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THE RIGID ANALYTIC PERIOD MAPPING, LUBIN-TATE SPACE, AND STABLE HOMOTOPY THEORY

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Abstract. The geometry of the Lubin-Tate space of deformations of a formal group is studied via an étale, rigid analytic map from the deformation space to projective space. This leads to a simple description of the equivariant canonical bundle of the deformation space which, in turn, yields a formula for the dualizing complex in stable homotopy theory.

Introduction

Ever since Quillen [22, 1] discovered the relationship between formal groups and complex cobordism, stable homotopy theory and the theory of formal groups have been intimately connected. Among other things the height filtration of formal groups has led to the chromatic filtration [21, 20, 23] which offers the best global perspective on stable homotopy theory available. From the point of view of homotopy theory this correspondence has been largely an organizational principle. It has always been easier to make calculations with the algebraic apparatus familiar to topologists. This is due to the fact that the geometry which comes up in studying formal groups is the geometry of *affine* formal schemes.

The point of this paper is to study the Lubin-Tate deformation spaces of 1-dimensional formal groups of finite height using a p-adic analogue of the classical period mapping. This is a rigid analytic, étale morphism, from the Lubin-Tate deformation space to projective space, which is equivariant for the natural group action. With this morphism the global geometry of projective space can be brought to bear on the study of formal groups. When applied to stable homotopy theory, this leads to a formula for the analogue of the Grothendieck-Serre dualizing complex.

1. Formal groups

1.1. The map to projective space. Let k be an algebraically closed field of characteristic $p > 0$, and let F_0 be a formal group of dimension 1 and finite height

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n over k. Lubin and Tate [18] studied the problem of deforming F_0 to a formal group F over R, where R is a complete, local Noetherian ring with residue field k. They defined deformations F and F' to be equivalent if there is an isomorphism $\phi : F \to F'$ over R which reduces to the identity morphism of F_0 and showed that the functor which assigns to R the equivalence classes of deformations of F_0 to R is representable by a smooth formal scheme X over the ring W of Witt vectors of k. Let $\widehat{\mathbb{G}}_a$ be the formal additive group. The essential cohomological calculations in their argument are:

(1) $\mathrm{Ext}^2(F, \widehat{\mathbb{G}}_a) = 0$, which assures that deformations from R/I to R exist when $I^2 = 0$;
(2) $\mathrm{Ext}^1(F, \widehat{\mathbb{G}}_a)$ is a free R-module of rank $= n-1$, which determines the dimension of the tangent space of the versal deformation;
(3) $\mathrm{Hom}(F, \widehat{\mathbb{G}}_a) = 0$, which gives the existence of the moduli space.

The space X is (noncanonically) isomorphic to the formal spectrum of the power series ring $W[\![u_1, \ldots, u_{n-1}]\!]$ in $(n-1)$ variables over W. The group of automorphisms of the special fibre, $G = \mathrm{Aut}(F_0)$, acts as formal automorphisms of X, as it acts as automorphisms of the functor represented by X [18]. An automorphism T of F_0 deforms to an isomorphism $T : F_a \to F_b$ of liftings over R (corresponding to points $a, b \in X(R)$) if and only if $T \cdot a = b$.

Let K be the quotient field of W, and let X_K be the rigid analytic space over K which is the “generic fibre” of the formal scheme X. Then X_K is (noncanonically) isomorphic to the open unit polydisc of dimension $= (n-1)$ over K, and G acts as rigid analytic automorphisms of X_K.

The algebra $D = \mathrm{End}(F_0) \otimes \mathbb{Q}_p$ is isomorphic to the division algebra of invariant $= \frac{1}{n}$ over $\mathbb{Q}_p$, and $\mathrm{End}(F_0)$ is the maximal $\mathbb{Z}_p$-order in D. Hence G is isomorphic to the group of units of this order, which is a maximal compact subgroup of the group $D^\times$.

Since the simple algebra

$$D \otimes_{\mathbb{Q}_p} K = M_n(K)$$

is split, G has a natural n-dimensional linear representation V_K over K. It follows that G acts by projective linear transformation on the projective space $\mathbb{P}(V_K)$ of hyperplanes in V_K. This action extends to a projective linear action of the group $D^\times$.

Our main result is the following:

Theorem 1. *The crystalline period mapping (to be defined below) is an étale, G-equivariant, rigid-analytic morphism $\Phi : X_K \to \mathbb{P}(V_K)$. Let L be a Tate algebra (= affinoid) over K, and let $R \subseteq L$ be the W-subalgebra of integral elements. Let F_a and F_b be deformations over R of F_0 corresponding to the points*

$$a, b \in X_K(L) = X(R).$$

An isogeny T of F_0 (viewed as an element of $D^\times$) deforms to an isogeny

$$f_T : F_a \to F_b$$

if and only if

$$T \cdot \Phi(a) = \Phi(b).$$

Remark. We have called Φ the crystalline period mapping to stress its analogy with the period mapping from a simply connected complex family of abelian varieties of complex dimension g to the Grassmannian of maximal isotropic subspaces in a complex symplectic space of dimension $2g$ [5]. Indeed both the crystalline and classical period mappings are given by solutions to the Picard-Fuchs equation, which is the differential equation given by the Gauss-Manin connection on the primitive elements in the first deRham cohomology groups of the fibres. In the complex case the image lies in an open orbit (Siegel space) for the real symplectic group by Riemann's positivity conditions. In our case the map Φ is surjective on points with values in the completion of an algebraic closure of K. The fibres of Φ can be identified [7, §23] with the cosets of $\mathrm{GL}_n(\mathbb{Z}_p)$ in the group

$$\left\{ g \in \mathrm{GL}_n(\mathbb{Q}_p) \mid \det g \in \mathbb{Z}_p^\times \right\}.$$

The proof of Theorem 1 and its extension to formal A-modules in the sense of Drinfeld [3] are given in our paper [7]. There we provided explicit formulae for Φ using coordinates on projective space; here we will sketch a more abstract approach. To specify a G-morphism

$$\Phi : X_K \to \mathbb{P}(V_K),$$

we must specify a G-equivariant, rigid analytic, line bundle $\mathcal{L}_K$ on X_K, as well as a homomorphism of $K[G]$-modules

$$V_K \to H^0(X_K, \mathcal{L}_K)$$

whose image has no base points. Then $\Phi(x)$ is the hyperplane W_x in V_K, mapping to sections of $\mathcal{L}_K$ which vanish at x.

Let F be the universal deformation of F_0 over X, and let $e : X \to F$ be the identity section of the formal morphism $\pi : F \to X$. Then

$$\omega = \omega(F) = e^* \Omega^1_{F/X}$$

defines the invertible sheaf on X of invariant differentials on F. The sheaf

$$\mathcal{L} = \omega^{-1} = \mathcal{L}\mathrm{ie}(F)$$

is a G-equivariant line bundle on X whose fibres are the tangent spaces of the corresponding deformations. Let $\mathcal{L}_K$ be the associated rigid analytic line bundle on X_K.

Let E be the universal additive extension of F over X; then E is a formal group of dimension $= n$ which lies in an exact sequence

$$0 \to N \to E \to F \to 0$$

of formal groups over X. The formal group N is isomorphic to the additive group $\widehat{\mathbb{G}}_a \otimes \mathrm{Ext}(F, \widehat{\mathbb{G}}_a)^\vee$ of dimension $= (n-1)$. Passing to Lie algebras gives an exact sequence

$$0 \to \mathcal{L}\mathrm{ie}(N) \to \mathcal{L}\mathrm{ie}(E) \to \mathcal{L}\mathrm{ie}(F) \to 0$$

of G-equivariant vector bundles on X.

The equivariant vector bundle $\mathcal{M} = \mathcal{L}\mathrm{ie}(E)$ of rank $= n$ on X is the covariant Dieudonné module of the formal group F (mod p). As such $\mathcal{M}$ is a crystal over X in the sense of Grothendieck [8, 19], it has an integrable connection

$$\nabla : \mathcal{M} \to \mathcal{M} \otimes \Omega^1_{X/W}$$

together with a Frobenius structure. A fundamental theorem of Dwork on the radius of convergence of solutions to p-adic differential equations (cf. [14, Proposition 3.1]) shows that the vector space

$$V_K = H^0(X_K, \mathcal{M}_K)^{\nabla}$$

of rigid analytic, horizontal sections has dimension n over K and that

$$V_K \otimes \mathcal{O}_{X_K} \simeq \mathcal{M}_K.$$

This vector space affords the natural n-dimensional representation of G. We stress that the horizontal sections in V_K are rigid analytic; the W-module $H^0(X, \mathcal{M})^{\nabla}$ of formal horizontal sections is zero once $n \geq 2$.

The surjection

$$\mathcal{M}_K \to \mathcal{L}_K \to 0$$

of rigid analytic vector bundles gives a map of $K[G]$-modules

$$V_K \to H^0(X_K, \mathcal{L}_K)$$

whose image is base point-free. This defines the crystalline period mapping Φ in the theorem. The fact that Φ is étale follows from an explicit computation of the determinant of the differential $d\Phi$ [7, §23].

Remark. (1) The crystal $\mathcal{M}$ can also be identified with the primitive elements in the first deRham cohomology group of F/X [16]. From this point of view the connection ∇ is that of Gauss-Manin.

(2) An analogue of the rigid analytic map Φ was introduced by Katz [15] in the study of the moduli of ordinary elliptic curves. Katz called his map L for logarithm. The map L was studied in the supersingular case by Katz [17] and Fujiwara [4], and this is essentially equivalent to a consideration of the map Φ in the case when F_0 has height $n = 2$. In this case the inverse images under Φ of the two points of $\mathbb{P}^1$ fixed by a maximal torus of G are the moduli of quasicanonical liftings of F_0, which were introduced in [6]. In the general case quasicanonical liftings have been studied by Jiu-Kang Yu [25].

(3) The components of the map Φ were also investigated in joint work of the first author with Ethan Devinatz [2]. That account is presented in a language that might be more familiar to topologists.

1.2. The G-action on X_K and the canonical line bundle. The group G is a (compact) p-adic Lie group. Let $\mathfrak{g}$ be its Lie algebra over $\mathbb{Q}_p$. Then

$$\mathfrak{g} \otimes K \simeq \mathfrak{gl}_n(K).$$

If $\gamma \in \mathfrak{g}$, then for $m \gg 0$ the element $\exp(p^m \cdot \gamma)$ lies in G and acts on X.

Proposition 2. *Let f be a rigid analytic function on X_K, and let γ be an element of $\mathfrak{g}$. Then the limit*

$$D_\gamma(f) = \lim_{m\to\infty} \frac{\exp(p^m\gamma)\circ f - f}{p^m}$$

exists in the Fréchet algebra A_K of rigid analytic functions on X_K. The map $f \mapsto D_\gamma(f)$ is a derivation of A_K over K, and the map $\gamma \mapsto D_\gamma$ defines a representation of Lie algebras $\mathfrak{g}\otimes K \to \mathrm{Der}_K(X_K)$.

Indeed, the corresponding facts are clear for the G-action on $\mathbb{P}(V_K)$, which is algebraic. Since Φ is étale, there is no obstruction to lifting the resulting vector fields to X_K. Informally speaking, the proposition shows that the action of G on X_K is "differentiable", which is not immediately apparent from the definition of the G-action on X. In fact, if the ring of formal functions $\mathcal{O}_X$ is made into a normed algebra in the obvious way (by choosing deformation parameters and giving them norm 1), then the action map

$$G \to \text{Bounded linear operators on } \mathcal{O}_X$$

is not even continuous. It is not difficult to find explicit formulae for the vector fields giving the differentiated action [7, §§24 and 25].

We can also use the map Φ to describe the canonical bundle of X over W

$$\Omega^{n-1} = \Omega^{n-1}_{X/W} = \bigwedge^{n-1} \Omega^1_{X/W}$$

in the category of G-equivariant line bundles on X.

Let $\Theta = \Theta_{X/W}$ be the tangent bundle of X over W. The deformation theory of Kodaira and Spencer (cf. [7, §17; 13, Corollary 4.8] gives an isomorphism of G-vector bundles

$$\Theta \simeq \mathrm{Hom}(\mathcal{L}\mathrm{ie}(N), \mathcal{L}\mathrm{ie}(F)).$$

Taking duals gives an isomorphism $\Omega^1 \simeq \mathcal{L}\mathrm{ie}(N)\otimes\omega$; hence,

$$\bigwedge^{n-1} \Omega^1 \simeq \bigwedge^{n-1} \mathcal{L}\mathrm{ie}(N)\otimes\omega^{\otimes(n-1)}.$$

Since the sequence

$$0 \to \mathcal{L}\mathrm{ie}(N) \to \mathcal{L}\mathrm{ie}(E) \to \mathcal{L}\mathrm{ie}(F) \to 0$$

of G-bundles is exact, we obtain a G-isomorphism

$$\Omega^{n-1} \simeq \bigwedge^{n} \mathcal{L}\mathrm{ie}(E)\otimes\omega^{\otimes n}.$$

It remains to identify the line bundle $\bigwedge^n \mathcal{L}\mathrm{ie}(E)$.

If $\mathcal{F}$ is an equivariant vector bundle on X and k is an integer, we let $\mathcal{F}' = \mathcal{F}[\det^k]$ be the equivariant bundle where the action of G on sections of $\mathcal{F}$ is twisted by the kth power of the reduced norm character $\det : G \to \mathbb{Z}_p^\times$:

$$g'(f) = \det(g)^k \cdot g(f).$$

By an analysis of the determinant of $d\Phi$, we show [7, §22] that Φ induces an isomorphism of equivariant line bundles on X:

$$\bigwedge^{n} \mathcal{L}\mathrm{ie}(E) \xrightarrow{\sim} \mathcal{O}_X[\det].$$

Hence we obtain

Corollary 3. *The canonical line bundle Ω^{n-1} of X is isomorphic, as a G-equivariant line bundle, to $\omega^{\otimes n}[\det]$.*

Remark. Since the map Φ is étale, this formula can also be deduced from the (more elementary) corresponding formula for the GL_n-equivariant bundle Ω^{n-1} of $\mathbb{P}(V_K)$. Indeed, $\Omega^{n-1}_{\mathbb{P}} = \mathcal{O}_{\mathbb{P}}(-n)[\det]$.

2. Duality in localized stable homotopy theory

In this section we will work in the category of p-local spectra (in the sense of stable homotopy theory). Unfortunately, it does not seem possible to preserve the standard notation of both algebraic geometry and algebraic topology and avoid a conflict. Throughout this section the symbols E and F will denote spectra.

The category of spectra is a triangulated category and appears in the abstract to be much like the category of sheaves (or complexes of sheaves) on a scheme. For many purposes it is sufficient to take this scheme to be a Riemann surface; but to understand the more refined apparatus of stable homotopy theory, it is necessary to take it to be a variety S having a unique subvariety S_n of each finite codimension n [10, 21].

From the point of view of (complexes of) sheaves the sphere spectrum corresponds to the sheaf of functions, the smash product of spectra corresponds to the (derived) tensor product of sheaves, and stable homotopy groups correspond to (hyper)cohomology groups. A more complete description of this analogy appears in Table 1.

There are two kinds of duality in stable homotopy theory. The *Spanier-Whitehead* dual of F,

$$DF = \mathrm{Map}[F, S^0],$$

is the spectrum of maps from F to the sphere spectrum. If F is finite, then the homotopy type of F is determined by a functorial (in E) isomorphism $E^*F \approx E_{-*}DF$. The *Brown-Comenetz* dual IF of F represents the functor

$$Y \mapsto \mathrm{Hom}\,(\pi_0 Y \wedge F, \mathbb{Q}/\mathbb{Z})\,.$$

If I denotes the Brown-Comenetz dual of S^0, then there is a weak equivalence $IF \approx \mathrm{Map}[F, I]$.

In the analogous situation of sheaves over a Riemann surface S, if F corresponds to a divisor D, then the Spanier-Whitehead dual of F corresponds to the divisor $-D$. By the Serre duality theorem the dual of $H^i(S, D)$ is $H^{1-i}(S, K-D)$, where K is the canonical sheaf. It follows that the Brown-Comenetz dual IF of F corresponds to the divisor $K - D$ and that the dualizing complex I corresponds to the divisor K or, more precisely, to the complex of sheaves consisting of the canonical sheaf in dimension -1 and zero elsewhere.

TABLE 1. Analogy between p-local spectra and quasicoherent sheaves (or complexes of sheaves) on a variety S having a unique subvariety S_n of each finite codimension n. The map $i_n : U_n \to S$ is the inclusion of $S \setminus S_n$

Stable Homotopy Theory	**Sheaf Theory**
p-local spectrum	quasicoherent sheaf or complex of quasicoherent sheaves
"finite" p-local spectrum	finite complex of coherent sheaves
smash product	tensor product
homotopy groups	hypercohomology
homotopy classes of maps	$\mathrm{R}\,\mathrm{Hom}$
function spectra	sheaf $\mathrm{R}\,\mathrm{Hom}(A^*, B^*)$
p-local sphere spectrum	$\mathcal{O}_S$
category $\mathcal{C}_n$	subcategory of coherent sheaves supported on S_n
Morava $K(n)$	total quotient field of $S_n \setminus (S_n \cap U_{n+1})$
functor L_n	functor $Ri_{n*} \circ i_n{}^*$
functor $L_{K(n)}$	completion of U_n along $U_n \cap S_{n-1}$
chromatic tower	Cousin complex

The importance of this formula for the dualizing complex is well known, and it is desirable to have an analogous description of the dualizing complex I. Since the category of finite spectra has infinite Krull dimension [10, 12], it is necessary to localize away from the primes of finite codimension. This leads to the chromatic tower.

2.1. The chromatic tower. In the situation of complexes of sheaves over the filtered variety S there is a standard procedure for constructing a resolution of a sheaf by sheaves whose support lies on one of the subvarieties $S_n \setminus S_{n+1}$. This is called the Cousin complex in [9]. In the context of stable homotopy theory, it is known as the chromatic resolution [24].

Fix a rational prime p, and let L_n denote localization with respect to the wedge $K(0) \vee \cdots \vee K(n)$ of the first $n+1$ Morava K-theories [23]. There are natural transformations

$$L_n \to L_{n-1} \tag{1}$$

and compatible transformations $1 \to L_n$. This results in the *chromatic tower*

$$\begin{array}{ccc} & & \vdots \\ & \nearrow & L_1 S \\ & & \downarrow \\ S & \longrightarrow & L_0 S \end{array}$$

When F is the p-localization of a finite spectrum, the map

$$\{\pi_k F\} \to \{\pi_k L_n F\}_n$$

is a pro-isomorphism for each k [11]. This means that all of the homotopy theory of finite spectra can be recovered from the chromatic tower.

The difference between L_n and L_{n-1} can be measured in two ways. The fibre of the transformation (1) is the functor M_n. It is known as the nth *monochromatic layer*. The difference is also measured by the functor $L_{K(n)}$, which is localization with respect to the nth Morava K-theory. It is believed that the stable homotopy type of the p-localization of a finite spectrum F can in fact be recovered from the collective knowledge of of the spectra $L_{K(n)}$. There are natural equivalences

$$L_{K(n)} M_n F \approx L_{K(n)} F,$$
$$M_n L_{K(n)} F \approx M_n F,$$

so the homotopy types of $L_{K(n)}F$ and $M_n F$ determine each other.

2.2. The Morava correspondence. As in §1 fix a formal group of height $n \geq 1$ over an algebraically closed field k of characteristic $p > 0$, and let G be its automorphism group and X the Lubin-Tate space of its deformations.

The homotopy types of the spectra $M_n F$ and $L_{K(n)}F$ are accessible through the Morava correspondence, which associates to each spectrum F, graded, equivariant, quasicoherent sheaves $\mathcal{K}_n(F)$ and $\mathcal{M}_n(F)$ over the Lubin-Tate space X, and spectral sequences

$$H^*_{\mathrm{cts}}(G; H^0 \mathcal{K}_n(F)) \Rightarrow W \otimes \pi_* L_{K(n)} F,$$
$$H^*_{\mathrm{cts}}(G; H^0 \mathcal{M}_n(F)) \Rightarrow W \otimes \pi_* M_n F.$$

These spectral sequences often collapse and always terminate at a finite stage which depends only on n. They converge to the graded group associated to a finite filtration of their abutment [11].

When $F = S^0$ is the sphere spectrum, the sheaf $\mathcal{K}_n(S^0)$ is the direct sum $\bigoplus_{n \in \mathbb{Z}} \mathcal{L}^{\otimes n}$. Its ring of global sections is

$$W[[u_1, \ldots, u_{n-1}]][u, u^{-1}]$$

with $u \in H^0(\omega)$ a generator. It is graded in such a way that $|u| = -2$ and $|u_i| = 0$.

Let

$$I_n = \mathrm{Map}[L_n S^0, I]$$

be the Brown-Comenetz dual of $L_n S^0$. There are maps $I_{n-1} \to I_n$, and the direct limit $\varinjlim I_n$ is the spectrum I. The cofiber of $I_{n-1} \to I_n$ is $L_{K(n)} I_n$. The spectrum $M_n I_n$ is the Brown-Comenetz dual of $L_{K(n)} S^0$. Knowledge of the homotopy type of $L_{K(n)} I_n$ completely describes Brown-Comenetz duality in the category of $K(n)$-local spectra and, by the above, ultimately gives a formula for the spectrum I.

2.3. Invertible spectra.

Definition 4. A $K(n)$-local spectrum F is *invertible* if there is a spectrum F' and a weak equivalence

$$L_{K(n)}(F \wedge F') \approx L_{K(n)} S^0.$$

The invertible spectra form a group Pic_n.

Theorem 5. (i) *A $K(n)$-local spectrum F is invertible if and only if the associated sheaf $\mathcal{K}_n(F)$ is the tensor product of an invertible sheaf with $\mathcal{K}_n(S^0)$.*

(ii) *If p is large compared with n ($2p-2 \geq \max\{n^2, 2n+2\}$ will do), then the homotopy type of an invertible spectrum is determined by its associated Morava module.*

Theorem 6. *Let $\mathcal{K}$ be the total quotient field of the sheaf $\mathcal{O}_X$. The Morava modules associated to $\Sigma^{-n^2} I_n$ are*

$$\mathcal{M}_n(\Sigma^{-n^2} I_n) = \mathcal{K}/\mathcal{O}_X \otimes \Omega_X^{n-1} \otimes \mathcal{K}_n(S^0),$$
$$\mathcal{K}_n(\Sigma^{-n^2-n} I_n) = \Omega_X^{n-1} \otimes \mathcal{K}_n(S^0).$$

In particular, the spectrum $L_{K(n)} I_n$ is invertible.

Together with Corollary 3 this determines the homotopy type of $L_{K(n)} I_n$ at large primes. In certain cases (§2.4) this can be put into a more concrete form.

2.4. Finite spectra and v_n self-maps.

Let $\mathcal{C}_0$ be the category of p-localizations of finite spectra, and let $\mathcal{C}_n \subseteq \mathcal{C}_0$ be the full subcategory of $K(n-1)$-acyclics. These categories fit into a sequence [12]

$$\cdots \subset \mathcal{C}_{n+1} \subset \mathcal{C}_n \subset \cdots \subset \mathcal{C}_0.$$

One of the main results of [12] is that each $F \in \mathcal{C}_n$ has, for $N \gg 0$, an "essentially unique" v_n self-map

$$v : \Sigma^{2p^N(p^n-1)} F \to F$$

satisfying

$$K(m)_* v = \begin{cases} 0 & \text{if } m \neq n, \\ \text{multiplication by } v_n^{p^N} & \text{otherwise.} \end{cases}$$

This gives for $N \gg 0$ a canonical equivalence

$$L_{K(n)} v : L_{K(n)} \Sigma^{2p^N(p^n-1)} F \xrightarrow{\sim} L_{K(n)} F.$$

Because of this one can "suspend" $K(n)$-local spectra by any element of

$$\varprojlim_N \mathbb{Z}/2p^N(p^n-1)\mathbb{Z}.$$

Corollary 7. *Let F be a spectrum in $\mathcal{C}_n$.*

(i) *If p is odd and $n=1$, then $I_1F = \Sigma^2 DF$.*

(ii) *If p is large with respect to n (as in Theorem 5) and $p1_F \sim *$, then $I_nF \approx \Sigma^\alpha L_{K(n)}DF$, where*

$$\alpha = \lim_{N\to\infty} 2p^{nN}\frac{p^n-1}{p-1} + n^2 - n.$$

(iii) *In particular, if F admits a self-map v satisfying*

$$K(n)_*v = v_n^{p^M},$$

then there is a homotopy equivalence

$$I_nF \approx \Sigma^{2p^{nM}(p^n-1)/(p-1)+n^2-n}DF.$$

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