chevalley square.

For $d \in \mathbf{D}_0$ consider the diagrams

$$\begin{array}{ccccc} T(\mathbf{D}\times\mathbf{D}') & \xrightarrow{T(\mathbf{D}\times F')} & T(\mathbf{D}\times\mathbf{C}') & \xrightarrow{T\langle d\quad \mathbf{C}'\rangle} & T\mathbf{C}' \\ {\scriptstyle T(F\times\mathbf{D}')}\downarrow & {\scriptstyle 1}\nearrow\!\!\!\nearrow & \downarrow{\scriptstyle T(F\times\mathbf{C}')} & {\scriptstyle T(\phi\times\mathbf{C}')}\nearrow\!\!\!\nearrow & \downarrow{\scriptstyle Tpr_{\mathbf{C}'}} \\ T(\mathbf{C}\times\mathbf{D}') & \xrightarrow[T(\mathbf{C}\times F')]{} & T(\mathbf{C}\times\mathbf{C}') & \xrightarrow[T(J\times\mathbf{C}')]{} & T((F\downarrow d)\times\mathbf{C}') \end{array}$$

$$\begin{array}{ccccc}
T(\mathbf{D}\times\mathbf{D}') & \xrightarrow{T\langle d\quad \mathbf{D}'\rangle} & T\mathbf{D}' & \xrightarrow{TF'} & T\mathbf{C}' \\
{\scriptstyle T(F\times\mathbf{D}')}\downarrow & {\scriptstyle T(\phi\times\mathbf{D}')}\nearrow\!\!\!\nearrow & \downarrow{\scriptstyle Tpr_{\mathbf{D}'}} & {\scriptstyle 1}\nearrow\!\!\!\nearrow & \downarrow{\scriptstyle Tpr_{\mathbf{C}'}} \\
T(\mathbf{C}\times\mathbf{D}') & \xrightarrow[T(J\times\mathbf{D}')]{} & T((F\downarrow d)\times\mathbf{D}') & \xrightarrow[T((F\downarrow d)\times F')]{} & T((F\downarrow d)\times\mathbf{C}')
\end{array}$$

whose compositions, of course, coincide. The right-hand square of the first and the left-hand square of the second are both B-C squares, by 2.2 applied, respectively to $T[\mathbf{C}']$ and $T[\mathbf{D}']$ while, since projection is an opfibration, 2.7 implies that the right-hand square of the second also has the B-C property. By H1, then, applied to $T[\mathbf{C}']$, the left-hand square shares it as well.

Note the symmetry in F and F'. Important special cases are those in which $\mathbf{C} = \mathbf{1}$ or $\mathbf{D} = \mathbf{1}$.

3. SEQUENTIAL HOMOTOPY COLIMITS

Let us turn our attention to the special case $T\text{-colim}_{\mathbf{N}}: T\mathbf{N} \to T\mathbf{1}$, where T is a left homotopy theory. A functor $\phi: \mathbf{N} \to \mathbf{N}$ is just an order-preserving map; it is final if it is unbounded. If so, we may define $\psi: \mathbf{N} \to \mathbf{N}$ by $\psi h = \sup\{m \mid \phi m \leq n\}$ and ψ is right-adjoint to ϕ, so that $L\phi \approx T\psi$. Thus $T\text{-colim}_{\mathbf{N}} = (T\text{-colim}_{\mathbf{N}})(L\phi) = (T\text{-colim}_{\mathbf{N}})(T\psi)$. But $\phi\psi = \mathrm{id}_{\mathbf{N}}$.

PROPOSITION 3.1. If $\phi: \mathbf{N} \to \mathbf{N}$ is final, then $(T\text{-colim}_{\mathbf{N}})(T\phi) \approx T\text{-colim}_{\mathbf{N}}$.

We shall see generalizations of this in IV 4.

The state of affairs in the bounded case is summed up in the following lemma.

LEMMA 3.2.

(i) If $W: \mathbf{N} \to T\mathbf{1}$ is eventually constant, then $W \approx \mathrm{dgm}_{\mathbf{N}}X$ for some X in $T\mathbf{N}$.

(ii) If $\mathrm{dgm}_{\mathbf{N}}X$, $\mathrm{dgm}_{\mathbf{N}}Y$ are eventually constant, then $T\mathbf{N}(X,Y) \to (T\mathbf{1})^{\mathbf{N}}(\mathrm{dgm}_{\mathbf{N}}X,\mathrm{dgm}_{\mathbf{N}}Y)$ is surjective.

(iii) If $\mathrm{dgm}_{\mathbf{N}}X$ is eventually constant, then $T\text{-colim}_{\mathbf{N}}X$ is its eventual value.

Suppose that $W_{n-1} \to W_n \to \cdots$ are all isomorphisms. Let $u: \mathbf{n} \to \mathbf{N}$ be the inclusion and v its left adjoint $vj = \min(n-1,j)$. By H2 there is an X' in $T\mathbf{n}$ with $\mathrm{dgm}_{\mathbf{n}}X' = (W_0 \to \cdots \to W_{n-1})$. But then, if $X = (Tv)X'$, $\mathrm{dgm}_{\mathbf{N}}X \approx W$. But also, H1 implies that $X \approx (Tv)(Tu)X$ from which (ii) follows at once. Finally, since $Tu = Lv$, $T\text{-colim}_{\mathbf{N}}X \approx (T\text{-colim}_{\mathbf{N}})(Tu)(Tv)X' \approx T\text{-colim}_{\mathbf{n}}X' = X_{n-1}$.

Next we shall give a computation of $T\text{-colim}_{\mathbf{N}}X$, due in essence to Milnor ((the "Milnor telescope", cf. [19]), in terms of coproducts and homotopy pushouts.

Let $\mathbf{A}_n$, n = 0,1,... be the ordered category

$$\begin{array}{ccccccccccc} & b_1 & & b_2 & & & & & b_n & & \\ \swarrow & & \searrow \ \swarrow & & \searrow \ \swarrow & \cdots & \searrow & \swarrow & & \searrow & \\ a_0 & & a_1 & & a_2 & & & a_{n-1} & & & a_n \end{array}$$

and $B_n \subset A_n$, $n = 1,2,\ldots$ the subcategory omitting a_n, and write $J\colon B_n \to A_n$, $J'\colon A_{n-1} \to B_n$ for the inclusions.

LEMMA 3.3. If X, in TA_n, has the diagram

$$\begin{array}{ccccccc} & X_1 & & & & X_n & \\ \swarrow & & \searrow^{id} & \cdots & {}^{id}\swarrow & & \searrow^{id} \\ X_0 & & X_1 & & X_{n-1} & & X_n \end{array}$$

then $T\text{-colim}_{A_n} X \approx X_0$.

The proof is by induction, starting with $A_0 \approx \mathbf{1}$. First, J has a right adjoint S, with $Sa_n = b_n$, so that $TS = LJ$. But $X = (TS)(TJ)X \approx (LJ)(TJ)X$, so that $(T\text{-colim}_{B_n})(TJ)X \approx (T\text{-colim}_{A_n})(LJ)(TJ)X \approx T\text{-colim}_{A_n} X$.

On the other hand, J' has a left adjoint S', with $S'b_n = a_{n-1}$, so that $TJ' = LS'$. Thus $(T\text{-colim}_{B_n})(TJ)X \approx (T\text{-colim}_{A_{n-1}})(LS')(TJ)X \approx (T\text{-colim}_{A_{n-1}})T(JS')X$, completing the induction.

Now set $A_\omega = \bigcup_n A_n$ and define $W\colon A_\omega \to N$ by $Wa_j = Wb_j = j$, for all j.

LEMMA 3.4. $(LW)(TW) \approx id_{TN}$.

We observe first that, for $n \in N_0$, $(W{\downarrow}n) \approx A_n$, while if X is in TN its image under $TN \to TA_\omega \to T(W{\downarrow}n)$ satisfies the hypothesis of 3.1 with the numbering reversed, so that its homotopy colimit is X_n. The conclusion now follows from 2.4.

Now let us write $M\colon A_\omega \to \underline{\Lambda}$ for the functor, i.e. the order-preserving map, $b_n \mapsto (0,0)$, $a_{2n} \mapsto (0,1)$, $a_{2n+1} \mapsto (1,0)$. This is a discrete opfibration. Thus if X is in TN we may compute $\text{dgm}_{\underline{\Lambda}}(LM)(TW)X$ as

$$\begin{array}{ccc} \coprod_n X_n & \xrightarrow{\ r\ } & \coprod_n X_{2n} \\ \big\downarrow{\scriptstyle \ell} & & \\ \coprod_n X_{2n+1} & & \end{array}$$

with $\ell\, inj_{2n+1} = inj_{2n+1}$, $\ell\, inj_{2n} = inj_{2n+1} \Pi X_{2n\to 2n+1}$ and r just reversing the parity.

If further we take $\underline{\Lambda}$ to P by $(i,j) \mapsto \max\{i,j\}$ with the two morphisms remaing distinct, then the diagram of the image of X becomes

(3.6) $$\coprod_n X_n \rightrightarrows \coprod_n X_n$$

with one morphism the identity and the other the familiar "shift".

But $\text{T-colim}_{A_\omega} = (\text{T-colim}_A)(LM)$, with a similar observation for P. This gives us Milnor's computation.

PROPOSITION 3.7. If X is in TN, then $\text{T-colim}_N X$ is a homotopy pushout of 3.5 and, also, a homotopy coequalizer of 3.6.

This allows us a gloss on 3.2.

LEMMA 3.8. If $W\colon N \to T1$ is eventually constant, then the homotopy coequalizer of $\coprod_n W_n \rightrightarrows \coprod_n W_n$ (cf. 3.6) is the eventual value of W (and is thus, in fact, the coequalizer).

We are now in a position to observe that the homotopy extension condition H2 holds more broadly than for finite free categories.

PROPOSITION 3.9. If T is a left homotopy theory, then $\text{dgm}_N\colon TN \to (T1)^N$ is a weak quotient functor.

Before turning to the proof we might remark that since T[C] is also a left homotopy theory, the same assertion holds for $\text{dgm}_N\colon T(C \times N) \to (TC)^N$, while if T is a right homotopy theory, then $T(N^{op}) \to (T1)^{N^{op}}$ is a weak quotient functor.

Now suppose $X\colon N \to T1$. Then by 3.2 we may construct a sequence $\hat{X}\colon N \to TN$ such that $\text{dgm}_N\hat{X}$ is

$$\begin{array}{c} \text{dgm}_N(\hat{X}_0) \\ \downarrow \\ \text{dgm}_N(\hat{X}_1) \\ \downarrow \\ \text{dgm}_N(\hat{X}_2) \\ \downarrow \\ \vdots \end{array} = \left(\begin{array}{ccccccccc} X_0 & \to & X_0 & \to & X_0 & \to & X_0 & \to & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ X_0 & \to & X_1 & \to & X_1 & \to & X_1 & \to & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ X_0 & \to & X_1 & \to & X_2 & \to & X_2 & \to & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \vdots & & \vdots & & \vdots & & \vdots & & \end{array}\right).$$

Let $\tilde{X}$ be the homotopy coequalizer of $\coprod_n \hat{X}_n \rightrightarrows \coprod_n \hat{X}_n$ as in 3.6. Then, using 3.8, we conclude that $\text{dgm}_N\tilde{X} \approx X$. This shows that dgm is surjective, up to isomorphism, on objects. An entirely analogous argument shows that it is full, once we observe that if $\text{dgm}_N W \approx X$ then the evident colimit-cone $\{\hat{X}_n \to W\}$ makes $W \approx \text{T-colim}_N\hat{X}$.

An evident corollary is the existence of privileged weak sequential colimits, as for those of finite free categories; we refer to them also as sequential homotopy colimits.

It is clear that the conclusion of H2 on 3.9 holds for a much larger class of categories. It would seem to be worthwhile to investigate this class, but this remains still to be done.

4. STEENROD FACTORIZATIONS AND THE DENSITY THEOREM

The construction used in II§4 to establish the existence of model-structures in the categories $K^{\mathbf{C}}$ has a counterpart in any homotopy theory, giving rise to "infinite factorizations" of morphisms into relatively simple pieces. These factorizations will be used repeatedly below; we shall also derive, using them, a density theorem which asserts that in a sense to be made explicit, the underlying homotopy category $\mathbf{T1}$ of a homotopy theory T generates the whole of T.

Suppose that T is a left homotopy theory and that $f: X \to Y$ in $\mathbf{TC}$. Taking $J: \mathbf{C}_0 \to \mathbf{C}$ for the inclusion of the discrete category of objects, we may construct in $\mathbf{TC}$ a commutative diagram

$$(4.1)\qquad \begin{array}{ccccc} (LJ)(TJ)X & \xrightarrow{(LJ)(TJ)f} & (LJ)(TJ)Y & & \\ \downarrow{\scriptstyle \varepsilon_X} & & \downarrow{\scriptstyle \hat{f}} & \searrow{\scriptstyle \varepsilon_Y} & \\ X & \xrightarrow{\quad \bar{f} \quad} & X' & \xrightarrow{f'} & Y \end{array}$$

in which the square is a privileged weak pushout and $f'\bar{f} = f$. Iterating this construction we get a sequence $X^{\#}: \mathbf{N} \to \mathbf{TC}$,

$$X^{\#} = (X_0 \xrightarrow{x_0} X_1 \xrightarrow{x_1} X_2 \longrightarrow \cdots)$$

together with a colimit cone $\{f_j: X_j \to Y\}$ by taking $f_0 = f$, $f_{j+1} = f'_j$, $x_j = \bar{f}_j$. We have moreover morphisms $(T\hat{f}_{j-1})\eta_{(TJ)Y}: (TJ)Y \to (TJ)X_j$ which, together with Tf_j, interpolate the constant sequence $(TJ)Y \to (TJ)Y \to \cdots$ into the sequence $(TJ)X_0 \to (TJ)X_1 \to \cdots$. It follows from 3.1 that $\text{T-colim}_{\mathbf{N}} X^{\#} \approx Y$, with the colimit cone $\{f_j\}$ giving the isomorphism. This interprets the assertion that f is the *right-infinite composition* of the sequence $X^{\#}$, which we shall refer to as the *left Steenrod factorization of* f.

A right homotopy theory has of course *right Steenrod factorizations,* defined dually.

We remark that the construction of Steenrod factorizations is not functorial. It would appear that a functorial factorization could be constructed with additional labor; what we have here is sufficient for our present purposes.

If T is a left homotopy theory and $\mathcal{D}$ is a class of objects in **TC**, then there is a smallest full replete subcategory of **TC** containing $\mathcal{D}$ and closed under coproducts and homotopy pushouts. It follows from 3.7 that this subcategory is closed under sequential homotopy colimits as well. We shall say that $\mathcal{D}$ is _dense_ if the subcategory of **TC** that it generates in this manner is all of **TC**. The existence of Steenrod factorizations proves, then, our density theorem.

THEOREM 4.2. If T is a left homotopy theory and $\mathcal{D}$ is a dense class of objects in **T1**, then $\{(Lc)X \mid X \in \mathcal{D},\ c\colon \mathbf{1} \to \mathbf{C}\}$ is dense in **TC**. In particular, $\{(Lc)X \mid X \in (\mathbf{T1})_0,\ c\colon \mathbf{1} \to \mathbf{C}\}$ is dense in **TC**.

It seems reasonable to say that the (Lc)X are the _free objects_ of **TC**. Thus the density theorem asserts that the free objects of **TC** are dense in **TC**. This of course is the sense in which **T1** generates all of T.

But it may well happen that **TC** is generated by something much smaller. For example, the singleton {*} is dense in $\underline{\Pi}\mathbf{1}$, so that the set $\{(L_{\underline{\Pi}}c)* \mid c\colon \mathbf{1} \to \mathbf{C}\}$ is dense in $\underline{\Pi}\mathbf{C}$.

5. POINTED HOMOTOPY THEORIES

A left homotopy theory T is _pointed_ if in **T1** the initial and terminal objects coincide. Since, by 1.4, for each **C** and c: **1** → **C**, Tc has both adjoints, the same must be true of each **TC**. By duality, the same conclusion holds for right homotopy theories as well. It will sometimes be useful to adopt, in pointed theories, the topologists' convention of writing X ∨ Y, $\bigvee_\alpha X_\alpha$ for coproducts.

We shall see here how to associate to each left homotopy theory T a pointed left homotopy theory $T^\bullet$ and a hypernatural transformation $T \to T^\bullet$.

$T^\bullet$ can be characterized as a subhyperfunctor of T[**2**]. For each **C** let $T^\bullet\mathbf{C}$ be the full subcategory of $T(\mathbf{2} \times \mathbf{C})$ containing those X for which X_0 is terminal in **TC**. This is clearly stable under all $T(\mathbf{2} \times F)$, F: **C** → **D**.

PROPOSITION 5.1. $T^\bullet$ is a left homotopy theory. If T is also a right homotopy theory, then $T^\bullet$ is a right homotopy theory as well.

The only axiom which does not obviously hold for $T^\bullet$ just because it does for T, is H3L, which is a consequence of the following lemma.

LEMMA 5.2. For each **C** the inclusion U: $T^\bullet\mathbf{C} \to T(\mathbf{2} \times \mathbf{C})$ has a left adjoint.

This adjoint we shall call by its classical name of _mapping cone_. Its construction illustrates how, with a little ingenuity, constructions familiar in $\mathcal{K}$ or Top can in fact be reproduced in a homotopy invariant setting.

We may assume, without loss of generality, that $\mathbf{C} = \mathbf{1}$. Consider the functors

$$\begin{array}{ccccccc} \mathbf{2} & \xrightarrow{I} & \underline{\Lambda} & \xrightarrow{J} & \mathbf{2}\times\mathbf{2} & \xleftarrow{K} & \mathbf{2} \\ i & \longmapsto & (i,0) & & (i,1) & \longleftarrow\!\!\shortmid & i \end{array}$$

J being inclusion. Since I is a discrete fibration, $T\mathbf{1}$ has a right adjoint RI, which has the property that for X in $T\mathbf{2}$

$$\mathrm{dgm}_{\underline{\Lambda}}(RI)X = \left(\begin{array}{ccc} & X_0 & \\ \swarrow & & \searrow \\ X_1 & & * \end{array}\right).$$

Since J is full $((\mathbf{C}J)(RI)X)_{10} = *$, so that (TK)(LJ)(RI)X is in $T^{\cdot}\mathbf{1}$. Thus (TK)(LJ)(RI) = UM where M: $T\mathbf{2} \to T^{\cdot}\mathbf{1}$. We claim that $M \dashv U$. For if Y is in $T^{\cdot}\mathbf{1}$ then, since RK = Tpr_1, (TJ)(RK)Y = (RI)Y. Thus $T\mathbf{2}(MX,Y) \approx T\underline{\Lambda}((RI)X,(RI)Y) \approx T\mathbf{2}(X,Y)$ since RI is a full imbedding.

Since U is the inclusion of a full subcategory, MU is isomorphic to the identity. We may recognize the unit of the adjunction in

$$\mathrm{dgm}_{\mathbf{2}}(LJ)(RI)X = (X \xrightarrow{\eta_X} UMX).$$

Returning to 5.1, if F: $\mathbf{C} \to \mathbf{D}$, then the left adjoint $L^{\cdot}F$ of $T^{\cdot}F$ is just $M_{\mathbf{D}}(LF)U_{\mathbf{C}}$, the notation being self-explanatory. This means of course that M is a left strong hypernatural transformation M: $T[\mathbf{2}] \to T^{\cdot}$ which preserves left homotopy Kan extensions.

A "forgetful" strict hypernatural transformation Θ: $T^{\cdot} \to T$ is defined by the composition

$$T^{\cdot}\mathbf{C} \xrightarrow{U_{\mathbf{C}}} T(\mathbf{2}\times\mathbf{C}) \xrightarrow{T(i_1\times\mathbf{C})} T\mathbf{C}.$$

This composition has the left adjoint $M_{\mathbf{C}}L(i_1 \times \mathbf{C})$ which, in conformity with the topologists usage we denote by $X \mapsto X^+$. It is easy to see that $\mathrm{dgm}(X^+) = (* \to * \sqcup X)$. Once again this is a left strong hypernatural transformation which preserves left homotopy Kan extensions.

The hypernatural transformation $X \mapsto X^+$ has a universal property whose precise statement is made a bit difficult by our somewhat exiguous provision of $\mathbf{2}$-category theory. It does not seem worthwhile, for our present purposes, to enlarge this. The idea is straightforward enough if we state it informally. Suppose T' is a pointed left homotopy theory and Φ: $\mathbf{1} \to T'$ is a left strong hypernatural transformation which preserves left Kan extensions. Then there is a Ψ: $T^{\cdot} \to T'$ with the same properties, essentially unique, such that for any

X in $\mathbf{TC}$, $\Phi_{\mathbf{C}}X \approx \Psi_{\mathbf{C}}X^+$. It is more or less evident that such a Ψ is given by $M'\Phi[2]U$, where M' is the mapping cone of T'. In particular, it is certainly the case that if T itself is pointed, then $\Theta\colon T^{\bullet} \approx T$.

We shall also restrict to an informal comment the observation that if $K^{\bullet}$ is the category of pointed simplicial sets, then, following the lines of Chapter II, we may define a hyperfunctor $\mathbf{C} \mapsto \mathrm{Ho}((K^{\bullet})^{\mathbf{C}})$, and that this hyperfunctor is essentially the same as $\underline{\Pi}^{\bullet}$.

The following lemmas will be needed below. Let $\mathbf{C}$ be a small category with initial object $\perp$. Then a functor $Q\colon \mathbf{2} \times \mathbf{C} \to \mathbf{C}$ is defined by $Qi_0 = \mathrm{const}_{\mathbf{C}} \perp$, $Qi_1 = \mathrm{id}_{\mathbf{C}}$.

LEMMA 5.3. If X is in T and $X_{\perp} = *$, then $(TQ)X$ is in $T^{\bullet}$ and $\Theta(TQ)X = X$.

In other words such an X has automatically the structure of a pointed object.

In a pointed left left homotopy theory, it is natural to introduce the construction of the smash product. Classically, this is the mapping cone of the canonical map $X \vee Y \to X \times Y$. In order to get a functorial construction, we proceed as follows.

Suppose $T^{\bullet}$ is a left homotopy theory. Then we may extend the product to a strict hypernatural transformation $\underline{\times}\colon \mathbf{TC} \times \mathbf{TD} \to T(\mathbf{C} \times \mathbf{D})$ by setting $X \underline{\times} Y = (T\mathrm{pr}_{\mathbf{C}})X \times T\mathrm{rp}_{\mathbf{D}}Y$. This $\underline{\times}$ is symmetric monoidal, which is to say coherently associative, symmetric, with unit $* \in (T\mathbf{1})_0$. It may or may not be the case that each $X\underline{\times}$ - preserves left homotopy Kan extensions. This is true for $\underline{\Pi}$ (cf. IV§2), but false in general for pointed theories, in which it does not preserve coproducts.

Now let $\mu\colon \mathbf{2} \times \mathbf{2} \to \mathbf{2}$ be given by $\mu(i,j) = ij$. We define $\wedge\colon T^{\bullet}\mathbf{C} \times T^{\bullet}\mathbf{D} \to T^{\bullet}(\mathbf{C} \times \mathbf{D})$ to be the composition

$$T^{\bullet}\mathbf{C}\times T^{\bullet}\mathbf{D} \xrightarrow{U\times U} T(\mathbf{2}\times\mathbf{C})\times T(\mathbf{2}\times\mathbf{D}) \xrightarrow{\underline{\times}} T(\mathbf{2}\times\mathbf{2}\times\mathbf{C}\times\mathbf{D}) \xrightarrow{L(\mu\times\mathbf{C}\times\mathbf{D})} T(\mathbf{2}\times\mathbf{C}\times\mathbf{D}) \xrightarrow{M} T(\mathbf{C}\times\mathbf{D}).$$

This is evidently a left strong hypernatural transformation. To see that it really does the job, it is sufficient to consider the case $\mathbf{C} = \mathbf{D} = \mathbf{1}$. If X and Y are in $T^{\bullet}\mathbf{1}$, then the diagram of $X \times Y$ is

$$\begin{array}{ccc} * & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ Y_1 & \longrightarrow & X_1 \times Y_1 \end{array}$$

and thus that of $(L\mu)(X \underline{\times} Y)$ is, by 2.4, just $X_1 \vee Y_1 \to X_1 \times Y_1$ as required.

The proof of the following statement presents no special difficulties but we shall nevertheless omit it.

PROPOSITION 5.4. If, in a left homotopy theory T, all $X\underline{\times}$-preserve left homotopy Kan extensions, then, in $T^{\cdot}$, $\wedge\!\!\!\wedge$ is symmetric monoidal and each $X\wedge\!\!\!\wedge$- preserves left homotopy Kan extensions.

Let us observe, finally, that all of the observations above dualize to right homotopy theories. The mapping cone becomes the "mapping fibre", while the pointed theory is $T_{\cdot} = T^{*\,\cdot\,*}$. We might observe that $\Pi_{\cdot}$ is the trivial theory; $\Pi_{\cdot}\mathbf{C} \approx \mathbf{1}$ for all $\mathbf{C}$. The <u>dual smash product</u>, i.e. the mapping fibre of $X \vee Y \to X \times Y$ has never, apparently, attracted to itself either a standard name or notation.

CHAPTER IV
TENSOR STRUCTURES

If C is a category with coproducts, then the category of sets operates on the left on C by sending (u,X), where u is a set and X is in C, to the copower of X indexed by u. Dually, if C has products, the category of sets operates on the right by sending (u,X) to the power of X indexed by u. In the jargon of category theory, this is expressed by saying that C is tensored or cotensored over the category of sets.

We shall see that every left homotopy theory is in an analogous way tensored over $\underline{\Pi}$ and every right homotopy theory cotensored over $\underline{\Pi}$. Explicitly, if T is a left homotopy theory, we shall define functors

$$\otimes_{\mathbf{G}}: \underline{\Pi}(\mathbf{C} \times \mathbf{G}^{op}) \times T(\mathbf{G} \times \mathbf{D}) \longrightarrow T(\mathbf{C} \times \mathbf{D})$$

which for fixed $\mathbf{G}$ constitute a hypernatural transformation. If T is instead a right homotopy theory, we shall have

$$\mathrm{Hom}_{\mathbf{C}}: \underline{\Pi}^*(\mathbf{C}^{op} \times \mathbf{G}) \times T(\mathbf{C},\mathbf{D}) \longrightarrow T(\mathbf{G} \times \mathbf{D}),$$

constituting for fixed $\mathbf{C}$ a hypernatural transformation.

The construction of these occupies §1 and §2. In a homotopy theory both, of course, exist and we begin in §3 to study the relations between them. Thus we shall cease to speak of left and right homotopy theories, and indeed from that point on the rest of this monograph will be concerned with homotopy theories instead.

1. THE FUNCTORS $\tilde{\otimes}_{\mathbf{G}}$

The tensor functors $\otimes_{\mathbf{G}}$ are obtained from functors $\tilde{\otimes}_{\mathbf{G}}$ by passage to a category of fractions. We devote ourselves here to the construction of $\tilde{\otimes}_{\mathbf{G}}$.

Let T be a left homotopy theory. If $W: \mathbf{C} \times \mathbf{G}^{op} \to \mathrm{Cat}$, we define $W\tilde{\otimes}_{\mathbf{G}}-: T\mathbf{G} \to T\mathbf{C}$ as the composition

$$T\mathbf{G} \xrightarrow{TP''} T(\mathbf{C} \ltimes W \rtimes \mathbf{G}) \xrightarrow{LP'} T\mathbf{C}$$

where P', P'' are the opfibration and fibration associated with $\mathbf{C} \ltimes W \rtimes \mathbf{G}$ (cf. I.2.4). (Thus $W\tilde{\otimes}_{\mathbf{G}}-$ depends on the choice of the adjoint LP' but of course only up to isomorphism.) If $f: W \to V$ in $\mathrm{Cat}^{\mathbf{C}\times\mathbf{G}^{op}}$, then $\mathbf{C} \ltimes f \rtimes \mathbf{G}: \mathbf{C} \ltimes W \rtimes \mathbf{G} \to \mathbf{C} \ltimes V \rtimes \mathbf{G}$ and we may construct the diagram

$$(1.1)\qquad \begin{array}{ccccc} TG & \xrightarrow{TP''_V} & T(\mathbf{C}\ltimes V\rtimes G) & \xrightarrow{LP'_V} & \mathbf{TC} \\ {\scriptstyle id}\downarrow & \overset{id}{\nearrow\!\!\!\nearrow} & \Big| \; {\scriptstyle T(\mathbf{C}\rtimes f\ltimes G)} & \overset{id_*}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle id} \\ TG & \xrightarrow{TP''_W} & T(\mathbf{C}\ltimes W\rtimes G) & \xrightarrow{LP'_W} & \mathbf{TC} \end{array}$$

in which the left-hand square is strictly commutative while the right-hand square is the left adjoint (I§4) of the strictly commutative square got by replacing LP'_V, LP'_W with TP'_V, TP'_W. The composition of the two squares in (1.1) gives a natural transformation $f\tilde{\otimes}_G-\colon (V\tilde{\otimes}_G-) \to (W\tilde{\otimes}_G-)$, making $\tilde{\otimes}_G$ a functor

$$\tilde{\otimes}_G\colon \mathrm{Cat}^{\mathbf{C}\times G^{op}} \times TG \longrightarrow \mathbf{TC}.$$

Since $T[\mathbf{D}]$, for any $\mathbf{D}$, is also a left homotopy theory, we have also constructed functors

$$\tilde{\otimes}_G\colon \mathrm{Cat}^{\mathbf{C}\times G^{op}} \times T(G\times \mathbf{D}) \longrightarrow T(\mathbf{C}\times \mathbf{D}).$$

We investigate next their dependence on $\mathbf{C}$ and $\mathbf{D}$.

PROPOSITION 1.2. $\tilde{\otimes}_G$ has the structure of a left strong hypernatural transformation.

We must provide, for $F\colon \mathbf{C}'\to\mathbf{C}$, $K\colon \mathbf{D}'\to\mathbf{D}$ a natural isomorphism as exhibited in the diagram

$$(1.3)\qquad \begin{array}{ccc} \mathrm{Cat}^{\mathbf{C}\times G^{op}}\times T(G\times\mathbf{D}) & \xrightarrow{\tilde{\otimes}_G} & T(\mathbf{C}\times\mathbf{D}) \\ {\scriptstyle (F\times G^{op})^*\times T(G\times K)}\downarrow & \overset{\tilde{\otimes}^1_G}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle T(F\times K)} \\ \mathrm{Cat}^{\mathbf{C}'\times G^{op}}\times T(G\times\mathbf{D}') & \xrightarrow[\tilde{\otimes}_G]{} & T(\mathbf{C}'\times\mathbf{D}'). \end{array}$$

Suppose $W\colon \mathbf{C}\times G^{op}\to \mathrm{Cat}$ and set $W' = (F\times G^{op})^*W$. Then

$$\begin{array}{ccc} (\mathbf{C}'\ltimes W'\rtimes G)\times\mathbf{D}' & \xrightarrow{\Phi\times K} & (\mathbf{C}\ltimes W\rtimes G)\times\mathbf{D} \\ {\scriptstyle P'\times\mathbf{D}'}\downarrow & & \downarrow{\scriptstyle P'\times\mathbf{D}} \\ \mathbf{C}'\times\mathbf{D}' & \xrightarrow{F\times K} & \mathbf{C}\times\mathbf{D} \end{array}$$

is a pullback in Cat and $P'\times\mathbf{D}$ is an opfibration. Let us construct (compare 1.1) the diagram

$$\begin{array}{ccccc} T(\mathbf{G}\times \mathbf{D}) & \xrightarrow{T(P''\times \mathbf{D})} & T((\mathbf{C}\ltimes W\rtimes \mathbf{G})\times \mathbf{D}) & \xrightarrow{L(P'\times \mathbf{D})} & T(\mathbf{C}\times \mathbf{D})\\ {\scriptstyle T(\mathbf{G}\times K)}\downarrow & \overset{id}{\nearrow} & \downarrow{\scriptstyle T(\Phi\times K)} & \overset{id_*}{\nearrow} & \downarrow{\scriptstyle T(F\times K)} \\ T(\mathbf{G}'\times \mathbf{D}) & \xrightarrow[T(P''\times \mathbf{D}')]{} & T((\mathbf{C}'\ltimes W'\rtimes \mathbf{G})\times \mathbf{D}') & \xrightarrow[L(P'\times \mathbf{D}')]{} & T(\mathbf{C}'\times \mathbf{D}')\end{array} \tag{1.4}$$

in which id_* is an isomorphism in virtue of (III.2.7).

The composition is a natural isomorphism $(W'\tilde{\otimes}_{\mathbf{G}}-)\colon T(\mathbf{G}\times K) \to T(F\times K)(W\tilde{\otimes}_{\mathbf{G}}-)$. It remains to be shown that this is natural in W and preserves compositions $\mathbf{D}'' \to \mathbf{D}' \to \mathbf{D}$, $\mathbf{C}'' \to \mathbf{C}' \to \mathbf{C}$. But both of these assertions follow from (I.4.3).

We distinguish some special cases. If $c\colon \mathbf{1} \to \mathbf{C}$, $d\colon \mathbf{1} \to \mathbf{D}$, then

$$\tilde{\otimes}^1_{\mathbf{G}}\colon W_c\tilde{\otimes}_{\mathbf{G}}X_d \approx (W\tilde{\otimes}_{\mathbf{G}}X)_{c,d}. \tag{1.5}$$

In the still more special case in which $\mathbf{G} = \mathbf{1}$ and W is discrete, i.e. $W\colon \mathbf{C} \to \text{Sets} \subset \text{Cat}$, we may identify $W_c \tilde{\otimes} X_d$ (omitting the subscript in $\tilde{\otimes}_{\mathbf{1}}$) as the W_c-indexed copower of X_d.

Also a special case, fixing $W\colon \mathbf{C}\times \mathbf{G}^{op} \to \text{Cat}$, $W\tilde{\otimes}_{\mathbf{G}}-\colon T[\mathbf{G}] \to T[\mathbf{C}]$ is a left strong hypernatural transformation.

PROPOSITION 1.6. $W\tilde{\otimes}_{\mathbf{G}}-$ preserves left homotopy Kan extensions.

This means that when $\mathbf{C} = \mathbf{C}'$, $F = id_{\mathbf{C}}$ the squares (1.3) have the Beck-Chevalley property. Referring to (1.4), the left-hand square enjoys it in virtue of (III.2.8) and the right-hand square by (I.4.5), the composition thus sharing it as well.

We pass now to consideration of the functors $-\tilde{\otimes}_{\mathbf{G}}X\colon \text{Cat}^{\mathbf{C}\times\mathbf{G}^{op}} \to T\mathbf{C}$.

LEMMA 1.7. $-\tilde{\otimes}_{\mathbf{G}}X$ preserves coproducts.

For $\mathbf{C} \ltimes (\coprod_\alpha W_\alpha) \rtimes \mathbf{G} \approx \coprod_\alpha \mathbf{C} \ltimes W_\alpha \rtimes \mathbf{G}$. The conclusion follows from HO.

If X is free, say $X = (Lg)A$ for $g\colon \mathbf{1} \to \mathbf{G}$ and A in $T\mathbf{1}$, then $-\tilde{\otimes}_{\mathbf{G}}X$ has a simple computation on discrete W.

LEMMA 1.8. If $W\colon \mathbf{C}\times \mathbf{G}^{op} \to \text{Sets}$, then $W\tilde{\otimes}_{\mathbf{G}}(Lg)A \approx W_g \tilde{\otimes} A$.

Without loss of generality, we may suppose $\mathbf{C} = \mathbf{1}$. Writing $I_g\colon W_g \to W \rtimes \mathbf{G}$ for the inclusion, we may construct the diagram

$$\begin{array}{ccccc} T\mathbf{1} & \xrightarrow{Lg} & T\mathbf{G} & \xrightarrow{W\otimes_{\mathbf{G}}-} & T\mathbf{1}\\ {\scriptstyle T\text{-const }Wg}\downarrow & \overset{id_*}{\nearrow} & \downarrow{\scriptstyle TP''} & \nearrow{\scriptstyle T\text{-colim}_{W\rtimes\mathbf{G}}} & \\ TW_g & \xrightarrow[LIg]{} & T(W\rtimes\mathbf{G}) & & \end{array}$$

where id_* is an isomorphism in virtue of (III.2.7) H4L and (I.4.5), and the triangle is strictly commutative as a special case of the definition of $\tilde{\otimes}_G$. Thus $W\tilde{\otimes}_G(Lg)A \approx \text{T-colim}_{W \rtimes G}(LIg)(\text{T-const}_{W_g})A \approx (\text{T-colim}_{W_g})(\text{T-const}_{W_g})A \approx W_g \tilde{\otimes} A$.

LEMMA 1.9. If $W = (W_{01} \leftarrow W_{00} \rightarrow W_{10})\colon \underline{\Lambda} \times \mathbf{C} \times \mathbf{G}^{op} \rightarrow \text{Sets}$ and, for all c, g, $W_{00cg} \rightarrow W_{01cg}$ is injective, then for any X in $T(\mathbf{G} \times \mathbf{D})$

(1.10) $$\left(\text{Lan}_{J \times \mathbf{C} \times \mathbf{G}^{op}} W\right) \tilde{\otimes}_G X \approx L(J \times \mathbf{C} \times \mathbf{D})(W \tilde{\otimes}_G X),$$

where $J\colon \underline{\Lambda} \rightarrow \mathbf{2} \times \mathbf{2}$ is the inclusion.

We might assume, without loss of generality, that $\mathbf{C} = \mathbf{1}$ and $\mathbf{D} = \mathbf{1}$. Let us for the moment make only the former assumption. Now both sides of (1.10) preserve left homotopy Kan extensions along functors $\mathbf{D} \rightarrow \mathbf{D}'$. Thus by the density theorem (III.4.2), it is enough to prove that (1.10) is an isomorphism in the special case $\mathbf{D} = \mathbf{1}$, $X = (Lg)A$ for $g\colon \mathbf{1} \rightarrow \mathbf{G}$, A in $\mathbf{T1}$, i.e. $(\text{Lan}_g W_g) \tilde{\otimes} A \approx (LJ)(W_g \tilde{\otimes} A)$.

But $W_g \approx (W_{01g} \leftarrow W_{00g} \longrightarrow W_{00g}) \sqcup (\emptyset \leftarrow \emptyset \rightarrow (W_{10g} - W_{00g})) \approx J^*pr_1^*(W_{00g} \rightarrow W_{01g}) \sqcup J^*pr_0^*(\emptyset \rightarrow (W_{10g} - W_{00g}))$. Since for each of these summands the isomorphisms is clear, it follows in the general case as well.

2. THE FUNCTORS $\otimes_G$, Hom_G

The principal step in the construction of these functors is made possible by Lemma 2.5 below. We need some preliminary observations. Recall (I§4) that $\Gamma\colon K \rightarrow \text{Cat}$ is the functor $V \longmapsto \underline{\Delta}^{op} \ltimes V$.

LEMMA 2.1. If T is a left homotopy theory and X is in $\mathbf{TG}$, then $(\Gamma\text{-}) \tilde{\otimes}_G X\colon K^{\mathbf{C}\times\mathbf{G}^{op}} \rightarrow \mathbf{TC}$ preserves coproducts.

It is enough to observe that $\Gamma\colon K^{\mathbf{C}\times\mathbf{G}^{op}} \rightarrow \text{Cat}^{\mathbf{C}\times\mathbf{G}^{op}}$ preserves them (compare 1.7).

LEMMA 2.2. If $W = (W_0 \leftarrow W_{00} \rightarrow W_{10})\colon \underline{\Lambda} \times \mathbf{C} \times \mathbf{G}^{op} \rightarrow K$ and $W_{00} \rightarrow W_{01}$ is a weak cofibration, then

$$\left(\Gamma\ \text{Lan}_{J \times \mathbf{C} \times \mathbf{G}^{op}} W\right) \tilde{\otimes}_G X \approx L(J \times \mathbf{C})(\Gamma W \tilde{\otimes}_G X).$$

For if $V'\colon \underline{\Delta}^{op} \times \mathbf{C} \times \mathbf{G}^{op} \rightarrow \text{Sets}$ is the transpose of $V\colon \mathbf{C} \times \mathbf{G}^{op} \rightarrow K$, then $(\Gamma V) \tilde{\otimes}_G X \approx (Lpr_{\mathbf{C}})(V' \tilde{\otimes}_G X)$, where $pr_{\mathbf{C}}\colon \underline{\Delta}^{op} \times \mathbf{C} \rightarrow \mathbf{C}$ is the projection. The lemma now follows from 1.9, applied to $W'_{01} \leftarrow W'_{00} \rightarrow W'_{10}$.

LEMMA 2.3. For any X in $T(\mathbf{C} \ltimes \underline{\Delta}^{op})$ the counit $\varepsilon_X\colon (Lpr_{\mathbf{C}})(Tpr_{\mathbf{C}})X \rightarrow X$ is an isomorphism.

We may remark parenthetically that this generalizes the statement that the geometric realization of a constant simplicial space has the homotopy type of the constant.

Without loss of generality, we may assume $\mathbf{C} = \mathbf{1}$. Let u: $\underline{\Delta}^{\#} \to \underline{\Delta}$ be the inclusion of the subcategory containing those ω: $[m] \to [n]$ such that $\omega 0 = 0$. Then u has the left adjoint q with $q[n] = [n+1]$, $(q\omega)0 = 0$, $(q\omega)(i+1) = \omega i + 1$. It follows that $(\text{T-colim}_{\underline{\Delta}^{op}})(\text{T-const}_{\underline{\Delta}^{op}}) \approx (\text{T-colim}_{\underline{\Delta}^{\# op}})(\text{T-const}_{\underline{\Delta}^{\# op}})$. But [0] is initial in $\underline{\Delta}^{\#}$, hence terminal in $\underline{\Delta}^{\# op}$.

LEMMA 2.4. For any n and any X in **T1**,

$$\Gamma\underline{\Delta}(-,[n]) \tilde{\otimes} X \longrightarrow \Gamma\underline{\Delta}(-,[0]) \tilde{\otimes} X$$

is an isomorphism.

For $\Gamma\underline{\Delta}(-,[n])$) $\tilde{\otimes} X \approx (\text{T-colim}_{\underline{\Delta}^{op}})(LP')(TP')(\text{T-const}_{\underline{\Delta}^{op}})$ where P': $\underline{\Delta}^{op} \times \underline{\Delta}(-,[n]) \to \underline{\Delta}^{op}$ is the canonical opfibration. But, recalling that $(\underline{\Delta}^{op} \times \underline{\Delta}(-,[n]))_0 = \coprod_k \underline{\Delta}([k],[n])$, we see that P' has the right adjoint r with $r[k]$: $[k] \to [n]$ the constant map with value n. Thus $LP' \approx Tr$, $(LP')(TP') \approx \text{id}$ and $\Gamma\underline{\Delta}(-,[n]) \tilde{\otimes} X \approx X$ by 2.3.

LEMMA 2.5. If f is a weak equivalence in $K^{\mathbf{C}\times\mathbf{G}^{op}}$ and X is in **TG**, then $\Gamma f \tilde{\otimes}_{\mathbf{G}} X$ is an isomorphism.

In view of (1.5) we may assume without loss of generality that $\mathbf{C} = \mathbf{1}$. We begin our argument in the case $\mathbf{G} = \mathbf{1}$. Consider the class of cofibrations f in K such that $\Gamma f \tilde{\otimes} X$ is an isomorphism. By (2.1,2) it is closed under coproducts, and under pushouts along cofibrations. By (2.4) it contains all cofibrations $\underline{\Delta}(-,[k]) \to \underline{\Delta}(-,[n])$. It follows from standard arguments that it contains all the anodyne maps of K and thus that the functor $(\Gamma -) \tilde{\otimes} X$: $K \to \mathbf{T1}$ inverts all weak equivalences.

Turning now to the general case, we consider the class of objects X in **TG** such that, whenever f is a weak equivalence in $K^{\mathbf{G}^{op}}$, then $\Gamma f \tilde{\otimes} X$ is an isomorphism. If g: $\mathbf{1} \to \mathbf{G}$ and it is in **T1**, then, for V in $K^{\mathbf{G}^{op}}$, $\Gamma V \tilde{\otimes}_{\mathbf{G}} (Lg)A \approx (Lpr_{\mathbf{C}})(V' \tilde{\otimes}_{\mathbf{G}} (Lg)A) \approx (Lpr_{\mathbf{C}})(V'_g \tilde{\otimes} A) \approx \Gamma V_g \tilde{\otimes} A$, as in 2.2, but using 1.9. Thus the class contains the free objects of **TG**. But if $X^0 \to X^1 \to \cdots$ is the left Steenrod factorization of an arbitrary $\emptyset \to X$ in **TG**, then inductively all X^n, and hence X itself, lie in the class.

We may accordingly define the functors $\otimes_{\mathbf{G}}$ by setting $W \otimes_{\mathbf{G}} X = \Gamma W \tilde{\otimes}_{\mathbf{G}} X$, $K^{\mathbf{C}\times\mathbf{G}^{op}} \times \mathbf{TG} \to \underline{\Pi}(\mathbf{C} \times \mathbf{G}^{op}) \times \mathbf{TG}$ being a fraction functor.

THEOREM 2.6. A left strong hypernatural transformation $\otimes_G$ is uniquely defined by commutativity in the diagrams

$$\begin{array}{ccc} K^{C\times G^{op}} \times T(G \times D) & \xrightarrow{(\Gamma-)\tilde{\otimes}_G-} & \\ \downarrow & & \searrow \\ \underline{\Pi}(C \times G^{op}) \times T(G \times D) & \xrightarrow[\otimes_G]{} & T(C \times D). \end{array}$$

Furthermore, $\otimes_G$ preserves left homotopy Kan extensions in either variable.

All that remains to be checked is that each $-\otimes_G X$ preserves homotopy Kan extensions, that is to say, the strongly commutative squares

$$\begin{array}{ccc} \underline{\Pi}(C \times G^{op}) \times TG & \xrightarrow{\otimes_G} & TC \\ {\scriptstyle \underline{\Pi}(F \times G^{op}) \times TG}\downarrow & \overset{\otimes_G^1}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle TF} \\ \underline{\Pi}(C' \times G^{op}) \times TG & \longrightarrow & TC' \end{array}$$

obtained from 1.3, with $D = 1$, by passing to the fraction categories, has the Beck-Chevalley property.

To see this we appeal once more to the density theorem, applied this time to the left homotopy theory $\underline{\Pi}[G^{op}]$. It follows from (2.1,2) that the class of W in $\underline{\Pi}(C' \times G^{op}) \simeq \underline{\Pi}[G^{op}]C'$ such that, for all X in TG, $(\otimes_G^1)_*: LF(W \otimes_G X) \to ((LF)W) \otimes_G X$ is an isomorphism, is closed under coproducts and homotopy pushouts.

Now suppose A is in $\underline{\Pi}G^{op}$ and $c: 1 \to C'$. Then

$$((LcA) \otimes_G X)_d \approx \coprod_{\phi:\ c\to d} A \otimes_G X \approx (Lc(A \otimes_G X))_d$$

so that $Lc(A \otimes_G X) \approx (LcA) \otimes_G X$. Thus $(LF)((Lc)A \otimes_G X) \approx (LF)(Lc)(A \otimes_G X) \approx (L(Fc)A) \otimes_G X \approx ((LF)(Lc)A) \otimes_G X$.

In other words, $(\Phi_G^1)_*$ is an isomorphism for all W = LcA, and thus for all W.

The cotensor structure for a right homotopy theory T is obtained by dualization. For T* (recall that $T^*C = (T(C^{op}))^{op}$) is a left homotopy theory, so that we have functors

$$\otimes_{C^{op}}: \underline{\Pi}(C \times G^{op}) \times T^*(C^{op} \times D^{op}) \to T^*(C^{op} \times D^{op})$$

or, denoting $(\otimes_{C^{op}})^{op}$ by Hom_C,

$$Hom_C: \underline{\Pi}^*(C^{op} \times G) \times T(C \times D) \to T(G \times D),$$

a right strong hypernatural transformation preserving right homotopy Kan extensions in either variable so that, for example, if W is in $\underline{\Pi}(\mathbf{C} \times \mathbf{G}^{op})$ which of course has the same objects as $\underline{\Pi}^*(\mathbf{C}^{op} \times \mathbf{G})$, and X is in $T\mathbf{C}$, then for any $F: \mathbf{C} \to \mathbf{C}'$

$$\mathrm{Hom}_{\mathbf{C}'}((LF)W,X) \approx (RF)\mathrm{Hom}_{\mathbf{C}}(W,X).$$

We may unpack this rather high-handed definition: suppose that $W: \mathbf{C} \times \mathbf{G}^{op} \to K$. Then $\mathrm{Hom}_{\mathbf{C}}(W,-)$ is the composition

$$T\mathbf{C} \xrightarrow{TP'} T(\mathbf{C} \ltimes {}^{op}\Gamma W \rtimes \mathbf{G}) \xrightarrow{RP''} T\mathbf{G} \tag{2.7}$$

where (I§4) ${}^{op}\Gamma: K \to \mathrm{Cat}$ is the functor $V \mapsto V \rtimes \mathbf{D} = (\Gamma V)^{op}$.

In the pointed case we may extend these operations to $\underline{\Pi}^\cdot$. Recall that $\underline{\Pi}^\cdot$ is a subhyperfunctor of $\underline{\Pi}[2]$, so that we may construct the composition

$$\underline{\Pi}^\cdot(\mathbf{C} \times \mathbf{G}^{op}) \times T\mathbf{G} \longrightarrow \underline{\Pi}(2 \times \mathbf{C} \times \mathbf{G}^{op}) \times T\mathbf{G} \xrightarrow{\otimes_{\mathbf{G}}} T(2 \times \mathbf{C}) \xrightarrow{M} T^\cdot\mathbf{C}$$

where M is the mapping cone functor of (III§5). If T is a pointed left homotopy theory, so that $T^\cdot = T$, we denote this composition by $\dot{\otimes}_{\mathbf{G}}$.

PROPOSITION 2.8. $\dot{\otimes}_{\mathbf{G}}$ is a left strong hypernatural transformation, preserving left homotopy Kan extensions in either variable.

This of course follows immediately from the properties of $\otimes_{\mathbf{G}}$ and M. We need only mention the dual,

$$\mathrm{Hom}^\cdot_{\mathbf{C}}: \underline{\Pi}^{\cdot *}(\mathbf{C}^{op} \times \mathbf{G}) \times T(\mathbf{C} \times \mathbf{D}) \to T(\mathbf{G} \times \mathbf{D})$$

for a pointed right homotopy theory T, with the expected properties.

We can of course recover $\otimes_{\mathbf{G}}$ from $\dot{\otimes}_{\mathbf{G}}$. It is easy to see that $W^+ \dot{\otimes}_{\mathbf{G}} X \approx W \otimes_{\mathbf{G}} X$; dually $\mathrm{Hom}^\cdot_{\mathbf{C}}(W^+,X) \approx \mathrm{Hom}_{\mathbf{C}}(W,X)$.

A number of observations remain to be made. We omit the arguments, which follow the lines already laid down above

PROPOSITION 2.9.

(i) $\otimes: \underline{\Pi}\mathbf{C} \times \underline{\Pi}\mathbf{D} \to \underline{\Pi}(\mathbf{C} \times \mathbf{D})$ is isomorphic to $\underline{\times}$.

(ii) $\dot{\otimes}: \underline{\Pi}^\cdot\mathbf{C} \times \underline{\Pi}^\cdot\mathbf{D} \to \underline{\Pi}^\cdot(\mathbf{C} \times \mathbf{D})$ is isomorphic to $\wedge\!\!\!\wedge$.

PROPOSITION 2.10. If V, W are, respectively, in $\underline{\Pi}(\mathbf{C} \times \mathbf{H}^{op})$, $\underline{\Pi}(\mathbf{H} \times \mathbf{G}^{op})$ and X is in $T(\mathbf{G})$, then $(V \otimes_{\mathbf{H}} W) \otimes_{\mathbf{G}} X \approx V \otimes_{\mathbf{H}} (W \otimes_{\mathbf{G}} X)$. The corresponding statement for the pointed case also holds.

3. TENSOR AND COTENSOR STRUCTURE IN HOMOTOPY THEORIES

If T is a homotopy theory, that is to say, is both a left and a right homotopy theory, then it has both a tensor and a cotensor structure over $\underline{\Pi}$. We investigate here the relation between them.

LEMMA 3.1. If T is a homotopy theory and X is in $\mathbf{T1}$, then, for any $\mathbf{C}$, $\gamma_{\mathbf{C}} \otimes X$: $\Gamma N\mathbf{C} \tilde{\otimes} X \to \mathbf{C} \tilde{\otimes} X$ is an isomorphism.

(For Γ, N, $\gamma_{\mathbf{C}}$ see (II 5) and recall that $\mathbf{C}\ \tilde{\otimes}- \approx (\text{T-colim}_{\mathbf{C}})(\text{T-const}_{\mathbf{C}})$.) We may begin by hoting that, for any $c \in \mathbf{C}_0$, $(c{\downarrow}\gamma_{\mathbf{C}}) = \Gamma N(c{\downarrow}\mathbf{C})$, with objects $(c \to u0 \to \cdots \to un)$: $[n+1] \to \mathbf{C}$ and morphisms ω^+: $(c \to u0 \to \cdots \to un) \to (c \to u\omega 0 \to \cdots \to u\omega m)$ for ω: $[m] \to [n]$. Moreover, the fibre $(\Gamma N\mathbf{C})_c$ of $\gamma_{\mathbf{C}}$ at c has objects $(u0 \to \cdots \to un)$: $[n] \to \mathbf{C}$ such that $u0 = c$ and morphisms ω: $(u0 \to \cdots \to un) \to (u\omega 0 \to \cdots \to u\omega m)$ such that $u0 \to u\omega 0$ is id_c. The canonical injection I: $(\Gamma N\mathbf{C})_c \to (c{\downarrow}\gamma_{\mathbf{C}})$ is given by $(u0 \to \cdots \to un) \mapsto (c \xrightarrow{\text{id}} u0 \to \cdots \to un)$, $\omega \mapsto \omega^+$. But in an obvious way $(c{\downarrow}\gamma_{\mathbf{C}})$ is included in $(\Gamma N\mathbf{C})_c$, the inclusion J being left adjoint to I with unit the degeneracy map s_0: $u = (u0 \to \cdots \to un) \to (c \xrightarrow{\text{id}} u0 \to \cdots \to un)$ and counit d_0^+: $(c \xrightarrow{\text{id}} u0 \to \cdots \to un) \to (u0 \to \cdots \to un)$. Thus $\gamma_{\mathbf{C}}$ is a fibration.

Since $(c{\downarrow}\mathbf{C})$ has initial object id_c, $N(c{\downarrow}\mathbf{C})$ is weakly contractible, so that $(\text{T-colim}_{(c{\downarrow}\gamma_{\mathbf{C}})})(\text{T-const}_{(c{\downarrow}\gamma_{\mathbf{C}})}) \approx N(c{\downarrow}\mathbf{C}) \otimes- \approx \text{id}_{\mathbf{T1}}$. Taking right adjoints, $(\text{T-lim}_{(c{\downarrow}\gamma_{\mathbf{C}})})(\text{T-const}_{(c{\downarrow}\gamma_{\mathbf{C}})}) \approx \text{id}_{\mathbf{T1}}$. Since $J \dashv I$ we have also $(\text{T-lim}_{(\Gamma N\mathbf{C})_c})(\text{T-const}_{(\Gamma N\mathbf{C})_c}) \approx \text{id}_{\mathbf{T1}}$. Finally, since $\gamma_{\mathbf{C}}$ is a fibration we have for any X in $\mathbf{TC}$

$$((R\gamma_{\mathbf{C}})(T\gamma_{\mathbf{C}})X)_c \approx (\text{T-lim}_{(\Gamma N\mathbf{C})_c})(\text{T-const}_{(\Gamma N\mathbf{C})_c}) \approx X_c,$$

or $(R\gamma_{\mathbf{C}})(T\gamma_{\mathbf{C}}) \approx \text{id}_{\mathbf{TC}}$. Taking left adjoints, $(L\gamma_{\mathbf{C}})(T\gamma_{\mathbf{C}}) \approx \text{id}_{\mathbf{TC}}$; (3.1) is an immediate consequence.

PROPOSITION 3.2. If f: $W \to V$ is a weak equivalence in $\text{Cat}^{\mathbf{C}\times\mathbf{G}^{op}}$, then, for any X in $\mathbf{TG}$, $f \tilde{\otimes}_{\mathbf{G}} X$: $W \tilde{\otimes}_{\mathbf{G}} X \to V \tilde{\otimes}_{\mathbf{G}} X$ is an isomorphism.

Without loss of generality, we may assume $\mathbf{C} = \mathbf{1}$. It is a consequence of (3.1) that $\Gamma NW \tilde{\otimes}_{\mathbf{G}} X \to W \tilde{\otimes}_{\mathbf{G}} X$ is an isomorphism: first, $\Gamma NW \tilde{\otimes}_{\mathbf{G}} (Lg)A \approx \Gamma NW_g \tilde{\otimes} A \approx W_g \tilde{\otimes} A \approx W \tilde{\otimes}_{\mathbf{G}} (Lg)A$ for g: $\mathbf{1} \to \mathbf{G}$, A in $\mathbf{T1}$, the general case following from the density theorem.

For $W \to V$ a weak equivalence means that $NW \to NV$ is a weak equivalence. Thus by (2.5) $NW \tilde{\otimes}_{\mathbf{G}} X = \Gamma NW \tilde{\otimes}_{\mathbf{G}} X \to \Gamma NV \tilde{\otimes}_{\mathbf{G}} X$ is an isomorphism; (3.2) follows.

We may remark at this point that we might have defined the tensor product $\text{Ho}(\text{Cat}^{\mathbf{C}\times\mathbf{G}^{op}}) \times \mathbf{TG} \to \mathbf{TC}$ as $\tilde{\otimes}_{\mathbf{G}}$, for T a homotopy theory. We should indeed mention a special case. Suppose W: $\mathbf{C} \times \mathbf{G}^{op} \to$ Sets. Thinking of Sets as a subcategory of Cat, W is an object of Ho($\mathrm{Cat}^{\mathbf{C}\times\mathbf{G}^{op}}$); identifying sets with discrete simplicial sets, W becomes an object of $\underline{\Pi}(\mathbf{C} \times \mathbf{G}^{op})$.

PROPOSITION 3.3. If W: $\mathbf{C} \times \mathbf{G}^{op} \to$ Sets, then $(W \mathbin{\tilde{\otimes}_{\mathbf{G}}} -) \approx (W \otimes_{\mathbf{G}} -)$: $T\mathbf{G} \to T\mathbf{C}$.

If W: $\mathbf{C} \times \mathbf{G}^{op} \to$ Cat we may write ${}^{op}W$ for the functor $({}^{op}W)_{c,g} = (W_{c,g})^{op}$.

PROPOSITION 3.4. $(W \otimes_{\mathbf{G}} -) \approx ({}^{op}W \otimes_{\mathbf{G}} -)$: $T\mathbf{G} \to T\mathbf{C}$.

For $\Gamma NW \approx \Gamma N({}^{op}W)$ by (II.5.7).

PROPOSITION 3.5. For W in $\underline{\Pi}(\mathbf{C} \times \mathbf{G}^{op})$, $(W \otimes_{\mathbf{G}} -) \dashv \mathrm{Hom}_{\mathbf{C}}(W,-)$.

For $(W \otimes_{\mathbf{G}} -) = (\Gamma W \mathbin{\tilde{\otimes}_{\mathbf{G}}} -) \approx (\Gamma N \Gamma W \mathbin{\tilde{\otimes}_{\mathbf{G}}} -) \approx ({}^{op}\Gamma W \mathbin{\tilde{\otimes}_{\mathbf{G}}} -)$. But from (2.7) we can write down immediately the left adjoint of $\mathrm{Hom}_{\mathbf{C}}(W,-)$ as the composition

$$T\mathbf{C} \xrightarrow{TP''} T(\mathbf{C} \ltimes {}^{op}\Gamma W \rtimes \mathbf{G}) \xrightarrow{LP'} T\mathbf{C},$$

which is to say, ${}^{op}\Gamma W \mathbin{\tilde{\otimes}_{\mathbf{G}}} -$.

The argument shows in fact that the adjunction is natural in W. If $f: W \to W'$ then for any X in $T\mathbf{G}$ the transpose of $(f \otimes_{\mathbf{G}} -)$: $(W \otimes_{\mathbf{G}} -) \to (W' \otimes_{\mathbf{G}} -)$ is just $\mathrm{Hom}_{\mathbf{C}}(f,-)$: $\mathrm{Hom}_{\mathbf{C}}(W',-) \to \mathrm{Hom}_{\mathbf{C}}(W,-)$.

If T is a pointed homotopy theory then in analogous fashion if W is in $\underline{\Pi}^{\cdot}(\mathbf{C} \times \mathbf{G}^{op})$ then $(W \mathbin{\dot{\otimes}_{\mathbf{G}}} -) \dashv \mathrm{Hom}^{\cdot}_{\mathbf{C}}(W,-)$.

It is a commonplace that the categories $\mathrm{Ho}K = \underline{\Pi}\mathbf{1}$ and $\mathrm{Ho}(K^{\cdot}) = \underline{\Pi}^{\cdot}\mathbf{1}$ are, respectively, cartesian-closed and monoidal-closed. Proposition 3.5, of course, specialized to $T = \underline{\Pi}$, gives a new proof of this fact as well as of a substantial generalization.

4. TENSOR STRUCTURE AND HOMOTOPY LIMITS

The tensor and cotensor structures over $\underline{\Pi}$ of a homotopy theory T were defined in terms of the homotopy Kan extensions. We assemble here some observations which move in the opposite direction, giving information about the latter in terms of the former. First of all is the "generalized homotopical Yoneda principle".

If $F: \mathbf{C} \to \mathbf{D}$ we may regard the functors $\mathbf{D}(F-,-)$: $\mathbf{C}^{op} \times \mathbf{D} \to$ Sets and $\mathbf{D}(-,F-)$: $\mathbf{D}^{op} \times \mathbf{C} \to$ Sets as objects $\mathbf{D}^F$ of $\underline{\Pi}(\mathbf{D} \times \mathbf{C}^{op})$ and $\mathbf{D}_F$ of $\underline{\Pi}(\mathbf{C} \times \mathbf{D}^{op})$.

PROPOSITION 4.1. If T is a homotopy theory and $F: \mathbf{C} \to \mathbf{D}$, then $LF \approx (\mathbf{D}^F \otimes_{\mathbf{C}} -)$: $T\mathbf{C} \to T\mathbf{D}$ and $RF \approx \mathrm{Hom}_{\mathbf{C}}(\mathbf{D}_F,-)$: $\Gamma\mathbf{C} \to T\mathbf{D}$.

We need only prove the first assertion, the second being the dual under $T \mapsto T^*$. By 3.3, $\mathbf{D}^F \otimes_{\mathbf{C}} -$ is up to isomorphism, the composition

$$\mathbf{T}\mathbf{C} \xrightarrow{\mathbf{T}P''} \mathbf{T}(\mathbf{D} \ltimes \mathbf{D}^F \rtimes \mathbf{C}) \xrightarrow{\mathbf{L}P'} \mathbf{T}\mathbf{D}.$$

But $\mathbf{D} \ltimes \mathbf{D}^F \rtimes \mathbf{C} = (F{\downarrow}\mathbf{D})$ so that its objects are morphisms $\phi\colon Fc \to d$ while its morphisms are commutative diagrams

$$\begin{array}{ccc} Fc & \xrightarrow{\phi} & d \\ {\scriptstyle F\gamma}\downarrow & & \downarrow{\scriptstyle \delta} \\ Fc' & \xrightarrow{\phi'} & d' \end{array}$$

in $\mathbf{D}$. Thus $c \mapsto \mathrm{id}_{Fc}$ defines a left adjoint S'' of P'' and $\mathbf{T}P'' \approx \mathbf{L}S''$, $(\mathbf{D}^F \otimes_{\mathbf{C}} -) \approx \mathbf{L}(P'S'')$. But $P'S'' = F$.

The "homotopical Yoneda principle" is the case $\mathbf{C} = \mathbf{D}$, $F = \mathrm{id}_{\mathbf{C}}$. The special case $\mathbf{D} = \mathbf{1}$, on the other hand, gives the following computations.

COROLLARY 4.2. $\mathbf{T}\text{-colim}_{\mathbf{C}} \approx * \otimes_{\mathbf{C}} -$, $\mathbf{T}\text{-lim}_{\mathbf{C}} \approx \mathrm{Hom}_{\mathbf{C}}(*,-)$.

For constant objects in $\mathbf{TC}$ we have an even more perspicuous computation. If A is in $\mathbf{T1}$ then $(\mathbf{T}\text{-const}_{\mathbf{C}})A \approx * \otimes (\mathbf{T}\text{-const}_{\mathbf{C}})A \approx (\underline{\Pi}\text{-const}_{\mathbf{C}*}) \otimes A$. But $- \otimes A$ preserves left homotopy Kan extensions. Referring to (II.6.6) we get the following results.

PROPOSITION 4.3. If A is in $\mathbf{T1}$ then $(\mathbf{T}\text{-colim}_{\mathbf{C}})(\mathbf{T}\text{-const}_{\mathbf{C}})A \approx N\mathbf{C} \otimes A$ and $(\mathbf{T}\text{-lim}_{\mathbf{C}})(\mathbf{T}\text{-const}_{\mathbf{C}})A \approx \mathrm{Hom}(N\mathbf{C},A)$.

In particular if $\mathbf{C}$ is weakly contractible, which means $N\mathbf{C} \approx *$ in $\underline{\Pi}\mathbf{1}$, this asserts that the homotopy limit and colimit over $\mathbf{C}$ of a constant is just the constant. We may use this fact to generalize the finality statement (III.3.1). Let us say that a functor $F\colon \mathbf{C} \to \mathbf{D}$ is <u>homotopically final</u> if, for any $d \in \mathbf{D}_0$, $(d{\downarrow}F)$ is weakly contractible. The dual notion is that of a <u>homotopically initial functor</u>. (We avoid as confusing the word "cofinal".)

PROPOSITION 4.4. If $F\colon \mathbf{C} \to \mathbf{D}$ is homotopically final, then $(\mathbf{T}\text{-colim}_{\mathbf{C}})(\mathbf{T}F) \approx \mathbf{T}\text{-colim}_{\mathbf{D}}$. Dually, if F is homotopically initial, then $(\mathbf{T}\text{-lim}_{\mathbf{C}})(\mathbf{T}F) \approx \mathbf{T}\text{-lim}_{\mathbf{D}}$.

We need only look at the adjoints. From (III.2.4) it follows that if F is homotopically final then $(\mathbf{R}F)(\mathbf{T}\text{-const}_{\mathbf{C}}) \approx \mathbf{T}\text{-const}_{\mathbf{D}}$.

If F is further a fibration or an opfibration, there is a stronger result. Note that a fibration $F\colon \mathbf{C} \to \mathbf{D}$ is final if and only if each $\mathbf{C}_d$ is weakly contractible.

PROPOSITION 4.5. If $F\colon \mathbf{C} \to \mathbf{D}$ is either a final fibration or an initial opfibration, then $(\mathbf{L}F)(\mathbf{T}F) \approx (\mathbf{R}F)(\mathbf{T}F) \approx \mathrm{id}_{\mathbf{TD}}$.

5. REGULAR HOMOTOPY THEORIES

Certain categories, most notably Sets and categories of algebras over sets, more generally topoi, have the property that finite limits commute with directed colimits: in abelian categories this is Grothendieck's axiom AB5. We shall not address the question of what is the correct analogue of this condition for a homotopy theory (as, cf. §4, homotopical finality is the analogue of finality), but shall rather discuss here a weaker property of the same character.

A homotopy theory T is <u>regular</u> if

(R1) $T\text{-colim}_{\mathbf{N}}\colon T\mathbf{N} \to T\mathbf{1}$ preserves finite products.

(R2) The square

$$\begin{array}{ccc} T(\underline{\Lambda}^{op} \times \mathbf{N}) & \xrightarrow{\ Lpr_{\underline{\Lambda}^{op}}\ } & T(\underline{\Lambda}^{op}) \\ {\scriptstyle Rpr_{\mathbf{N}}}\downarrow & & \downarrow{\scriptstyle T\text{-lim}_{\underline{\Lambda}^{op}}} \\ T\mathbf{N} & \xrightarrow[\ T\text{-colim}_{\mathbf{N}}\ ]{} & T\mathbf{1} \end{array}$$

commutes up to isomorphism or rather, to be pedantic, the squares to which it is left and right adjoint have the Beck-Chevalley property.

Thus this is not always the case as attested, for example by the representable homotopy theory $\mathbf{C} \mapsto (Ab^{op})^{\mathbf{C}}$. The standard example is, however, regular.

PROPOSITION 5.1. $\underline{\Pi}$ is a regular homotopy theory.

It is easy to check that a sequence $X\colon \mathbf{N} \to K$ such that each $X_n \to X_{n+1}$ is injective is strongly cofibrant in $K^{\mathbf{N}}$, so that $\text{colim}_{\mathbf{N}}X$ represents $\underline{\Pi}\text{-colim}_{\mathbf{N}}X$. If X and Y both have this property, then $X \times Y$ shares it as well, and $\text{colim}_{\mathbf{N}}(X \times Y) \approx (\text{colim}_{\mathbf{N}}X) \times (\text{colim}_{\mathbf{N}}Y)$, which proves R1.

Now any functor $\underline{\Lambda}^{op} \times \mathbf{N} \to K$ can be seen to be weakly equivalent to an X such that, for each n, $(X_{01n} \twoheadrightarrow X_{11n} \twoheadleftarrow X_{10n})$ is strongly fibrant, which is to say, both maps are fibrations in K, and for each i, j the sequence $(X_{ij0} \to X_{ij1} \to X_{ij2} \to \cdots)$ is strongly cofibrant. If Y_n is the limit, i.e. the pullback of $(X_{01n} \to X_{11n} \to X_{10n})$, then $Y_n \approx T\text{-lim}_{\underline{\Lambda}^{op}}(X_{01n} \to X_{11n} \leftarrow X_{10n})$ and the sequence $(Y_0 \to Y_1 \to \cdots)$ is again strongly cofibrant. Furthermore, in virtue of (III.2.8), $\text{colim}_{\mathbf{N}}X \approx (Lpr_{\underline{\Lambda}^{op}})X$. But also $\text{colim}_{\mathbf{N}}X$ is strongly fibrant in $K^{\underline{\Lambda}^{op}}$; this follows easily from Kan's criterion for fibrancy. Thus

$$(T\text{-colim}_{\mathbf{N}})(Rpr_{\mathbf{N}})X \approx \text{colim}_{\mathbf{N}}Y \approx \lim_{\underline{\Lambda}^{op}}(\text{colim}_{\mathbf{N}}X \approx (T\text{-lim}_{\underline{\Lambda}^{op}})(T\text{-colim}_{\mathbf{N}}).$$

Using, once more, (III.28) we see that our rather conservative definition of regularity has more liberal consequences.

PROPOSITION 5.2. If T is a regular homotopy theory, then $T^{\cdot}$ is regular. Also, for any $\mathbf{C}$, $T[\mathbf{C}]$ is regular.

In particular, $\underline{\Pi}^{\cdot}$ is regular.

We shall want below an internalized version of regularity which is a consequence of the definition.

PROPOSITION 5.3. If T is a regular homotopy theory and K is a finite simplicial set, then

$$\begin{array}{ccc} TN & \xrightarrow{\mathrm{Hom}(K,-)} & TN \\ {\scriptstyle T\text{-colim}_N}\downarrow & & \downarrow{\scriptstyle T\text{-colim}_N} \\ T\mathbf{1} & \xrightarrow{\mathrm{Hom}(K,-)} & T\mathbf{1} \end{array}$$

commutes up to isomorphism.

A finite simplicial set K, i.e. a simplicial set with finitely many nondegenerate simplices, is of course equal for some n to its n-skeleton K^n, and each skeleton is obtained from the previous one by a pushout

$$\begin{array}{ccc} \coprod \underline{\Delta}(-,[n])^{n-1} & \longrightarrow & \coprod \underline{\Delta}(-,[n]) \\ \downarrow & & \downarrow \\ K^{n-1} & \longrightarrow & K^n \end{array}$$

where the coproducts, indexed by the nondegenerate simplices, are finite. The inductive hypothesis, that the asserted commutativity holds for finite simplicial sets of lower dimension, together with R1, tells us that $\mathrm{Hom}(K^{n-1},-)$ and $\mathrm{Hom}(\coprod \underline{\Delta}(-,[n])^{n-1},-)$ commute with $T\text{-colim}_N$. Since $\underline{\Delta}(-,[n])$ is weakly contractible, $\mathrm{Hom}(\underline{\Delta}(-,[n]),-)$ is isomorphic to the identity. Thus, by R2, $\mathrm{Hom}(K^n,-)$ commutes with $T\text{-colim}_N$.

An easy generalization of this is also useful. Let us say that an object of $\underline{\Pi}G$ is <u>finitary</u> if it is contained in the smallest class containing all $(Lg)K$, $g\colon \mathbf{1} \to G$, K finite and closed under homotopy pushouts (and hence under finite coproducts).

COROLLARY 5.4. If W, in TG, is finitary and T is regular, then $\mathrm{Hom}_G(W,-)$ commutes with $T\text{-colim}_N$.

We need hardly state that when T is a pointed homotopy theory, the statements analogous to 5.3,4 involving $\mathrm{Hom}^{\cdot}$ instead of Hom are true as well.

CHAPTER V

LOCALIZATION

The notion of localization in a category was recalled in (I§1). It adapts with little change to a homotopy theory. This is the subject matter of §1, which concludes with the theorem that a localization of a homotopy theory is once more a homotopy theory. In §2 we present a short list of localizations with a few indications of how the localization theorem, i.e. the existence of the appropriate adjoint functors, may be proved.

The remainder of the chapter is devoted to general localization theorems, as well as E. H. Brown's representability theorem, which is properly speaking out of context here, but patently in the same spirit as the localization theorem of Bousfield type which it accompanies.

1. LOCALIZATION IN A HOMOTOPY THEORY

Localization theory in a category C begins with Galois correspondences inj, inj* between full subcategories of C^2 and C. In a homotopy theory T we simply start the same way, using the category **T1** for C. Our aim however is to understand the relation between the homotopy completeness of **T1** witnessed by its occurrence within T and the localization theory of **T1**. For this purpose it is useful to make the following conventions. Clearly it is no significant restriction to confine ourselves to full replete subcategories of $(\mathbf{T1})^2$. But by H1,2 these correspond bijectively, via dgm_2, to full replete subcategories of **T2** and thus, as in I§3, to full replete maximal subhyperfunctors of T[**2**]. We shall accordingly, when discussing localization in a homotopy theory T, regard the operations inj, inj*, loc, loc* as operating on full replete maximal subhyperfunctors of T[**2**] and T, but recall that these are specified by subcategories of **T2**, or even $(\mathbf{T1})^2$, and **T1**, so that the definitions of inj , ... remain those of (I§1).

We may even carry our conflation one step further and recall that a full replete subcategory of $(\mathbf{T1})^2$ corresponds to a class of morphisms in **T1**. This permits us to state informally an observation which has indeed nothing to do with homotopy theory, but might have been made in (I§1).

PROPOSITION 1.1. Suppose $D \subset T$ and f, g are composable morphisms in **T1**. Then

(i) if any two of f, g, fg are in loc*D, so also is the third;

(ii) if f, g are in inj*D, then so is fg;

(iii) if fg is in inj*D, then so is g.

To this we may immediately add a closure property which does depend on T being a homotopy theory (or at least a left homotopy theory).

PROPOSITION 1.2. Any inj*D is closed under right-infinite composition.

For a sequence $X_0 \xrightarrow{x_0} X_1 \xrightarrow{x_1} \cdots$ in **T1** has a homotopy colimit, say $\overline{X}$. If each x_i is in inj*D, then so also is their right-infinite composition $X_0 \to \overline{X}$.

Some additional closure properties are immediately to hand.

PROPOSITION 1.3.

(i) All inj S and loc S, for $S \subset T[\mathbf{2}]$, are closed under products.

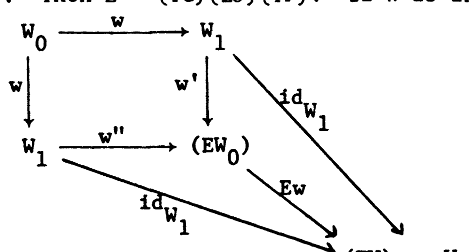
(ii) All inj*D and loc*D, where $D \subset T$, are closed under coproducts.

(iii) If, in a homotopy pushout in **T1**, one of the initial morphisms is in inj*D, then so is the one opposite it.

We shall (obviously) say that inj*D is closed under homotopy pushouts.

To proceed farther, it is useful to introduce, in a homotopy theory T, the <u>generalized</u> <u>Einhängung</u> E: **T2** → **T2**. For this purpose, we introduce (briefly) the category **E** consisting of **2** × **2** with a terminal element $\top$ adjoined, so that $(1,1) < \top$. Let **E'** be the subcategory which omits (1,1) and J: **E'** → **E** the inclusion and define F: **E'** → **2** by F(0,0) = 0, F(1,0) = F(0,1) = $F\top$ = 1, G: **2** → **E** by G0 = (1,1), G1 = $\top$. Then E = (TG)(LJ)(TF). If W is in **T2** then (LF)(TF)W has the diagram

(1.4)



in which the square is a homotopy pushout. If $W_1 = *$ then $(EW)_0$ is just the classical Einhängung or unreduced suspension.

The form of this construction makes it clear that E is in fact a left strong hypernatural transformation $T[\mathbf{2}] \to T[\mathbf{2}]$ preserving left homotopy Kan extensions. Its point is the following lemma.

LEMMA 1.5. If $D \subset T$ then W is in loc*D if and only if both W and EW are in inj*D.

The utility of E depends on yet another property.

LEMMA 1.6. Consider the diagrams in **T1**,

$$\begin{array}{ccc} W_0 & \xrightarrow{w} & W_1 \\ \downarrow & & \downarrow \\ X_0 & \xrightarrow{x} & X_1 \end{array} \quad , \quad \begin{array}{ccc} (EW)_0 & \xrightarrow{Ew} & W_1 \\ \downarrow & & \downarrow \\ (EX)_0 & \xrightarrow{Ex} & X_1 \end{array} .$$

If the left-hand one is a homotopy pushout, so also is the right-hand one.

This follows from (III.1.3), which shows that the left-hand square in

$$\begin{array}{ccccc} W_0 & \longrightarrow & (EW)_0 & \xrightarrow{Ew} & W_1 \\ \downarrow & & \downarrow & & \downarrow \\ X_0 & \longrightarrow & (EX)_0 & \longrightarrow & X_1 \end{array}$$

is a homotopy pushout.

Let us say that a full replete maximal subhyperfunctor $S \subset T[\mathbf{2}]$ is <u>stable</u> if it is closed under E (equivalently, $S\mathbf{1}$ is closed under E). We shall also say that $D \subset T$ is <u>stable</u> if loc*D is stable.

LEMMA 1.7.

(i) $D \subset T$ is stable if and only if loc*D is closed under homotopy pushouts.

(ii) If $S \subset T[\mathbf{2}]$ is stable, then loc S is stable and loc S = inj S.

This is an immediate consequence of (1.5,6).

LEMMA 1.8. If $D \subset T$ is stable then loc*D is closed under right-infinite composition.

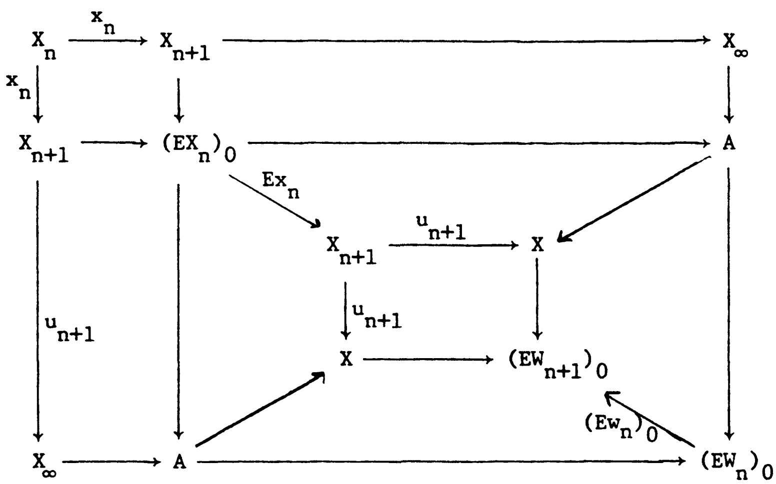
This says that loc*D shares the property of inj*D asserted in (1.2). But loc*D is much smaller, and the closure-property correspondingly deeper.

Suppose X, in **TN**, has the diagram $X_0 \xrightarrow{x_0} X_1 \xrightarrow{x_1} X_2 \to \cdots$ with each x_i in loc*D. We may construct a W in $T(\mathbf{2} \times \mathbf{N})$ such that $\mathrm{dgm}_{\mathbf{N}} W = (W_0 \xrightarrow{w_0} W_1 \to \cdots)$ and

$$\mathrm{dgm}_{\mathbf{2}\times\mathbf{N}} W = \left(\begin{array}{ccccc} X_0 & \xrightarrow{x_0} & X_1 & \longrightarrow & \cdots \\ \downarrow u_0 & & \downarrow u_1 & & \\ X_\infty & \xrightarrow{id} & X_\infty & \longrightarrow & \cdots \end{array} \right)$$

where $X_\infty = T\text{-}\mathrm{colim}_{\mathbf{N}} X$ and the $\{u_i\}$ constitute the corresponding colimit cone, so that u_0 is the right-infinite composition of the x_i. By (1.1) W_0 is in inj*D so that it is sufficient to show that EW_0 is also in inj*D.

But $E(W_0)$ is the right-infinite composition of $(EW_0)_0 \xrightarrow{(EW_0)_0} (EW_1)_0 \xrightarrow{(EW_1)_0} (EW_2)_0 \to \cdots$. Thus we need only show that each $(EW_n)_0$ is in inj*D. This follows from 1.7, applied to the diagram



in which all the squares are homotopy pushouts.

LEMMA 1.9. If $S \subset T[2]$ is stable, $\mathcal{F}$ is a finite free category and X is in $(\text{loc } S)\mathcal{F}$, then $T\text{-lim}_{\mathcal{F}}X$ is in $(\text{loc } S)\mathbf{1}$.

We must show that if W is in $S\mathbf{1}$ then $T\mathbf{1}(W_1, T\text{-lim}_{\mathcal{F}}X) \to T\mathbf{1}(W_0, T\text{-lim}_{\mathcal{F}}X)$ is surjective, By adjointness this is the same as showing that $T\mathcal{F}(T\text{-const}_{\mathcal{F}}W_1, X) \to T\mathcal{F}(T\text{-const}_{\mathcal{F}}X_0, X)$ is surjective.

Given f, let us construct the homotopy pushout

$$\begin{array}{ccc} T\text{-const } W_0 & \xrightarrow{f} & X \\ \downarrow & & \downarrow x \\ T\text{-const } W_1 & \longrightarrow & X' \end{array} .$$

By (1.7) each x_c is in loc*loc S. Thus there is a unique morphism $\text{dgm}_{\mathcal{F}}X' \to \text{dgm}_{\mathcal{F}}X$, which we may write as $\text{dgm}_{\mathcal{F}}r$, such that $(\text{dgm}_{\mathcal{F}}r)(\text{dgm}_{\mathcal{F}}x) = 1$. But then rx is an isomorphism and $(rx)^{-1}rf' \mapsto f$.

In particular, loc S is closed under homotopy pullbacks. The stronger statement follows.

PROPOSITION 1.10. If $S \subset T[2]$ is stable, then $\text{loc } S \subset T$ is closed under right homotopy Kan extensions.

It is sufficient to show that if X is in (loc S)**C**, then T-$\lim_{\mathbf{C}}$X is in (loc S)**1**. Let $\cdots \to X_2 \to X_1 \to X_0 = *$ be the right Steenrod factorization of X (cf. III§4). Then there are homotopy pullbacks

$$\begin{array}{ccc} X_{n+1} & \longrightarrow & X_n \\ \downarrow & & \downarrow \\ \prod_c (Rc)X_c & \longrightarrow & \prod_c (Rc)X_{n,c} . \end{array}$$

so that all X_n are in (loc S)**F** and all $X_{n,c}$ in (loc S)**1**. But (T-$\lim_{\mathbf{C}}$)(Rc)A = A. Thus, inductively, for all n, T-$\lim_{\mathbf{C}}X^n$ is in (loc S)**1**. Using the dual of (III.3.5) we conclude that this is the case for X, the homotopy limit of $\cdots \to X_1 \to X_0$, as well.

Let us summarize what we know about the hyperfunctor loc S.

PROPOSITION 1.11. If S ⊂ T[2] is stable then loc S ⊂ T satisfies axioms H0-2, H3R and H4L.

H0-2 are true just because loc S is a full replete subhyperfunctor of T, while H3R is an immediate consequence of 1.10 and H4L simply a consequence of the fact that loc S ⊂ T preserves products.

We shall say that T <u>admits a localization with respect to a stable</u> S ⊂ T[2] if each inclusion J**C**: (loc S)**C** → T**C** has a left adjoint $\mathrm{Loc}_S\mathbf{C}$. This looks stronger than the condition of (I§1), which would ask for an adjoint only for **C** = **1**. But notice that for a representable homotopy theory $\mathbf{C} \mapsto C^{\mathbf{C}}$ with C complete and cocomplete the two conditions coincide.

THEOREM 1.12. If S ⊂ T[2] is a stable subhyperfunctor and T admits a localization with respect to S then loc S is a homotopy theory. The inclusion S ⊂ T is a strict hypernatural transformation which preserves right homotopy Kan extensions and the localizing functor Loc_S yields a left strong hypernatural transformation which preserves left homotopy Kan extensions.

All that is left to do is to show that loc S satisfies H3L. But if **F**: **C** → **D** then (Loc_S**D**)(L**F**)(J**C**) is left adjoint to (loc S)**F**, the restriction of T**F**, for if X is in (loc S)**C** and Y is in (loc S)**D** then

$$\begin{aligned} ((\mathrm{loc}\ S)\mathbf{D})((\mathrm{loc}_S\mathbf{D})(L\mathbf{F})(J\mathbf{C})X,Y) &\approx (T\mathbf{D})((L\mathbf{F})(J\mathbf{C})X,(J\mathbf{D})Y) \\ &\approx (T\mathbf{C})((J\mathbf{C})X,(T\mathbf{F})(J\mathbf{D})Y) \\ &\approx (T\mathbf{C})((J\mathbf{C})X,(J\mathbf{C}((\mathrm{loc}\ S)\mathbf{F})Y) \\ &\approx ((\mathrm{loc}\ S)\mathbf{C})(X,((\mathrm{loc}\ S)\mathbf{F})Y) . \end{aligned}$$

The notions of colocality and colocalization in a homotopy theory T are defined in strictly dual fashion: they are just locality and localization in T*. They have of course all the dual properties; in particular a colocalization of a homotopy theory is again a homotopy theory. We need not spell out the details. Some examples appear immediately below.

2. EXAMPLES OF LOCALIZATIONS AND COLOCALIZATIONS

Here is a list of some more-or-less well-known localizations and colocalizations or, to be more precise, of well-known things which are one or the other: it is not perhaps equally well-known that they are localizations or that they constitute homotopy theories. In each case we give the subhyperfunctor with respect to which they are (co-) localizations and in most cases we make some remarks on the proof of the localization theorem which assures the existence of the (co-) localizing functor.

(2.1) For any homotopy theory T, $T^{\cdot}$ is the localization of T[**2**] relative to the subhyperfunctor $S \subset T[\mathbf{2} \times \mathbf{2}]$ generated by the image of

$$T[\mathbf{2}]\mathbf{1} \xrightarrow{T[\mathbf{2}]\text{-const}_{\mathbf{2}}} T[\mathbf{2}]\mathbf{2}.$$

In other words the objects of S**1** have diagrams

$$\begin{array}{ccc} T\text{-const}_{\mathbf{2}}V & \longrightarrow & T\text{-const}_{\mathbf{2}}W \\ \| & & \| \\ (L0)V & & (L0)W \end{array}$$

Thus S is clearly stable. Since it contains objects of the type just described with $V = \emptyset$, if X is in (loc S)**1** $\subset$ T[**2**]**1** = T**2**, then $(T\mathbf{2})(T\text{-const}_{\mathbf{2}}W,X) = (T\mathbf{1})(W,X_0) \approx (T\mathbf{2})(\emptyset,X)$. Thus X_0 is terminal. The localizing functor is of course just the mapping cone functor of (III§5).

(2.2) In $\underline{\Pi}^{\cdot}$ let S be generated by the singleton $\{* \to S^0\}$ where $S^0 = *^+$ in $\underline{\Pi}^{\cdot}\mathbf{1}$ is the 0-sphere. Then S is costable: the homotopy pullback of $* \to S^0$ with itself is just $*$. It is evident that (coloc S)**1**, the colocal objects in $\underline{\Pi}^{\cdot}\mathbf{1}$, consists of the pointed connected simplicial sets. The colocalizing functor here is just the "component of the basepoint". The homotopy theory of connected pointed simplicial sets thus constructed we denote by $\underline{\Pi}_0$.

(2.3) In $\underline{\Pi}_0$ let S be generated by the maps $* \to K(G,1)$ where G ranges over all groups. Since the homotopy pullback in $\underline{\Pi}_0$ of $* \to K(G,\mathbf{1})$ with itself is once more just $*$, this S is costable. The colocal objects are the pointed simply-connected simplicial sets. The colocalizing functor is the universal covering space. The colocal homotopy theory is $\underline{\Pi}_1$.

(2.4). As a variant of (2.3) we might colocalize with respect to the $* \to K(A,1)$ with A abelian. The colocal objects are the ones with perfect fundamental groups and the colocalizing functor is the connected covering corresponding to the largest perfect subgroup of the fundamental group.

(2.5) In $\underline{\Pi}_1$ take S to be generated by the set $\{* \to K(\mathbb{Q}/\mathbb{Z},2), \ldots, * \to K(\mathbb{Q}/\mathbb{Z},n)\}$. This is costable because the loop-space of $K(\mathbb{Q}/\mathbb{Z},q)$ is $K(\mathbb{Q}/\mathbb{Z},q-1)$. The colocal objects are the simply connected simplicial sets with $H^q(X,\mathbb{Q}/\mathbb{Z}) = 0$, $q = 2,\ldots,n$, thus the n-connected simplicial sets. For the colocalizing functor see (2.7) below. The corresponding homotopy theory is $\underline{\Pi}_n$.

(2.6) In $\underline{\Pi}_0$, let S be generated by $\{S^k \to * \mid k > n\}$; its stability is clear. The local objects are the pointed connected simplicial sets whose homotopy vanishes in degrees greater than n. A localizing functor may be constructed by taking a fibrant simplicial set to the image of its canonical map into its n^{th} coskeleton: this defines a functor which extends to all the functor categories. The corresponding homotopy theory is $\underline{\Pi}_0^n$; the localizing functor $b_n: \underline{\Pi}_0\mathbf{1} \to \underline{\Pi}_0^n\mathbf{1}$ is usually called the "n^{th} Postnikov base". Indeed $\underline{\Pi}_0^n$ is, for each n, a localization of $\underline{\Pi}_0^{n+1}$; the units of the adjunctions involved constitute the Postnikov system.

(2.7) The existence of the colocalizing functor in (2.5) may be extracted from that of the localizing functor in (2.6). The method of construction described there actually gives functors $\hat{b}_n: \underline{\Pi}_0\mathbf{C} \to \underline{\Pi}_0(\mathbf{2} \times \mathbf{C})$ such that $\mathrm{dgm}_{\mathbf{Z}}(\hat{b}_n X) = (X \to b_n X)$. The colocalizing functor is then the composition of $\hat{b}_n$ with the mapping fibre functor, the dual of the mapping cone.

(2.8) In the same spirit we can construct the homotopy theory $\underline{\Pi}_m^n$ of m-connected, (n+1)-coconnected simplicial sets which is equally well a localization of $\underline{\Pi}_m$ or a colocalization of $\underline{\Pi}_0^n$. In particular, $\underline{\Pi}_0^1$ is equivalent to the representable homotopy theory of the category of groups, while $\underline{\Pi}_n^{n+1}$ is equivalent to the representable homotopy theory of the category of abelian groups.

(2.9) In $\underline{\Pi}$ suppose h is a homology theory satisfying the Eilenberg-Steenrod axioms, possibly without the dimension axiom, and Milnor's infinite-additivity axiom. Let S be generated by the class of all maps f such that hf is an isomorphism. Its stability is an easy consequence of the properties of homology theories. The local objects are just the h-local objects in the sense of Adams [1], following Sullivan [26]. Bousfield [3] has shown the existence of a left adjoint to $(\mathrm{loc}\ S)\mathbf{1} \subset \underline{\Pi}\mathbf{1}$. His argument shows equally well the existence of adjoints for all $\mathbf{C}$, so that the h-local simplicial sets once more constitute a homotopy theory.

3. COMPACT OBJECTS, THE LITTLE BOUSFIELD LEMMA AND BROWN'S THEOREM

If T is a left homotopy theory an object W of $T\mathbf{1}$ is compact if for any sequence $X_0 \to X_1 \to \cdots$ in $T\mathbf{1}$ with homotopy colimit, say, $\overline{X}$ the canonical map

$$\mathrm{colim}_{\mathbf{N}} T\mathbf{1}(W, X_n) \longrightarrow T\mathbf{1}(W, \overline{X})$$

is bijective. An object of $T\mathbf{C}$ is compact just when it is compact as an object of $T[\mathbf{C}]\mathbf{1}$.

LEMMA 3.1. If W is compact in $T\mathbf{1}$ and $c\colon \mathbf{1} \to \mathbf{C}$ then $(Lc)W$ is compact in $T\mathbf{C}$.

Thus in $\underline{\Pi}\mathbf{1}$ all finite simplicial sets are compact; in general finitary objects of $\underline{\Pi}\mathbf{C}$ are compact (IV 5).

The following lemma, due in spirit to Bousfield [3] allows us to prove in some cases the existence of localizing functors.

LEMMA 3.2. Let T be a homotopy theory and let $S \subset T[\mathbf{2}]$ be a stable sub-hyperfunctor such that $S\mathbf{1}$ is a set with the property that, for every W in $S\mathbf{1}$, W_0 is compact. Then T admits a localization with respect to S.

In view of (1.6) and (3.1) it will be sufficient to show that $(\mathrm{loc}\ S)\mathbf{1} \subset T\mathbf{1}$ has a left adjoint. But also, $\mathrm{loc}\ S = \mathrm{inj}\ S$.

Now suppose X is in $T\mathbf{1}$. We define an effacement X' of X with respect to S by means of a homotopy pushout

$$\begin{array}{ccc} \coprod W_0 & \xrightarrow{\ \coprod w\ } & \coprod W_1 \\ \downarrow & & \downarrow \\ X & \xrightarrow{\ \xi\ } & X' \end{array}$$

where the coproducts are over the set of pairs (W,f), W in $S\mathbf{1}$, $f\colon W_0 \to X$.

If $f\colon W_0 \to X$, then there is an $f'\colon W_1 \to X'$ with $f'w = \xi f$. Moreover, if Y is in $(\mathrm{loc}\ S)\mathbf{1}$, then any $g\colon X \to Y$ extends uniquely to $g'\colon X' \to Y$, i.e. ξ is in $(\mathrm{loc}^*\mathrm{loc}\ S)\mathbf{1}$.

Using this construction we build a sequence $X = X_0 \to X_1 \to X_2 \to \cdots$ with $X_{n+1} = X_n'$. Since each W_0, for W in $S\mathbf{1}$, is compact, $T\text{-}\mathrm{colim}_{\mathbf{N}} X = \overline{X}$ is injective with respect to $S\mathbf{1}$. But, by 1.8, $X_0 \to \overline{X}$ is in $(\mathrm{loc}^*\mathrm{loc}\ S)\mathbf{1}$ and is consequently the reflection of X into $(\mathrm{loc}\ S)\mathbf{1}$.

This is the little Bousfield lemma because of the compactness condition. This can be relaxed, though not eliminated, if some variant condition is imposed. It is not quite clear what this should be in general, and we leave the question open here. Thus we shall be unable to prove the existence of a localization in general in example 2.9. However, the lemma is quite strong enough to serve in 2.6, the Postnikov system case.

E. H. Brown's representability theorem does not really concern localizations. Its proof however is remarkably similar to that of the Bousfield lemma (or rather, given the history, the latter is similar to the former). We shall need some definitions. If a category C has coproducts and weak pushouts, a functor $F: C^{op} \to$ Sets is half-exact if its preserves products and weak pullbacks, i.e. takes coproducts in C to products and weak pushouts to weak pullbacks. A set $\mathcal{W} \in C_0$ is left-adequate if the representable functors $C(W,-)$, $W \in \mathcal{W}$ jointly reflect isomorphism, which is to say that all $C(W,f)$ are bijective if and only if f is an isomorphism.

LEMMA 3.3. If T is a left homotopy theory and $\mathcal{W} \in (T\mathbf{1})_0$ is a left adequate set, then for any $\mathbf{C}$ the set $\{(Lc)W \mid W \in \mathcal{W},\ c: \mathbf{1} \to \mathbf{C}\}$ is left adequate in $T\mathbf{C}$.

THEOREM 3.4. If T is a left homotopy theory such that there is in $T\mathbf{1}$ a left adequate set of compact objects, then for any $\mathbf{C}$ a functor $(T\mathbf{C})^{op} \to$ Sets is representable if and only if it is half-exact.

The proof is in essence that of Brown [5], cf. also [12]. In view of (3.1,3) we may without loss of generality suppose $\mathbf{C} = \mathbf{1}$. Let V be a small full subcategory of $T\mathbf{1}$ containing $\mathcal{W}$. Then we can find an X_0 in $T\mathbf{1}$ and a natural transformation $T\mathbf{1}(-,X_0) \to F$, or equivalently by Yoneda's lemma an element of FX_0, such that, for all $V \in \mathbf{V}_0$, $T\mathbf{1}(V,F_0) \to FV$ is surjective. Indeed it is sufficient to take $X_0 = \coprod_{V \in \mathbf{V}_0,\ v \in FV} V$.

Now if, for some X, $\phi: T\mathbf{1}(-,X) \to F$, the kernel-pair (also called the equivalence relation) of ϕ defines a half-exact functor R and natural transformations $R \rightrightarrows T\mathbf{1}(-,x)$. We have just seen that there is a Y and a natural transformation $T(\mathbf{1})(-,Y) \to R$ with all $T\mathbf{1}(V,T) \to RV$ surjective. Comparing with $R \rightrightarrows T\mathbf{1}(-,X)$ we have, by Yoneda's lemma, morphisms $Y \rightrightarrows X$. Let $X \to X^{\#}$ be their homotopy coequalizer. Then $T\mathbf{1}(-,X) \to F$ factors as $T\mathbf{1}(-,X) \to T\mathbf{1}(-,X^{\#})F \to F$.

Using this construction we build a sequence $X_0 \to X_1 \to \cdots$ with $X_{n+1} = X_n^{\#}$ and natural transformations $T\mathbf{1}(-,X_n) \to F$. If $\overline{X}$ is the homotopy colimit of the sequence, then all of these factor through some $T\mathbf{1}(-,\overline{X}) \to F$ because of (III.3.3). It is clear that, for all $V \in \mathbf{V}_0$, $T\mathbf{1}(V,\overline{X}) \to FV$ is bijective.

But if $\mathbf{V} \subset \mathbf{V}'$ and V' yields by the same construction $T\mathbf{1}(-,\overline{X}) \to F$ then, since $\mathcal{W} \subset \mathbf{V}_0$ is left adequate, $\overline{X} \approx \overline{X}'$. In other words $\overline{X}$ is independent of V, so that $T\mathbf{1}(-,\overline{X}) \approx F$.

The conclusion is by no means generally true. It fails for example in $\underline{\Pi}$ ([12]). When the hypothesis holds, as it does for example in $\underline{\Pi}_0$ when

$\{S^n \mid n \geq 1\}$ is by Whitehead's theorem left adequate, if may allow us to conclude that categories $T\mathbf{C}$ are monoidal closed. Using only the classical case we see that for any pointed T, each $T\mathbf{C}$ is enriched over $\underline{\Pi}_0\mathbf{1}$. We shall not however pursue these matters here.

4. UNIFORM LOCALIZATION

If T is a homotopy theory and $\mathcal{W}$ is a full subcategory of $\underline{\Pi}(\mathbf{C} \times \mathbf{2})$, we shall say that an X in $T\mathbf{C}$ is <u>uniformly local with respect to</u> $\mathcal{W}$ if it is local with respect to $W \otimes A$ for all W in $\mathcal{W}$, A in $T\mathbf{1}$. These X generate the full replete maximal subhyperfunctor $\operatorname{uloc} \mathcal{W} \subset T[\mathbf{C}]$. To describe it in another way, let $S \subset T[\mathbf{C} \times 2]$ be the full replete maximal subhyperfunctor generated by the $W \otimes A$. Then $\operatorname{uloc} \mathcal{W} = \operatorname{loc} S$: uniform locality is a special case of locality.

However, if X is uniformly local with respect to $\mathcal{W}$, then for A in $T\mathbf{1}$, W in $\mathcal{W}$,

$$\begin{aligned} T\mathbf{1}(A,\operatorname{Hom}_{\mathbf{C}}(W_1,X)) &\approx T\mathbf{C}(W_1 \otimes A,X) \\ &\approx T\mathbf{C}(W_0 \otimes A,X) \\ &\approx T\mathbf{1}(A,\operatorname{Hom}_{\mathbf{C}}(W_0,X)). \end{aligned}$$

In other words, X is uniformly local if and only if, for all W in $\mathcal{W}$, $\operatorname{Hom}_{\mathbf{C}}(W_1,X) \to \operatorname{Hom}_{\mathbf{C}}(W_0,X)$ is an isomorphism. This will imply that all uniform localizations are stable.

LEMMA 4.1. If, in $\underline{\Pi}\mathbf{C}$, $v: V_0 \to V_1$ is a homotopy pushout of $w: W_0 \to W_1$ and X, in $T\mathbf{C}$, is uniformly local with respect to w, then X is also uniformly local with respect to v.

For

$$\begin{array}{ccc} \operatorname{Hom}_{\mathbf{C}}(V_1,X) & \longrightarrow & \operatorname{Hom}_{\mathbf{C}}(W_1,X) \\ \downarrow & & \downarrow \\ \operatorname{Hom}_{\mathbf{C}}(V_0,X) & \longrightarrow & \operatorname{Hom}_{\mathbf{C}}(W_0,X) \end{array}$$

is a homotopy pullback in $T\mathbf{1}$, hence $\operatorname{Hom}_{\mathbf{C}}(V_1,X) \approx \operatorname{Hom}_{\mathbf{C}}(V_0,X)$.

COROLLARY 4.2. If $\mathcal{W} \subset \underline{\Pi}(\mathbf{C} \times \mathbf{2})$, then $\operatorname{uloc} \mathcal{W}$ is stable.

Let $\hat{\mathcal{W}} = \mathcal{W} \cup E\mathcal{W} \cup E^2\mathcal{W} \cup \cdots$. Since $(EW) \otimes A \approx E(W \otimes A)$, $\{W \otimes A \mid W \in \hat{\mathcal{W}}, A \in (T\mathbf{1})_0\}$ is stable. But 4.1 implies that $\operatorname{uloc} \hat{\mathcal{W}} = \operatorname{uloc} \mathcal{W}$.

COROLLARY 4.3. If $\mathcal{W} \subset \underline{\Pi}(\mathbf{C} \times \mathbf{2})$ is stable then X, in $T\mathbf{C}$, is uniformly local with respect to $\mathcal{W}$ if and only if, for all $W \in \mathcal{W}$, $\operatorname{Hom}_{\mathbf{C}}(W_1,X) \to \operatorname{Hom}_{\mathbf{C}}(W_0,X)$ is a split epimorphism.

For then $\{W \otimes A\}$ is stable, and it is sufficient for locality that X be injective.

We are now in a position to prove under modest hypotheses a localization theorem for uniform localizations.

THEOREM 4.4. Let T be a regular homotopy theory and suppose that $\mathcal{W} \subset \underline{\Pi}(\mathbf{C} \times \mathbf{2})$ is small and that for all $W \in \mathcal{W}_0$, W_0 and W_1 are finitary. Then $T[\mathbf{C}]$ admits a uniform localization with respect to $\mathcal{W}$.

Without loss of generality we may assume that $\mathcal{W}$ is stable, since $\hat{\mathcal{W}}$, as above, satisfies the same hypotheses. The argument will simply "internalize" the one used for the Bousfield lemma (3.4). Suppose that X_0 is in $T\mathbf{C}$. We construct a sequence $X_0 \to X_1 \to \cdots$ by means of homotopy pushouts in $T\mathbf{C}$

$$\begin{array}{ccc} \coprod_{\mathcal{W}_0} W_0 \otimes \mathrm{Hom}_{\mathbf{C}}(W_0,X_n) & \xrightarrow{\coprod w \otimes \mathrm{Hom}_{\mathbf{C}}(W_0,X_n)} & \coprod_{\mathcal{W}_0} W_1 \otimes \mathrm{Hom}_{\mathbf{C}}(W_0,X_n) \\ \Big\downarrow \varepsilon_n & & \Big\downarrow \theta_n \\ X_n & \xrightarrow{\quad x_n \quad} & X_{n+1} \end{array}$$

where the ε_n are the counits of the adjunctions $(W_0 \otimes -) \dashv \mathrm{Hom}_{\mathbf{C}}(W_0,-)$.

By (III.3.9) there is an X in $T(\mathbf{C} \times \mathbf{N})$ such that $\mathrm{dgm}_{\mathbf{N}} X = (X_0 \to X_1 \to \cdots)$. If we construct a homotopy pushout in $T(\mathbf{C} \times \mathbf{N})$

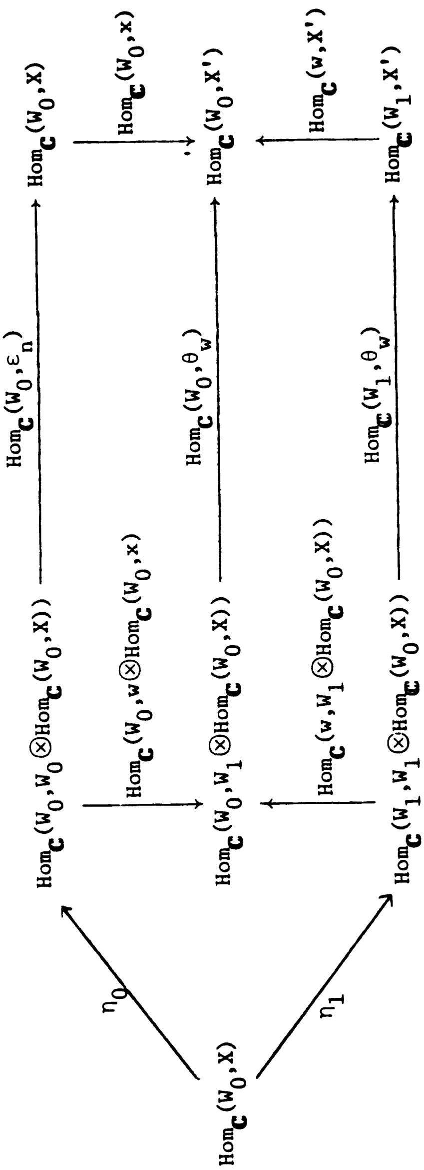
$$\begin{array}{ccc} \coprod_{\mathcal{W}_0} W_0 \otimes \mathrm{Hom}_{\mathbf{C}}(W_0,X) & \xrightarrow{\coprod w \otimes \mathrm{Hom}_{\mathbf{C}}(W_0,X)} & \coprod W_1 \otimes \mathrm{Hom}_{\mathbf{C}}(W_0,X) \\ \Big\downarrow \varepsilon & & \Big\downarrow \theta \\ X & \xrightarrow{\quad x \quad} & X' \end{array}$$

then induction with respect to n shows that $\mathrm{dgm}\, X' = (X_1 \xrightarrow{x_1} X_2 \xrightarrow{x_2} \cdots)$ and we may suppose that $X' = T(\mathbf{C} \times \sigma)X$ where $\sigma\colon \mathbf{N} \to \mathbf{N}$ is the successor function and that $\mathrm{dgm}_{\mathbf{N}} x = (x_0, x_1, \ldots)$.

For any W, $\theta_W\colon W_1 \otimes \mathrm{Hom}_{\mathbf{C}}(W_0,X) \to X'$ has the transpose $\tilde{\theta}_W\colon \mathrm{Hom}_{\mathbf{C}}(W_0,X) \to \mathrm{Hom}_{\mathbf{C}}(W_1,X')$. Let us consider the diagram on the following page in which η_0, η_1, ε_0 are the appropriate units and counit of the indicated adjunctions. The left-hand "square" commutes because of the naturality of the $\otimes$- Hom adjunction (cf. IV.3.5), the others for evident reasons. Thus, reading across the top and bottom

$$\mathrm{Hom}_{\mathbf{C}}(W,X')\tilde{\theta}_W = \mathrm{Hom}_{\mathbf{C}}(W_0,x).$$

Since T is regular and W_0, W_1 are finitary, we may pass to the colimit getting


Hom_C(W_0,X)
Hom_C(W_0,x)
Hom_C(W_0,X')
Hom_C(w,X')
Hom_C(W_1,X')
Hom_C(W_0,ε_n)
Hom_C(W_0,θ_w)
Hom_C(W_1,θ_w)
Hom_C(W_0,W_0⊗Hom_C(W_0,X))
Hom_C(W_0,w⊗Hom_C(W_0,x)
Hom_C(W_0,W_1⊗Hom_C(W_0,X))
Hom_C(w,W_1⊗Hom_C(W_0,X))
Hom_C(W_1,W_1⊗Hom_C(W_0,X))
η_0
η_1
Hom_C(W_0,X)

$$\mathrm{Hom}_{\mathbf{C}}(w,\text{T-colim}_{\mathbf{N}}X')(\text{T-colim}_{\mathbf{N}}\tilde{\theta}_w) \approx \mathrm{Hom}_{\mathbf{C}}(W_0,\text{T-colim}_{\mathbf{N}}x).$$

But σ is final, so that $\text{T-colim}_{\tilde{\mathbf{N}}}x$ is an isomorphism. Thus $\mathrm{Hom}_{\mathbf{C}}(W_1,(\text{T-colim}_{\mathbf{N}}x)^{-1})(\text{T-colim}_{\mathbf{N}}\tilde{\theta}_w)$ is a right inverse of $\mathrm{Hom}_{\mathbf{C}}(w,\text{T-colim}_{\mathbf{N}}X)$ and $\text{T-colim}_{\mathbf{N}}X$ is uniformly local. Since each x_n is clearly in loc*(uloc $\mathcal{W}$), $X_0 \mapsto (\text{T-colim}_{\mathbf{N}}X)$ is the localizing functor required.

This theorem provides easy generalizations of some of the examples of §2. We shall bypass them in favor of an application, which seems to be novel even in the classical case, to homotopical algebra.

REMARK. We might have defined uniform localization with respect to a $\mathcal{W} \subset \underline{\Pi}(\mathbf{C} \times \mathbf{D}^{op} \times \mathbf{2})$, X in TC being uniformly local if it is local with respect to $W \otimes_{\mathbf{D}} A$ for all W in $\mathcal{W}$, A in TD. In fact this would provide no greater generality, since X has this property if and only if it is uniformly local with respect to $\bigcup_{d:\ \mathbf{1}\to\mathbf{D}} \underline{\Pi}(\mathbf{C} \times d \times \mathbf{2})\mathcal{W}$.

CHAPTER VI
HOMOTOPICAL ALGEBRA

By "homotopical algebra" we mean not what was conveyed by the name of Quillen's monograph [21] but rather a homotopical version of the notion of universal algebra introduced by Lawvere [16]. The basic idea is due to Segal (cf. [23]) as is one of the major theorems which we quote below.

We shall begin with a brief and partial review of Lawvere's algebraic theories and algebra categories, define categories of homotopical algebras and show, using our uniform localization theorem, that they are again homotopy theories. This seems not to be a well-known fact and its consequences remain to be explored.

In §2 we go on to give a brief list of such homotopy theories of homotopical algebras; in §3 we consider in more detail the theories of homotopical groups and state a version of Segal's theorem on "special $\underline{\Delta}$-spaces" which relates homotopical groups and loop-spaces in the classical theory. We conclude with some observations on multiple algebraic structures, leading to a modest characterization of iterated loop-spaces.

It will be clear that this material is still inchoate. Many fundamental problems remain unsolved. We attempt no catalogue of them, but shall mention one or two in passing.

1. UNIVERSAL ALGEBRA AND HOMOTOPICAL ALGEBRA

Let us denote by Φ the full subcategory of Sets whose objects are the natural numbers $0 = \emptyset$, $1 = \{0\}$, $2 = \{0,1\}$, ... An <u>algebraic theory</u> is a category H with the same objects, containing Φ as a subcategory in such a way that the inclusion preserves coproducts, so that $m + n$ is the coproduct in either category.

If C is a category with finite products an <u>H-algebra in C</u> is a product-preserving functor $X\colon H^{op} \to C$. For such an X, $X_n \approx (X_1)^n$. The elements of $H(1,n)$ are called the <u>n-ary operations</u> of H; they induce morphisms $(X_1)^n \to X_1$. The full subcategory of $C^{H^{op}}$ containing the H-algebras is $\mathrm{Alg}(H,C)$; its morphism are the <u>algebra-homomorphisms</u>. Evaluation at 1 is a faithful functor. If X is an H-algebra, X_1 is the <u>underlying object</u> of X.

The theory H is easily recovered from $\mathrm{Alg}(H,\mathrm{Sets})$. The underlying object functor has a left adjoint F, the <u>free algebra</u> functor. The category H

is isomorphic to the full subcategory of Alg(H,Sets) containing the objects F0, F1, F2, ... , the imbedding of Φ in H being given by the restriction of F to Φ.

If C has finite coproducts then an H-coalgebra in C is a functor Y: H $\to$ C preserving coproducts. The corresponding category of coalgebras can be identified as $(\mathrm{Alg}(H,C^{op}))^{op}$.

If T is a homotopy theory and H is an algebraic theory, then a homotopical H-algebra in T is an object X of $T(H^{op})$ such that $\mathrm{dgm}_{H^{op}}X\colon H^{op} \to T1$ is an H-algebra in T1. Homotopical coalgebras are defined dually.

We may also characterize these homotopical algebras in another way. In $\underline{\Pi}(H^{op} \times 2)$ let W be the small full stable subcategory generated by those objects W such that

$$\mathrm{dgm}_2 W \approx (L_m * \sqcup L_n * \xrightarrow{(L_i*\ \ L_j*)} L_{m+n}*)$$

where i: m $\to$ m + n, j: n $\to$ m + n are the injections of the coproduct. Since $\mathrm{Hom}_{H^{op}}(L_m*,X) = X_m$, it is clear that X, in TH^{op}, is a homotopical algebra if and only if it is uniformly local with respect to W. We are thus led to define Hoalg(H,T) not as a category but as the subhyperfunctor uloc W $\subset$ $T[H^{op}]$. The subhyperfunctor Hocoalg(H,T) $\subset$ T[H] is defined dually.

THEOREM 1.1. If T is a regular homotopy theory and H is an algebraic theory, then Hoalg(H,T) is a uniform localization of $T[H^{op}]$ and is thus once again a homotopy theory.

For L_m* is certainly finitary in $\underline{\Pi}H^{op}$ and the hypotheses of the uniform localization theorem (V.4.4) are thus satisfied.

Dualization needs some caution. If T is coregular, i.e. if T* is regular, we may conclude that Hocoalg(H,T) is a colocalization. We have no warrant for believing that both algebras and coalgebras will be well-behaved in the same homotopy theory.

As a corollary of (1.1) we may derive the existence of a free homotopical algebra functor left adjoint to the underlying-object functor. Let us denote by U the inclusion of Hoalg(H,T)1 in TH^{op} and by Loc its left adjoint. Then (T1)U, i.e. $X \mapsto X_1$, is the underlying object functor and F_H = Loc(L1): T1 $\to$ Hoalg(H,X)1 is its left adjoint.

The analogy with ordinary algebra stops at this point. The underlying object functor need not be faithful, and it is certainly not to be expected that Hoalg(H,T)1 should be triplable over T1, as Alg(H,Sets), for example, is triplable over Sets. The homotopical analogue of triplability remains to be discovered.

One final point should be mentioned. If T is a regular homotopy theory and H is an algebraic theory, then the homotopy colimit in $T(H^{op})$ of a sequence in Hoalg(H,T)**1** is itself still a homotopical algebra. In other words the inclusion U preserves sequential homotopy colimits. But of course it preserves finite products and homotopy pullbacks as well.

PROPOSITION 1.2. If T is regular than Hoalg(H,T) is a regular homotopy theory.

2. SOME SIMPLE CASES

In spite of the concluding remark of §1, the analogies between algebra and homotopical algebra are not to be ignored. For example, Φ itself is an algebraic theory and, for any C with finite products, Alg(Φ,C) is equivalent to C, the equivalence being given by the underlying object functor. The same statement is true for homotopical algebras.

PROPOSITION 2.1. If T is a homotopy theory then Hoalg(Φ,T) $\approx$ T $\approx$ Hocoalg(Φ,T).

The underlying-object functor $X \mapsto X_1$, for X in Hoalg(Φ,T)**C** $\subset$ $T(\Phi^{op} \times \mathbf{C})$ has the inverse R1, since if A is in T**C** then $((R1)A)_n \approx A^{\Phi^{op}(n,1)} = A^{\Phi(1,n)} = A^n$, so that (R1)A is already an algebra, while the unit and counit of the adjunction are isomorphisms.

For the next example let $\Phi^{\cdot}$ be the theory of pointed finite sets. The free algebra on n is $n^+ = n \sqcup \{*\}$.

PROPOSITION 2.2. Hoalg($\Phi^{\cdot}$,T) $\approx$ $T^{\cdot}$, Hocoalg($\Phi^{\cdot}$,T) $\approx$ $T_{\cdot}$.

The equivalence in either case is given by the functor p: $\mathbf{2}^{op} \to \Phi^{\cdot}$ defined by p0 = 0, p1 = 1 and p(1 → 0) = λ, the unique map $1^+ \to 0^+$ in Sets. We need not supply the details.

Notice that this implies that Hoalg($\Phi^{\cdot}$,$T^{\cdot}$) $\approx$ Hoalg($\Phi^{\cdot}$,T). This, in fact, depends only on one property of $\Phi^{\cdot}$. In any algebraic theory the object 0 is initial. A theory is <u>pointed</u> if it is terminal as well. This is equivalent to the assertion that there is a unique 0-ary operation λ: 1 → 0 in the theory. Thus $\Phi^{\cdot}$ and the theory of monoids are pointed, Φ and the theory of rings are not, since Φ has no 0-ary operations while the theory of rings has one for each positive integer.

PROPOSITION 2.3. If H is a pointed algebraic theory, then Hoalg(H,T) $\approx$ Hoalg(H,$T^{\cdot}$) and Hocoalg(H,T) $\approx$ Hocoalg(H,$T_{\cdot}$).

The equivalences are provided by the functors

$$H \xrightarrow{\langle 1 \quad id_H \rangle} \mathbf{2}^{op} \times H \xrightarrow{q} H \text{ where } q(i,n) = in.$$

Next let us fix a monoid M, which we shall also regard as a category with one object, and let $\hat{M}$ be the theory of left M-sets. The free left M-set generated by n is just $n \otimes pM$, the n^{th} copower of the principal left M-set. Thus $\hat{M}(m,n) \approx \Phi(m,n) \otimes M^{op}$.

PROPOSITION 2.4. $Hoalg(\hat{M},T) \approx T[M]$.

The isomorphism $\hat{M}(1,1) \approx M^{op}$ gives an injection $\mu: M^{op} \to \hat{M}$ and thus $T(\mu^{op}): T(\hat{M}^{op}) \to TM$. But if X is in TM, then $((R\mu^{op})X)_n \approx X^n$, so that the restriction of $T(\mu^{op})$ to $Hoalg(\hat{M},T)$ is an equivalence.

3. HOMOTOPICAL GROUPS

Let G be the algebraic theory of groups and let $\mu: 1 \to 2$ and $\lambda: 1 \to 0$ in G be, respectively, the binary operation "multiplication" and the unique 0-ary operation "unit". There is an imbedding $J: \underline{\Delta} \to G$, well known from the classical theory of the bar construction, given by $[n] \to n$ and

$$(3.1) \qquad (d_i^n: [n-1] \to [n]) \longmapsto \begin{cases} \phi + (n-1) & , \ i = 0 \\ (i = 1) + \mu + (n-1-i), & i = 1,\dots,n-1 \\ (n-1) + \phi & , \ i = n \end{cases}$$

$$(s_i^n: [n+1] \to [n]) \longmapsto \quad i + \lambda + (n-i) \qquad , \ i = 0,\dots,n$$

where $\phi: 0 \to 1$ is the unique map.

We may remark parenthetically that, as 3.1 shows, all that is really used is the monoid structure.

Let us now consider $LJ: \underline{\Pi\Delta} \to \underline{\Pi}G$ and compute, for $n = 0,1,\dots$, $((LJ)^*)_n = \underline{\Pi}\text{-colim}_{(J\downarrow n)}*$. It is easy to see that $(J\downarrow n) = \Gamma NF_n$ where F is the free-group functor. Thus, by (II.6.5), $((LJ)*)_n \approx \underline{\Pi}\text{-colim}_{F_n}*$. In particular $((LJ)*)_0 \approx *$. Thus, if $Q: \mathbf{2} \times G \to G$ is the functor $(i,n) \mapsto$ in of (III.5.3), $(\underline{\Pi}Q)(LJ)_* \in \underline{\Pi}(\mathbf{2} \times G)_0$ actually lies in $\underline{\Pi}^{\cdot}G$.

Furthermore, $((\underline{\Pi}Q)(LJ)*)_n$, the "classifying space" of F_n, is by classical arguments the coproduct in $\underline{\Pi}^{\cdot}\mathbf{1}$ of n copies of the circle. We are thus led to denote $(\underline{\Pi}Q)(LJ)*$ by S^1 and observe that it belongs to $Hocoalg(G,\underline{\Pi}^{\cdot})\mathbf{1}$. In other words S^1 is a homotopical cogroup in $\underline{\Pi}^{\cdot}$, with $S_n^1 \approx S_1^1 \vee \cdots \vee S_1^1$.

Suppose now that T is a pointed homotopy theory and that X is in TC. Then the functor $(- \dot{\otimes} X): \underline{\Pi}^{\cdot} \to T[C]$ preserves coproducts. Thus $(S^1 \dot{\otimes} -): T \to T[G]$ takes its values in the subhyperfunctor $Hocoalg(G,T)$. The corestriction $\Sigma: T \to Hocoalg(G,T)$ is conventionally called the suspension.

Dually, $Hom^{\cdot}(-,X): \underline{\Pi}^{\cdot *} \to T[C]$ takes coproducts in $\underline{\Pi}^{\cdot}$ to products in T[C], so that $Hom^{\cdot}(S^1,-): T \to T[G^{op}]$ has its values in $Hoalg(G,T)$. The corestriction in this case is the loop-space Ω: T $\to$ Hoalg(G,T).

Thus the suspension and the loop-space come already endowed with the structures of homotopical cogroup and homotopical group. From the properties of $\dot{\otimes}$ and Hom$^\cdot$ we can read off properties of Σ and Ω.

PROPOSITION 3.2. The suspension and loop-space are, respectively, left and right strong hypernatural transformations, preserving on the one hand left, and on the other right homotopy Kan extensions.

We have in fact been a bit pedantic in the last few paragraphs in failing to profit from the start from the formal duality between these operations. Using subscripts to indicate the homotopy theory in question, we need only to have observed that

$$\Sigma_{T^*} = (\Omega_T)^*, \tag{3.3}$$

Since $S^1 \dot{\otimes}$ - and Hom(S^1,-) have adjoints ($S^1 \dot{\otimes}$ -) $\dashv$ Hom$^\cdot$(S^1,-) and ($S^1 \dot{\otimes}_{op}$ -) $\dashv$ Hom$^\cdot$(S^1,-), the suspension and loop-space also have adjoints, viz. the restrictions of these. The left-adjoint B: Hoalg(G,T) $\to$ T of the loop-space is conventionally called the bar-construction. The right-adjoint of the suspension seems to have no conventional name or notation. Let us denote it for the nonce by D: Hocoalg(G,T) $\to$ T. These may be thought of as, respectively, a "formal de-looping" and a "formal de-suspending" operation.

Indeed in the standard pointed theory, B is not merely a "formal" delooping. We may quote in this connection a theorem with a long history, going back to Stasheff [24], but in its present form due essentially to Segal [23].

THEOREM 3.4. The unit id $\to$ ΩB in Hoalg(G,$\underline{\Pi}^\cdot$) is an isomorphism. The composition BΩ: $\underline{\Pi}^\cdot \to \underline{\Pi}^\cdot$ is the colocalization at the pointed connected theory $\underline{\Pi}_0$. Thus Hoalg(G,$\underline{\Pi}^\cdot$) $\approx$ $\underline{\Pi}_0$.

We shall not give the proof, which is easy given Segal's results. The interesting question, which remains open, is for which pointed homotopy theories the corresponding statement is true.

It is apparently false in $\underline{\Pi}^{\cdot *}$, where it would assert inter alia that ΣD $\approx$ id ([14]).

4. HOMOTOPICAL MULTIALGEBRAS

If K is an algebraic theory and C is a category with finite products, then Alg(K,C) also has finite products and, H being another algebraic theory, we may construct the category Alg(H,Alg(K,C)). This category may also be identified in the following way. The category H $\times$ K is not of course an algebraic theory, but is certainly supplied with coproducts (m,n) + (m',n') =

(m + m, n + n'). We may say that X: $(H \times K)^{op} \to C$ is an H,K-bialgebra if it preserves products. The full subcategory Alg(H,K;C) of $C^{(H\times K)^{op}}$ containing these these is isomorphic to Alg(H,Alg(K,C)). There is an obvious generalization to the category Alg($H_1,\ldots,H_n$;C) of $H_1,\ldots,H_n$-multialgebra in C.

In fact all this terminology and notation does not occur in the usual expositions of universal algebra for the very good reason that, given H and K; there is a unique algebraic theory $H \circ K$ such that Alg(H,Alg(K,C)) $\approx$ Alg($H \circ K$,C). Possibly the best known example is $G \circ G = A = G \circ A$, where G is the theory of groups and A the theory of abelian groups.

The situation in a homotopy theory is quite different. Given H, K and a regular homotopy theory T, we may first construct Hoalg(K,T), which by 1.2 is a regular homotopy theory, and thus also Hoalg(H,Hoalg(K,T)). Taking $T = \underline{\Pi}^{\cdot}$ we see immediately that this is not in general Hoalg($H \circ K$,T). Thus, in contrast to the purely algebraic situation, there is some point in defining, for algebraic theories $H_1,\ldots,H_n$ and a homotopy theory T, the category Hoalg($H_1,\ldots,H_n$;T) of <u>homotopical</u> $H_1,\ldots,H_n$ <u>multialgebras</u> as the full replete maximal subhyperfunctor of $T[H_1^{op} \times \cdots \times H_n^{op}]$ containing those objects whose diagrams are in Alg($H_1,\ldots,H_n$;T1). <u>Homotopical multicoalgebras</u>, Hocoalg($H_1,\ldots,H_n$;T) $\subset T[H_1 \times \cdots \times H_n]$ are defined dually.

PROPOSITION 4.1. If T is a regular homotopy theory, then Hoalg($H_1,\ldots,H_n$;T) is a uniform localization of $T[H_1^{op} \times \cdots \times H_n^{op}]$. It is a regular homotopy theory equivalent to Hoalg(H_1,Hoalg($H_2,\ldots,H_n$;T)).

Let us consider the special case $H_1 = \cdots = H_n = G$, the theory of groups. It is plain that $S^1 \dot{\otimes} \cdots \dot{\otimes} S^1$ (n factors) is a homotopical G,...,G-multicoalgebra in $\underline{\Pi}^{\cdot}$; let us denote it by S^n. Then $\mathrm{Hom}^{\cdot}(S^n,-)\colon T \to T[G^{op} \times \cdots \times G^{op}]$ corestricts to $\Omega^n\colon T \to$ Hoalg(G,...,G;T). From the adjointness of $\dot{\otimes}$ and $\mathrm{Hom}^{\cdot}$ (IV.3.5) we see that Ω^n is in fact the n-fold loop-space

$$\begin{array}{ccccccc} T & \xrightarrow{\ \Omega\ } & \mathrm{Hoalg}(G,T) & \xrightarrow{\ \Omega\ } & \mathrm{Hoalg}(G,G;T) & \longrightarrow & \cdots \\ & \searrow^{S^1\dot{\otimes}-} & \cap & & \cap & & \\ & & T[G^{op}] & \xrightarrow{S^1\dot{\otimes}-} & TP[G^{op} \times G^{op}] & \longrightarrow & \cdots . \end{array}$$

Its left adjoint is of course B^n, the iterated bar construction.

Segal's theorem (3.4) has an evident corollary.

THEOREM 4.2. In Hoalg(G,...,G;$\underline{\Pi}^{\cdot}$), id $\to \Omega^n B^n$ is an isomorphism. The composition $B^n\Omega^n\colon \underline{\Pi}^{\cdot} \to \underline{\Pi}^{\cdot}$ is the colocalization at $\underline{\Pi}_n$. Thus Hoalg(G,...,G;$\underline{\Pi}^{\cdot}$) $\approx \underline{\Pi}_n$.

Theorems of this type have been described as "recognition principles" for n-fold loop-spaces.

Much evidently remains to be done, but we shall end our discussion here.

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RIGHT-hand, ODD-numbered pages should be headed with the TITLE of the paper (in shortened form if necessary).

Exceptions: PAGE 1 and any other page that carries a display title require NO RUNNING HEADS.

Page Numbers should be at the top of the page, on the same line with the running heads.

LEFT-hand, EVEN numbers – flush with left margin;

RIGHT-hand, ODD numbers – flush with right margin.

Exceptions: PAGE 1 and any other page that carries a display title should have page number, centered below the text, on blue line provided.

FRONT MATTER PAGES should be numbered with Roman numerals (lower case), positioned below text in same manner as described above.

MEMOIRS FORMAT

It is suggested that the material be arranged in pages as indicated below.
Note: Starred items (*) are requirements of publication.

Front Matter (first pages in book, preceding main body of text).

Page i – *Title, *Author's name.

Page iii – Table of contents.

Page iv – *Abstract (at least 1 sentence and at most 300 words).

1980 Mathematics Subject Classification (1985 Revision). This classification represents the primary and secondary subjects of the paper, and the scheme can be found in Annual Subject Indexes of MATHEMATICAL REVIEWS beginning in 1984.

Key words and phrases, if desired. (A list which covers the content of the paper adequately enough to be useful for an information retrieval system.)

Page v, etc. – Preface, introduction, or any other matter not belonging in body of text.

Page 1 – Chapter Title (dropped 1 inch from top line, and centered).

Beginning of Text.

Footnotes: *Received by the editor date.
Support information – grants, credits, etc.

Last Page (at bottom) – Author's affiliation.

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