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Homological algebra and (semi)stable homotopy

Alex Heller

CUNY, Department of Mathematics, 33 West 42nd Street, New York, NY 10036-8099, USA

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Abstract

If A is a complete and cocomplete abelian category, which we allow ourselves to conflate with the corresponding representable homotopy theory then the 2-functors $Hoch A$, taking the small category C to the homotopy category of chain complexes over A^C and $Hoch^+ A$, with value the homotopy category of *positive* chain complexes, are both homotopy theories (in the sense of my monograph, A.M.S. Memoirs 383), the former being *stable* in the sense that the suspension hyperfunctor is an equivalence, while the latter is *semistable*. The hyperfunctors $res : A \to Hoch A$ and $res^+ : A \to Hoch^+ A$ which take an X in A^C to a chain complex concentrated in degree 0 may be characterized as "resolvent". Then the two chain-complex theories associated to A are, respectively, the universal resolvent stabilization and semistabilization of A.

In other words, a "universal problem" of stabilization leads, for abelian categories, to the construction of chain complexes, just as a corresponding problem for topological spaces leads to the construction of spectra.

0. Introduction

This is a sequel to [5], which was concerned with cocontinuous (and hence, dually, continuous) stabilizations of regular (and, dually, coregular) homotopy theories. A stabilization is to be construed as a hyperfunctor of some specified type into a stable homotopy theory. It often turns out, then, that there is among them a universal one. Thus for example Boardman's stable homotopy category belongs to the universal cocontinuous stabilization (referred to in [5] as the left stabilization) of the standard pointed homotopy theory. Insofar as there is a moral to all of this it is that "stabilizations" may be of more than one sort. We consider here stabilizations of representable homotopy theories whose representing categories are abelian and characterize as universal stabilizations the hyperfunctors into homotopy fraction-theories of chain complexes which associate to each object the corresponding chain complex concentrated in degree 0.

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In other words, *homological algebra*, which is to say the study of the homotopy category of chain complexes over an abelian category, may be seen to be the necessary result of an attempt to stabilize the *algebra* of the underlying abelian category. To be more precise, this is what happens if we demand that the stabilization be *resolvent* (a technical condition explained in Section 1 below). For certain abelian-representable homotopy theories – namely, those in which, in the representing categories, injective and projective objects coincide – yet another sort of stabilization exists and has indeed been adumbrated in the earlier literature, e.g. in [2].

Needless to say, we have adopted throughout the language of [5], its conventions about small homotopy theories and in particular the convention of often neglecting not only the proofs but even the statements of duals, to which, nevertheless, the reader should remain alert.

1. Exactness and unipotence

By a *short exact sequence* in a pointed homotopy theory T we mean an object of $T[\mathbf{2} \times \mathbf{2}, (0,1)]$, thus having the diagram

$$\begin{tikzcd} \bullet \arrow[r, "u"] \arrow[d] & \bullet \arrow[d, "v"] \\ 0 \arrow[r] & \bullet \end{tikzcd}$$

in which, furthermore, $v = (cof)u$, $u = (fib)v$. By extension, we may also refer to the diagram as a short exact sequence.

If $T = A$ is an *abelian-representable* homotopy theory, i.e. has the property $AC = (A\mathbf{1})^C$ with $A\mathbf{1}$ abelian, complete and cocomplete then a short exact sequence in A, i.e. in some AC, is just a short exact sequence in the usual sense.

A hyperfunctor $\Phi : T \to T'$ is *exact* if it preserves short exact sequences. For a hyperfunctor between abelian-representable homotopy theories this implies that it preserves cofibres and fibres and thus homotopy pulllbacks and pushouts as well. This is not however true in general.

If Φ, as above, is exact and $X \to Y \to X'$ is a short exact sequence in T then its image under Φ determines a morphism $\Phi X' \to \Sigma\Phi X$. We shall say that Φ is *unipotent* if given short exact sequences $X_1 \to Y_0 \to X_0, \ldots, X_{n+1} \to Y_n \to X_n, \ldots$ the homotopy colimit of the resulting sequence

$$X_0 \to \Sigma X_1 \to \Sigma^2 X_2 \cdots$$

is 0. A homotopy theory is unipotent if its identity hyperfunctor is.

Thus, for example, any representable pointed homotopy theory is unipotent, since its suspension is the trivial hyperfunctor. Standard pointed homotopy theory is also unipotent, since any sequence like the one above has homotopy colimit 0. The only unipotent stable homotopy theory, however, is the trivial one – for obvious reasons.

Exactness is of course a self-dual notion. The dual of unipotency is *counipotency*.

2. Semistable homotopy theories; resolvent hyperfunctors

A pointed homotopy theory T is *left semistable* if the unit $\eta: id_T \to \Omega\Sigma$ of the adjunction $\Sigma \dashv \Omega$ is an isomorphism, *right semistable* if the counit is an isomorphism. Thus T is stable if and only if it is both left and right semistable.

In a left semistable homotopy theory $\Omega^2\Sigma^2$ is also isomorphic to the identity. Thus any object is a double loop-space and thus has the structure of an abelian group, so that left semistable homotopy theories are always additive, i.e. enriched over the category of abelian groups.

We recall that in $T[\mathbf{2}]$ the hyperfunctors $cof \dashv fib$ satisfy $cof^3 \approx \Sigma, fib^3 \approx \Omega$.

Lemma 2.1. *If T is left semistable then the unit $\eta: id \to (fib)(cof)$ is an isomorphism.*

For the composition of any three consecutive arrows in

$$id \xrightarrow{\eta} (fib)(cof) \xrightarrow{(fib)\eta(cof)} (cof)^2(fib)^2$$
$$\xrightarrow{(fib)^2\eta(cof)^2} (fib)^3(cof)^3 \xrightarrow{(fib)^3\eta(cof)^3} (fib)^4(cof)^4$$

is by the semistability an isomorphism, hence also that of the two central ones, and hence all of them.

Theorem 2.2. *If T and T' are left semistable homotopy theories and $\Phi : T \to T'$ then* (i) *Φ preserves homotopy pushouts if and only if it is exact*; (ii) *Φ is cocontinuous if and only if it is exact and preserves coproducts.*

The first assertion is an immediate consequence of the fact that in a semistable homotopy theory every morphism is the fibre of a short exact sequence. For the latter, consider in any TC the set of objects X such that $T-colim_C\Phi X \to \Phi(T-colim_C)X$ is an isomorphism. This clearly contains all L_cW for $c: \mathbf{1} \to C, W \in T\mathbf{1}$. But it is closed under coproducts and homotopy pushouts, so that by the density theorem [4] it is all of TC.

A hyperfunctor $\Phi: T \to T'$ is *resolvent* if it is exact, unipotent and coproduct-preserving. The dual notion is *coresolvency*. A *resolvent* (*semi*)-*stabilization* of a pointed homotopy theory T is a resolvent hyperfunctor $\Phi: T \to S$ with S (left semi-) stable. Observe that if $\Psi: S \to S'$ is a cocontinuous hyperfunctor between left semistable homotopies then $\Psi\Phi$ is again resolvent.

If T is a pointed homotopy theory and S is a left semistable one we shall denote by $Res(T,S)$ the category of resolvent hyperfunctors $T \to S$ and hypernatural

transformations between them. If $\Phi : T \to S$ is such a hyperfunctor then composition with it determines a functor $l\mathscr{S}^{cc}(S,S') \to Res(T,S')$, where $l\mathscr{S}^{cc}$ is the 2-category of left semi-stable homotopy theories, cocontinuous hyperfunctors and hypernatural transformations between them. We say that Φ is a *universal resolvent semistabilization* if this functor is an equivalence of categories. If such a resolvent semistablization exists it is unique, up to a unique isomorphism class of equivalences. The notion of *resolvent stabilization*
is defined analogously, as are the dual notions of *coresolvent (semi-)stabilizations.*

Our principal result will assert the existence of universal resolvent semistabilizations and stabilizations for certain abelian-representable homotopy theories. These will be constructed using chain complexes, to which we now turn.

3. Chain complexes in representable-abelian homotopy theories

If T is a pointed hypercategory then, in accordance with the usual convention, chT is the full subhypercategory of $S[\mathbf{Z}^{\mathrm{op}}]$, where $\mathbf{Z}$ is, once again, the ordered set of integers, containing those X such that for all n, $X_{n+1} \to X_{n-1}$ is 0. We distinguish in chT the subhypercategories

$$ch^{-}T \subset ch^{\mathrm{b}}T \subset chT \supset ch^{\mathrm{bb}}T \supset ch^{+}T$$

containing, respectively, those chain-complexes whose non-zero terms are concentrated in degrees ≤ 0, those whose non-zero terms have degrees bounded above, those whose non-zero terms have degrees bounded below and those whose non-zero terms are concentrated in degrees ≥ 0.

If A is an abelian-representable homotopy theory then $(chA)\mathbf{1}$ is complete and cocomplete abelian and represents the hypercategory chA, which is thus a homotopy theory. If A has exact products or coproducts, enough projectives or enough injectives, then so also does chA. Similarly, $ch^{+}A, ch^{-}A$ are homotopy theories and share the other properties with A. The remaining two, however, are in general neither complete or cocomplete and are thus not homotopy theories.

The homology hyperfunctor $H: chA \to A[\mathbf{Z}_0]$ is defined in the usual way. We say that a morphism ϕ in chA is a *homology equivalence* if $H\phi$ is an isomorphism and denote the class of homology equivalences in each AC by $\mathscr{E}_C$. These classes are obviously preserved by the functors AF for $F: C \to D$ in CAT and thus define a fraction hypercategory $HochA$. The hypercategories $Hoch^{-}A, \dots, Hoch^{+}A$ are constructed analogously.

If A is an abelian-representable homotopy theory with exact coproducts and enough projectives we propose to introduce in each of the categories $(chA)C, (ch^{\mathrm{bb}}A)C, (ch^{+}A)C$ a closed Quillen model structure with $\mathscr{E}_C, \mathscr{E}_C^{\mathrm{bb}}, \mathscr{E}_C^{+}$ as the weak equivalences. In the case of $(ch^{\mathrm{bb}})A$ this has already been done Quillen (cf. [7], see also [6]): fibrations are epimorphisms; cofibrations are monomorphisms whose cokernels are projective in each degree. For *positive* chain complexes, i.e. in $ch^{+}A$, we vary this by taking as fibrations those morphisms which are epic in all degrees ≥ 0. In chA fibrations are, once again, epimorphisms, while cofibrations are the injections

$$X^0 \to colim(X^0 \to X^1 \to X^2 \to \cdots),$$

where each $X^n \to X^{n+1}$ is a cofibration in $ch^{\mathrm{bb}}A$.

Lemma 3.1. *The structures thus described are closed Quillen model structures.*

The argument is in essence that of Quillen (loc. cit.), supplemented by the following observation. If X is in chA then $X = colim(X^0 \to X^1 \to \cdots)$ where

$$X_k^n = \begin{cases} X_k & \text{if } k \geq n, \\ B_{n-1}X & \text{if } x = n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Using the model structure in $ch^{\mathrm{bb}}A$ we can construct there a sequence $Y^0 \to Y^1 \to \cdots$ with Y^0 cofibrant and all $Y^k \to Y^{k-1}$ cofibrations, provided with a morphism into $X^0 \to X^1 \to \cdots$ such that each $Y^k \to X^k$ is a homology equivalence. But then for each k, $H_k(Y^n)$ is eventually constant, so that, A having exact coproducts, the familiar construction of Milnor shows that the colimit of the sequence of Y's, which is cofibrant in chA, maps by a homology equivalence into X.

Theorem 3.2. *If A is an abelian-representable homotopy theory with exact coproducts and enough projectives then $Hoch^+A$ is a left semistable homotopy theory and $HochA$ is a stable homotopy theory.*

The demonstration that they are homotopy theories follows sufficiently closely that of [4] for the case of simplicial sets that it seems unnecessary to repeat it here, except to observe that, all objects being fibrant, the cofibrant ones are bifibrant, so that homology equivalences between them are characterized in terms of chain homotopies, which are preserved by all additive functors and thus in particular by Kan extensions. The semistability of ch^+A results from the computation of Σ and Ω. The former is just the shift in degree; the latter truncates the complex:

$$(\Omega X)_k = \begin{cases} X_{k+1}, & k > 0, \\ Z_1X, & k = 0, \\ 0, & k < 0. \end{cases}$$

We have, for brevity, omitted the discussion of the dual case, in which A has exact products and enough injectives.

In the diagram

$$\begin{tikzcd} A \arrow[r, "\mathrm{R}^+"] & Ch^+A \arrow[r] \arrow[d, "\Gamma"] & ChA \arrow[d] \\ & Hoch^+A \arrow[r] & HochA \end{tikzcd}$$

in which R^+ is the hyperfunctor which associates to an X in A the complex consisting of X concentrated in degree 0 and the others are inclusions or fraction-hyperfunctors all the arrows preserve coproducts and are exact. Since $Hoch^+A$ is unipotent so also are the composites $res^+ : A \to Hoch^+A$ and $res : A \to HochA$. Thus res^+, res are resolvent. We may now state precisely our main theorem.

Theorem 3.3. *If A is an abelian-representable homotopy theory with exact coproducts and enough projectives then $res^+ : A \to Hoch^+A$ and $res : A \to HochA$ are, respectively a universal resolvent semistabilization and a universal resolvent stabilization of A.*

The proof will require an extension of the notion of chain complex to a wider class of homotopy theories.

4. Chain complexes: the general case

In general if T is a pointed homotopy theory it need not be the case that chT is one as well. We are led therefore to the following construction. Let us recall, from [5], the ordered set (thought of as a category)

$$V = \{(i,j) \mid |i-j| \le 1\} \subset \mathbf{Z} \times \mathbf{Z},$$

together with its subsets

$$\dot{V} = \{(i,j) \mid |i-j| = 1\}, \qquad V^- = \{(i, i-1)\} \subset \dot{V}.$$

We define $ChT = T[\dot{V}, V^-]$. As both a localization and a colocalization of $T[\dot{V}]$ this is once more a homotopy theory. An X in ChT has, with respect to $\dot{V}$, the diagram

$$\begin{tikzcd} \cdots X_{-2,-1} \arrow[r] \arrow[dr] & X_{-1,0} \arrow[r] \arrow[dr] & X_{0,1} \arrow[r] \arrow[dr] & X_{1,2} \cdots \\ \cdots X_{-1,-2} \arrow[r] \arrow[ur] & X_{0,-1} \arrow[r] \arrow[ur] & X_{1,0} \arrow[r] \arrow[ur] & X_{2,1} \cdots \end{tikzcd}$$

with all terms in the second row vanishing.

If $\omega : Z^{\mathrm{op}} \to \dot{V}$ is the map $n \mapsto (-n-1, -n)$ then $T[\omega] \colon ChT \to chT$. If T is a pointed representable homotopy theory then $T[\omega]$ is an equivalence of hypercategories.

Just as in the abelian-representable case we have a hyperfunctor $\mathrm{R} \colon T \to ChT$ with X in T going into the chain-complex with X concentrated in bidegree $(-1,0)$. This is, in the general case, less obvious than in the representable one. It may be constructed as the inverse of the equivalence $T[-1,0]: T[\dot{V}, \dot{V} - \{(-1,0)\}] \to T$ followed by the inclusion into ChT. Clearly, R is exact and preserves coproducts. The hyperfunctor $D : T \to ChT$, which takes X into the complex with diagram

$$\begin{tikzcd}
\cdots\ 0 \arrow[r] \arrow[dr] & X \arrow[r, "idX"] \arrow[dr] & X \arrow[r] \arrow[dr] & 0\ \cdots \\
\cdots\ 0 \arrow[r] \arrow[ur] & 0 \arrow[r] \arrow[ur] & 0 \arrow[r] \arrow[ur] & 0\ \cdots
\end{tikzcd}$$

may be constructed as the left adjoint of evaluation at $(-2,-1)$.

The usual apparatus of cycles boundaries and homology in abelian chain complexes is of course lacking in the general case. We shall make use instead of a hyperfunctor $Tot : Ch^+T \to T$ called *totalization* for reasons to be explained in Section 5 below. It is convenient to introduce it as the composition

$$ChT = T[\dot{V}, V^-] \xrightarrow{L[J]} T[V, V^-] \xrightarrow{T[0,0]} T$$

where $J : \dot{V} \to V$ is the inclusion. It may as usual also be computed as a suitable homotopy colimit. It is easy to see that it enjoys the following properties.

Proposition 4.1. *$Tot : Ch^+T \to T$ is cocontinuous. If $\sigma : Ch^+T \to Ch^+T$ denotes the degree shift then for $n \geq 0$, $Tot(\sigma^n)D = 0$ and $Tot(\sigma^n)\mathrm{R}^+ \approx \Sigma^n$.*

5. Totalization in chain-homotopy theories

If A is an abelian-representable homotopy theory with exact coproducts and enough projectives then Ch^+Hoch^+A is the fraction-theory of the homotopy theory ch^+ch^+A of *double complexes* in A with respect to the classes of morphisms inverted by the homology hyperfunctor H_{II} with respect to the second index.

The chain totalization hyperfunctor

$$tot : ch^+ch^+A \to ch^+A,$$

with $(totX)_n = \sum_{i+j=n} X_{(i,j)}$ and so forth is familiar from standard homological algebra, as is the fact that it sends H_{II}-equivalences into homology equivalences in ch^+A and thus determines a hyperfunctor $Ho(tot) : Ch^+Hoch^+A \to Hoch^+A$.

Lemma 5.1. *$Tot \approx Ho(tot)$.*

This is proved by induction for chain-complexes with degrees bounded above, using the fact that both functors agree on double complexes concentrated in degrees $(i,0)$ and vanish on those of the form DX. But any double complex is the homotopy colimit of its skeletons with respect to the second degree. Since both hyperfunctors are cocontinuous, their domains and codomains being semistable, the conclusion follows.

This accounts, of course, for *Tot* being called "totalization" too.

Corollary 5.2. *$Tot(ch^+)(res^+)\colon ch^+A \to Hoch^+A$ is isomorphic to the canonical fraction-functor Γ.*

6. Proof of the main theorem

We begin with the semistable case. If A is as described in Theorem 3.3 and S is a left semistable homotopy theory we shall construct a functor $Res(A,S) \to l\mathscr{S}^{cc}(Hoch^+A, S)$ which is inverse, up to isomorphism, to composition with $res^+ : A \to Hoch^+A$.

Lemma 6.1. *If $\Phi\colon A \to S$ is resolvent the composition $TotCh^+\Phi\colon Ch^+A \to Ch^+S$ inverts homology equivalences. Thus $TotCh^+\Phi = \hat{\Phi}\Gamma$, where $\hat{\Phi} : HoCh^+A \to S$.*

For $Tot(HoCh^+\Phi)$ is exact, preserves coproducts and vanishes on the "contractible" complexes $\sigma^n DW$. Induction shows that if $X \in Ch^+A$ is bounded and acyclic then $Tot(HoCh^+\Phi)X = 0$. But any acyclic X is the sequential homotopy colimit of the complexes $X^{(n)}$ which agree with X in degrees smaller than n and have, in degree n, the value Z_nX, and this homotopy colimit is, for trivial reasons, preserved by $Tot(Hoch^+\Phi)$. An easy mapping-cylinder argument yields the conclusion.

The functor we are looking for is $\Phi \mapsto \hat{\Phi}$. For

$$\hat{\Phi}res^+ = \hat{\Phi}\Gamma\mathrm{R}^+ = TotCh^+\Phi\mathrm{R}^+ = Tot\mathrm{R}^+\Phi = \Phi$$

while for $\Psi : Hoch^+A \to S$, in virtue of Lemma 5.1,

$$\Psi\widehat{res}^+\Gamma = Tot(Ch^+\Psi)(Ch^+res^+)\Gamma = \Psi Tot(Ch^+res^+) = \Psi\Gamma.$$

We may now consider the stable case. For A as in Theorem 3.3, $Hoch^+A$ is always a regular homotopy theory and thus has by [5, Theorem 8.1] a cocontinuous stabilization. But this stabilization is evidently given by the inclusion $Hoch^+A \to HochA$. The conclusion now follows.

7. Frobenius stabilization

An abelian category which has both enough projectives and enough injectives, and in which these classes coincide, is called a *Frobenius category*. Examples include modules over Frobenius rings, and in particular over group algebras of finite groups and the so-called Freyd completion or completion with respect to images of any of the categories SC where S is a stable homotopy theory ([3], cf. also [1]).

In the interest of economy we shall refer to the injectives or projectives of such a category as *ambijectives*. A congruence in a Frobenius category may be introduced as the set of pairs whose difference factors through an ambijective. It has long been recognized that the quotient with respect to such a congruence shares some of the properties of a homotopy category [2]. We next adduce a precise version of this observation, introducing in the process yet another example of stabilization of an abelian-representable homotopy theory, which is distinguished from our resolvent stabilizations above by the failure of unipotence.

Let us say that an abelian-representable homotopy theory A is a *Frobenius homotopy theory* if $A\mathbf{1}$ is a Frobenius category. It does not follow that each AC is a Frobenius category. However we may identify in each AC, i.e. in $(A\mathbf{1})^C$, the class of *locally ambijective* objects, i.e those $X : C \to A\mathbf{1}$ such that X_c is ambijective for each $c \in C$. In AC we define $\mathscr{E}_C$ to be the class of morphisms whose kernels and cokernels are locally ambijective, and define the hypercategory $\mathscr{F}A$ by $(\mathscr{F}A)C = AC[\mathscr{E}_C^{-1}]$. This fraction category coincides with the quotient category when $C = \mathbf{1}$.

Theorem 7.1. *If A is a Frobenius homotopy theory with exact products and coproducts then $\mathscr{F}A$ is a stable homotopy theory.*

It seems unnecessary to include the proof, which follows familiar lines, introducing into each AC two Quillen model structures, one with epimorphisms as fibrations, the other with monomorphisms as cofibrations.

The fraction hyperfunctor $frob : A \to \mathscr{F}A$ is exact and preserves both products and coproducts. It is, however, neither resolvent nor coresolvent. As may be seen by looking at injective and projective resolutions in A, both unipotence and counipotence fail in any nontrivial A.

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