A CONVENIENT CATEGORY FOR DIRECTED HOMOTOPY

L. FAJSTRUP AND J. ROSICKÝ*

ABSTRACT. We propose a convenient category for directed homotopy consisting of preordered topological spaces generated by cubes. Its main advantage is that, like the category of topological spaces generated by simplices suggested by J. H. Smith, it is locally presentable.

1. Introduction

We propose a convenient category for doing directed homotopy whose main advantage is its local presentability. It is based on the suggestion of J. H. Smith to use Δ-generated topological spaces as a convenient category for usual homotopy. His suggestion was written down by D. Dugger [7] but it turns out that it is not clear how to prove that the resulting category is locally presentable. We will present the missing proof and, in fact, we prove a more general result saying that for each fibre-small topological category $\mathcal{K}$ and each small full subcategory $\mathcal{I}$, the category $\mathcal{K}_{\mathcal{I}}$ of $\mathcal{I}$-generated objects in $\mathcal{K}$ is locally presentable. In the case of J. H. Smith, we take as $\mathcal{K}$ the category **Top** of topological spaces and continuous maps and as $\mathcal{I}$ the full subcategory consisting of simplices Δ_n, $n = 0, 1, \dots, n, \dots$. Recall that a category $\mathcal{K}$ is topological if it is equipped with a faithful functor $U : \mathcal{K} \to$ **Set** to the category of sets such that one can mimick the formation of "initially generated topological spaces" (see [2]). The category **d-Space** of d-spaces (in the sense of [11]) is topological and its full subcategory generated by suitably ordered cubes is our proposed convenient category for directed homotopy.

The idea of suitably generated topological spaces is quite old and goes back to [18] and [17] where the aim was to get a cartesian closed

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replacement of **Top**. The classical choice of $\mathcal{I}$ is the category of compact Hausdorff spaces. The insight of Smith is that the smallness of $\mathcal{I}$ makes $\mathbf{Top}_{\mathcal{I}}$ locally presentable. By [18] 3.3, $\mathbf{Top}_{\Delta}$ is even cartesian closed.

2. Locally presentable categories

A category $\mathcal{K}$ is *locally λ-presentable* (where λ is a regular cardinal) if it is cocomplete and has a set $\mathcal{A}$ of λ-presentable objects such that every object of $\mathcal{K}$ is a λ-directed colimit of objects from $\mathcal{A}$. A category which is locally λ-presentable for some regular cardinal λ is called *locally presentable*. Recall that an object K is λ-presentable if its hom-functor $\hom(K, -) : \mathcal{K} \to \mathbf{Set}$ preserves λ-filtered colimits. We will say that K is *presentable* if it is λ-presentable for some regular cardinal λ. A useful characterization is that a category $\mathcal{K}$ is locally presentable if and only if it is cocomplete and has a small dense full subcategory consisting of presentable objects (see [3], 1.20).

A distinguished advantage of locally presentable categories are the following two results. Recall that, given morphisms $f : A \to B$ and $g : C \to D$ in a category $\mathcal{K}$, we write

$$f \square g \qquad (f \perp g)$$

if, in each commutative square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "v"'] & D \end{tikzcd}$$

there is a (unique) diagonal $d : B \to C$ with $df = u$ and $gd = v$.

For a class $\mathcal{H}$ of morphisms of $\mathcal{K}$ we put

$$\begin{aligned} \mathcal{H}^{\square} &= \{g | f \square g \text{ for each } f \in \mathcal{H}\}, \\ {}^{\square}\mathcal{H} &= \{f | f \square g \text{ for each } g \in \mathcal{H}\}, \\ \mathcal{H}^{\perp} &= \{g | f \perp g \text{ for each } f \in \mathcal{H}\}, \\ {}^{\perp}\mathcal{H} &= \{f | f \perp g \text{ for each } g \in \mathcal{H}\}. \end{aligned}$$

The smallest class of morphisms of $\mathcal{K}$ containing isomorphisms and being closed under transfinite compositions and pushouts of morphisms from $\mathcal{H}$ is denoted as $\operatorname{cof}(\mathcal{H})$ while the smallest class of morphisms of $\mathcal{K}$ closed under all colimits (in the category $\mathcal{K}^{\to}$ of morphisms of $\mathcal{K}$) and containing $\mathcal{H}$ is denoted as $\operatorname{colim}(\mathcal{H})$.

Given two classes $\mathcal{L}$ and $\mathcal{R}$ of morphisms of $\mathcal{K}$, the pair $(\mathcal{L}, \mathcal{R})$ is called a *weak factorization system* if

(1) $\mathcal{R} = \mathcal{L}^{\square}$, $\mathcal{L} = {}^{\square}\mathcal{R}$

and

(2) any morphism h of $\mathcal{K}$ has a factorization $h = gf$ with $f \in \mathcal{L}$ and $g \in \mathcal{R}$.

The pair $(\mathcal{L}, \mathcal{R})$ is called a *factorization system* if condition (1) is replaced by

(1') $\mathcal{R} = \mathcal{L}^{\perp}$, $\mathcal{L} = {}^{\perp}\mathcal{R}$.

While the first result below can be found in [4] (or [1]), we are not aware of any published proof of the second one.

Theorem 2.1. *Let $\mathcal{K}$ be a locally presentable category and $\mathcal{C}$ a set of morphisms of $\mathcal{K}$. Then $(\mathrm{cof}(\mathcal{C}), \mathcal{C}^{\square})$ is a weak factorization system in $\mathcal{K}$.*

Theorem 2.2. *Let $\mathcal{K}$ be a locally presentable category and $\mathcal{C}$ a set of morphisms of $\mathcal{K}$. Then $(\mathrm{colim}(\mathcal{C}), \mathcal{C}^{\perp})$ is a factorization system in $\mathcal{K}$.*

Proof. It is easy to see (and well known) that

$$\mathrm{colim}(\mathcal{C}) \subseteq {}^{\perp}(\mathcal{C}^{\perp}).$$

It is also easy to see that $g : C \to D$ belongs to $\mathcal{C}^{\perp}$ if and only if it is orthogonal in $\mathcal{K} \downarrow D$ to each morphism $f : (A, vf) \to (B, v)$ with $f \in \mathcal{C}$. By [3], 4.4, it is equivalent to g being injective to a larger set of morphisms of $\mathcal{K} \downarrow D$. Since this larger set is constructed using pushouts and pushouts in $\mathcal{K} \downarrow D$ are given by pushouts in $\mathcal{K}$, $g : C \to D$ belongs to $\mathcal{C}^{\perp}$ if and only if it is injective in $\mathcal{K} \downarrow D$ to each morphism $f : (A, vf) \to (B, v)$ with $f \in \bar{\mathcal{C}}$ where $\bar{\mathcal{C}}$ is given as follows Given $f \in \mathcal{C}$, we form the pushout of f and f and consider a unique morphism f^* making the following diagram commutative

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[d, "p_2"] \arrow[ddr, "\mathrm{id}_B"] & \\
B \arrow[r, "p_1"'] \arrow[drr, "\mathrm{id}_B"'] & A^* \arrow[dr, "f^*"] & \\
 & & B
\end{tikzcd}$$

Then f^* belongs to $\operatorname{colim}(\mathcal{C})$ because it is the pushout of $f : f \to \mathrm{id}_B$ and $f : f \to \mathrm{id}_B$ in $\mathcal{K}^{\to}$ and $f, \mathrm{id}_B \in \operatorname{colim}(\mathcal{C})$:

$$\begin{tikzcd}
B \arrow[rrr, "\mathrm{id}_B"] \arrow[ddd, "\mathrm{id}_B"'] & & & B \arrow[ddd, "\mathrm{id}_B"] \\
& A \arrow[ul, "f"'] \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[ur, "\mathrm{id}_B"'] \arrow[d, "p_2"] & \\
& B \arrow[r, "p_1"'] \arrow[dl, "\mathrm{id}_B"] & A^* \arrow[dr, "f^*"] & \\
B \arrow[rrr, "\mathrm{id}_B"'] & & & B
\end{tikzcd}$$

Since $\bar{\mathcal{C}}$ is a set, $(\operatorname{cof}(\bar{\mathcal{C}}), \bar{\mathcal{C}}^{\square})$ is a weak factorization system (by 2.1). We have shown that

$$\bar{\mathcal{C}}^{\square} = \mathcal{C}^{\perp}$$

and

$$\bar{\mathcal{C}} \subseteq \operatorname{colim}(\mathcal{C}).$$

The consequence is that

$$\operatorname{cof}(\bar{\mathcal{C}}) \subseteq \operatorname{colim} \mathcal{C}.$$

It follows from the fact that each pushout of a morphism f belongs to $\operatorname{colim}(\{f\})$ (see [13], (the dual of) M13) and a transfinite composition of morphisms belongs to their colimit closure. In fact, given a smooth chain of morphisms $(f_{ij} : K_i \to K_j)_{i<j<\lambda}$ (i.e., λ is a limit ordinal, $f_{jk} f_{ij} = f_{ik}$ for $i < j < k$ and $f_{ij} : K_i \to K_j$ is a colimit cocone for any limit ordinal $j < \lambda$), let $f_i : K_i \to K$ be a colimit cocone. Then f_0, which is the transfinite composition of f_{ij} is a colimit in $\mathcal{K}^{\to}$ of the chain

$$\begin{tikzcd}
K_0 \arrow[r, "\mathrm{id}_{K_0}"] \arrow[d, "f_{00}"'] & K_0 \arrow[r] \arrow[d, "f_{01}"] & \cdots\cdots & K_0 \arrow[d, "f_0"] \\
K_0 \arrow[r, "f_{01}"'] & K_1 \arrow[r] & \cdots\cdots & K
\end{tikzcd}$$

Thus we have

$$\operatorname{cof}(\bar{\mathcal{C}}) \subseteq {}^{\perp}(\mathcal{C}^{\perp}).$$

Conversely

$${}^{\perp}(\mathcal{C}^{\perp}) \subseteq {}^{\square}(\mathcal{C}^{\perp}) = {}^{\square}(\bar{\mathcal{C}}^{\square}) = \operatorname{cof}(\bar{\mathcal{C}}).$$

We have proved that $(\operatorname{colim}(\mathcal{C}), \mathcal{C}^{\perp})$ is a factorization system. $\square$

3. Generated spaces

A functor $U : \mathcal{K} \to \mathbf{Set}$ is called *topological* if each cone

$$(f_i : X \to UA_i)_{i\in I}$$

in **Set** has a unique U-initial lift $(\bar{f}_i : A \to A_i)_{i\in I}$ (see [2]). It means that

(1) $UA = X$ and $U\bar{f}_i = f_i$ for each $i \in I$ and
(2) given $h : UB \to X$ with $f_i h = U\bar{h}_i$, $\bar{h}_i : B \to A_i$ for each $i \in I$ then $h = U\bar{h}$ for $\bar{h} : B \to A$.

Each topological functor is faithful and thus the pair $(\mathcal{K}, U)$ is a concrete category. Such concrete categories are called topological. The motivating example of a topological category is **Top**.

Example 3.1. (1) A preordered set $(A, \leq)$ is a set A equipped with a reflexive and transitive relation $\leq$. It means that it satisfies the formulas

$$(\forall x)(x \leq x)$$

and

$$(\forall x, y, z)(x \leq y \wedge y \leq z \to x \leq z).$$

Morphisms of preordered sets are isotone maps, ie., maps preserving the relation $\leq$. The category of preordered sets is topological. The U-initial lift of a cone $(f_i : X \to UA_i)_{i\in I}$ is given by putting $a \leq b$ on X if and only if $f_i(a) \leq f_i(b)$ for each $i \in I$.

(2) An ordered set is a preordered set $(A, \leq)$ where $\leq$ is also antisymmetric, i.e., if it satisfies

$$(\forall x, y)(x \leq y \wedge y \leq x \to x = y).$$

The category of ordered sets is not topological because the underlying functor to sets does not preserve colimits.

All three formulas from the example are strict universal Horn formulas and the difference between the first two and the third one is that antisymmetry uses the equality. It was shown in [16] that this situation is typical. But one has to use the logic $L_{\infty,\infty}$ (see [6]). It means that one has a class of relation symbols whose arities are arbitrary cardinal numbers and one uses conjunctions of an arbitrary set of formulas and quantifications over an arbitrary set of variables. A relational universal strict Horn theory T without equality then consists of formulas

$$(\forall x)(\varphi(x) \to \psi(x))$$

where x is a set of variables and φ, ψ are conjunctions of atomic formulas without equality. The category of models of a theory T is denoted by $\mathbf{Mod}(T)$.

Theorem 3.2. *Each fibre-small topological category $\mathcal{K}$ is isomorphic (as a concrete category) to a category of models of a relational universal strict Horn theory T without equality.*

This result was proved in [16], 5.3. A theory T can consist of a proper class of formulas. When T is a set, $\mathbf{Mod}(T)$ is locally presentable (see [3], 5.30). The theory for **Top** is given by an ultrafilter convergence (see [16], 5.4) and it was presented by Manes [15]. This theory is not a set of formulas. The category **Top** is far from being locally presentable because it does not have a small dense full subcategory (see [3], 1.24(7)) and no non-discrete space is presentable ([3], 1.14(6)).

A cone $(\bar{f}_i : A \to A_i)_{i\in I}$ is *U-initial* if it satisfies condition (2) above. Topological functors can be characterized as functors U such that each cocone $(f_i : UA_i \to X)_{i\in I}$ has a unique U-final lift $(\bar{f}_i : A_i \to A)_{i\in I}$ (see [2], 21.9). It means that

(1') $UA = X$ and $U\bar{f}_i = f_i$ for each $i \in I$ and

(2') given $h : X \to UB$ with $hf_i = U\bar{h}_i$, $\bar{h}_i : A_i \to B$ for each $i \in I$ then $h = U\bar{h}$ for $\bar{h} : A \to B$.

A cocone $(\bar{f}_i : A_i \to A)_{i\in I}$ is called *U-final* if it satisfies the condition (2').

Definition 3.3. Let $(\mathcal{K}, U)$ be a topological category and $\mathcal{I}$ a full subcategory of $\mathcal{K}$. An object K of $\mathcal{K}$ is called *$\mathcal{I}$-generated* if the cocone $(C \to K)_{C\in\mathcal{I}}$ consisting of all morphisms from objects of $\mathcal{I}$ to K is U-final.

Let $\mathcal{K}_{\mathcal{I}}$ denote the full subcategory of $\mathcal{K}$ consisting of $\mathcal{I}$-generated objects. Using the terminology of [2], $\mathcal{K}_{\mathcal{I}}$ is the *final closure* of $\mathcal{I}$ in $\mathcal{K}$ and $\mathcal{I}$ is *finally dense* in $\mathcal{K}_{\mathcal{I}}$.

Remark 3.4. Let $\mathcal{I}$ be a full subcategory of **Top**. A topological space X is *$\mathcal{I}$-generated* if it has the property that a subset $S \subseteq X$ is open if and only if $f^{-1}(S)$ is open for every continuous map $f : Z \to X$ with $Z \in \mathcal{I}$. Thus we get $\mathcal{I}$-generated spaces of [7] in this case.

We follow the terminology of [7] although it is somewhat misleading because, in the classical case of $\mathcal{I}$ consisting of compact Hausdorff spaces, the resulting $\mathcal{I}$-generated spaces are called k-spaces. A compactly generated space should also be weakly Hausdorff (see, e.g., [12]).

Proposition 3.5. *Let $(\mathcal{K}, U)$ be a topological category and $\mathcal{I}$ a full subcategory. Then $\mathcal{K}_{\mathcal{I}}$ is coreflective in $\mathcal{K}$ and contains $\mathcal{I}$ as a dense subcategory.*

Proof. By [2], 21.31, $\mathcal{K}_{\mathcal{I}}$ is coreflective in $\mathcal{K}$. Since $\mathcal{I}$ is finally dense in $\mathcal{K}_{\mathcal{I}}$, it is dense. $\square$

The coreflector $R : \mathcal{K} \to \mathcal{K}_{\mathcal{I}}$ assigns to K the smallest $\mathcal{I}$-generated object on UK.

A concrete category $(\mathcal{K}, U)$ is called *fibre-small* provided that, for each set X, there is only a set of objects K in $\mathcal{K}$ with $UK = X$.

Theorem 3.6. *Let $(\mathcal{K}, U)$ be a fibre-small topological category and let $\mathcal{I}$ be a full small subcategory of $\mathcal{K}$. Then the category $\mathcal{K}_{\mathcal{I}}$ is locally presentable.*

Proof. By 3.2, $\mathcal{K}$ is concretely isomorphic to $\mathbf{Mod}(T)$ where T is a relational universal strict Horn theory without equality. We can express T as a union of an increasing chain

$$T_0 \subseteq T_1 \subseteq \dots T_i \subseteq \dots$$

of subsets T_i indexed by all ordinals. The inclusions $T_i \subseteq T_j$, $i \leq j$ induce functors $H_{ij} : \mathbf{Mod}(T_j) \to \mathbf{Mod}(T_i)$ given by reducts. Analogously, we get functors $H_i : \mathbf{Mod}(T) \to \mathbf{Mod}(T_i)$ for each i. All these functors are concrete (i.e., preserve underlying sets) and have left adjoints

$$F_{ij} : \mathbf{Mod}(T_i) \to \mathbf{Mod}(T_j)$$

and

$$F_i : \mathbf{Mod}(T_i) \to \mathbf{Mod}(T).$$

These left adjoints are also concrete and $F_i(A)$ is given by the U-initial lift of the cone

$$f : U_i(A) \to U(B)$$

consisting of all maps f such that $f : A \to H_i(B)$ is a morphism in $\mathbf{Mod}(T_i)$. The functors F_{ij} are given in the same way. Since these left adjoints are concrete, they are faithfull and it immediately follows from their construction that they are also full. Thus we have expressed $\mathbf{Mod}(T)$ as a union of an increasing chain of full coreflective subcategories

$$\mathbf{Mod}(T_0) \subseteq \mathbf{Mod}(T_1) \subseteq \dots \mathbf{Mod}(T_i) \subseteq \dots$$

indexed by all ordinals. Moreover, all these coreflective subcategories are locally presentable.

Let $\mathcal{I}$ be a full small subcategory of $\mathcal{K}$. Then there is an ordinal i such that $\mathcal{I} \subseteq \mathbf{Mod}(T_i)$. Consequently, $\mathcal{K}_{\mathcal{I}} \subseteq \mathbf{Mod}(T_i)$ and thus $\mathcal{K}_{\mathcal{I}}$ is a full coreflective subcategory of a locally presentable $\mathbf{Mod}(T_i)$ having a small dense full subcategory $\mathcal{I}$. Since $\mathcal{I}$ is a set, there is a regular cardinal λ such that all objects from $\mathcal{I}$ are λ-presentable in $\mathbf{Mod}(T_i)$ (see [3], 1.16). Since $\mathcal{K}_{\mathcal{I}}$ is closed under colimits in $\mathbf{Mod}(T_i)$, each object from $\mathcal{I}$ is λ-presentable in $\mathcal{K}_{\mathcal{I}}$. Hence $\mathcal{K}_{\mathcal{I}}$ is locally λ-presentable. $\square$

Corollary 3.7. *Let $\mathcal{I}$ be a small full subcategory of* **Top**. *Then the category* $\mathbf{Top}_{\mathcal{I}}$ *is locally presentable.*

Remark 3.8. Let $\mathcal{K}$ be a category such that the coreflective closure $\mathcal{K}_{\mathcal{I}}$ of each small full subcategory $\mathcal{I}$ of $\mathcal{K}$ is locally presentable. Then $\mathcal{K}$ is a union of a chain

$$\mathcal{K}_0 \subseteq \mathcal{K}_1 \subseteq \dots \mathcal{K}_i \subseteq$$

of full coreflective subcategories which are locally presentable. It suffices to express $\mathcal{K}$ as a union of a chain

$$\mathcal{I}_0 \subseteq \mathcal{I}_1 \subseteq \dots \mathcal{I}_i \subseteq$$

of small full subcategories and pass to

$$\mathcal{K}_{\mathcal{I}_0} \subseteq \mathcal{K}_{\mathcal{I}_1} \subseteq \dots \mathcal{K}_{\mathcal{I}_i} \subseteq$$

Theorem 3.9. *Let $\mathcal{I}$ be a full subcategory of* **Top** *containing discs D_n and spheres S_n, $n = 0, 1, \dots$. Then the category* $\mathbf{Top}_{\mathcal{I}}$ *admits a cofibrantly generated model structure, where cofibrations and weak equivalences are the same as in* **Top**.

Proof. Analogous to [12], 2.4.23. □

4. Generated ordered spaces

In order to get our convenient category for directed homotopy, we have to replace **Top** by a suitable category of ordered topological spaces. We have considered two such categories:

- The category **PTop** of preordered topological spaces. Its objects are topological spaces whose underlying set is preordered. Morphisms are continuous maps f s.t. $x \leq y \Rightarrow f(x) \leq f(y)$.
- The category **d-Space** of topological spaces X with a set of paths $\vec{P}(X) \subset X^I$ (see 4.1).

These are all topological categories, i.e., the forgetful functor to **Set** is topological, and they are directed. We would like to have directed loops in the category, i.e., the circle S^1 with counterclockwise direction. In **PTop** we require transitivity, and hence, a relation relating pairs of points on the circle $e^{i\theta} \leq e^{i\phi}$ when $\theta \leq \phi$, will be the trivial relation in **PTop**

In **d-Space**, [11] the directions are represented in the allowed paths and not as a relation on the space itself. On a d-space, $(X, \vec{P}(X))$ the relation $x \leq y$ if there is $\gamma \in \vec{P}(X)$ s.t. $\gamma(0) = x$ and $\gamma(1) = y$ is gives a functor from **d-Space** to **PTop**. In the other direction, the increasing continuous maps from $\vec{I}$ to a space in **PTop** will give a set of dipaths, hence a functor to **d-Space**.

Definition 4.1. The objects in **d-Space** are pairs $(X, \vec{P}(X))$, where X is a topological space and $\vec{P}(X) \subset X^I$ satisfies

- All constant paths are in $\vec{P}(X)$
- $\vec{P}(X)$ is closed under concatenation and increasing reparametrization.

$\vec{P}(X)$ is called the set of dipaths or directed paths.

A morphism $f : (X, \vec{P}(X)) \to (Y, \vec{P}(Y))$ is a continuous map $f : X \to Y$ s.t. $\gamma \in \vec{P}(X)$ implies $f \circ \gamma \in \vec{P}(Y)$

In **d-Space** we do have directed circles.

Theorem 4.2. d-Space *is a topological category.*

Proof. Let T be a relational universal strict Horn theory without equality giving **Top** and using relation symbols R_j, $j \in J$. We add a new continuum-ary relation symbol R whose interpretation is the set of directed paths. We add to T the following axioms:

(1) $(\forall x)R(x)$ where x is the constant,
(2) $(\forall x, y, z)(\bigwedge_{0<i\leq\frac{1}{2}} z_t = x_t \wedge \bigwedge_{0<i\leq\frac{1}{2}} z_{\frac{1}{2}+i} = y_i \wedge x_1 = y_0 \wedge R(x) \wedge R(y) \to R(z))$,
(3) $(\forall x)(R(x) \to R(xt)$ where t is an increasing reparametrization,
(4) $(\forall x)(R(x) \to R_j(xa))$ where $j \in J$ and I satisfies R_j for a.

The resulting relational universal strict Horn theory axiomatizes d-spaces. In fact, (1) makes each constant path directed, (2) says that directed paths are closed under concatenation, (3) says that they are closed under increasing reparametrization and (4) says that they are continuous. $\square$

Remark 4.3. (i) A d-space is called *saturated* if it satisfies the converse implication to (3):

(5) $(\forall x)(R(xt) \to R(x)$ where t is an increasing reparametrization

It means that a path is directed whenever some of its increasing reparametrizations is directed. Thus saturated d-spaces also form a topological category.

(ii) There is, of course, a direct proof of 4.2. By [2], 21.9, it suffices to see that the forgetful functor $U : \textbf{d-Space} \to \textbf{Set}$ satisfies: For any cocone $(f_i : UA_i \to X)$ there is a unique *U-final lift* $(\bar{f}_i : A_i \to A)$, i.e., there is a unique **d-Space** structure on X such that $h : X \to UB$ is a d-morphism whenever $h \circ f_i$ is a d-morphism for all i. The topology is defined by V open if and only if $f_i^{-1}(V)$ open for all i. Let $\vec{P}(A)$ be the closure under concatenation and increasing reparametrization of the set of all constant paths and all $f_i \circ \gamma$ where $\gamma \in \vec{P}(A_i)$. It is not hard to see, that this is a U-final lift.

Corollary 4.4. *Let $\mathcal{I}$ be a small full subcategory of* **d-Space**. *Then the category* $\mathbf{d\text{-}Space}_{\mathcal{I}}$ *is locally presentable.*

Definition 4.5. Let $\mathcal{B}$ be the full subcategory of **d-Space** with objects all cubes $I_1 \times I_2 \times \ldots \times I_n$ where I_k is either the unit interval with the trivial order (i.e., $a \leq b$ for all a, b) or the unit interval with the standard order. The (pre)order on $I_1 \times I_2 \times \ldots \times I_n$ is the product relation. The dipaths are the increasing paths wrt. this relation.

Notation 4.6. Let I denote the unit interval with the trivial order and let $\vec{I}$ denote the unit interval with the standard order.

Corollary 4.7. *The category* $\mathbf{d\text{-}Space}_{\mathcal{B}}$ *is locally presentable.*

We consider the category $\mathbf{d\text{-}Space}_{\mathcal{B}}$ a suitable framework for studying the directed topology problems arising in concurrency. One reason for this is, that the geometric realization of a cubical complex is in $\mathbf{d\text{-}Space}_{\mathcal{B}}$. These are geometric models of Higher Dimensional Automata, see [9]. In [9], the directions on the spaces are given via a *local partial order* and not as d-spaces, but the increasing paths wrt. the local partial order are precisely the dipaths in the d-space structure.

For directed homotopy theory, this category is also suitable:

Definition 4.8. Let $f, g : X \to Y$ be d-maps. A d-homotopy [11] is a d-map $H : X \times \vec{I} \to Y$ s.t. $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$; the d-homotopy equivalence relation is the reflexive transitive hull of this relation. A d-homotopy of dipaths γ, μ with common initial and final points is a d-map $H : \vec{I} \times \vec{I} \to Y$ s.t. $H(t, 0) = \gamma(t)$, $H(t, 1) = \mu(t)$ and $H(0, s) = \gamma(0) = \mu(0)$ and $H(1, s) = \gamma(1) = \mu(1)$.

A dihomotopy [9] is unordered along the homotopy coordinate: $H : X \times I \to Y$. This gives an equivalence relation without closing off. Dihomotopic dipaths are defined as above - with fixed endpoints.

Since we allowed both the trivially ordered interval and the naturally ordered interval in $\mathcal{B}$, the category $\mathbf{d\text{-}Space}_{\mathcal{B}}$ is convenient for both kinds of directed homotopy.

Globes have been considered as models for higher dimensional automata, in [10]. A globe on a non-empty (d-)space X is the unreduced suspension $X \times \vec{I}/(x, 1) \sim *_1, (x, 0) \sim *_0$. If X is in $\mathbf{d\text{-}Space}_{\mathcal{B}}$ then clearly so is the globe of X as a coequalizer. The globe of the empty set is the d-space of two disjoint points, which is also in $\mathbf{d\text{-}Space}_{\mathcal{B}}$

The elementary globes, the globe of an unordered ball, are equivalent to the globe of an unordered cube, which is in our category.

5. Dicoverings

In [8], dicoverings, i.e., coverings of directed topological spaces are introduced as a counterpart of coverings in the undirected case. The categorical framework there is (subcategories of) locally partially ordered spaces. It turns out, that it is not obvious which category, one should choose to get universal dicoverings. With the framework here, we have a setting which on the one hand is much more general than the almost combinatorial one of cubical sets, and on the other hand, it is not as general as locally partially ordered topological spaces, where dicovering theory is certainly not well behaved. In [8] we consider dicoverings with respect to a basepoint, a fixed initial point.

Definition 5.1. Let $p : Y \to X$ be a morphism in **d-Space**, let $x_0 \in X$. Then p is a dicovering wrt. x_0 if for all $y_0 \in p^{-1}(x_0)$ and all $\gamma \in \vec{P}(X)$ with $\gamma(0) = x_0$, there is a unique lift $\hat{\gamma}$ with $\hat{\gamma}(0) = y_0$:

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d, hook] & Y \arrow[d, "p"] \\ \vec{I} \arrow[r, "\gamma"] \arrow[ur, dashed, "\hat{\gamma}"] & X \end{tikzcd}$$

And for all $H : I \times \vec{I} \to X$ with $H(s, 0) = x_0$, there is a unique lift $\hat{H}$:

$$\begin{tikzcd} (I \times \{0\}, I \times \{0\}) \arrow[r] \arrow[d, hook] & (Y, y_0) \arrow[d, "p"] \\ (I \times \vec{I}, I \times \{0\}) \arrow[r, "H"] \arrow[ur, dashed, "\hat{H}"] & (X, x_0) \end{tikzcd}$$

In the present framework however, we will consider lifting properties wrt. all initial points: Let J be the coequalizer

$$\begin{tikzcd} I \arrow[r, bend left, "g"] \arrow[r, bend right, "f"'] & I \times \vec{I} \arrow[r] & J \end{tikzcd}$$

where $f(x) = (0, 0)$ and $g(x) = (x, 0)$.

Definition 5.2. Let $p : Y \to X$ be a morphism in **d-Space**. Then p is a dicovering, if for all $\gamma \in \vec{P}(X)$ there is a unique lift $\hat{\gamma}$

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d, hook] & Y \arrow[d, "p"] \\ \vec{I} \arrow[r, "\gamma"] \arrow[ur, dashed, "\hat{\gamma}"] & X \end{tikzcd}$$

and for all $H : J \to X$ there is a unique lift

$$\begin{tikzcd} * \arrow[r] \arrow[d, hook] & Y \arrow[d, "p"] \\ J \arrow[r, "\gamma"'] \arrow[ur, dashed, "\hat{\gamma}"] & X \end{tikzcd}$$

where $*$ is the point $(x, 0) \in J$.

Hence, a dicovering is a morphism $p : Y \to X$ which has the unique right lifting property with respect to the inclusions $\mathcal{C} = \{0 \to \vec{I}, * \to J\}$. Hence

Proposition 5.3. *A morphism $p : Y \to X$ in* **d-Space** *is a dicovering if and only if it is in $\mathcal{C}^{\perp}$.*

Definition 5.4. A *universal dicovering* of $X \in$ **d-Space**$_{\mathcal{B}}$ is a morphism $\pi : \tilde{X} \to X$ such that for any dicovering $p : Y \to X$ in **d-Space**$_{\mathcal{B}}$, there is a unique morphism $\phi : \tilde{X} \to Y$ such that $\pi = p \circ \phi$.

Corollary 5.5. *Let $X \in$* **PTop**$_{\mathcal{B}}$*. Then there is a universal dicovering $\pi : \tilde{X} \to X$, and it is unique.*

Proof. This follows from 2.2, since **d-Space**$_{\mathcal{B}}$ is locally presentable. Let

$$0 \xrightarrow{w} \tilde{X} \xrightarrow{u} X$$

be the $(\mathrm{colim}(\mathcal{C}), \mathcal{C}^{\perp})$ factorization of the unique morphism from the initial object 0 (the empty set) to X. Then $u : \tilde{X} \to X$ is a universal dicovering of X. In fact, each dicovering

$$v : Y \to X$$

has a unique factorization through u. It suffices to apply the unique right lifting property to

$$\begin{tikzcd} \tilde{X} \arrow[r, "u"] & X \\ 0 \arrow[u] \arrow[r] & Y \arrow[u, "v"'] \end{tikzcd}$$

□

In [8], we construct a "universal" dicovering $\pi : \tilde{X} \to X$ by endowing the set of dihomotopy classes of dipaths initiating in a fixed point x_0 with a topology and a local partial order. If all points in X are reachable by a directed path from x_0 and if X is in **d-Space**$_{\mathcal{B}}$ the construction here and the underlying d-space of the locally partially ordered space $\tilde{X}$ in [8] should coincide, but we do not have a proof of this yet.

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L. Fajstrup
Department of Mathematics
University of Aalborg
Fredrik Bajers Vej 7G, DK9220 Aalborg Øst, Denmark
Denmark
fajstrup@math.aau.dk

J. Rosický
Department of Mathematics and Statistics
Masaryk University, Faculty of Sciences,
Janáčkovo nám. 2a, 60200 Brno, Czech Republic
rosicky@math.muni.cz