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Some remarks on projective Mackey functors

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Abstract

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We consider certain problems in the algebra of Mackey functors for a finite group raised by equivariant topology. Our main theorems are: A Mackey functor has projective dimension 0, 1 or ∞ (Theorem 2.1). The G-theory of Mackey functors is calculable in terms of the G-theories of the Weyl groups (Theorem 3.1). For Mackey functors with the group order invertible the K-theory is the sum of the K-theories of the Weyl groups (Theorem 6.1); a similar result holds unstably (Theorem 6.4). For a group of prime order the projective class group of Mackey functors is the sum of the class group of the group ring and that of the Burnside ring (Theorem 7.1).

Introduction

The algebra of Mackey functors for a finite group G was distilled by Dress for his work on induction theorems [7], and good algebraic use has been made of them since. In the study of equivariant cohomology theories, Mackey functors play the same rôle that abelian groups play in nonequivariant cohomology, which is of course fundamental. This topological application suggests certain natural algebraic questions and we consider some of them here. Firstly there is the homological algebra of Mackey functors, of direct relevance to the cohomology of Moore spectra, and secondly the projective class group, which is the natural home of the finiteness obstruction of Wall [17] in the stable equivariant context.

The rest of the paper is layed out as follows.

Section 1: Recollection of the definition of Mackey functors, the central

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example of the Burnside Mackey functor and introduction of the class of free functors, of which projective functors are the direct summands.

Section 2: The analogue of Rim's theorem stating that Mackey functors have projective dimension 0, 1 or ∞.

Section 3: How to calculate the Grothendieck group G_0 of all finitely generated Mackey functors.

Section 4: The endomorphism rings of the free Mackey functors: identification in favourable cases.

Section 5: The projective class group of all Mackey functors and the class group of Mackey functors with restricted isotropy type.

Section 6: Mackey functors with the group order invertible.

Section 7: Calculation of G_0 and K_0 for the category of Mackey functors for groups of prime order: the latter is the sum of the class groups of the group ring and the Burnside ring.

Section 8: Some examples to illustrate the delicacy of the calculation in general, certain rank functions and their connection with Dress's theory of defect sets.

Appendix A: The immediate topological applications.

Appendix B: There is no stable G Moore spectrum for $\mathbb{Z}$ if $G \neq 1$.

The present account is an expansion of part of an earlier preprint [8] in which the topological aspect was more prominent. The obstruction theory of [8] is superseded by that of Costenoble and Waner [3] and we intend to return to the remaining topological questions elsewhere.

We are grateful to the referee for suggesting we investigate the behaviour of Mackey functors in which the group order is invertible; this led to various clarifications, and also to Section 6.

1. Free and projective Mackey functors

We first recall basic facts from the theory of Mackey functors from the work of Dress [7] (see also [10]). For this we let **Ab** denote the category of abelian groups, and we let **G-set** denote the category of finite left G-sets and G-maps between them.

Definition 1.1. A *Mackey functor* consists of a covariant and a contravariant functor $L : \mathbf{G\text{-}set} \to \mathbf{Ab}$ which agree on objects (we shall therefore write $L(U)$ for the value of either functor on a G-set U and α_* and α^* for the co- and contravariant values on a G-map $\alpha : U \to V$) which satisfy the following two conditions:

(1) the functors convert disjoint unions into direct sums and direct products and

(2) (the Mackey axiom) for any pullback diagram

$$\begin{array}{ccc} R & \xrightarrow{\alpha} & S \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ T & \xrightarrow[\delta]{} & U \end{array}$$

of G-sets we have $\gamma^*\delta_* = \alpha_*\beta^*$.

We denote the category of Mackey functors and natural transformations by $\mathfrak{M}_G$, and we may refer to the covariant structure maps as induction or transfer and to the contravariant ones as restriction. Since **Ab** is an abelian category, so is $\mathfrak{M}_G$. Later on we shall also want to refer to the category of Mackey functors with values in the category of k-modules for some ring k; by analogy with the notation for the group ring we let $k\mathfrak{M}_G$ denote this category.

Remark. There are other ways of packaging Definition 1.1. Lewis bases his study [10] on the Burnside category $\mathcal{B}$ of Lindner [13] whose objects are finite G-sets, but which has morphisms corresponding to both inductions and restrictions. The morphisms from a G-set A to a G-set B in $\mathcal{B}$ are formal differences of diagrams $A \xleftarrow{a} T \xrightarrow{b} B$ in **G-set**, with composition via pullback. A Mackey functor is then a contravariant additive functor $M : \mathcal{B} \to \mathbf{Ab}$, the image of the above morphism corresponds to $a_* b^*$ in the present account. Various theorems about Mackey functors thus become represented as properties of the Burnside category: for example it is self-dual. The approach has many formal advantages, but we have chosen to stay closer to the language of the classical examples.

One may then show [12, V.9.9] that the Burnside category is equivalent to the full subcategory of the equivariant stable homotopy category generated by the G-sets: this observation shows the way to a definition of Mackey functors when G is a compact Lie group.

We turn to the most important example of a Mackey functor.

Definition 1.2. The *Burnside functor* is defined by letting $\mathbb{A}(U)$ be the Grothendieck group of finite G-sets over U. For a G-map $\alpha : U \to V$ the map α_* is defined by composition with α and the map α^* is defined by pullback along α. The Mackey axiom is an easily verified fact about pullbacks of composites.

Remarks. (1) $\mathbb{A}(U)$ has a ring structure induced by fibre product over U, and $\alpha^* : \mathbb{A}(V) \to \mathbb{A}(U)$ is a map of rings. One then verifies that $\alpha_* : \mathbb{A}(U) \to \mathbb{A}(V)$ is a map of $\mathbb{A}(V)$ modules.

(2) By considering the fibre over $1H$ we see that $\mathbb{A}(G/H)$ is naturally isomorphic to the Burnside ring of left H-sets [6]. The maps α_* and α^* correspond to the induction and restriction maps.

Next we recall that if L is a Mackey functor and U is a G-set we may form the associated Mackey functor L_U with coefficients in U defined by $L_U(S) = L(U \times S)$.

Definition 1.3. A *free* Mackey functor is any one isomorphic to a sum of functors $\mathbb{A}_U$ for suitable finite G-sets U. We see that if such a functor is finitely generated it is isomorphic to a single factor $\mathbb{A}_U$ for some U since $\mathbb{A}_S \oplus \mathbb{A}_T \cong \mathbb{A}_{S+T}$.

There is some topological justification for this terminology, but for the present purposes an algebraic one is more appropriate: the following proposition is well known.

Proposition 1.4. *For any Mackey functor L and G-set U the map*

$$\mathfrak{M}_G(\mathbb{A}_U, L) \to L(U)$$

obtained by evaluation at the diagonal $\Delta_U : U \to U \times U$ is an isomorphism. It is natural in L and if $f : \mathbb{A}_T \to \mathbb{A}_U$ corresponds to the G-set $\{\alpha, \beta\} : Z \to U \times T$ over $U \times T$ in $\mathbb{A}_U(T)$, then the induced map f^* is the composite of $\alpha^* : L(U) \to L(Z)$ and $\beta_* : L(Z) \to L(T)$.

Proof. The fundamental observation is that the G-set $\{\alpha, \beta\}$ is $\beta_* \alpha^*(\Delta_U)$. This is easily verified from the definitions.

Accordingly if the evaluation of $\phi : \mathbb{A}_U \to L$ at Δ_U is zero, ϕ is zero on all G-sets $\{\alpha, \beta\}$. Since $\mathbb{A}_U(T)$ is generated by differences of such G-sets over $U \times T$ it follows that the evaluation map is injective.

To see that every element of $L(U)$ is the image of Δ_U under a suitable map $\phi : \mathbb{A}_U \to L$ we need to know that for any $x \in L(U)$ the assignment $\phi(\Delta_U) = x$ does extend to a natural transformation of Mackey functors. Indeed by definition of $\mathbb{A}_U(T)$ an element is uniquely expressible as a difference of disjoint G-sets each uniquely expressible in the form $\{\alpha, \beta\}$. By the fundamental observation above we are forced to take $\phi(\{\alpha, \beta\}) = \beta_* \alpha^*(x)$, and we must check this commutes with restriction and induction maps. Since induced maps $\gamma : T \to T'$ take G-sets to G-sets and $\mathbb{A}_U(T)$ has a basis of G-sets, it suffices to check that $\phi\gamma_*(\{\alpha, \beta\}) = \gamma_*(\beta_* \alpha^*(x))$ and similarly for γ^*. We leave this to the reader. □

Corollary 1.5. *It follows that there are enough projectives, and that any projective is a summand of a free one.* □

Corollary 1.6. *If we have an isomorphism $\mathbb{A}_X \cong \mathbb{A}_Y$ of Mackey functors, then we have an isomorphism $X \cong Y$ of G-sets.*

Proof. Perhaps the easiest way to see this is to argue by induction on the order of H that X and Y have the same multiple of G/H using the Mackey functors $R(\mathbb{Z}N_G(H)/H)$ of [9]. $\square$

2. Projective dimension of Mackey functors

In this section we shall prove the analogue for Mackey functors of Rim's theorem on $\mathbb{Z}G$-modules [15].

Theorem 2.1. *If K is a Mackey functor of finite projective dimension, then K has projective dimension 0 or 1.*

Proof. Suppose that L is a Mackey functor for which $\mathrm{Ext}^n(K, L) \neq 0$. For some free functor F we have an exact sequence $0 \to L' \to F \to L \to 0$. We shall see that F is of injective dimension 1 and hence if $n \geq 2$, $\mathrm{Ext}^n(K, L) \cong \mathrm{Ext}^{n+1}(K, L')$. It follows that if K has projective dimension ≥ 2 it has infinite projective dimension.

Lemma 2.2. (i) *The Mackey functors $\mathbb{A}_T \otimes \mathbb{Q}$ and $\mathbb{A}_T \otimes \mathbb{Q}/\mathbb{Z}$ are injective.*
(ii) *The Mackey functor $\mathbb{A}_T$ is of injective dimension* 1.
(iii) *An arbitrary free Mackey functor is of injective dimension* 1.

Proof. We first note that it is sufficient to prove (i). Indeed (ii) follows from (i) since $\mathrm{Hom}(\cdot, \mathbb{A}_T)$ is not exact. For (iii) it is enough to show $\mathrm{Ext}^n(K, F) = 0$ for $n \geq 2$ when F is free. When K is finitely generated this follows from (ii), and the general case follows by passing to direct limits.

We turn now to (i) and check that if we apply $\mathfrak{M}_G(\cdot, \mathbb{A}_T \otimes B)$ to the exact sequence $\mathbb{A}_U \to \mathbb{A}_V \to \mathbb{A}_W$ we obtain an exact sequence if B is an injective $\mathbb{Z}$-module. Of course by Proposition 1.4 this gives the sequence

$$\mathbb{A}(U \times T) \otimes B \leftarrow \mathbb{A}(V \times T) \otimes B \leftarrow \mathbb{A}(W \times T) \otimes B\ . \tag{1}$$

On the other hand, by definition of exactness we know that the sequence

$$\mathbb{A}(U \times T) \to \mathbb{A}(V \times T) \to \mathbb{A}(W \times T) \tag{2}$$

is exact. We shall give a formal duality argument to show that the sequence (1) is obtained from the sequence (2) by applying $\mathrm{Hom}_{\mathbb{Z}}(\cdot, B)$ and is therefore exact as required since B is injective.

Consider the map $e_U : \mathbb{A}_{U \times U} \to \mathbb{A}$ characterised by $e_U(\Delta_{U \times U}) = \Delta_U$. We obtain a pairing $\langle \cdot, \cdot \rangle : \mathbb{A}_U(*) \times \mathbb{A}_U(*) \to \mathbb{A}(*)$ as the composite of the map $\mathbb{A}_U(*) \times \mathbb{A}_U(*) \to \mathbb{A}_{U \times U}(*)$ induced by cartesian product, with the map $(e_U)_*$. The following lemma completes the proof, since we may compose the pairing with the various mark homomorphisms into $\mathbb{Z}$.

Lemma 2.3. (i) *Suppose* $f: \mathbb{A}_V \to \mathbb{A}_U$ *is a map. For any* $x \in \mathbb{A}(U)$, $y \in \mathbb{A}(V)$ *we have*

$$\langle f^* x, y \rangle = \langle x, f_* y \rangle .$$

(ii) *The pairing* $\langle \cdot, \cdot \rangle$ *is nondegenerate in the sense that* $\langle x, \cdot \rangle = 0$ *only if* $x = 0$.

Proof. (i) Using bilinearity of the pairing we may assume f corresponds to a G-set $\{\alpha, \beta\}$ over $U \times V$, and hence, since $f_* = \alpha_* \beta^*$, and $f^* = \beta_* \alpha^*$, we may assume by symmetry in the definitions that f is induced by a map of G-sets $f: V \to U$. Using bilinearity again, it is enough to verify the statement when $x: X \to U$, and $y: Y \to V$ are G-sets. We check that for a G-set $\{\gamma, \delta\}: Z \to U \times U$ over $U \times U$, $(e_U)_*(\{\gamma, \delta\})$ is the G-set $\{t \in Z \mid \gamma(t) = \delta(t)\}$ and so we have $\langle x, y \rangle = \{(\xi, \eta) \in X \times Y \mid x(\xi) = y(\eta)\}$. Now a short calculation verifies the result in this case.

(ii) Using bilinearity and the description of $\langle x, y \rangle$ above we reduce to the case when U is a transitive G-set G/H, and find that in this case if $x, y \in A(H)$ then $\langle x, y \rangle = \mathrm{ind}_H^G(xy)$ in $A(G)$. The result is now covered by Lemma 1 of [6]. $\square\square\square$

3. G_0 of the category of Mackey functors

In this section we consider the Grothendieck group $G_0(\mathfrak{M}_G)$ formed by taking one generator $[L]$ for each finitely generated Mackey functor and one relation $[L] = [L'] + [L'']$ for each short exact sequence $0 \to L' \to L \to L'' \to 0$. It is also useful to consider Mackey functors whose values are modules over a ring k. Our interest really lies with K_0 which we treat in Section 5: in general this behaves quite differently from G_0, but we show in Section 6 that for the category of Mackey functors with $|G|$ invertible K-theory and G-theory coincide.

Theorem 3.1. *If G is cyclic of prime power order or if $|G|$ is invertible in k, we have an isomorphism*

$$G_i(k\mathfrak{M}_G) \cong \bigoplus_{(H)} G_i(kW_G(H)) ,$$

where the sum extends over conjugacy classes of subgroups of G. In general we obtain a succession of exact sequences for calculating $G_*(k\mathfrak{M}_G)$ in terms of $G_*(kW_G(H))$ for the various subgroups H.

This refines the immediate consequence of [9, Theorem 12] or [18, (3.5)] that $G_0(\mathfrak{M}_G)$ is generated by the functors $RV = H^0 V$ or $H_0 V$ for the various simple $W_G(H)$-modules V and subgroups H of G. We warn however that neither H^0 nor H_0 are exact functors.

Proof. For the present we allow G to be any finite group; the need for restrictions will steadily emerge. We consider the set **All** of all subgroups of G, and more generally subsets $\mathscr{C}$ closed under conjugacy and passage to larger subgroups, and refer to them as cofamilies. Next we let $\mathfrak{M}_G/\mathscr{C}$ denote the full subcategory of $\mathfrak{M}_G$ consisting of functors M supported on $\mathscr{C}$ (i.e. if $K \not\in \mathscr{C}$ then $M(G/K) = 0$). Evidently we may choose a filtration

$$\emptyset = \mathscr{C}_0 \subseteq \mathscr{C}_1 \subseteq \cdots \subseteq \mathscr{C}_n = \mathbf{All}$$

of the lattice **All** by cofamilies $\mathscr{C}_i$ so that $\mathscr{C}_{i+1}$ is obtained from $\mathscr{C}_i$ by adding a single conjugacy class (H_i).

Accordingly we study a single link $\mathscr{C} \subseteq \mathscr{D} = \mathscr{C} \cup (H)$ in the chain. Theorem 3.1 is a corollary of the more general result (Theorem 3.5) applying to such a link. In particular, if we let $N = N_G(H)$ be the normalizer of H and $W = W_G(H) = N/H$ the Weyl group, we have the sequence

$$k\mathfrak{M}_G/\mathscr{C} \xrightarrow{i} k\mathfrak{M}_G/\mathscr{D} \xrightarrow{e} kW_G(H)\text{-mod} \tag{3}$$

of categories and functors, where i is the inclusion and $e(M) = M(G/H)$.

Lemma 3.2. *The map e of* (3) *is a quotient map of categories and i is the inclusion of the torsion subcategory. The same holds on the full subcategories of finitely generated objects.*

Proof. It is clear that $\mathfrak{M}_G/\mathscr{C}$ is the subcategory of objects taken to zero by e. It is also clear that e is exact.

Now it is well known that any W-module V generates a W-Mackey functor H_0V defined by $H_0V(W/K) = H_0(K; V)$. In particular $H_0V(W/1) = V$, and in fact the functor H_0 is left adjoint to $e : \mathfrak{M}_W \to \mathbb{Z}W\text{-mod}$ with the unit being the identity [18]. We shall explain in Propositions 3.6 and 3.7 and Corollary 3.8 how the methods of [9] allow us to use this to construct a left adjoint to e at the level of G-Mackey functors which has the identity as unit. We may now apply an easy formal lemma to complete the proof.

Lemma 3.3. *If $T : \mathscr{A} \to \mathscr{B}$ is an exact map of abelian categories with a left adjoint H so that the unit is a natural isomorphism, then T is a localisation of categories, and gives rise to a long exact sequence of K-theory.* □ □

Corollary 3.4. *With $\mathscr{D} = \mathscr{C} \cup (H)$,* (3) *induces a long exact sequence in G-theory, which ends*

$$\cdots \to G_1(kW_G(H)) \to G_0(k\mathfrak{M}_G/\mathscr{C})$$
$$\to G_0(k\mathfrak{M}_G/\mathscr{D}) \to G_0(kW_G(H)) \to 0\,. \qquad \square$$

So far the discussion has been reasonably general, but in order to obtain a splitting of the exact sequence of Corollary 3.4 we seem to need a hypothesis.

Theorem 3.5. *If H is not the intersection of two elements of $\mathscr{C}$ or if $|W_G(H)|$ is invertible in k, then the exact sequence of Corollary 3.4 splits to give*

$$G_i(k\mathfrak{M}_G/\mathscr{D}) \cong G_i(kW_G(H)) \oplus G_i(k\mathfrak{M}_G/\mathscr{C}) .$$

Proof. If the group order is invertible in k then the functor H is exact, and hence provides a splitting of e.

If H is not the intersection of two elements of $\mathscr{C}$ the map i is split by the map $s : \mathfrak{M}_G/\mathscr{D} \to \mathfrak{M}_G/\mathscr{C}$ defined by $s(M)(G/H) = 0$ and $s(M)(G/K) = M(G/K)$ if K is not conjugate to H. The point is that $s(M)$ need not satisfy the Mackey axiom if H is the intersection of two elements of $\mathscr{C}$. $\square$

Example. If $G \cong C_p \times C_p$ for some prime p then Theorem 3.5 shows that if $\mathscr{N}\mathscr{T}$ is the cofamily of nontrivial subgroups then

$$G_i(\mathfrak{M}_G/\mathscr{N}\mathscr{T}) = \bigoplus_{H \in \mathscr{N}\mathscr{T}} G_i(\mathbb{Z}G/H) ,$$

but we see no reason why the sequence

$$\cdots \to G_0(\mathfrak{M}_G/\mathscr{N}\mathscr{T}) \to G_0(\mathfrak{M}_G) \to G_0(\mathbb{Z}G) \to 0$$

should split.

We now summarise the results of [9] that we need, together with their reverses: the verification is straightforward.

Proposition 3.6. *If $\iota : H \to G$ is the inclusion of a subgroup, we have functors $\iota_* : \mathfrak{M}_G \to \mathfrak{M}_H$ defined by $(\iota_* M)(T) = M(G \times_H T)$ for an H-set T and $\iota^* : \mathfrak{M}_H \to \mathfrak{M}_G$ defined by $(\iota^* P)(S) = P(S)$ for a G-set S. These are each both left and right adjoint to one another. Thus*

$$\mathfrak{M}_G(\iota^* P, M) \cong \mathfrak{M}_H(P, \iota_* M) \quad \textit{and} \quad \mathfrak{M}_H(\iota_* M, P) \cong \mathfrak{M}_G(M, \iota^* P) .$$

The unit of the first and the counit of the second are induced by the inclusion of H-sets $j : T \to G \times_H T$ using appropriate variance. The counit of the first and the unit of the second are induced by the action map $a : G \times_H S \to S$ of G-sets. The fact that these units and counits are maps of Mackey functors come from the Mackey axiom applied to the pullback diagram

$$\begin{tikzcd} G \times_H S \arrow[r] \arrow[d] & G \times_H S' \arrow[d] \\ S \arrow[r] & S' \end{tikzcd}$$

of G-sets and the pullback diagram

$$\begin{tikzcd} T \arrow[r] \arrow[d] & T' \arrow[d] \\ G \times_H T \arrow[r] & G \times_H T' \end{tikzcd}$$

of H-sets. The triangular identities follow from diagrams of G-sets and H-sets which commute because the neutral element of G acts as the identity. □

Proposition 3.7. *If* $\varepsilon : G \to J$ *is a surjective map with kernel* K, *there are functors* $\varepsilon^* : \mathfrak{M}_G \to \mathfrak{M}_J$ *defined by* $(\varepsilon^* M)(U) = M(U)$ *for J-sets* U *and* $\varepsilon_* : \mathfrak{M}_J \to \mathfrak{M}_G$ *defined by* $(\varepsilon_* Q)(S) = Q(S^K)$ *for G-sets S. In this case however the functors are only left and right adjoint to each other on suitably restricted subcategories. Thus*

$$^K\mathfrak{M}_G(\varepsilon_* Q, M) \cong \mathfrak{M}_J(Q, \varepsilon^* M)$$

and

$$\mathfrak{M}_J(\varepsilon^* M, Q) \cong {}_K\mathfrak{M}_G(M, \varepsilon_* Q)\,,$$

where ${}_K\mathfrak{M}_G$ *is the full subcategory of* $\mathfrak{M}_G$ *in which the covariant maps* π_* *are zero whenever* $\pi : G/H \to G/H'$ *is the projection associated to a pair of subgroups with* $K \subseteq H'$ *but* $K \not\subseteq H$ *and where* $^K\mathfrak{M}_G$ *is the full subcategory of* $\mathfrak{M}_G$ *in which all contravariant maps* π^* *as above are zero. In this case the unit of the first and the counit of the second are the identity, and the counit of the first and the unit of the second are induced by the map* $S^K \to S$ *of G-sets. We note that since K is normal* $(G/H)^K \to G/H$ *is either the identity (if* $K \subseteq H$*) or the inclusion of the empty set (if* $K \not\subseteq H$*). The counit and unit induced by these maps are maps of Mackey functors by choice of the restricted subcategories, and the triangular identities are obvious.* □

Using the composites of the above adjunctions we obtain units and counits for adjunctions $H_0 \vdash e$ and $e \vdash H^0$ where we have allowed context to supply omitted ε's and ι's. Now the unit of the first adjunction and the counit of the second are the identity map; on the other hand the counit of the first and the unit of the second are not morphisms unless we restrict to suitable subcategories. However, one may verify that these subcategories may be taken to be considerably larger than the construction as a composite might suggest. This is essential to our applications, and also to the use of $R = H^0$ in [9, Theorem 12].

Corollary 3.8. *We have adjunctions*

$$^{[H]}\mathfrak{M}_G(H_0V, M) \cong \mathbb{Z}W_G(H)\text{-mod}(V, eM)$$

and

$$\mathbb{Z}W_G(H)\text{-mod}(eM, V) \cong {}_{[H]}\mathfrak{M}_G(M, H^0V)\,,$$

where $^{[H]}\mathfrak{M}_G$ *denotes the full subcategory of Mackey functors for which all proper restrictions from* G/H *are zero and* ${}_{[H]}\mathfrak{M}_G$ *denotes the subcategory for which all proper inductions to* G/H *are zero.* $\square$

The relevance of this is ensured by the fact that since (H) is minimal in $\mathcal{D}$ we have $\mathfrak{M}_G/\mathcal{D} \subseteq {}^{[H]}\mathfrak{M}_G \cap {}_{[H]}\mathfrak{M}_G$.

4. The endomorphism rings of indecomposable free functors

The study of projective Mackey functors which are summands of a multiple of $\mathbb{A}_X$ is equivalent to that of projective modules over the endomorphism ring $\mathscr{E}_X$ of $\mathbb{A}_X$. We therefore devote a short section to consideration of this ring. The first observation is that by Proposition 1.4 we have the additive isomorphism

$$\mathscr{E}_X \cong \mathbb{A}_X(X) = \mathbb{A}(X \times X)\,.$$

We shall make the ring structure explicit from this point of view, but we begin with the warning that it does *not* coincide with the ring structure via fibre product alluded to above. For example with $X = G$ we find $\mathscr{E}_G \cong \mathbb{Z}G$ whilst the fibre product ring structure is a product of $|G|$ copies of $\mathbb{Z}$.

The following basic lemma is easily verified.

Lemma 4.1. *If* $x = \{\alpha, \beta\} : S \to X \times X$ *and* $y = \{\gamma, \delta\} : T \to X \times X$ *are G-sets, then the product xy in* $\mathscr{E}_X$ *is the G-set*

$$\{\alpha\pi_1, \delta\pi_2\} : S \times_{(\beta,\gamma)} T \to X \times X\,. \qquad \square$$

We now turn to the particular case that $X = G/L$ is a transitive G-set. We begin by recalling that the maps $G/(H \cap {}^{\theta}K) \to G/H \times G/K$ given by $1(H \cap {}^{\theta}K) \mapsto (1H, \theta K)$ combine to give an isomorphism $\sum_{\theta\in\Theta} G/(H \cap {}^{\theta}K) \cong G/H \times G/K$, where Θ is a transversal of the double coset space $H\backslash G/K$ in G. In particular, for any $\gamma \in G$ and $H \subseteq L \cap {}^{\gamma}L$ we have a G-set

$$x(H, \gamma) : G/H \to G/L \cap {}^{\gamma}L \to G/L \times G/L$$

defined by $1H \mapsto (1L, \gamma L)$. As γ runs through a transversal of $L\backslash G/L$ in G and H runs through the $(L \cap {}^{\gamma}L)$-conjugacy classes of subgroups of $L \cap {}^{\gamma}L$ the elements $x(H, \gamma)$ give a $\mathbb{Z}$-basis of $\mathscr{E}_{G/L}$. The relation between the x's for various choices of representatives is as follows.

Lemma 4.2. (a) *If H is conjugate to H' in $L \cap {}^{\gamma}L$, then $x(H, \gamma) = x(H', \gamma)$.*
(b) *For $l \in L$ we have* (i) $x({}^{l}H, l\gamma) = x(H, \gamma)$ *and* (ii) $x(H, \gamma l) = x(H, \gamma)$. □

We now consider the product $x(H, \gamma)x(K, \delta)$, using Lemma 4.1. Indeed, it is easy to see that $G/H \times_{(\gamma, 1)} G/K$ contains the element $(1H, \theta K)$ precisely if $\gamma L = \theta L$, and that the corresponding term in the product is precisely $x(H \cap {}^{\theta}K, \theta\delta)$. Thus we find

$$x(H, \gamma)x(K, \delta) = \sum_{\{\theta \in \Theta \mid \theta L = \gamma L\}} x(H \cap {}^{\theta}K, \theta\delta) . \tag{4}$$

Now if $L = N$ is a normal subgroup all isotropy groups in $G/N \times G/N$ are N and $x(H, \gamma)$ depends only on $\bar{\gamma} = \gamma N$ and on the N-conjugacy class of $H \subseteq N$. We thus write $x(H)\bar{\gamma}$ for $x(H, \gamma)$ and the product formula becomes

$$x(H)\bar{\gamma}\, x(K)\bar{\delta} = \sum_{\{\theta \in \Theta \mid \bar{\theta} = \bar{\gamma}\}} x(H \cap {}^{\theta}K)\overline{\gamma\delta} . \tag{5}$$

This can be interpreted as a definition of the semidirect product in the following summary.

Corollary 4.3. *If N is a normal subgroup of G, then*

$$\mathscr{E}_{G/N} \cong A(N) \rtimes \mathbb{Z}[G/N] . \qquad \square$$

Remark 4.4. We may use the rings $\mathscr{E}_{G/L}$ to give another long exact sequence of G-theory. Indeed the evaluation map $\mathfrak{M}_G \to \mathscr{E}_{G/L}$-mod is a quotient map with torsion class consisting of Mackey functors vanishing at G/L. To show this we may use the following construction on an $\mathscr{E}_{G/L}$-module V. We note that for any free $\mathscr{E}_{G/L}$-module F there is a sum MF of Mackey functors $\mathbb{A}_{G/L}$ with $MF(G/L) \cong F$, and furthermore

$$\mathscr{E}_{G/L}\text{-mod}(F_1, F_0) = \mathfrak{M}_G(MF_1, MF_0) .$$

Thus if $F_1 \to F_0 \to V \to 0$ is exact we may define a Mackey functor $MV := \operatorname{cok}(MF_1 \to MF_0)$. It is then easy to check that

$$\mathscr{E}_{G/L}\text{-mod}(V, N(G/L)) = \mathfrak{M}_G(MV, N) .$$

However, we note that MV will usually not be supported on the cofamily generated by L.

5. K_0 of the category of Mackey functors

The projective class group $K_0(\mathfrak{M}_G)$ of $\mathfrak{M}_G$ is formed as usual by taking one generator for each isomorphism class of finitely generated projective Mackey functors and imposing the relation $[P \oplus Q] = [P] + [Q]$. Equivalently it is the Quillen K_0 of the category $\mathbf{P}(\mathfrak{M}_G)$ of finitely generated projective Mackey functors.

Since there are many different types of free Mackey functor it is natural to consider for each G-set X the category $\mathbf{P}(\mathfrak{M}_G; X)$ of those functors which are direct summands in multiples of $\mathbb{A}_X$. It is clear that this only depends on the isotropy groups which occur in X and that if X contains all isotropy types, $\mathbf{P}(\mathfrak{M}_G) = \mathbf{P}(\mathfrak{M}_G; X)$. From the correspondence of projectives to idempotents we see from Proposition 1.4 that we can express the K-theory of $\mathfrak{M}_G$ in terms of that of the rings studied in Section 4.

Lemma 5.1. *We have a natural isomorphism*

$$K_0(\mathbf{P}(\mathfrak{M}_G; X)) \cong K_0(\mathscr{E}_X)\,. \qquad \square$$

Evidently if $X = Y + Z$ is a sum, the ring $\mathscr{E}_X$ is naturally written as a matrix

$$\begin{pmatrix} \mathfrak{M}_G(\mathbb{A}_Y, \mathbb{A}_Y) & \mathfrak{M}_G(\mathbb{A}_Z, \mathbb{A}_Y) \\ \mathfrak{M}_G(\mathbb{A}_Y, \mathbb{A}_Z) & \mathfrak{M}_G(\mathbb{A}_Z, \mathbb{A}_Z) \end{pmatrix} = \begin{pmatrix} \mathscr{E}_Y & \mathbb{A}(Y \times Z) \\ \mathbb{A}(Z \times Y) & \mathscr{E}_Z \end{pmatrix}$$

and in the best imaginable case the K-theory of $\mathscr{E}_X$ would simply be the sum of the diagonal entries $\mathscr{E}_Y$ and $\mathscr{E}_Z$.

The corresponding constructions for Mackey functors are perhaps a little more obvious. There is evidently an exact map

$$\theta : \bigoplus_{(H)} \mathbf{P}(\mathfrak{M}_G; G/H) \to \mathbf{P}(\mathfrak{M}_G) \tag{6}$$

and it is natural to ask how close this is to inducing an isomorphism of K-theory. We show in Section 6 that it induces an isomorphism of K_0 if G is of prime order.

Remark 5.2. From Corollary 1.6 we see that the natural map $A(G) \to K_0(\mathfrak{M}_G)$ is injective. In a further effort to split θ we introduce certain 'ranks' Q_H in Section 8.

6. Mackey functors with the group order invertible

We give two approaches to the easy life away from the group order, one which works at the level of K-theory, and one which identifies all projective indecomposables exactly. Strictly speaking the first approach is unnecessary, but it illustrates the use of the Cartan map and shows the form that transfer maps take in the theory.

First, let us study $K_0(\mathfrak{M}_G)$ by comparing it to $G_0(\mathfrak{M}_G)$ which we understand quite well after Theorem 3.1, using the Cartan map. As before we let $k\mathfrak{M}_G$ denote the category of k-module valued Mackey functors for a ring k; of course the inclusion of the category of finitely generated projectives in all finitely generated Mackey functors as usual induces a map $K_*(k\mathfrak{M}_G) \to G_*(k\mathfrak{M}_G)$. Quillen's Resolution Theorem gives criteria under which this map is an isomorphism, and it is certainly satisfied if all functors are of finite projective dimension. Of course this is never the case integrally, but a transfer argument establishes it if the group order is invertible in the subring k of $\mathbb{Q}$. The analogue of this for the group ring kG is well known and can be proved by a simplified version of the argument below.

Theorem 6.1. *If $k \subseteq \mathbb{Q}$ is a ring in which $|G|$ is invertible, then the natural map*

$$K_i(k\mathfrak{M}_G) \xrightarrow{\cong} G_i(k\mathfrak{M}_G)$$

is an isomorphism, and hence in particular, by Theorem 3.1,

$$K_0(k\mathfrak{M}_G) \cong \bigoplus_{(H)} K_0(kW_G(H)) .$$

Proof. As remarked above, it is enough to show that every k-Mackey functor has finite projective dimension. Since $|G|$ is invertible in k this follows from the construction of a suitable transfer.

Indeed if $\iota : H \to G$ is the inclusion of a subgroup we have functors

$$\iota_* : \mathfrak{M}_G(M, N) \to \mathfrak{M}_H(\iota_* M, \iota_* N) \quad \text{and}$$
$$\iota^* : \mathfrak{M}_H(P, Q) \to \mathfrak{M}_G(\iota^* P, \iota^* Q)$$

as in Proposition 3.6. We may consider the composites and obtain endomorphisms of $\mathfrak{M}_G(M, N)$ and $\mathfrak{M}_H(P, Q)$ by means of the unit and counit of the adjunctions in Proposition 3.6; it is easy to calculate the result.

Lemma 6.2. (a) *$\iota^*\iota_*$ induces multiplication by $[G/H]$ on $\mathfrak{M}_G(M, N)$; thus a natural transformation $f : M \to N$ is taken to the one whose value at a G-set S is the composite of $M(S) \xrightarrow{f} N(S)$ with multiplication by $[G/H]$ in either $M(S)$ or $N(S)$.*

(b) *$\iota_*\iota^*$ induces the identity map of $\mathfrak{M}_H(P, Q)$.* □

We note that these maps all pass to Ext groups, so that if we know that $\mathrm{Ext}^n_{\mathfrak{M}_H}(\iota_* M, \iota_* N) = 0$ and that $[G/H]$ acts invertibly on $\mathrm{Ext}^n_{\mathfrak{M}_G}(M, N)$ we may conclude that $\mathrm{Ext}^n_{\mathfrak{M}_G}(M, N) = 0$.

Now we turn to the proof that k-Mackey functors for G are of finite projective dimension. We argue by induction on the group order, supposing for all proper subgroups K of G that all k-Mackey functors are of finite projective dimension if $|K|$ is invertible in k; this is well known for the trivial subgroup.

By Theorem 12 of [9] it is enough to show for some n that $\mathrm{Ext}^n_{\mathfrak{M}_G}(RV, N) = 0$ for every subgroup H and every $kW_G(H)$-module V. Furthermore, by Proposition 8 of [9] $[G/H]$ acts as $|N_G(H)/H|$ on RV, which is an isomorphism by the hypothesis on k. If H is a proper subgroup the result follows by induction. On the other hand, if $H = G$ it is easy to describe the functors RV; they are simply defined by $RV(G/G) = V$ whilst $RV(G/H) = 0$ if $H \neq G$. Certainly the k-module V has a resolution of length 1, and if V is k-free RV is the summand of $\mathbb{A} \otimes k$ specified by the idempotent e with $\phi_H(e) = 0$ if $H \neq G$, and hence is projective. □

A similar result holds for any regular ring k in which $|G|$ is invertible.

We next turn to the projective Mackey functors themselves.

Lemma 6.3. *Suppose H is a subgroup of G, $|W_G(H)|$ is invertible in k and V is a $kW_G(H)$-module.*

(a) *The norm map $H_0V \to H^0V$ is an isomorphism of Mackey functors.*

(b) *If V is projective as a k-module, then it is projective as a $kW_G(H)$-module.*

Proof. It is easy to verify that the norm map gives a map of $W_G(H)$-Mackey functors, and hence of G-Mackey functors. Part (a) is then immediate by a transfer argument in Tate cohomology, and part (b) (which is Maschke's theorem) follows by taking the $W_G(H)$-average of a k splitting. □

Theorem 6.4. *If $|G|$ is invertible in k, all projective Mackey functors can be written as a sum of the projective Mackey functors $H_0V \cong H^0V$ for various subgroups H and k-projective $kW_G(H)$-modules V.*

Proof. By [9, Theorem 12] it is enough to show that the Mackey functors $H_0V \cong H^0V$ are projective. We therefore show that any surjective map $L \twoheadrightarrow H_0V$ may be split.

We combine the use of idempotents in the localised Burnside ring with the fact that H_0 is left adjoint to evaluation on the category of Mackey functors vanishing below H. Indeed, since V is projective, the map $L(G/H) \twoheadrightarrow V$ of $W_G(H)$-modules is split, so that if L vanishes below H the splitting gives rise to a map of Mackey functors. This is still a splitting since a map of H_0V is determined by its behaviour at G/H.

If L does not vanish below H we consider the idempotent e of $A(G)[1/|G|]$ defined by $\phi_K(e) = 0$ iff K is contained in H. Of course $L = eL \oplus (1-e)L$, eL is zero beneath H and $H_0V = eH_0V$; the map $eL \twoheadrightarrow H_0V$ is split by the above argument. □

7. K_0 and G_0 of the category of Mackey functors for $G = C_p$

We show here that the methods of Sections 3–6 are adequate for groups of prime order p.

Theorem 7.1. (a) *The map θ of Section 5 induces an isomorphism of K_0*

$$K_0(\mathfrak{M}_{C_p}) \cong K_0(\mathbb{Z}C_p) \oplus K_0(A(C_p)) .$$

(b) *We have the isomorphism*

$$G_0(\mathfrak{M}_{C_p}) \cong G_0(\mathbb{Z}C_p) \oplus G_0(\mathbb{Z}) .$$

Remarks. (a) Of course by another theorem of Rim, $K_0(\mathbb{Z}C_p) \cong K_0(\mathbb{Z}[\zeta_p])$, where $\zeta_p = e^{2\pi i/p}$ [14, 15], and this group is much studied, but far from understood. In particular it is nontrivial for the prime 23 and all sufficiently large primes. On the other hand it is not too hard to calculate $K_0(A(C_p)) \cong \mathbb{Z} \oplus \mathbb{Z}/((p-1)/2)$ if p is odd (and $\mathbb{Z}$ for $p = 2$) [5].

(b) Both G_0 groups on the right-hand side are isomorphic to the corresponding K_0-groups, since $G_0(\mathbb{Z}C_p) \cong G_0(\mathbb{Z}[\zeta_p])$ [2, (39.26)] and Dedekind domains are regular.

Proof. Part (b) is immediate from Theorem 3.1. For part (a) we use Lemma 5.1 and then a succession of Mayer–Vietoris sequences to simplify the rings involved. The first arises from a pullback square which contains those used by Milnor and tom Dieck and Petrie as its diagonal entries. For this purpose we let $A = A(C_p)$ and $C = \mathbb{Z} \times \mathbb{Z}$ be its integral closure in its total ring of fractions. Under this inclusion $[C_p/1] \mapsto (0, p)$ so that in particular $pC \subseteq A$. We then have

$$\mathscr{E}_{C_p/1 + C_p/C_p} \cong \begin{pmatrix} \mathbb{Z}C_p & \mathbb{Z} \\ \mathbb{Z} & A(C_p) \end{pmatrix},$$

where the structure maps from $\mathbb{Z}$ to $\mathbb{Z}C_p$ and $A(C_p)$ correspond to the norm element and $[C_p/1]$ respectively. With the further convention that $\mathbb{Z}'$ also denotes the integers and the structure maps $\mathbb{Z}' \to \mathbb{Z}$ are the identity and the structure maps $\mathbb{Z} \to \mathbb{Z}'$ are multiplication by p, the reader may verify that we have a pullback diagram of rings

$$\begin{array}{ccc} \begin{pmatrix} \mathbb{Z}C_p & \mathbb{Z} \\ \mathbb{Z} & A(C_p) \end{pmatrix} & \longrightarrow & \begin{pmatrix} \mathbb{Z}[\zeta_p] & 0 \\ 0 & A/pC \end{pmatrix} \\ \big\downarrow & & \big\downarrow \\ \begin{pmatrix} \mathbb{Z}' & \mathbb{Z} \\ \mathbb{Z} & C \end{pmatrix} & \longrightarrow & \begin{pmatrix} \mathbb{Z}/p & 0 \\ 0 & C/pC \end{pmatrix} \end{array}$$

The two rings on the right are understood, and so the Mayer–Vietoris sequence reduces us to the study of the lower left-hand ring. To analyse this we use a second pullback square

$$\begin{array}{ccc}\begin{pmatrix}\mathbb{Z}' & \mathbb{Z}\\ \mathbb{Z} & C\end{pmatrix} & \longrightarrow & \begin{pmatrix}\mathbb{Z}/p & 0\\ 0 & \mathbb{Z}\end{pmatrix}\\ \downarrow & & \downarrow\\ \begin{pmatrix}\mathbb{Z}' & \mathbb{Z}\\ \mathbb{Z} & \mathbb{Z}'\end{pmatrix} & \longrightarrow & \begin{pmatrix}\mathbb{Z}/p & 0\\ 0 & 0\end{pmatrix}\end{array}$$

Again the right-hand rings are understood and so we are reduced to study of the lower left-hand ring. We use a third and final pullback diagram

$$\begin{array}{ccc}\begin{pmatrix}\mathbb{Z}' & \mathbb{Z}\\ \mathbb{Z} & \mathbb{Z}'\end{pmatrix} & \longrightarrow & \begin{pmatrix}\mathbb{Z}' & \mathbb{Z}'\\ \mathbb{Z}' & \mathbb{Z}'\end{pmatrix}\\ \downarrow & & \downarrow\\ \begin{pmatrix}\mathbb{Z}'/p & 0\\ \mathbb{Z}/p & \mathbb{Z}'/p\end{pmatrix} & \longrightarrow & \begin{pmatrix}\mathbb{Z}'/p & \mathbb{Z}'/p\\ \mathbb{Z}'/p & \mathbb{Z}'/p\end{pmatrix}\end{array}$$

The two right-hand rings are just ordinary matrix rings and hence understood by Morita equivalence isomorphism $K_0(M_2(R)) \cong K_0(R)$. The lower left-hand ring has the nilpotent ideal

$$\begin{pmatrix}0 & 0\\ \mathbb{Z}/p & 0\end{pmatrix}$$

by which the quotient is $\mathbb{Z}/p \times \mathbb{Z}/p$, whose K-theory is understood. The detailed analysis of the Mayer–Vietoris sequences gives an exact sequence

$$0 \to \mathbb{Z}/\left(\frac{p-1}{2}\right) \to K_0(\mathfrak{M}_{C_p}) \to \mathbb{Z}^2 \oplus K_0(\mathbb{Z}[\zeta_p]) \to 0$$

which can be split by the use of θ. □

8. Further examples and connections with defect sets

We want to explain the relation of the present considerations to Dress's theory of defect sets. Since $\mathbb{A}_{G/H}$ is generated at G/H by $\Delta_{G/H}$ it is natural to consider the singular submodule $S_H(L)$ of $L(G/H)$ defined by $S_H(L) = S_H^{\text{ind}}(L) + S_H^{\text{res}}(L)$, where

$$S_H^{\text{ind}}(L) = \sum_{J \subset H} \text{ind}_J^H(L(G/J)),$$

$$S_H^{\text{res}}(L) = \sum_{K \supset H} \text{res}_H^K(L(G/K)).$$

We then define the indecomposable quotient by

$$Q_H(L) = L(G/H)/S_H(L)\,,$$

and it measures the number of necessary generators at G/H. The point is that we then have

$$\begin{aligned}
&Q_H(\mathbb{A}_{G/K}) = 0 \quad && \text{if } K \text{ is not conjugate to } H\,,\\
&Q_H(L \oplus M) = Q_H(L) \oplus Q_H(M)\,,\\
&Q_H(L) = 0 \;\Leftrightarrow && \text{there is a } G\text{-set } X \text{ with } G/H \not\subseteq X\\
& && \text{and an epimorphism } \mathbb{A}_X \twoheadrightarrow M\,.
\end{aligned} \tag{7}$$

Lemma 8.1. (a) *The subgroup $S_H(\mathbb{A}_{G/H})$ is an ideal in $\mathscr{E}_{G/H}$.*

(b) *The subgroup $S_H(L)$ is an $\mathscr{E}_{G/H}$-submodule of $L(G/H)$.*

(c) *The $\mathscr{E}_{G/H}$-module $Q_H(L)$ is a module over the ring $Q(\mathscr{E}_{G/H}) := \mathscr{E}_{G/H}/S_H(\mathbb{A}_{G/H})$.* □

The two extreme cases $H = G$ and $H = 1$ behave in very different ways.

Proposition 8.2. (a) *For all groups $Q_G(\mathbb{A}_{G/G}) \neq 0$.*

(b) *For all sufficiently complicated groups $Q_1(\mathbb{A}_{G/1}) = 0$.*

Proof. (a) This is Dress's theorem that the defect set of the Burnside functor is trivial [7, Theorem 3].

(b) It is easy to check that $S_1(\mathbb{A}_{G/1}) = \sum_{H \neq 1} (\mathbb{Z}G)^H$ is the singular submodule $S(\mathbb{Z}G)$ studied in [1], where it was shown to be almost always equal to $\mathbb{Z}G$ (for example if G has p-rank ≥ 2 for two distinct primes p). □

The other systematic result that is not hard to prove is that p-groups behave quite well.

Proposition 8.3. *If G is a p-group, then $Q_H(\mathbb{A}_{G/H}) \neq 0$ for all subgroups H.*

Proof. Since $\Delta_{G/H}$ generates $\mathbb{A}_{G/H}$, every element of $\mathbb{A}_{G/H}(X)$ can be expressed as a sum of transfers and restrictions of it. If $Q_H(\mathbb{A}_{G/H}) = 0$, then $\Delta_{G/H}$ is itself a sum of proper transfers and restrictions. Combining these two facts we see that $\Delta_{G/H}$ is a sum of terms $\mathrm{ind}_J^H \mathrm{res}_J^H(\Delta_{G/H})$ and $\mathrm{res}_H^K \mathrm{ind}_H^K(\Delta_{G/H})$ for various groups $J \subset H$ and $K \supset H$. Now we have the augmentation

$$\varepsilon : \mathscr{E}_{G/H} \to \mathbb{Z}$$

obtained by counting the number of orbits in a G-set over $G/H \times G/H$. Clearly

$\varepsilon(\Delta_{G/H}) = 1$, however for a p-group the augmentations of $\mathrm{ind}_J^H \mathrm{res}_J^H(x)$ and $\mathrm{res}_H^K \mathrm{ind}_H^K(x)$ are both divisible by p. □

The relevance of these results to projectives is in the existence of nontrivial stably free projectives.

Remark 8.4. If $Q_H(\mathbb{A}_{G/H}) = 0$, then by (7) we have $\mathbb{A}_X \cong \mathbb{A}_{G/H} \oplus P$ for some projective P and some G-set X not containing G/H. Accordingly P is stably free, but by Corollary 1.6 it is not free.

The functors Q also go some way towards splitting the map θ of Section 5. Indeed for each subgroup H we have the diagram

$$\begin{tikzcd} K_0(\mathbf{P}(\mathfrak{M}_G; G/H)) \arrow[r, "\theta_H"] \arrow[d] & K_0(\mathfrak{M}_G) \arrow[dl, "Q_H"] \\ K_0(Q(\mathscr{E}_{G/H})) & \end{tikzcd}$$

Hence

$$\ker(\theta) \subseteq \bigoplus_{(H)} \ker(\bar{Q}_H : K_0(\mathscr{E}_{G/H}) \to K_0(Q(\mathscr{E}_{G/H}))) ,$$

where $\bar{Q}_H = Q_H \theta_H$. Considering the case of a group of prime order we see that $Q(\mathscr{E}_{C_p/1}) = \mathbb{Z}[\zeta_p]$ and $Q(\mathscr{E}_{C_p/C_p}) = \mathbb{Z}$, so that whilst this argument gives some information it also loses some.

Appendix A. Applications to stable equivariant homotopy theory

In this section we state the analogues for G-spectra [12] of Wall's theorems characterising the finite dimensionality and finiteness of CW-complexes up to homotopy equivalence from cohomological data [17]. It is not necessary to give proofs since Wall's proofs translate directly into the present context. However, we shall give definitions so that the reader understands what this context entails.

We work in a stable homotopy category of G-spectra, such as that of Lewis and May; we are interested in when a G-spectrum X is equivalent to a G-CW-spectrum K which is either

(i) finite (i.e. formed from a finite number of G-cells),

(ii) finite dimensional (i.e. formed using G-cells in a finite range of dimensions), or

(iii) (a,b)-dimensional (i.e. constructed using d-cells only for $a \leq d \leq b$).

We aim to give cohomological criteria so we briefly recall some facts about homology and cohomology. The fact that suspension by a real representation is an equivalence of the stable category has two important consequences. Firstly, cohomology can be graded over RO(G), although we shall only refer to integer gradings. More important to us is the existence of transfers, so that all cohomology groups are Mackey functor valued. Accordingly, ordinary cohomology [11] has coefficients in a Mackey functor. In the nonequivariant context the importance of $\mathbb{Z}$ arises since the degree of a map between spheres of the same dimension classifies the map. Equivariantly we have Segal's theorem that $\pi_0^G(S^0) = A(G)$, so that the rôle of $\mathbb{Z}$ is taken by the Mackey functor $\mathbb{A} = \pi_0^{\bullet}(S^0)$ that we have come to know so well. Accordingly any unspecified coefficients are in $\mathbb{A}$.

Since one can form an Eilenberg–MacLane spectrum $H\mathbb{A}$ representing ordinary cohomology with $\mathbb{A}$ coefficients by attaching cells of dimension ≥ 2 to S^0 so as to kill homotopy groups in positive dimensions, we immediately have the Hurewicz theorem stating that the Hurewicz map

$$\pi_n^G(X) \to H_n^G(X; \mathbb{A})$$

is an isomorphism if X is $(n-1)$-connected. The importance of the free Mackey functors comes about since they are the homology and homotopy functors of a wedge of G-cells of dimension n, i.e. of $\Sigma^n(T_+)$ for discrete G-sets T:

$$\mathbb{A}_T \cong \pi_n^{\bullet}(\Sigma^n(T_+)) \cong H_n^{\bullet}(\Sigma^n(T_+)) .$$

As usual one has two Eckmann–Hilton dual types of building blocks associated with a Mackey functor L: the Eilenberg–MacLane spectra with nonzero homotopy groups only in a single dimension and the Moore spectra, bounded below and with homology concentrated in a single dimension.

It is easy to construct an Eilenberg–MacLane spectrum HL for L in dimension 0: one takes an exact sequence $\mathbb{A}_{T_1} \to \mathbb{A}_{T_0} \to L \to 0$, constructs a 1-skeleton as the mapping cone of the corresponding map $(T_1)_+ \to (T_0)_+$ and then kills higher homotopy groups. It is also easy to see by the Hurewicz and Whitehead theorems that HL is unique up to equivalence.

Similarly *provided L has projective dimension* ≤ 1, it is easy to construct a Moore spectrum of type ML (i.e. a bounded below spectrum X with $H_0^{\bullet}(X; \mathbb{A}) = L$) and to show it is unique. There are examples of Mackey functors L of infinite dimension for which no Moore spectrum exists (for instance the Mackey functor of the trivial G-module $\mathbb{Z}$ has no Moore spectrum if $G \neq 1$ (see Appendix B)). We do not have examples where L is of infinite dimension and a spectrum of type ML does exist.

It is clear by considering the spectral sequence of the skeletal filtration that we have for any Mackey functor K the isomorphism

$$H_G^*(ML; K) \cong \mathrm{Ext}_{\mathfrak{M}_G}^{\bullet}(L, K) ,$$

where ML denotes an arbitrary G-spectrum of type ML if such a thing exists. We may now state the theorems, which concern a G-spectrum X.

Theorem A.1. *If $a < b$ then X is equivalent to an (a, b)-dimensional spectrum if and only if the following four conditions are satisfied:*

(i) *X is bounded below,*

(ii) *$H_n^{\bullet}(X; \mathbb{A}) = 0$ for $n < a$,*

(iii) *$H_n^{\bullet}(X; \mathbb{A}) = 0$ for $n > b$,*

(iv) *$H^{b+1}(X; L) = 0$ for all Mackey functors L.*

Remark A.2. If X satisfies (i)–(iv) with $a = b$, then X is equivalent to a Moore spectrum $\Sigma^a MP$ with P a projective Mackey functor. This is equivalent to an $(a, a+1)$-dimensional spectrum (since it is a retract of a wedge of spheres), but need not be equivalent to an (a, a)-dimensional spectrum.

The proof of the theorem proceeds by inductive construction of skeleta for X and then uses (iv) and the Eilenberg swindle to ensure that the process finishes in dimension b.

We now turn to more delicate questions.

Theorem A.3. *If P is a finitely generated projective Mackey functor, then*

(a) *there is a G-CW-spectrum M of type MP with finite skeleta,*

(b) *M may be taken to be of finite dimension iff P is stably free,*

(c) *if P is stably free, P may be taken to be of dimension 1; it may be taken to be of dimension 0 iff P is free.*

In view of (b) it is perhaps not so surprising that the finiteness obstruction lies in the projective class group. Indeed if X satisfies the conditions of Theorem A.1 we may take an equivalent finite dimensional G-CW-spectrum K and consider the homology spectral sequence of the skeletal filtration. This is a chain complex of free functors and if its homology groups are finitely generated there is a chain equivalent complex $P.$ of finite length whose terms P_i are finitely generated projective functors. One may define the finiteness obstruction $\sigma(X) = \sum_i (-1)^i [P_i]$ in $\tilde{K}_0(\mathfrak{M}_G) := K_0(\mathfrak{M}_G)/A(G)$ and one may check that it only depends on the homotopy type of X.

Theorem A.4. *If $a < b$ and X satisfies the four conditions of Theorem A.1 and if in addition $H_i^{\bullet}(X; \mathbb{A})$ is finitely generated for each i, then X is equivalent to a finite (a, b)-dimensional G-CW-spectrum iff the finiteness obstruction $\sigma(X) \in \tilde{K}_0(\mathfrak{M}_G)$ is equal to zero.*

Remarks A.5. (a) In the nonequivariant case the finiteness obstruction is strictly an unstable phenomenon. In the equivariant case the existence of an obstruction to finiteness is well known [12, p. 121], but the cited observation appeals to unstable geometric arguments. The present discussion completely reduces the problem to a difficult algebraic one.

(b) In particular we see from Section 7 that for a group of prime order p, if $p=2$ or 3 there is no obstruction to finiteness, but for primes $p \geq 5$ the obstruction lies in the nonzero group $\mathbb{Z}/((p-1)/2) \oplus \tilde{K}_0(\mathbb{Z}[\zeta_p])$.

(c) The case $a=b$ is dealt with in Theorem A.3.

(d) It is easy to see from the proof that for each $\sigma \in \bar{K}(\mathfrak{M}_G)$ the $(b-1)$-type of X contains a G-spectrum Y with $\sigma(Y)=\sigma$.

Remarks A.6. If we work in the category of spectra localised so as to invert a set π of primes, then the appropriate home for the finiteness obstruction is $\bar{K}_0(\mathbb{Z}[\pi^{-1}]\mathfrak{M}_G)$. For instance in the category of rational spectra the obstruction lies in the quotient of $K_0(\mathbb{Q}\mathfrak{M}_G) = \bigoplus_{(H)} K_0(\mathbb{Q}W_G(H))$ by the subgroup generated by free functors, which is eminently approachable. In particular it is easy to verify that for groups of prime order the finiteness obstructions for any multiple of an idempotent summand of S^0 or G_+ is not zero. It is amusing to compare this with Remark A.5(b).

Appendix B. There is no stable G-Moore spectrum for $\mathbb{Z}$ if $G \neq 1$

It is enough to deal with the case of a group $G = \langle g \rangle$ of prime order p, since every nontrivial group has a subgroup of prime order. It turns out that $\mathbb{Z}$ has a periodic resolution of period 4 (even if $p=2$). Indeed we may write it explicitly using Proposition 1.4 to name the maps. We let x denote G as an element of $A(G)$, $N = 1 + g + \cdots + g^{p-1}$ and $1' \in \mathbb{A}(G)$ denote the identity map. The resolution is then

$$\cdots \to \mathbb{A} \xrightarrow{p-x} \mathbb{A} \xrightarrow{N} \mathbb{A}_G \xrightarrow{1-g} \mathbb{A}_G \xrightarrow{1'} \mathbb{A} \xrightarrow{p-x} \mathbb{A} \to \mathbb{Z} \to 0 .$$

Now if a spectrum X of type ML exists we may take the homology spectral sequence of the skeletal filtration: it collapses to give a resolution of L by free Mackey functors. Furthermore one may easily check that if a spectrum X of type ML exists there is an equivalent G-CW-spectrum with any specified resolution of L as its homology spectral sequence [3, 5.2]. To show there is no spectrum of type ML it is therefore enough to show that a particular resolution cannot be realised.

We note that if $M\mathbb{Z}$ is realisable then so is ML when L is any one of the kernels occurring in the resolution: we just collapse an appropriate skeleton of the realisation. It is convenient to work with $L = \mathrm{im}(p-x)$ and to use the appropriate truncation of the above periodic resolution. We thus take $X^{(0)} = S^0$ and attach a 1-cell by the unstable map $1' : G_+ \to S^0$ taking all of G to the nonbase point. Thus $X^{(1)}$ is the cage space consisting of the north and south poles together with p lines of longitude joining them. Now it is easy to check that we may take $X^{(2)} = S^\eta$ where η is the two-dimensional real representation in which g acts as rotation by $2\pi/p$. Furthermore we may check that this is the only possible choice for $X^{(2)}$, since the lift of $(1-g) : \Sigma G_+ \to \Sigma G_+$ to a map $\Sigma G_+ \to X^{(1)}$ is unique

($[\Sigma G_+, S^0]^G = \mathbb{Z}/2$, whilst $[\Sigma G_+, X^{(1)}]^G$ is torsion free). To continue the construction we must lift $N : \Sigma^2 S^0 \to \Sigma^2 G_+$ to a map $\Sigma^2 S^0 \to X^{(2)}$, but a short calculation shows this is impossible. In fact we have a long exact sequence

$$\begin{aligned}[S^2, S^\eta]^G \to [S^2, \Sigma^2 G_+]^G \to [S^2, \Sigma X^{(1)}]^G \\ \to [S^2, \Sigma S^\eta]^G \to [S^2, \Sigma^3 G_+]^G = 0\end{aligned}$$

and one may check that $[S^1, S^\eta]^G = \mathbb{Z}/2$ whilst $[S^1, X^{(1)}]^G = \mathbb{Z}/2 \oplus \mathbb{Z}/p$.

Remark. Our argument is sufficiently elementary that the obstruction theory of [3] is unnecessary. We also note that Costenoble and Waner [3, 4] are more concerned with realising Bredon coefficient systems as Moore 1-spectra with G-action. In particular they do not give any examples of the present type: it is obvious that S^0 is a Moore 1-spectrum for $\mathbb{Z}$.

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