THE SPECIAL FIBER OF THE MOTIVIC DEFORMATION OF THE STABLE HOMOTOPY CATEGORY IS ALGEBRAIC

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Abstract. For each prime p, we define a t-structure on the category $S^{0,0}/\tau\text{-}\mathbf{Mod}^b_{\clubsuit}$ of harmonic $\mathbb{C}$-motivic left module spectra over $S^{0,0}/\tau$, whose MGL-homology has bounded Chow degree, such that its heart is equivalent to the abelian category of p-completed $\mathrm{BP}_*\mathrm{BP}$-comodules that are concentrated in even degrees. We prove that $S^{0,0}/\tau\text{-}\mathbf{Mod}^b_{\clubsuit}$ is equivalent to $\mathcal{D}^b(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod})$ as stable ∞-categories equipped with t-structures.

As an application, for each prime p, we prove that the motivic Adams spectral sequence for $S^{0,0}/\tau$, which converges to the motivic homotopy groups of $S^{0,0}/\tau$, is isomorphic to the algebraic Novikov spectral sequence, which converges to the classical Adams-Novikov E_2-page for the sphere spectrum S^0. This isomorphism of spectral sequences allows Isaksen and the second and third authors to compute the stable homotopy groups of spheres at least to the 90-stem, with ongoing computations into even higher dimensions.

Contents

1. Introduction

Motivic homotopy theory, introduced by Voevodsky and Morel [43, 44, 46, 59–61, 63], is a successful application of abstract homotopy theory to solve problems in number theory and algebraic geometry (see [48, 57, 62] for example).

Over $\operatorname{Spec}\mathbb{C}$, one may view the p-completed stable motivic homotopy category as a deformation of the p-completed classical stable homotopy category. The parameter of the deformation is given by an element τ in $\pi_{0,-1}$ of the p-completed motivic sphere spectrum, which can be intuitively viewed as the standard coordinate $t \mapsto e^{2\pi i t}$ on $\mathbb{G}_m$. Formally speaking, following Hu-Kriz-Ormsby [23], the element τ is the inverse limit of the Bockstein pre-images of the Morel classes [45] of roots of unity. Dugger-Isaksen [11] have identified the generic fiber "$\tau = 1$" with the classical stable homotopy category, and the first main result of this paper identifies the special fiber "$\tau = 0$" with the derived category of $\mathrm{BP}_*\mathrm{BP}$-comodules, which is entirely algebraic in nature. Moreover, under this identification, the motivic Adams-Novikov spectral sequence corresponds to the τ-Bockstein spectral sequence. This deformation induces a deformation of motivic Adams spectral sequences. The second main result of this paper identifies the motivic Adams spectral sequence at the special fiber "$\tau = 0$" with the algebraic Novikov spectral sequence, which is again entirely algebraic. This deformation makes it possible for Isaksen, the second and third authors [27] to compute classical stable homotopy groups of spheres at least to the 90-stem, with ongoing computations into even higher dimensions.

1.1. **Main results.**

In this paper, we prove two results in the stable motivic homotopy theory over $\operatorname{Spec}\mathbb{C}$, with connections to chromatic homotopy theory and applications to classical homotopy theory.

The first result identifies the special fiber "$\tau = 0$" of the motivic deformation with the derived category of $\mathrm{BP}_*\mathrm{BP}$-comodules. We prove an ∞-category version of a conjecture due to the first author and Isaksen in 2016 [16]. Note that the derived category in the following Theorem 1.1 is understood as a stable ∞-category in the sense of Lurie in Higher Algebra [34, Section 1.3.2].

Theorem 1.1 (Theorem 1.11)**.** *There is an equivalence of stable ∞-categories equipped with t-structures at each prime p*

$$\mathcal{D}^b(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}) \simeq S^{0,0}/\tau\text{-}\mathbf{Mod}^b_{\clubsuit},$$

between the bounded derived category of p-completed $\mathrm{BP}_*\mathrm{BP}$-comodules that are concentrated in even degrees, and the category of harmonic motivic left module spectra over $S^{0,0}/\tau$, whose MGL-homology has bounded Chow degree, with morphisms the $S^{0,0}/\tau$-linear maps.

Here $S^{0,0}/\tau$ is a motivic E_∞-ring spectrum, which is also known as the cofiber of τ. The motivic spectrum MGL is the algebraic cobordism spectrum introduced by Voevodsky [63] and studied by Levine-Morel [33], Panin-Pimenov-Röndigs [49] and many others. A motivic left module spectrum over $S^{0,0}/\tau$ is *harmonic*, if it is cellular and the map to its MGL-completion induces an isomorphism on $\pi_{*,*}$. See Definition 1.5 for a precise definition. The Chow degree is twice the topological degree minus the motivic weight.

The derived category of p-completed $\mathrm{BP}_*\mathrm{BP}$-comodules that are concentrated in even degrees is also known as the derived category of quasi-coherent sheaves on the moduli stack of formal groups over $\mathbb{Z}_p$. This connection is foundational to chromatic homotopy theory, and is due to Quillen [52] and Goerss-Hopkins [17, 19]. Our theorem further connects these categories to motivic homotopy theory.

By an Ind-object argument, we have an unbounded version of Theorem 1.1 that connects to Hovey's [20] derived category $\mathbf{Stable}(\mathrm{BP}_*\mathrm{BP})$.

Corollary 1.2. *There is an equivalence of stable ∞-categories at each prime p*

$$\mathbf{Stable}(\mathrm{BP}_*\mathrm{BP}) \simeq S^{0,0}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}$$

between Hovey's unbounded derived category of $\mathrm{BP}_*\mathrm{BP}$*-comodules defined in* [20] *and the category of cellular motivic left module spectra over $S^{0,0}/\tau$.*

After the announcement of Theorem 1.1, alternative proofs of certain versions of Corollary 1.2 have appeared in work of Pstrągowski [51] and Krause [29].

The second result identifies the motivic Adams spectral sequence at the special fiber "$\tau = 0$" with the algebraic Novikov spectral sequence. It can be used to systematically compute a huge number of classical Adams differentials that are hard to obtain by other methods.

It is known to Isaksen [26, Proposition 6.2.5] and the first author [15, Corollary 3.14] that there is an isomorphism between the motivic homotopy groups of $S^{0,0}/\tau$ and the classical Adams-Novikov E_2-page. Our second result shows that there is an isomorphism of spectral sequences that converge to them.

Theorem 1.3 (Theorem 1.14)**.** *For each prime p, there is an isomorphism of spectral sequences between the motivic Adams spectral sequence for $S^{0,0}/\tau$ and the algebraic Novikov spectral sequence for the classical sphere spectrum S^0.*

Based on Theorem 1.3, Isaksen, the second and third authors [27] have computed classical stable stems at least to the 90-stem, with ongoing computations into even higher dimensions. Computations of many historically difficult differentials in the range up to the 45-stem are included in the appendix.

We'd like to comment that, in contrast to the original motivations of motivic homotopy theory, Isaksen and his collaborators [24–26, 28] have recently begun to reverse the information flow and applied stable motivic homotopy theory to obtain computational results in the classical stable homotopy theory. Our Theorems 1.1 and 1.3 have the same spirit and further deepen the connections to chromatic homotopy theory. Using motivic homotopy theory, we build up a new connection between the classical Adams spectral sequence and the Adams-Novikov spectral sequence, that allows us to compute stable stems in a much larger range than was previously possible.

1.2. **The stable ∞-category of motivic spectra over $S^{0,0}/\tau$.** We work with the stable ∞-category of motivic spectra over $\operatorname{Spec}\mathbb{C}$ localized at a fixed prime p [61–63]. To be precise, we use the category of $\mathcal{S}$-modules constructed by Hu [22], which is a symmetric monoidal ∞-category in the sense of Lurie [34, Section 2.1.2]. We denote it by

$$\mathbb{C}\textbf{-mot-Spectra}.$$

Voevodsky [59–62] constructed the *mod p* motivic Eilenberg-Mac Lane spectrum $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$ that represents the *mod p* motivic cohomology. Its value at a point is

$$\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}{}_{*,*} = \mathbb{F}_p[\tau],$$

where τ is in bidegree $(0,-1)$.

Let $S^{p,q}$ be the $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$-completed motivic sphere spectrum in bidegree (p,q). The class τ can be lifted to a map between $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$-completed motivic sphere spectra

$$\tau : S^{0,-1} \longrightarrow S^{0,0}$$

that induces a nonzero map on *mod p* motivic homology. The reader should be warned that τ does not further lift to a map between uncompleted motivic sphere spectra. See Dugger-Isaksen and Hu-Kriz-Ormsby [11, 23] for more details. We denote the cofiber of τ by $S^{0,0}/\tau$.

$$S^{0,-1} \xrightarrow{\tau} S^{0,0} \longrightarrow S^{0,0}/\tau \longrightarrow S^{1,-1}$$

The $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$-completed motivic sphere spectrum is an E_∞-ring object in the symmetric monoidal ∞-category $\mathbb{C}$**-mot-Spectra**. We denote by

$$S^{0,0}\textbf{-Mod}$$

the stable ∞-category of motivic module spectra over $S^{0,0}$.

Convention 1.4. All smash products in $S^{0,0}$**-Mod** are understood taken over the $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$-completed sphere spectrum $S^{0,0}$.

We have suspension functors $\Sigma^{p,q}(-) = S^{p,q} \wedge -$ in the category $S^{0,0}$-**Mod** for any $p, q \in \mathbb{Z}$. In particular, the suspension functor $\Sigma^{1,0}$ gives the triangulation translation functor.

Following Dugger-Isaksen [12, Definition 2.10], we define the category

$$S^{0,0}\text{-}\mathbf{Mod}_{\mathrm{cell}}$$

of cellular $S^{0,0}$ module spectra as the smallest stable subcategory of $S^{0,0}$-**Mod** containing all sphere spectra $S^{p,q}$ for all $p, q \in \mathbb{Z}$ that is closed under arbitrary colimits. Recall from Lurie [34, Definition 1.1.3.2] that a stable subcategory of a stable ∞-category is a full subcategory containing a zero object and stable under the formation of fibers and cofibers. The reader should be warned that not all motivic spectra in $S^{0,0}$-**Mod** are weak equivalent to a cellular object.

Given an E_∞-ring object $R \in S^{0,0}$-**Mod**, denote by

$$R\text{-}\mathbf{Mod}$$

the stable ∞-category of left modules over R in $S^{0,0}$-**Mod**, and by

$$R\text{-}\mathbf{Mod}_{\mathrm{cell}}$$

the smallest full subcategory containing R that is closed under arbitrary colimits and suspension by $\Sigma^{p,q}$ for all $p, q \in \mathbb{Z}$.

It is a theorem of the first author [15] that $S^{0,0}/\tau$ is an E_∞-ring object in $S^{0,0}$-**Mod**. We therefore have defined stable ∞-categories

$$S^{0,0}/\tau\text{-}\mathbf{Mod} \text{ and } S^{0,0}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}.$$

We can view the ring map

$$S^{0,0} \longrightarrow S^{0,0}/\tau$$

to exhibit $S^{0,0}/\tau$ as the special fiber of the deformation parametrized by τ. The generic fiber of this deformation is $\tau^{-1}S^{0,0}$.

Let MGL be the motivic algebraic cobordism spectrum introduced by Voevodsky [63] and studied by Levine-Morel [33], Panin-Pimenov-Röndigs [49] and many others. Let

$$\mathrm{MU}^{\mathrm{mot}}$$

be the $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$-completion of MGL, which is an E_∞-ring object in $S^{0,0}$-$\mathbf{Mod}_{\mathrm{cell}}$ (See [22, Theorem 14.2] for example). Its motivic homotopy groups are computed by Hu-Kriz-Ormsby and Dugger-Isaksen [11, 23].

$$\pi_{*,*}\mathrm{MU}^{\mathrm{mot}} = \mathbb{Z}_p[\tau][x_1, x_2, \cdots],$$

where x_i is in bidegree $(2i, i)$. Since $\pi_{*,*}$MGL is much more complicated, we will mostly work with $\mathrm{MU}^{\mathrm{mot}}$ instead of MGL in our paper.

It is understood that the $\mathrm{MU}^{\mathrm{mot}}$-homology of X, where $X \in S^{0,0}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}$, is computed as

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}X = \pi_{*,*}(\mathrm{MU}^{\mathrm{mot}} \wedge X),$$

where the smash product is understood taken over the $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$-completed sphere spectrum $S^{0,0}$. Note that the $\mathrm{MU}^{\mathrm{mot}}$-homology of X is $\mathrm{MGL}_{*,*}X$ when taking X as its underlying motivic spectrum in $\mathbb{C}$-**mot**-**Spectra**.

The spectrum $\mathrm{MU}^{\mathrm{mot}}/\tau := S^{0,0}/\tau \wedge \mathrm{MU}^{\mathrm{mot}}$ is an E_∞-ring object in $S^{0,0}\text{-}\mathbf{Mod}_{\mathrm{cell}}$. Its motivic homotopy groups are

$$\pi_{*,*}(\mathrm{MU}^{\mathrm{mot}}/\tau) = \mathbb{Z}_p[x_1, x_2, \cdots] = \mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau.$$

Forgetting the motivic weight, the bigraded ring $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$ can be identified as the single graded ring MU_* completed at the prime p.

Definition 1.5. Let X be a motivic spectrum in $S^{0,0}/\tau\text{-}\mathbf{Mod}$. We say that X is *harmonic*, if X is cellular and the map

$$X \longrightarrow X^{\wedge}_{\mathrm{MGL}}$$

induces an isomorphism on $\pi_{*,*}$. We denote by

$$S^{0,0}/\tau\text{-}\mathbf{Mod}_{\clubsuit}$$

the full stable ∞-subcategory of harmonic $S^{0,0}/\tau$-module spectra.

Here the MGL-nilpotent completion $X^{\wedge}_{\mathrm{MGL}}$ is understood taken in $\mathbb{C}$-**mot**-**Spectra**. For a precise definition, see [11, 23]. One could also define the $\mathrm{MU}^{\mathrm{mot}}$-completion $X^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}$ in $S^{0,0}\text{-}\mathbf{Mod}$. It is clear that for X in $S^{0,0}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}$, the two completions $X^{\wedge}_{\mathrm{MGL}}$ and $X^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}$ are equivalent.

It is clear that the spectrum $\mathrm{MU}^{\mathrm{mot}}/\tau$ is harmonic. See Section 4.1 for more examples.

To describe t-structures on certain stable ∞-categories of motivic spectra, such as $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}$ and $S^{0,0}/\tau\text{-}\mathbf{Mod}_{\clubsuit}$, we recall the definition of the Chow degree. Recall that by Lurie's definition [34, Definition 1.2.1.4], a t-structure on a stable ∞-category is a t-structure on its homotopy category, which is a triangulated category.

Definition 1.6. Let $G_{*,*}$ be a bigraded abelian group that are the homotopy groups of any motivic spectrum. The *Chow degree* of an element

$$g \in G_{s,w}$$

is defined as $s - 2w$.

We say that $G_{*,*}$ is *concentrated in Chow degrees* I, where I is a set of integers, if all nonzero elements in $G_{*,*}$ are concentrated in Chow degrees belonging to I.

For example, the homotopy groups of $\mathrm{MU}^{\mathrm{mot}}/\tau$ are concentrated in Chow degree 0, while the homotopy groups of $\mathrm{MU}^{\mathrm{mot}}$ are concentrated in non-negative even Chow degrees.

Definition 1.7.

(1) We define
$$\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{b}_{\mathrm{cell}}$$
as the stable full subcategory of $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}$ spanned by objects whose homotopy groups are concentrated in bounded Chow degrees.

(2) We define
$$\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{b,\geq 0}_{\mathrm{cell}},$$
$$\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{b,\leq 0}_{\mathrm{cell}},$$
$$\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}}$$
as the full subcategories of $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{b}_{\mathrm{cell}}$ spanned by objects whose homotopy groups are concentrated in nonnegative, nonpositive and zero Chow degrees respectively.

(3) We define
$$S^{0,0}/\tau\text{-}\mathbf{Mod}^{b}_{\clubsuit}$$
as the stable full subcategory of $S^{0,0}/\tau\text{-}\mathbf{Mod}_{\clubsuit}$ spanned by objects whose $\mathrm{MU}^{\mathrm{mot}}$-homology groups are concentrated in bounded Chow degrees.

(4) We define
$$S^{0,0}/\tau\text{-}\mathbf{Mod}^{b,\geq 0}_{\clubsuit},$$
$$S^{0,0}/\tau\text{-}\mathbf{Mod}^{b,\leq 0}_{\clubsuit},$$
$$S^{0,0}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\clubsuit}$$
as the full subcategories of $S^{0,0}/\tau\text{-}\mathbf{Mod}^{b}_{\clubsuit}$ spanned by objects whose $\mathrm{MU}^{\mathrm{mot}}$-homology groups are concentrated in nonnegative, nonpositive and zero Chow degrees respectively.

Definition 1.8. We define
$$\mathrm{MU}_*\text{-}\mathbf{Mod},$$
$$\mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod}$$
as the abelian categories of modules over the p-completed ring MU_* and comodules over the p-completed Hopf algebroid $\mathrm{MU}_*\mathrm{MU}$, that are concentrated in even degrees. We define
$$\mathcal{D}^b(\mathrm{MU}_*\text{-}\mathbf{Mod}),$$

$$\mathcal{D}^b(\mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod})$$

to be their bounded derived categories.

Note that the abelian category of modules over p-completed MU_* is *not* equivalent to the abelian category of modules over p-completed BP_*. However, the abelian category of comodules over p-completed $\mathrm{MU}_*\mathrm{MU}$ is equivalent to the abelian category of comodules over p-completed $\mathrm{BP}_*\mathrm{BP}$. We will work with MU and $\mathrm{MU}^{\mathrm{mot}}$ since they are E_∞-ring objects in the corresponding categories while BP is not, due to a recent result of Lawson [32]. More details about these facts are discussed in Section 6.

Theorem 1.9.

(1) The full subcategories $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b,\geq 0}$ *and* $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b,\leq 0}$ *define a t-structure on* $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b}$.

(2) The functor

$$\pi_{*,*}:\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{\heartsuit}\longrightarrow \mathrm{MU}_*\text{-}\mathbf{Mod}$$

is an equivalence.

(3) There exists an equivalence of stable ∞-categories

$$\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b}\longrightarrow \mathcal{D}^b(\mathrm{MU}_*\text{-}\mathbf{Mod}),$$

that preserves the equipped t-structures and extends the functor $\pi_{*,*}$ on the heart.

Remark 1.10. Note that the functor $\pi_{*,*}$ naturally lands in the category of modules over the bigraded ring $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$. Since all elements of this bigraded ring are concentrated in Chow degree 0, it can be identified as the single graded ring MU_* by forgetting the motivic weight. A similar comment applies to the following theorem as well.

Theorem 1.11.

(1) The full subcategories $S^{0,0}/\tau\text{-}\mathbf{Mod}_{\clubsuit}^{b,\geq 0}$ *and* $S^{0,0}/\tau\text{-}\mathbf{Mod}_{\clubsuit}^{b,\leq 0}$ *define a t-structure on* $S^{0,0}/\tau\text{-}\mathbf{Mod}_{\clubsuit}^{b}$.

(2) The functor

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}:S^{0,0}/\tau\text{-}\mathbf{Mod}_{\clubsuit}^{\heartsuit}\longrightarrow \mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod}$$

is an equivalence.

(3) There exists an equivalence of stable ∞-categories

$$S^{0,0}/\tau\text{-}\mathbf{Mod}_{\clubsuit}^{b}\longrightarrow \mathcal{D}^b(\mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod}),$$

that preserves the equipped t-structures and extends the functor $\mathrm{MU}^{\mathrm{mot}}_{*,*}$ on the heart.

Remark 1.12. We'd like to comment that, from a motivic deformation perspective, our Theorem 1.11 gives a new connection between the moduli stack of formal groups and the classical stable homotopy theory.

From the deformation

$$S^{0,0}/\tau \xleftarrow{\text{special fiber}} S^{0,0} \xrightarrow{\text{generic fiber}} \tau^{-1}S^{0,0}$$

parametrized by τ, we have two adjunctions of stable ∞-categories:

$$(S^{0,0}\text{-}\mathbf{Mod})_{\tau=0} \rightleftarrows S^{0,0}\text{-}\mathbf{Mod} \rightleftarrows (S^{0,0}\text{-}\mathbf{Mod})_{\tau=1}$$

By Dugger-Isaksen [11], on the generic fiber, the full subcategory of cellular objects in

$$(S^{0,0}\text{-}\mathbf{Mod})_{\tau=1} := \tau^{-1}S^{0,0}\text{-}\mathbf{Mod}$$

is equivalent to the classical stable homotopy category.

Our main theorem shows that, on the special fiber, the full subcategory of cellular objects in the category

$$(S^{0,0}\text{-}\mathbf{Mod})_{\tau=0} := S^{0,0}/\tau\text{-}\mathbf{Mod}$$

is equivalent to the derived category of comodules over the p-completed Hopf algebroid $\mathrm{MU}_*\mathrm{MU}$. By Quillen's theorem [52], the latter can be identified with the derived category of quasi-coherent sheaves on the moduli stack of formal groups over $\mathbb{Z}_p$.

Remark 1.13. We'd also like to comment that, in our proof of Theorem 1.11, we set up a strongly convergent motivic Adams-Novikov spectral sequence in the category $S^{0,0}/\tau\text{-}\mathbf{Mod}^b_{\mathrm{cell}}$,

$$\mathrm{Ext}^{*,*,*}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y) \Longrightarrow [\Sigma^{*,*}X, Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}]_{S^{0,0}/\tau}\,.$$

This is stated as Theorem 5.5 in Section 5. Classically, the Adams-Novikov spectral sequence is set up such that the first variable is the sphere spectrum. It seems to be a folklore theorem without published reference that there exists an Adams-Novikov spectral sequence when the first variable X is arbitrary. Our construction can be modified to the classical situation and would provide such a reference. See Section 5 for more discussion.

1.3. **The motivic Adams spectral sequence and the algebraic Novikov spectral sequence.** The following Theorem 1.14 establishes an isomorphism between the algebraic Novikov spectral sequence and the motivic Adams spectral sequence for $S^{0,0}/\tau$.

Theorem 1.14. *At each prime p, there is an isomorphism of tri-graded spectral sequences: the motivic Adams spectral sequence for $S^{0,0}/\tau$, which converges to the motivic homotopy groups of $S^{0,0}/\tau$, and the re-graded algebraic Novikov spectral sequence, which converges to the Adams-Novikov E_2-page for sphere.*

Moreover, this isomorphism between the abutments preserves the composition products and higher compositions in the respective categories.

The indexes are indicated in the following diagram:

$$\begin{tikzcd}
\mathrm{Ext}^{s,2w}_{\mathrm{BP}_*\mathrm{BP}/I}(\mathbb{F}_p, I^{a-s}/I^{a-s+1}) \arrow[r, "\cong"] \arrow[d, Rightarrow, "\textbf{Algebraic Novikov SS}"] & \mathrm{Ext}^{a,2w-s+a,w}_{A^{\mathrm{mot}}_{*,*}}(\mathbb{F}_p[\tau], \mathbb{F}_p) \arrow[d, Rightarrow, "\textbf{Motivic Adams SS}"] \\
\mathrm{Ext}^{s,2w}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*) \arrow[r, "\cong"] & \pi_{2w-s,w}(S^{0,0}/\tau).
\end{tikzcd}$$

Here $I = (p, v_1, v_2, \dots)$ *is the augmentation ideal of* BP_* *and* $A^{\mathrm{mot}}_{*,*}$ *is the motivic mod* p *dual Steenrod algebra.*

The isomorphism between the abutments is known to Isaksen [26, Proposition 6.2.5] and the first author [15, Corollary 3.14]. Our Theorem 1.14 also implies that the isomorphism preserves the filtrations on the E_∞-pages.

There has been huge interest in obtaining information on the stable homotopy groups of spheres by comparing the Adams spectral sequence with the Adams-Novikov spectral sequence. See [40, 41, 47, 53] for example. An important connection and technique of studying both spectral sequences is the following Miller square [40].

$$\begin{tikzcd}
 & \mathrm{Ext}^{s,t}_{P_*}(\mathbb{F}_p, I^{a-s}/I^{a-s+1}) \arrow[dl, Rightarrow, "\textbf{Cartan-Eilenberg SS}"'] \arrow[dr, Rightarrow, "\textbf{Algebraic Novikov SS}"] & \\
\mathrm{Ext}^{a,t}_{A_*}(\mathbb{F}_p, \mathbb{F}_p) \arrow[dr, Rightarrow, "\textbf{Adams SS}"'] & & \mathrm{Ext}^{s,t}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*) \arrow[dl, Rightarrow, "\textbf{Adams-Novikov SS}"] \\
 & \pi_* S^0 &
\end{tikzcd}$$

By a change-of-ring isomorphism, the E_2-page of the Cartan-Eilenberg spectral sequence, which computes the Adams E_2-page, is isomorphic to the algebraic Novikov spectral sequence, which computes the Adams-Novikov E_2-page. Note that for p odd, the Cartan-Eilenberg spectral sequence collapses for degree reasons.

To explore this square, Miller [40] smashes together the Adams resolution and the Adam-Novikov resolution, and gets a comparison theorem on the d_2-differentials in the algebraic Novikov spectral sequence and the Adams spectral sequence. The following theorem is due to Miller [40, Theorem 4.2] and Novikov [47]. See also Andrews-Miller [1] for a discussion.

Theorem 1.15. *Let p be an odd prime. Suppose that in the Cartan-Eilenberg spectral sequence, an element z in $\mathrm{Ext}_{P_*}^{s,t}(\mathbb{F}_p, I^{a-s}/I^{a-s+1})$ converges to an element z' in $\mathrm{Ext}_{A_*}^{a,t}(\mathbb{F}_p, \mathbb{F}_p)$. Then we have*

$$-d_2^{\mathbf{ASS}} z' \text{ is detected by } d_2^{\mathbf{algNSS}} z,$$

in the Cartan-Eilenberg spectral sequence, where $d_2^{\mathbf{ASS}}$ is a d_2-differential in the Adams spectral sequence, and $d_2^{\mathbf{algNSS}}$ is a d_2-differential in the algebraic Novikov spectral sequence.

Based on Miller square and Theorem 1.15, Miller [40] proves the Telescope Conjecture at chromatic height 1 at odd primes.

To understand the connection between **higher** differentials in the Adams and algebraic Novikov spectral sequences, it would be desirable to establish new connections between them.

For example, suppose in general that we have two spectral sequences

$$E_2 \Rightarrow E_\infty, \; E'_2 \Rightarrow E'_\infty$$

that are not necessarily connected by a homomorphism of spectral sequences. To compare them, it would be useful to have a third spectral sequence

$$E''_2 \Rightarrow E''_\infty$$

making a zig-zag diagram of spectral sequences.

$$\begin{tikzcd}
E_2 \arrow[d, Rightarrow] & E''_2 \arrow[l] \arrow[r] \arrow[d, Rightarrow] & E'_2 \arrow[d, Rightarrow] \\
E_\infty & E''_\infty \arrow[l] \arrow[r] & E'_\infty
\end{tikzcd}$$

This is the one of the major techniques used by the second and third authors in [64] to explore Mahowald square [36] and compute differentials in the Adams spectral sequences.

Following this philosophy, for Miller square [40], a basic question would be: Which spectral sequence can we put in between these two spectral sequences and have a zig-zag diagram?

$$\begin{tikzcd}
\mathrm{Ext}^{s,t}_{P_*}(\mathbb{F}_p, I^{a-s}/I^{a-s+1}) \arrow[d, Rightarrow, "\textbf{Algebraic Novikov SS}"'] & ? \arrow[l] \arrow[r] \arrow[d, Rightarrow] & \mathrm{Ext}^{a,t}_{A_*}(\mathbb{F}_p, \mathbb{F}_p) \arrow[d, Rightarrow, "\textbf{Adams SS}"] \\
\mathrm{Ext}^{s,t}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*) & ? \arrow[l] \arrow[r] & \pi_* S^0
\end{tikzcd}$$

Our Theorem 1.14 shows that we can achieve a zig-zag diagram in the motivic world.

In fact, consider the $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$-completed motivic sphere spectrum $S^{0,0}$. Inverting τ, we get the classical p-completed sphere S^0 by Dugger-Isaksen [11]. On the other hand, moding out by τ, we get $S^{0,0}/\tau$. Then the naturality of the Adams spectral sequences gives us a zig-zag diagram.

$$\begin{tikzcd}
\mathrm{Ext}^{s,t}_{A^{mot}_{*,*}}(\mathbb{F}_p[\tau], \mathbb{F}_p) \arrow[d, Rightarrow, "\textbf{Motivic Adams SS}"'] & \mathrm{Ext}^{s,t}_{A^{mot}_{*,*}}(\mathbb{F}_p[\tau], \mathbb{F}_p[\tau]) \arrow[l] \arrow[r] \arrow[d, Rightarrow, "\textbf{Motivic Adams SS}"'] & \mathrm{Ext}^{a,t}_{A_*}(\mathbb{F}_p, \mathbb{F}_p) \arrow[d, Rightarrow, "\textbf{Adams SS}"] \\
\pi_{*,*} S^{0,0}/\tau & \pi_{*,*} S^{0,0} \arrow[l] \arrow[r, "\textbf{Re}"] & \pi_* S^0
\end{tikzcd}$$

By Theorem 1.14, the left side spectral sequence, which is the motivic Adams spectral sequence for $S^{0,0}/\tau$, is isomorphic to the algebraic Novikov spectral sequence.

More generally, we have the following motivic square.

$$\begin{tikzcd}
& \mathrm{Ext}^{*,*,*}_{A^{mot}_{*,*}}(\mathbb{F}_p[\tau], \mathbb{F}_p)[\tau] \arrow[dl, Rightarrow, "\textbf{Algebraic } \tau\textbf{-Bockstein SS}"'] \arrow[dr, Rightarrow, "\textbf{Motivic Adams SS}"] & \\
\mathrm{Ext}^{*,*,*}_{A^{mot}_{*,*}}(\mathbb{F}_p[\tau], \mathbb{F}_p[\tau]) \arrow[dr, Rightarrow, "\textbf{Motivic Adams SS}"'] & & \pi_{*,*} S^{0,0}/\tau[\tau] \arrow[dl, Rightarrow, "\tau\textbf{-Bockstein SS}"] \\
& \pi_{*,*} S^{0,0} &
\end{tikzcd}$$

Let's compare the motivic square with Miller square.

For the lower right side, it is proved by Isaksen [26] that the motivic Adams-Novikov spectral sequence for $S^{0,0}$ is isomorphic to the τ-Bockstein spectral sequence, and that it is rigid, in the sense that it contains the same information as the classical Adams-Novikov spectral sequence. Each nontrivial differential in the classical Adams-Novikov spectral sequence corresponds to a family of nontrivial differentials in the motivic Adams-Novikov spectral sequence, that are connected to each other by multiplication by τ. We can recover all nonzero differentials in the motivic Adams-Novikov spectral sequence by knowing all nonzero differentials in the classical Adams-Novikov spectral sequence, and vice versa.

For the upper left side, the relation of the two spectral sequences in the motivic square and Miller square is the same as the relation on the lower right side. The algebraic τ-Bockstein spectral sequence can be thought as a motivic version of the Cartan-Eilenberg spectral sequence, and contains the same information, in the same sense as the lower right side situation.

For the upper right side, our Theorem 1.14 says that the two spectral sequences are isomorphic.

Therefore, for three out of the four sides, the motivic square contains exactly the same information as the ones in Miller square.

For the remaining lower left side, Dugger-Isaksen [11] shows that the τ-inverted motivic Adams spectral sequence is isomorphic to the τ-inverted classical Adams spectral sequence. This means that the difference between the motivic square and Miller square lies in the τ-torsion information. Therefore, when comparing the higher differentials in the classical and motivic Adams spectral sequences, the τ-torsion information is necessary to make the zig-zag strategy work.

Now, to compute a nontrivial classical Adams differential, for any r, start with an algebraic Novikov d_r-differential. Theorem 1.14 gives us a motivic Adams d_r-differential for $S^{0,0}/\tau$. Pulling back to the bottom cell of $S^{0,0}/\tau$ of the source element gives us a motivic Adams $d_{r'}$-differential for the motivic sphere with $r' \leq r$. Using the Betti realization functor, we then obtain a classical Adams $d_{r'}$-differential!

In practice, Isaksen, the second and the third authors [27] extend the computation of classical and motivic stable stems into a large range using the following steps.

(1) Use a computer to carry out the entirely algebraic computation of the cohomology of the $\mathbb{C}$-motivic Steenrod algebra. These groups serve as the input to the $\mathbb{C}$-motivic Adams spectral sequence.
(2) Use a computer to carry out the entirely algebraic computation of the algebraic Novikov spectral sequence that converges to the cohomology of the Hopf algebroid $(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP})$. This includes all differentials, and the multiplicative structure of the cohomology of $(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP})$.

(3) Use Theorem 1.14 to identify the algebraic Novikov spectral sequence with the motivic Adams spectral sequence that computes the homotopy groups of $S^{0,0}/\tau$. This includes an identification of the cohomology of $(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP})$ with the homotopy groups of $S^{0,0}/\tau$.
(4) Use the inclusion of the bottom cell and the projection to the top cell to pull back and push forward Adams differentials for $S^{0,0}/\tau$ to Adams differentials for the motivic sphere.
(5) Apply a variety of ad hoc arguments to deduce additional Adams differentials for the motivic sphere. The most important method involves shuffling Toda brackets.
(6) Use a long exact sequence in homotopy groups to deduce hidden τ-extensions in the motivic Adams spectral sequence for the sphere.
(7) Invert τ to obtain the classical Adams spectral sequence and the classical stable homotopy groups.

We'd like to highlight a few consequences of our stem-wise computations.

Example 1.16. Consider the following four differentials in the classical Adams spectral sequence for the 2-completed sphere.

(1) There is a d_3 differential in the 15-stem

$$d_3(h_0h_4) = h_0d_0.$$

This is proved by May and Mahowald-Tangora in [37, 38] by comparing with Toda's unstable computations [58].

(2) There is a d_4 differential in the 38-stem

$$d_4(h_3h_5) = h_0x.$$

This is proved in Mahowald-Tangora [37] by an ad-hoc method using a certain finite CW spectrum.

(3) There is a d_3 differential in the 38-stem

$$d_3(e_1) = h_1t.$$

This is proved by Bruner in [7] by power operations in the Adams spectral sequence.

(4) There is a d_3 differential in the 61-stem

$$d_3(D_3) = B_3.$$

This is proved by the second and third authors [64] using the RP^∞-technique. The proof of this differential in [64] is a significant part of the proof that the 61-sphere has a unique smooth structure.

It turns out that all these four differentials can be proved by our method. They all correspond to nontrivial differentials in the algebraic Novikov spectral sequence with the same length, and therefore are all consequences of purely algebraic computations and our Theorem 1.14.

Remark 1.17. For some of the differentials computed by Isaksen, the second and the third authors [27] using Theorem 1.14, our method gives the only proof. For example, we prove an Adams d_3-differential in the 68-stem

$$d_3(d_2) = h_0^2 Q_3,$$

which shows the non-existence of the homotopy class κ_2 in π_{68}. As another example, we prove an Adams d_5-differential in the 92-stem

$$d_5(g_3) = h_6 d_0^2,$$

which shows the non-existence of the homotopy class $\overline{\kappa}_3$ in π_{92}. Since both the elements d_2 and g_3 lie in a nonzero Sq^0-family in the 4-line of the classical Adams E_2-page, the two new nontrivial differentials serves as new evidence of Minami's new Doomsday Conjecture.

Remark 1.18. Theorem 1.14 can also be used to compute nontrivial extensions and Toda brackets. For example, there is an η-extension from $h_3 d_1$ to N in the 46-stem. This is proved by the second and third authors [65, Proposition 1.3(2)] using the RP^∞-technique. As another example, there is a Toda bracket

$$\langle \theta_4, 2, \sigma^2 \rangle$$

in the 45-stem. It is computed by Isaksen in [26, Lemma 4.2.91] by ad hoc methods. This Toda bracket computation is crucial in the third author's proof [66] that

$$2\theta_5 = 0$$

in the 62-stem. Both the nontrivial η-extension and the Toda bracket computations are present in the motivic homotopy groups of $S^{0,0}/\tau$. By Theorem 1.14, they can be computed by the product and Massey product structure on the classical Adams-Novikov E_2-page. In particular, the corresponding 3-fold Massey product can be verified in the algebraic Novikov spectral sequence using May's convergence theorem [39]. Therefore, both the nontrivial η-extension and the Toda bracket computations are consequences of purely algebraic computations and our Theorem 1.14.

1.4. **Organization.** This paper is organized in two parts.

In Part 1, we prove the equivalence of stable ∞-categories in Theorem 1.9 and Theorem 1.11. Our proofs use a theorem of Lurie in Higher Algebra [34] on the relation between a stable ∞-category with a t-structure and the derived category of its heart. We recall Lurie's theorem in Section 2, and prove Theorem 1.9 and Theorem 1.11 in Section 3 and 4. We also prove Corollary 1.2 in the end of Section 4. We introduce the absolute Adams-Novikov spectral sequence in the category $S^{0,0}/\tau\text{-}\mathbf{Mod}^b_{\clubsuit}$ in Section 5, which is necessary for our proof of Theorem 1.11.

In Part 2, we prove the isomorphism of spectral sequences in Theorem 1.14. In Section 9, we recall the construction of the algebraic Novikov spectral sequence and discuss the regrading. In Section 10, we check that, through the equivalence of stable ∞-categories in Theorem 1.11, the algebraic Novikov tower in the derived category of $\mathrm{BP}_*\mathrm{BP}$-comodules corresponds to the motivic Adams tower of $S^{0,0}/\tau$ in the category of $S^{0,0}/\tau$-modules. In Section 10, we re-compute certain low filtration and historically more difficult differentials in the range up to the 45-stem at the prime 2, as an illustration of the power of the isomorphism of spectral sequences in Theorem 1.14.

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Part 1. Equivalence of stable ∞-categories

The question of when the homotopy category of module spectra over a certain ring spectrum is equivalent to the derived category of an abelian category as a triangulated category has been studied in many context by many people. For example, Schwede and Shipley [55] studied the case for the Eilenberg-Mac Lane spectrum HR, where R is a commutative ring, Patchkoria [50] studied the case for the complex periodic K-theory localized at an odd prime, Greenlees [18] studied the case for the rational S^1-equivarant sphere spectrum, and Deligne and Goncharov [10] studied the case for the rational motivic Eilenberg-Mac Lane spectrum $\mathrm{H}\mathbb{Q}^{\mathrm{mot}}$. The answers are positive in these cases. On the other hand, Schwede [54] showed that the classical stable homotopy category is not a derived category.

The goal of Part 1 is to prove that the homotopy category of harmonic $S^{0,0}/\tau$-spectra whose $\mathrm{MU}^{\mathrm{mot}}$-homology are concentrated in bounded Chow degrees is equivalent to the bounded derived category of $\mathrm{MU}_*\mathrm{MU}$-comodules that are concentrated in even degrees. In fact, we prove Theorem 1.11 that There exists an equivalence of stable ∞-categories that preserves the equipped t-structures

$$S^{0,0}/\tau\text{-}\mathbf{Mod}^b_{\clubsuit} \longrightarrow \mathcal{D}^b(\mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod}).$$

We apply a theorem of Lurie in Higher Algebra [34, Proposition 1.3.3.7] on the relation between a stable ∞-category with a t-structure and the derived category of its heart. As a warm-up, we prove Theorem 1.9 that there exists an equivalence of stable ∞-categories that preserves the equipped t-structures

$$\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^b_{\mathrm{cell}} \longrightarrow \mathcal{D}^b(\mathrm{MU}_*\text{-}\mathbf{Mod}).$$

The structure of Part 1 is organized as follows. In Section 2, We recall the definition of t-structures on stable ∞-categories and Lurie's theorem. We modify Lurie's theorem to our situation that are used to prove Theorem 1.9 and Theorem 1.11. Using Dugger-Isaksen's [12] universal coefficient spectral sequence in the category $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^b_{\mathrm{cell}}$, we prove Theorem 1.9 in Section 3. Based on an absolute version of the Adams-Novikov spectral sequence in the category $S^{0,0}/\tau\text{-}\mathbf{Mod}^b_{\clubsuit}$, we prove Theorem 1.11 in Section 4. We also prove Corollary 1.2 in the end of Section 4. We set up the absolute Adams-Novikov tower and its associated spectral sequence that is necessary in the proof of Theorem 1.11 in Section 5. We present a brief account of the well known Morita equivalence of the two abelian categories of $\mathrm{MU}_*\mathrm{MU}$-comodules and $\mathrm{BP}_*\mathrm{BP}$-comodules in Section 6.

2. Lurie's Theorem on t-structures

In [34, Proposition 1.3.3.7], Lurie proves a theorem on the relation between a stable ∞-category with a t-structure and the derived category of its heart. In this section, we state a corrollary of Lurie's theorem as Proposition 2.1, and its dual version Proposition 2.2. Both propositions are used in Section 3 and 4. We will first state Proposition 2.1 and 2.2, and then recall relevant definitions and Lurie's theorem. We use Lurie's theorem to prove Proposition 2.1 in the end of this section.

Let $\mathcal{C}$ be a stable ∞-category. Denote by Σ its translation automorphism, by $h\mathcal{C}$ its homotopy category, and by $[-,-]_{\mathcal{C}}$ the abelian group of homotopy classes of maps in $\mathcal{C}$. When it is clear from the context, we will also denote it by $[-,-]$.

Proposition 2.1. *Let $\mathcal{C}$ be a stable ∞-category. Suppose that*

(1) there exists a bounded t-structure on $\mathcal{C}$,

(2) the abelian category $\mathcal{A} = h\mathcal{C}^{\heartsuit}$ has enough projective objects,

(3) for any pair of objects $X, Y \in \mathcal{A}$, if X is projective, then the abelian groups $[\Sigma^{-i}X, Y]_{\mathcal{C}}$ vanish for $i > 0$.

Then there exists an equivalence of stable ∞-categories

$$F : \mathcal{D}^b(\mathcal{A}) \longrightarrow \mathcal{C}$$

extending the inclusion $f : N(\mathcal{A}) \simeq \mathcal{C}^{\heartsuit} \subseteq \mathcal{C}$, and which preserves t-structures. Here $N(\mathcal{A})$ is the nerve of the abelian category $\mathcal{A}$ and $\mathcal{D}^b(\mathcal{A})$ is the stable full subcategory of $\mathcal{D}^-(\mathcal{A})$ spanned by objects with bounded homology.

Note that the right hand side of the equivalence $\mathcal{C}$ is also bounded with respect to its t-structure. Considering the opposite category, we have the following dual version of Proposition 2.1.

Proposition 2.2. *Let $\mathcal{C}$ be a stable ∞-category. Suppose that*

(1) there exists a bounded t-structure on $\mathcal{C}$,

(2) the abelian category $\mathcal{A} = h\mathcal{C}^{\heartsuit}$ has enough injective objects,

(3) for any pair of objects $X, Y \in \mathcal{A}$, if Y is injective, then the abelian groups $[\Sigma^{-i}X, Y]_{\mathcal{C}}$ vanish for $i > 0$.

Then there exists an equivalence of stable ∞-categories

$$G : \mathcal{D}^b(\mathcal{A}) \longrightarrow \mathcal{C}$$

extending the inclusion $g : N(\mathcal{A}) \simeq \mathcal{C}^{\heartsuit} \subseteq \mathcal{C}$, and which preserves t-structures. Here $N(\mathcal{A})$ is the nerve of the abelian category $\mathcal{A}$ and $\mathcal{D}^b(\mathcal{A})$ is the stable subcategory of $\mathcal{D}^+(\mathcal{A})$ spanned by objects with bounded homology.

We recall from [34, Definition 1.2.1.4] that a t-structure on a stable ∞-category $\mathcal{C}$ is defined as a t-structure on its homotopy category $h\mathcal{C}$, which is a triangulated category. More precisely, we have the following definition.

Definition 2.3. A *t-structure* on a stable ∞-category $\mathcal{C}$ is a pair of two full subcategories $\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}$ that are stable under isomorphisms, satisfying the following three properties

(1) for $X \in \mathcal{C}_{\geq 0}$ and $Y \in \Sigma^{-1}\mathcal{C}_{\leq 0}$, we have $[X, Y]_{\mathcal{C}} = 0$,

(2) there are inclusions $\Sigma\mathcal{C}_{\geq 0} \subseteq \mathcal{C}_{\geq 0}$, $\Sigma^{-1}\mathcal{C}_{\leq 0} \subseteq \mathcal{C}_{\leq 0}$,

(3) for any $X \in \mathcal{C}$, there exists a fiber sequence

$$X_{\geq 0} \longrightarrow X \longrightarrow X_{\leq -1},$$

with $X_{\geq 0} \in \mathcal{C}_{\geq 0}$ and $X_{\leq -1} \in \Sigma^{-1}\mathcal{C}_{\leq 0}$.

Note that as in [34], we use homological indexing convention.

Definition 2.4. Let $\mathcal{C}$ and $\mathcal{C}'$ be stable ∞-categories equipped with t-structures. We say that an exact functor $f : \mathcal{C} \to \mathcal{C}'$ is *right t-exact*, if it carries $\mathcal{C}_{\geq 0}$ to $\mathcal{C}'_{\geq 0}$. An exact functor $f : \mathcal{C} \to \mathcal{C}'$ is *left t-exact*, if it carries $\mathcal{C}_{\leq 0}$ to $\mathcal{C}'_{\leq 0}$. A functor is *t-exact* if it is both left and right t-exact.

Definition 2.5. Denote by $\mathcal{C}_{\geq n}$ and $\mathcal{C}_{\leq n}$ the ∞-categories $\Sigma^n\mathcal{C}_{\geq 0}$ and $\Sigma^n\mathcal{C}_{\leq 0}$ respectively. For every integer n, the subcategories $\mathcal{C}_{\geq n}$ and $\mathcal{C}_{\leq n}$ sit in adjunctions

$$\begin{tikzcd} \mathcal{C}_{\geq n} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift left, "\tau_{\geq n}"] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, shift left, "\tau_{\leq n}"] \arrow[r, phantom, "\perp"] & \mathcal{C}_{\leq n} \arrow[l, shift left] \end{tikzcd},$$

where $\tau_{\geq n}$ and $\tau_{\leq n}$ are called the *n^{th}-truncation functors*.

Sometimes the truncation functors are post-composed with the inclusion functors, so they land in $\mathcal{C}$.

Definition 2.6. Denote by $\mathcal{C}^+$ and $\mathcal{C}^-$ the stable full subcategories spanned by *left-bounded* and *right-bounded* objects in $\mathcal{C}$

$$\mathcal{C}^+ := \operatorname{hocolim}\Big(\mathcal{C}_{\leq 0} \hookrightarrow \mathcal{C}_{\leq 1} \hookrightarrow \cdots\Big),$$

$$\mathcal{C}^- := \operatorname{hocolim}\Big(\mathcal{C}_{\geq 0} \hookrightarrow \mathcal{C}_{\geq -1} \hookrightarrow \cdots\Big),$$

and by $\mathcal{C}^{\mathrm{b}} := \mathcal{C}^+ \cap \mathcal{C}^-$ be the stable subcategory of *bounded objects*. We say that the t-structure is *left-bounded*, *right-bounded*, or *bounded*, if the inclusions of $\mathcal{C}^+$, $\mathcal{C}^-$ or $\mathcal{C}^{\mathrm{b}}$ respectively, in $\mathcal{C}$, is an equivalence.

The intersection $\mathcal{C}^{\heartsuit} = \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}$ is called the *heart* of the t-structure.

The ∞-category $\mathcal{C}^{\heartsuit}$ is always equivalent to (the nerve of) its homotopy category $h\mathcal{C}^{\heartsuit}$, which is an abelian category. Following [34], we abuse the notation by identifying $\mathcal{C}^{\heartsuit}$ with the abelian category $h\mathcal{C}^{\heartsuit}$.

Definition 2.7. Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. We define the *left completion* $\widehat{\mathcal{C}}$ of $\mathcal{C}$ to be a homotopy limit of the tower

$$\cdots \longrightarrow \mathcal{C}_{\leq 2} \xrightarrow{\tau_{\leq 1}} \mathcal{C}_{\leq 1} \xrightarrow{\tau_{\leq 0}} \mathcal{C}_{\leq 0} \xrightarrow{\tau_{\leq 0}} \cdots$$

We say that $\mathcal{C}$ is *left-complete* if the functor $\mathcal{C} \longrightarrow \widehat{\mathcal{C}}$ is an equivalence. The left completion $\widehat{\mathcal{C}}$ is again a stable ∞-category and inherits a t-structure from $\mathcal{C}$.

Two important examples of stable ∞-categories with t-structures are the ∞-category of spectra (as discussed in Section 1.4 of [34]) and the derived ∞-category of an abelian category (as discussed in Section 1.3 of [34]).

Example 2.8. Denote by **Spectra** the ∞-category of spectra and the two full subcategories

$$\mathbf{Spectra}_{\geq 0} = \{X \in \mathbf{Spectra} \mid \pi_n X = 0 \text{ for } n < 0\},$$
$$\mathbf{Spectra}_{\leq 0} = \{X \in \mathbf{Spectra} \mid \pi_n X = 0 \text{ for } n > 0\}.$$

define a t-structure. Left and right bounded objects correspond to connective and coconnective spectra respectively, and its heart can be identified with the abelian category of abelian groups. Moreover, as proved in [34, Proposition 1.4.3.6], it is left-complete.

Example 2.9. Suppose that $\mathcal{A}$ is an abelian category with enough projective objects. There exists an associated *derived* ∞*-category* $\mathcal{D}^{-}(\mathcal{A})$, whose objects can be identified with (right-bounded) chain complexes with values in $\mathcal{A}$. This ∞-category $\mathcal{D}^{-}(\mathcal{A})$ is stable and its homotopy category $h\mathcal{D}^{-}(\mathcal{A})$ can be identified as the usual derived category as triangulated categories.

It admits a natural t-structure defined by

- $\mathcal{D}^{-}(\mathcal{A})_{\geq 0}$ is the full subcategory spanned by the complexes whose homology vanishes in negative degrees,
- $\mathcal{D}^{-}(\mathcal{A})_{\leq 0}$ is the full subcategory spanned by the complexes whose homology vanishes in in positive degrees.

As proved in [34, Propsition 1.3.3.16], this t-structure is left complete and right bounded. Moreover, as proved in [34, Propsition 1.3.3.12], the derived ∞-category $\mathcal{D}^{-}(\mathcal{A})$ has a universal property in the sense that if $\mathcal{C}$ is any stable ∞-category equipped with a left complete t-structure, then any right exact functor $\mathcal{A} \to \mathcal{C}^{\heartsuit}$ extends (in an essentially unique way) to a right t-exact functor $\mathcal{D}^{-}(\mathcal{A}) \to \mathcal{C}$.

We have the following recognition criterion due to Lurie [34, Proposition 1.3.3.7].

Proposition 2.10. *Let $\mathcal{C}$ be a stable ∞-category equipped with a left complete t-structure, whose heart $\mathcal{A} = h\mathcal{C}^{\heartsuit}$ has enough projective objects. Then there exists an essential unique t-exact functor*

$$F : \mathcal{D}^{-}(\mathcal{A}) \longrightarrow \mathcal{C}$$

extending the inclusion $f : N(\mathcal{A}) \simeq \mathcal{C}^{\heartsuit} \subseteq \mathcal{C}$.

Moreover, the following two conditions are equivalent:

- *The functor F is fully faithful.*
- *For any pair of objects $X, Y \in \mathcal{A}$, if X is projective, then the abelian groups $[\Sigma^{-i}X, Y]_{\mathcal{C}}$ vanish for $i > 0$.*

If the conditions are satisfied, then the essential image of F is the full subcategory $\mathcal{C}^{-} = \bigcup_n \mathcal{C}_{\geq n}$ of right bounded objects in $\mathcal{C}$.

Remark 2.11. It is clear that if we restrict the functor F on the bounded stable subcategory $\mathcal{D}^b(\mathcal{A})$, then it gives an equivalence of stable ∞-categories

$$F : \mathcal{D}^b(\mathcal{A}) \longrightarrow \mathcal{C}^b$$

that preserves t-structures.

Remark 2.12. Lurie's theorem is exactly the reason we are working with stable ∞-categories instead of triangulated categories. Given a triangulated category equipped with a t-structure, there in general does not exist a functor from the derived category of the heart to the original triangulated category extending the identity functor on the heart (see [14] for more details for example). However, if the triangulated category comes from the homotopy category of a stable ∞-category, then such a functor always exists. Moreover, Lurie's theorem gives us a recognition criterion in terms of homological algebra to see when such a functor is also an equivalence and preserves t-structures.

Now we use Lurie's theorem to prove Proposition 2.1.

Proof. As explained in Remark 1.2.1.18 in [34], for any stable ∞-category $\mathcal{C}$ with a t-structure, the functor $\mathcal{C} \to \widehat{\mathcal{C}}$ induces an equivalence

$$\mathcal{C}^{+} \to (\widehat{\mathcal{C}})^{+}.$$

For the stable ∞-category $\mathcal{C}$ with a bounded t-structure in the statement of Proposition 2.1, we consider its left-completion $\widehat{\mathcal{C}}$ so we could apply Proposition 2.10. Therefore, the equivalence in the statement of Proposition 2.1 comes from the following zigzag of equivalences:

$$\mathcal{C} \longleftarrow \mathcal{C}^{+} \longrightarrow (\widehat{\mathcal{C}})^{+} \longleftarrow (\widehat{\mathcal{C}})^{b} \longleftarrow \mathcal{D}^b(\mathcal{A}),$$

where the first equivalence comes from the fact that the t-structure on $\mathcal{C}$ is bounded, the third equivalence comes from the fact that the t-structure on $\widehat{\mathcal{C}}$ is right bounded since $\mathcal{C}$ is, and the last equivalence comes from Lurie's theorem and Remark 2.11. $\square$

3. An algebraic model for cellular $\mathrm{MU}^{\mathrm{mot}}/\tau$-modules

In this section, we use Proposition 2.1 to prove Theorem 1.9. Namely, there exists a t-exact equivalence of stable ∞-categories

$$\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b} \longrightarrow \mathcal{D}^{b}(\mathrm{MU}_*\text{-}\mathbf{Mod}),$$

whose restriction on the heart is given by

$$\pi_{*,*} : \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{\heartsuit} \longrightarrow \mathrm{MU}_*\text{-}\mathbf{Mod}.$$

In Section 3.1, we first recall the universal coefficient spectral sequence in the category $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}$, which is constructed by Dugger-Isaksen [12]. This is stated as Theorem 3.2. Using this spectral sequence, we prove the equivalence on the heart as Proposition 3.5 in Section 3.2. Then, using this spectral sequence again, we show in Section 3.3 that the full subcategories

$$\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b,\geq 0},\ \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b,\leq 0}$$

define a t-structure. This concludes the equivalence of stable ∞-categories as Proposition 3.6 and Theorem 3.7.

We will use in Section 5 the above equivalence of stable ∞-categories to construct enough motivic spectra to build $\mathrm{MU}^{\mathrm{mot}}/\tau$-based Adams resolutions in the category $S^{0,0}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}$.

3.1. The category $\mathbf{MU}^{\mathbf{mot}}/\tau\text{-}\mathbf{Mod}_{\mathbf{cell}}$ and the universal coefficient spectral sequence.

We begin with two adjunctions. The first adjunction

$$S^{0,0}\text{-}\mathbf{Mod}_{\mathrm{cell}} \underset{U}{\overset{S^{0,0}/\tau\wedge -}{\rightleftarrows}} S^{0,0}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}, \tag{3.1}$$

between cellular $S^{0,0}$-modules and cellular $S^{0,0}/\tau$-modules is induced by the E_∞-ring map

$$S^{0,0} \longrightarrow S^{0,0}/\tau.$$

Since $\mathrm{MU}^{\mathrm{mot}}/\tau$ is a cellular E_∞-$S^{0,0}/\tau$-algebra, the above adjunction (3.1) extends to

$$S^{0,0}\text{-}\mathbf{Mod}_{\mathrm{cell}} \underset{U}{\overset{S^{0,0}/\tau\wedge -}{\rightleftarrows}} S^{0,0}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}} \underset{U}{\overset{\mathrm{MU}^{\mathrm{mot}}\wedge -}{\rightleftarrows}} \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}. \tag{3.2}$$

Definition 3.1. Define

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod}^0$$

as the full subcategory of $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod}$ spanned by all modules $M_{*,*}$ that are concentrated in Chow degree 0, i.e., $M_{s,w} = 0$ whenever $s \neq 2w$.

We thus have a commutative diagram

$$\begin{tikzcd}
\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}} \arrow[r, "\pi_{*,*}"] & \mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod} \\
\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}} \arrow[r, "\pi_{*,*}"] \arrow[u, hook] & \mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod}^0. \arrow[u, hook]
\end{tikzcd}$$

As explained in Remark 1.10, forgetting the motivic weight we have an equivalence

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}\text{-}\mathbf{Mod}^0 \cong \mathrm{MU}_*\text{-}\mathbf{Mod}.$$

To show that the restriction of $\pi_{*,*}$ to the heart induces an equivalence

$$\pi_{*,*}\colon h\left(\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}}\right) \xrightarrow{\cong} \mathrm{MU}_*\text{-}\mathbf{Mod},$$

we recall the universal coefficient spectral sequence constructed by Dugger-Isaksen in [12]. This spectral sequence is our main tool to compute homotopy classes of maps in the stable ∞-category $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}$.

Theorem 3.2 (Universal Coefficient spectral sequence)**.** *For any $X, Y \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}$, there is a conditionally convergent spectral sequence*

$$E_2^{s,t,w} = \mathrm{Ext}^{s,t,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau}(\pi_{*,*}X, \pi_{*,*}Y) \Longrightarrow \left[\Sigma^{t-s,w}X, Y\right]_{\mathrm{MU}^{\mathrm{mot}}/\tau}.$$

Moreover, if both $\pi_{*,*}X$ and $\pi_{*,*}Y$ are concentrated in bounded Chow degrees, then the spectral sequence convergences strongly and collapses at a finite page.

Proof. We refer to [12, Proposition 7.7] for the precise construction of the spectral sequence and the proof of conditional convergence. For the second statement of the theorem, we recall a few facts from the proof of [12, Proposition 7.7].

The E_1-page arises from a free resolution over $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$:

$$0 \longleftarrow \pi_{*,*}X \twoheadleftarrow \pi_{*,*}F_0 \longleftarrow \pi_{*,*}F_1 \longleftarrow \cdots,$$

and is given by

$$E_1^{s,t,w} := \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau}(\pi_{*,*}(\Sigma^{t,w}F_s), \pi_{*,*}Y).$$

The E_2-page is the cohomology of this chain complex, giving the claimed Ext groups.

Suppose that $\pi_{*,*}X$ and $\pi_{*,*}Y$ are concentrated in Chow degrees $[a,b]$ and $[c,d]$ respectively, where $a \le b$ and $c \le d$. Since $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$ is concentrated in Chow degree 0, we can choose all $\pi_{*,*}(F_s)$ such that they are concentrated in Chow degrees $[a,b]$. Therefore $\pi_{*,*}(\Sigma^{t,w}F_s)$ is concentrated in Chow degrees

$$[a+(t-2w),\ b+(t-2w)]$$

for all $s \ge 0$.

In order for the group $E_1^{s,t,w}$ to be nonzero, we must have

$$c \le b+(t-2w), \quad d \ge a+(t-2w).$$

For a fixed weight w, this gives that

$$t \in [c-b+2w, d-a+2w].$$

Since later pages $E_r^{s,t,w}$ are iterated subquotients of $E_1^{s,t,w}$, their t-degrees are all concentrated in $[c-b+2w, d-a+2w]$.

Recall that the d_r-differential has the form

$$E_r^{s,t,w} \xrightarrow{d_r} E_r^{s+r,t+r-1,w}.$$

In particular, it changes the t-degrees by $r-1$. Since the t-degrees of all possible nonzero elements in the E_1-page satisfy $t \in [c-b+2w, d-a+2w]$, we must have $d_r = 0$ when

$$r-1 > (d-a+2w)-(c-b+2w) = (b-a)+(d-c)$$

for degree reasons. In other words, the spectral sequence collapses at the $E_{(b-a)+(d-c)+2}$ page.

Therefore, under the condition that both $\pi_{*,*}X$ and $\pi_{*,*}Y$ are concentrated in bounded Chow degrees, this spectral sequence convergences strongly and collapses at a finite page. $\square$

Recall from Definition 1.7 that

$$\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b,\ge 0},$$

$$\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b,\le 0},$$

$$\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{\heartsuit}$$

are the full subcategories of $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b}$ that are spanned by objects whose homotopy groups are concentrated in nonnegative, nonpositive and zero Chow degrees respectively.

Corollary 3.3. *Given* $X \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b,\geq 0}$ *and* $Y \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b,\leq 0}$. *The abelian group of homotopy classes of bidegree* $(0,0)$ *can be computed algebraically by the isomorphism*

$$[X,Y]_{\mathrm{MU}^{\mathrm{mot}}/\tau} \longrightarrow \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}{}_{*,*}/\tau}(\pi_{*,*}X, \pi_{*,*}Y)$$

that is induced by applying $\pi_{*,*}$.

Proof. Consider the the E_2-page of the universal coefficient spectral sequence, the tri-degrees that converge to the bidegree $(0,0)$ are of the form $(t,t,0)$ for $t \geq 0$, i.e., the parts $E_2^{s,t,w} = E_2^{t,t,0}$.

By the proof of Theorem 3.2, the t-degrees of all possible nonzero elements in the E_1-page and therefore E_2-page satisfy $t \leq d - a + 2w = d - a$. Since $\pi_{*,*}X$ and $\pi_{*,*}Y$ are concentrated in nonnegative and nonpositive bounded Chow degrees, we have $d = a = 0$. Therefore, we have $t \leq 0$.

Combining both facts, we have established that the only possible nonzero elements in the E_2-page that converge to the bidegree $(0,0)$ are in

$$E_2^{0,0,0} = \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}{}_{*,*}/\tau}(\pi_{*,*}X, \pi_{*,*}Y).$$

To show that all elements in $E_2^{0,0,0}$ survive in the spectral sequence, firstly note that they are not targets of any nonzero differentials since they are in s-degree 0. Secondly, all d_r-differentials for $r \geq 2$ increase the t-degree. Since the t-degrees of all nonzero elements are non-positive, the elements in $E_2^{0,0,0}$ do not support nonzero differentials. This completes the proof. $\square$

3.2. **The equivalence on the heart.** Now we are ready to show that the functor $\pi_{*,*}$ induces an equivalence on the heart. The following is a special case of Corollary 3.3.

Corollary 3.4. *The functor*

$$\pi_{*,*}\colon \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{\heartsuit} \longrightarrow \mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod}^{0}$$

is fully faithful. Here the right hand side is understood as a discrete ∞*-category.*

Proof. For two objects $X, Y \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{\heartsuit}$, by Corollary 3.3, the edge homomorphism

$$[X,Y]_{\mathrm{MU}^{\mathrm{mot}}/\tau} \xrightarrow{\pi_{*,*}} \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*}/\tau}(\pi_{*,*}X, \pi_{*,*}Y)$$

is an isomorphism. This shows that $\pi_{*,*}$ is fully faithful. $\square$

Since we are dealing with cellular objects, the only objects that have zero homotopy groups are contractible. To show the equivalence on the heart, we only need to show the essential surjectivity of $\pi_{*,*}$.

Proposition 3.5. *The functor*

$$\pi_{*,*}\colon \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{\heartsuit} \longrightarrow \mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod}^{0}$$

is an equivalence of ∞-categories.

Proof. We need to show that for any module $M \in \mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod}^{0}$, it can be realized as the homotopy groups of an object in $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{\heartsuit}$.

Suppose that M is a free $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$-module that is concentrated in Chow degree 0.

$$M \cong \bigoplus_{i\in I} \Sigma^{2k_i,k_i}\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$$

Here $\Sigma^{2k_i,k_i}\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$ is a free bigraded rank 1 module over $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$ with a shift of bidegree $(2k_i,k_i)$. We can realize M as the homotopy groups of the wedge

$$\bigvee_{i\in I} \Sigma^{2k_i,k_i}\mathrm{MU}^{\mathrm{mot}}/\tau$$

with the same index set, which is cellular.

For an arbitrary $M \in \mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod}^{0}$, we can pick a free resolution

$$0 \longleftarrow M \longleftarrow F_0 \xleftarrow{f_1} F_1 \xleftarrow{f_2} F_2 \longleftarrow \cdots \tag{3.3}$$

in $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod}^{0}$.

Each F_i can be realized by

$$Z_i \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{\heartsuit}$$

and by Corollary 3.4, each map f_i can be realized by a map $g_i \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{\heartsuit}$ as in

$$Z_0 \xleftarrow{g_1} Z_1 \xleftarrow{g_2} Z_2 \longleftarrow \cdots .$$

We claim that we can construct a tower

$$X_1 \longrightarrow X_2 \longrightarrow \cdots ,$$

with the property that the homotopy groups of X_i are concentrated in Chow degrees 0 and i. The Chow degree 0 part is isomorphic to $M = \mathrm{Coker} f_1$, and the Chow degree i part is isomorphic to $\Sigma^{i,0}\mathrm{Ker} f_i$. Note that since each f_i is a map in $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod}^{0}$, both bigraded modules $\mathrm{Coker} f_1$ and $\mathrm{Ker} f_i$ are concentrated in Chow degree 0. In other words, they are given by

$$\bigoplus_{l=-\infty}^{+\infty} \pi_{2l+k,l}(X_i) = \begin{cases} M = \mathrm{Coker} f_1 & \text{if } k = 0 \\ \Sigma^{i,0}\mathrm{Ker} f_i & \text{if } k = i \\ 0 & \text{otherwise.} \end{cases}$$

We prove this claim inductively.

In fact, we can choose X_1 to be the cofiber of

$$g_1 : Z_1 \longrightarrow Z_0.$$

This gives us a long exact sequence on homotopy groups

$$\cdots \longrightarrow \pi_{*+1,*}X_1 \longrightarrow \pi_{*,*}Z_1 \xrightarrow{f_1} \pi_{*,*}Z_0 \longrightarrow \pi_{*,*}X_1 \longrightarrow \cdots .$$

Since both $\pi_{*,*}Z_0$ and $\pi_{*,*}Z_1$ are concentrated in Chow degree 0, we must have that $\pi_{*,*}X_1$ is concentrated in Chow degree 0 and 1. We can compute directly from the long exact sequence that the Chow degree 0 and 1 parts of $\pi_{*,*}X_1$ are isomorphic to $M = \mathrm{Coker} f_1$ and $\Sigma^{1,0}\mathrm{Ker} f_1$ respectively.

Suppose now that we have constructed the tower up to X_i. We have a homomorphism

$$\pi_{*,*}Z_{i+1} \cong F_{i+1} \twoheadrightarrow \mathrm{Im} f_{i+1} \cong \mathrm{Ker} f_i \hookrightarrow \pi_{*,*}(\Sigma^{-i,0}X_i)$$

in $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod}^0$. Here the first map is induced by f_{i+1} and the second map corresponds to the Chow degree i part of $\pi_{*,*}X_i$.

By Corollary 3.3, this homomorphism can be realized as a map

$$Z_{i+1} \longrightarrow \Sigma^{-i,0}X_i.$$

Define X_{i+1} as the $\Sigma^{i,0}$-suspension of its cofiber, so we have a cofiber sequence

$$\Sigma^{i,0}Z_{i+1} \longrightarrow X_i \longrightarrow X_{i+1}.$$

By the associated long exact sequence in homotopy groups, we have that $\pi_{*,*}X_{i+1}$ is concentrated in Chow degrees 0 and $i+1$. The Chow degree 0 part is isomorphic to M, and the Chow degree $i+1$ part is isomorphic to $\Sigma^{i+1,0}\mathrm{Ker} f_{i+1}$ as required.

Having the tower

$$X_1 \longrightarrow X_2 \longrightarrow \cdots ,$$

we define X as its homotopy colimit

$$X := \mathrm{hocolim}\,(X_1 \longrightarrow X_2 \longrightarrow \cdots).$$

The homotopy groups of X are computed by the colimit

$$\pi_{*,*}X \cong \mathrm{colim}\,(\pi_{*,*}X_1 \longrightarrow \pi_{*,*}X_2 \longrightarrow \cdots) = M$$

and are in particular concentrated in Chow degree 0.

Therefore we have proved that any module $M \in \mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod}^0$ can be realized as a spectrum $X \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}}$. $\square$

3.3. **The t-structure, and the equivalence of the categories.** We prove that the full subcategories previously defined satisfy the required axioms for the t-structure.

Proposition 3.6. *The pair of full subcategories*

$$\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b,\geq 0},\ \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b,\leq 0}$$

defines a bounded t-structure on

$$\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b}.$$

Proof. There are three axioms to check for the t-structure.

The second axiom

$$\Sigma^{1,0}(\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b,\geq 0}) \subseteq \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b,\geq 0},$$

$$\Sigma^{-1,0}(\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b,\leq 0}) \subseteq \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b,\leq 0}$$

follows directly from the definition of the Chow degree.

For the first axiom, we need to show that

$$[X, Y]_{\mathrm{MU}^{\mathrm{mot}}/\tau} = 0$$

for any objects

$$X \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b,\geq 0},\ Y \in \Sigma^{-1,0}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b,\leq 0}.$$

By Corollary 3.3, we have that

$$[X, Y]_{\mathrm{MU}^{\mathrm{mot}}/\tau} \cong \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau}(\pi_{*,*}X, \pi_{*,*}Y)$$

Since $\pi_{*,*}X$ is concentrated in non-negative Chow degrees and $\pi_{*,*}Y$ is concentrated in negative Chow degrees, the right hand side is zero.

For the third axiom, we need to show that for any

$$X \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b},$$

there exists a fiber sequence

$$X_{\geq 0} \longrightarrow X \longrightarrow X_{\leq -1}$$

such that

$$X_{\geq 0} \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b,\geq 0},\ X_{\leq -1} \in \Sigma^{-1,0}(\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b,\leq 0}).$$

Suppose that $\pi_{*,*}(X)$ is concentrated in Chow degrees $[n, m]$, where $m \geq n$.

If $n \geq 0$, we can take the fiber sequence

$$X \longrightarrow X \longrightarrow *.$$

If $n < 0$, then $\pi_{*,*}(\Sigma^{-n,0}X)$ is concentrated in Chow degrees $[0, m-n]$. Consider the Chow degree 0 part of $\pi_{*,*}(\Sigma^{-n,0}X)$, namely

$$\bigoplus_k \pi_{2k,k}(\Sigma^{-n,0}X).$$

By Proposition 3.5 there is a spectrum

$$X_n \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{\heartsuit}$$

realizing this bigraded $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$-module

$$\pi_{*,*}X_n \cong \bigoplus_k \pi_{2k,k}(\Sigma^{-n,0}X).$$

Consider the projection map

$$\pi_{*,*}(\Sigma^{-n,0}X) \twoheadrightarrow \bigoplus_k \pi_{2k,k}(\Sigma^{-n,0}X) \cong \pi_{*,*}X_n.$$

Note that

$$\Sigma^{-n,0}X \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b,\geq 0}.$$

Therefore, by Corollary 3.3, the projection map can be realized by a map

$$\Sigma^{-n,0}X \longrightarrow X_n.$$

Denote by $X_{[n+1,m]}$ the $\Sigma^{n,0}$-suspension of its fiber. This gives a fiber sequence

$$X_{[n+1,m]} \longrightarrow X \longrightarrow \Sigma^{n,0}X_n.$$

From the long exact sequence in homotopy groups, we have that $\pi_{*,*}X_{[n+1,m]}$ is concentrated in Chow degrees $[n+1, m]$, and that the map $X_{[n+1,m]} \longrightarrow X$ induces an isomorphism of homotopy groups in Chow degrees $[n+1, m]$.

Iterating this process, we can construct a finite sequence of maps in $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b}$

$$X_{[0,m]} \longrightarrow X_{[-1,m]} \longrightarrow \cdots \longrightarrow X_{[n+1,m]} \longrightarrow X,$$

where $X_{[0,m]} \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b,\geq 0}$.

Now define $X_{\geq 0} \coloneqq X_{[0,m]}$, and $X_{\leq -1}$ to be the cofiber of the above composition of maps. This gives the desired cofiber sequence

$$X_{\geq 0} \longrightarrow X \longrightarrow X_{\leq -1},$$

with

$$X_{\leq -1} \in \Sigma^{-1,0}(\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b,\leq 0}),$$

since the map $X_{\geq 0} \longrightarrow X$ induces an isomorphism of homotopy groups in Chow degrees $[0, m]$ by construction. $\square$

Having this t-structure on $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^b_{\mathrm{cell}}$, the main result of this sections follows from Proposition 2.1.

Theorem 3.7. *There is a t-exact equivalence of stable ∞-categories*

$$\mathcal{D}^b(\mathrm{MU}_*\text{-}\mathbf{Mod}) \xrightarrow{\cong} \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^b_{\mathrm{cell}}.$$

Proof. It is clear that the t-structure is bounded. By Proposition 3.5, and the equivalence

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}\text{-}\mathbf{Mod}^0 \cong \mathrm{MU}_*\text{-}\mathbf{Mod},$$

the heart can be identified as modules over MU_*. Therefore, it has enough projective objects.

It remains to show that for any two objects

$$X, Y \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}}$$

with $\pi_{*,*}X$ projective over $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$, we have that

$$[\Sigma^{-i,0}X, Y]_{\mathrm{MU}^{\mathrm{mot}}/\tau} = 0$$

for $i > 0$.

We apply the Universal Coefficient spectral sequence in Theorem 3.2,

$$\mathrm{Ext}^{s,t,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau}(\pi_{*,*}X, \pi_{*,*}Y) \Longrightarrow [\Sigma^{t-s,w}X, Y]_{\mathrm{MU}^{\mathrm{mot}}/\tau}.$$

Since $\pi_{*,*}X$ is projective over $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$, the E_2-page of the spectral sequence is concentrated on the line $s = 0$, and therefore collapses at the E_2-page.

Moreover, since both $\pi_{*,*}X$ and $\pi_{*,*}Y$ are concentrated in Chow degree 0, the E_2-page is also concentrated in Chow degree 0, namely $t - 2w = 0$ in this case.

We are interested in the case $t - s = -i < 0$ and $w = 0$. By the above analysis, the corresponding tri-degrees in the E_2-page are all 0 in our case. Therefore, we must have that

$$[\Sigma^{-i,0}X, Y]_{\mathrm{MU}^{\mathrm{mot}}/\tau} = 0.$$

This completes the proof. ∎

4. An algebraic model for cellular $S^{0,0}/\tau$-modules

After the warmup in Section 3, we use Proposition 2.2 to prove Theorem 1.11. Namely, There exists a t-exact equivalence of stable ∞-categories

$$S^{0,0}/\tau\text{-}\mathbf{Mod}^b_{\clubsuit} \longrightarrow \mathcal{D}^b(\mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod}),$$

whose restriction on the heart is given by

$$\mathrm{MU}^{\mathrm{mot}}_{*,*} : S^{0,0}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\clubsuit} \longrightarrow \mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod}.$$

The structure of this section is similar to that of Section 3.

In Section 4.1, we discuss the category of harmonic $S^{0,0}/\tau$-modules. We will also recall certain facts on the category of $\mathrm{MU}_*\mathrm{MU}$-comodules, such as the Landweber's Filtration Theorem. Instead of using the universal coefficient spectral sequence in the category $\mathrm{MU}^{\mathrm{mot}}/\tau$-$\mathbf{Mod}_{\mathrm{cell}}$, we will use the absolute Adams-Novikov spectral sequence in the category of harmonic $S^{0,0}/\tau$-modules. This spectral sequence is constructed in Section 5. Using this spectral sequence, we prove the equivalence on the heart as Proposition 4.11 in Section 4.2. Then again using this spectral sequence, we show in Section 4.3 that the full subcategories

$$S^{0,0}/\tau\text{-}\mathbf{Mod}_{\clubsuit}^{b,\geq 0},\ S^{0,0}/\tau\text{-}\mathbf{Mod}_{\clubsuit}^{b,\leq 0}$$

define a t-structure and conclude the equivalence of stable ∞-categories as Proposition 4.12 and Theorem 4.13.

4.1. **The categories $S^{0,0}/\tau$-$\mathbf{Mod}_{\clubsuit}$ and $\mathbf{MU_*MU}$-Comod.** We first recall from Definition 1.5 that a $S^{0,0}/\tau$-module spectrum Y is *harmonic* if it is cellular and the natural map

$$Y \longrightarrow Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}$$

is an isomorphism on $\pi_{*,*}$. It is clear that in the category $S^{0,0}/\tau$-$\mathbf{Mod}_{\mathrm{cell}}$, being harmonic is closed under taking suspensions, finite products and fibers. The category of harmonic $S^{0,0}/\tau$-module spectra is denoted by $S^{0,0}/\tau$-$\mathbf{Mod}_{\clubsuit}$.

We have the following examples and non-examples of harmonic $S^{0,0}/\tau$-module spectra.

Example 4.1.

(1) Any finite cellular object in $S^{0,0}/\tau$-$\mathbf{Mod}$ is harmonic.

(2) Any finite cellular object in $\mathrm{MU}^{\mathrm{mot}}/\tau$-$\mathbf{Mod}$ is harmonic.

(3) The η-inverted cofiber of τ is cellular but *not* harmonic.
Here η is the Hopf map in $\pi_{1,1}S^{0,0}$. Post-composing with the unit map $S^{0,0} \to S^{0,0}/\tau$, we also denote its Hurewicz image in $\pi_{1,1}S^{0,0}/\tau$ by η. It is non-nilpotent in the ring $\pi_{*,*}S^{0,0}/\tau$. The η-inverted cofiber of τ

$$\eta^{-1}S^{0,0}/\tau \coloneqq \mathrm{hocolim}\left(S^{0,0}/\tau \to \Sigma^{-1,-1}S^{0,0}/\tau \to \Sigma^{-2,-2}S^{0,0}/\tau \to \cdots\right)$$

is a cellular object in $S^{0,0}/\tau$-$\mathbf{Mod}$. Since η maps to zero in $\pi_{1,1}\mathrm{MU}^{\mathrm{mot}}$, the completion $(\eta^{-1}S^{0,0}/\tau)^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}$ is contractible. Therefore, the spectrum $\eta^{-1}S^{0,0}/\tau$ is not harmonic.

We need the following Lemma 4.2 in the proof of Proposition 4.11, whose proof we postpone until the end of Section 5.

Lemma 4.2. *Suppose that $\{Y_\alpha\}$ is a filtered system in $S^{0,0}/\tau$-$\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}}$ such that each Y_α is harmonic. Then the homotopy colimit of $\{Y_\alpha\}$ in $S^{0,0}/\tau$-$\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}}$ is also harmonic.*

We also recall that for a $S^{0,0}/\tau$-module X, its $\mathrm{MU}^{\mathrm{mot}}$-homology is defined as

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}X \coloneqq \pi_{*,*}(\mathrm{MU}^{\mathrm{mot}} \wedge X) \cong \pi_{*,*}(\mathrm{MU}^{\mathrm{mot}}/\tau \wedge_{S^{0,0}/\tau} X).$$

Following computations of $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}$ from Hu-Kriz-Ormsby and Dugger-Isaksen [11, 23], we have the $\mathrm{MU}^{\mathrm{mot}}$-homology of $\mathrm{MU}^{\mathrm{mot}}/\tau$

$$\pi_{*,*}(\mathrm{MU}^{\mathrm{mot}}/\tau \wedge_{S^{0,0}/\tau} \mathrm{MU}^{\mathrm{mot}}/\tau) \cong \mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau[b_1, b_2, \ldots] \cong \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau$$

where $|b_i| = (2i, i)$, and is in Chow degree 0.

Definition 4.3. Denote by

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod}$$

the abelian category of left comodules over the Hopf algebroid $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau$, and by

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod}^0$$

its full subcategory spanned by all comodules M whose underlying $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$-modules are concentrated in Chow degree 0.

We thus have a commutative diagram

$$\begin{tikzcd}
S^{0,0}/\tau\text{-}\mathbf{Mod}_{\clubsuit} \arrow[r, "\mathrm{MU}^{\mathrm{mot}}_{*,*}"] & \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod} \\
S^{0,0}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\clubsuit} \arrow[u, hook] \arrow[r, "\mathrm{MU}^{\mathrm{mot}}_{*,*}"] & \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod}^0. \arrow[u, hook]
\end{tikzcd}$$

Forgetting the motivic weight, we have the equivalence

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod}^0 \cong \mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod}.$$

Recall that we have the adjunction between modules and comodules

$$U\colon \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod} \rightleftarrows \mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod}\colon \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau \otimes_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau} -. \tag{4.1}$$

Note that the forgetful functor is a *left* adjoint, while the tensor-up functor is a *right* adjoint. We refer to [20, Section 1.1] for more details.

Using the ring map $S^{0,0}/\tau \longrightarrow \mathrm{MU}^{\mathrm{mot}}/\tau$, we can form the commutative diagram

$$\begin{tikzcd}
\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}} \arrow[rr, "\pi_{*,*}"] & & \mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod} \\
S^{0,0}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}} \arrow[u, "\mathrm{MU}^{\mathrm{mot}}/\tau \wedge_{C\tau} -"] \arrow[rr, "\mathrm{MU}^{\mathrm{mot}}_{*,*}"] & & \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod}. \arrow[u, "\mathrm{U}"']
\end{tikzcd}$$

For the category of comodules over $\mathrm{MU}_*\mathrm{MU}$, we recall the Landweber's Filtration Theorem. Recall from [30], [31] that there are elements $v_n \in \mathrm{MU}_*$ with $v_0 = p$, giving the invariant prime ideals $I_n = (v_0, \ldots, v_n) \trianglelefteq \mathrm{MU}_*$. Moreover, these elements satisfy the formula

$$\eta_R(v_n) \equiv v_n \mod I_{n-1},$$

and so MU_*/I_n is canonically a comodule over $\mathrm{MU}_*\mathrm{MU}$. This gives a short exact sequence of comodules

$$0 \longrightarrow \mathrm{MU}_*/I_n \xrightarrow{\cdot v_n} \mathrm{MU}_*/I_n \longrightarrow \mathrm{MU}_*/I_{n+1} \longrightarrow 0,$$

for every $n \geq 0$. Landweber's Filtration Theorem ([30], [31]) states that any comodule M over $\mathrm{MU}_*\mathrm{MU}$ whose underlying MU_*-module is finitely presented, can be reconstructed by finitely many extensions of suspensions of MU_*/I_n's.

Theorem 4.4 (Landweber's Filtration Theorem)**.** *Suppose that S is a subset of* $\mathrm{MU}_*\mathrm{MU}$-**Comod** *such that*

(1) it contains MU_* *and* MU_*/I_n*'s for all $n \geq 0$,*

(2) and it is closed under suspensions and extensions.

Then S contains all comodules over $\mathrm{MU}_*\mathrm{MU}$ *whose underlying* MU_**-modules are finitely presented.*

There are two more facts that we will use on the category $\mathrm{MU}_*\mathrm{MU}$-**Comod**. The first one is the following lemma. For a proof, see Miller-Ravenel [42, Lemma 2.11] and Hovey [20] for example.

Lemma 4.5. *Any comodules over* $\mathrm{MU}_*\mathrm{MU}$ *is a filtered colimit of finitely presented comodules.*

The second one is a standard fact.

Lemma 4.6. *The category* $\mathrm{MU}_*\mathrm{MU}$-**Comod** *has enough injective objects.*

In Section 3, an important tool for the category $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{b}_{\mathrm{cell}}$ is the universal coefficient spectral sequence. For the category $S^{0,0}/\tau\text{-}\mathbf{Mod}^{b}_{\clubsuit}$, we will construct the absolute Adams-Novikov spectral sequence, namely, for any two objects X and Y in this category, there is a strongly convergent spectral sequence that collapses at a finite page.

$$\mathrm{Ext}^{s,t,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y) \Longrightarrow [\Sigma^{t-s,w}X, Y]_{S^{0,0}/\tau},$$

with differentials

$$d_r\colon E^{s,t,w}_r \longrightarrow E^{s+r,t+r-1,w}_r.$$

The existence of this absolute Adams-Novikov spectral sequence in the category $S^{0,0}/\tau\text{-}\mathbf{Mod}^{b}_{\clubsuit}$ is proved as Theorem 5.6 in Section 5.

Using the absolute Adams-Novikov spectral sequence, we will prove the following Corollary 4.7 and 4.8 in Section 5.3.

Corollary 4.7. *Given $X \in S^{0,0}/\tau\text{-}\mathbf{Mod}^{b,\geq 0}_{\clubsuit}$ and $Y \in S^{0,0}/\tau\text{-}\mathbf{Mod}^{b,\leq 0}_{\clubsuit}$, the abelian group of homotopy classes of degree $(0,0)$ maps can be computed algebraically by the isomorphism*

$$[X, Y]_{S^{0,0}/\tau} \longrightarrow \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y)$$

that is induced by applying $\mathrm{MU}^{\mathrm{mot}}_{*,*}$.

Corollary 4.8. *Given $X, Y \in S^{0,0}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\clubsuit}$, for any bidegree (t,w), there is an isomorphism*

$$[\Sigma^{t,w}X, Y]_{S^{0,0}/\tau} \cong \mathrm{Ext}^{2w-t,2w,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y).$$

4.2. **The equivalence on the heart.** Now we are ready to show that the functor $\mathrm{MU}^{\mathrm{mot}}_{*,*}$ induces an equivalence on the heart. The following is a special case of Corollary 4.7.

Corollary 4.9. *The functor*

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}\colon S^{0,0}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\clubsuit} \xrightarrow{\cong} \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod}^{0}$$

is fully faithful. Here the right hand side is understood as a discrete ∞-category.

Proof. For objects $X, Y \in S^{0,0}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\clubsuit}$, by Corollary 4.7, the edge homomorphism

$$[X, Y]_{S^{0,0}/\tau} \xrightarrow{\mathrm{MU}^{\mathrm{mot}}_{*,*}} \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}\left(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y\right)$$

is an isomorphism. This shows that $\mathrm{MU}^{\mathrm{mot}}_{*,*}$ is fully faithful. $\square$

Since we are dealing with cellular objects, the only objects that have zero homotopy groups are contractible. To show the equivalence on the heart, we only need to show the essential surjectivity of $\mathrm{MU}^{\mathrm{mot}}_{*,*}$.

Unlike the case for modules over $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$, we do not have free resolutions for comodules over $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau$. We will instead use Landweber's Filtration Theorem to realize all comodules that are finitely presented, and then extend the result using filtered colimits.

We start with the following 2-out-of-3 Lemma.

Lemma 4.10. *Consider any short exact sequence in* $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod}^0$

$$0 \longrightarrow M' \xrightarrow{f'} M \xrightarrow{f''} M'' \longrightarrow 0. \tag{4.2}$$

If any two of the three comodules M', M, M'' are realizable in $S^{0,0}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\clubsuit}$, then so is the third.

Proof. There are three cases that we need to prove.

(1) Suppose that both comodules M' and M are realizable by

$$M' \cong \mathrm{MU}^{\mathrm{mot}}_{*,*}X', \ M \cong \mathrm{MU}^{\mathrm{mot}}_{*,*}X.$$

By Corollary 4.9, the algebraic map f' is also realizable as the $\mathrm{MU}^{\mathrm{mot}}_{*,*}$-homology of a map

$$X' \xrightarrow{F'} X.$$

Since $S^{0,0}/\tau\text{-}\mathbf{Mod}^{b}_{\clubsuit}$ is closed under taking cofibers, we can realize the comodule M'' by the $\mathrm{MU}^{\mathrm{mot}}_{*,*}$-homology of the cofiber of F'. In fact, the associated long exact sequence on the $\mathrm{MU}^{\mathrm{mot}}_{*,*}$-homology tells us

$$M'' \cong \mathrm{MU}^{\mathrm{mot}}_{*,*}X'',$$

where X'' is the the cofiber of F'.

(2) Suppose that both comodules M and M'' are realizable. Then we realize the algebraic map and take the fiber instead. The same argument shows that it realizes M'.

(3) Suppose that both comodules M' and M'' are realizable by

$$M' \cong \mathrm{MU}^{\mathrm{mot}}_{*,*}X', \ M'' \cong \mathrm{MU}^{\mathrm{mot}}_{*,*}X''.$$

In this case, the short exact sequence (4.2) corresponds to an element in

$$\mathrm{Ext}^{1,0,0}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(M'', M').$$

By Corollary 4.8, this algebraic element can be realized by a map

$$F : \Sigma^{-1,0}X'' \longrightarrow X'.$$

Define X to be the cofiber of the map F. Then X realizes M. In fact, F is detected on the 1-line of the Adams-Novikov spectral sequence. By the construction of the absolute Adams-Novikov spectral sequence, it is characterized by the short exact sequence on the $\mathrm{MU}^{\mathrm{mot}}$-homology groups that corresponds to the extension (4.2).

This completes the proof. $\square$

Now we prove the equivalence on the heart.

Proposition 4.11. *The functor*

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}\colon S^{0,0}/\tau\text{-}\mathbf{Mod}_{\clubsuit}^{\heartsuit} \xrightarrow{\cong} \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod}^{0}$$

is an equivalence of categories.

Proof. We only need to show that the functor $\mathrm{MU}^{\mathrm{mot}}_{*,*}$ is essentially surjective. In other words, for any comodule $M \in \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod}^{0}$, we show that it can be realized as a harmonic $S^{0,0}/\tau$-module X, whose $\mathrm{MU}^{\mathrm{mot}}_{*,*}$-homology is M. This follows from Lemma 4.10, 4.5, 4.2 and Landweber's Filtration Theorem via the equivalence

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod}^{0} \cong \mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod}.$$

In fact, first note that MU_* corresponds $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$, and is therefore realized by $S^{0,0}/\tau$. By Lemma 4.10, we can inductively realized comodules MU_*/I_n for all $n \geq 0$. Then by Landweber's Filtration Theorem and Lemma 4.10, we can realized all finitely presented comodules.

For any comodule $M \in \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod}^{0}$, or equivalently a comodule over $\mathrm{MU}_*\mathrm{MU}$, we can write it as a filtered colimit of finitely presented ones M_α,

$$M \cong \operatorname{colim} M_\alpha.$$

By the above discussion, we can realize each M_α by $X_\alpha \in S^{0,0}/\tau\text{-}\mathbf{Mod}_{\clubsuit}^{\heartsuit}$. Moreover, by Corollary 4.9, we can realize the whole filtered system $\{M_\alpha\}$ by a filtered system $\{X_\alpha\}$. Taking the homotopy colimit, we define

$$X \coloneqq \operatorname{hocolim} X_\alpha.$$

By Lemma 4.2, X is harmonic. Since $\mathrm{MU}^{\mathrm{mot}}_{*,*}$ commutes with filtered colimits, we have that the comodule M is realized by X. This completes the proof. $\square$

4.3. **The t-structure and the equivalence of the categories.** We prove that two full subcategories satisfy the required axioms for the t-structure.

Proposition 4.12. *The pair of full subcategories*

$$S^{0,0}/\tau\text{-}\mathbf{Mod}_{\clubsuit}^{b,\geq 0},\ S^{0,0}/\tau\text{-}\mathbf{Mod}_{\clubsuit}^{b,\leq 0}$$

defines a bounded t-structure on $S^{0,0}/\tau\text{-}\mathbf{Mod}_{\clubsuit}^{b}$.

Proof. The proof is analogous to the proof of 3.6 .There are three axioms to check for the t-structure.

The second axiom

$$\Sigma^{1,0}(S^{0,0}/\tau\text{-}\mathbf{Mod}_{\text{cell}}^{b,\geq 0}) \subseteq S^{0,0}/\tau\text{-}\mathbf{Mod}_{\text{cell}}^{b,\geq 0},$$

$$\Sigma^{-1,0}(S^{0,0}/\tau\text{-}\mathbf{Mod}_{\text{cell}}^{b,\leq 0}) \subseteq S^{0,0}/\tau\text{-}\mathbf{Mod}_{\text{cell}}^{b,\leq 0}$$

follows directly from the definition of the Chow degree.

For the first axiom, we need to show that

$$[X, Y]_{S^{0,0}/\tau} = 0$$

for any objects

$$X \in S^{0,0}/\tau\text{-}\mathbf{Mod}_{\clubsuit}^{b,\geq 0},\ Y \in \Sigma^{-1,0}S^{0,0}/\tau\text{-}\mathbf{Mod}_{\clubsuit}^{b,\leq 0}.$$

By Corollary 4.7, we have that

$$[X, Y]_{S^{0,0}/\tau} \longrightarrow \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y).$$

Since $\mathrm{MU}^{\mathrm{mot}}_{*,*}X$ is concentrated in non-negative Chow degrees and $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y$ is concentrated in negative Chow degrees, the right hand side is zero.

For the third axiom, we need to show that for any

$$X \in S^{0,0}/\tau\text{-}\mathbf{Mod}_{\clubsuit}^{b},$$

there exists a fiber sequence

$$X_{\geq 0} \longrightarrow X \longrightarrow X_{\leq -1}$$

such that

$$X_{\geq 0} \in S^{0,0}/\tau\text{-}\mathbf{Mod}_{\clubsuit}^{b,\geq 0},\ X_{\leq -1} \in \Sigma^{-1,0}(S^{0,0}/\tau\text{-}\mathbf{Mod}_{\clubsuit}^{b,\leq 0}).$$

Suppose that $\mathrm{MU}^{\mathrm{mot}}_{*,*}X$ is concentrated in Chow degrees $[n, m]$, where $m \geq n$.

If $n \geq 0$, we can take the fiber sequence

$$X \longrightarrow X \longrightarrow *.$$

If $n < 0$, then $\mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{-n,0}X)$ is concentrated in Chow degrees $[0, m-n]$. Consider the Chow degree 0 part of $\mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{-n,0}X)$, namely the $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau$-comodule

$$\bigoplus_k \mathrm{MU}^{\mathrm{mot}}_{2k,k}(\Sigma^{-n,0}X).$$

By Proposition 4.11 there is a spectrum

$$X_n \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\clubsuit}^{\heartsuit}$$

realizing this bigraded $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$-module

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}X_n \cong \bigoplus_k \mathrm{MU}^{\mathrm{mot}}_{2k,k}(\Sigma^{-n,0}X).$$

Consider the projection map

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{-n,0}X) \twoheadrightarrow \bigoplus_k \mathrm{MU}^{\mathrm{mot}}_{2k,k}(\Sigma^{-n,0}X) \cong \mathrm{MU}^{\mathrm{mot}}_{*,*}X_n.$$

Note that

$$\Sigma^{-n,0}X \in S^{0,0}/\tau\text{-}\mathbf{Mod}^{b,\geq 0}_{\clubsuit}.$$

Therefore, by Corollary 4.8, the projection map can be realized by a map

$$\Sigma^{-n,0}X \longrightarrow X_n.$$

Denote by $X_{[n+1,m]}$ the $\Sigma^{n,0}$-suspension of its fiber. This gives a fiber sequence

$$X_{[n+1,m]} \longrightarrow X \longrightarrow \Sigma^{n,0}X_n.$$

From the long exact sequence in $\mathrm{MU}^{\mathrm{mot}}$-homology groups, we have that $\mathrm{MU}^{\mathrm{mot}}_{*,*}X_{[n+1,m]}$ is concentrated in Chow degrees $[n+1,m]$, and that the map $X_{[n+1,m]} \longrightarrow X$ induces an isomorphism of $\mathrm{MU}^{\mathrm{mot}}$-homology groups in Chow degrees $[n+1,m]$.

Iterating this process, we can construct a finite sequence of maps in $S^{0,0}/\tau\text{-}\mathbf{Mod}^{b}_{\clubsuit}$

$$X_{[0,m]} \longrightarrow X_{[-1,m]} \longrightarrow \cdots \longrightarrow X_{[n+1,m]} \longrightarrow X,$$

where $X_{[0,m]} \in S^{0,0}/\tau\text{-}\mathbf{Mod}^{b,\geq 0}_{\clubsuit}$.

Now define $X_{\geq 0} \coloneqq X_{[0,m]}$, and $X_{\leq -1}$ to be the cofiber of the above composition of maps. This gives the desired cofiber sequence

$$X_{\geq 0} \longrightarrow X \longrightarrow X_{\leq -1},$$

with

$$X_{\leq -1} \in \Sigma^{-1,0}(S^{0,0}/\tau\text{-}\mathbf{Mod}^{b,\leq 0}_{\clubsuit}),$$

since the map $X_{\geq 0} \longrightarrow X$ induces an isomorphism of $\mathrm{MU}^{\mathrm{mot}}$-homology groups in Chow degrees $[0,m]$ by construction.

□

Having this t-structure on $S^{0,0}/\tau\text{-}\mathbf{Mod}^{b}_{\clubsuit}$, the main result of this sections follows from Proposition 2.2.

Theorem 4.13. *There is a t-exact equivalence of stable ∞-categories*

$$\mathcal{D}^b(\mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod}) \xrightarrow{\cong} S^{0,0}/\tau\text{-}\mathbf{Mod}^{b}_{\clubsuit}.$$

Proof. The proof is analogous to the proof of Theorem 3.7. It is clear that the t-structure is bounded. By Proposition 4.11, and the equivalence

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod}^0 \cong \mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod},$$

the heart can be identified as comodules over $\mathrm{MU}_*\mathrm{MU}$. By Lemma 4.6, it has enough injective objects.

It remains to show that for objects

$$X,\ Y \in S^{0,0}/\tau\text{-}\mathbf{Mod}_{\clubsuit}^{\heartsuit}$$

with $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y$ injective over $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau$, we have that

$$[\Sigma^{-i,0}X, Y]_{S^{0,0}/\tau} = 0$$

for any $i > 0$.

We apply the absolute Adams-Novikov spectral sequence

$$\mathrm{Ext}^{s,t,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}\left(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y\right) \Longrightarrow \left[\Sigma^{t-s,w}X, Y\right]_{S^{0,0}/\tau}$$

in the category $S^{0,0}/\tau\text{-}\mathbf{Mod}_{\clubsuit}^{b}$, as in Corollary 5.6.

Since $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y$ is an injective $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau$-comodule, the E_2-page of the spectral sequence is concentrated on the line $s = 0$, and therefore collapses at the E_2-page.

Moreover, since both $\mathrm{MU}^{\mathrm{mot}}_{*,*}X$ and $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y$ are concentrated in Chow degree 0, the E_2-page is also concentrated in Chow degree 0, namely $t - 2w = 0$ in this case.

We are interested in the case $t - s = -i < 0$ and $w = 0$. By the above analysis, the corresponding tri-degrees in the E_2-page are all 0 in our case. Therefore, we must have that

$$[\Sigma^{-i,0}X, Y]_{S^{0,0}/\tau} = 0.$$

This completes the proof. $\square$

Remark 4.14. We comment on the bi-grading in the equivalence of stable ∞-categories in Theorem 4.13 through some examples.

(1) It is clear that $S^{0,0}/\tau$ corresponds to MU_* in the derived category of $\mathrm{MU}_*\mathrm{MU}$-comodules.

(2) Consider $\Sigma^{2,1}S^{0,0}/\tau$. Since its $\mathrm{MU}^{\mathrm{mot}}$-homology is concentrated in Chow degree 0, it lives in the heart. Therefore, by the t-exactness, it corresponds to a cochain complex that is concentrated in cohomological degree 0. A direct computation show that it corresponds to $\Sigma^2\mathrm{MU}_*$. We also denote this object in the category $\mathcal{D}^b(\mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod})$ by $\Sigma^{2,1}\mathrm{MU}_*$.

(3) Consider $\Sigma^{1,0}S^{0,0}/\tau$. Its $\mathrm{MU}^{\mathrm{mot}}$-homology is concentrated in Chow degree -1. By the t-exactness, it corresponds to the cochain complex that is concentrated in cohomological degree -1, with the comodule MU_* in that cohomological degree. We also denote this object by $\Sigma^{1,0}\mathrm{MU}_*$.

(4) In general, denote by $\Sigma^{m,n}\mathrm{MU}_*$ the object in the category $\mathcal{D}^b(\mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod})$ that $\Sigma^{m,n}S^{0,0}/\tau$ corresponds to. Then $\Sigma^{m,n}\mathrm{MU}_*$ is a cochain complex that is concentrated in cohomological degree $2n-m$, with the comodule $\Sigma^{2n}\mathrm{MU}_*$ in that cohomological degree.

Now we prove Corollary 1.2.

Proof of Corollary 1.2. Let $S^{0,0}/\tau\text{-}\mathbf{Mod}_{\mathrm{fin}}$ be the category of finite cellular motivic left module spectra over $S^{0,0}/\tau$, and $\mathcal{D}^b(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod})_{\mathrm{fin}}$ be the full subcategory of $\mathcal{D}^b(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod})$ consisting of objects generated by BP_* and its shifts (by both internal and homological degrees) under finite colimits.

Since $S^{0,0}/\tau$ is harmonic, and corresponds to BP_* under the equivalence in Theorem 1.1, we have an equivalence of stable ∞-categories equipped with t-structures at each prime p

$$\mathcal{D}^b(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod})_{\mathrm{fin}} \simeq S^{0,0}/\tau\text{-}\mathbf{Mod}_{\mathrm{fin}}.$$

By Theorem 5.3.5.11 of Lurie's Higher Topos Theory [35], if $\mathcal{C}$ is a full subcategory of an ∞-category $\mathcal{D}$, whose elements are compact, and generate $\mathcal{D}$ under filtered colimits, then $\mathcal{D}$ is equivalent to the ∞-category $\mathrm{Ind}(\mathcal{C})$ of Ind-objects of $\mathcal{C}$.

It follows that

$$S^{0,0}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}} \simeq \mathrm{Ind}(S^{0,0}/\tau\text{-}\mathbf{Mod}_{\mathrm{fin}}).$$

On the other hand, BP_* generates $\mathcal{D}^b(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod})_{\mathrm{fin}}$ under finite colimits. Moreover, it is proved by Hovey in [20, Section 6] that objects in the category $\mathcal{D}^b(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod})_{\mathrm{fin}}$ are compact, and generate $\mathbf{Stable}(\mathrm{BP}_*\mathrm{BP})$ under filtered colimits. It then follows from Theorem 5.3.5.11 of [35] that

$$\mathbf{Stable}(\mathrm{BP}_*\mathrm{BP}) \simeq \mathrm{Ind}(\mathcal{D}^b(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod})_{\mathrm{fin}}).$$

Therefore, we have an equivalence of stable ∞-categories at each prime p

$$\mathbf{Stable}(\mathrm{BP}_*\mathrm{BP}) \simeq S^{0,0}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}.$$

$\square$

5. The absolute Adams-Novikov spectral sequence

In Section 3, we use the universal coefficient spectral sequence

$$E_2^{s,t,w} = \mathrm{Ext}^{s,t,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau}(\pi_{*,*}X, \pi_{*,*}Y) \Longrightarrow \left[\Sigma^{t-s,w}X, Y\right]_{\mathrm{MU}^{\mathrm{mot}}/\tau}$$

of Theorem 3.2 to compute homotopy classes of maps in $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^b_{\mathrm{cell}}$. This is a very convenient tool since both the t-structure on $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^b_{\mathrm{cell}}$ and the E_2-page of universal coefficient spectral sequence are defined in terms of homotopy groups. The bounds in the t-structure corresponds to vanishing areas in the spectral sequence.

For the category $S^{0,0}/\tau\text{-}\mathbf{Mod}^b_{\clubsuit}$, the t-structure is defined in terms of $\mathrm{MU}^{\mathrm{mot}}$-homology. We therefore need a version of the motivic Adams-Novikov spectral sequence that computes $S^{0,0}/\tau$-linear maps.

Recall from Dugger-Isaksen [11, Section 8] or Hu-Kriz-Ormsby [23] the usual $\mathrm{MU}^{\mathrm{mot}}$-based motivic Adams-Novikov spectral sequence

$$\mathrm{Ext}^{*,*,*}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}}(\mathrm{MU}^{\mathrm{mot}}_{*,*}S^{0,0}, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y) \Longrightarrow \pi_{*,*}Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}.$$

This spectral sequence is not what we need. We need a spectral sequence of the form

$$\mathrm{Ext}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y) \Longrightarrow [X, Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}]_{S^{0,0}/\tau},$$

for the following two reasons.

Firstly, we need a spectral sequence computing homotopy classes of maps in the category $S^{0,0}/\tau\text{-}\mathbf{Mod}^b_{\clubsuit}$, instead of homotopy classes of maps between the underlying motivic spectra.

Secondly, we need the first variable X to be a general spectrum than just the unit object $S^{0,0}/\tau$. Classically, it seems to be a folklore theorem without published reference that there exists an Adams-Novikov spectral sequence when the first variable X is arbitrary. When the first variable X is the sphere spectrum, we can use the standard cosimplicial cobar Adams-Novikov resolution for the second variable Y to set up this spectral sequence. This is done in [53, Chapter 2] classically and in [11, Section 8] and [23] motivically. Such a resolution induces a resolution of $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y$ by relative injective comodules. It computes the E_2-page as an Ext-group only when the first variable $\mathrm{MU}^{\mathrm{mot}}_{*,*}X$ is a projective module over $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$ [53, Corollary A1.2.12]. Since our first variable X is arbitrary, the E_2-page in general does not have a description as an Ext-group.

Instead of using the canonical Adams-Novikov tower that produces a resolution of $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y$ by relative injectives, we construct an **absolute** Adams-Novikov tower that produces a resolution of $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y$ by **absolute** injectives. The first step is Lemma 5.1 and Lemma 5.2, where we produce enough $S^{0,0}/\tau$-modules whose $\mathrm{MU}^{\mathrm{mot}}$-homology are injective comodules. The second step is Lemma 5.3, where we show that we can algebraically resolve comodules in $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau$-**Comod** by these injective comodules. The third step is Proposition 5.4, where we topologically realize the algebraic construction to produce an absolute Adams-Novikov tower in the category $S^{0,0}/\tau$-$\mathbf{Mod}^b_{\mathrm{cell}}$. Finally, in Theorem 5.5, we construct the absolute Adams-Novikov spectral sequence and analyze its convergence.

5.1. **The absolute Adams-Novikov tower.** Recall that in the abelian category $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau$-**Comod**, monomorphisms, epimorphisms and exactness are defined on the underlying abelian groups.

The following Lemma 5.1 is a consequence of Proposition 3.5 and the homology version of Dugger-Isaksen's the universal coefficient spectral sequence [12, Proposition 7.7].

Lemma 5.1. *For any injective module $N \in \mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$-$\mathbf{Mod}^0$, there exists $I \in \mathrm{MU}^{\mathrm{mot}}/\tau$-$\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}}$ such that*

$$\pi_{*,*}I \cong N,$$

and that

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}I \cong \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau \otimes_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau} N,$$

which is an injective $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau$-comodule.

Proof. By Proposition 3.5, for $N \in \mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$-$\mathbf{Mod}^0$, there exists an essentially unique cellular $\mathrm{MU}^{\mathrm{mot}}/\tau$-module I with the property that $\pi_{*,*}I \cong N$.

For the second condition, we have the equivalences

$$\begin{aligned}\mathrm{MU}^{\mathrm{mot}}/\tau \wedge_{C\tau} I &\simeq \mathrm{MU}^{\mathrm{mot}}/\tau \wedge_{C\tau} \left(\mathrm{MU}^{\mathrm{mot}}/\tau \wedge_{\mathrm{MU}^{\mathrm{mot}}/\tau} I\right)\\ &\simeq \left(\mathrm{MU}^{\mathrm{mot}}/\tau \wedge_{C\tau} \mathrm{MU}^{\mathrm{mot}}/\tau\right) \wedge_{\mathrm{MU}^{\mathrm{mot}}/\tau} I.\end{aligned}$$

Since $\mathrm{MU}^{\mathrm{mot}}/\tau$ is cellular, the homotopy groups of the last term can be computed by the homology version of Dugger-Isaksen's universal coefficient spectral sequence [12, Proposition 7.7]

$$\mathrm{Tor}^{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau}_{s,t,w}\left(\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau, \pi_{*,*}I\right) \Longrightarrow \pi_{t+s,w}\left(\mathrm{MU}^{\mathrm{mot}}/\tau \wedge_{C\tau} \mathrm{MU}^{\mathrm{mot}}/\tau \wedge_{\mathrm{MU}^{\mathrm{mot}}/\tau} I\right)$$

in the category $\mathrm{MU}^{\mathrm{mot}}/\tau$-$\mathbf{Mod}_{\mathrm{cell}}$.

Note that $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau$ is free and $\pi_{*,*}I$ is injective over $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$, the spectral sequence is therefore concentrated on the line $s = 0$ and collapses at the E_2-page. This gives the following isomorphisms of comodules

$$\begin{aligned}
\mathrm{MU}^{\mathrm{mot}}_{*,*}I &\cong \pi_{*,*}(\mathrm{MU}^{\mathrm{mot}}/\tau \wedge_{C\tau} I)\\
&\cong \pi_{*,*}(\mathrm{MU}^{\mathrm{mot}}/\tau \wedge_{C\tau} \mathrm{MU}^{\mathrm{mot}}/\tau \wedge_{\mathrm{MU}^{\mathrm{mot}}/\tau} I)\\
&\cong \mathrm{Tor}_{0,*,*}^{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau}\left(\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau, \pi_{*,*}I\right)\\
&\cong \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau \otimes_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau} \pi_{*,*}I\\
&\cong \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau \otimes_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau} N.
\end{aligned}$$

It is known that the comodules induced from injective modules are injective as comodules (see [53, Lemma A1.2.2] for example), this shows that $\mathrm{MU}^{\mathrm{mot}}_{*,*}I$ is an injective $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau$-comodule. □

Lemma 5.1 is our source of motivic $S^{0,0}/\tau$-modules whose $\mathrm{MU}^{\mathrm{mot}}/\tau$-homology is injective as a comodule.

Lemma 5.2. *Suppose that I satisfies the conclusions of Lemma 5.1. Then for any $X \in S^{0,0}/\tau$-$\mathbf{Mod}^b_{\mathrm{cell}}$, the abelian group of homotopy classes of degree $(0,0)$ maps can be computed algebraically by the following isomorphism*

$$[X, I]_{S^{0,0}/\tau} \cong \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}I).$$

Proof. The lemma follows from the following isomorphisms

$$\begin{aligned}
[X, I]_{S^{0,0}/\tau} &\cong [\mathrm{MU}^{\mathrm{mot}}/\tau \wedge_{C\tau} X, I]_{\mathrm{MU}^{\mathrm{mot}}/\tau}\\
&\cong \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \pi_{*,*}I)\\
&\cong \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, N)\\
&\cong \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau \otimes_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau} N)\\
&\cong \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}I).
\end{aligned}$$

In fact, the first isomorphism follows from the adjunction (3.2) between $S^{0,0}/\tau$-modules and $\mathrm{MU}^{\mathrm{mot}}/\tau$-modules. The third and last isomorphisms follow from Lemma 5.1. The fourth isomorphism follows from a change-of-ring isomorphism. It remains to show the second isomorphism.

Since both I and $\mathrm{MU}^{\mathrm{mot}}/\tau \wedge_{C\tau} X$ belong to $\mathrm{MU}^{\mathrm{mot}}/\tau$-$\mathbf{Mod}_{\mathrm{cell}}$, the set of homotopy classes of maps

$$[\mathrm{MU}^{\mathrm{mot}}/\tau \wedge_{C\tau} X, I]_{\mathrm{MU}^{\mathrm{mot}}/\tau}$$

can be computed by the universal coefficient spectral sequence of Theorem 3.2

$$\mathrm{Ext}^{s,t,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \pi_{*,*}I) \Longrightarrow [\Sigma^{t-s,w}\mathrm{MU}^{\mathrm{mot}}/\tau \wedge_{C\tau} X, I]_{\mathrm{MU}^{\mathrm{mot}}/\tau}.$$

Since $\pi_{*,*}I \cong N$ is an injective $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$-module, the spectral sequence is concentrated on the line $s = 0$ and collapses at the E_2-page. This gives the second isomorphism. $\square$

Lemma 5.3. *For any* $M \in \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau$-**Comod** *that is concentrated in Chow degree* k*, there exists a monomorphism*

$$M \hookrightarrow \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau \otimes_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau} N,$$

where N *is injective in* $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$-**Mod** *and is concentrated in Chow degree* k.

Proof. This proof is the standard way of showing that an abelian category has enough injectives, by inducing them from $\mathbb{Z}$-modules (or $\mathbb{Z}_p$ in our case). We start with the monomorphism of bigraded $\mathbb{Z}_p$-modules

$$M \hookrightarrow \prod_{x \in M \setminus 0} \Sigma^{|x|}\mathbb{Q}_p/\mathbb{Z}_p, \tag{5.1}$$

where $\Sigma^{|x|}$ denotes a shift by the bidegree of x. The target is an injective $\mathbb{Z}_p$-module that is concentrated in Chow degree k. Adjoint the above map through the two adjunctions

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod} \underset{\text{ext.}}{\overset{\text{res.}}{\rightleftarrows}} \mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod} \underset{\text{coext.}}{\overset{\text{res.}}{\rightleftarrows}} \mathbb{Z}_p\text{-}\mathbf{Mod},$$

we have a monomorphism

$$M \hookrightarrow \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau \otimes_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau} \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau, \prod_{x \in M \setminus 0} \Sigma^{|x|}\mathbb{Q}_p/\mathbb{Z}_p) \tag{5.2}$$

of comodules. The target is injective since right adjoints preserve injectives. The map (5.2) is a monomorphism, since post-composing it with the two counit maps recovers the monomorphism (5.1). $\square$

Proposition 5.4. *Any* $Y \in S^{0,0}/\tau$-$\mathbf{Mod}^b_{\mathrm{cell}}$ *admits an* **absolute** *Adams-Novikov tower*

$$\begin{array}{ccccccc} Y = & Y_0 & \longleftarrow & Y_1 & \longleftarrow & Y_2 & \longleftarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ & I_0 & & I_1 & & I_2 & \end{array}$$

in the category $S^{0,0}/\tau$-$\mathbf{Mod}^b_{\mathrm{cell}}$*, such that*

(1) each map $Y_s \longrightarrow Y_{s-1}$ *induces a zero homomorphism in* $\mathrm{MU}^{\mathrm{mot}}$*-homology,*

(2) each cofiber I_s *is a finite product of suspensions of objects that satisfy the conclusions of Lemma 5.1.*

Moreover, any map $f\colon X \longrightarrow Y$ *in* $S^{0,0}/\tau$-$\mathbf{Mod}^b_{\mathrm{cell}}$ *can be lifted to a map of absolute Adams-Novikov towers.*

Proof. Suppose that $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y$ is concentrated in Chow degrees $[a,b]$, namely

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}Y \cong \bigoplus_{k=a}^{b}\bigoplus_{l=-\infty}^{+\infty} \mathrm{MU}^{\mathrm{mot}}_{2l+k,l}Y.$$

By Lemma 5.3, for every $k \in [a,b]$, there exists a monomorphism

$$\bigoplus_{l=-\infty}^{+\infty} \mathrm{MU}^{\mathrm{mot}}_{2l+k,l}(Y) \cong \bigoplus_{l=-\infty}^{+\infty} \mathrm{MU}^{\mathrm{mot}}_{2l,l}(\Sigma^{-k,0}Y) \hookrightarrow \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau \otimes_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau} N_{0,k}$$

where $N_{0,k}$ is injective module that is concentrated in Chow degree 0. By Lemma 5.1, there exists a spectrum $I_{0,k} \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}}$ such that

$$\pi_{*,*}I_{0,k} \cong N_{0,k},$$

and that

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}I_{0,k} \cong \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau \otimes_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau} N_{0,k}.$$

By Lemma 5.2, we have that

$$\begin{aligned}
[\Sigma^{-k,0}Y, I_{0,k}]_{S^{0,0}/\tau} &\cong \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{-k,0}Y), \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau \otimes_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau} N_{0,k}) \\
&\cong \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\bigoplus_{l=-\infty}^{+\infty} \mathrm{MU}^{\mathrm{mot}}_{2l,l}(\Sigma^{-k,0}Y), \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau \otimes_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau} N_{0,k}).
\end{aligned}$$

The second isomorphism follows from the fact that $N_{0,k}$ is concentrated in Chow degree 0. Therefore, the algebraic map of comodules

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{-k,0}Y) \twoheadrightarrow \bigoplus_{l=-\infty}^{+\infty} \mathrm{MU}^{\mathrm{mot}}_{2l,l}(\Sigma^{-k,0}Y) \hookrightarrow \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau \otimes_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau} N_{0,k},$$

where the first map is the project map to the Chow degree 0 part, can be realized as a $S^{0,0}/\tau$-linear map

$$\Sigma^{-k,0}Y \longrightarrow I_{0,k}.$$

Combine these maps for all $k \in [a,b]$, we obtain a map

$$Y \longrightarrow \prod_{k=a}^{b} \Sigma^{k,0} I_{0,k}.$$

Note that this map induces a monomorphism in $\mathrm{MU}^{\mathrm{mot}}$-homology.

Denote the finite product by

$$I_0 := \prod_{k=a}^{b} \Sigma^{k,0} I_{0,k},$$

and the fiber of the map $Y \to I_0$ by Y_1, as in

$$\begin{tikzcd} Y \arrow[d] & Y_1 \arrow[l] \\ I_0. & \end{tikzcd}$$

By the associated long exact sequence in $\mathrm{MU}^{\mathrm{mot}}$-homology, the map $Y_1 \longrightarrow Y$ induces the zero map in $\mathrm{MU}^{\mathrm{mot}}$-homology, and $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y_1$ is concentrated in Chow degrees $[a-1, b-1]$. So in particular we have

$$Y_1 \in S^{0,0}/\tau\text{-}\mathbf{Mod}^b_{\mathrm{cell}}.$$

We can repeat the procedure, producing an absolute Adams-Novikov tower

$$\begin{tikzcd} Y \arrow[d] & Y_1 \arrow[l] \arrow[d] & Y_2 \arrow[l] \arrow[d] & \cdots \arrow[l] \\ I_0 & I_1 & I_2 & \end{tikzcd}$$

satisfying the desired properties.

We now prove the second claim of the theorem. For any $S^{0,0}/\tau$-linear map $f_0 : X_0 \to Y_0$, we may assume that $\mathrm{MU}^{\mathrm{mot}}_{*,*}X_0$ and $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y_0$ are both concentrated in Chow degrees $[a, b]$. Denote the first step of their tower by

$$\begin{tikzcd} X_0 \arrow[r, "f_0"] \arrow[d] & Y_0 \arrow[d] \\ I_0 & J_0, \end{tikzcd}$$

where I_0 and J_0 are the finite products of suspensions of objects that satisfy the conclusions of Lemma 5.1. Applying $\mathrm{MU}^{\mathrm{mot}}_{*,*}$, we have the following diagram of $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau$-comodules

$$\begin{tikzcd} \mathrm{MU}^{\mathrm{mot}}_{*,*}X_0 \arrow[r, "f_{0*,*}"] \arrow[d, hook] & \mathrm{MU}^{\mathrm{mot}}_{*,*}Y_0 \arrow[d, hook] \\ \mathrm{MU}^{\mathrm{mot}}_{*,*}I_0 \arrow[r, dashed, "\phi"] & \mathrm{MU}^{\mathrm{mot}}_{*,*}J_0. \end{tikzcd}$$

Here the existence of the homomorphism ϕ is due to the universal property of injective objects in the category $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod}$.

Note that

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}I_0 = \mathrm{MU}^{\mathrm{mot}}_{*,*}\Big(\prod_{k=a}^{b} \Sigma^{k,0} I_{0,k}\Big) = \prod_{k=a}^{b} \mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{k,0} I_{0,k}),$$

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}J_0 = \mathrm{MU}^{\mathrm{mot}}_{*,*}(\prod_{k=a}^{b}\Sigma^{k,0}J_{0,k}) = \prod_{k=a}^{b}\mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{k,0}J_{0,k}).$$

The Chow degree k parts of $\mathrm{MU}^{\mathrm{mot}}_{*,*}I_0$ and $\mathrm{MU}^{\mathrm{mot}}_{*,*}J_0$ are given by

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{k,0}I_{0,k}),\ \mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{k,0}J_{0,k}).$$

Therefore, the homomorphism ϕ is given by the product of homomorphisms

$$\phi_k : \mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{k,0}I_{0,k}) \longrightarrow \mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{k,0}J_{0,k})$$

for each $k \in [a,b]$.

Since $J_{0,k}$ satisfies the conclusions of Lemma 5.1, we have that

$$\begin{aligned}
[\Sigma^{k,0}I_{0,k}, \Sigma^{k,0}J_{0,k}]_{S^{0,0}/\tau} &\cong [I_{0,k}, J_{0,k}]_{S^{0,0}/\tau} \\
&\cong \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}I_{0,k}, \mathrm{MU}^{\mathrm{mot}}_{*,*}J_{0,k}) \\
&\cong \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{k,0}I_{0,k}), \mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{k,0}J_{0,k})),
\end{aligned}$$

where the second isomorphism is given by Lemma 5.2. Therefore the homomorphism ϕ_k can be realized by a $S^{0,0}/\tau$-linear map

$$g_{0,k} : \Sigma^{k,0}I_{0,k} \longrightarrow \Sigma^{k,0}J_{0,k}.$$

Taking the product of $g_{0,k}$ for all $k \in [a,b]$, we define a map $g_0 : I_0 \to J_0$. Then g_0 realizes ϕ, and we have the diagram

$$\begin{tikzcd}
X_0 \arrow[r, "f_0"] \arrow[d] & Y_0 \arrow[d] \\
I_0 \arrow[r, "g_0"] & J_0.
\end{tikzcd}$$

To see that the square commutes up to homotopy, note that

$$\begin{aligned}
[X, J_0]_{S^{0,0}/\tau} &\cong [X, \prod_{k=a}^{b} \Sigma^{k,0} J_{0,k}]_{S^{0,0}/\tau} \\
&\cong \prod_{k=a}^{b} [X, \Sigma^{k,0} J_{0,k}]_{S^{0,0}/\tau} \\
&\cong \prod_{k=a}^{b} [\Sigma^{-k,0} X, J_{0,k}]_{S^{0,0}/\tau} \\
&\cong \prod_{k=a}^{b} \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{-k,0} X), \mathrm{MU}^{\mathrm{mot}}_{*,*} J_{0,k}) \\
&\cong \prod_{k=a}^{b} \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*} X, \mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{k,0} J_{0,k})) \\
&\cong \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*} X, \mathrm{MU}^{\mathrm{mot}}_{*,*}(\prod_{k=a}^{b} \Sigma^{k,0} J_{0,k})) \\
&\cong \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*} X, \mathrm{MU}^{\mathrm{mot}}_{*,*} J_0),
\end{aligned}$$

where the fourth isomorphism is given by Lemma 5.2.

Therefore, the commutativity of this square follows from the commutativity of the corresponding square in $\mathrm{MU}^{\mathrm{mot}}$-homology.

The commutative diagram in $S^{0,0}/\tau$-$\mathbf{Mod}^{b}_{\mathrm{cell}}$ induces a map $f_1 : X_1 \to Y_1$ between the fibers, so the following diagram commutes up to homotopy

$$\begin{tikzcd}
X \arrow[d] \arrow[dd, bend right=50, "f_0"'] & X_1 \arrow[l] \arrow[d] \arrow[dd, dashed, bend right=50, "f_1"'] & X_2 \arrow[l] & \cdots \arrow[l] \\
I_0 \arrow[dd, bend right=50, "g_0"'] & I_1 & & \\
Y \arrow[d] & Y_1 \arrow[l] \arrow[d] & Y_2 \arrow[l] & \cdots \arrow[l] \\
J_0 & J_1. & &
\end{tikzcd}$$

Iterating this process produces the desired map of absolute Adams-Novikov towers. $\square$

5.2. **The spectral sequence.** Every absolute Adams-Novikov tower gives rise to a $S^{0,0}/\tau$-linear Adams-Novikov spectral sequence. In the following Theorem 5.5, we identify the E_2-page of the spectral sequence and its abutment. We also show that it does not depend on the absolute Adams-Novikov tower, and converges strongly for objects with bounded Chow degree.

Theorem 5.5. *For $X, Y \in S^{0,0}/\tau\text{-}\mathbf{Mod}^b_{\mathrm{cell}}$, there is an absolute Adams-Novikov spectral sequence*

$$E_2^{s,t,w} \cong \mathrm{Ext}^{s,t,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}\left(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y\right) \Longrightarrow \left[\Sigma^{t-s,w}X, Y^\wedge_{\mathrm{MU}^{\mathrm{mot}}}\right]_{S^{0,0}/\tau},$$

with differentials

$$d_r\colon E_r^{s,t,w} \longrightarrow E_r^{s+r,t+r-1,w},$$

that does not depend on the absolute Adams-Novikov tower. Here $Y^\wedge_{\mathrm{MU}^{\mathrm{mot}}}$ is the $\mathrm{MU}^{\mathrm{mot}}$*-completion of Y. Moreover, this spectral sequence converges strongly and collapses at a finite page.*

Proof. There are 4 parts in this proof. In part (1), we show the existence of the spectral sequence from the absolute Adams-Novikov tower constructed in Proposition 5.4. In part (2), we show that this spectral sequence converges strongly and collapses at a finite page. In part (3), We show that the E_2-page of this spectral sequence does not depend on the absolute Adams-Novikov tower. In part (4), We show that the spectral sequence converges strongly and collapses at a finite page.

(1) By Proposition 5.4, there exists an absolute Adams-Novikov tower.

$$\begin{tikzcd} Y \arrow[d] & Y_1 \arrow[l] \arrow[d] & Y_2 \arrow[l] \arrow[d] & \cdots \arrow[l] \\ I_0 & I_1 & I_2. \end{tikzcd}$$

This gives a sequence of maps in $S^{0,0}/\tau\text{-}\mathbf{Mod}^b_{\mathrm{cell}}$:

$$Y \longrightarrow I_0 \xrightarrow{d_1} \Sigma^{1,0}I_1 \xrightarrow{d_1} \Sigma^{2,0}I_2 \longrightarrow \cdots.$$

By the construction of the tower, the $\mathrm{MU}^{\mathrm{mot}}$-homology of this sequence is an absolute injective resolution of the comodule $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y$.

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}Y \longrightarrow \mathrm{MU}^{\mathrm{mot}}_{*,*}I_0 \xrightarrow{d_1} \mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{1,0}I_1) \xrightarrow{d_1} \mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{2,0}I_2) \longrightarrow \cdots$$

Applying the functor $[X, -]_{S^{0,0}/\tau}$ to the absolute Adams-Novikov tower, we obtain an exact couple that gives the desired Adams-Novikov spectral sequence. The E_1-page is given by

$$E_1^{s,t,w} := [\Sigma^{t,w} X, \Sigma^{s,0} I_s]_{S^{0,0}/\tau},$$

and the d_1 differentials are given in the cochain complex

$$[X, I_0]_{S^{0,0}/\tau} \xrightarrow{d_1} [X, \Sigma^{1,0} I_1]_{S^{0,0}/\tau} \xrightarrow{d_1} [X, \Sigma^{2,0} I_2]_{S^{0,0}/\tau} \longrightarrow \cdots .$$

The cohomology of this cochain complex computes the E_2-page, which we now identify.

Since each I_j is a finite product of suspensions of objects that satisfy the conclusions of Lemma 5.1, we have

$$[X, \Sigma^{j,0} I_j]_{S^{0,0}/\tau} \cong \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*} X, \mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{j,0} I_j)),$$

by Lemma 5.2 and the proof of Proposition 5.4. Then the cochain complex can be identified as

$$\mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*} X, \mathrm{MU}^{\mathrm{mot}}_{*,*} I_0) \xrightarrow{d_1} \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*} X, \mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{1,0} I_1))$$

$$\xrightarrow{d_1} \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*} X, \mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{2,0} I_2)) \longrightarrow \cdots .$$

The d_1-differentials agree with the ones obtained by standard methods (see [53, Chapter 2] for example). Therefore the E_2-page is given by

$$E_2^{s,t,w} \cong \mathrm{Ext}^{s,t,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*} X, \mathrm{MU}^{\mathrm{mot}}_{*,*} Y).$$

(2) We next show that the spectral sequence converges strongly and collapses at a finite page, under the hypotheses that both $\mathrm{MU}^{\mathrm{mot}}_{*,*} X$ and $\mathrm{MU}^{\mathrm{mot}}_{*,*} Y$ are concentrated in bounded Chow degrees. This argument is similar to the one given in the proof of Theorem 3.2.

Suppose that $\mathrm{MU}^{\mathrm{mot}}_{*,*} X$ and $\mathrm{MU}^{\mathrm{mot}}_{*,*} Y$ are concentrated in Chow degrees $[a, b]$ and $[c, d]$ respectively, where $a \leq b$ and $c \leq d$. Then $\mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{t,w} X)$ is concentrated in Chow degrees

$$[a + (t - 2w), b + (t - 2w)].$$

From the construction of the absolute Adams-Novikov tower in Proposition 5.4, it follows that $\mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{s,0} I_s)$ are concentrated in Chow degrees $[c, d]$ for all $s \geq 0$. In order for the group

$$E_1^{s,t,w} \cong [\Sigma^{t,w} X, \Sigma^{s,0} I_s]_{S^{0,0}/\tau} \cong \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{t,w} X), \mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{s,0} I_s)),$$

to be nonzero, we must have

$$c \leq b + (t - 2w), \quad d \geq a + (t - 2w).$$

For a fixed weight w, this gives that

$$t \in [c-b+2w, d-a+2w].$$

Since later pages $E_r^{s,t,w}$ are iterated subquotients of $E_1^{s,t,w}$, their t-degrees are all concentrated in $[c-b+2w, d-a+2w]$.

It is standard to check that the d_r-differentials has the form

$$d_r\colon E_r^{s,t,w} \longrightarrow E_r^{s+r,t+r-1,w}.$$

In particular, it changes the t-degrees by $r-1$. Since the t-degrees of all possible nonzero elements in the E_1-page satisfy $t \in [c-b+2w, d-a+2w]$, we must have $d_r = 0$ when

$$r-1 > (d-a+2w) - (c-b+2w) = (b-a) + (d-c)$$

for degree reasons. In other words, the spectral sequence collapses at the $E_{(b-a)+(d-c)+2}$ page.

Therefore, under the condition that both $\mathrm{MU}^{\mathrm{mot}}_{*,*}X$ and $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y$ are concentrated in bounded Chow degrees, this spectral sequence convergences strongly and collapses at a finite page.

(3) We show that the E_2-page of this spectral sequence does not depend on the absolute Adams-Novikov tower. Consider two absolute injective resolutions of $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y$ from two absolute Adams-Novikov towers $\{Y_s, I_s\}$ and $\{Y'_s, I'_s\}$ of Y. By Proposition 5.4, the identity map $\mathrm{id}\colon Y \longrightarrow Y$ produces a map of towers, and in particular compatible maps $g_s\colon I_s \longrightarrow I'_s$ for all $s \geq 0$. These maps induce a lift of the identity map between the two absolute injective resolutions of $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y$ as in the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{MU}^{\mathrm{mot}}_{*,*}Y \arrow[r] \arrow[d, equal, "\mathrm{id}"] & \mathrm{MU}^{\mathrm{mot}}_{*,*}I_0 \arrow[r] \arrow[d, "\mathrm{MU}^{\mathrm{mot}}_{*,*}(g_0)"] & \mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{1,0}I_1) \arrow[r] \arrow[d, "\mathrm{MU}^{\mathrm{mot}}_{*,*}(g_1)"] & \cdots \\
0 \arrow[r] & \mathrm{MU}^{\mathrm{mot}}_{*,*}Y \arrow[r] & \mathrm{MU}^{\mathrm{mot}}_{*,*}I'_0 \arrow[r] & \mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{1,0}I'_1) \arrow[r] & \cdots .
\end{tikzcd}$$

The maps $\mathrm{MU}^{\mathrm{mot}}_{*,*}(g_s)$ induce isomorphisms on the E_2-pages, and therefore an isomorphism of spectral sequences by standard arguments in homological algebra (see [5, Theorem 5.3] and [53, Section 2.2] for example).

(4) For the convergence problem, let Y/Y_s be the cofiber of the map $Y_s \to Y$ in the absolute Adams-Novikov tower, and define the homotopy colimit in $S^{0,0}/\tau\text{-}\mathbf{Mod}^b_{\mathrm{cell}}$

$$\widehat{Y} = \mathrm{holim}(Y/Y_s).$$

By [5], the spectral sequence converges conditionally to

$$[X, \widehat{Y}]_{S^{0,0}/\tau}.$$

To identify it as $[X, Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}]_{S^{0,0}/\tau}$, since X is cellular, we only need to show that $\widehat{Y}$ has the same homotopy groups as the $Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}$.

Take $X = S^{0,0}/\tau$. Since $\mathrm{MU}^{\mathrm{mot}}_{*,*}S^{0,0}/\tau = \mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$ is free over itself, we can use the canonical $\mathrm{MU}^{\mathrm{mot}}/\tau$-Adams resolution [53, Definition 2.2.10] for Y in this case. Now we compare the canonical $\mathrm{MU}^{\mathrm{mot}}/\tau$-based Adams-Novikov tower of Y with the absolute Adams-Novikov tower of Y.

As we did in the proof of Proposition 5.4, we have a map of towers from the canonical $\mathrm{MU}^{\mathrm{mot}}/\tau$-based one to the absolute one. As we did in part (3), the identity map on Y induces a homomorphism from the canonical cobar resolution of $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y$ to the absolute injective resolution of $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y$, so in particular a homomorphism of relative injective resolutions.

This induces a homomorphism from the usual Adams-Novikov spectral sequence to the absolute Adams-Novikov spectral sequence, with an isomorphism on the E_2-page. It is therefore an isomorphism of spectral sequences and we have a weak equivalence

$$\widehat{Y} \xrightarrow{\simeq} Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}.$$

Since any cellular $S^{0,0}/\tau$-module X can be written in terms of filtered colimits and cofibers of suspensions of $S^{0,0}/\tau$'s, there is an isomorphism

$$[X, \widehat{Y}]_{S^{0,0}/\tau} \xrightarrow{\cong} [X, Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}]_{S^{0,0}/\tau}.$$

Therefore, the absolute Adams-Novikov spectral sequence computes $[X, Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}]_{S^{0,0}/\tau}$. □

When Y is harmonic, the weak equivalence $Y \to Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}$ gives the following corollary.

Corollary 5.6. *For any $X, Y \in S^{0,0}/\tau\text{-}\mathbf{Mod}^{b}_{\clubsuit}$, there is an absolute Adams-Novikov spectral sequence*

$$E_2^{s,t,w} = \mathrm{Ext}^{s,t,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}\left(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y\right) \Longrightarrow \left[\Sigma^{t-s,w}X, Y\right]_{S^{0,0}/\tau},$$

with differentials

$$d_r \colon E_r^{s,t,w} \longrightarrow E_r^{s+r,t+r-1,w},$$

that converges strongly and collapses at a finite page.

Remark 5.7. The above arguments can be applied to the case of the classical stable homotopy category to construct the general Adams-Novikov spectral sequence, which seems to be a folklore theorem without published references.

For classical spectra X and Y, there is a conditionally convergent spectral sequence

$$\mathrm{Ext}^{s,t}_{\mathrm{MU}_*\mathrm{MU}}(\mathrm{MU}_*X, \mathrm{MU}_*Y) \Longrightarrow [\Sigma^{t-s}X, Y^{\wedge}_{\mathrm{MU}}],$$

where MU_*X does *not* have to be projective over MU_*. The difference is in the proof of Lemma 5.1, where we apply the Brown representability theorem instead.

5.3. **Proofs of Lemma 4.2, Corollary 4.7 and Corollary 4.8.** We give the proofs of Lemma 4.2, Corollary 4.7 and Corollary 4.8 in this section.

Corollary 4.7 states that if $X \in S^{0,0}/\tau\text{-}\mathbf{Mod}_{\clubsuit}^{b,\geq 0}$ and $Y \in S^{0,0}/\tau\text{-}\mathbf{Mod}_{\clubsuit}^{b,\leq 0}$, then the abelian group of homotopy classes of degree $(0,0)$ maps can be computed algebraically by the isomorphism

$$[X,Y]_{S^{0,0}/\tau} \longrightarrow \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y)$$

that is induced by applying $\mathrm{MU}^{\mathrm{mot}}_{*,*}$.

Proof of Corollary 4.7. The proof is similar to the one of Corollary 3.3.

Consider the the E_2-page of the absolute Adams-Novikov spectral sequence, the tridegrees that converge to the bidegree $(0,0)$ are of the form $(t,t,0)$ for $t \geq 0$, i.e., the parts $E_2^{s,t,w} = E_2^{t,t,0}$.

By the proof of Theorem 5.5, the t-degrees of all possible nonzero elements in the E_1-page and therefore E_2-page satisfy $t \leq d - a + 2w = d - a$. Since $\mathrm{MU}^{\mathrm{mot}}_{*,*}X$ and $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y$ are concentrated in nonnegative and nonpositive bounded Chow degrees, we have $d = a = 0$. Therefore, we have $t \leq 0$.

Combining both facts, we have established that the only possible nonzero elements in the E_2-page that converge to the bidegree $(0,0)$ are in

$$E_2^{0,0,0} = \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y).$$

To show that all elements in $E_2^{0,0,0}$ survive in the spectral sequence, firstly note that they are not targets of any nonzero differentials since they are in s-degree 0. Secondly, all d_r-differentials for $r \geq 2$ increase the t-degree. Since the t-degrees of all nonzero elements are non-positive, the elements in $E_2^{0,0,0}$ do not support nonzero differentials. There are no hidden extensions due to degree reasons. This completes the proof. $\square$

Corollary 4.8 states that given $X, Y \in S^{0,0}/\tau\text{-}\mathbf{Mod}_{\clubsuit}^{\heartsuit}$, for any bidegree (t,w), there is an isomorphism

$$[\Sigma^{t,w}X, Y]_{S^{0,0}/\tau} \cong \mathrm{Ext}^{2w-t,2w,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y).$$

Proof of Corollary 4.8. Consider the the E_2-page of the absolute Adams-Novikov spectral sequence. Since both $\mathrm{MU}^{\mathrm{mot}}_{*,*}X$ and $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y$ are concentrated in Chow degree 0, the E_2-page

$$E_2^{s,t,w} = \mathrm{Ext}^{s,t,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}\left(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y\right)$$

is concentrated in degrees $t = 2w$. Since all differentials preserves the motivic weights w, this spectral sequence collapses at the E_2-page. There are no hidden extensions due to degree reasons. Therefore, we have the isomorphism

$$[\Sigma^{t,w}X, Y]_{S^{0,0}/\tau} \cong \mathrm{Ext}^{2w-t,2w,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y).$$

□

We now prove Lemma 4.2, which states that if $\{Y_\alpha\}$ is a filtered system in $S^{0,0}/\tau$-$\mathbf{Mod}^{\heartsuit}$ such that each Y_α is harmonic, then the homotopy colimit $\{Y_\alpha\}$ in $S^{0,0}/\tau$-$\mathbf{Mod}^{\heartsuit}$ is also harmonic.

Proof of Lemma 4.2. Consider the absolute Adams-Novikov spectral sequence of Theorem 5.5

$$\mathrm{Ext}^{s,t,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}\left(\mathrm{MU}^{\mathrm{mot}}_{*,*}S^{0,0}/\tau, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y\right) \Longrightarrow \left[\Sigma^{t-s,w}S^{0,0}/\tau, Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}\right]_{S^{0,0}/\tau} \cong \pi_{t-s,w}Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}$$

in the case that $X = S^{0,0}/\tau$ and $Y = \operatorname{hocolim} Y_\alpha$. Since both $S^{0,0}/\tau$ and Y are in the heart, the E_2-page is concentrated in degrees $t = 2w$. Since all differentials preserves the motivic weights w, this spectral sequence collapses at the E_2-page. There are no hidden extensions due to degree reasons. Therefore, we have the isomorphism

$$\pi_{t,w}Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}} \cong [\Sigma^{t,w}S^{0,0}/\tau, Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}]_{S^{0,0}/\tau} \cong \mathrm{Ext}^{2w-t,2w,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}S^{0,0}/\tau, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y).$$

Since $\mathrm{MU}^{\mathrm{mot}}_{*,*}S^{0,0}/\tau \cong \mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$ is free over $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$, one can use the canonical cobar resolution for $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y$. Since it is functorial and commutes with filtered colimits, the isomorphism

$$\operatorname{colim} \mathrm{MU}^{\mathrm{mot}}_{*,*}Y_\alpha \cong \mathrm{MU}^{\mathrm{mot}}_{*,*}(Y)$$

induces an isomorphism

$$\operatorname{colim} \mathrm{Ext}^{*,*,*}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}\left(\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y_\alpha\right) \cong \mathrm{Ext}^{*,*,*}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}\left(\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y\right).$$

Therefore, we have the following isomorphisms

$$\begin{aligned}
\pi_{t,w}Y &\cong \pi_{t,w}(\text{holim } Y_\alpha) \\
&\cong \text{colim } \pi_{t,w}Y_\alpha \\
&\cong \text{colim } [\Sigma^{t,w}S^{0,0}/\tau, Y_\alpha]_{S^{0,0}/\tau} \\
&\cong \text{colim}\, \mathrm{Ext}^{2w-t,2w,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}\left(\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y_\alpha\right) \\
&\cong \mathrm{Ext}^{2w-t,2w,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}\left(\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y\right) \\
&\cong [\Sigma^{t,w}S^{0,0}/\tau, Y^\wedge_{\mathrm{MU}^{\mathrm{mot}}}]_{S^{0,0}/\tau} \\
&\cong \pi_{t,w}Y^\wedge_{\mathrm{MU}^{\mathrm{mot}}},
\end{aligned}$$

where the fourth isomorphism is given by Corollary 4.8, since each Y_α is harmonic. This shows that Y is harmonic. □

Remark 5.8. Lemma 4.2 can be generalized to the case when there is a uniform bound on the Chow degrees of $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y_\alpha$ for all α.

6. MU$_*$MU-comodules and BP$_*$BP-comodules

In Part 1, we work with $\mathrm{MU}^{\mathrm{mot}}$, the motivic analogue of p-completed MU instead of BP. This is convenient since MU is an E_∞-ring spectra while a recent of result of Lawson [32] shows that BP is not. However, in Part 2, we work with BP$_*$BP-comodules instead of MU$_*$MU-comodules since they are more convenient for computational purpose. We present a brief account of the well known Morita equivalence of the two abelian categories of MU$_*$MU-comodules and BP$_*$BP-comodules in this section.

Let

$$\mathrm{MU}_*\mathrm{BP} = \pi_*(\mathrm{MU} \wedge \mathrm{BP}),$$
$$\mathrm{BP}_*\mathrm{MU} = \pi_*(\mathrm{BP} \wedge \mathrm{MU}).$$

Then MU$_*$BP is a MU$_*$MU-BP$_*$BP-bi-comodule, and BP$_*$MU is a BP$_*$BP-MU$_*$MU-bi-comodule.

Lemma 6.1.

(1) MU$_*$BP *is a relative injective left* MU$_*$MU-*comodule and a relative injective right* BP$_*$BP-*comodule.*

(2) BP$_*$MU *is a relative injective right* MU$_*$MU-*comodule and a relative injective left* BP$_*$BP-*comodule.*

Proof. Note that MU is a wedge of suspensions of BP's. Therefore, as a left MU$_*$MU-comodule, MU$_*$BP is a direct summand of MU$_*$MU. As a right BP$_*$BP-comodule, MU$_*$BP is a direct sum of BP$_*$BP. This proves the lemma. □

Lemma 6.2.

(1) There is an isomorphism of $\mathrm{MU}_*\mathrm{MU}$-$\mathrm{MU}_*\mathrm{MU}$-*bi-comodules*

$$\mathrm{MU}_*\mathrm{BP}\square_{\mathrm{BP}_*\mathrm{BP}}\mathrm{BP}_*\mathrm{MU} \cong \mathrm{MU}_*\mathrm{MU}.$$

(2) There is an isomorphism of $\mathrm{BP}_*\mathrm{BP}$-$\mathrm{BP}_*\mathrm{BP}$-*bi-comodules*

$$\mathrm{BP}_*\mathrm{MU}\square_{\mathrm{MU}_*\mathrm{MU}}\mathrm{MU}_*\mathrm{BP} \cong \mathrm{BP}_*\mathrm{BP}.$$

Proof. We recall the following dual version of the BP-based Adams-Novikov spectral sequence.

Let X and Y be spectra with free BP-homology. Then we have a spectral sequence converging to $\pi_*(X \wedge Y)$ with the E_2-page

$$\mathrm{Cotor}^{*,*}_{\mathrm{BP}_*\mathrm{BP}}(X_*\mathrm{BP}, \mathrm{BP}_*Y).$$

Since X and Y have free BP-homology, we have that

$$\mathrm{BP}_*(X \wedge Y) \cong \mathrm{BP}_*X \otimes_{\mathrm{BP}_*} \mathrm{BP}_*Y.$$

By definition (see Appendix 1 of Ravenel's green book [53]), the primitives of $\mathrm{BP}_*X \otimes_{\mathrm{BP}_*} \mathrm{BP}_*Y$ are canonically isomorphic to

$$X_*\mathrm{BP}\square_{\mathrm{BP}_*\mathrm{BP}}\mathrm{BP}_*Y,$$

whose derived functors are $\mathrm{Cotor}^{*,*}_{\mathrm{BP}_*\mathrm{BP}}(X_*\mathrm{BP}, \mathrm{BP}_*Y)$.

Since MU has free BP-homology, we take $X = Y = \mathrm{MU}$ and consider the BP-based Adams-Novikov spectral sequence that converges to $\pi_*(\mathrm{MU} \wedge \mathrm{MU})$.

By Lemma 6.1, $\mathrm{MU}_*\mathrm{BP}$ is a relative injective right $\mathrm{BP}_*\mathrm{BP}$-comodule, and $\mathrm{BP}_*\mathrm{MU}$ is a relative injective left $\mathrm{BP}_*\mathrm{BP}$-comodule. Therefore, all the higher derived functors vanish and the Adams-Novikov spectral sequence collapses at the E_2-page. We have that

$$\mathrm{MU}_*\mathrm{BP}\square_{\mathrm{BP}_*\mathrm{BP}}\mathrm{BP}_*\mathrm{MU} \cong \pi_*(\mathrm{MU} \wedge \mathrm{MU}) = \mathrm{MU}_*\mathrm{MU}.$$

Similarly, we consider the dual version of MU-based Adams-Novikov spectral sequence. Then a similar argument with the fact that BP has free MU-homology and Lemma 6.1 shows that

$$\mathrm{BP}_*\mathrm{MU}\square_{\mathrm{MU}_*\mathrm{MU}}\mathrm{MU}_*\mathrm{BP} \cong \pi_*(\mathrm{BP} \wedge \mathrm{BP}) = \mathrm{BP}_*\mathrm{BP}.$$

$\square$

Combining Lemma 6.1, Lemma 6.2 and the fact that both $\mathrm{MU}_*\mathrm{MU}\square_{\mathrm{MU}_*\mathrm{MU}}-$ and $\mathrm{BP}_*\mathrm{BP}\square_{\mathrm{BP}_*\mathrm{BP}}-$ are naturally equivalent to the identity functors, we have the following well known proposition.

Proposition 6.3. *There exist an exact equivalence of categories*

$$\mathrm{MU}_*\mathrm{BP}\square_{\mathrm{BP}_*\mathrm{BP}}- : \mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod} \longleftrightarrow \mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod} : \mathrm{BP}_*\mathrm{MU}\square_{\mathrm{MU}_*\mathrm{MU}} - .$$

7. Further questions

The category of cellular modules over $S^{0,0}/\tau$ measures the difference between cellular modules over the $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$-completed motivic sphere spectrum $S^{0,0}$ and cellular modules over the classical p-completed sphere spectrum S^0.

Definition 7.1. Let $S^{0,0}$-$\mathbf{Mod}_{\mathrm{fin}}$ be the category of finite cellular modules over $S^{0,0}$, and S^0-$\mathbf{Mod}_{\mathrm{fin}}$ be the category of classical finite cellular modules over S^0. Let $S^{0,0}$-$\mathbf{Mod}_{\mathrm{fin}}^{\tau\text{-tor}}$ be the full subcategory of $S^{0,0}$-$\mathbf{Mod}_{\mathrm{fin}}$ that is generated by $S^{0,0}/\tau$-$\mathbf{Mod}_{\mathrm{fin}}$ under cofibers, i.e. the smallest full subcategory containing objects of $S^{0,0}/\tau$-$\mathbf{Mod}_{\mathrm{fin}}$ and closed under taking cofibers.

It is straightforward to prove the following Proposition 7.2 from Dugger-Isaksen [11, Sections 3.2 and 3.4] and Isaksen [26, Proposition 3.0.2].

Proposition 7.2. *The sequence*

$$S^{0,0}\text{-}\mathbf{Mod}_{\mathrm{fin}}^{\tau\text{-tor}} \longrightarrow S^{0,0}\text{-}\mathbf{Mod}_{\mathrm{fin}} \xrightarrow{\mathbf{Re}} S^0\text{-}\mathbf{Mod}_{\mathrm{fin}}$$

is an exact sequence of stable ∞-categories in the sense of Blumberg-Gepner-Tabuada [4, Section 5], *where* **Re** *is the Betti realization functor constructed in Dugger-Isaksen* [13, Theorem 1.4].

In the sense of Proposition 7.2, our theorem 1.1 gives a decomposition of the cellular stable motivic category into more classical categories.

In particular, we can apply the non-connective algebraic K-theory functor $\mathbb{K}$ constructed in Blumberg-Gepner-Tabuada [4, Section 9], and get a cofiber sequence of non-connective algebraic K-theory spectra, since the functor $\mathbb{K}$ sends exact sequence of stable ∞-categories into cofiber sequences:

$$\mathbb{K}(S^{0,0}\text{-}\mathbf{Mod}_{\mathrm{fin}}^{\tau\text{-tor}}) \longrightarrow \mathbb{K}(S^{0,0}\text{-}\mathbf{Mod}_{\mathrm{fin}}) \xrightarrow{\mathbf{Re}} \mathbb{K}(S^0\text{-}\mathbf{Mod}_{\mathrm{fin}})$$

Since the Betti realization functor admits a section, the above cofiber sequence actually splits

$$\mathbb{K}(S^{0,0}\text{-}\mathbf{Mod}_{\mathrm{fin}}) \simeq \mathbb{K}(S^0\text{-}\mathbf{Mod}_{\mathrm{fin}}) \vee \mathbb{K}(S^{0,0}\text{-}\mathbf{Mod}_{\mathrm{fin}}^{\tau\text{-tor}}).$$

Note that the spectrum $\mathbb{K}(S^0\text{-}\mathbf{Mod}_{\mathrm{fin}})$ for the p-completed sphere spectrum is described by Bökstedt-Hsiang-Madsen [6].

To understand the spectrum $\mathbb{K}(S^{0,0}\text{-}\mathbf{Mod}_{\mathrm{fin}}^{\tau\text{-tor}})$, we consider the inclusion functor

$$S^{0,0}/\tau\text{-}\mathbf{Mod}_{\mathrm{fin}} \longrightarrow S^{0,0}\text{-}\mathbf{Mod}_{\mathrm{fin}}^{\tau\text{-tor}}. \tag{7.1}$$

We propose the following question.

Question 7.3. Does this inclusion functor (7.1) induce an equivalence on non-connective algebraic K-theory spectra?

Let $\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}_{\mathrm{fin}}$ be the subcategory of $\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}$ whose underlying BP_*-module is finitely presented. If the answer to Question 7.3 is yes, then by the theorem of heart due to Barwick [3] and Theorem 1.1, we have the following isomorphism for all i,

$$\mathbb{K}_i(S^{0,0}\text{-}\mathbf{Mod}_{\mathrm{fin}}) \cong \mathbb{K}_i(S^0\text{-}\mathbf{Mod}_{\mathrm{fin}}) \oplus \mathbb{K}_i(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}_{\mathrm{fin}}).$$

If we further regard the category $\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}_{\mathrm{fin}}$ as the category $\mathbf{Coh}(\mathcal{M}_{FG})$ of coherent sheaves over the moduli stack $\mathcal{M}_{FG}$ of formal groups [17], and the answer to Question 7.3 is yes, then there is an isomorphism for all i,

$$\mathbb{K}_i(S^{0,0}) \cong \mathbb{K}_i(S^0) \oplus \mathbb{K}_i(\mathbf{Coh}(\mathcal{M}_{FG})).$$

Part 2. Equivalence of spectral sequences

8. Main theorem of Part 2

The goal of Part 2 of this paper is to prove the following Theorem 8.1.

Theorem 8.1. *At each prime p, there is an isomorphism of tri-graded spectral sequences between the motivic Adams spectral sequence for $S^{0,0}/\tau$, which converges to the motivic homotopy groups of $S^{0,0}/\tau$, and the regraded algebraic Novikov spectral sequence, which converges to the Adams-Novikov E_2-page for the sphere spectrum.*

The indexes are indicated in the following diagram:

$$\begin{tikzcd}
\mathrm{Ext}^{s,2w}_{\mathrm{BP}_*\mathrm{BP}/I}(\mathrm{BP}_*/I, I^{a-s}/I^{a-s+1}) \arrow[r, "\cong"] \arrow[d, Rightarrow, "\textbf{Algebraic Novikov SS}"'] & \mathrm{Ext}^{a,2w-s+a,w}_{A^{mot}_{*,*}}(\mathbb{F}_p[\tau], \mathbb{F}_p) \arrow[d, Rightarrow, "\textbf{Motivic Adams SS}"] \\
\mathrm{Ext}^{s,2w}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*) \arrow[r, "\cong"] & \pi_{2w-s,w}S^{0,0}/\tau.
\end{tikzcd}$$

Here $I = (p, v_1, v_2, \cdots)$ is the augmentation ideal of BP_* and $A^{mot}_{*,*}$ is the motivic mod p dual Steenrod algebra.

The isomorphism between the abutments is known to Isaksen [26, Proposition 6.2.5] and the first author [15, Corollary 3.14].

Proposition 8.2 (Isaksen, Gheorghe)**.** *The motivic Adams-Novikov spectral sequence for $S^{0,0}/\tau$ collapses for filtration reasons, and there is an isomorphism of graded rings*

$$\pi_{2w-s,w}(S^{0,0}/\tau) \cong \mathrm{Ext}^{s,2w}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*)$$

Moreover, this isomorphism preserves higher products (Toda brackets and Massey products respectively).

Moreover, we have that

Proposition 8.3. *The isomorphism of spectral sequences in Theorem 8.1 on the abutments preserves composition products and higher products in the respective categories as in Proposition 8.2.*

Proof. Since the multiplicative structure on the abutments comes from composition of morphisms in both categories $S^{0,0}/\tau$-$\mathbf{Mod}^b_{\clubsuit}$ and $\mathcal{D}^b(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod})$, it follows from functorality of the equivalence of categories in Theorem 4.13. $\square$

A consequence of Proposition 8.3 is that The isomorphism of spectral sequences in Theorem 8.1 also preserves the filtrations on the E_∞-pages. Note that we do not prove that it also preserves the multiplicative structure on the E_2-pages. We shall prove it in future work.

Part 2 is organized in the following way. In Section 9, we recall the construction of the algebraic Novikov spectral sequence, discuss the regrading, and its associated tower. In Section 10, we prove in Theorem 8.1 that the two spectral sequences are isomorphic. We check that, through the equivalence of stable ∞-categories in Theorem 4.13 of Part 1, the algebraic Novikov tower in the derived category of $\mathrm{BP}_*\mathrm{BP}$-comodules corresponds to the motivic Adams tower of $S^{0,0}/\tau$ in the category of $S^{0,0}/\tau$-modules. In Section 10, we re-compute certain low filtration and historically more difficult differentials in the range up to the 45-stem at the prime 2, as an illustration of the power of the isomorphism of spectral sequences in Theorem 8.1.

9. Re-grading of the algebraic Novikov spectral sequence

The algebraic Novikov spectral sequence is introduced by Novikov [47] and Miller [41]. Ravenel's green book [53] and Andrews-Miller's paper [1] are also good references for this material. It is usually graded in the way that it starts with the E_1-page. To compare it with the motivic Adams spectral spectral, which is studied by Morel, Dugger-Isaksen and Hu-Kriz-Ormsby [11, 23, 43], we re-grade the algebraic Novikov spectral sequence in this section.

9.1. **The algebraic Novikov spectral sequence.** We first recall that the algebraic Novikov spectral sequence comes from filtering the cobar complex by powers of the augmentation ideal I on BP_* and $\mathrm{BP}_*\mathrm{BP}$.

Definition 9.1. Let M be a module over BP_*. For an element $m \in M$, we say that it has I-filtration at least $n \geq 0$ if $m \in I^n M$.

Since I is an invariant ideal of $\mathrm{BP}_*\mathrm{BP}$, the I-filtration gives a filtration of comodules on any $\mathrm{BP}_*\mathrm{BP}$-comodule. It is clear that the I-filtration on BP_*-modules and $\mathrm{BP}_*\mathrm{BP}$-comodules is a decreasing filtration.

We can form the associated graded $E_0\mathrm{BP}_*$ of BP_* with respect to the I-filtration

$$E_0\mathrm{BP}_* = \mathbb{F}_p[\bar{v}_0, \bar{v}_1, \cdots].$$

Here $\bar{v}_0$ is represented by p, and $\bar{v}_i$ is represented by v_i. We also have the associated graded $E_0\mathrm{BP}_*\mathrm{BP}$ of $E_0\mathrm{BP}_*\mathrm{BP}$ with respect to the I-filtration

$$E_0\mathrm{BP}_*\mathrm{BP} = E_0\mathrm{BP}_*[\bar{t}_1, \bar{t}_2, \cdots].$$

Here $\bar{t}_i$ is represented by t_i.

Theorem 9.2. *(Novikov [47], Miller [41]) There exists a tri-graded spectral sequence with*

$$E_1^{s,i,t} = \mathrm{Ext}^{s,t}_{E_0\mathrm{BP}_*\mathrm{BP}}(E_0\mathrm{BP}_*, E_0\mathrm{BP}_*),$$

where i is the I-filtration, and

$$d_r : E_r^{s,i,t} \longrightarrow E_r^{s+1,i+r,t},$$

converging to

$$\mathrm{Ext}^{s,t}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*).$$

9.2. **The cobar construction.** To discuss the algebraic Novikov spectral sequence in details, we first review the cobar complex. For a general reference regarding this material, see [53] for example.

Recall that $(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP})$ forms a Hopf algebroid with structure maps

$$\Delta_{\mathrm{BP}_*\mathrm{BP}} : \mathrm{BP}_*\mathrm{BP} \longrightarrow \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{BP}$$

$$\epsilon : \mathrm{BP}_*\mathrm{BP} \longrightarrow \mathrm{BP}_*$$

For any left $\mathrm{BP}_*\mathrm{BP}$-comodule M and right $\mathrm{BP}_*\mathrm{BP}$-comodule N, with structure maps

$$\Delta_M : M \longrightarrow \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} M,$$

$$\Delta_N : N \longrightarrow N \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{BP},$$

we have the two-sided cobar construction

$$C^\bullet(N, \mathrm{BP}_*\mathrm{BP}, M)$$

which is a cosimplicial $\mathbb{Z}_p$-module with

$$C^n(N, \mathrm{BP}_*\mathrm{BP}, M) = N \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{BP}^{\otimes n} \otimes_{\mathrm{BP}_*} M.$$

The coface maps

$$d_i : C^{n-1}(N, \mathrm{BP}_*\mathrm{BP}, M) \longrightarrow C^n(N, \mathrm{BP}_*\mathrm{BP}, M)$$

where $0 \leq i \leq n$, and codegeneracy maps

$$s_i : C^{n+1}(N, \mathrm{BP}_*\mathrm{BP}, M) \longrightarrow C^n(N, \mathrm{BP}_*\mathrm{BP}, M)$$

where $0 \leq i \leq n$, are given by

$$\begin{aligned}
d_0 &= \Delta_N \otimes \mathrm{id}^{\otimes n}_{\mathrm{BP}_*\mathrm{BP}} \otimes \mathrm{id}_M \\
d_i &= \mathrm{id}_N \otimes \mathrm{id}^{\otimes i-1}_{\mathrm{BP}_*\mathrm{BP}} \otimes \Delta_{\mathrm{BP}_*\mathrm{BP}} \otimes \mathrm{id}^{\otimes n-i-1}_{\mathrm{BP}_*\mathrm{BP}} \otimes \mathrm{id}_M \qquad && \text{if } 1 \leq i \leq n-1 \\
d_n &= \mathrm{id}_N \otimes \mathrm{id}^{\otimes n}_{\mathrm{BP}_*\mathrm{BP}} \otimes \Delta_M \\
s_i &= \mathrm{id}_N \otimes \mathrm{id}^{\otimes i-1}_{\mathrm{BP}_*\mathrm{BP}} \otimes \epsilon \otimes \mathrm{id}^{\otimes n-i}_{\mathrm{BP}_*\mathrm{BP}} \otimes \mathrm{id}_M && \text{for all } 0 \leq i \leq n
\end{aligned}$$

We have the associated cochain complex $C^*(N, \mathrm{BP}_*\mathrm{BP}, M)$,

$$N \otimes_{\mathrm{BP}_*} M \longrightarrow N \otimes \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} M \longrightarrow N \otimes \mathrm{BP}_*\mathrm{BP}^{\otimes 2} \otimes_{\mathrm{BP}_*} M \longrightarrow \cdots$$

where the differentials are given by alternating sum of the d_i's in the cobar construction.

We also have the normalized cochain complex $\overline{C}^*(N, \mathrm{BP}_*\mathrm{BP}, M)$,

$$N \otimes_{\mathrm{BP}_*} M \longrightarrow N \otimes \overline{\mathrm{BP}_*\mathrm{BP}} \otimes_{\mathrm{BP}_*} M \longrightarrow N \otimes \overline{\mathrm{BP}_*\mathrm{BP}}^{\otimes 2} \otimes_{\mathrm{BP}_*} M \longrightarrow \cdots$$

where $\overline{\mathrm{BP}_*\mathrm{BP}}$ is the kernel of the counit map $\epsilon : \mathrm{BP}_*\mathrm{BP} \longrightarrow \mathrm{BP}_*$, and the differentials are induced by the ones in $C^*(N, \mathrm{BP}_*\mathrm{BP}, M)$.

Take $N = \mathrm{BP}_*\mathrm{BP}$, we consider the two-sided cobar construction

$$C^\bullet(\mathrm{BP}_*\mathrm{BP}, \mathrm{BP}_*\mathrm{BP}, M).$$

There is a coaction of $\mathrm{BP}_*\mathrm{BP}$ on $C^\bullet(\mathrm{BP}_*\mathrm{BP}, \mathrm{BP}_*\mathrm{BP}, M)$, where we use the map $\Delta_{\mathrm{BP}_*\mathrm{BP}}$ on the left of each level,

$$\mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{BP}^{\otimes n} \otimes_{\mathrm{BP}_*} M \xrightarrow{\Delta_{\mathrm{BP}_*\mathrm{BP}} \otimes \cdots} \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} (\mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{BP}^{\otimes n} \otimes_{\mathrm{BP}_*} M)$$

Since all cosimplicial structure maps preserve the coaction of $\mathrm{BP}_*\mathrm{BP}$, the two-sided cobar construction

$$C^\bullet(\mathrm{BP}_*\mathrm{BP}, \mathrm{BP}_*\mathrm{BP}, M)$$

is actually a cosimplicial left $\mathrm{BP}_*\mathrm{BP}$-comodule.

The normalized cochain complex

$$\overline{C}^*(\mathrm{BP}_*\mathrm{BP}, \mathrm{BP}_*\mathrm{BP}, M)$$

gives a relative injective resolution of M. The primitives on each level gives

$$\overline{C}^*(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP}, M),$$

whose homology computes

$$\mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(M) := \mathrm{Ext}^{*,*}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, M).$$

Under the I-filtration, we can view the associated graded of the cobar complex

$$\overline{C}^*(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP}, \mathrm{BP}_*)$$

as the E_0-page of the algebraic Novikov spectral sequence. It turns out that it is isomorphic to

$$\overline{C}^*(\mathrm{BP}_*/I, \mathrm{BP}_*\mathrm{BP}/I, E_0\mathrm{BP}_*)$$

The coaction of $\mathrm{BP}_*\mathrm{BP}/I$ on $E_0\mathrm{BP}_*$ is given by the following composite

$$E_0\mathrm{BP}_* \xrightarrow{\eta_L} E_0\mathrm{BP}_*\mathrm{BP} \xleftarrow[i \otimes \eta_R]{\cong} \mathrm{BP}_*\mathrm{BP}/I \otimes_{\mathrm{BP}_*/I} E_0\mathrm{BP}_*$$

where $i : \mathrm{BP}_*\mathrm{BP}/I \longrightarrow E_0\mathrm{BP}_*\mathrm{BP}$ is the inclusion of the I-filtration 0 part.

Therefore we can identify the E_1-page of the algebraic Novikov spectral sequence as the following

$$E_1^{s,i,t} = \mathrm{Ext}^{s,t}_{\mathrm{BP}_*\mathrm{BP}/I}(\mathrm{BP}_*/I, I^i/I^{i+1})$$

9.3. **The re-grading.** For the statement of the main theorem of Part 2 - Theorem 8.1, we re-grade the algebraic Novikov spectral sequence.

Definition 9.3. We define the a-filtration of the algebraic Novikov spectral sequence

$$a = i + s.$$

In terms of an element $x \in \overline{C}^s(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP}, \mathrm{BP}_*)$, this means that x has a-filtration at least $s + i$ if and only if

$$x \in \overline{C}^s(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP}, I^i).$$

Remark 9.4. We comment on the tri-degrees in the cobar resolution for BP_*. The t-degree is the internal degree, which is preserved by differentials. The cohomological s-degree is the number of bar's in the cobar resolution. The i-filtration degree is the number of v's in the cobar resolution. Finally our new a-filtration degree is the sum of numbers of bar's and v's in the cobar resolution.

After the re-grading, the d_r differentials, which used to raise the i-filtration degree by r, now raise the a-filtration degree by $r+1$. This is due to the fact that they also raise the s-degree by 1. We therefore rename the d_r's as d_{r+1}'s. The re-graded algebraic Novikov spectral sequence therefore starts with the E_2-page, instead of the E_1-page.

In summary, we have the following re-graded algebraic Novikov spectral sequence.

Corollary 9.5. *There exists a tri-graded spectral sequence with*

$$E_2^{s,a,t} = \mathrm{Ext}^{s,t}_{\mathrm{BP}_*\mathrm{BP}/I}(\mathrm{BP}_*/I, I^{a-s}/I^{a-s+1}),$$

and

$$d_r : E_r^{s,a,t} \longrightarrow E_r^{s+1,a+r,t},$$

converging to

$$\mathrm{Ext}^{s,t}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*).$$

9.4. **Cartan-Eilenberg spectral sequence.** The E_2-page of the algebraic Novikov spectral sequence has another interpretation–it is isomorphic to the E_2-page of the Cartan-Eilenberg spectral sequence that converges to the E_2-page of the Adams E_2-page. For a general reference regarding this material, see [53] for example.

Recall that the mod p dual Steenrod algebra A_* is

$$A_* = \begin{cases} \mathbb{F}_p[\zeta_1, \zeta_2, \cdots] & |\zeta_i| = 2^i - 1 & \text{for } p = 2, \\ \Lambda_{\mathbb{F}_p}[\tau_0, \tau_1, \cdots] \otimes \mathbb{F}_p[\xi_1, \xi_2, \cdots] & |\xi_i| = 2p^i - 2,\ |\tau_i| = 2p^i - 1 & \text{for } p > 2. \end{cases}$$

For $p = 2$, we set

$$\xi_i = \zeta_i^2$$

Let $\bar{\xi}_i$ be the conjugate of ξ_i. Let P_* be the following sub-Hopf algebra of A_*

$$P_* = \mathbb{F}_p(\bar{\xi}_1, \bar{\xi}_2, \cdots)$$

and Λ_* be the quotient

$$\Lambda_* = A_* \otimes_{P_*} \mathbb{F}_p = \begin{cases} \Lambda_{\mathbb{F}_p}[\zeta_1, \zeta_2, \cdots] & \text{for } p = 2, \\ \Lambda_{\mathbb{F}_p}[\tau_0, \tau_1, \cdots] & \text{for } p > 2. \end{cases}$$

We have an epimorphism

$$\mathrm{BP}_*\mathrm{BP} \longrightarrow P_*$$

that sends t_i to $\bar{\xi}_i$. This gives an epimorphism of Hopf algebroids

$$(\mathrm{BP}_*,\ \mathrm{BP}_*\mathrm{BP}) \longrightarrow (\mathbb{F}_p,\ P_*),$$

which gives the following isomorphisms

$$\mathrm{BP}_*\mathrm{BP}/I \cong E_0\mathrm{BP}_*\mathrm{BP} \otimes_{E_0\mathrm{BP}_*} \mathbb{F}_p \cong P_*.$$

Therefore, the E_1-page of the algebraic Novikov spectral sequence is isomorphic to

$$E_1^{s,i,t} = \mathrm{Ext}_{P_*}^{s,t}(\mathbb{F}_p, I^i/I^{i+1}),$$

by the change-of-ring isomorphism.

The above Ext group can be regarded as the E_2-page of the Cartan-Eilenberg spectral sequence.

Recall that there is an extension of Hopf algebras

$$P_* \longrightarrow A_* \longrightarrow \Lambda_*,$$

which produces a Cartan-Eilenberg spectral sequence with

$$E_2^{s,i,t} = \mathrm{Ext}_{P_*}^{s}(\mathbb{F}_p, \mathrm{Ext}_{\Lambda_*}^{i,t}(\mathbb{F}_p, \mathbb{F}_p)),$$

and

$$d_r : E_r^{s,i,t} \longrightarrow E_r^{s+r,i-r+1,t},$$

converging to

$$\mathrm{Ext}_{A_*}^{s+i,t}(\mathbb{F}_p, \mathbb{F}_p).$$

Since Λ_* is an exterior algebra, the Ext group

$$\mathrm{Ext}_{\Lambda_*}^{*,*}(\mathbb{F}_p, \mathbb{F}_p)$$

is isomorphic to a polynomial algebra. It can be further identified as the Adams E_2-page for BP, and therefore the Adams E_∞-page for BP since it collapses at the Adams E_2-page. This is isomorphic to

$$E_0\mathrm{BP}_* = \mathbb{F}_p[\bar{v}_0, \bar{v}_1, \cdots],$$

with $[\tau_i]$ (or $[\zeta_i]$ when $p = 2$) corresponds to $\bar{v}_i$, and the Adams filtration corresponds the I-filtration. Moreover, the coaction of P_* on $\mathrm{Ext}_{\Lambda_*}(\mathbb{F}_p, \mathbb{F}_p)$ is isomorphic to the coaction of $\mathrm{BP}_*\mathrm{BP}/I$ on $E_0\mathrm{BP}_*$.

Therefore, we can identify the E_2-page of the Cartan-Eilenberg spectral sequence as the E_2-page of the re-graded algebraic Novikov spectral sequence with the re-grading

$$a = s + i.$$

9.5. **Miller's square.** In fact, the 4 spectral sequences that we have discussed fit into the following Miller's square [40]:

$$\begin{tikzcd}
 & \mathrm{Ext}_{P_*}^{s,t}(\mathbb{F}_p, I^{a-s}/I^{a-s+1}) \arrow[dl, Rightarrow, "\textit{\textbf{Cartan-Eilenberg SS}}"'] \arrow[dr, Rightarrow, "\textit{\textbf{algebraic Novikov SS}}"] & \\
\mathrm{Ext}_{A_*}^{a,t}(\mathbb{F}_p, \mathbb{F}_p) \arrow[dr, Rightarrow, "\textit{\textbf{Adams SS}}"'] & & \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}^{s,t}(\mathrm{BP}_*, \mathrm{BP}_*) \arrow[dl, Rightarrow, "\textit{\textbf{Adams-Novikov SS}}"] \\
 & \pi_* S^0 &
\end{tikzcd}$$

Remark 9.6. This square is not "commutative": the $*$ in $\pi_* S^0$ is $t - a$ when converging from the Adams spectral sequence, and is $t - s$ when converging from the Adams-Novikov spectral sequence. In general, an element in the stable homotopy groups of sphere does not necessarily have the same Adams and Adams-Novikov filtration. As a first example, the multiplication by p map has Adams filtration 1 and Adams-Novikov filtration 0.

9.6. **The algebraic Novikov tower.** For later reference, we write down the algebraic Novikov tower using the newly defined a-filtration explicitly.

Definition 9.7.

- Let $C_0^* = \overline{C}^*(\mathrm{BP}_*\mathrm{BP}, \mathrm{BP}_*\mathrm{BP}, \mathrm{BP}_*)$ be the normalized cobar resolution of BP_* over $\mathrm{BP}_*\mathrm{BP}$, with
$$C_0^s = \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} \overline{\mathrm{BP}_*\mathrm{BP}}^{\otimes s} \otimes_{\mathrm{BP}_*} \mathrm{BP}_*,$$
where $\overline{\mathrm{BP}_*\mathrm{BP}}$ is the kernel of the counit map $\epsilon : \mathrm{BP}_*\mathrm{BP} \longrightarrow \mathrm{BP}_*$.
- For $m \geq 1$, let C_m^* be the sub cochain complex of C_0^* defined by the a-filtration, namely, at cohomological degree s, we have
$$C_m^s = \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} \overline{\mathrm{BP}_*\mathrm{BP}}^{\otimes s} \otimes_{\mathrm{BP}_*} I^{m-s}$$
It is understood that $I^r = \mathrm{BP}_*$ for $r \leq 0$. Therefore, for $s \geq m$, we have $C_m^s = C_0^s$.
- For $m \geq 0$, let Q_m^* be the quotient cochain complex of the inclusion map
$$C_{m+1}^* \xrightarrow{i_m} C_m^*.$$
More explicitly, it has the form
$$Q_m^s = \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} \overline{\mathrm{BP}_*\mathrm{BP}}^{\otimes s} \otimes_{\mathrm{BP}_*} I^{m-s}/I^{m-s+1}.$$

Therefore, we have a tower of cochain complexes, which induces the following tower in the derived category of $\mathrm{BP}_*\mathrm{BP}$-comodules:

$$\begin{tikzcd}
\overline{C}^*(\mathrm{BP}_*\mathrm{BP}, \mathrm{BP}_*\mathrm{BP}, \mathrm{BP}_*) \arrow[r, equal] & C_0^* \arrow[d, "q_0"] & C_1^* \arrow[l, "i_0"'] \arrow[d, "q_1"] & C_2^* \arrow[l, "i_1"'] \arrow[d, "q_2"] & \cdots \arrow[l, "i_2"'] \\
& Q_0^* & Q_1^* & Q_2^* &
\end{tikzcd}$$

Note that for each cochain complex Q_m^*, we have
$$I^{m-s}/I^{m-s+1} = \mathrm{BP}_*$$
when $s \geq m+1$. In other words, the cochain complex Q_m^* has the same cochain groups and differetnials as the normalized cobar resolution at degrees least $m+1$. Therefore, for $s \geq m+2$,
$$H^s Q_m^* = H^s C_0^* = 0.$$
In particular, the cochain complex Q_m^* has bounded cohomology. This implies that each cochain complex C_m^* also has bounded cohomology.

Therefore, although the cochain complexes Q_m^* and C_m^* are unbounded, they live in the category $\mathcal{D}^b(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod})$.

Applying the functor

$$\mathbf{R}^{*,*}\operatorname{Hom}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, -),$$

where $\mathbf{R}^{*,*}\operatorname{Hom}_{\mathrm{BP}_*\mathrm{BP}}(-,-)$ is the derived homomorphisms in the category $\mathcal{D}^b(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod})$, we get a spectral sequence with the E_1-page

$$\mathbf{R}^{*,*}\operatorname{Hom}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, Q_m^*),$$

converging to

$$\mathbf{R}^{*,*}\operatorname{Hom}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*) = \operatorname{Ext}^{*,*}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*).$$

By construction, this is the regraded algebraic Novikov spectral sequence.

10. The equivalence to the motivic Adams spectral sequence

Having the tower in the bounded derived category of $\mathrm{BP}_*\mathrm{BP}$-comodules that gives the regraded algebraic Novikov spectral sequence, we use the equivalence of stable ∞-categories in Theorem 4.13 and Proposition 6.3 in Part 1

$$\mathcal{D}^b(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}) \xrightarrow{\cong} \mathcal{D}^b(\mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod}) \xrightarrow{\cong} S^{0,0}/\tau\text{-}\mathbf{Mod}^b_{\clubsuit}.$$

to get a tower in the category $S^{0,0}/\tau\text{-}\mathbf{Mod}^b_{\clubsuit}$:

$$\begin{tikzcd} S^{0,0}/\tau \arrow[r, equal] & X_0 \arrow[d, "f_0"] & X_1 \arrow[l, "g_0"'] \arrow[d, "f_1"] & X_2 \arrow[l, "g_1"'] \arrow[d, "f_2"] & \cdots \arrow[l, "g_2"'] \\ & K_0 & K_1 & K_2 & \end{tikzcd}$$

We show that the above tower in the category $S^{0,0}/\tau\text{-}\mathbf{Mod}^b_{\clubsuit}$ is indeed a motivic Adams tower for $S^{0,0}/\tau$ in the sense of Dugger-Isaksen [11].

10.1. Characterization of Adams towers.

Recall that we denote by $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$ the motivic mod p Eilenberg-Mac Lane spectrum. It is shown by Hu-Kriz-Ormsby [23] and Hoyois [21] that $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$ is cellular.

Recall the following criterion for a tower in $S^{0,0}/\tau\text{-}\mathbf{Mod}^b_{\clubsuit}$ to be an Adams tower.

Definition 10.1. A tower in $S^{0,0}/\tau\text{-}\mathbf{Mod}^b_{\clubsuit}$

$$\begin{tikzcd} S^{0,0}/\tau \arrow[r, equal] & X_0 \arrow[d, "f_0"] & X_1 \arrow[l, "g_0"'] \arrow[d, "f_1"] & X_2 \arrow[l, "g_1"'] \arrow[d, "f_2"] & \cdots \arrow[l, "g_2"'] \\ & K_0 & K_1 & K_2 & \end{tikzcd}$$

is a *motivic Adams tower* if

(1) each motivic spectrum K_m is $S^{0,0}/\tau$ smashed with a wedge of suspensions of $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$.
(2) each map $f_m : X_m \longrightarrow K_m$ induces an epimorphism on the $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$-cohomology. Or equivalently, each map $g_m : X_{m+1} \longrightarrow X_m$ induces the zero map on the $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$-cohomology.

Note that by the adjunction between modules over $S^{0,0}$ and $S^{0,0}/\tau$ and that $S^{0,0}/\tau$ is Spanier-Whitehead dual to itself up to a bidegree shift, it is equivalent to check that each map g_m induces the zero map on

$$[-, S^{0,0}/\tau \wedge \mathrm{H}\mathbb{F}_p^{\mathrm{mot}}]_{S^{0,0}/\tau}$$

in Condition (2).

From the general discussions by Christensen [9], all such towers are equivalent to each other in the sense that there exist towers maps that induce canonical isomorphisms on the E_2-pages.

Note that Dugger-Isaksen [11] uses the cobar construction to define the motivic Adams spectral sequence for $S^{0,0}/\tau$, which satisfies the two conditions in Definition 10.1. Therefore, the motivic Adams spectral sequence for $S^{0,0}/\tau$ by Dugger-Isaksen [11] is canonically isomorphic to the motivic Adams spectral sequence defined by any motivic Adams tower satisfying the two conditions in Definition 10.1.

In the rest of this section, we check that the two conditions in Definition 10.1 are satisified by our tower in $S^{0,0}/\tau\text{-}\mathbf{Mod}_{\clubsuit}^{b}$ obtained from the algebraic Novikov tower in the category $\mathcal{D}^b(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod})$ and Theorem 4.13.

10.2. **Proof of the first condition.** To check the first condition, we first identify the $\mathrm{BP}_*\mathrm{BP}$-comodule that corresponds to $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$ under the equivalences in Proposition 4.11 and Proposition 6.3.

$$S^{0,0}/\tau\text{-}\mathbf{Mod}_{\clubsuit}^{\heartsuit} \xrightarrow[\cong]{\mathrm{MU}^{\mathrm{mot}}_{*,*}} \mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod} \xrightarrow[\cong]{\mathrm{BP}_*\mathrm{MU}\square_{\mathrm{MU}_*\mathrm{MU}}-} \mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}.$$

Lemma 10.2. *Under the above equivalences in Proposition 4.11 and Proposition 6.3, the* $\mathrm{BP}_*\mathrm{BP}$*-comodule*

$$\mathrm{BP}_*\mathrm{BP}/I = \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} \mathbb{F}_p$$

corresponds to

$$\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}/\tau = S^{0,0}/\tau \wedge \mathrm{H}\mathbb{F}_p^{\mathrm{mot}}.$$

Proof. Since $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$ is an $\mathrm{MU}^{\mathrm{mot}}$-module, and both $\mathrm{MU}^{\mathrm{mot}}$ and $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$ are cellular, the homotopy groups of

$$\mathrm{MU}^{\mathrm{mot}}/\tau \wedge_{S^{0,0}/\tau} \mathrm{H}\mathbb{F}_p^{\mathrm{mot}}/\tau$$

can be computed by the homology version of Dugger-Isaksen's universal coefficient spectral sequence [12, Proposition 7.7] in the category $\mathrm{MU}^{\mathrm{mot}}/\tau$-$\mathbf{Mod}_{\mathrm{cell}}$.

$$\mathrm{Tor}_{s,t,w}^{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau}\left(\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau, \mathbb{F}_p\right) \Longrightarrow \pi_{t+s,w}\left(\mathrm{MU}^{\mathrm{mot}}/\tau \wedge_{C\tau} \mathrm{MU}^{\mathrm{mot}}/\tau \wedge_{\mathrm{MU}^{\mathrm{mot}}/\tau} \mathrm{H}\mathbb{F}_p^{\mathrm{mot}}/\tau\right).$$

Note that $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau$ is free over $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$, this spectral sequence is therefore concentrated on the line $s = 0$ and collapses at the E_2-page. This gives the following isomorphisms

$$\begin{aligned}
\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}/\tau &\cong \pi_{*,*}(\mathrm{MU}^{\mathrm{mot}}/\tau \wedge_{C\tau} \mathrm{H}\mathbb{F}_p^{\mathrm{mot}}/\tau) \\
&\cong \pi_{*,*}(\mathrm{MU}^{\mathrm{mot}}/\tau \wedge_{C\tau} \mathrm{MU}^{\mathrm{mot}}/\tau \wedge_{\mathrm{MU}^{\mathrm{mot}}/\tau} \mathrm{H}\mathbb{F}_p^{\mathrm{mot}}/\tau) \\
&\cong \mathrm{Tor}_{0,*,*}^{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau}\left(\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau, \mathbb{F}_p\right) \\
&\cong \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau \otimes_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau} \mathbb{F}_p \\
&\cong \mathrm{MU}_*\mathrm{MU} \otimes_{\mathrm{MU}_*} \mathbb{F}_p.
\end{aligned}$$

Therefore, under the equivalences in Proposition 4.11 and Proposition 6.3, the $S^{0,0}/\tau$-module $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}/\tau$ corresponds to

$$\begin{aligned}
\mathrm{BP}_*\mathrm{MU}\square_{\mathrm{MU}_*\mathrm{MU}}(\mathrm{MU}_*\mathrm{MU} \otimes_{\mathrm{MU}_*} \mathbb{F}_p) &\cong (\mathrm{BP}_*\mathrm{MU}\square_{\mathrm{MU}_*\mathrm{MU}}\mathrm{MU}_*\mathrm{MU}) \otimes_{\mathrm{MU}_*} \mathbb{F}_p \\
&\cong \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{MU}_*} \mathbb{F}_p \\
&\cong \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} \mathbb{F}_p \\
&\cong \mathrm{BP}_*\mathrm{BP}/I.
\end{aligned}$$

Here the second isomorphism follows from Lemma 6.2. This completes the proof. □

Corollary 10.3. *Suppose that N is a BP_*-module that is concentrated in even degrees and is annihilated by I. Then any comodule of the form*

$$\mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} N$$

corresponds to a wedge of suspensions of $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}/\tau$.

Proof. Since $\mathrm{BP}_*/I = \mathbb{F}_p$, any BP_*-module which is annihilated by I is a direct sum of copies of $\mathbb{F}_p$ in different degrees. Therefore, N is isomorphic to a direct sum of copies of $\mathbb{F}_p$ in even degrees.

By Lemma 10.2, the comodule $\mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} \mathbb{F}_p$ corresponds to $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}/\tau$. Therefore, we have the comodule $\mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} \Sigma^{2n}\mathbb{F}_p$ corresponds to

$$\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}/\tau \wedge S^{2n,n}$$

Therefore, the comodule $\mathrm{BP}_*\mathrm{BP}\otimes_{\mathrm{BP}_*}N$ corresponds to a wedge of suspensions of $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}/\tau$. □

Now we prove that our tower satisfies Condition (1) of Definition 10.1.

Proposition 10.4.

(1) All differentials in the cochain complexes Q_m^* are zero, and therefore Q_m^* splits as a direct sum of cochain complexes that are concentrated in one cohomological degree.

(2) Furthermore, each Q_m^s corresponds to a wedge of suspensions of $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}/\tau$, and hence so is Q_m^*.

Therefore, each K_i is a wedge of suspensions of $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}/\tau$.

Proof. In Q_m^*, all differentials $Q_m^s \longrightarrow Q_m^{s+1}$ have the form

$$\mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} \overline{\mathrm{BP}_*\mathrm{BP}}^{\otimes s} \otimes_{\mathrm{BP}_*} I^{m-s}/I^{m-s+1} \longrightarrow \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} \overline{\mathrm{BP}_*\mathrm{BP}}^{\otimes s+1} \otimes_{\mathrm{BP}_*} I^{m-s-1}/I^{m-s}.$$

Since differentials in cobar complex does not decrease the I-filtration, all induced differentials in Q_m^* are zero due to the I-filtration.

It is clear that I^{m-s}/I^{m-s+1} is annihilated by I, so is

$$\overline{\mathrm{BP}_*\mathrm{BP}}^{\otimes s} \otimes_{\mathrm{BP}_*} I^{m-s}/I^{m-s+1}.$$

Since they are concentrated in even degrees, by Corollary 10.3, the comodules

$$Q_m^s = \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} \overline{\mathrm{BP}_*\mathrm{BP}}^{\otimes s} \otimes_{\mathrm{BP}_*} I^{m-s}/I^{m-s+1}$$

corresponds to a wedge of suspensions of $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}/\tau$. □

10.3. **Proof of the second condition.** To prove Condition (2) of Definition 10.1, we first prove the following Lemma 10.5.

Lemma 10.5. *Suppose that X is in the category $S^{0,0}/\tau$-$\mathbf{Mod}^b_{\clubsuit}$ and that $C^*_{\mathrm{BP}_*\mathrm{BP}}(X)$ is the cochain complex of $\mathrm{BP}_*\mathrm{BP}$-comodules representing the image of X under the equivalence in Theorem 4.13 in Part 1. Let $C^*_{\mathrm{BP}_*}(X)$ be its underlying cochain complex of BP_*-modules.*

Then the $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$-cohomology of X can be computed as

$$\mathbf{R}^{*,*}\operatorname{Hom}_{\mathrm{BP}_*}(C^*_{\mathrm{BP}_*}(X), \Sigma^{-1,1}\mathbb{F}_p),$$

where $\mathbf{R}^{*,*}\operatorname{Hom}_{\mathrm{BP}_*}(-,-)$ is the derived homomorphism in the derived category of BP_*-modules, and $\Sigma^{-1,1}\mathbb{F}_p$ is the cochain complex $\Sigma^2\mathbb{F}_p$ that is concentrated in cohomological degree 3 (see Remark 4.14 for the explanation of the bigrading).

We will see in the proof that, if we compute

$$[X, \mathrm{H}\mathbb{F}_p^{\mathrm{mot}}/\tau]_{S^{0,0}/\tau},$$

instead of the $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$-cohomology of X, the conclusion will not have the bidegree shift.

Proof. The $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$-cohomology of X is given by

$$\begin{aligned}
[\Sigma^{*,*}X,\ \mathrm{H}\mathbb{F}_p]_{S^{0,0}} &\cong \pi_{*,*}\operatorname{Hom}_{S^{0,0}}(X,\ \mathrm{H}\mathbb{F}_p)\\
&\cong \pi_{*,*}\operatorname{Hom}_{S^{0,0}}(S^{0,0}/\tau \wedge_{S^{0,0}/\tau} X,\ \mathrm{H}\mathbb{F}_p)\\
&\cong \pi_{*,*}\operatorname{Hom}_{S^{0,0}/\tau}(X,\ \operatorname{Hom}_{S^{0,0}}(S^{0,0}/\tau,\ \mathrm{H}\mathbb{F}_p))\\
&\cong [\Sigma^{*,*}X,\ \operatorname{Hom}_{S^{0,0}}(S^{0,0}/\tau,\ \mathrm{H}\mathbb{F}_p)]_{S^{0,0}/\tau}\\
&\cong [\Sigma^{*,*}X,\ D(S^{0,0}/\tau)\wedge_{S^{0,0}}\mathrm{H}\mathbb{F}_p]_{S^{0,0}/\tau}\\
&\cong [\Sigma^{*,*}X,\ \Sigma^{-1,1}S^{0,0}/\tau\wedge_{S^{0,0}}\mathrm{H}\mathbb{F}_p]_{S^{0,0}/\tau}\\
&\cong \mathbf{R}^{*,*}\operatorname{Hom}_{\mathrm{BP}_*\mathrm{BP}}(C^*_{\mathrm{BP}_*\mathrm{BP}}(X),\ \Sigma^{-1,1}\mathrm{BP}_*\mathrm{BP}\otimes_{\mathrm{BP}_*}\mathbb{F}_p)\\
&\cong \mathbf{R}^{*,*}\operatorname{Hom}_{\mathrm{BP}_*}(C^*_{\mathrm{BP}_*}(X),\ \Sigma^{-1,1}\mathbb{F}_p)
\end{aligned}$$

Here the third isomorphism follows from the standard tensor-hom adjuction in a topological category, the sixth isomorphism follows the fact that the Spanier-Whitehead dual $D(S^{0,0}/\tau)$ of $S^{0,0}/\tau$ is $\Sigma^{-1,1}S^{0,0}/\tau$ (see [15, Proposition 4.3] for a proof for example), the seventh isomorphism follows from Theorem 4.13, and the last isomorphism follows from the adjunction of the derived functor of $\mathrm{BP}_*\mathrm{BP}\otimes_{\mathrm{BP}_*} -$ and the forgetful functor between the derived categories of BP_*-modules and $\mathrm{BP}_*\mathrm{BP}$-comodules. ☐

We also need the following Lemma 10.6, whose proof is technical, and is postponed to the last subsection of this section.

Lemma 10.6. *The following homomorphisms*

$$\operatorname{Ext}^{*,*}_{\mathrm{BP}_*}(I^{m+1},\mathbb{F}_p)\longrightarrow \operatorname{Ext}^{*,*}_{\mathrm{BP}_*}(I^{m},\mathbb{F}_p),$$

that are induced by the inclusions $I^{m+1}\longrightarrow I^m$ are zero for all $m\geq 0$.

Now we prove that our tower satisfies Condition (2) of Definition 10.1.

Proposition 10.7. *Each map $g_m : X_{m+1}\longrightarrow X_m$ induces the zero map on the $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$-cohomology.*

Proof. Consider the normalized cobar complex

$$C_0^* = \overline{C}^*(\mathrm{BP}_*\mathrm{BP},\mathrm{BP}_*\mathrm{BP},\mathrm{BP}_*).$$

The cochain complex of BP_*-modules

$$0\longrightarrow \mathrm{BP}_*\longrightarrow C_0^*$$

is a long exact sequence of free BP_*-modules.

Therefore, as a cochain complex of BP_*-modules, C_0^* splits as a direct sum of cochain complexes

$$C_0^* = \bigoplus_j D_{0,j}^*,$$

where $D_{0,0}^*$ is a direct sum of cochain complexes of the form

$$0 \longrightarrow \mathrm{BP}_* \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

shifted by even internal degrees, and for $j \geq 1$, $D_{0,j}^*$ is a direct sum of cochain complexes of the form

$$\cdots \longrightarrow 0 \longrightarrow \overline{\mathrm{BP}_*\mathrm{BP}}^{\otimes j} \otimes_{\mathrm{BP}_*} \mathrm{BP}_* \xrightarrow{\mathrm{id}} \overline{\mathrm{BP}_*\mathrm{BP}}^{\otimes j} \otimes_{\mathrm{BP}_*} \mathrm{BP}_* \longrightarrow 0 \longrightarrow \cdots$$

shifted by even internal degrees. This is due to the facts that as BP_*-modules, $\mathrm{BP}_*\mathrm{BP}$ splits as copies of BP_* shifted by even degrees, and that any bounded below long exact sequence of projective modules splits in this way.

In particular, each $D_{0,j}^s$ is a free BP_*-module, and $H^*D_{0,j}^* = 0$ for $j \geq 1$.

The a-filtration is compatible with the splitting $C_0^* = \bigoplus_j D_{0,j}^*$, since it is defined by the action of powers of the ideal I, which only depends on the underlying BP_*-module structure.

Recall that for $m \geq 1$, each $S^{0,0}/\tau$-module spectrum X_m corresponds to the cochain complex C_m^*, where

$$C_m^s = \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} \overline{\mathrm{BP}_*\mathrm{BP}}^{\otimes s} \otimes_{\mathrm{BP}_*} I^{m-s}$$

We have the splitting of cochain complexes of BP_*-modules

$$C_m^* = \bigoplus_j D_{m,j}^*,$$

where

$$D_{m,j}^s = D_{0,j}^s \otimes_{\mathrm{BP}_*} I^{m-s}.$$

For example, the cochain complex $D_{4,2}^*$ is a direct sum of cochain complexes of the form

$$\cdots \longrightarrow 0 \longrightarrow \overline{\mathrm{BP}_*\mathrm{BP}}^{\otimes 2} \otimes_{\mathrm{BP}_*} I^3 \longrightarrow \overline{\mathrm{BP}_*\mathrm{BP}}^{\otimes 2} \otimes_{\mathrm{BP}_*} I^2 \longrightarrow 0 \longrightarrow \cdots$$

shifted by even internal degrees. Therefore, we have the cohomology of these cochain complexes of BP_*-modules

$$H^*D_{m,0}^* \cong \bigoplus_* \Sigma^{2*} I^m$$

concentrated in cohomological degree 0, and for $j \geq 1$

$$H^*D_{m,j}^* \cong \bigoplus_* \Sigma^{2*}\overline{\mathrm{BP}_*\mathrm{BP}}^{\otimes j} \otimes_{\mathrm{BP}_*} I^{m-2}/I^{m-3}$$

concentrated in cohomological degree j.

Now we consider the maps

$$\mathbf{R}^{*,*}\operatorname{Hom}_{\mathrm{BP}_*}(D^*_{m+1,j},\mathbb{F}_p) \longrightarrow \mathbf{R}^{*,*}\operatorname{Hom}_{\mathrm{BP}_*}(D^*_{m,j},\mathbb{F}_p) \tag{10.1}$$

that are induced by the inclusions

$$D^*_{m+1,j} \longrightarrow D^*_{m,j}.$$

For $j \geq 1$, these maps can be identified as

$$\mathbf{R}^{*,*}\operatorname{Hom}_{\mathrm{BP}_*}(H^*D^*_{m+1,j},\mathbb{F}_p) \longrightarrow \mathbf{R}^{*,*}\operatorname{Hom}_{\mathrm{BP}_*}(H^*D^*_{m,j},\mathbb{F}_p).$$

This is due to the fact that for objects in the heart of $\mathcal{D}^b(\mathrm{BP}_*\text{-}\mathbf{Mod})$, morphisms between them in the derived category are the same as the ones between their homology. Since $H^*D^*_{m+1,j}$ and $H^*D^*_{m,j}$ are concentrated in the same cohomological degree, and it is clear that the maps

$$H^*D^*_{m+1,j} \longrightarrow H^*D^*_{m,j}$$

are all zero, we have that the maps (10.1) are all zero for $j \geq 1$.

For $j = 0$, these maps can be rewritten as

$$\operatorname{Ext}^{*,*}_{\mathrm{BP}_*}(I^m,\mathbb{F}_p) \longrightarrow \operatorname{Ext}^{*,*}_{\mathrm{BP}_*}(I^{m+1},\mathbb{F}_p).$$

By Lemma 10.6, they are all zero. Therefore, the maps

$$\mathbf{R}^{*,*}\operatorname{Hom}_{\mathrm{BP}_*}(C^*_{m+1},\mathbb{F}_p) \longrightarrow \mathbf{R}^{*,*}\operatorname{Hom}_{\mathrm{BP}_*}(C^*_{m},\mathbb{F}_p)$$

are all zero, since they are zero on each direct summand.

Shifting the bidegrees of $\mathbb{F}_p$ to $\Sigma^{-1,1}\mathbb{F}_p$, we have that the maps

$$\mathbf{R}^{*,*}\operatorname{Hom}_{\mathrm{BP}_*}(C^*_{m+1},\Sigma^{-1,1}\mathbb{F}_p) \longrightarrow \mathbf{R}^{*,*}\operatorname{Hom}_{\mathrm{BP}_*}(C^*_{m},\Sigma^{-1,1}\mathbb{F}_p)$$

are all zero. Note that by construction of our tower in the category $S^{0,0}/\tau\text{-}\mathbf{Mod}^b_{\clubsuit}$, we have that

$$C^*_{\mathrm{BP}_*}(X_m) = C^*_m$$

for all m. Therefore, by Lemma 10.5, each map $g_m : X_{m+1} \longrightarrow X_m$ induces the zero map on the $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$-cohomology. This completes the proof. □

Combining Proposition 10.4 and 10.7, we have shown that our tower satisfies Conditions (1) and (2) of Definition 10.1, and therefore is a motivic Adams tower for $S^{0,0}/\tau$. This proves that there exists an isomorphism between the regraded algebraic Novikov spectral sequence and the motivic Adams spectral sequence for $S^{0,0}/\tau$.

10.4. **Proof of Lemma 10.6.** We prove Lemma 10.6 in this subsection.

The following Noetherian version of Lemma 10.6 is well known (see [56] for example). Suppose that

$$R = \mathbb{Z}_p[x_1, x_2, \cdots]$$

$$m = (p, x_1, x_2, \cdots)$$

$$R_t = \mathbb{Z}_p[x_1, x_2, \cdots, x_t]$$

$$m_t = (p, x_1, x_2, \cdots, x_t)$$

Then we have

Proposition 10.8. *The following maps*

$$\mathrm{Tor}^{R_t}_{*,*}(m_t^{n+1}, \mathbb{F}_p) \longrightarrow \mathrm{Tor}^{R_t}_{*,*}(m_t^n, \mathbb{F}_p)$$

$$\mathrm{Ext}^{*,*}_{R_t}(m_t^{n+1}, \mathbb{F}_p) \longrightarrow \mathrm{Ext}^{*,*}_{R_t}(m_t^n, \mathbb{F}_p)$$

that are induced by the inclusion maps $m_t^{n+1} \longrightarrow m_t^n$, are both zero for all $n \geq 0$ and $t \geq 1$.

Proof. See [56] for a proof of the first statement.

The second statement follows from the first one, since

$$\mathrm{Ext}^{*,*}_{R_t}(M, \mathbb{F}_p) = \mathrm{Hom}_{R_t}(\mathrm{Tor}^{R_t}_{*,*}(M, \mathbb{F}_p), \mathbb{F}_p)$$

is true for the polynomial $\mathbb{Z}_p$-algebra R_t and any module M over R_t. □

Proof of Lemma 10.6. Note that the statement in Lemma 10.6 also follows from the corresponding statement for Tor, since

$$\mathrm{Ext}^{*,*}_{R}(M, \mathbb{F}_p) = \mathrm{Hom}_{R}(\mathrm{Tor}^{R}_{*,*}(M, \mathbb{F}_p), \mathbb{F}_p)$$

is also true for the polynomial $\mathbb{Z}_p$-algebra R and any module M over R.

Therefore, we only need to prove the dual statement that

$$\mathrm{Tor}^{R}_{*,*}(m^{n+1}, \mathbb{F}_p) \longrightarrow \mathrm{Tor}^{R}_{*,*}(m^n, \mathbb{F}_p)$$

is zero.

Take a free resolution F_* of $\mathbb{F}_p$ over R. Since R is free over R_t, we can also regard F_* as a free resolution of $\mathbb{F}_p$ over R_t. Since

$$R = \mathrm{colim}\ R_t,\ m^n = \mathrm{colim}\ m_t^n,$$

we have

$$\mathrm{colim}\ F_* \otimes_{R_t} m_t^n = F_* \otimes_R m^n.$$

Therefore, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Tor}^{R_t}_{*,*}(m_t^n, \mathbb{F}_p) \arrow[r, equal] & H_*(F_* \otimes_{R_t} m_t^n) \arrow[r] & H_*(F_* \otimes_R m^n) \arrow[r, equal] & \mathrm{Tor}^{R}_{*,*}(m^n, \mathbb{F}_p) \\
\mathrm{Tor}^{R_t}_{*,*}(m_t^{n+1}, \mathbb{F}_p) \arrow[r, equal] & H_*(F_* \otimes_{R_t} m_t^{n+1}) \arrow[r] \arrow[u, "0"] & H_*(F_* \otimes_R m^{n+1}) \arrow[r, equal] \arrow[u] & \mathrm{Tor}^{R}_{*,*}(m^{n+1}, \mathbb{F}_p)
\end{tikzcd}$$

Since direct limits are exact, this completes the proof. □

11. Appendix - computation of some classical Adams differentials

In this appendix, we illustrate the power of the isomorphism of spectral sequences in Theorem 8.1, by re-computing certain low filtration and historically more difficult differentials in the range up to the 45-stem at the prime 2. We follow notations in Isaksen's Stable Stems [26] and Isaksen, the second and third author's More Stable Stems [27].

When computing nontrivial differentials in the classical Adams spectral sequence, it is usually harder to give proofs for the ones whose sources are in low Adams filtrations. There are at least two reasons for this. Firstly, there are more potential targets that it could hit, so it means more possibilities to check and rule out. Secondly, on the other hand, elements in high Adams filtrations can usually be detected by certain known spectrum in small chromatic height - for instance, elements above the 1/3-line can be detected by the $K(1)$-local sphere and many elements around the 1/5-line can be detected by the spectrum of topological modular forms. This gives ways to compare with Adams spectral sequences of other spectra.

Up to the 45-stem, we list the following 10 nontrivial differentials, whose sources are in low Adams filtrations. Five of them are d_2-differentials, four of them are d_3-differentials, and one of them is a d_4-differential.

Historically, the first four of them were proved by May in his thesis, by comparing with Toda's unstable computation. The next two are obtained by the Hopf invariant one problem and by comparing with the J-spectrum. Note that the elements Δh_2^2 and $h_0\Delta h_2^2$ were historically called r and s, and there is a nontrivial extension in the May spectral sequence that gives us a relation $s = h_0 r$. The last four, except the one on $d_3(e_1)$, were proved by Barratt-Mahowald-Tangora [2] using ad hoc methods. In fact, the differentials $d_3(h_2h_5) = h_0 p$ and $d_2(c_2) = h_0 f_1$ are both closely related to the nontrivial ν-extension from h_4^2 to the element p, and the differential $d_4(h_3h_5) = h_0 x$ is closely related to the nontrivial σ-extension from h_4^2 to the element x. For the element e_1, Barratt-Mahowald-Tangora [2] erroneously thought it was a permanent cycle. It was later proved by Bruner [7] using power operations that it supports a nontrivial differential $d_3(e_1) = h_1 t$.

Adams differential	Stem of the source	Filtration of the source
$d_2(h_4) = h_0 h_3^2$	15	1
$d_3(h_0 h_4) = h_0 d_0$	15	2
$d_2(e_0) = h_1^2 d_0$	17	4
$d_2(f_0) = h_0^2 e_0$	18	4
$d_2(h_5) = h_0 h_4^2$	31	1
$d_3(h_0^3 h_5) = h_0 \Delta h_2^2$	31	4
$d_3(h_2 h_5) = h_0 p = h_1 d_1$	34	2
$d_4(h_3 h_5) = h_0 x$	38	2
$d_3(e_1) = h_1 t$	38	4
$d_2(c_2) = h_0 f_1$	41	3

Now using Theorem 8.1, we compare them with the computations of the motivic Adams spectral sequence of $S^{0,0}/\tau$. All five d_2-differentials are present in the motivic Adams spectral sequence of $S^{0,0}/\tau$. This gives immediate proofs for all five d_2-differentials.

Moreover, the three out of the four d_3-differentials except $d_3(h_2h_5)$ are present in the motivic Adams spectral sequence of $S^{0,0}/\tau$. To be careful, one also need to rule out the possibility of shorter differentials - d_2's in these cases. This can be done by multiplying h_0 to the proposed d_2-differentials and get contradictions.

For the d_3-differential $d_3(h_2h_5) = h_1d_1$, one can show the following three statements are equivalent, by considering the long exact sequence of motivic homotopy groups associated to the cofiber map of τ.

(1) There is a differential $d_3(h_2h_5) = \tau h_1 d_1$ in the motivic Adams spectral sequence of $S^{0,0}$.

(2) In homotopy groups, $\{h_2h_5\}$ maps to $\{h_1d_1\}$ under the quotient map from $S^{0,0}/\tau$ to its top cell $S^{1,-1}$.

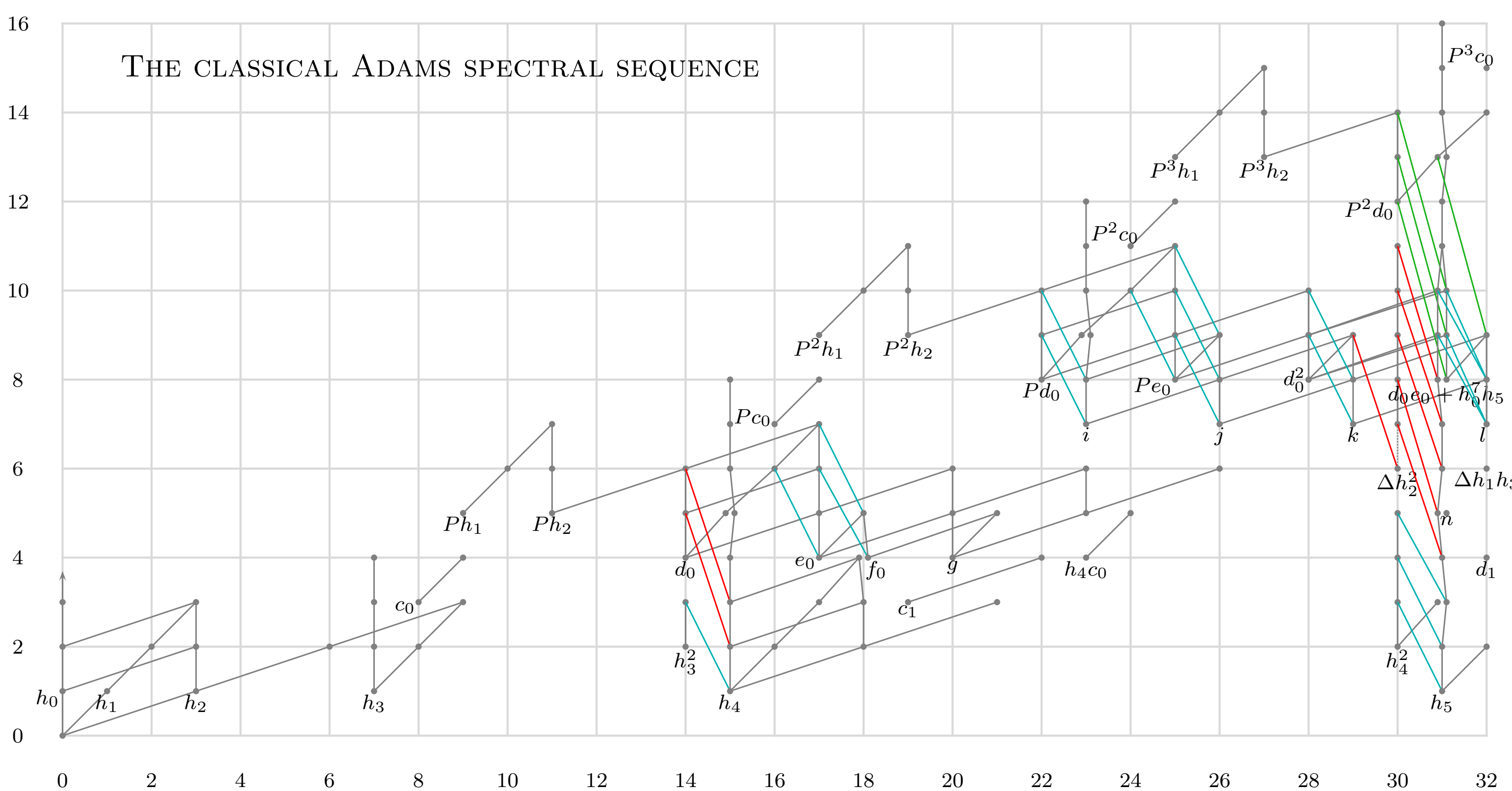
(3) There is an η-extension from h_2h_5 to $\overline{h_1^2 d_1}$ in $\pi_{*,*}S^{0,0}/\tau$, where $\overline{h_1^2 d_1}$ is the element in the motivic Adams E_2-page of $S^{0,0}/\tau$ that corresponds to $h_1^2 d_1$ in that of the top cell $S^{1,-1}$.

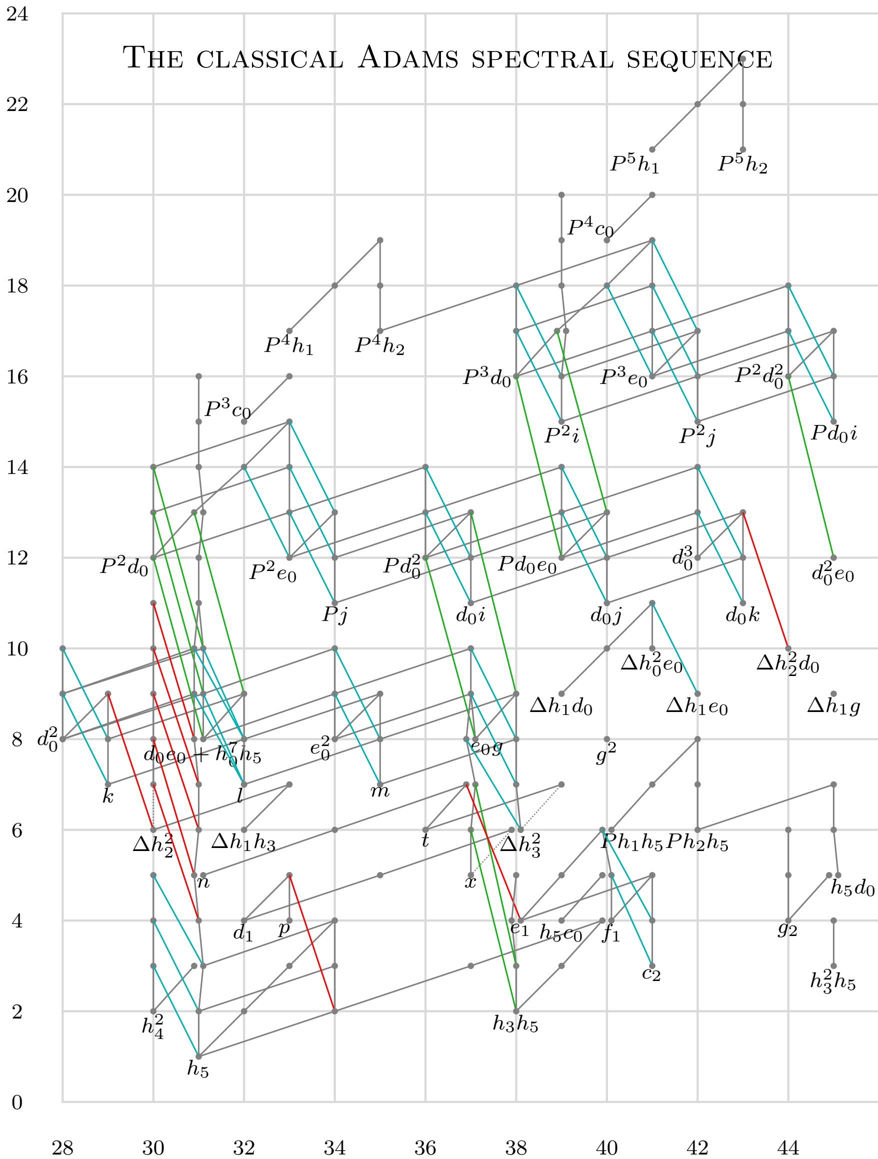
The statement (3) can be checked in the E_∞-page of the motivic Adams spectral sequence for $S^{0,0}/\tau$, which is isomorphic to the classical Adams-Novikov E_2-page. This gives a proof for the d_3-differential in the motivic Adams Adams spectral sequence for $S^{0,0}$, and hence the classical d_3-differential. Note that the statement (2) is proved by Isaksen in Table 42 of Stable Stems [26].

At last, the d_4-differential $d_4(h_3h_5)$ is also present in the motivic Adams spectral sequence for $S^{0,0}/\tau$. To pull it back and get the d_4-differential in the motivic sphere, one need to rule out the possibilities of nonzero d_2's and d_3's. For degree reasons, there is no possible d_2's. To rule out the only d_3 possibility that $d_3(h_3h_5) = x$, note that since $h_3x = h_0^2g_2$, this would give another d_3-differential by multiplying by h_3: $d_3(h_3^2h_5) = h_0^2g_2$. However, there is no such d_3 in the motivic Adams spectral sequence for $S^{0,0}/\tau$, which gives a contradiction.

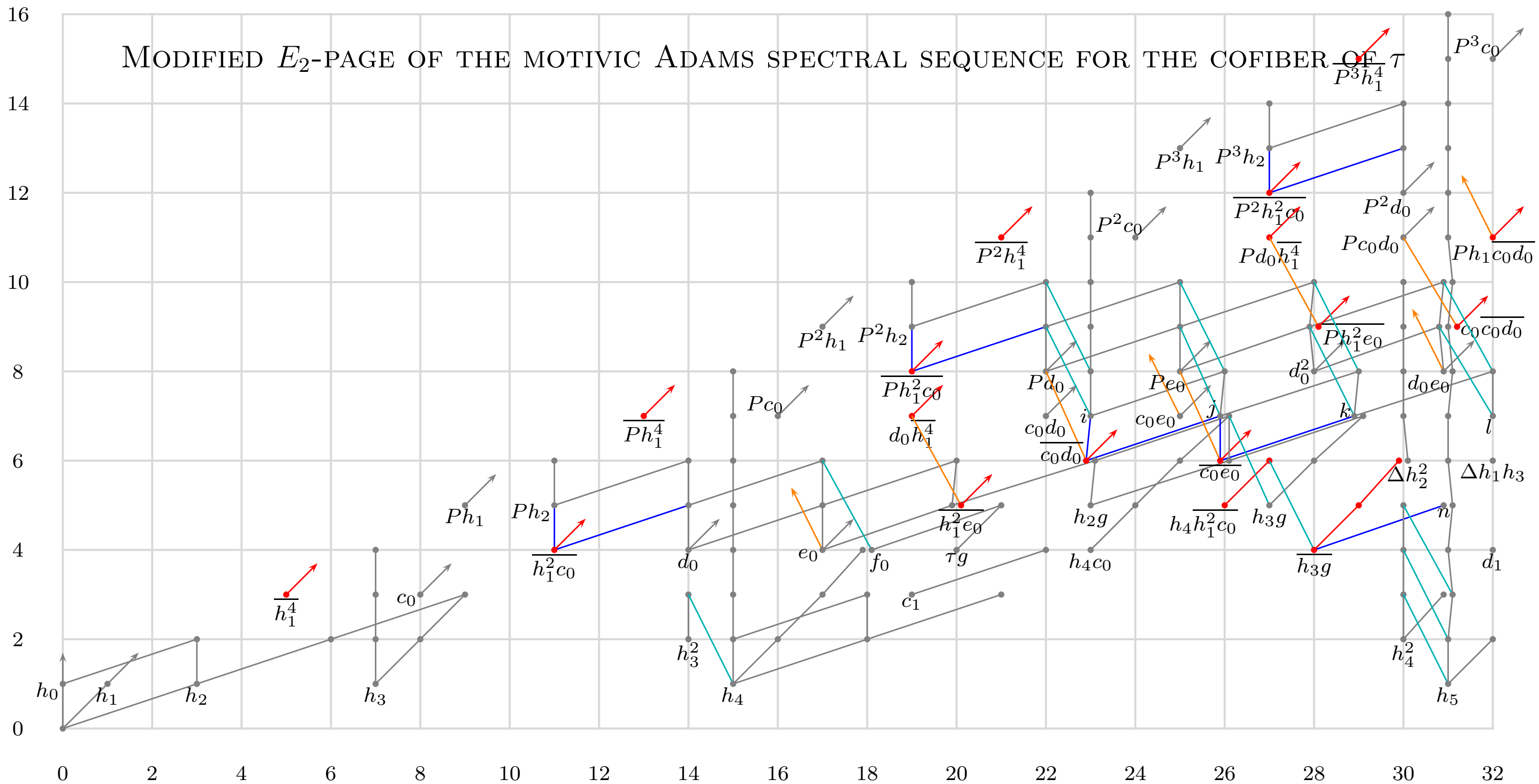
In sum, we reprove all 10 nontrivial low filtration differentials up to the 45-stem without much effort. In fact, among all nontrivial differentials up to the 45-stem, there is only one that cannot be proved by our motivic $S^{0,0}/\tau$-method: $d_3(\Delta h_2^2) = h_1d_0^2$. This can be proved by other methods, such as the ad hoc method by Barratt-Mahowald-Tangora [2], the power operation method by Bruner [8], the method of detection by tmf, and the RP^∞-technique in [67].

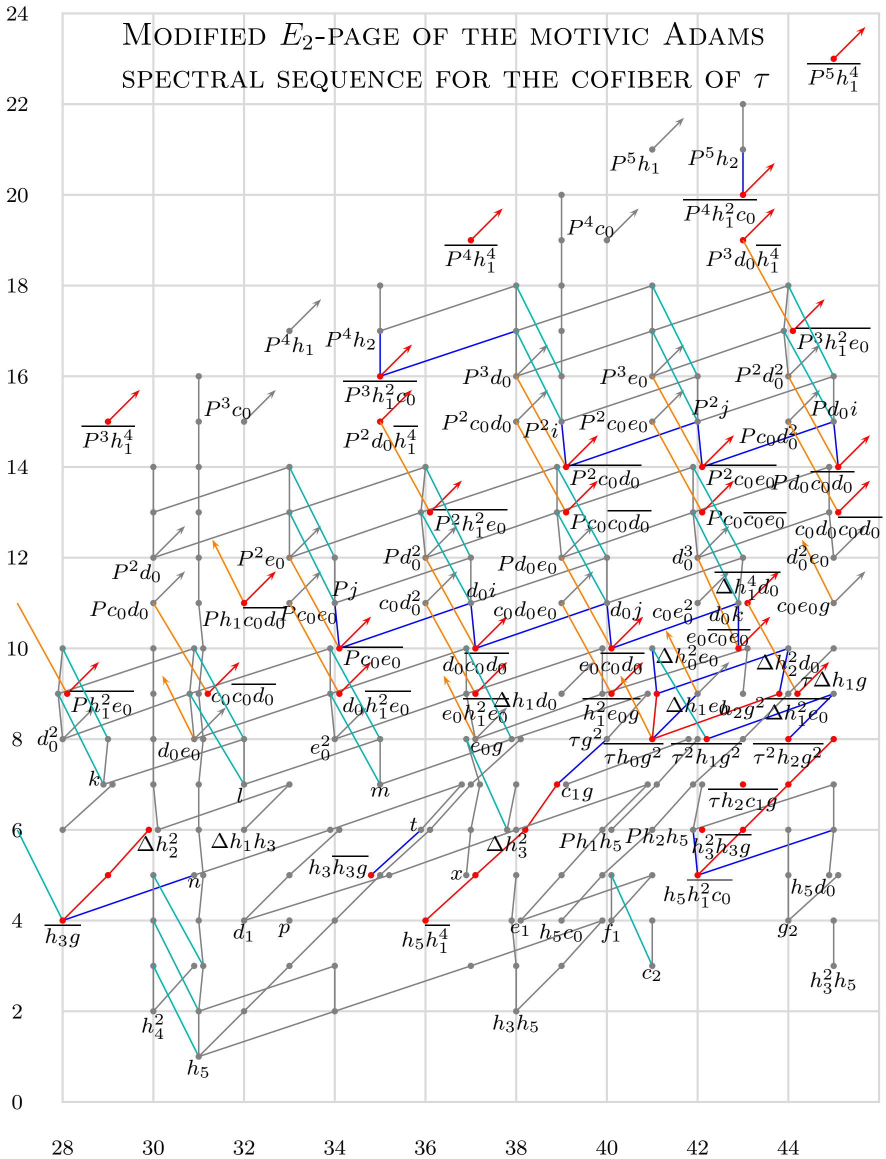
We include the following Isaksen's charts for the reader's reference of the differentials that are discussed in this appendix.

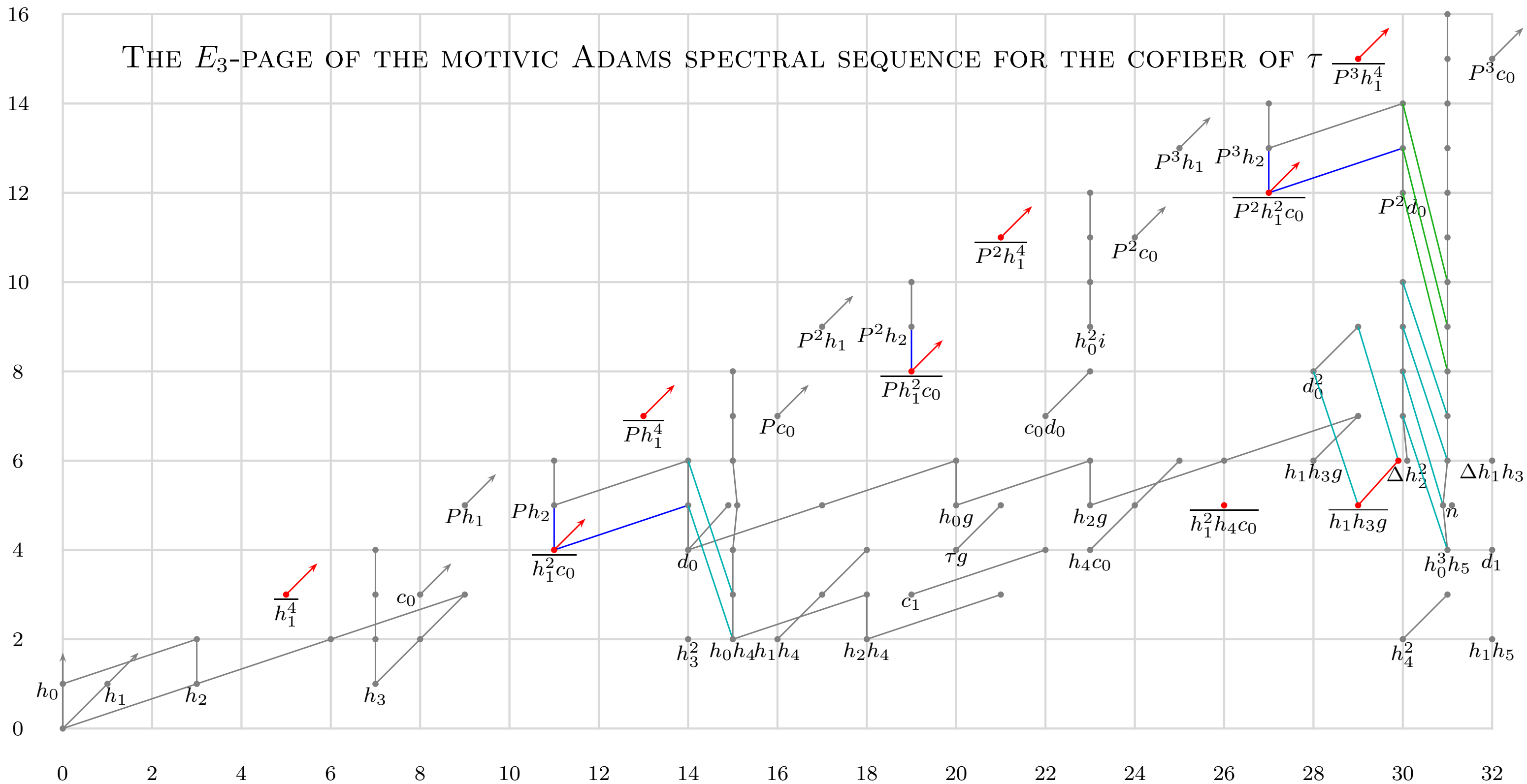

THE CLASSICAL ADAMS SPECTRAL SEQUENCE
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
32
h_0
h_1
h_2
h_3
c_0
Ph_1
Ph_2
h_3^2
h_4
d_0
e_0
f_0
Pc_0
P^2h_1
P^2h_2
c_1
g
h_4c_0
Pd_0
P^2c_0
i
Pe_0
j
P^3h_1
P^3h_2
d_0^2
k
P^2d_0
h_4^2
Δh_2^2
$d_0e_0 + h_0^7h_5$
n
h_5
P^3c_0
l
Δh_1h_3
d_1


THE CLASSICAL ADAMS SPECTRAL SEQUENCE
24
22
20
18
16
14
12
10
8
6
4
2
0
28
30
32
34
36
38
40
42
44
P^5h_1
P^5h_2
P^4c_0
P^4h_1
P^4h_2
P^3d_0
P^3e_0
$P^2d_0^2$
P^3c_0
P^2i
P^2j
Pd_0i
P^2d_0
P^2e_0
Pd_0^2
Pd_0e_0
d_0^3
$d_0^2e_0$
Pj
d_0i
d_0j
d_0k
$\Delta h_0^2e_0$
$\Delta h_2^2d_0$
Δh_1d_0
Δh_1e_0
Δh_1g
d_0^2
$d_0e_0+h_0^7h_5$
e_0^2
e_0g
g^2
k
l
m
Δh_2^2
Δh_1h_3
t
Δh_3^2
Ph_1h_5
Ph_2h_5
n
x
h_5d_0
d_1
p
e_1
h_5c_0
f_1
g_2
c_2
$h_3^2h_5$
h_4^2
h_3h_5
h_5

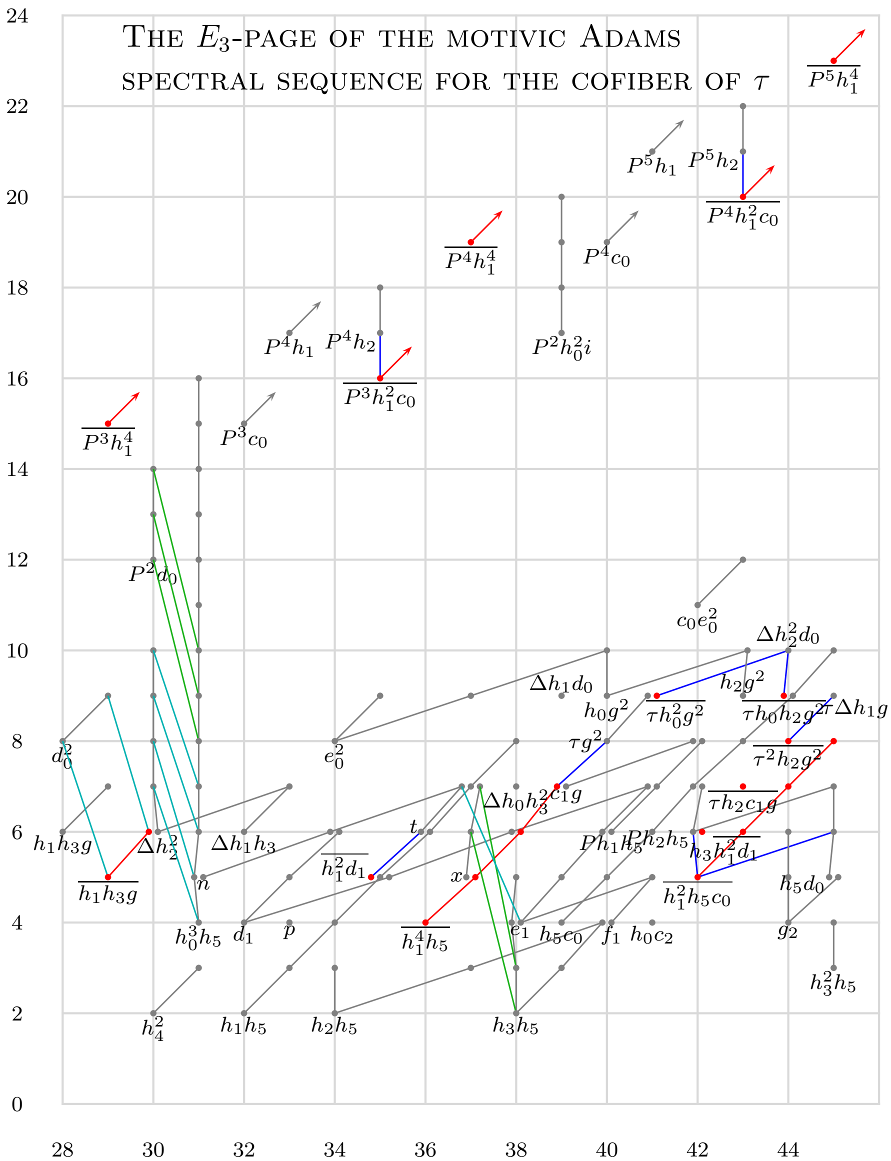
Modified E_2-page of the motivic Adams spectral sequence for the cofiber of τ




MODIFIED E_2-PAGE OF THE MOTIVIC ADAMS SPECTRAL SEQUENCE FOR THE COFIBER OF τ



The E_3-page of the motivic Adams spectral sequence for the cofiber of τ
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32
0 2 4 6 8 10 12 14 16
h_0
h_1
h_2
h_3
$\overline{h_1^4}$
c_0
Ph_1
Ph_2
$\overline{h_1^2 c_0}$
$\overline{Ph_1^4}$
d_0
h_3^2
$h_0 h_4$
$h_1 h_4$
Pc_0
$h_2 h_4$
c_1
$P^2 h_1$
$P^2 h_2$
$\overline{Ph_1^2 c_0}$
$h_0 g$
τg
$\overline{P^2 h_1^4}$
$c_0 d_0$
$h_0^2 i$
$h_2 g$
$h_4 c_0$
$P^2 c_0$
$\overline{h_1^2 h_4 c_0}$
$P^3 h_1$
$P^3 h_2$
$\overline{P^2 h_1^2 c_0}$
d_0^2
$h_1 h_3 g$
$\overline{h_1 h_3 g}$
$\overline{P^3 h_1^4}$
$P^2 d_0$
Δh_2^2
h_4^2
$h_0^3 h_5$
n
$P^3 c_0$
$\Delta h_1 h_3$
d_1
$h_1 h_5$


The E_3-page of the motivic Adams
spectral sequence for the cofiber of τ
$\overline{P^5h_1^4}$
P^5h_1
P^5h_2
$\overline{P^4h_1^2c_0}$
$\overline{P^4h_1^4}$
P^4c_0
P^4h_1
P^4h_2
$P^2h_0^2i$
$\overline{P^3h_1^2c_0}$
$\overline{P^3h_1^4}$
P^3c_0
P^2d_0
$c_0e_0^2$
$\Delta h_2^2d_0$
Δh_1d_0
h_2g^2
h_0g^2
$\overline{\tau h_0^2g^2}$
$\overline{\tau h_0h_2g^2}$
Δh_1g
τg^2
$\overline{\tau^2h_2g^2}$
d_0^2
e_0^2
$\Delta h_0h_3^2$
c_1g
$\overline{\tau h_2c_1g}$
h_1h_3g
Δh_2^2
Δh_1h_3
t
Ph_1h_5
Ph_2h_5
$h_3\overline{h_1^2d_1}$
$\overline{h_1h_3g}$
$\overline{h_1^2d_1}$
x
n
$\overline{h_1^2h_5c_0}$
h_5d_0
$h_0^3h_5$
d_1
p
$\overline{h_1^4h_5}$
e_1
h_5c_0
f_1
h_0c_2
g_2
$h_3^2h_5$
h_4^2
h_1h_5
h_2h_5
h_3h_5
0
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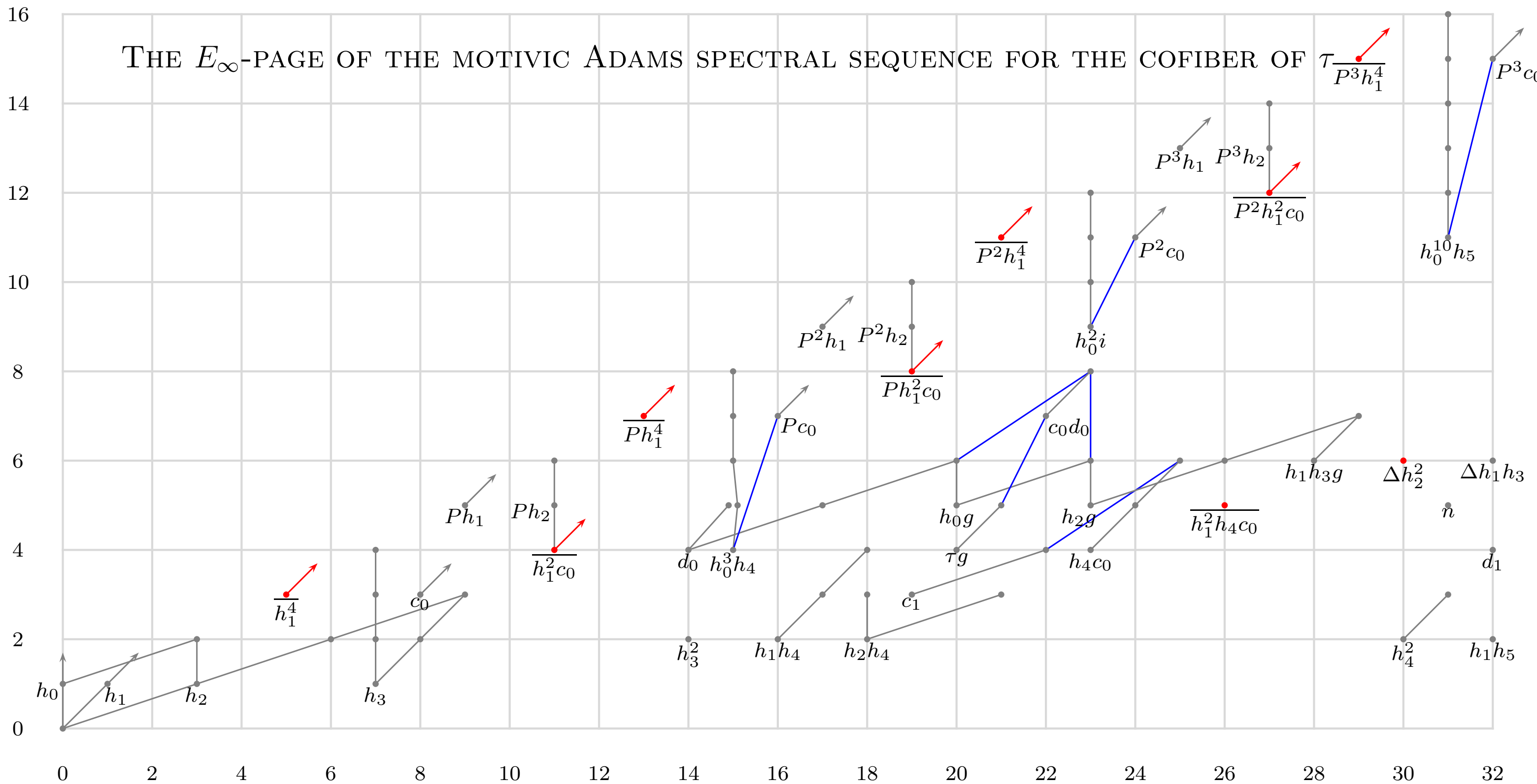
THE E_∞-PAGE OF THE MOTIVIC ADAMS SPECTRAL SEQUENCE FOR THE COFIBER OF τ
h_0
h_1
h_2
h_3
$\overline{h_1^4}$
c_0
Ph_1
Ph_2
$\overline{h_1^2c_0}$
$\overline{Ph_1^4}$
h_3^2
d_0
$h_0^3h_4$
Pc_0
h_1h_4
P^2h_1
P^2h_2
h_2h_4
c_1
$\overline{Ph_1^2c_0}$
τg
h_0g
$\overline{P^2h_1^4}$
c_0d_0
h_4c_0
h_2g
h_0^2i
P^2c_0
P^3h_1
$\overline{h_1^2h_4c_0}$
P^3h_2
$\overline{P^2h_1^2c_0}$
h_1h_3g
$\overline{P^3h_1^4}$
Δh_2^2
h_4^2
$h_0^{10}h_5$
n
P^3c_0
Δh_1h_3
d_1
h_1h_5
0
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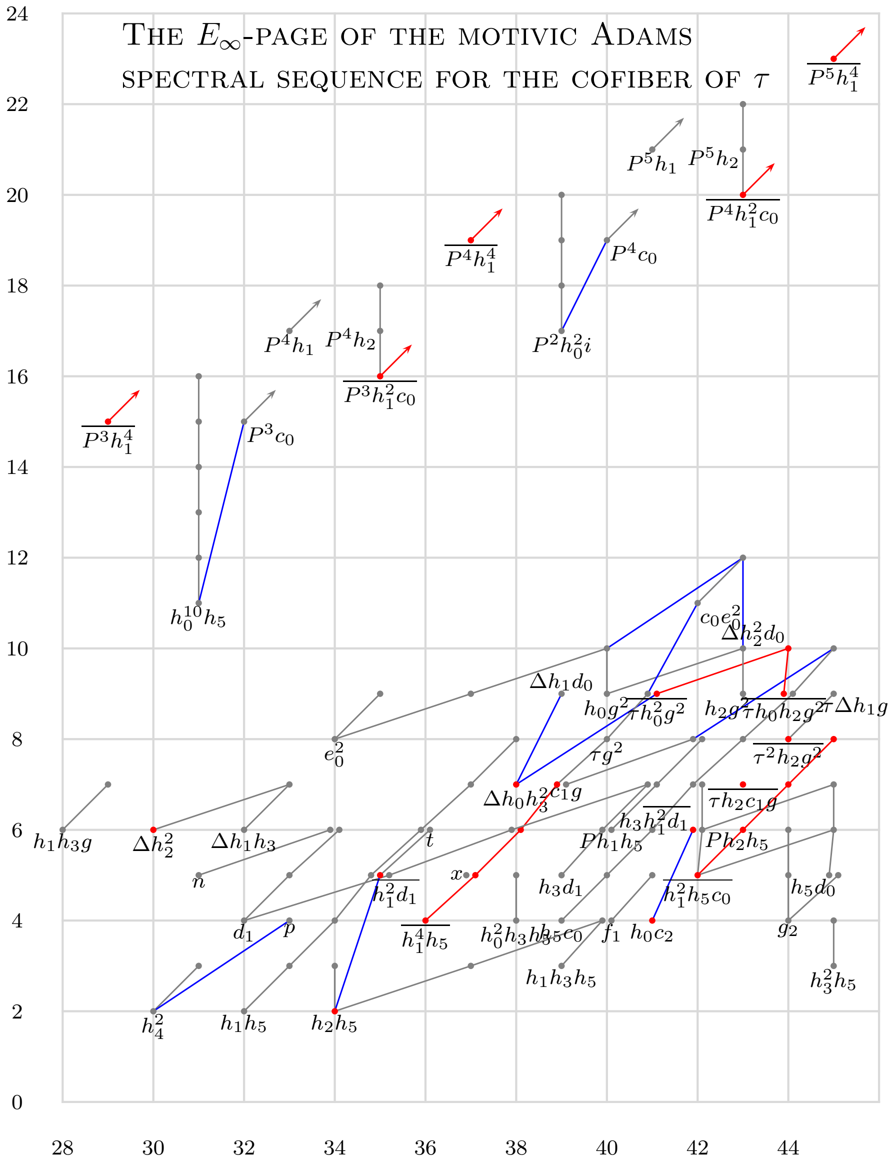
32


THE E_∞-PAGE OF THE MOTIVIC ADAMS
SPECTRAL SEQUENCE FOR THE COFIBER OF τ
$\overline{P^5h_1^4}$
P^5h_1
P^5h_2
$\overline{P^4h_1^2c_0}$
$\overline{P^4h_1^4}$
P^4c_0
P^4h_1
P^4h_2
$P^2h_0^2i$
$\overline{P^3h_1^2c_0}$
$\overline{P^3h_1^4}$
P^3c_0
$h_0^{10}h_5$
$c_0e_0^2$
$\Delta h_2^2d_0$
Δh_1d_0
h_0g^2
$\overline{\tau h_0^2g^2}$
h_2g^2
$\overline{\tau h_0h_2g^2}$
Δh_1g
e_0^2
τg^2
$\overline{\tau^2h_2g^2}$
$\Delta h_0h_3^2$
c_1g
$\overline{\tau h_2c_1g}$
$h_3\overline{h_1^2d_1}$
h_1h_3g
Δh_2^2
Δh_1h_3
t
Ph_1h_5
Ph_2h_5
n
x
$\overline{h_1^2d_1}$
h_3d_1
$\overline{h_1^2h_5c_0}$
h_5d_0
d_1
p
$\overline{h_1^4h_5}$
$h_0^2h_3h_5$
h_5c_0
f_1
h_0c_2
g_2
$h_1h_3h_5$
$h_3^2h_5$
h_4^2
h_1h_5
h_2h_5
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