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A resolution of the $K(2)$-local sphere at the prime 3

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Abstract

We develop a framework for displaying the stable homotopy theory of the sphere, at least after localization at the second Morava K-theory $K(2)$. At the prime 3, we write the spectrum $L_{K(2)}S^0$ as the inverse limit of a tower of fibrations with four layers. The successive fibers are of the form E_2^{hF} where F is a finite subgroup of the Morava stabilizer group and E_2 is the second Morava or Lubin-Tate homology theory. We give explicit calculation of the homotopy groups of these fibers. The case $n = 2$ at $p = 3$ represents the edge of our current knowledge: $n = 1$ is classical and at $n = 2$, the prime 3 is the largest prime where the Morava stabilizer group has a p-torsion subgroup, so that the homotopy theory is not entirely algebraic.

The problem of understanding the homotopy groups of spheres has been central to algebraic topology ever since the field emerged as a distinct area of mathematics. A period of calculation beginning with Serre's computation of the cohomology of Eilenberg-MacLane spaces and the advent of the Adams spectral sequence culminated, in the late 1970s, with the work of Miller, Ravenel, and Wilson on periodic phenomena in the homotopy groups of spheres and Ravenel's nilpotence conjectures. The solutions to most of these conjectures by Devinatz, Hopkins, and Smith in the middle 1980s established the primacy of the "chromatic" point of view and there followed a period in which the community absorbed these results and extended the qualitative picture of stable homotopy theory. Computations passed from center stage, to some extent, although there has been steady work in the wings – most notably by Shimomura and his coworkers, and Ravenel, and more lately by Hopkins and

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his coauthors in their work on topological modular forms. The amount of interest generated by this last work suggests that we may be entering a period of renewed focus on computations.

In a nutshell, the chromatic point of view is based on the observation that much of the structure of stable homotopy theory is controlled by the algebraic geometry of formal groups. The underlying geometric object is the moduli stack of formal groups. Much of what can be proved and conjectured about stable homotopy theory arises from the study of this stack, its stratifications, and the theory of its quasi-coherent sheaves. See for example, the table in Section 2 of [11].

The output we need from this geometry consists of two distinct pieces of data. First, the chromatic convergence theorem of [21, §8.6] says the following. Fix a prime p and let $E(n)_*$, $n \geq 0$ be the Johnson-Wilson homology theories and let L_n be localization with respect to $E(n)_*$. Then there are natural maps $L_nX \to L_{n-1}X$ for all spectra X, and if X is a p-local finite spectrum, then the natural map

$$X \longrightarrow \mathrm{holim} L_nX$$

is a weak equivalence.

Second, the maps $L_nX \to L_{n-1}X$ fit into a good fiber square. Let $K(n)_*$ denote the n-th Morava K-theory. Then there is a natural commutative diagram

$$\begin{array}{ccc} L_nX & \longrightarrow & L_{K(n)}X \\ \downarrow & & \downarrow \\ L_{n-1}X & \longrightarrow & L_{n-1}L_{K(n)}X \end{array} \tag{0.1}$$

which for any spectrum X is a homotopy pull-back square. It is somewhat difficult to find this result in the literature; it is implicit in [13].

Thus, if X is a p-local finite spectrum, the basic building blocks for the homotopy type of X are the Morava K-theory localizations $L_{K(n)}X$.

Both the chromatic convergence theorem and the fiber square of (0.1) can be viewed as analogues of phenomena familiar in algebraic geometry. For example, the fibre square can be thought of as an analogue of a Mayer-Vietoris situation for a formal neighborhood of a closed subscheme and its open complement (see [1]). The chromatic convergence theorem can be thought of as a result which determines what happens on a variety S with a nested sequence of closed sub-schemes S_n of codimension n by what happens on the open subvarieties $U_n = S - S_n$ (See [9, §IV.3], for example.) This analogy can be made precise using the moduli stack of p-typical formal group laws for S and, for S_n, the substack which classifies formal groups of height at least n. Again see [11]; also, see [19] for more details.

In this paper, we will write (for $p = 3$) the $K(2)$-local stable sphere as a very small homotopy inverse limit of spectra with computable and computed homotopy groups. Specifying a Morava K-theory always means fixing a prime p and a formal group law of height n; we unapologetically focus on the case $p = 3$ and $n = 2$ because this is at the edge of our current knowledge. The homotopy type and homotopy groups for $L_{K(1)}S^0$ are well understood at all primes and are intimately connected with the J-homomorphism; indeed, this calculation was one of the highlights of the computational period of the 1960s. If $n = 2$ and $p > 3$, the Adams-Novikov spectral sequence (of which more is said below) calculating $\pi_*L_{K(2)}S^0$ collapses and cannot have extensions; hence, the problem becomes algebraic, although not easy. Compare [26].

It should be noticed immediately that for $n = 2$ and $p = 3$ there has been a great deal of calculations of the homotopy groups of $L_{K(2)}S^0$ and closely related spectra, most notably by Shimomura and his coauthors. (See, for example, [23], [24] and [25].) One aim of this paper is to provide a conceptual framework for organizing those results and produce further advances.

The $K(n)$-local category of spectra is governed by a homology theory built from the Lubin-Tate (or Morava) theory E_n. This is a commutative ring spectrum with coefficient ring

$$(E_n)_* = W(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]][u^{\pm 1}]$$

with the power series ring over the Witt vectors in degree 0 and the degree of u equal to -2. The ring

$$(E_n)_0 = W(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]]$$

is a complete local ring with residue field $\mathbb{F}_{p^n}$. It is one of the rings constructed by Lubin and Tate in their study of deformations for formal group laws over fields of characteristic p. See [17].

As the notation indicates, E_n is closely related to the Johnson-Wilson spectrum $E(n)$ mentioned above.

The homology theory $(E_n)_*$ is a complex-oriented theory and the formal group law over $(E_n)_*$ is a universal deformation of the Honda formal group law Γ_n of height n over the field $\mathbb{F}_{p^n}$ with p^n elements. (Other choices of formal group laws of height n are possible, but all yield essentially the same results. The choice of Γ_n is only made to be explicit; it is the usual formal group law associated by homotopy theorists to Morava K-theory.) Lubin-Tate theory implies that the graded ring $(E_n)_*$ supports an action by the group

$$\mathbb{G}_n = \mathrm{Aut}(\Gamma_n) \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p).$$

The group $\mathrm{Aut}(\Gamma_n)$ of automorphisms of the formal group law Γ_n is also known as the Morava stabilizer group and will be denoted by $\mathbb{S}_n$. The Hopkins-Miller theorem (see [22]) says, among other things, that we can lift this action to an action on the spectrum E_n itself. There is an Adams-Novikov spectral sequence

$$E_2^{s,t} := H^s(\mathbb{S}_n, (E_n)_t)^{\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)} \Longrightarrow \pi_{t-s} L_{K(n)} S^0.$$

(See [12] for a basic description.) The group $\mathbb{G}_n$ is a profinite group and it acts continuously on $(E_n)_*$. The cohomology here is continuous cohomology. We note that by [5] $L_{K(n)}S^0$ can be identified with the homotopy fixed point spectrum $E_n^{h\mathbb{G}_n}$ and the Adams-Novikov spectral sequence can be interpreted as a homotopy fixed point spectral sequence.

The qualitative behaviour of this spectral sequence depends very much on qualitative cohomological properties of the group $\mathbb{S}_n$, in particular on its cohomological dimension. This in turn depends very much on n and p.

If $p-1$ does not divide n (for example, if $n < p-1$) then the p-Sylow subgroup of $\mathbb{S}_n$ is of cohomological dimension n^2. Furthermore, if $n^2 < 2p-1$ (for example, if $n = 2$ and $p > 3$) then this spectral sequence is sparse enough so that there can be no differentials or extensions.

However, if $p-1$ divides n, then the cohomological dimension of $\mathbb{S}_n$ is infinite and the Adams-Novikov spectral sequence has a more complicated behaviour. The reason for infinite cohomological dimension is the existence of elements of order p in $\mathbb{S}_n$. However, in this case at least the virtual cohomological dimension remains finite, in other words there are finite index subgroups with finite cohomological dimension. In terms of resolutions of the trivial module $\mathbb{Z}_p$, this means that while there are no projective resolutions of the trivial $\mathbb{S}_n$-module $\mathbb{Z}_p$ of finite length, one might still hope that there exist "resolutions" of $\mathbb{Z}_p$ of finite length in which the individual modules are direct sums of modules which are permutation modules of the form $\mathbb{Z}_p[[\mathbb{G}_2/F]]$ where F is a finite subgroup of $\mathbb{G}_n$. Note that in the case of a discrete group which acts properly and cellularly on a finite dimensional contractible space X such a "resolution" is provided by the complex of cellular chains on X.

This phenomenon is already visible for $n = 1$ in which case $\mathbb{G}_1 = \mathbb{S}_1$ can be identified with $\mathbb{Z}_p^\times$, the units in the p-adic integers. Thus $\mathbb{G}_1 \cong \mathbb{Z}_p \times C_{p-1}$ if p is odd while $\mathbb{G}_1 \cong \mathbb{Z}_2 \times C_2$ if $p = 2$. In both cases there is a short exact sequence

$$0 \to \mathbb{Z}_p[[\mathbb{G}_1/F]] \to \mathbb{Z}_p[[\mathbb{G}_1/F]] \to \mathbb{Z}_p \to 0$$

of continuous $\mathbb{G}_1$-modules (where F is the maximal finite subgroup of $\mathbb{G}_1$). If p is odd this sequence is a projective resolution of the trivial module while for $p = 2$ it is only a resolution by permutation modules. These resolutions are the algebraic analogues of the fibrations (see [12])

$$L_{K(1)}S^0 \simeq E_1^{h\mathbb{G}_1} \to E_1^{hF} \to E_1^{hF} . \tag{0.2}$$

We note that p-adic complex K-theory $K\mathbb{Z}_p$ is in fact a model for E_1, the homotopy fixed points $E_1^{hC_2}$ can be identified with 2-adic real K-theory $KO\mathbb{Z}_2$

if $p = 2$ and $E_1^{hC_{p-1}}$ is the Adams summand of $K\mathbb{Z}_p$ if p is odd, so that the fibration of (0.2) indeed agrees with that of [12].

In this paper we produce a resolution of the trivial module $\mathbb{Z}_p$ by (direct summands of) permutation modules in the case $n = 2$ and $p = 3$ and we use it to build $L_{K(2)}S^0$ as the top of a finite tower of fibrations where the fibers are (suspensions of) spectra of the form E_2^{hF} where $F \subseteq \mathbb{G}_2$ is a finite subgroup.

In fact, if $n = 2$ and $p = 3$, only two subgroups appear. The first is a subgroup $G_{24} \subseteq \mathbb{G}_2$; this is a finite subgroup of order 24 containing a normal cyclic subgroup C_3 with quotient G_{24}/C_3 isomorphic to the quaternion group Q_8 of order 8. The other group is the semidihedral group SD_{16} of order 16. The two spectra we will see, then, are $E_2^{hG_{24}}$ and $E_2^{hSD_{16}}$.

The discussion of these and related subgroups of $\mathbb{G}_2$ occurs in Section 1 (see 1.1 and 1.2). The homotopy groups of these spectra are known. We will review the calculation in Section 3.

Our main result can be stated as follows (see Theorems 5.4 and 5.5).

THEOREM 0.1. *There is a sequence of maps between spectra*

$$\begin{aligned} L_{K(2)}S^0 \to E_2^{hG_{24}} &\to \Sigma^8 E_2^{hSD_{16}} \vee E_2^{hG_{24}} \to \Sigma^8 E_2^{hSD_{16}} \vee \Sigma^{40} E_2^{hSD_{16}} \\ &\to \Sigma^{40} E_2^{hSD_{16}} \vee \Sigma^{48} E_2^{hG_{24}} \to \Sigma^{48} E_2^{hG_{24}} \end{aligned}$$

with the property that the composite of any two successive maps is zero and all possible Toda brackets are zero modulo indeterminacy.

Because the Toda brackets vanish, this "resolution" can be refined to a tower of spectra with $L_{K(2)}S^0$ at the top. The precise result is given in Theorem 5.6. There are many curious features of this resolution, of which we note here only two. First, this is not an Adams resolution for E_2, as the spectra E_2^{hF} are not E_2-injective, at least if 3 divides the order of F. Second, there is a certain superficial duality to the resolution which should somehow be explained by the fact that $\mathbb{S}_n$ is a virtual Poincaré duality group, but we do not know how to make this thought precise.

As mentioned above, this result can be used to organize the already existing and very complicated calculations of Shimomura ([24], [25]) and it also suggests an independent approach to these calculations. Other applications would be to the study of Hopkins's Picard group (see [12]) of $K(2)$-local invertible spectra.

Our method is by brute force. The hard work is really in Section 4, where we use the calculations of [10] in an essential way to produce the short resolution of the trivial $\mathbb{G}_2$-module $\mathbb{Z}_3$ by (summands of) permutation modules of the form $\mathbb{Z}_3[[\mathbb{G}_2/F]]$ where F is finite (see Theorem 4.1 and Corollary 4.2). In Section 2, we calculate the homotopy type of the function spectra $F(E^{hH_1}, E^{hH_2})$ if H_1 is a closed and H_2 a finite subgroup of $\mathbb{G}_n$; this will allow us to construct the required maps between these spectra and to make the Toda bracket calculations. Here the work of [5] is crucial. These calculations also explain the role of the suspension by 48 which is really a homotopy theoretic phenomenon while the other suspensions can be explained in terms of the algebraic resolution constructed in Section 4.

1. Lubin-Tate theory and the Morava stabilizer group

The purpose of this section is to give a summary of what we will need about deformations of formal group laws over perfect fields. The primary point of this section is to establish notation and to run through some of the standard algebra needed to come to terms with the $K(n)$-local stable homotopy category.

Fix a perfect field k of characteristic p and a formal group law Γ over k. A *deformation* of Γ to a complete local ring A (with maximal ideal $\mathfrak{m}$) is a pair (G, i) where G is a formal group law over A, $i : k \to A/\mathfrak{m}$ is a morphism of fields and one requires $i_*\Gamma = \pi_*G$, where $\pi : A \to A/\mathfrak{m}$ is the quotient map. Two such deformations (G, i) and (H, j) are $\star$-isomorphic if there is an isomorphism $f : G \to H$ of formal group laws which reduces to the identity modulo $\mathfrak{m}$. Write $\mathrm{Def}_\Gamma(A)$ for the set of $\star$-isomorphism classes of deformations of Γ over A.

A common abuse of notation is to write G for the deformation (G, i); i is to be understood from the context.

Now suppose the height of Γ is finite. Then the theorem of Lubin and Tate [17] says that the functor $A \mapsto \mathrm{Def}_\Gamma(A)$ is representable. Indeed let

$$E(\Gamma, k) = W(k)[[u_1, \ldots, u_{n-1}]] \tag{1.1}$$

where $W(k)$ denotes the Witt vectors on k and n is the height of Γ. This is a complete local ring with maximal ideal $\mathfrak{m} = (p, u_1, \ldots, u_{n-1})$ and there is a canonical isomorphism $q : k \cong E(\Gamma, k)/\mathfrak{m}$. Then Lubin and Tate prove there is a deformation (G, q) of Γ over $E(\Gamma, k)$ so that the natural map

$$\mathrm{Hom}_c(E(\Gamma, k), A) \to \mathrm{Def}_\Gamma(A) \tag{1.2}$$

sending a continuous map $f : E(\Gamma, k) \to A$ to $(f_*G, \bar{f}q)$ (where $\bar{f}$ is the map on residue fields induced by f) is an isomorphism. Continuous maps here are very simple: they are the *local* maps; that is, we need only require that $f(\mathfrak{m})$ be contained in the maximal ideal of A. Furthermore, if two deformations are $\star$-isomorphic, then the $\star$-isomorphism between them is unique.

We would like to now turn the assignment $(\Gamma, k) \mapsto E(\Gamma, k)$ into a functor. For this we introduce the category $\mathcal{FGL}_n$ of height n formal group laws over perfect fields. The objects are pairs (Γ, k) where Γ is of height n. A morphism

$$(f, j) : (\Gamma_1, k_1) \to (\Gamma_2, k_2)$$

is a homomorphism of fields $j : k_1 \to k_2$ and an isomorphism of formal group laws $f : j^*\Gamma_1 \to \Gamma_2$.

Let (f, j) be such a morphism and let G_1 and G_2 be the fixed universal deformations over $E(\Gamma_1, k_1)$ and $E(\Gamma_2, k_2)$ respectively. If $\widetilde{f} \in E(\Gamma_2, k_2)[[x]]$ is any lift of $f \in k_2[[x]]$, then we can define a formal group law H over $E(\Gamma_2, k_2)$ by requiring that $\widetilde{f} : H \to G_2$ is an isomorphism. Then the pair (H, j) is a deformation of Γ_1, hence we get a homomorphism $E(\Gamma_1, k_1) \to E(\Gamma_2, k_2)$ classifying the $\star$-isomorphism class of H – which, one easily checks, is independent of the lift $\widetilde{f}$. Thus if Rings_c is the category of complete local rings and local homomorphisms, we get a functor

$$E(\cdot, \cdot) : \mathcal{FGL}_n \longrightarrow \mathrm{Rings}_c.$$

In particular, note that any morphism in $\mathcal{FGL}_n$ from a pair (Γ, k) to itself is an isomorphism. The automorphism group of (Γ, k) in $\mathcal{FGL}_n$ is the "big" Morava stabilizer group of the formal group law; it contains the subgroup of elements (f, id_k). This formal group law and hence also its automorphism group is determined up to isomorphism by the height of Γ if k is separably closed.

Specifically, let Γ be the Honda formal group law over $\mathbb{F}_{p^n}$; thus the p-series of Γ is

$$[p](x) = x^{p^n}.$$

From this formula it immediately follows that any automorphism $f : \Gamma \to \Gamma$ over any finite extension field of $\mathbb{F}_{p^n}$ actually has coefficients in $\mathbb{F}_{p^n}$; thus we obtain no new isomorphisms by making such extensions. Let $\mathbb{S}_n$ be the group of automorphisms of this Γ over $\mathbb{F}_{p^n}$; this is the classical Morava stabilizer group. If we let $\mathbb{G}_n$ be the group of automorphisms of $(\Gamma, \mathbb{F}_{p^n})$ in $\mathcal{FGL}_n$ (the big Morava stabilizer group of Γ), then one easily sees that

$$\mathbb{G}_n \cong \mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p).$$

Of course, $\mathbb{G}_n$ acts on $E(\Gamma, \mathbb{F}_{p^n})$. Also, we note that the Honda formal group law is defined over $\mathbb{F}_p$, although it will not get its full group of automorphisms until changing base to $\mathbb{F}_{p^n}$.

Next we put in the gradings. This requires a paragraph of introduction. For any commutative ring R, the morphism $R[[x]] \to R$ of rings sending x to 0 makes R into an $R[[x]]$-module. Let $\mathrm{Der}_R(R[[x]], R)$ denote the R-module of continuous R-derivations; that is, continuous R-module homomorphisms

$$\partial : R[[x]] \longrightarrow R$$

so that

$$\partial(f(x)g(x)) = \partial(f(x))g(0) + f(0)\partial(g(x)).$$

If ∂ is any derivation, write $\partial(x) = u$; then, if $f(x) = \sum a_i x^i$,

$$\partial(f(x)) = a_1\partial(x) = a_1 u.$$

Thus ∂ is determined by u, and we write $\partial = \partial_u$. We then have that $\mathrm{Der}_R(R[[x]], R)$ is a free R-module of rank one, generated by any derivation ∂_u so that u is a unit in R. In the language of schemes, ∂_u is a generator for the tangent space at 0 of the formal scheme $\mathbb{A}^1_R$ over $\mathrm{Spec}(R)$.

Now consider pairs (F, u) where F is a formal group law over R and u is a unit in R. Thus F defines a smooth one dimensional commutative formal group scheme over $\mathrm{Spec}(R)$ and ∂_u is a chosen generator for the tangent space at 0. A morphism of pairs

$$f : (F, u) \longrightarrow (G, v)$$

is an isomorphism of formal group laws $f : F \to G$ so that

$$u = f'(0)v.$$

Note that if $f(x) \in R[[x]]$ is a homomorphism of formal group laws from F to G, and ∂ is a derivation at 0, then $(f^*\partial)(x) = f'(0)\partial(x)$. In the context of deformations, we may require that f be a $\star$-isomorphism.

This suggests the following definition: let Γ be a formal group law of height n over a perfect field k of characteristic p, and let A be a complete local ring. Define $\mathrm{Def}_\Gamma(A)_*$ to be equivalence classes of pairs (G, u) where G is a deformation of Γ to A and u is a unit in A. The equivalence relation is given by $\star$-isomorphisms transforming the unit as in the last paragraph. We now have that there is a natural isomorphism

$$\mathrm{Hom}_c(E(\Gamma, k)[u^{\pm 1}], A) \cong \mathrm{Def}_\Gamma(A)_*.$$

We impose a grading by giving an action of the multiplicative group scheme $\mathbb{G}_m$ on the scheme $\mathrm{Def}_\Gamma(\cdot)_*$ (on the right) and thus on $E(\Gamma, k)[u^{\pm 1}]$ (on the left): if $v \in A^\times$ is a unit and (G, u) represents an equivalence class in $\mathrm{Def}_\Gamma(A)_*$ define an new element in $\mathrm{Def}_\Gamma(A)_*$ by $(G, v^{-1}u)$. In the induced grading on $E(\Gamma, k)[u^{\pm 1}]$, one has $E(\Gamma, k)$ in degree 0 and u in degree -2.

This grading is essentially forced by topological considerations. See the remarks before Theorem 20 of [27] for an explanation. In particular, it is explained there why u is in degree -2 rather than 2.

The rest of the section will be devoted to what we need about the Morava stabilizer group. The group $\mathbb{S}_n$ is the group of units in the endomorphism ring $\mathcal{O}_n$ of the Honda formal group law of height n. The ring $\mathcal{O}_n$ can be described as follows (See [10] or [20]). One adjoins a noncommuting element S to the Witt vectors $\mathbb{W} = W(\mathbb{F}_{p^n})$ subject to the conditions that

$$Sa = \phi(a)S \qquad \text{and} \qquad S^n = p$$

where $a \in \mathbb{W}$ and $\phi : \mathbb{W} \to \mathbb{W}$ is the Frobenius. (In terms of power series, S corresponds to the endomorphism of the formal group law given by $f(x) = x^p$.) This algebra $\mathcal{O}_n$ is a free $\mathbb{W}$-module of rank n with generators $1, S, \ldots S^{n-1}$

and is equipped with a valuation ν extending the standard valuation of $\mathbb{W}$; since we assume that $\nu(p) = 1$, we have $\nu(S) = 1/n$. Define a filtration on $\mathbb{S}_n$ by

$$F_k\mathbb{S}_n = \{x \in \mathbb{S}_n \mid \nu(x-1) \geq k\}.$$

Note that k is a fraction of the form a/n with $a = 0, 1, 2, \dots$. We have

$$F_0\mathbb{S}_n/F_{1/n}\mathbb{S}_n \cong \mathbb{F}_{p^n}^\times\ ,$$
$$F_{a/n}\mathbb{S}_n/F_{(a+1)/n}\mathbb{S}_n \cong \mathbb{F}_{p^n}, \qquad a \geq 1$$

and

$$\mathbb{S}_n \cong \lim_a \mathbb{S}_n/F_{a/n}\mathbb{S}_n.$$

If we define $S_n = F_{1/n}\mathbb{S}_n$, then S_n is the p-Sylow subgroup of the profinite group $\mathbb{S}_n$. Note that the Teichmüller elements $\mathbb{F}_{p^n}^\times \subseteq \mathbb{W}^\times \subseteq \mathcal{O}_n^\times$ define a splitting of the projection $\mathbb{S}_n \to \mathbb{F}_{p^n}^\times$ and, hence, $\mathbb{S}_n$ is the semi-direct product of $\mathbb{F}_{p^n}^\times$ and the p-Sylow subgroup.

The action of the Galois group $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ on $\mathcal{O}_n$ is the obvious one: the Galois group is generated by the Frobenius ϕ and

$$\phi(a_0 + a_1S + \cdots + a_{n-1}S^{n-1}) = \phi(a_0) + \phi(a_1)S + \cdots + \phi(a_{n-1})S^{n-1}.$$

We are, in this paper, concerned mostly with the case $n = 2$ and $p = 3$. In this case, every element of $\mathbb{S}_2$ can be written as a sum

$$a + bS, \qquad a, b \in W(\mathbb{F}_9) = \mathbb{W}$$

with $a \not\equiv 0 \bmod 3$. The elements of S_2 are of the form $a + bS$ with $a \equiv 1 \bmod 3$.

The following subgroups of $\mathbb{S}_2$ will be of particular interest to us. The first two are choices of maximal finite subgroups.[1] The last one (see 1.3) is a closed subgroup which is, in some sense, complementary to the center.

1.1. Choose a primitive eighth root of unity $\omega \in \mathbb{F}_9$. We will write ω for the corresponding element in $\mathbb{W}$ and $\mathbb{S}_2$. The element

$$s = -\frac{1}{2}(1 + \omega S)$$

is of order 3; furthermore,

$$\omega^2 s \omega^6 = s^2.$$

Hence the elements s and ω^2 generate a subgroup of order 12 in $\mathbb{S}_2$ which we label G_{12}. As a group, it is abstractly isomorphic to the unique nontrivial semi-direct product of cyclic groups

$$C_3 \rtimes C_4.$$

[1] The first author would like to thank Haynes Miller for several lengthy and informative discussions about finite subgroups of the Morava stabilizer group.

Any other subgroup of order 12 in $\mathbb{S}_2$ is conjugate to G_{12}. In the sequel, when discussing various representations, we will write the element $\omega^2 \in G_{12}$ as t.

We note that the subgroup $G_{12} \subseteq \mathbb{S}_2$ is a normal subgroup of a subgroup G_{24} of the larger group $\mathbb{G}_2$. Indeed, there is a diagram of short exact sequences of groups

$$\begin{tikzcd}
1 \arrow[r] & G_{12} \arrow[r] \arrow[d, "\subseteq"'] & G_{24} \arrow[r] \arrow[d, "\subseteq"'] & \mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3) \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & \mathbb{S}_2 \arrow[r] & \mathbb{G}_2 \arrow[r] & \mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3) \arrow[r] & 1.
\end{tikzcd}$$

Since the action of the Galois group on $\mathbb{S}_2$ does not preserve any choice of G_{12}, this is not transparent. In fact, while the lower sequence is split the upper sequence is not. More concretely we let

$$\psi = \omega\phi \in \mathbb{S}_2 \rtimes \mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3) = \mathbb{G}_2$$

where ω is our chosen 8th root of unity and ϕ is the generator of the Galois group. Then if s and t are the elements of order 3 and 4 in G_{12} chosen above, we easily calculate that $\psi s = s\psi$, $t\psi = \psi t^3$ and $\psi^2 = t^2$. Thus the subgroup of $\mathbb{G}_2$ generated by G_{12} and ψ has order 24, as required. Note that the 2-Sylow subgroup of G_{24} is the quaternion group Q_8 of order 8 generated by t and ψ and that indeed

$$1 \longrightarrow G_{12} \longrightarrow G_{24} \longrightarrow \mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3) \longrightarrow 1$$

is not split.

1.2. The second subgroup is the subgroup SD_{16} generated by ω and ϕ. This is the semidirect product

$$\mathbb{F}_9^\times \rtimes \mathbb{Z}/2 \ ,$$

and it is also known as the semidihedral group of order 16.

1.3. For the third subgroup, note that the evident right action of $\mathbb{S}_n$ on $\mathcal{O}_n$ defines a group homomorphism $\mathbb{S}_n \to \mathrm{GL}_n(\mathbb{W})$. The determinant homomorphism $\mathbb{S}_n \to \mathbb{W}^\times$ extends to a homomorphism

$$\mathbb{G}_n \to \mathbb{W}^\times \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \ .$$

For example, if $n = 2$, this map sends $(a + bS, \phi^e)$, $e \in \{0, 1\}$, to

$$(a\phi(a) - pb\phi(b), \phi^e)$$

where ϕ is the Frobenius. It is simple to check (for all n) that the image of this homomorphism lands in

$$\mathbb{Z}_p^\times \times \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \subseteq \mathbb{W}^\times \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \ .$$

If we identify the quotient of $\mathbb{Z}_p^\times$ by its maximal finite subgroup with $\mathbb{Z}_p$, we get a "reduced determinant" homomorphism

$$\mathbb{G}_n \to \mathbb{Z}_p \ .$$

Let $\mathbb{G}_n^1$ be the kernel of this map and $\mathbb{S}_n^1$ resp. S_n^1 be the kernel of its restriction to $\mathbb{S}_n$ resp. S_n. In particular, any finite subgroup of $\mathbb{G}_n$ is a subgroup of $\mathbb{G}_n^1$. One also easily checks that the center of $\mathbb{G}_n$ is $\mathbb{Z}_p^\times \subseteq \mathbb{W}^\times \subseteq \mathbb{S}_n$ and that the composite

$$\mathbb{Z}_p^\times \to \mathbb{G}_n \to \mathbb{Z}_p^\times$$

sends a to a^n. Thus, if p does not divide n, we have

$$\mathbb{G}_n \cong \mathbb{Z}_p \times \mathbb{G}_n^1.$$

2. The $K(n)$-local category and the Lubin-Tate theories E_n

The purpose of this section is to collect together the information we need about the $K(n)$-local category and the role of the functor $(E_n)_*(\cdot)$ in governing this category. *But attention*! — $(E_n)_*X$ is *not* the homology of X defined by the spectrum E_n, but a completion thereof; see Definition 2.1 below.

Most of the information in this section is collected from [3], [4], and [15].

Fix a prime p and let $K(n)$, $1 \leq n < \infty$, denote the n-th Morava K-theory spectrum. Then $K(n)_* \cong \mathbb{F}_p[v_n^{\pm 1}]$ where the degree of v_n is $2(p^n - 1)$. This is a complex oriented theory and the formal group law over $K(n)_*$ is of height n. As is customary, we specify that the formal group law over $K(n)_*$ is the graded variant of the Honda formal group law; thus, the p-series is

$$[p](x) = v_n x^{p^n}.$$

Following Hovey and Strickland, we will write $\mathcal{K}_n$ for the category of $K(n)$-local spectra. We will write $L_{K(n)}$ for the localization functor from spectra to $\mathcal{K}_n$.

Next let K_n be the extension of $K(n)$ with $(K_n)_* \cong \mathbb{F}_{p^n}[u^{\pm 1}]$ with the degree of $u = -2$. The inclusion $K(n)_* \subseteq (K_n)_*$ sends v_n to $u^{-(p^n-1)}$. There is a natural isomorphism of homology theories

$$(K_n)_* \otimes_{K(n)_*} K(n)_*X \xrightarrow{\cong} (K_n)_*X$$

and $K(n)_* \to (K_n)_*$ is a faithfully flat extension; thus the two theories have the same local categories and weakly equivalent localization functors.

If we write F for the graded formal group law over $K(n)_*$ we can extend F to a formal group law over $(K_n)_*$ and define a formal group law Γ over $\mathbb{F}_{p^n} = (K_n)_0$ by

$$x +_\Gamma y = \Gamma(x, y) = u^{-1}F(ux, uy) = u^{-1}(ux +_F uy).$$

Then F is chosen so that Γ is the Honda formal group law.

We note that – as in [4] – there is a choice of the universal deformation G of Γ such that the p-series of the associated graded formal group law G_0 over $E(\Gamma, \mathbb{F}_{p^n})[u^{\pm 1}]$ satisfies

$$[p](x) = v_0 x +_{G_0} v_1 x^p +_{G_0} v_2 x^{p^2} +_{G_0} \dots$$

with $v_0 = p$ and

$$v_k = \begin{cases} u^{1-p^k} u_k & 0 < k < n; \\ u^{1-p^n} & k = n; \\ 0 & k > n. \end{cases}$$

This shows that the functor $X \mapsto (E_n)_* \otimes_{BP_*} BP_* X$ (where $(E_n)_*$ is considered a BP_*-module via the evident ring homomorphism) is a homology theory which is represented by a spectrum E_n with coefficients

$$\pi_*(E_n) \cong E(\Gamma, \mathbb{F}_{p^n})[u^{\pm 1}] \cong \mathbb{W}[[u_1, \dots, u_{n-1}]][u^{\pm 1}] \ .$$

The inclusion of the subring $E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}]$ into $(E_n)_*$ is again faithfully flat; thus, these two theories have the same local categories. We write $\mathcal{L}_n$ for the category of $E(n)$-local spectra and L_n for the localization functor from spectra to $\mathcal{L}_n$.

The reader will have noticed that we have avoided using the expression $(E_n)_* X$; we now explain what we mean by this. The $K(n)$-local category $\mathcal{K}_n$ has internal smash products and (arbitrary) wedges given by

$$X \wedge_{\mathcal{K}_n} Y = L_{K(n)}(X \wedge Y)$$

and

$$\bigvee_{\mathcal{K}_n} X_\alpha = L_{K(n)}(\bigvee X_\alpha) \ .$$

In making such definitions, we assume we are working in some suitable model category of spectra, and that we are taking the smash product between cofibrant spectra; that is, we are working with the derived smash product. The issues here are troublesome, but well understood, and we will not dwell on these points. See [6] or [14]. If we work in our suitable categories of spectra the functor $Y \mapsto X \wedge_{\mathcal{K}_n} Y$ has a right adjoint $Z \mapsto F(X, Z)$.

We define a version of $(E_n)_*(\cdot)$ intrinsic to $\mathcal{K}_n$ as follows.

Definition 2.1. Let X be a spectrum. Then we define $(E_n)_* X$ by the equation

$$(E_n)_* X = \pi_* L_{K(n)}(E_n \wedge X).$$

We remark immediately that $(E_n)_*(\cdot)$ is not a homology theory in the usual sense; for example, it will not send arbitrary wedges to sums of abelian groups. However, it is tractable, as we now explain. First note that E_n itself is $K(n)$-local; indeed, Lemma 5.2 of [15] demonstrates that E_n is a finite wedge of spectra of the form $L_{K(n)}E(n)$. Therefore if X is a finite CW spectrum, then $E_n \wedge X$ is already in $\mathcal{K}_n$, so

$$(E_n)_*X = \pi_*(E_n \wedge X). \tag{2.1}$$

Let $I = (i_0, \dots, i_{n-1})$ be a sequence of positive integers and let

$$\mathfrak{m}^I = (p^{i_0}, u_1^{i_1}, \dots, u_{n-1}^{i_{n-1}}) \subseteq \mathfrak{m} \subseteq (E_n)_*$$

where $\mathfrak{m} = (p, u_1, \dots, u_{n-1})$ is the maximal ideal in E_*. These form a system of ideals in $(E_n)_*$ and produce a filtered diagram of rings $\{(E_n)_*/\mathfrak{m}^I\}$; furthermore

$$(E_n)_* = \lim_I (E_n)_*/\mathfrak{m}^I.$$

There is a cofinal diagram $\{(E_n)_*/\mathfrak{m}^J\}$ which can be realized as a diagram of spectra in the following sense: using nilpotence technology, one can produce a diagram of finite spectra $\{M_J\}$ and an isomorphism

$$\{(E_n)_*M_J\} \cong \{(E_n)_*/\mathfrak{m}^J\}$$

as diagrams. See §4 of [15]. Here $(E_n)_*M_J = \pi_*E_n \wedge M_J = \pi_*L_K(n)(E_n \wedge M_J)$. The importance of this diagram is that (see [15, Prop. 7.10]) for each spectrum X

$$L_{K(n)}X \simeq \mathrm{holim}_J M_J \wedge L_nX. \tag{2.2}$$

This has the following consequence, immediate from Definition 2.1: there is a short exact sequence

$$0 \to \lim{}^1(E_n)_{k+1}(X \wedge M_J) \to (E_n)_kX \to \lim(E_n)_k(X \wedge M_J) \to 0.$$

This suggests $(E_n)_*X$ is closely related to some completion of $\pi_*(E_n \wedge X)$ and this is nearly the case. The details are spelled out in Section 8 of [15], but we will not need the full generality there. In fact, all of the spectra we consider here will satisfy the hypotheses of Proposition 2.2 below.

If M is an $(E_n)_*$-module, let $M^\wedge_\mathfrak{m}$ denote the completion of M with respect to the maximal ideal of $(E_n)_*$. A module of the form

$$(\bigoplus_\alpha \Sigma^{k_\alpha}(E_n)_*)^\wedge_\mathfrak{m}$$

will be called *pro-free.*

PROPOSITION 2.2. *If X is a spectrum so that $K(n)_*X$ is concentrated in even degrees, then*

$$(E_n)_*X \cong \pi_*(E_n \wedge X)^\wedge_\mathfrak{m}$$

and $(E_n)_*X$ is pro-free as an $(E_n)_*$-module.

See Proposition 8.4 of [15].

As with anything like a flat homology theory, the object $(E_n)_*X$ is a comodule over some sort of Hopf algebroid of co-operations; it is our next project to describe this structure. In particular, this brings us to the role of the Morava stabilizer group. We begin by identifying $(E_n)_*E_n$.

Let $\mathbb{G}_n$ be the (big) Morava stabilizer group of Γ, the Honda formal group law of height n over $\mathbb{F}_{p^n}$. For the purposes of this paper, a *Morava module* is a complete $(E_n)_*$-module M equipped with a continuous $\mathbb{G}_n$-action subject to the following compatibility condition: if $g \in \mathbb{G}_n$, $a \in (E_n)_*$ and $x \in M$, then

$$g(ax) = g(a)g(x) \ . \tag{2.3}$$

For example, if X is any spectrum with $K(n)_*X$ concentrated in even degrees, then $(E_n)_*X$ is a complete $(E_n)_*$-module (by Proposition 2.2) and the action of $\mathbb{G}_n$ on E_n defines a continuous action of $\mathbb{G}_n$ on $(E_n)_*X$. This is a prototypical Morava module.

Now let M be a Morava module and let

$$\mathrm{Hom}^c(\mathbb{G}_n, M)$$

be the abelian group of continuous maps from $\mathbb{G}_n$ to M where the topology on M is defined via the ideal $\mathfrak{m}$. Then

$$\mathrm{Hom}^c(\mathbb{G}_n, M) \cong \lim{}_i \mathrm{colim}_k \mathrm{map}(\mathbb{G}_n/U_k, M/\mathfrak{m}^i M) \tag{2.4}$$

where U_k runs over any system of open subgroups of $\mathbb{G}_n$ with $\bigcap_k U_k = \{e\}$. To give $\mathrm{Hom}^c(\mathbb{G}_n, M)$ a structure of an $(E_n)_*$-module let $\phi : \mathbb{G}_n \to M$ be continuous and $a \in (E_n)_*$. The we define $a\phi$ by the formula

$$(a\phi)(x) = a\phi(x) \ . \tag{2.5}$$

There also is a continuous action of $\mathbb{G}_n$ on $\mathrm{Hom}^c(\mathbb{G}_n, M)$: if $g \in \mathbb{G}_n$ and $\phi : \mathbb{G}_n \to M$ is continuous, then one defines $g\phi : \mathbb{G}_n \to M$ by the formula

$$(g\phi)(x) = g\phi(g^{-1}x) \ . \tag{2.6}$$

With this action, and the action of $(E_n)_*$ defined in (2.5), the formula of (2.3) holds. Because M is complete (2.4) shows that $\mathrm{Hom}^c(\mathbb{G}_n, M)$ is complete.

Remark 2.3. With the Morava module structure defined by equations 2.5 and 2.6, the functor $M \to \mathrm{Hom}^c(\mathbb{G}_n, M)$ has the following universal property. If N and M are Morava modules and $f : N \to M$ is a morphism of continuous $(E_n)_*$ modules, then there is an induced morphism

$$\begin{aligned} N &\longrightarrow \mathrm{Hom}^c(\mathbb{G}_n, M) \\ \alpha &\mapsto \phi_\alpha \end{aligned}$$

with $\phi_\alpha(x) = xf(x^{-1}\alpha)$. This yields a natural isomorphism

$$\mathrm{Hom}_{(E_n)_*}(N, M) = \mathrm{Hom}_{\mathrm{Morava}}(N, \mathrm{Hom}^c(\mathbb{G}_n, M))$$

from continuous $(E_n)_*$ module homomorphisms to morphisms of Morava modules.

There is a *different*, but isomorphic natural Morava module structure on $\mathrm{Hom}^c(\mathbb{G}_n, -)$ so that this functor becomes a true right adjoint of the forget functor from Morava modules to continuous $(E_n)_*$-modules. However, we will not need this module structure at any point and we supress it to avoid confusion.

For example, if X is a spectrum such that $(E_n)_*X$ is $(E_n)_*$-complete, the $\mathbb{G}_n$-action on $(E_n)_*X$ is encoded by the map

$$(E_n)_*X \to \mathrm{Hom}^c(\mathbb{G}_n, (E_n)_*X)$$

adjoint (in the sense of the previous remark) to the identity.

The next result says that this is essentially all the stucture that $(E_n)_*X$ supports. For any spectrum X, $\mathbb{G}_n$ acts on

$$(E_n)_*(E_n \wedge X) = \pi_* L_{K(n)}(E_n \wedge E_n \wedge X)$$

by operating in the left factor of E_n. The multiplication $E_n \wedge E_n \to E_n$ defines a morphism of $(E_n)_*$-modules

$$(E_n)_*(E_n \wedge X) \to (E_n)_*X$$

and by composing we obtain a map

$$\phi : (E_n)_*(E_n \wedge X) \to \mathrm{Hom}^c(\mathbb{G}_n, (E_n)_*(E_n \wedge X)) \to \mathrm{Hom}^c(\mathbb{G}_n, (E_n)_*X)\ .$$

If $(E_n)_*X$ is complete, this is a morphism of Morava modules.

We now record:

Proposition 2.4. *For any cellular spectrum X with $(K_n)_*X$ concentrated in even degrees the morphism*

$$\phi : (E_n)_*(E_n \wedge X) \to \mathrm{Hom}^c(\mathbb{G}_n, (E_n)_*X)$$

is an isomorphism of Morava modules.

Proof. See [5] and [27] for the case $X = S^0$. The general case follows in the usual manner. First, it's true for finite spectra by a five lemma argument. For this one needs to know that the functor

$$M \mapsto \mathrm{Hom}^c(\mathbb{G}_n, M)$$

is exact on finitely generated $(E_n)_*$-complete modules. This follows from (2.4). Then one argues the general case, by noting first that by taking colimits over finite cellular subspectra

$$\phi : (E_n)_*(E_n \wedge M_J \wedge X) \to \mathrm{Hom}^c(\mathbb{G}_n, (E_n)_*(M_J \wedge X))$$

is an isomorphism for any J and any X. Note that $E_n \wedge M_J \wedge X$ is $K(n)$-local for any X; therefore, $L_{K(n)}$ commutes with the homotopy colimits in question. Finally the hypothesis on X implies

$$(E_n)_*(E_n \wedge X) \cong \lim (E_n)_*(E_n \wedge M_J \wedge X).$$

and thus we can conclude the result by taking limits with respect to J. $\square$

We next turn to the results of Devinatz and Hopkins ([5]) on homotopy fixed point spectra. Let $\mathcal{O}_{\mathbb{G}_n}$ be the orbit category of $\mathbb{G}_n$. Thus an object in $\mathcal{O}_{\mathbb{G}_n}$ is an orbit $\mathbb{G}_n/H$ where H is a closed subgroup and the morphisms are continuous $\mathbb{G}_n$-maps. Then Devinatz and Hopkins have defined a functor

$$\mathcal{O}_{\mathbb{G}_n}^{op} \to \mathcal{K}$$

sending $\mathbb{G}_n/H$ to a $K(n)$-local spectrum E_n^{hH}. If H is finite, then E_n^{hH} is the usual homotopy fixed point spectrum defined by the action of $H \subseteq \mathbb{G}_n$. By the results of [5], the morphism ϕ of Proposition 2.4 restricts to an isomorphism (for any closed H)

$$(E_n)_* E_n^{hH} \xrightarrow{\cong} \mathrm{Hom}^c(\mathbb{G}_n/H, (E_n)_*). \tag{2.7}$$

We would now like to write down a result about the function spectra $F((E_n)^{hH}, E_n)$. First, some notation. If E is a spectrum and $X = \lim_i X_i$ is an inverse limit of a sequence of finite sets X_i then define

$$E[[X]] = \mathrm{holim}_i E \wedge (X_i)_+.$$

Proposition 2.5. *Let H be a closed subgroup of $\mathbb{G}_n$. Then there is a natural weak equivalence*

$$E_n[[\mathbb{G}_n/H]] \xrightarrow{\simeq} F((E_n)^{hH}, E_n).$$

Proof. First let U be an open subgroup of $\mathbb{G}_n$. Functoriality of the homotopy fixed point spectra construction of [5] gives us a map $E_n^{hU} \wedge \mathbb{G}_n/U_+ \to E_n$ where as usual $\mathbb{G}_n/U_+$ denotes $\mathbb{G}_n/U$ with a disjoint base point added. Together with the product on E_n we obtain a map

$$E_n \wedge E_n^{hU} \wedge \mathbb{G}_n/U_+ \to E_n \wedge E_n \to E_n \tag{2.8}$$

whose adjoint induces an equivalence

$$L_{K(n)}(E_n \wedge E_n^{hU}) \to \prod_{\mathbb{G}_n/U} E_n \tag{2.9}$$

realizing the isomorphism of (2.7) above. Note that this is a map of E_n-module spectra. Let $F_{E_n}(-, E_n)$ be the function spectra in the category of left E_n-module spectra. (See [6] for details.) If we apply $F_{E_n}(-, E_n)$ to the equivalence of (2.9) we obtain an equivalence of E_n-module spectra

$$F_{E_n}(\prod_{\mathbb{G}_n/U} E_n, E_n) \to F_{E_n}(E_n \wedge E_n^{hU}, E_n).$$

This equivalence can then be written as

$$E_n \wedge (\mathbb{G}_n/U)_+ \to F(E_n^{hU}, E_n); \tag{2.10}$$

furthermore, an easy calculation shows that this map is adjoint to the map of (2.8).

More generally, let H be any closed subgroup of $\mathbb{G}_n$. Then there exists a decreasing sequence U_i of open subgroups U_i with $H = \bigcap_i U_i$ and by [5] we have

$$E_n^{hH} \simeq L_{K(n)}\mathrm{hocolim}_i E_n^{hU_i} .$$

Thus, the equivalence of (2.10) and by passing to the limit we obtain the desired equivalence. □

Now note that if X is a profinite set with continuous H-action and if E is an H-spectrum then $E[[X]]$ is an H-spectrum via the diagonal action. It is this action which is used in the following result.

PROPOSITION 2.6. 1) *Let H_1 be a closed subgroup and H_2 a finite subgroup of $\mathbb{G}_n$. Then there is a natural equivalence*

$$E_n[[\mathbb{G}_n/H_1]]^{hH_2} \xrightarrow{\simeq} F(E_n^{hH_1}, E_n^{hH_2}) .$$

2) *If H_1 is also an open subgroup then there is a natural decomposition*

$$E_n[[\mathbb{G}_n/H_1]]^{hH_2} \simeq \prod_{H_2\backslash \mathbb{G}_n/H_1} E_n^{hH_x}$$

where $H_x = H_2 \cap xH_1x^{-1}$ is the isotropy subgroup of the coset xH_1 and $H_2\backslash\mathbb{G}_n/H_1$ is the (finite) set of double cosets.

3) *If H_1 is a closed subgroup and $H_1 = \bigcap_i U_i$ for a decreasing sequence of open subgroups U_i then*

$$F(E_n^{hH_1}, E_n^{hH_2}) \simeq \mathrm{holim}_i E_n[[\mathbb{G}_n/U_i]]^{hH_2} \simeq \mathrm{holim}_i \prod_{H_2\backslash\mathbb{G}_n/U_i} E_n^{hH_{x,i}}$$

where $H_{x,i} = H_2 \cap xU_ix^{-1}$ is, as before, the isotropy subgroup of the coset xU_i.[2]

Proof. The first statement follows from Proposition 2.5 by passing to homotopy fixed point spectra with respect to H_2 and the second statement is then an immediate consequence of the first. For the third statement we write $\mathbb{G}_n/H_1 = \lim_i \mathbb{G}_n/U_i$ and pass to the homotopy inverse limit. □

[2]We are grateful to P. Symonds for pointing out that the naive generalization of the second statement does not hold for a general closed subgroup.

We will be interested in the E_n-Hurewicz homomorphism

$$\pi_0 F(E_n^{hH_1}, E_n^{hH_2}) \to \mathrm{Hom}_{(E_n)_*E_n}((E_n)_*E_n^{hH_1}, (E_n)_*E_n^{hH_2})$$

where $\mathrm{Hom}_{(E_n)_*E_n}$ denotes morphisms in the category of Morava modules. Let

$$(E_n)_*[[\mathbb{G}_n]] = \lim_i (E_n)_*[\mathbb{G}_n/U_i]$$

denote the completed group ring and give this the structure of a Morava module by letting $\mathbb{G}_n$ act diagonally.

PROPOSITION 2.7. *Let H_1 and H_2 be closed subgroups of $\mathbb{G}_n$ and suppose that H_2 is finite. Then there is an isomorphism*

$$\big((E_n)_*[[\mathbb{G}_n/H_1]]\big)^{H_2} \xrightarrow{\cong} \mathrm{Hom}_{(E_n)_*E_n}((E_n)_*E_n^{hH_1}, (E_n)_*E_n^{hH_2})$$

such that the following diagram commutes

$$\begin{tikzcd}
\pi_* E_n[[\mathbb{G}_n/H_1]]^{hH_2} \arrow[r] \arrow[d, "\cong"'] & \big((E_n)_*[[\mathbb{G}_n/H_1]]\big)^{H_2} \arrow[d, "\cong"] \\
\pi_* F(E_n^{hH_1}, E_n^{hH_2}) \arrow[r] & \mathrm{Hom}_{(E_n)_*E_n}((E_n)_*E_n^{hH_1}, (E_n)_*E_n^{hH_2})
\end{tikzcd}$$

where the top horizontal map is the edge homomorphism in the homotopy fixed point spectral sequence, the left-hand vertical map is induced by the equivalence of Proposition 2.6 *and the bottom horizontal map is the E_n-Hurewicz homomorphism.*

Proof. First we assume that H_2 is the trivial subgroup and H_1 is open, so that $\mathbb{G}_n/H_1$ is finite. Then there is an isomorphism

$$(E_n)_*[[\mathbb{G}_n/H_1]] \to \mathrm{Hom}_{(E_n)_*}(\mathrm{Hom}^c(\mathbb{G}_n/H_1, (E_n)_*), (E_n)_*)$$

which is the unique linear map which sends a coset to evaluation at that coset. Applying Remark 2.3 we obtain an isomorphism of Morava modules

$$(E_n)_*[[\mathbb{G}_n/H_1]] \to \mathrm{Hom}_{(E_n)_*}(\mathrm{Hom}^c(\mathbb{G}_n/H_1, (E_n)_*), \mathrm{Hom}^c(\mathbb{G}_n, (E_n)_*)).$$

This isomorphism can be extended to a general closed subgroup H_1 by writing H_1 as the intersection of a nested sequence of open subgroups (as in the proof of Proposition 2.5) and taking limits. Then we use the isomorphisms of (2.7) to identify $(E_n)_*E_n^{hH_i}$ with $\mathrm{Hom}^c(\mathbb{G}_n/H_i, (E_n)_*)$. This defines the isomorphism we need, and it is straightforward to see that the diagram commutes. To end the proof, note that the case of a general finite subgroup H_2 follows by passing to H_2-invariants. $\square$

3. The homotopy groups of E_2^{hF} at $p = 3$

To construct our tower we are going to need some information about $\pi_* E_2^{hF}$ for various finite subgroups of the stabilizer group $\mathbb{G}_2$. Much of what we say here can be recovered from various places in the literature (for example, [8], [18], or [7]) and the point of view and proofs expressed are certainly those of Mike Hopkins. What we add here to the discussion in [7] is that we pay careful attention to the Galois group. In particular we treat the case of the finite group G_{24}.

Recall that we are working at the prime 3. We will write E for E_2, so that we may write E_* for $(E_2)_*$.

In Remark 1.1 we defined a subgroup

$$G_{24} \subseteq \mathbb{G}_2 = \mathbb{S}_2 \rtimes \mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3)$$

generated by elements s, t and ψ of orders 3, 4 and 4 respectively. The cyclic subgroup C_3 generated by s is normal, and the subgroup Q_8 generated by t and ψ is the quaternion group of order 8.

The first results are algebraic in nature; they give a nice presentation of E_* as a G_{24}-algebra. First we define an action of G_{24} on $\mathbb{W} = W(\mathbb{F}_9)$ by the formulas:

$$s(a) = a \qquad t(a) = \omega^2 a \qquad \psi(a) = \omega\phi(a) \tag{3.1}$$

where ϕ is the Frobenius. Note the action factors through $G_{24}/C_3 \cong Q_8$. Restricted to the subgroup $G_{12} = \mathbb{S}_2 \cap G_{24}$ this action is $\mathbb{W}$-linear, but over G_{24} it is simply linear over $\mathbb{Z}_3$. Let χ denote the resulting G_{24}-representation and χ' its restriction to Q_8.

This representation is a module over a twisted version of the group ring $\mathbb{W}[G_{24}]$. The projection

$$G_{24} \longrightarrow \mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3)$$

defines an action[3] of G_{24} on $\mathbb{W}$ and we use this action to twist the multiplication in $\mathbb{W}[G_{24}]$. We should really write $\mathbb{W}_\phi[G_{24}]$ for this twisted group ring, but we forebear, so as to not clutter notation. Note that $\mathbb{W}[Q_8]$ has a similar twisting, but $\mathbb{W}[G_{12}]$ is the ordinary group ring.

Define a G_{24}-module ρ by the short exact sequence

$$0 \to \chi \to \mathbb{W}[G_{24}] \otimes_{\mathbb{W}[Q_8]} \chi' \to \rho \to 0 \tag{3.2}$$

where the first map takes a generator e of χ to

$$(1 + s + s^2)e \in \mathbb{W}[G_{24}] \otimes_{\mathbb{W}[Q_8]} \chi'.$$

[3] This action is different from that of the representation defined by the formulas of 3.1.

LEMMA 3.1. *There is a morphism of G_{24}-modules*

$$\rho \longrightarrow E_{-2}$$

so that the induced map

$$\mathbb{F}_9 \otimes_{\mathbb{W}} \rho \to E_0/(3, u_1^2) \otimes_{E_0} E_{-2}$$

is an isomorphism. Furthermore, this isomorphism sends the generator e of ρ to an invertible element in E_*.

Proof. We need to know a bit about the action of $\mathbb{G}_2$ on E_*. The relevant formulas have been worked out by Devinatz and Hopkins. Let $\mathfrak{m} \subseteq E_0$ be the maximal ideal and $a + bS \in \mathbb{S}_2$. Then Proposition 3.3 and Lemma 4.9 of [4] together imply that, modulo $\mathfrak{m}^2 E_{-2}$

$$(a + bS)u \equiv au + \phi(b)uu_1 \tag{3.3}$$

$$(a + bS)uu_1 \equiv 3bu + \phi(a)uu_1 \ . \tag{3.4}$$

In some cases we can be more specific. For example, if $\alpha \in \mathbb{F}_9^\times \subseteq \mathbb{W}^\times \subseteq \mathbb{G}_2$, then the induced map of rings

$$\alpha_* : E_* \to E_*$$

is the $\mathbb{W}$-algebra map defined by the formulas

$$\alpha_*(u) = \alpha u \qquad \text{and} \qquad \alpha_*(uu_1) = \alpha^3 uu_1 \ . \tag{3.5}$$

Finally, since the Honda formal group is defined over $\mathbb{F}_3$ the action of the Frobenius on $E_* = \mathbb{W}[[u_1]][u^{\pm 1}]$ is simply extended from the action on $\mathbb{W}$. Thus we have

$$\psi(x) = \omega_* \phi(x) \tag{3.6}$$

for all $x \in E_2$.

The formulas (3.3) up to (3.6) imply that $E_0/(3, u_1^2) \otimes_{E_0} E_{-2}$ is isomorphic to $\mathbb{F}_9 \otimes_{\mathbb{W}} \rho$ as a G_{24}-module and, further, that we can choose as a generator the residue class of u. In [7] (following [18], who learned it from Hopkins) we found a class $y \in E_{-2}$ so that

$$y \equiv \omega u \qquad \text{mod} \qquad (3, u_1) \tag{3.7}$$

and so that

$$(1 + s + s^2)y = 0.$$

This element might not yet have the correct invariance property with respect to ψ; to correct this, we average and set

$$\begin{aligned} x = \frac{1}{8}(&y + \omega^{-2} t_*(y) + \omega^{-4}(t^2)_*(y) + \omega^{-6}(t^3)_*(y) \\ &+ \omega^{-1}\psi_*(y) + \omega^{-7}(\psi t)_*(y) + \omega^{-5}(\psi t^2)_*(y) + \omega^{-3}(\psi t^3)_*(y)). \end{aligned}$$

We can now send the generator of ρ to x. Note also that the formulas (3.3) up to (3.7) imply that

$$x \equiv \frac{1}{8}(\omega u + \omega^3 u) \text{ modulo } (3, u_1^2)\ . \qquad \square$$

We now make a construction. The morphism of G_{24}-modules constructed in this last lemma defines a morphism of $\mathbb{W}$-algebras

$$S(\rho) = S_{\mathbb{W}}(\rho) \longrightarrow E_*$$

sending the generator e of ρ to an invertible element in E_{-2}. The symmetric algebra is over $\mathbb{W}$ and the map is a map of $\mathbb{W}$-algebras. The group G_{24} acts through $\mathbb{Z}_3$-algebra maps, and the subgroup G_{12} acts through $\mathbb{W}$-algebra maps. If $a \in \mathbb{W}$ is a multiple of the unit, then $\psi(a) = \phi(a)$.

Let

$$N = \prod_{g \in G_{12}} ge \in S(\rho); \tag{3.8}$$

then N is invariant by G_{12} and $\psi(N) = -N$ so that we get a morphism of graded G_{24}-algebras

$$S(\rho)[N^{-1}] \longrightarrow E_*$$

(where the grading on the source is determined by putting ρ in degree -2). Inverting N inverts e, but in an invariant manner. This map is not yet an isomorphism, but it is an inclusion onto a dense subring. The following result is elementary (cf. Proposition 2 of [7]):

LEMMA 3.2. *Let $I = S(\rho)[N^{-1}] \cap \mathfrak{m}$. Then completion at the ideal I defines an isomorphism of G_{24}-algebras*

$$S(\rho)[N^{-1}]_I^{\wedge} \cong E_*.$$

Thus the input for the calculation of the E_2-term $H^*(G_{24}, E_*)$ of the homotopy fixed point spectral sequence associated to $E_2^{hG_{24}}$ will be discrete. Indeed, let $A = S(\rho)[N^{-1}]$. Then the essential calculation is that of $H^*(G_{24}, A)$. For this we begin with the following. For any finite group G and any G module M, let

$$\mathrm{tr}_G = \mathrm{tr} : M \longrightarrow M^G = H^0(G, M)$$

be the transfer: $\mathrm{tr}(x) = \sum_{g \in G} gx$. In the following result, an element listed as being in bidegree (s,t) is in $H^s(G, A_t)$.

If $e \in \rho$ is the generator, define $d \in A$ to be the multiplicative norm with respect to the cyclic group C_3 generated by s: $d = s^2(e)s(e)e$. By construction d is invariant with respect to C_3.

Lemma 3.3. *Let $C_3 \subseteq G_{12}$ be the normal subgroup of order three. Then there is an exact sequence*

$$A \xrightarrow{\mathrm{tr}} H^*(C_3, A) \to \mathbb{F}_9[a, b, d^{\pm 1}]/(a^2) \to 0$$

where a has bidegree $(1, -2)$, b has bidegree $(2, 0)$ and d has bidegree $(0, -6)$. Furthermore the action of t and ψ is described by the formulas

$$t(a) = -\omega^2 a \qquad t(b) = -b \qquad t(d) = \omega^6 d$$

and

$$\psi(a) = \omega a \qquad \psi(b) = b \qquad \psi(d) = \omega^3 d\ .$$

Proof. This is the same argument as in Lemma 3 of [7], although here we keep track of the Frobenius.

Let F be the G_{24}-module $\mathbb{W}[G_{24}] \otimes_{\mathbb{W}[Q_8]} \chi'$; thus equation 3.2 gives a short exact sequence of G_{24}-modules

$$0 \to S(F) \otimes \chi \to S(F) \to S(\rho) \to 0\ . \tag{3.9}$$

In the first term, we set the degree of χ to be -2 in order to make this an exact sequence of graded modules. We use the resulting long exact sequence for computations. We may choose $\mathbb{W}$-generators of F labelled x_1, x_2, and x_3 so that if s is the chosen element of order 3 in G_{24}, then $s(x_1) = x_2$ and $s(x_2) = x_3$. Furthermore, we can choose x_1 so that it maps to the generator e of ρ and is invariant under the action of the Frobenius. Then we have

$$S(F) = \mathbb{W}[x_1, x_2, x_3]$$

with the x_i in degree -2. Under the action of C_3 the orbit of a monomial in $\mathbb{W}[x_1, x_2, x_3]$ has three elements unless that monomial is a power of $\sigma_3 = x_1x_2x_3$ – which, of course, maps to d. Thus, we have a short exact sequence

$$S(F) \xrightarrow{\mathrm{tr}} H^*(C_3, S(F)) \to \mathbb{F}_9[b, d] \to 0$$

where b has bidegree $(2, 0)$ and d has bidegree $(0, -6)$. Here $b \in H^2(C_3, \mathbb{Z}_3) \subseteq H^2(C_3, \mathbb{W})$ is a generator and $\mathbb{W} \subseteq S(F)$ is the submodule generated by the algebra unit. Note that the action of t is described by

$$t(d) = \omega^6 d \qquad \text{and} \qquad t(b) = -b\ .$$

The last is because the element t acts nontrivially on the subgroup $C_3 \subseteq G_{24}$ and hence on $H^2(C_3, \mathbb{W})$. Similarly, since the action of the Frobenius on d is trivial and ψ acts trivially on C_3, we have

$$\psi(d) = \omega^3 d \qquad \text{and} \qquad \psi(b) = b\ .$$

The short exact sequence (3.9) and the fact that $H^1(C_3, S(F)) = 0$ now imply that there is an exact sequence

$$S(\rho) \xrightarrow{\mathrm{tr}} H^*(C_3, S(\rho)) \to \mathbb{F}_9[a, b, d]/(a^2) \to 0\ .$$

The element a maps to

$$b \in H^2(C_3, S_0(F) \otimes \chi) = H^2(C_3, \chi)$$

under the boundary map (which is an isomorphism)

$$H^1(C_3, \rho) = H^1(C_3, S_1(\rho)) \to H^2(C_3, \chi);$$

thus a has bidegree $(1, -2)$ and the actions of t and ψ are twisted by χ:

$$t(a) = -\omega^2 a = \omega^6 a \qquad \text{and} \qquad \psi(a) = \omega a \ . \qquad \square$$

We next write down the invariants E_*^F for the various finite subgroups F of G_{24}. To do this, we work up from the symmetric algebra $S(\rho)$, and we use the presentation of the symmetric algebra as given in the exact sequence (3.9). Recall that we have written $S(F) = \mathbb{W}[x_1, x_2, x_3]$ where the normal subgroup of order three in G_{24} cyclically permutes the x_i. This action by the cyclic group extends in an obvious way to an action of the symmetric group Σ_3 on three letters; thus we have an inclusion of algebras

$$\mathbb{W}[\sigma_1, \sigma_2, \sigma_3] = \mathbb{W}[x_1, x_2, x_3]^{\Sigma_3} \subseteq S(F)^{C_3}.$$

There is at least one other obvious element invariant under the action of C_3: set

$$\epsilon = x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_1 - x_2^2 x_1 - x_1^2 x_3 - x_3^2 x_2 \ . \tag{3.10}$$

This might be called the "anti-symmetrization" (with respect to Σ_3) of $x_1^2 x_2$.

Lemma 3.4. *There is an isomorphism*

$$\mathbb{W}[\sigma_1, \sigma_2, \sigma_3, \epsilon]/(\epsilon^2 - f) \cong S(F)^{C_3}$$

where f is determined by the relation

$$\epsilon^2 = -27\sigma_3^2 - 4\sigma_2^3 - 4\sigma_3\sigma_1^3 + 18\sigma_1\sigma_2\sigma_3 + \sigma_1^2\sigma_2^2 \ .$$

Furthermore, the actions of t and ψ are given by

$$t(\sigma_1) = \omega^2\sigma_1 \qquad t(\sigma_2) = -\sigma_2 \qquad t(\sigma_3) = \omega^6\sigma_3 \qquad t(\epsilon) = \omega^2\epsilon$$

and

$$\psi(\sigma_1) = \omega\sigma_1 \qquad \psi(\sigma_2) = \omega^2\sigma_2 \qquad \psi(\sigma_3) = \omega^3\sigma_3 \qquad \psi(\epsilon) = \omega^3\epsilon \ .$$

Proof. Except for the action of ψ, this is Lemma 4 of [7]. The action of ψ is straightforward $\square$

This immediately leads to the following result.

PROPOSITION 3.5. *There is an isomorphism*

$$\mathbb{W}[\sigma_2, \sigma_3, \epsilon]/(\epsilon^2 - g) \cong S(\rho)^{C_3}$$

where g is determined by the relation

$$\epsilon^2 = -27\sigma_3^2 - 4\sigma_2^3$$

with the actions of t and ψ as given above in Lemma 3.4. *Under this isomorphism σ_3 maps to d.*

Proof. This follows immediately from Lemma 3.4, the short exact sequence (3.9), and the fact (see the proof of Lemma 3.3) that $H^1(C_3, S(F)) = 0$. Together these imply that

$$S(\rho)^{C_3} \cong S(F)^{C_3}/(\sigma_1) \; . \qquad \square$$

The next step is to invert the element N of (3.8). This element is the image of σ_3^4; thus, we are effectively inverting the element $d = \sigma_3 \in S(\rho)^{C_3}$. We begin with the observation that if we divide

$$\epsilon^2 = -27\sigma_3^2 - 4\sigma_2^3$$

by σ_3^6 we obtain the relation

$$(\frac{\epsilon}{\sigma_3^3})^2 + 4(\frac{\sigma_2}{\sigma_3^2})^3 = -\frac{27}{\sigma_3^4} \; .$$

Thus if we set

$$c_4 = -\frac{\omega^2\sigma_2}{\sigma_3^2}, \quad c_6 = \frac{\omega^3\epsilon}{2\sigma_3^3}, \quad \Delta = -\frac{\omega^6}{4\sigma_3^4} = \frac{\omega^2}{4\sigma_3^4} \tag{3.11}$$

then we get the expected relation[4]

$$c_6^2 - c_4^3 = 27\Delta \; .$$

Furthermore, c_4, c_6, and Δ are all invariant under the action of the entire group G_{24}. (Indeed, the powers of ω are introduced so that this happens.)

To describe the group cohomology, we define elements

$$\alpha = \frac{\omega a}{d} \in H^1(C_3, (S(\rho)[N^{-1}])_4)$$

[4]This is the relation appearing in theory of modular forms [2], except here 2 is invertible so we can replace 1728 by 27. There is some discussion of the connection in [8]. The relation could be arrived at more naturally by choosing, as our basic formal group law, one arising from the theory of elliptic curves, rather than the Honda formal group law.

and

$$\beta = \frac{\omega^3 b}{d^2} \in H^2(C_3, (S(\rho)[N^{-1}])_{12}).$$

These elements are fixed by t and ψ and, for degree reasons, acted on trivially by c_4 and c_6. The following is now easy.

PROPOSITION 3.6. 1) *The inclusion*

$$\mathbb{Z}_3[c_4, c_6, \Delta^{\pm 1}]/(c_6^2 - c_4^3 = 27\Delta) \to S(\rho)[N^{-1}]^{G_{24}}$$

is an isomorphism of algebras.

2) *There is an exact sequence*

$$S(\rho)[N^{-1}] \xrightarrow{\mathrm{tr}} H^*(G_{24}, S(\rho)[N^{-1}]) \to \mathbb{F}_3[\alpha, \beta, \Delta^{\pm 1}]/(\alpha^2) \to 0$$

and c_4 and c_6 act trivially on α and β.

Then a completion argument, as in Theorem 6 of [7] or [18] implies the next result.

THEOREM 3.7. 1) *There is an isomorphism of algebras*

$$(E_*)^{G_{24}} \cong \mathbb{Z}_3[[c_4^3\Delta^{-1}]][c_4, c_6, \Delta^{\pm 1}]/(c_6^2 - c_4^3 = 27\Delta) \ .$$

2) *There is an exact sequence*

$$E_* \xrightarrow{\mathrm{tr}} H^*(G_{24}, E_*) \to \mathbb{F}_3[\alpha, \beta, \Delta^{\pm 1}]/(\alpha^2) \to 0$$

and c_4 and c_6 act trivially on α and β.

Remark 3.8. The same kind of reasoning can be used to obtain the group cohomologies $H^*(F, E_*)$ for other finite subgroups of $\mathbb{G}_2$. First define an element

$$\delta = \sigma_3^{-1} \in S(\rho)[N^{-1}] \ .$$

Then $\Delta = (\omega^2/4)\delta^4$; thus $-\Delta$ has a square root:

$$(-\Delta)^{1/2} = \frac{\omega^3}{2}\delta^2 \ .$$

The elements t and ψ of G_{24} act on δ by the formulas

$$t(\delta) = \omega^2\delta \qquad \text{and} \qquad \psi(\delta) = \omega^5\delta \ .$$

The element $(-\Delta)^{1/2}$ is invariant under the action of t^2 and ψ (whereas the evident square root of Δ is not fixed by ψ).

Let C_{12} be the cyclic subgroup of order 12 in G_{24} generated by s and ψ. This subgroup has a cyclic subgroup C_6 of order 6 generated by s and $t^2 = \psi^2$.

We have

$$
\begin{aligned}
(E_*)^{C_3} &\cong \mathbb{W}[[c_4^3\Delta^{-1}]][c_4, c_6, \delta^{\pm 1}]/(c_6^2 - c_4^3 = 27\Delta)\\
(E_*)^{C_{12}} &\cong \mathbb{Z}_3[[c_4^3\Delta^{-1}]][c_4, c_6, (-\Delta)^{\pm 1/2}]/(c_6^2 - c_4^3 = 27\Delta)\\
(E_*)^{C_6} &\cong \mathbb{W} \otimes_{\mathbb{Z}_3} (E_*)^{C_{12}}\\
(E_*)^{G_{12}} &\cong \mathbb{W}[[c_4^3\Delta^{-1}]][c_4, c_6, \Delta^{\pm 1}]/(c_6^2 - c_4^3 = 27\Delta) \cong \mathbb{W} \otimes_{\mathbb{Z}_3} (E_*)^{G_{24}}.
\end{aligned}
$$

Furthermore, for all these groups, the analogue of Theorem 3.7.2 holds. For example, there are exact sequences

$$
\begin{aligned}
&E_* \xrightarrow{\mathrm{tr}} H^*(C_3, E_*) \to \mathbb{F}_9[\alpha, \beta, \delta^{\pm 1}]/(\alpha^2) \to 0\\
&E_* \xrightarrow{\mathrm{tr}} H^*(C_{12}, E_*) \to \mathbb{F}_3[\alpha, \beta, (-\Delta)^{\pm 1/2}]/(\alpha^2) \to 0\\
&E_* \xrightarrow{\mathrm{tr}} H^*(C_6, E_*) \to \mathbb{F}_9[\alpha, \beta, (-\Delta)^{\pm 1/2}]/(\alpha^2) \to 0\\
&E_* \xrightarrow{\mathrm{tr}} H^*(G_{12}, E_*) \to \mathbb{F}_9[\alpha, \beta, (\Delta)^{\pm 1}]/(\alpha^2) \to 0
\end{aligned}
$$

and c_4 and c_6 act trivially on α and β.

These results allow one to competely write down the various homotopy fixed point spectral sequences for computing $\pi_* E^{hF}$ for the various finite groups in question. The differentials in the spectral sequence follow from Toda's classical results and the following easy observation: every element in the image of the transfer is a permanent cycle. We record:

Lemma 3.9. *In the spectral sequence*

$$H^*(G_{24}, E_*) \Longrightarrow \pi_* E^{hG_{24}}$$

the only nontrivial differentials are d_5 and d_9. They are determined by

$$d_5(\Delta) = a_1\alpha\beta^2 \qquad \textit{and} \qquad d_9(\alpha\Delta^2) = a_2\beta^5$$

where a_1 and a_2 are units in $\mathbb{F}_3$.

Proof. These are a consequence of Toda's famous differential (see [29]) and nilpotence. See Proposition 7 of [7] or, again, [18]. There it is done for G_{12} rather than G_{24}, but because G_{12} is of index 2 in G_{24} and we are working at the prime 3, this is sufficient. □

The lemma immediately calculates the differentials in the other spectral sequences; for example, if one wants homotopy fixed points with respect to the C_3-action, we have, up to units,

$$d_5(\delta) = \delta^{-3}\alpha\beta^2 \qquad \text{and} \qquad d_9(\alpha\delta^2) = \delta^{-6}\beta^5.$$

It is also worth pointing out that the d_5-differential in Lemma 3.9 and some standard Toda bracket manipulation (see the proof of Theorem 8 in [7]) implies the relation $(\Delta\alpha)\alpha = \pm\beta^3$ which holds in $\pi_{27}(E^{hG_{24}})$.

The above discussion is summarized in the following main homotopy theoretic result of this section.

THEOREM 3.10. *In the spectral sequence*

$$H^*(C_3, E_*) \Longrightarrow \pi_* E^{hC_3}$$

we have an inclusion of subrings

$$E_\infty^{0,*} \cong \mathbb{W}[[c_4^3\Delta^{-1}]][c_4, c_6, c_4\delta^{\pm1}, c_6\delta^{\pm1}, 3\delta^{\pm1}, \delta^{\pm3}]/(c_4^3 - c_6^2 = 27\Delta) \subseteq E_2^{0,*}.$$

In positive filtration E_∞ is additively generated by the elements α, $\delta\alpha$, $\alpha\beta$, $\delta\alpha\beta$, β^j, $1 \leq j \leq 4$ and all multiples of these elements by $\delta^{\pm3}$. These elements are of order 3 and are annihilated by c_4, c_6, $c_4\delta^{\pm1}$, $c_6\delta^{\pm1}$ and $3\delta^{\pm1}$. Furthermore the following multiplicative relation holds in $\pi_{30}(E^{hC_3})$: $\delta^3(\delta\alpha)\alpha = \pm\omega^6\beta^3$.

For the case of the cyclic group C_6 of order 6 generated by s and t^2, one notes that $t^2(\delta) = -\delta$ and the spectral sequence can now be read off Theorem 3.10. This also determines the case of C_{12}, the group generated by s and ψ. We leave the details to the reader but state the result in the case of G_{24}.

THEOREM 3.11. *In the spectral sequence*

$$H^*(G_{24}, E_*) \Longrightarrow \pi_* E^{hG_{24}}$$

we have an inclusion of subrings

$$E_\infty^{0,*} \cong \mathbb{Z}_3[[c_4^3\Delta^{-1}]][c_4, c_6, c_4\Delta^{\pm1}, c_6\Delta^{\pm1}, 3\Delta^{\pm1}, \Delta^{\pm3}]/(c_4^3 - c_6^2 = 27\Delta) \subseteq E_2^{0,*}.$$

In positive filtration E_∞ is additively generated by the elements α, $\Delta\alpha$, $\alpha\beta$, $\Delta\alpha\beta$, β^j, $1 \leq j \leq 4$ and all multiples of these elements by $\Delta^{\pm3}$. These elements are of order 3 and are annihilated by c_4, c_6, $c_4\Delta^{\pm1}$, $c_6\Delta^{\pm1}$ and $3\Delta^{\pm1}$. Furthermore $(\Delta\alpha)\alpha = \pm\beta^3$ in $\pi_{30}(E^{hG_{24}})$.

Remark 3.12. 1) Note that E^{hC_3} is periodic of period 18 and that δ^3 detects a periodicity class. The spectra E^{hC_6} and $E^{hC_{12}}$ are periodic with period 36 and $(-\Delta)^{3/2}$ detects the periodicity generator. Finally $E^{hG_{12}}$ and $E^{hG_{24}}$ are periodic with period 72 and Δ^3 detects the periodicity generator.

2) By contrast, the Morava module $E_*E^{hG_{24}}$ is of period 24. To see this, note that the isomorphism of (2.7) supplies an isomorphism of Morava modules

$$E_*E^{hG_{24}} \cong \mathrm{Hom}^c(\mathbb{G}_2/G_{24}, E_*)$$

with $\mathbb{G}_2$ acting diagonally on the right-hand side. Then G_{24}-invariance of Δ implies that there is a well defined automorphism of Morava modules given by $\varphi \mapsto (g \mapsto \varphi(g)g_*(\Delta))$.

3) If F has order prime to 3, then $\pi_*(E^{hF}) = (E_*)^F$ is easy to calculate. For example, using (3.5) and (3.6) we obtain

$$\pi_*(E^{hSD_{16}}) = \mathbb{Z}_3[v_1][v_2^{\pm 1}]\hat{} \subseteq E_*$$

with $v_1 = u_1 u^{-2}$ and $v_2 = u^{-8}$ and completion is with respect to the ideal generated by $(v_1^4 v_2^{-1})$. Similarly

$$\pi_*(E^{hQ_8}) = \mathbb{Z}_3[v_1][\omega^2 u^{\pm 4}]\hat{} \subseteq E_*$$

where completion is with respect to the ideal $(v_1^2\omega^2 u^4)$. Note that $E^{hSD_{16}}$ is periodic of order 16 and E^{hQ_8} is periodic of order 8.

We finish this section by listing exactly the computational results we will use in building the tower in Section 5.

Corollary 3.13. 1) *Let $F \subseteq \mathbb{G}_2$ be any finite subgroup. Then the edge homomorphism*

$$\pi_0 E^{hF} \longrightarrow (E_0)^F$$

is an isomorphism of algebras.

2) *Let $F \subseteq \mathbb{G}_2$ be any finite subgroup. Then*

$$\pi_{24} E^{hF} \longrightarrow (E_{24})^F$$

is an injection.

Proof. If F has order prime to 3, both of these statements are clear. If 3 divides the order of F, then the 3-Sylow subgoup of F is conjugate to C_3; hence $\pi_* E^{hF}$ is a retract of $\pi_* E^{hC_3}$ and the result follows from Theorem 3.10. □

Corollary 3.14. *Let $F \subseteq \mathbb{G}_2$ be a finite subgroup. If the order of F is prime to* 3, *then*

$$\pi_1 E^{hF} = 0\ .$$

For all finite subgroups F,

$$\pi_{25} E^{hF} = 0\ .$$

Proof. Again apply Theorem 3.10. □

Corollary 3.15. *Let $F \subseteq \mathbb{G}_2$ be any finite subgroup containing the central element $\omega^4 = -1$. Then*

$$\pi_{26} E^{hF} = 0\ .$$

Proof. Equation 3.5 implies that $(E_*)^F$ will be concentrated in degrees congruent to 0 mod 4. Combining this observation with Theorem 3.10 proves the result. □

4. The algebraic resolution

Let G be a profinite group. Then its p-adic group ring $\mathbb{Z}_p[[G]]$ is defined as $\lim_{U,n} \mathbb{Z}_p/(p^n)[G/U]$ where U runs through all open subgroups of G. Then $\mathbb{Z}_p[[G]]$ is a complete ring and we will only consider continuous modules over such rings.

In this section we will construct our resolution of the trivial $\mathbb{Z}_3[[\mathbb{G}_2]]$-module $\mathbb{Z}_3$. In 1.3 we wrote down a splitting of the group $\mathbb{G}_2$ as $\mathbb{G}_2^1 \times \mathbb{Z}_3$ and this splitting allows us to focus on constructing a resolution of $\mathbb{Z}_3$ as a $\mathbb{Z}_3[[\mathbb{G}_2^1]]$-module.

Recall that we have selected a maximal finite subgroup $G_{24} \subseteq \mathbb{G}_2^1$; it is generated by an element s of order three, $t = \omega^2$, and $\psi = \omega\phi$ where ω is a primitive eighth root of unity and ϕ is the Frobenius. As before C_3 denotes the normal subgroup of order 3 in G_{24} generated by s, Q_8 denotes the subgroup of G_{24} of order 8 generated by t and ψ and SD_{16} the subgroup of $\mathbb{G}_2^1$ generated by ω and ψ.

The group Q_8 is a subgroup of SD_{16} of index 2. Let χ be the sign representation (over $\mathbb{Z}_3$) of SD_{16}/Q_8; we regard χ as representation of SD_{16} using the quotient map. (Note that this is not the same χ as in §3!) In this section, the induced $\mathbb{Z}_3[[\mathbb{G}_2^1]]$-module

$$\chi \uparrow_{SD_{16}}^{\mathbb{G}_2^1} \stackrel{\text{def}}{=} \mathbb{Z}_3[[\mathbb{G}_2^1]] \otimes_{\mathbb{Z}_3[SD_{16}]} \chi \tag{4.1}$$

will play an important role. If τ is the trivial representation of SD_{16}, there is an isomorphism of SD_{16}/Q_8-modules

$$\mathbb{Z}_3[SD_{16}/Q_8] \cong \chi \oplus \tau \ . \tag{4.2}$$

Thus, we have that $\chi \uparrow_{SD_{16}}^{\mathbb{G}_2^1}$ is a direct summand of the induced module

$$\mathbb{Z}_3[[\mathbb{G}_2^1/Q_8]] = \mathbb{Z}_3[[\mathbb{G}_2^1]] \otimes_{\mathbb{Z}_3[Q_8]} \mathbb{Z}_3 \ .$$

The following is the main algebraic result of the paper. It will require the entire section to prove.

THEOREM 4.1. *There is an exact sequence of* $\mathbb{Z}_3[[\mathbb{G}_2^1]]$-*modules*

$$0 \to \mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]] \to \chi \uparrow_{SD_{16}}^{\mathbb{G}_2^1} \to \chi \uparrow_{SD_{16}}^{\mathbb{G}_2^1} \to \mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]] \to \mathbb{Z}_3 \to 0 \ .$$

Some salient features of this "resolution" (we will use this word even though not all of the modules are projective) are that each module is a summand of a permutation module $\mathbb{Z}_3[[\mathbb{G}_2^1/F]]$ for a finite subgroup F and that each module is free over K, where $K \subseteq \mathbb{G}_2^1$ is a subgroup so that we can decompose the 3-Sylow subgroup S_2^1 of $\mathbb{G}_2$ as $K \rtimes C_3$. Important features of K include that it is a torsion-free 3-adic Poincaré duality group of dimension 3. (See before Lemma 4.10 for more on K.)

Since $\mathbb{G}_2 \cong \mathbb{G}_2^1 \times \mathbb{Z}_3$, we may tensor the resolution of Theorem 4.1 with the standard resolution for $\mathbb{Z}_3$ to get the following as an immediate corollary,

COROLLARY 4.2. *There is an exact sequence of* $\mathbb{Z}_3[[\mathbb{G}_2]]$-*modules*

$$\begin{aligned} 0 \to \mathbb{Z}_3[[\mathbb{G}_2/G_{24}]] \to \mathbb{Z}_3[[\mathbb{G}_2/G_{24}]] \oplus \chi \uparrow_{SD_{16}}^{\mathbb{G}_2} \to \chi \uparrow_{SD_{16}}^{\mathbb{G}_2} \oplus \chi \uparrow_{SD_{16}}^{\mathbb{G}_2} \\ \to \chi \uparrow_{SD_{16}}^{\mathbb{G}_2} \oplus \mathbb{Z}_3[[\mathbb{G}_2/G_{24}]] \to \mathbb{Z}_3[[\mathbb{G}_2/G_{24}]] \to \mathbb{Z}_3 \to 0 \ . \end{aligned}$$

As input for our calculation we will use

$$H^*(S_2^1) := H^*(S_2^1, \mathbb{F}_3) = \mathrm{Ext}^*_{\mathbb{Z}_3[[S_2^1]]}(\mathbb{Z}_3, \mathbb{F}_3) \ ,$$

as calculated by the second author in [10]. This is an effective starting point because of the following lemma and the fact that

$$\mathrm{Ext}^q_{\mathbb{Z}_3[[S_2^1]]}(M, \mathbb{F}_3) \cong \mathrm{Tor}_q^{\mathbb{Z}_3[[S_2^1]]}(\mathbb{F}_3, M)^* \tag{4.3}$$

for any profinite continuous $\mathbb{Z}_3[[S_2^1]]$-module M. Here $(-)^*$ means $\mathbb{F}_p$-linear dual.

A profinite group G is called finitely generated if there is a finite set of elements $X \subseteq G$ so that the subgroup generated by X is dense. This is true of all the groups in this paper. If G is a p-profinite group and $I \subseteq \mathbb{Z}_p[[G]]$ is the kernel of the augmentation $\mathbb{Z}_p[[G]] \to \mathbb{F}_p$, then

$$\mathbb{Z}_p[[G]] \cong \lim_n \mathbb{Z}_p[[G]]/I^n.$$

A $\mathbb{Z}_p[[G]]$-module M will be called *complete* if it is I-adically complete, i.e. if $M \cong \lim_n M/I^n M$.

LEMMA 4.3. *Let G be a finitely generated p-profinite group and $f : M \to N$ a morphism of complete $\mathbb{Z}_p[[G]]$-modules. If*

$$\mathbb{F}_p \otimes f : \mathbb{F}_p \otimes_{\mathbb{Z}_p[[G]]} M \to \mathbb{F}_p \otimes_{\mathbb{Z}_p[[G]]} N$$

is surjective, *then f is surjective. If*

$$\mathrm{Tor}(\mathbb{F}_p, f) : \mathrm{Tor}_q^{\mathbb{Z}_p[[G]]}(\mathbb{F}_p, M) \to \mathrm{Tor}_q^{\mathbb{Z}_p[[G]]}(\mathbb{F}_p, N)$$

is an isomorphism for $q = 0$ and onto for $q = 1$, then f is an isomorphism.

Proof. This is an avatar of Nakayama's lemma. To see this, suppose K is some complete $\mathbb{Z}_p[[G]]$-module so that $\mathbb{F}_p \otimes_{\mathbb{Z}_p[[G]]} K = 0$. Then an inductive argument shows

$$\mathbb{Z}_p[[G]]/I^n \otimes_{\mathbb{Z}_p[[G]]} K = 0$$

for all n; hence $K = 0$. This is the form of Nakayama's lemma we need.

The result is then proved using the long exact sequence of Tor groups: the weaker hypothesis implies that the cokernel of f is trivial; the stronger hypothesis then implies that the kernel of f is trivial. $\square$

We next turn to the details about $H^*(S_2^1;\mathbb{F}_3)$ from [10]. (See Theorem 4.3 of that paper.) We will omit the coefficients $\mathbb{F}_3$ in order to simplify our notation. The key point here is that the cohomology of the group S_2^1 is detected on the centralizers of the cyclic subgroups of order 3. There are two conjugacy classes of such subgroups of order 3 in S_2^1; namely, C_3 and $\omega C_3\omega^{-1}$. The element $s = s_1$ is our chosen generator for C_3; thus we choose as our generator for $\omega C_3\omega^{-1}$ the element $s_2 = \omega s\omega^{-1}$. The Frobenius ϕ also conjugates C_3 to $\omega C_3\omega^{-1}$ and a short calculation shows that

$$\phi(s_1) = \phi(s) = s_2^2 \ . \tag{4.4}$$

The centralizer $C(C_3)$ in S_2^1 is isomorphic to $C_3 \times \mathbb{Z}_3$, $\omega\phi$ commutes with $C(C_3)$ and conjugation by ω^2 sends $x \in C(C_3)$ to its inverse x^{-1} (see [7]). In particular, for every $x \in C(C_3)$ we have

$$\omega x\omega^{-1} = \phi(x)^{-1}\phi^{-1} \in C(\omega C_3\omega^{-1}) \ . \tag{4.5}$$

Note that $C(\omega C_3\omega^{-1}) = \omega C(C_3)\omega^{-1}$. Write $E(X)$ for the exterior algebra on a set X. Then

$$H^*(C(C_3)) \cong \mathbb{F}_3[y_1] \otimes E(x_1, a_1)$$

and

$$H^*(C(\omega C_3\omega^{-1})) \cong \mathbb{F}_3[y_2] \otimes E(x_2, a_2) \ .$$

We know that C_4 (which is generated by ω^2) acts on

$$H^*(C(C_3)) \cong \mathbb{F}_3[y_1] \otimes E(x_1, a_1)$$

sending all three generators to their negative. This action extends to an action of SD_{16} on the product

$$H^*(C(C_3)) \times H^*(C(\omega C_3\omega^{-1}))$$

as follows. By (4.4) and (4.5) the action of the generators ω and ϕ of SD_{16} is given by

$$\begin{aligned} &\omega_*(x_1) = x_2, &&\omega_*(y_1) = y_2, &&\omega_*(a_1) = a_2,\\ &\phi_*(x_1) = -x_2, &&\phi_*(y_1) = -y_2, &&\phi_*(a_1) = -a_2. \end{aligned} \tag{4.6}$$

$$\begin{aligned} &\omega_*(x_2) = -x_1, &&\omega_*(y_2) = -y_1, &&\omega_*(a_2) = -a_1,\\ &\phi_*(x_2) = -x_1, &&\phi_*(y_2) = -y_1, &&\phi_*(a_2) = -a_1 \ . \end{aligned} \tag{4.7}$$

THEOREM 4.4 ([10]). 1) *The inclusions $\omega^iC(C_3)\omega^{-i} \to S_2^1$, $i = 0, 1$ induce an SD_{16}-equivariant homomorphism*

$$H^*(S_2^1) \to \prod_{i=1}^{2} \mathbb{F}_3[y_i] \otimes E(x_i, a_i)$$

which is an injection onto the subalgebra generated by x_1, x_2, y_1, y_2, $x_1a_1 - x_2a_2$, y_1a_1 *and* y_2a_2.

2) *In particular,* $H^*(S_2^1)$ *is free as a module over* $\mathbb{F}_3[y_1+y_2]$ *on generators* 1, x_1, x_2, y_1, $x_1a_1 - x_2a_2$, y_1a_1, y_2a_2, *and* $y_1x_1a_1$.

We will produce the resolution of Theorem 4.1 from this data and by splicing together the short exact sequences of Lemma 4.5, 4.6, and 4.7 below. Most of the work will be spent in identifying the last module; this is done in Theorem 4.9.

In the following computations, we will write

$$\mathrm{Ext}(M) = \mathrm{Ext}^*_{\mathbb{Z}_3[[S_2^1]]}(M, \mathbb{F}_3) \ .$$

This graded vector space is a module over

$$H^*(S_2^1) = \mathrm{Ext}^*_{\mathbb{Z}_3[[S_2^1]]}(\mathbb{Z}_3, \mathbb{F}_3) = \mathrm{Ext}(\mathbb{Z}_3) \ ,$$

and, hence is also a module over the sub-polynomial algebra of $H^*(S_2^1)$ generated by $y_1 + y_2$. If M is actually a continuous $\mathbb{Z}_3[[\mathbb{G}_2^1]]$-module, then $\mathrm{Ext}(M)$ has an action by $SD_{16} \cong \mathbb{G}_2^1/S_2^1$ which extends the action by $H^*(S_2^1)$ in the obvious way: if $\alpha \in SD_{16}$, $a \in H^*(S_2^1)$ and $x \in \mathrm{Ext}(M)$, then

$$\alpha(ax) = \alpha(a)\alpha(x) \ .$$

We can write this another way. Let's define $\mathrm{Ext}(\mathbb{Z}_3) \odot \mathbb{F}_3[SD_{16}]$ to be the algebra constructed by taking

$$\mathrm{Ext}(\mathbb{Z}_3) \otimes \mathbb{F}_3[SD_{16}]$$

with twisted product

$$(a \otimes \alpha)(b \otimes \beta) = a\alpha(b) \otimes \alpha\beta \ .$$

The above remarks imply that if M is a $\mathbb{Z}_3[[\mathbb{G}_2^1]]$-module, then $\mathrm{Ext}(M)$ is an $\mathrm{Ext}(\mathbb{Z}_3) \odot \mathbb{F}_3[SD_{16}]$-module.

This structure behaves well with respect to long exact sequences. If

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is a short exact sequence of $\mathbb{Z}_3[[\mathbb{G}_2^1]]$-modules we get a long exact sequence in Ext which is a long exact sequence of $\mathrm{Ext}(\mathbb{Z}_3) \odot \mathbb{F}_3[SD_{16}]$-modules. As a matter of notation, if $x \in \mathrm{Ext}(\mathbb{Z}_3)$ we will write $\overline{x} \in \mathrm{Ext}(M)$ if x is the image of $\overline{x}$ under some unambiguous and injective sequence of boundary homomorphisms of long exact sequences.

Lemma 4.5. *There is a short exact sequence of* $\mathbb{Z}_3[[\mathbb{G}_2^1]]$*-modules*

$$0 \to N_1 \to \mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]] \xrightarrow{\epsilon} \mathbb{Z}_3 \to 0$$

where the map ϵ is the augmentation. If we write z for $x_1a_1 - x_2a_2 \in H^2(S_1^2)$ then $\mathrm{Ext}(N_1)$ *is a module over* $\mathbb{F}_3[y_1+y_2]$ *on generators* e, $\overline{z}$, $\overline{y_1a_1}$, $\overline{y_2a_2}$, *and* $\overline{y_1x_1a_1}$ *of degrees* 0, 1, 2, 2, *and* 3 *respectively. The last four generators are free and* $(y_1+y_2)e = 0$. *The action of* SD_{16} *is determined by the action on* $\mathrm{Ext}(\mathbb{Z}_3)$ *and the facts that*

$$\omega_*(e) = -e = \phi_*(e) \ .$$

Proof. As a $\mathbb{Z}_3[[S_2^1]]$-module, there is an isomorphism

$$\mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]] \cong \mathbb{Z}_3[[S_2^1/C_3]] \oplus \mathbb{Z}_3[[S_2^1/\omega C_3\omega^{-1}]].$$

Hence, by the Shapiro lemma there is an isomorphism

$$\mathrm{Ext}(\mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]]) \cong H^*(C_3,\mathbb{F}_3) \times H^*(\omega C_3\omega^{-1},\mathbb{F}_3)$$

and the map $\mathrm{Ext}(\mathbb{Z}_3) \to \mathrm{Ext}(\mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]])$ corresponds via this isomorphism to the restriction map. The result now follows from Theorem 4.4. □

Recall that χ is the rank one (over $\mathbb{Z}_3$) representation of SD_{16} obtained by pulling back the sign representation along the quotient map $\varepsilon : SD_{16} \to SD_{16}/Q_8 \cong \mathbb{Z}/2$.

Lemma 4.6. *There is a short exact sequence of* $\mathbb{Z}_3[[\mathbb{G}_2^1]]$*-modules*

$$0 \to N_2 \to \chi \uparrow_{SD_{16}}^{\mathbb{G}_2^1} \to N_1 \to 0 \ .$$

The cohomology module $\mathrm{Ext}(N_2)$ *is a freely generated module over* $\mathbb{F}_3[y_1+y_2]$ *on generators* $\overline{z}$, $\overline{y_1a_1}$, $\overline{y_2a_2}$, *and* $\overline{y_1x_1a_1}$ *of degrees* 0, 1, 1, *and* 2 *respectively. The action of* ω *is determined by the action on* $\mathrm{Ext}(\mathbb{Z}_3)$.

Proof. By the previous result the SD_{16}-module $\mathbb{F}_3 \otimes_{\mathbb{Z}_3[[S_2^1]]} N_1$ is one dimensional over $\mathbb{F}_3$ generated by the dual (with respect to (4.3)) of the class of e and the action is given by the sign representation along ε. Lift e to an element $d \in N_1$. Then SD_{16} may not act correctly on d, but we can average d to obtain an element c on which SD_{16} acts correctly and which reduces to the same element in $\mathbb{F}_3 \otimes_{\mathbb{Z}_3[[S_2^1]]} N_1$; indeed,

$$c = \frac{1}{16} \sum_{\alpha \in SD_{16}} \varepsilon(\alpha)^{-1}\alpha_*(d) \ .$$

This defines the morphism

$$\chi \uparrow_{SD_{16}}^{\mathbb{G}_2^1} \to N_1 \ .$$

Lemma 4.3 now implies that this map is surjective and we obtain the exact sequence we need. For the calculation of $\mathrm{Ext}(N_2)$ note that we have an isomorphism of S_2^1-modules

$$\chi \uparrow_{SD_{16}}^{\mathbb{G}_2^1} \cong \mathbb{Z}_3[[S_2^1]] \ .$$

The result now follows from the previous lemma and the long exact sequence. $\square$

LEMMA 4.7. *There is a short exact sequence of* $\mathbb{Z}_3[[\mathbb{G}_2^1]]$*-modules*

$$0 \to N_3 \to \chi \uparrow_{SD_{16}}^{\mathbb{G}_2^1} \to N_2 \to 0$$

where $\mathrm{Ext}(N_3)$ *is a free module over* $\mathbb{F}_3[y_1+y_2]$ *on generators* $\overline{y_1a_1}$, $\overline{y_2a_2}$, $\overline{y_1x_1a_1}$, *and* $\overline{y_2x_2a_2}$ *of degree* 0, 0, 1 *and* 1 *respectively. In fact, the iterated boundary homomorphisms*

$$\mathrm{Ext}^*(N_3) \to \mathrm{Ext}^{*+3}(\mathbb{Z}_3) = H^{*+3}(S_2^1, \mathbb{F}_3)$$

define an injection onto an $\mathrm{Ext}(\mathbb{Z}_3) \odot \mathbb{F}_3[SD_{16}]$*-submodule isomorphic to*

$$\mathrm{Ext}(\mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]]) \ .$$

Proof. The SD_{16}-module $\mathbb{F}_3 \otimes_{\mathbb{Z}_3[[S_2^1]]} N_2$ is $\mathbb{F}_3 \otimes_{\mathbb{Z}_3} \chi$ generated by the class dual to $\bar{z}$. As in the proof the last lemma, we can now form a surjective map

$$\chi \uparrow_{SD_{16}}^{\mathbb{G}_2^1} \to N_2 \ ,$$

and this map defines N_3. The calculation of $\mathrm{Ext}(N_3)$ follows from the long exact sequence. $\square$

To make use of this last result we prove a lemma.

LEMMA 4.8. *Let* $A = \mathrm{Ext}(\mathbb{Z}_3) \odot \mathbb{F}_3[SD_{16}]$ *and* $M = \mathrm{Ext}(\mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]])$, *regarded as an* A*-module. Then* M *is a simple* A*-module*; *in fact*,

$$\mathrm{End}_A(M) \cong \mathbb{F}_3 \ .$$

Proof. Let e_1 and e_2 in

$$\mathrm{Ext}(\mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]]) \cong H^*(C_3, \mathbb{F}_3) \times H^*(\omega C_3 \omega^{-1}, \mathbb{F}_3)$$

be the evident two generators in degree 0. If $f : M \to M$ is any A-module endomorphism, we may write

$$f(e_1) = ae_1 + be_2$$

where $a, b \in \mathbb{F}_3$. Then (using the notation of Theorem 4.4) we have

$$0 = f(y_2 e_1) = b y_2 e_2.$$

Since $y_2 e_2 \neq 0$, we have $b = 0$. Also, since $\omega_*(e_1) = e_2$, we have $f(e_2) = ae_2$. Finally, since every homogeneous element of M is of the from $x_1 y_1^i e_1 + x_2 y_2^i e_2$, we have $f = a\mathrm{id}_M$. $\square$

This means that in order to prove the following result, we need only produce a map $f : N_3 \to \mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]]$ of $\mathbb{G}_2^1$-modules which induces a nonzero map on Ext groups.

THEOREM 4.9. *There is an isomorphism of* $\mathbb{Z}_3[[\mathbb{G}_2^1]]$*-modules*

$$N_3 \xrightarrow{\cong} \mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]] \ .$$

This requires a certain amount of preliminaries, and some further lemmas. We are looking for a diagram (see diagram (4.12) below) which will build and detect the desired map.

The first ingredient of our calculation is a spectral sequence. Let us write

$$0 \to C_3 \to C_2 \to C_1 \to C_0 \to \mathbb{Z}_3 \to 0$$

for the resolution obtained by splicing together the short exact sequences of Lemma 4.5, 4.6, and 4.7:

$$0 \to N_3 \to \chi \uparrow_{SD_{16}}^{\mathbb{G}_2^1} \to \chi \uparrow_{SD_{16}}^{\mathbb{G}_2^1} \to \mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]] \to \mathbb{Z}_3 \to 0 \ .$$

By extending the resolution $C_\bullet \to \mathbb{Z}_3$ to a bicomplex of projective $\mathbb{Z}_3[[\mathbb{G}_2^1]]$-modules, we get, for any $\mathbb{Z}_3[[\mathbb{G}_2^1]]$-module M and any closed subgroup $H \subseteq \mathbb{G}_2^1$, a first quadrant cohomology spectral sequence

$$E_1^{p,q} = \mathrm{Ext}^p_{\mathbb{Z}_3[[H]]}(C_q, M) \Longrightarrow H^{p+q}(H, M) \ . \tag{4.8}$$

In particular, because $E_1^{0,q} = 0$ for $q > 3$, there is an edge homomorphism

$$\mathrm{Hom}_{\mathbb{Z}_3[[H]]}(N_3, M) = \mathrm{Hom}_{\mathbb{Z}_3[[H]]}(C_3, M) \to H^3(H, M) \ . \tag{4.9}$$

Dually, there are homology spectral sequences

$$E^1_{p,q} = \mathrm{Tor}_p^{\mathbb{Z}_3[[H]]}(M, C_q) \Longrightarrow H_{p+q}(H, M) \tag{4.10}$$

with an edge homomorphism

$$H_3(H, M) \to M \otimes_{\mathbb{Z}_3[[H]]} N_3 \ . \tag{4.11}$$

That said, we remark that the important ingredient here is that $\mathbb{G}_2^1$ contains a subgroup K which is a Poincaré duality group of dimension three and which has good cohomological properties. The reader is referred to [28] for a modern discussion of a duality theory in the cohomology of profinite groups.

To define K, we use the filtration on the 3-Sylow subgroup $S_2^1 = F_{1/2}\mathbb{S}_2^1$ of $\mathbb{G}_2^1$ described in the first section. There is a projection

$$S_2^1 \to F_{1/2}\mathbb{S}_2^1/F_1\mathbb{S}_2^1 \cong \mathbb{F}_9 \ .$$

We follow this by the map $\mathbb{F}_9 \to \mathbb{F}_9/\mathbb{F}_3 \cong C_3$ to define a group homorphism $S_2^1 \to C_3$; then, we define $K \subset S_2^1$ to be the kernel. The chosen subgroup $C_3 \subseteq S_2^1$ of order 3 provides a splitting of $S_2^1 \to C_3$; hence S_2^1 can be written as a semi-direct product $K \rtimes C_3$. Note that every element of order three in S_2^1 maps to a nonzero element in C_3 so that K is torsion free.

From [10], we know a good deal about K, some of which is recorded in the following lemma. Let $j : K \to S_2^1$ denote the inclusion.

LEMMA 4.10. *The group K is a 3-adic Poincaré duality group of dimension 3, and if $[K] \in H_3(K, \mathbb{Z}_3)$ is a choice of fundamental class, then*

$$j_*[K] \in H_3(S_2^1, \mathbb{Z}_3)$$

is a nonzero SD_{16}-invariant generator of infinite order.

Proof. The fact that K is a Poincaré duality group is discussed in [10]; this discussion is an implementation of the theory of Lazard [16]. We must now address the statements about $j_*[K]$. For this, we first compute with cohomology, and we use the results and notation of Theorem 4.4.

It is known (see Proposition 4.3 and 4.4 of [10]), that the Lyndon-Serre-Hochschild spectral sequence

$$H^p(C_3, H^q(K, \mathbb{F}_3)) \Longrightarrow H^{p+q}(S_2^1, \mathbb{F}_3)$$

collapses and that $H^0(C_3, H^q(K, \mathbb{F}_3))$ is one dimensional for $0 \leq q \leq 3$; in particular, $j^* : H^3(S_2^1, \mathbb{F}_3) \to H^3(K, \mathbb{F}_3)$ is onto. Since the composites

$$H^1(C_3, \mathbb{F}_3) \to H^1(S_2^1, \mathbb{F}_3) \to H^1(w^i C_3 w^{-i}, \mathbb{F}_3)$$

of the inflation with the restriction maps are isomorphisms for $i = 1, 2$, the image of the generator of $H^1(C_3, \mathbb{F}_3)$ is some linear combination $ax_1 + bx_2$ with both $a \neq 0$ and $b \neq 0$. This implies that $j^*(x_i y_i) = 0$ for $i = 1, 2$; for example

$$aj^*(x_1 y_1) = j^*(y_1(ax_1 + bx_2)) = 0.$$

But since $j^* : H^3(S_2^1, \mathbb{F}_3) \to H^3(K, \mathbb{F}_3)$ is onto and $H^3(S_2^1, \mathbb{F}_3)$ is generated by $x_i y_i$ and $a_i y_i$ for $i = 1, 2$ it is impossible that $j^*(a_i y_i)$ is trivial for both $i = 1, 2$. Because K is a Poincaré duality group of dimension 3 we also know that the Bockstein $\beta : H^2(K, \mathbb{F}_3) \to H^3(K, \mathbb{F}_3)$ is zero; hence

$$j^*(a_1 y_1 - a_2 y_2) = j^*(\beta(x_1 a_1 - x_2 a_2)) = 0$$

and therefore

$$j^*(a_1 y_1) = j^*(a_2 y_2) \neq 0\ .$$

This shows that $H^3(S_2^1, \mathbb{F}_3) \to H^3(K, \mathbb{F}_3)$ is onto and factors through the SD_{16}-coinvariants, or dually $H_3(K, \mathbb{F}_3) \to H_3(S_2^1, \mathbb{F}_3)$ is an injection and lands in the SD_{16}-invariants. Furthermore, $H_3(K, \mathbb{F}_3)$ even maps to the kernel of the Bockstein $\beta : H_3(S_2^1, \mathbb{F}_3) \to H_2(S_2^1, \mathbb{F}_3)$ and the induced map

$$H_3(K, \mathbb{F}_3) \to \frac{\text{Ker } \beta : H_3(S_2^1, \mathbb{F}_3) \to H_2(S_2^1, \mathbb{F}_3)}{\text{Im}\beta : H_4(S_2^1, \mathbb{F}_3) \to H_3(S_2^1, \mathbb{F}_3)} \cong H_3(S_2^1; \mathbb{Z}_3) \otimes_{\mathbb{Z}_3} \mathbb{F}_3$$

is an isomorphism, yielding that $j_*[K]$ is a generator of infinite order. $\square$

We will use cap products with the elements $[K]$ and $j_*[K]$ to construct a commutative diagram for detecting maps $N_3 \to \mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]]$. In the form we use the cap product, it has a particularly simple expression. Let G be a profinite group and M a continuous $\mathbb{Z}_p[[G]]$-module. If $a \in H^n(G, M)$ and $x \in H_n(G, \mathbb{Z}_p)$ we may define $a \cap x \in H_0(G, M)$ as follows: choose a projective resolution $Q_\bullet \to \mathbb{Z}_p$ and represent a and x by a cocycle $\alpha : Q_n \to M$ and a cycle $y \in \mathbb{Z}_p \otimes_{\mathbb{Z}_p[[G]]} Q_n$ respectively. Then α descends to a map

$$\overline{\alpha} : \mathbb{Z}_p \otimes_{\mathbb{Z}_p[[G]]} Q_n \longrightarrow \mathbb{Z}_p \otimes_{\mathbb{Z}_p[[G]]} M$$

and $a \cap x$ is represented by $\overline{\alpha}(y)$. It is a simple matter to check that this is well-defined; in particular, if $y = \partial z$ is a boundary, then $\alpha(y) = 0$ because α is a cocycle. The usual naturality statements apply, which we record in a lemma. Note that part (2) is a special case of part (1) (with $K = G$).

LEMMA 4.11. 1) *If $\varphi : K \to G$ is a continuous homomorphism of profinite groups, and $x \in H_n(K, \mathbb{Z}_p)$ and $a \in H^n(G, M)$, then*

$$\varphi_*(\varphi^* a \cap x) = a \cap \varphi_*(x).$$

2) *Suppose $K \subseteq G$ is the inclusion of a normal subgroup and M is a G-module. Then G/K acts on $H^*(K, M)$ and $H_*(K, \mathbb{Z}_p)$ and for $g \in G/K$, $a \in H^n(K, M)$, and $x \in H^*(K, \mathbb{Z}_p)$*

$$g_*(g^* a \cap x) = a \cap g_*(x).$$

Here is our main diagram. Let $i : K \to \mathbb{G}_2^1$ be the inclusion.

(4.12)
$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(N_3, \mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]]) \arrow[r, "\text{edge}"] \arrow[d] & H^3(\mathbb{G}_2^1, \mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]]) \arrow[d, "\cap i_*[K]"] \\
\mathrm{Hom}_{\mathbb{Z}_3[SD_{16}]}(\mathbb{Z}_3 \otimes_{\mathbb{Z}_3[[S_2^1]]} N_3, \mathbb{Z}_3 \otimes_{\mathbb{Z}_3[[S_2^1]]} \mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]]) \arrow[r, "\text{ev}"] \arrow[d] & H_0(\mathbb{G}_2^1, \mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]]) \arrow[d] \\
\mathrm{Hom}_{\mathbb{Z}_3[SD_{16}]}(\mathbb{F}_3 \otimes_{\mathbb{Z}_3[[S_2^1]]} N_3, \mathbb{F}_3 \otimes_{\mathbb{Z}_3[[S_2^1]]} \mathbb{F}_3[[\mathbb{G}_2^1/G_{24}]]) \arrow[r, "\text{ev}"] & H_0(\mathbb{G}_2^1, \mathbb{F}_3[[\mathbb{G}_2^1/G_{24}]]).
\end{tikzcd}$$

We now annotate this diagram. The maps labelled ev are defined by evaluating a homomorphism at the image of $j_*[K]$ under the edge homomorphism

$$H_3(S_2^1, \mathbb{Z}_3) \to \mathbb{Z}_3 \otimes_{\mathbb{Z}_3[[S_2^1]]} N_3$$

resp.

$$H_3(S_2^1, \mathbb{Z}_3) \to \mathbb{Z}_3 \otimes_{\mathbb{Z}_3[[S_2^1]]} N_3 \to \mathbb{F}_3 \otimes_{\mathbb{Z}_3[[S_2^1]]} N_3$$

of (4.11). We now take the image of that element under the projection map to the coinvariants. Similar remarks apply to $\mathbb{F}_3$-coefficients.

The diagram commutes, by the definition of cap product. Theorem 4.9 now follows from the final two Lemmas 4.12 and 4.13 below; in fact, once we have proved these lemmas, diagram (4.12) will then show that we can choose a morphism of continuous $\mathbb{G}_2^1$-modules

$$f : N_3 \longrightarrow \mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]]$$

so that

$$\mathbb{F}_3 \otimes f : \mathbb{F}_3 \otimes_{\mathbb{Z}_3[[S_2^1]]} N_3 \longrightarrow \mathbb{F}_3 \otimes_{\mathbb{Z}_3[[S_2^1]]} \mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]]$$

is nonzero. Then Lemmas 4.7, 4.8 and 4.3 imply that f is an isomorphism.

Lemma 4.12. *The homomorphism*

$$\cap i_*[K] : H^3(\mathbb{G}_2^1, \mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]]) \to H_0(\mathbb{G}_2^1, \mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]])$$

is an isomorphism.

Proof. Recall that we have denoted the inclusion $K \to S_2^1$ by j. We begin by demonstrating that

$$\cap j_*[K] : H^3(S_2^1, \mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]]) \to H_0(S_2^1, \mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]])$$

is an isomorphism. Since the action of C_3 on $H_3(K, \mathbb{Z}_3) \cong \mathbb{Z}_3$ is necessarily trivial we see that $[K]$ is C_3-invariant and Lemma 4.11 supplies a commutative diagram

$$\begin{tikzcd}
H^3(S_2^1, \mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]]) \arrow[r, "j^*"] \arrow[d, "\cap j_*[K]"'] & H^3(K, \mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]])^{C_3} \arrow[d, "\cap[K]"] \\
H_0(S_2^1, \mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]]) & H_0(K, \mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]])^{C_3}. \arrow[l, "j_*"']
\end{tikzcd}$$

The morphism $\cap[K]$ is an isomorphism by Poincaré duality. As $\mathbb{Z}_3[[S_2^1]]$-modules, we have

$$\mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]] \cong \mathbb{Z}_3[[S_2^1/C_3]] \oplus \mathbb{Z}_3[[S_2^1/\omega C_3 \omega^{-1}]]$$

on generators eG_{24} and ωG_{24} in $\mathbb{G}_2^1/G_{24}$; hence, as K-modules

$$\mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]] \cong \mathbb{Z}_3[[K]] \oplus \mathbb{Z}_3[[K]]$$

which shows that j induces an isomorphism on $H_0(-,\mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]])$. We claim that C_3 acts trivially on H_0 and thus j_* is an isomorphism. In fact, it is clear that the C_3-action fixes the coset eG_{24}; furthermore $\omega C_3\omega^{-1}$ is another complement to K in S_2^1 and therefore

$$C_3\omega C_3 \subset KC_3\omega C_3 = S_2^1\omega C_3 = K\omega C_3\omega^{-1}\omega C_3 \subset K\omega C_3\ ,$$

and hence the class of ωC_3 in H_0 is also fixed.

In addition, since $H^q(K,\mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]]) = 0$ if $q \neq 3$, the Lyndon-Serre-Hochschild spectral sequence shows that j^* is an isomorphism.

To finish the proof, we continue in the same manner. Let $r : S_2^1 \to \mathbb{G}_2^1$ be the inclusion, so that $i = rj : K \to \mathbb{G}_2^1$. By Lemma 4.10, $j_*[K]$ is SD_{16}-invariant and then 4.11 supplies once more a commutative diagram

$$\begin{tikzcd}
H^3(\mathbb{G}_2^1,\mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]]) \arrow[r, "r^*"] \arrow[d, "\cap i_*[K]"'] & H^3(S_2^1,\mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]])^{SD_{16}} \arrow[d, "\cap j_*[K]"] \\
H_0(\mathbb{G}_2^1,\mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]]) & H_0(S_2^1,\mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]])^{SD_{16}} \arrow[l, "r_*"']
\end{tikzcd}\ .$$

We have just shown that $\cap j_*[K]$ is an isomorphism. The map r_* sends invariants to coinvariants and, since the order of SD_{16} is prime to 3, is an isomorphism. Again, because the order of SD_{16} is prime to 3 the spectral sequence of the extension $S_2^1 \to \mathbb{G}_2^1 \to SD_{16}$ collapses at E_2 and therefore the map r^* is an isomorphism. This completes the proof. $\square$

LEMMA 4.13. *The edge homomorphism*

$$\mathrm{Hom}_{\mathbb{G}_2^1}(N_3,\mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]]) \to H^3(\mathbb{G}_2^1,\mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]])$$

is surjective.

Proof. We examine the spectral sequence of (4.8):

$$E_1^{p,q} \cong \mathrm{Ext}^p_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(C_q,\mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]]) \Longrightarrow H^{p+q}(\mathbb{G}_2^1,\mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]]).$$

We need only show that

$$\mathrm{Ext}^p_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(C_q,\mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]]) = 0$$

for $p+q = 3$ and $q < 3$. If $q = 1$ or 2, then $C_q = \chi\uparrow_{SD_{16}}^{\mathbb{G}_2^1}$. Now $\chi\uparrow_{SD_{16}}^{\mathbb{G}_2^1}$ is projective as a $\mathbb{Z}_3[[\mathbb{G}_2^1]]$-module and therefore $\mathrm{Ext}^p_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(C_q,\mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]])$ is trivial for $p > 0$.

If $q = 0$, then $C_0 = \mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]]$ and by the Shapiro lemma we get an isomorphism

$$\mathrm{Ext}^3_{\mathbb{G}_2^1}(C_0,\mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]]) = H^3(G_{24},\mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]]) \cong H^3(C_3,\mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]])^{Q_8}\ .$$

The profinite C_3-set $\mathbb{G}_2^1/G_{24}$ is an inverse limit of finite C_3-sets X_i and thus we get an exact sequence

$$0 \to \lim_i{}^1 H^2(C_3, \mathbb{Z}_3[X_i]) \to H^3(C_3, \mathbb{Z}_3[[\mathbb{G}_2^1/C_3]]) \to \lim_i H^3(C_3, \mathbb{Z}_3[X_i]) \to 0 \ .$$

Now each X_i is made of a finite number of C_3-orbits. The contribution of each orbit to $H^3(C_3, -)$ is trivial and to $H^2(C_3, -)$ it is either trivial or $\mathbb{Z}/3$. Therefore $\lim_i$ is clearly trivial and $\lim_i^1$ is trivial because the Mittag-Leffler condition is satisfied. □

5. The tower

In this section we write down the five stage tower whose homotopy inverse limit is $L_{K(2)}S^0 = E_2^{h\mathbb{G}_2}$ and the four stage tower whose homotopy inverse limit is $E_2^{h\mathbb{G}_2^1}$. As before we will write $E = E_2$ and we recall that we have fixed the prime 3.

To state our results, we will need a new spectrum. Let χ be the representation of the subgroup $SD_{16} \subseteq \mathbb{G}_2$ that appeared in (4.1) and let e_χ be an idempotent in the group ring $\mathbb{Z}_3[SD_{16}]$ that picks up χ. The action of SD_{16} on E gives us a spectrum E^χ which is the telescope associated to this idempotent: $E^\chi := e_\chi E$.

Then we have isomorphisms of Morava modules

$$\begin{aligned} E_*E^\chi &\cong \mathrm{Hom}_{\mathbb{Z}_3[SD_{16}]}(\chi, E_*E)) \cong \mathrm{Hom}_{\mathbb{Z}_3[SD_{16}]}(\chi, \mathrm{Hom}^c(\mathbb{G}_2, E_*)) \\ &\cong \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2]]}(\chi \uparrow_{SD_{16}}^{\mathbb{G}_2}, \mathrm{Hom}^c(\mathbb{G}_2, E_*)) \cong \mathrm{Hom}^c_{\mathbb{Z}_3}(\chi \uparrow_{SD_{16}}^{\mathbb{G}_2}, E_*) \ . \end{aligned}$$

We recall that $\mathrm{Hom}^c(\mathbb{G}_2, E_*)$ is a Morava module via the diagonal $\mathbb{G}_2$-action, and a $\mathbb{Z}_3[SD_{16}]$-module via the translation action on $\mathbb{G}_2$. The group $\mathrm{Hom}^c_{\mathbb{Z}_3}(-,-)$ is the group of all homomorphisms which are continuous with respect to the obvious p-adic topologies.

It is clear from (4.2) that E^χ is a direct summand of $E_2^{hQ_8}$ and a module spectrum over $E_2^{hSD_{16}}$. In fact, it is easy to check that the homotopy of E^χ is free of rank 1 as a $\pi_*(E^{hSD_{16}})$-module on a generator $\omega^2 u^4 \in \pi_8(E^{hQ_8}) \subset \pi_*(E)$: this generator detemines a map of module spectra from $\Sigma^8 E^{hSD_{16}}$ to E^χ which is an equivalence. From now on we will use this equivalence to replace E^χ by $\Sigma^8 E^{hSD_{16}}$. We note that E^χ is periodic with period 16.

Lemma 5.1. *There is an exact sequence of Morava modules*

$$\begin{aligned} 0 \to E_* \to E_*E^{hG_{24}} &\to E_*\Sigma^8 E^{hSD_{16}} \oplus E_*E^{hG_{24}} \\ &\to E_*\Sigma^8 E^{hSD_{16}} \oplus E_*\Sigma^{40} E^{hSD_{16}} \\ &\to E_*\Sigma^{40} E^{hSD_{16}} \oplus E_*\Sigma^{48} E^{hG_{24}} \to E_*\Sigma^{48} E^{hG_{24}} \to 0 \ . \end{aligned}$$

Proof. Take the exact sequence of continuous $\mathbb{G}_2$-modules of Corollary 4.2 and apply $\mathrm{Hom}^c_{\mathbb{Z}_3}(\bullet, E_*)$. Then use the isomorphism

$$E_*\Sigma^8 E^{hSD_{16}} = \mathrm{Hom}^c_{\mathbb{Z}_p}(\chi \uparrow^{\mathbb{G}_2}_{SD_{16}}, E_*)$$

above and the isomorphisms

$$E_*E^{hF} \cong \mathrm{Hom}^c(\mathbb{G}_2/F, E_*)$$

supplied by (2.7) to get an exact sequence of Morava modules

$$\begin{aligned} 0 \to E_* \to E_*E^{hG_{24}} \to E_*\Sigma^8 E^{hSD_{16}} \oplus E_*E^{hG_{24}} \to E_*\Sigma^8 E^{hSD_{16}} \oplus E_*\Sigma^8 E^{hSD_{16}} \\ \to E_*\Sigma^8 E^{hSD_{16}} \oplus E_*E^{hG_{24}} \to E_*E^{hG_{24}} \to 0\ . \end{aligned}$$

Finally, we use that $\Sigma^8 E^{hSD_{16}} \simeq \Sigma^{40} E^{hSD_{16}}$ because $E^{hSD_{16}}$ is periodic of period 16 and $E_*E^{hG_{24}} \cong E_*\Sigma^{48}E^{hG_{24}}$ as Morava modules (see Remark 3.12.2). □

Remark 5.2. In the previous lemma, replacing $\Sigma^8 E^{hSD_{16}}$ by $\Sigma^{40}E^{hSD_{16}}$ is merely aesthetic: it emphasizes some sort of duality. However, $E^{hG_{24}}$ and $\Sigma^{48}E^{hG_{24}}$ are different spectra, even though $E_*E^{hG_{24}} \cong E_*\Sigma^{48}E^{hG_{24}}$. This substitution is essential to the solution to the Toda bracket problem which arises in Theorem 5.5.

In the same way, one can immediately prove

LEMMA 5.3. *There is an exact sequence of Morava modules*

$$0 \to E_*E^{h\mathbb{G}_2^1} \to E_*E^{hG_{24}} \to E_*\Sigma^8 E^{hSD_{16}} \to E_*\Sigma^{40}E^{hSD_{16}} \to E_*\Sigma^{48}E^{hG_{24}} \to 0.$$

THEOREM 5.4. *The exact sequence of Morava modules*

$$\begin{aligned} 0 \to E_* \to E_*E^{hG_{24}} \to E_*\Sigma^8 E^{hSD_{16}} \oplus E_*E^{hG_{24}} \to E_*\Sigma^8 E^{hSD_{16}} \oplus E_*\Sigma^{40} E^{hSD_{16}} \\ \to E_*\Sigma^{40} E^{hSD_{16}} \oplus E_*\Sigma^{48}E^{hG_{24}} \to E_*\Sigma^{48}E^{hG_{24}} \to 0 \end{aligned}$$

can be realized in the homotopy category of $K(2)$-local spectra by a sequence of maps

$$\begin{aligned} L_{K(2)}S^0 \to E^{hG_{24}} \to \Sigma^8 E^{hSD_{16}} \vee E^{hG_{24}} \\ \to \Sigma^8 E^{hSD_{16}} \vee \Sigma^{40}E^{hSD_{16}} \to \Sigma^{40}E^{hSD_{16}} \vee \Sigma^{48}E^{hG_{24}} \to \Sigma^{48}E^{hG_{24}} \end{aligned}$$

so that the composite of any two successive maps is null homotopic.

Proof. The map $L_{K(2)}S^0 \to E^{hG_{24}}$ is the unit map of the ring spectrum $E^{hG_{24}}$. To produce the other maps and to show that the successive composites are null homotopic, we use the diagram of Proposition 2.7. It is enough to show that the E_n-Hurewicz homomorphism

$$\pi_0 F(X,Y) \to \mathrm{Hom}_{E_*E}(E_*X, E_*Y)$$

is onto when X and Y belong to the set $\{\Sigma^8 E^{hSD_{16}}, E^{hG_{24}}\}$. (Notice that the other suspensions cancel out nicely, since $E^{hSD_{16}}$ is 16-periodic.) Since $\Sigma^8 E^{hSD_{16}}$ is a retract of E^{hQ_8}, it is sufficient to show that

$$\pi_0 F(E^{hK_1}, E^{hK_2}) \to \mathrm{Hom}_{E_*E}(E_*E^{hK_1}, E_*E^{hK_2})$$

is onto for K_1 and K_2 in the set $\{Q_8, G_{24}\}$. Using Proposition 2.6 and the short exact lim-lim^1 sequence for the homotopy groups of holim we see that it is enough to note that $(E_1)^K = 0$ and

$$\pi_0 E^{hK} \to (E_0)^K$$

is surjective whenever $K \subseteq K_2 \cap xK_1x^{-1}$. The first part is trivial and for the second part we can appeal to Corollary 3.13.

To show that the successive compositions are zero, we proceed similarly, again using Proposition 2.7, but now we have to show that various Hurewicz maps are injective. In this case, the suspensions do not cancel out, and we must show

$$\pi_0 F(E^{hK_1}, \Sigma^{48k} E^{hK_2}) \to \mathrm{Hom}_{E_*E}(E_*E^{hK_1}, E_*\Sigma^{48k} E^{hK_2})$$

is injective for $k = 0$ and 1, at least for K_1 of the form $\mathbb{G}_2$, G_{24}, or Q_8 and K_2 of the form G_{24} or Q_8. Since all the spectra involved in Proposition 2.6 are 72-periodic, the lim-lim^1 sequence for the homotopy groups of holim shows once more that it is sufficient to note that E_{24k+1} is trivial, that $\pi_{24k+1}(E^{hK})$ is finite and

$$\pi_{24k} E^{hK} \to (E_{24k})^K \tag{5.1}$$

is injective for $k = 0$ and 1 and $K \subseteq K_2 \cap xK_1x^{-1}$.

Again the first part is trivial while the other parts follow from Theorem 3.10 and Corollary 3.13. Note that for $k = 1$ the map in (5.1) need not be an isomorphism. □

The following result will let us build the tower.

THEOREM 5.5. *In the sequence of spectra*

$$\begin{aligned} L_{K(2)}S^0 &\to E^{hG_{24}} \to \Sigma^8 E^{hSD_{16}} \vee E^{hG_{24}} \\ &\to \Sigma^8 E^{hSD_{16}} \vee \Sigma^{40} E^{hSD_{16}} \to \Sigma^{40} E^{hSD_{16}} \vee \Sigma^{48} E^{hG_{24}} \to \Sigma^{48} E^{hG_{24}} \end{aligned}$$

all the possible Toda brackets are zero modulo their indeterminacy.

Proof. There are three possible three-fold Toda brackets, two possible four-fold Toda brackets and one possible five-fold Toda bracket. All but the last lie in zero groups.

Because $\Sigma^8 E^{hSD_{16}}$ is a summand of E^{hQ_8}, the three possible three-fold Toda brackets lie in

$$\pi_1 F(E^{h\mathbb{G}_2}, E^{hQ_8} \vee \Sigma^{32} E^{hQ_8}), \quad \pi_1 F(E^{hG_{24}}, \Sigma^{32} E^{hQ_8} \vee \Sigma^{48} E^{hG_{24}}),$$
$$\pi_1 F(E^{hQ_8} \vee E^{hG_{24}}, \Sigma^{48} E^{hG_{24}})$$

which are all zero by Proposition 2.6, Corollary 3.14 and Corollary 3.15. The most interesting calculation is the middle of these three and the most interesting part of that calculation is

$$\pi_1 F(E^{hG_{24}}, \Sigma^{48} E^{hG_{24}}) \cong \pi_{25}(\mathrm{holim}_i \prod_{G_{24}\backslash \mathbb{G}_n/U_i)} E^{hH_{x,i}}) \ .$$

This is zero by Corollary 3.14 and Corollary 3.15 (note that the element $-1 = \omega^4 \in G_{24}$ is in the center of $\mathbb{G}_2$ and it is in $H_{x,i}$ for every x); however, notice that *without* the suspension by 48 this group is nonzero.

The two possible four-fold Toda brackets lie in

$$\pi_2 F(E^{h\mathbb{G}_2}, \Sigma^{32} E^{hQ_8} \vee \Sigma^{48} E^{hG_{24}}) \qquad \text{and} \qquad \pi_2 F(E^{hG_{24}}, \Sigma^{48} E^{hG_{24}}).$$

We claim these are also zero groups. All of the calculations here present some interest. For example, consider

$$\pi_2 F(E^{hG_{24}}, \Sigma^{48} E^{hG_{24}}) \cong \pi_{26}(\mathrm{holim}_i \prod_{G_{24}\backslash \mathbb{G}_n/U_i)} E^{hH_{x,i}}) \ .$$

where $H_{x,i} = xU_i x^{-1} \cap G_{24}$. Since the element $-1 = \omega^4 \in G_{24}$ is in the center of $\mathbb{G}_2$, it is in $H_{x,i}$ for every x and the result follows from Corollary 3.15 and the observation that $\pi_{27}(E^{hH_{x,i}})$ is finite.

Finally, the five-fold Toda bracket lies in

$$\pi_3 F(E^{h\mathbb{G}_2}, \Sigma^{48} E^{hG_{24}}) \cong \pi_{27} E^{hG_{24}} \cong \mathbb{Z}/3 \ .$$

Thus, we do not have the zero group; however, we claim that the map $E^{h\mathbb{G}_2} \to E^{hG_{24}}$ at the beginning of our sequence supplies a surjective homomorphism

$$\pi_* F(E^{hG_{24}}, \Sigma^{48} E^{hG_{24}}) \to \pi_* F(E^{h\mathbb{G}_2}, \Sigma^{48} E^{hG_{24}}) \ .$$

This implies that the indeterminancy of the five-fold Toda bracket is the whole group, completing the proof.

To prove this claim, note the $E^{h\mathbb{G}_2} \to E^{hG_{24}}$ is the inclusion of the homotopy fixed points by a larger subgroup into a smaller one. Thus Proposition 2.6 yields a diagram

$$\begin{tikzcd}
F(E^{hG_{24}}, E^{hG_{24}}) \arrow[r] \arrow[d, "\simeq"'] & F(E^{\mathbb{G}_2}, E^{hG_{24}}) \arrow[d, "\simeq"] & \\
E[[\mathbb{G}_2/G_{24}]]^{hG_{24}} \arrow[r] & E[[\mathbb{G}_2/\mathbb{G}_2]]^{hG_{24}} \arrow[r, "\simeq"] & E^{hG_{24}}
\end{tikzcd}$$

and the contribution of the coset eG_{24} in $E[[\mathbb{G}_2/G_{24}]]^{hG_{24}}$ shows that the horizontal map is a split surjection of spectra. $\square$

The following result is now an immediate consequence of Theorems 5.4 and 5.5:

THEOREM 5.6. *There is a tower of fibrations in the $K(n)$-local category*

$$\begin{tikzcd} L_{K(2)}S^0 \arrow[r] & X_3 \arrow[r] & X_2 \arrow[r] & X_1 \arrow[r] & E^{hG_{24}} \\ \Sigma^{44}E^{hG_{24}} \arrow[u] & \Sigma^{45}E^{hG_{24}} \vee \Sigma^{37}E^{hSD_{16}} \arrow[u] & \Sigma^{6}E^{hSD_{16}} \vee \Sigma^{38}E^{hSD_{16}} \arrow[u] & \Sigma^{7}E^{hSD_{16}} \vee \Sigma^{-1}E^{hG_{24}} \arrow[u] & \end{tikzcd}.$$

Using Lemma 5.3 and the very same program, we may produce the following result. The only difference will be that the Toda brackets will all lie in zero groups.

THEOREM 5.7. *There is a tower of fibrations in the $K(n)$-local category*

$$\begin{tikzcd} E^{h\mathbb{G}_2^1} \arrow[r] & Y_2 \arrow[r] & Y_1 \arrow[r] & E^{hG_{24}} \\ \Sigma^{45}E^{hG_{24}} \arrow[u] & \Sigma^{38}E^{hSD_{16}} \arrow[u] & \Sigma^{7}E^{hSD_{16}} \arrow[u] & \end{tikzcd}.$$

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