FROM ELLIPTIC CURVES TO HOMOTOPY THEORY

MIKE HOPKINS AND MARK MAHOWALD

ABSTRACT. A surprising connection between elliptic curves over finite fields and homotopy theory has been discovered by Hopkins. In this note we will follow this development for the prime 2 and discuss the homotopy which developed from this.

A preliminary report

1. Introduction

The path which we wish to follow begins with elliptic curves over finite fields and in particular over $\mathbb{F}_4$. From such a curve we get a formal group which will have height 2. The Lubin-Tate deformation theory constructs a formal group over the ring $\mathbb{W}_{\mathbb{F}_4}[[a]][u, u^{-1}]$. It can be shown that this ring is the homotopy of a spectrum, E_2, which is MU orientable. The group of automorphisms of the formal group over $\mathbb{F}_4$ acts on this ring. The Hopkins-Miller theory constructs a lift of this action to an action on the spectrum E_2. This group is a profinite group, called the Morava stabilizer group S_2. There is a finite subgroup G of S_2 of order 24 which is the automorphism group of the elliptic curve. This finite group acts on E_2 and we define $EO_2 = E_2^{hG}$. It is the torsion homotopy of this spectrum which illuminates much of the homotopy of spheres in the known range.

We begin with the curve, $x^3 + y^2 + y = 0$ in $\mathbb{P}^2(\mathbb{F}_4)$. In the elliptic curve literature this is called a supersingular curve. It is non-singular and has one point on the line at infinity. If we represent $\mathbb{F}_4$ as the set $\{0, 1, \rho, \rho^2\}$ where $1 + \rho + \rho^2 = 0$, then the solution set in the affine plane consists of eight points. If $x = 0$ then $y = 0$ or 1. If $x \in {\mathbb{F}_4}^+$ then $y = \rho^i$ for $i = 1, 2$. The group of the elliptic curve is $\mathbb{F}_3 \oplus \mathbb{F}_3$.

The group of affine transformations of ${\mathbb{F}_4}^2$ consists of matrices

$$\begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

Date: July 20, 1998.

Those which leave the equation of the curve alone satisfy

$$a = \alpha \in {\mathbb{F}_4}^{+} \tag{1.1}$$
$$b = 0 \tag{1.2}$$
$$c^3 = f + f^2 \tag{1.3}$$
$$d = ac^2 \tag{1.4}$$
$$e = 1 \tag{1.5}$$

It is easy to verify that this group G has order 24 and is $SL(\mathbb{F}_3, 2)$. If we include the Galois action we get a $\mathbb{Z}/2$ extension of this group. Let G_{16} be the 2 primary part. We have the following result, which is well known. It will illuminate the latter calculations.

Theorem 1.1. *If we suppress the topological degree, then*

$$H^*(G_{16}, \mathbb{Z}/2) \simeq \mathrm{Ext}^*_{A(1)}(\mathbb{Z}/2, \mathbb{Z}/2).$$

Our program will be to construct a formal group from the group of this elliptic curve. Then G will be a group of automorphisms of this formal group. We will lift the curve to the ring $\mathbb{W}_{\mathbb{F}_4}[[a]]$ as

$$y^2 + a\ xy + y = x^3.$$

We can lift G as a group of automorphisms of this curve. Then the formal group associated to this curve will be the universal formal group given by the Lubin-Tate theory. The E_2 term of the Adams-Novikov spectral sequence to calculate EO_{2*} will be $H^*(G; \mathbb{W}_{\mathbb{F}_4}[[a]][u, u^{-1}])^{Gal}$.

2. The formal group

The material of this section is standard. We will include it for completeness for the homotopy theory reader who might not be familiar with the algebraic theory of elliptic curves.

The formal group constructed from an elliptic curve is constructed by resolving the multiplication on the curve around the point at infinity which is taken as the unit of the group. First we construct a parametric represention in terms of an uniformizer at infinity. Let

$$w = y^{-1} \tag{2.6}$$
$$z = x/y \tag{2.7}$$

Then the equation of the curve becomes $w = z^3 + w^2$. We have not noted signs since we are working over $\mathbb{F}_4$.

Proposition 2.1. (i) $w(z) = \Sigma_{i \geq 0} z^{3\ 2^i}$.
(ii) $x(z) = z/w(z) = z^{-2} + z + z^4 + z^{10} + \cdots$
(iii) $y(z) = 1/w(z) = z^{-3} + 1 + z^3 + z^9 + \cdots$

This is an easy calculation. At this point one can follow the discussion in Silverman [14] page 114. This discussion is considerably simplified by the fact that the field has characteristic 2. This gives the following result.

Proposition 2.2. *The formal group constructed from the elliptic curve,* $x^3 + y^2 + y = 0$ *over* $\mathbb{F}_4$ *has as the first few terms*

$$F(u,v) = u + v + u^2v^2 + u^4v^6 + u^6v^4 + u^4v^{12} + u^{12}v^4 + u^8v^8 + \cdots .$$

The next term has degree 22. This is a formal group of height 2 and the 2-series is $z^4(\Sigma_{i\geq 0} z^{12(2^i-1)})$.

Next we want to lift this formal group to a formal group over the ring $\mathbb{W}_{\mathbb{F}_4}[[a]][u, u^{-1}]$ which gives the above curve under the quotient map to $\mathbb{F}_4$. We will do this by just lifting the elliptic curve. The formal group is then constructed in the usual way as is done in [14]. The equation of the lifted curve is

$$x^3 = y^2 + a\ u\ x\ y + u^3 y.$$

We want to lift our group as a group of affine transformations which leave the curve alone. Thus we want to make the substitutions

$$x \mapsto \alpha x + u^2 r \tag{2.8}$$

$$y \mapsto y + u\ s\ x + u^3 t \tag{2.9}$$

In order to preserve the curve we require that the coefficient of x^2, x, and the constant term all be zero. This gives

$$3r = s^2 + s\ a \tag{2.10}$$

$$s = 3r^2 - 2ast - a(rs + t) \tag{2.11}$$

$$t = r^3 - art - t^2 \tag{2.12}$$

The group G is generated by $\alpha \in {\mathbb{F}_4}^+$ and a pair (β, γ) which satisfies the equation $\beta^3 + \gamma + \gamma^2 = 0$. We can take two generators, $\alpha = \rho, (\beta, \gamma) = (0,0)$ and $\alpha = 1, (\beta, \gamma) = (1, \rho)$ and lift these. The rest of group will be various products of these. It is clear how to lift the first. We will concentrate on the second. We want to find infinite series for r, s and t which reduce to $1, 1,$ and ρ modulo the maximum ideal. We begin with these equations and successively substitute into the above equations giving

$$r(a) = (1/3)(1 + a) \tag{2.13}$$

$$s(a) = (1/3)(1 + 2a + a^2) - 2a\rho - a((1/3)(1 + a) + \rho) \tag{2.14}$$

$$t(a) = (1/3)^3(1 + a)^3 - (1/3)a(1 + a)s(a) - \rho^2 \tag{2.15}$$

$$r(a) = (1/3)(s(a)^2 + as(a)) \tag{2.16}$$

$$s(a) = 3r(a)^2 - 2as(a)t(a) - a(r(a)s(a) + t(a)) \tag{2.17}$$

$$t(a) = r(a)^3 - ar(a)t(a) - t(a)^2 \tag{2.18}$$

Each time we substitute the formula for the classes on the right hand side from the formulas above. After each process we have correct liftings modulo the maximum ideal raised to one higher power. That this works is just Hensel's Lemma. Compare [14], page 112.

What we have constructed is a map $G \to \mathbb{W}_{\mathbb{F}_4}[[a]][u, u^{-1}]$. This is the beginning of a co-simplical complex

$$\mathbb{W}_{\mathbb{F}_4}[[a]][u, u^{-1}] \Rightarrow \mathrm{Hom}(G, \mathbb{W}_{\mathbb{F}_4}[[a]][u, u^{-1}]) \cdots$$

The action of G on $\mathbb{W}_{\mathbb{F}_4}[[a]][u, u^{-1}]$ is the only additional part to add. That defines

$$a \mapsto a + 2s \tag{2.19}$$

$$u^3 \mapsto u^3 + ar + 2t. \tag{2.20}$$

The homology of this co-simplical complex is the E_2 term of the Adams-Novikov spectral sequence to calculate the homotopy of the Hopkins-Miller spectrum EO_2. We will do this calculation in several ways but the key will be to show that it is something which is already known.

3. The elliptic curve Hopf algebroid

The Weierstrass form of an elliptic curve is usually written

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

A change of coordinates does not change the curve and so substituting

$$\begin{aligned} x &= x' + r \\ y &= y' + sx' + t \end{aligned}$$

gives us the same curve. The coefficients transfer according to the following table. (Compare [14].)

$$\begin{aligned} a_1' &= a_1 + 2s \\ a_2' &= a_2 - sa_1 + 3r - s^2 \\ a_3' &= a_3 + ra_1 + 2t \\ a_4' &= a_4 - sa_3 + 2ra_2 - (t + rs)a_1 + 3r^2 - 2st \\ a_6' &= a_6 + ra_4 + r^2a_2 + r^3 - ta_3 - t^2 - rta_1 \end{aligned}$$

These formulas are very suggestive of the structure formulas which result from MU_* resolutions. Indeed, we can take these formulas to be the definition of η_R and get a Hopf algebroid

$$(A, \Lambda) = (\mathbb{Z}[a_1, a_2, a_3, a_4, a_6], \mathbb{Z}[a_1, a_2, a_3, a_4, a_6, s, r, t]).$$

The two maps from $A \to \Lambda$ are the inclusion and the one given by the table above. In books such as [14] the classes c_4 and c_6 are usually given and they represent classes in the homology in dimension zero of the simplical complex constructed from the above Hopf algebroid. The formulas for them are

$$\begin{aligned} c_4 &= (a_1^2 + 4a_2)^2 - 24(2a_4 + a_1a_3) \\ c_6 &= -(a_1^2 + 4a_2)^3 + 36(a_1^2 + 4a_2)(2a_4 + a_1a_3) - 21(a_3^2 + 4a_6) \end{aligned}$$

Notice that $c_4^3 - c_6^2$ is divisible by 1728. Let $\Delta = (c_4^3 - c_6^2)/1728$. The zero dimensional homology of the above chain complex is

$$\mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - 1728\Delta).$$

One of our questions is the computation of the rest of this chain complex. We will do this by getting another interpretation of the chain complex. For this interpretation we will have a complete calculation. Before we do this we want to connect this resolution with the Lubin-Tate theory.

In section 2 we consider the elliptic curve over $\mathbb{W}_{\mathbb{F}_4}[[a]][u, u^{-1}]$ given by the equation $y^2 + a\ u\ xy + u^3 y = x^3$. Thus we have a map $f : A[\rho]/(\rho^2 + \rho + 1) \to \mathbb{W}_{\mathbb{F}_4}[[a]][u, u^{-1}]$ defined by

$$\begin{aligned} a_1 &\mapsto && au \\ a_3 &\mapsto && u^3 \\ a_i &\mapsto && 0, \text{otherwise} \end{aligned}$$

Theorem 3.1. *After completing $A[\rho]/(\rho^2 + \rho + 1)$ at the ideal $(2, a_1)$ and inverting Δ, the map f induces an isomorphism between the two chain complexes.*

Corollary 3.2. *The E_2 term of the Adams-Novikov spectral sequence to compute the homotopy of EO_2 is the homology of the Hopf algebroid (A, Λ) completed at the ideal $(2, a_1)$ with Δ inverted.*

In the next section we will show that this computation is well known.

4. Ring spectrum resolutions

Using Bott periodicity we have a map $\gamma : \Omega SU(4) \to BU$. Let T be the resulting Thom complex. As is the case with any ring spectrum we can construct a resolution $\mathbb{T}$

$$S^0 \to T \overset{\to}{\underset{\to}{}} T \wedge T \overset{\to}{\underset{\to}{\to}} T \wedge T \wedge T \cdots .$$

This is acyclic from its definition. The first step in understanding such resolutions is the following version of the Thom isomorphism theorem.

Proposition 4.1. *There is a homotopy equivalence $T \wedge \Omega SU(4)_+ \cong T \wedge T$ This homotopy equivalence is induced by a map between the base spaces*

$$\Omega SU(4) \times \Omega SU(4) \overset{\Delta, id}{\to} \Omega SU(4) \times \Omega SU(4) \times \Omega SU(4) \overset{id, \mu}{\to} \Omega SU(4) \times \Omega SU(4).$$

Here, Δ is the map which sends $x \to (x, -x)$ and μ is the loop space multiplication.

The map in Thom complexes induce by this composite is $T \wedge \Omega SU(4)_+ \cong T \wedge T$. Let $\bar{T}$ be the cofiber of the unit map. Then $T \wedge \bar{T} \cong T \wedge \Omega SU(4)$. It is the T_* homotopy of $\Omega SU(4)$ which describes the T Hopf algebroid. One of the main results of [12] is following.

Proposition 4.2. *The map $d = \eta_L - \eta_R$ can be viewed as a map $T \to T \wedge \Omega SU(4)$ which is induced by the diagonal*

$$\Delta : \Omega SU(4) \to \Omega SU(4) \times \Omega SU(4).$$

Let $b_i \in H_{2i}(\mathbb{C}P)$ be the homology generators. We will identify these classes with their image in $H_*(\Omega SU(4))$. Thus

$$H_*(\Omega SU(4)) \cong \mathbb{Z}[b_1, b_2, b_3].$$

The homotopy classes in $\pi_*(T)$ which are in the Hurewicz image are multiples of primitive classes. On the other hand the classes b_i are not primitive for $i > 1$. We have:

$$\begin{aligned}
\Delta b_1 &= b_1 \otimes 1 + 1 \otimes b_1 \\
\Delta b_2 &= b_2 \otimes 1 + b_1 \otimes b_1 + 1 \otimes b_2 \\
\Delta b_3 &= b_3 \otimes 1 + b_2 \otimes b_1 + b_1 \otimes b_2 + 1 \otimes b_3
\end{aligned}$$

Thus we can define primitive classes as follows:

$$\begin{aligned}
m_1 &= b_1 \\
m_2 &= 2b_2 - b_1^2 \\
m_3 &= 3(b_3 - b_1 b_2) + b_1^3
\end{aligned}$$

This allows us to define homotopy classes

$$\begin{aligned}
a_1 &= 2m_1 \\
a_2 &= 3m_2 - m_1^2 \\
a_3 &= 2m_3
\end{aligned}$$

We define additional classes

$$\begin{aligned}
a_4 &= 3m_2^2 - 2m_1 m_3 \\
a_6 &= m_2^3 - m_3^2
\end{aligned}$$

Then we calculate da_i by the following rules:

- compute Δa_i
- drop each class of the form $x \otimes 1$.
- classes of the form $x \otimes m_1$ are written as xs
- classes of the form $x \otimes m_2$ are written as xr
- classes of the form $x \otimes m_3$ are written as xt
- classes of the form $x \otimes y$ must have $x \in \mathbb{Z}[a_1, a_2, a_3]$. We write them as xy.

If $\Omega SU(4)$ stably split as a wedge of spheres, then $T \wedge \Omega SU(4)$ would give the free splitting of $T \wedge T$ into a wedge of T's. This is what $A[s, r, t]$ represents. But $\Omega SU(4)$ does not split in this manner. It would be enough if the pieces into which $\Omega SU(4)$ does split would be trivial T_* modules but that is not true either. The further splitting produces an extra term, $a_1 r$ in the expression for $\eta_R a_3$. This gives us:

$$\begin{aligned}
\eta_R a_1 &= a_1 + 2s \\
\eta_R a_2 &= a_2 + 3r - a_1 s - s^2 \\
\eta_R a_3 &= a_3 + 2t + a_1 r \\
\Delta a_4 &= 3(m_2 \otimes 1 + 1 \otimes m_2)^2 \\
&\quad - 2(m_3 \otimes 1 + 1 \otimes m_3 + m_1 \otimes m_2)(m_1 \otimes 1 + 1 \otimes m_1) \\
\eta_R a_4 &= a_4 + 2a_2 r + 3r^2 - a_3 s - st - a_1 t - a_1 sr \\
\eta_R a_6 &= a_6 + a_4 r + a_2 r^2 + r^3 - a_3 t - t^2 - a_1 rt
\end{aligned}$$

Thus, we have reproduced the formulas constructed in the previous section from the change of variables formulas. We still need to get the setting where the polynomial algebra $\mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$ does represent the homotopy of something.

Let E be any spectrum. If we smash the resolution $\mathbb{T}$ with E, we still have an acyclic complex with augmentation E. If we apply homotopy, we get a complex whose homology is the E_2 term of a spectral sequence to compute the homotopy of E. We need a spectrum E so that $\pi_*(E \wedge T) \cong \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$. Hopkins and Miller [7] have constructed a spectrum which almost works. In a latter section a connected version of the Hopkins-Miller spectrum eo_2 is constructed. It has the key properties:

Theorem 4.3. *Let $D(A_1)$ be a spectrum whose cohomology, as a module over the Steenrod algebra is free on Sq^2 and Sq^4. Then localized at 2, $eo_2 \wedge D(A_1) \cong BP\langle 2\rangle$. Let X be the spectrum whose cohomology, as a module over the mod 3 Steenrod algebra is free on P^1, the localized at 3, $eo_2 \wedge X \cong BP\langle 2\rangle \wedge (S^0 \vee S^8)$*

Corollary 4.4. $\pi_*(eo_2 \wedge T) \cong \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$ *and*

$$\pi_*(eo_2 \wedge T \wedge T) \cong \mathbb{Z}[a_1, a_2, a_3, a_4, a_6] \otimes \mathbb{Z}[s, r, t].$$

An easy calculation gives us the following.

Theorem 4.5. *If we apply the functor $\mathrm{Ext}_{A(2)}(\_, \mathbb{Z}/2)$ to the resolution $\mathbb{T}$, we get*

$$\mathbb{Z}/2[v_0, v_1, a_2, v_3, a_4, a_6] \to \mathbb{Z}/2[v_0, v_1, a_2, v_3, a_4, a_6, s, r, t] \to \cdots$$

where a_i has filtration 0. This chain complex will compute $\mathrm{Ext}_{A(2)}(\mathbb{Z}/2, \mathbb{Z}/2)$. In particular, this implies $H^*(eo_2, \mathbb{Z}/2) \simeq A \otimes_{A(2)} \mathbb{Z}/2$.

5. An outline of the calculation

In the rest of this paper we will discuss the homotopy of the spectrum constructed by Hopkins and Miller [7] which they labeled EO_2. We will also be quite interested in the Hurewicz image in $\pi_*(S^0)$.

Theorem 5.1. *The action of S_2 on E_{2*} lifts to an E_∞ ring action of S_2 on E_2. Furthermore, S_2 has a subgroup of order 24, $GL(\mathbb{F}_3, 2)$. This group can be extended by the Galois group, $\mathbb{Z}/2$. The group of order 48 acts on EO_2 and the homotopy fixed point set of this action defines EO_2. In addition,*

$$EO_2 \wedge D(A_1) = E_2$$

.

We will take this result as an axiom for the rest of this paper. We will calculate the homotopy of EO_2 in two ways. First we will construct a spectral sequence which untangles the formula $EO_2 \wedge D(A_1) = E_2$. This is done in the next section. Next we will consider the connected cover of EO_2 and show that it essentially has $A \otimes_{A(2)} \mathbb{Z}/2$ as its cohomology. We then have an Adams spectral sequence calculation which has been known for about twenty years. This approach allows one to have available a rather large collection of spaces whose EO_{2*} homology has been computed. See for example [2].

These results also give a counter example to the main result of [3] which asserted that $A \otimes_{A(2)} \mathbb{Z}/2$ could not be the cohomology of a spectrum. The error in that paper can be traced to a homotopy calculation in [4] which was in error. The correction of the appropriate homotopy calculation is done in [9].

In the last section we discuss homotopy classes in the spheres which can be constructed by this spectrum.

6. The homotopy of EO_2

Our first calculation of EO_{2*} uses the formula

$$EO_2 \wedge D(A_1) = E_2$$

The CW complex $D(A_1)$ is constructed by the following lemma where we use the notation $M_\alpha = S^0 \cup e^{|\alpha|+1}$.

Lemma 6.1. *There is a map*

$$\gamma : \Sigma^5 M_\eta \wedge M_\nu \to M_\eta \wedge M_\nu$$

Proof. This is a straightforward calculation in $\pi_*(S^0)$. □

We can use the definition of $D(A_1)$ and the formula in (2.1) to construct a spectral sequence. Abstractly, we think of $D(A_1)$ as constructed out of three mapping cones, M_η, M_ν and M_γ where γ is defined in the Lemma. Thus we have a contracting homomorphism in $P(h_1, h_2, h_{2,0}) \otimes H_*(D(A_1))$ with $d\, h_1 = e_\eta$, $d\, h_2 = e_\nu$ and $d\, h_{2,0} = e_\gamma$. We will use the defining equation (2.1) to give us a free E_2 resolution. For the moment we want to think of this as an unfiltered but graded object. There is a total differential whose homology is EO_{2*}. If we assign filtration 0 to h_1 and v_1 and filtration one to each of $h_2, h_{2,0}, v_2$ then the corresponding E_1 will be $bo_*[h_2, h_{2,0}, v_2]$. If we recognize the bo structure of the set $< h_2, h_{2,0}, v_2 >$ then the corresponding resolution is just the Kozul resolution of [2], section 5 (page 319ff). Of course, v_2 should be inverted and $2 = v_0$. This gives the following result.

Theorem 6.2. *There is a spectral sequence with*

$$E_2^{s,t} = v_2^{-1}\mathrm{Ext}^{s,t}_{A(2)}(\mathbb{Z}/2, \mathbb{Z}/2)$$

which converges to $E_0(EO_{2*})$.

We will explore this in more detail in latter sections. In particular we will want to understand the differentials.

Using the Adams Novikov differentials from the same starting point we get another spectral sequence. In this case we assign filtration 0 to v_1 and v_2, filtration 1 to h_1, h_2 and $h_{2,0}$. We will also work over the integers.

In order to state the answer in a compact form we introduce several homotopy patterns.

Figure 6.3. *The following diagram defines A. The solid circles represent $\mathbb{Z}/2$'s and the open circles represent a $\mathbb{Z}/8$ ($\mathbb{Z}/4$) in stem 0 (stem 3). In stem 3 there is an extension to the $\mathbb{Z}/2$ giving a $\mathbb{Z}/8$ in this case too.*

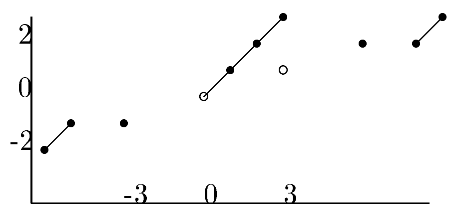
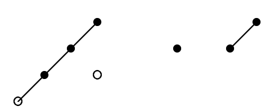


Figure 6.4. *The following diagram defines B. The diagram starts in filtration $(0, 0)$. The starting circle represents a $\mathbb{Z}$ and the circle in dimension 3 represents a $\mathbb{Z}/4$ with an extension to the $\mathbb{Z}/2$ giving a $\mathbb{Z}/8$.*



Theorem 6.5. *There is a spectral sequence which converges to EO_{2*} and the E_2 is given by*

$$h_{2,0}^4 P(h_{2,0}^4, v_2^4, v_2^{-4}) \otimes A \oplus (P(v_2^4, v_2^{-4}) \otimes B) \oplus v_1^4 bo[v_2^4, v_2^{-4}]$$

where A is the module of Figure 6.3 and B is the module of Figure 6.4.

7. The Bockstein spectral sequence

In this section we will give the proof of theorem 6.5. We start with the following resolution, $\mathbb{Z}[v_1, v_2, h_1, h_2, h_{2,0}]$ where the classes v_i have filtration 0 and the other classes have filtration 1. The dimension of the classes is

$(2,6,1,3,5)$ respectively. The Novikov differentials give the following formulas:

$$v_2 \to v_1 t_1^2 + v_0 t_2 + v_1^2 t_1 \tag{7.21}$$
$$v_2^2 \to v_1^2 t_1 v_2 + v_0^2 v_2 t_2 + v_1^4 t_1^2 \tag{7.22}$$
$$h_{2,0} \to h_1 h_2 \tag{7.23}$$
$$h_{2,0}^2 \to h_2^3 + v_2 h_1^3 \tag{7.24}$$

It is worth noting how the formulas in the Hopf algebroid give these formulas. The class v_2 corresponds to a_3. The differential on $a_3 = 2t + a_1 r$. When we substitute the differential on a_2 which sets $3r = a_1 s + s^2$ we see that the differential on $a_3 = 2t + a_1^2 s/3 + a_1 s^2/3$. Setting $a_1 = v_1,\ s = t_1$ shows that the two formulas are the same. The class $h_{2,0}$ is represented by t_2 here and t in the Hopf algebroid.

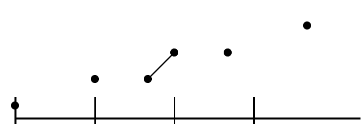
We will break the calculation into several steps introducing various classes one at a time.

Step 1. We first look at just $\mathbb{Z}[h_2, h_{2,0}]$ and apply the $h_{2,0}^2$ differential. This leaves

$$\langle 1, h_2, h_2^2, h_{20}, h_{2,0}h_2, h_{2,0}h_2^2 \rangle \mathbb{Z}[h_{2,0}^4].$$

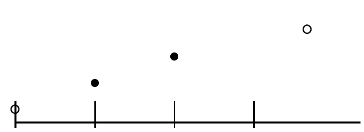
Much of the rest of the calculation is free over $h_{2,0}^4$ and we will drop mention of it until we need it again. We can write this calculation pictorially as the following figure.

Figure 7.1. *Each dot represents a* $\mathbb{Z}[v_1, v_2, h_1, h_{2,0}^4]$.



Step 2. Next we want to add h_1. This amounts to taking the above figure and tensoring it with $\mathbb{Z}[h_1]$. We then feed in the differential defined by $h_{2,0}$. This gives us:

Figure 7.2.

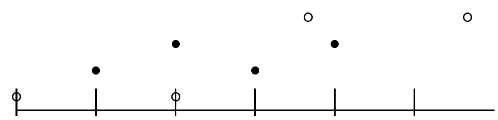


Each circle represents a class which is free on $\mathbb{Z}[h_1]$ and the dots represent a $\mathbb{Z}$. Of course the picture is free over $\mathbb{Z}[v_1, v_2, h_{2,0}^4]$.

Step 3 Next we will tensor this picture with $\mathbb{Z}[v_1] \otimes \Lambda[v_2]$. We use the differential defined by v_2 to kill $v_1 h_2$. We also use the differential on v_2

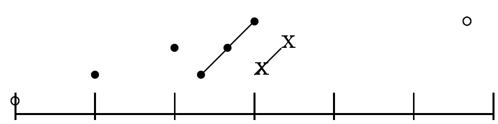
and $h_{2,0}$ to construct the Massey product, $a = v_1 h_{2,0} + h_1 v_2$. The starting picture is:

Figure 7.3.



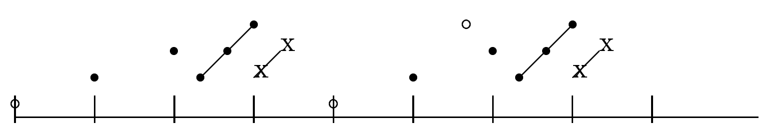
We have $h_{2,0} \to h_1 h_2$ and $v_2 \to v_1 h_2$. This defines a Massey product $a = v_1 h_{2,0} + h_1 v_2$. Now $h_2 h_{2,0} \to h_1 h_2^2$ and $v_2 h_2 \to v_1 h_2^2$ leaving $h_2 a$. But $v_2 h_{2,0} \to h_2 a$. Also, $h_{2,0} h_2^2 \to h_1 h_2^3 = h_1^3 a$ and $v_2 h_2^2 \to v_1 h_2^3 = v_1 h_1^2 a$ and these leave $h_2^2 a$ which is the target of $h_2 v_2 h_{2,0}$. This leaves the following picture.

Figure 7.4. *The open circles represent $P(h_1, v_1)$ free classes. The dots represent $\mathbb{Z}$. The x classes represent $\mathbb{Z}[v_1]$ classes.*



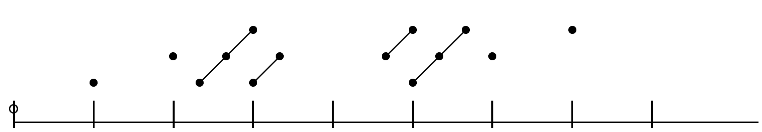
Step 4. Next we add a copy of the above based on v_2^2. The starting picture gives:

Figure 7.5.



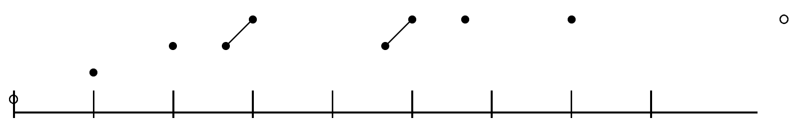
We have $v_2^2 \to v_1^2 a$ leaving $h_1^2 v_2^2 \otimes P(h_1, v_1)$. Also $v_2 h_{2,0} h_2^2 \to h_1^4 v_2^2$ leaving $h_1^2 v_2^2 P(v_1)\langle 1, h_1\rangle$. Finally $v_2^2 a \to h_1^2 v_2^2 v_1^2$ leaving just $v_2^2 h_1^2 a$. This gives

Figure 7.6. *The open circles represent $P(h_1, v_1)$ and the dots represent $\mathbb{Z}$.*



Step 5. The torsion Bocksteins are now easy. $a \to 2h_2^2$, $v_1 \to 2h_1$, $v_2^2 h_2 \to 2h_1^2 v_2^2$ and $v_2^2 h_2^2 \to 2h_1^3 v_1 v_2^2$. This leaves:

Figure 7.7. *The open circles represent $P(h_1, v_1)$ free classes. In addition the class corresponding to h_2 and $v_2^2 h_1^2 a$ represents a $\mathbb{Z}$.*



Step 6. Next we add $h_{2,0}^4$ and we have $v_2^3 h_{2,0} h_2^2 \to v_1^4 h_{2,0}^4$ and $v_2^2 h_1^2 a \to 8h_{2,0}^4$. This last formula uses several substitutions to complete.

$$\begin{aligned} v_2^2 h_1^2 a &\to 4h_{2,0} v_2 a h_1^2 \\ &= 4h_{2,0}^2 v_1 a h_1 = 4h_{2,0}^3 v_1^2 h_1 = 8h_{2,0}^4. \end{aligned}$$

The first formula is a consequence of the calculation in the bo_* spectral sequence. See, for example, [10]. This completes the calculation and the proof of Theorem 6.5.

It is interesting to compare the filtration of the differential which kills $v_1^4 h_{2,0}^4$. It is an Adams-Novikov d_1 but it would be and Adams differential d_2. As we shall see later, it requires the Adams spectral sequence to use d_2's,d_3's and d_4's to recover from this.

8. The Adams-Novikov spectral sequence

In this section we will compute the Adams-Novikov differentials and thus calculate the associated graded homotopy of EO_2. The starting point is the following. We will show latter that it is a d_1 in the usual Adams spectral sequence.

Proposition 8.1. *In the Adams-Novikov spectral sequence for EO_2 we have $d_5 v_2^4 = h_2 h_{2,0}^4$.*

Proof. We begin with a calculation in stable homotopy.

Lemma 8.2. $\nu^2 \bar{\kappa} \in \langle \eta_4 \sigma, \eta, 2\iota \rangle$.

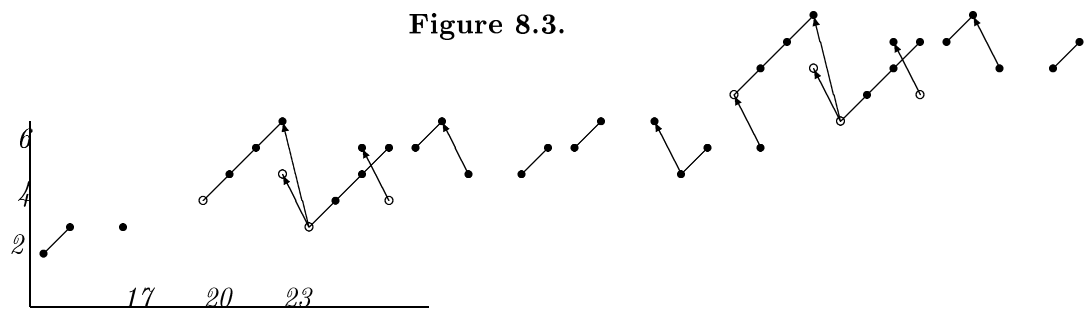
We will first use this lemma to complete the proof of the proposition. We note that $h_2^2 h_{2,0}^4$ is the Adams-Novikov name for $\nu^2 \bar{\kappa}$. By checking the above calculation, we see that $\eta_4 \sigma = 0$ in EO_{2*}. Thus the bracket of the lemma must go to zero in EO_2. Hence the class of $h_2^2 h_{2,0}^4$ must be in the indeterminacy of the bracket. It is easy to see that only zero is in the indeterminacy and so $h_2^2 h_{2,0}^4$ must project to the zero class. The only way this can happen is for $d_5 v_2^4 h_2 = h_2^2 h_{2,0}^4$. dividing by h_2 gives the proposition. ☐

Now we will prove the Lemma. This is essentially an Adams d_1. First recall that $\eta_4 \sigma$ is represented by $h_1 h_4 c_0$ in the Adams E_2. In order to form a bracket such as $\langle \eta_4 \sigma, \eta, 2\iota \rangle$ we need to know why $h_1^2 h_4 c_0 = 0$. The easiest approach is to use the lambda algebra and the calculations of [15]. We see that $\lambda_2 \lambda_3 \lambda_5 \lambda_7 \lambda_7 = h_1 h_4 c_0$. Then from [15] we see that $\lambda_8 \lambda_9 \lambda_3 \lambda_3 \lambda_3$ hits

$\lambda_1\lambda_2\lambda_3\lambda_5\lambda_7\lambda_7$. Thus $\lambda_0\lambda_8\lambda_9\lambda_3\lambda_3\lambda_3 \in \langle h_1h_4c_0, h_1, h_0\rangle$. Up to the addition of some boundaries, this is just $\lambda_6\lambda_6\lambda_5\lambda_3\lambda_3\lambda_3$. This is equivalent to the leading term name of $\nu^2\bar{\kappa}$. This completes the proof.

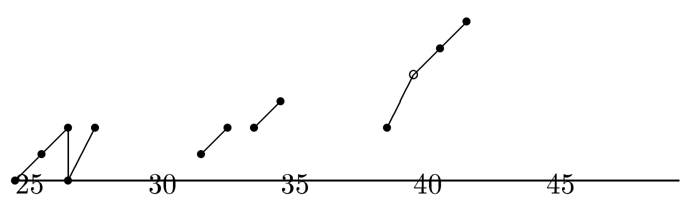
The following figure illustrates this first differential. We place the chart for v_2^4 in filtration 3 so it is easier to see just what is happening.

Figure 8.3.



We can collect the result of this computation in the following chart. We have listed some exotic multiplications which we will prove in the rest of this section.

Figure 8.4. *The class in dimension 4 is a $\mathbb{Z}/4$. Lines which connect adjacent elements but are of length 2 represent exotic extensions. There are three such. One is multiplication by 2 in stem 27. The other two are multiplications by η, one in stem 27 and the other in stem 39. The complete calculation has this chart multiplied by $\mathbb{Z}[\bar{\kappa}]$.*



We need to establish some of the compositions which are non-zero in this homotopy module. We introduce some notation. We let ι, η, ν, to represent the generators of the 0, 1, and 3 stems. This is consistent with the traditional names of these classes in the homotopy of spheres. The elements in the 8, 14 and 20 stem we will label $\epsilon, \kappa, \bar{\kappa}$ respectively. For other classes, we will use the symbol a_i for an element in the ith stem. The exotic extensions referred to above then are covered by the following proposition.

Proposition 8.5. *The following compositions are non-zero: $\eta a_{27}, 2a_{27}, \eta a_{39}$.*

Proof. First note that $2a_{27}$ is just the standard extension which comes from the 3 stem where $4\nu = \eta^3$. Next, the class $a_{28} = \epsilon\bar{\kappa}$. This is a filtration preserving calculation. The definition of ϵ forces $\epsilon \in \langle \nu, 2\nu, \eta\rangle$. When we multiply this bracket by $\bar{\kappa}$ we see that $\epsilon\bar{\kappa} = \langle \bar{\kappa}, \nu, 2\nu \rangle \eta$. This bracket clearly represents a_{27}. Notice that we can not form this latter bracket in spheres but need the differential on v_2^4 in order to form the bracket.

In the homotopy of spheres we have the bracket relation $\langle \nu, \kappa\eta, \eta \rangle = 2\bar{\kappa}$. This follows easily from the Adams spectral sequence where there is a d_2 which makes $\kappa\eta^2 = 0$. In the usual naming, we have $d_2 e_0 = h_1^2 d_0$. We also have $h_2 e_0 = h_0 g$. This establishes this relationship. Now if we multiply both sides by $\bar{\kappa}$ we have $\bar{\kappa}\langle \nu, \kappa\eta, \eta \rangle = 2\bar{\kappa}$. But $\bar{\kappa}\langle \nu, \kappa\eta, \eta \rangle = \langle \bar{\kappa}, \nu, \eta\kappa \rangle \eta$. This is the relation we wanted. □

Next we want to establish the special ν multiplications.

Proposition 8.6. *We have the following compositions:* $\nu a_{25} = a_{28}, \nu a_{32} = a_{35}, \nu a_{39} = a_{42}$.

Proof. A bracket construction for a_{25} is $a_{25} = \langle \bar{\kappa}, \nu, \eta \rangle$. If we multiply this on the right by ν we have $\langle \bar{\kappa}, \nu, \eta \rangle \nu = \bar{\kappa}\langle \nu, \eta, \nu \rangle = \bar{\kappa}\epsilon = a_{28}$. In a similar way we see that $a_{32} = \langle \bar{\kappa}, \nu, \epsilon \rangle$. If we multiply both sides by ν we get $\langle \bar{\kappa}, \nu, \epsilon \rangle \nu = \bar{\kappa}\langle \nu, \epsilon, \nu \rangle$. But in spheres $\langle \nu, \epsilon, \nu \rangle = \eta\kappa$ and this gives the relation. In the above proposition we showed $a_{39} = \langle \bar{\kappa}, \nu, \eta\kappa \rangle$. Multiplying this by ν we have $a_{39}\nu = \bar{\kappa}\langle \nu, \eta\kappa, \nu \rangle = \bar{\kappa}^2\eta^2$ and this is the relationship we wanted. □

In a very similar fashion we establish the following. We will skip the proof.

Proposition 8.7. *We have the following ϵ compositions.* $\epsilon a_{25} = a_{33}, \epsilon a_{27} = a_{35}, \epsilon a_{32} = 2a_{40}, \epsilon a_{34} = a_{42}, \epsilon a_{39} = a_{47} = \bar{\kappa} a_{25}, \epsilon a_{40} = a_{48}$.

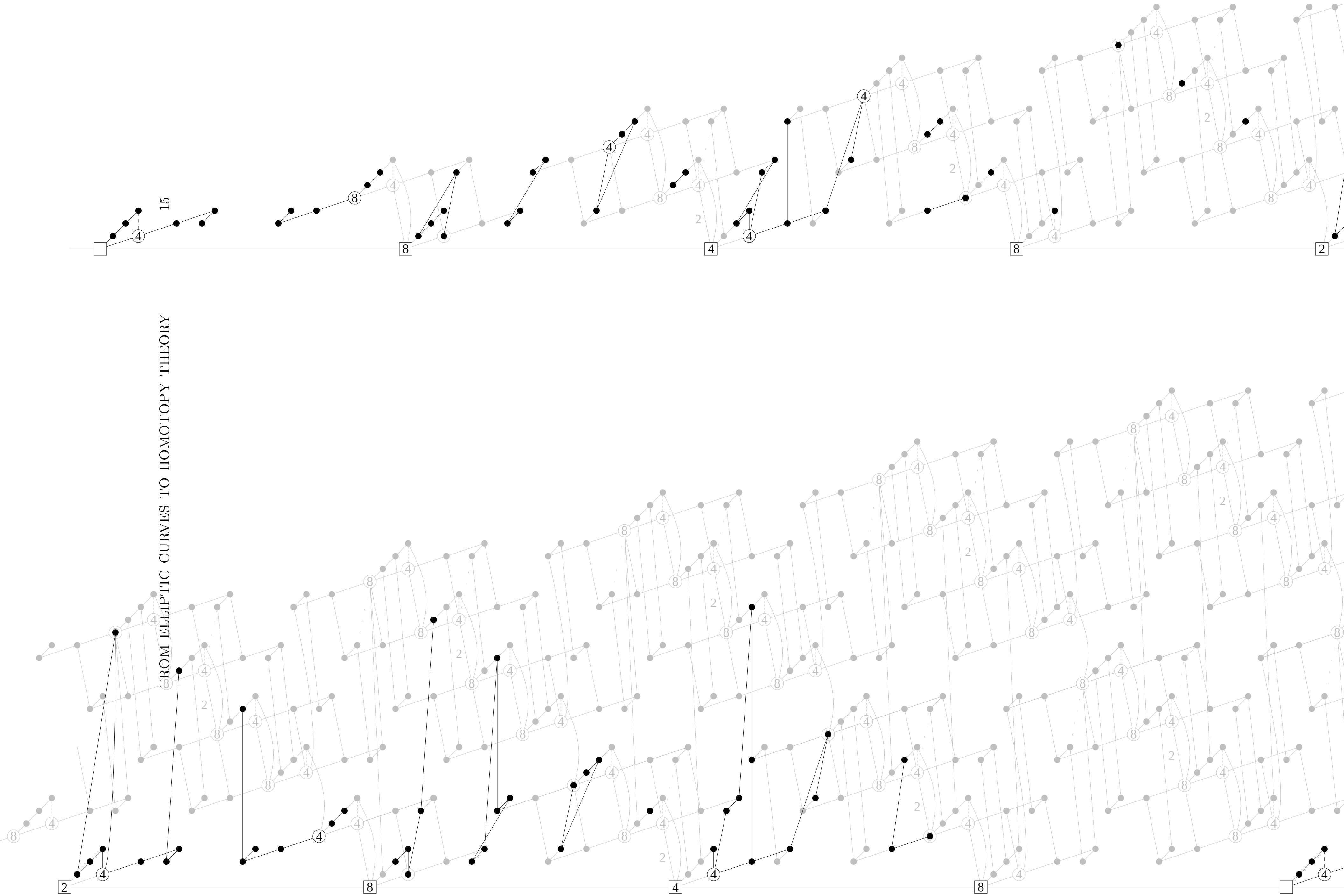
With these extensions established, the rest of the spectral sequence is quite easy. We have the following theorem.

Theorem 8.8. *We have the following differentials:*

$$d_5 v_2^8 = \bar{\kappa} a_{27} \ (= 2\nu\bar{\kappa} v_2^4 = 2 d_5 v_2^4)$$

and

$$d_7 v_2^{16} = \eta^2 a_{25} \bar{\kappa} v_2^8 \ (= 2 v_2^8 d_5 v_2^8)$$



9. The connected cover of EO_2

In this section we will construct the connected cover of EO_2 and get some of its properties. We begin with the following which is proved in [7].

Theorem 9.1. $v_1^{-1}EO_2 = KO[[v_2/v_1^3]][v_2^4, v_2^{-4}]$

Our strategy to construct the the connected cover of EO_2 will be to construct the following map.

$$f : bo[v_2^4/v_1^{12}] \to v_1^{-1}EO_2[0, \cdots, \infty].$$

With this map we will consider the pull back square as defining the spectrum Y

$$\begin{array}{ccc} Y & \longrightarrow & bo[v_2^4/v_1^{12}] \\ \big\downarrow & & \big\downarrow \\ EO_2[0, \cdots, \infty] & \longrightarrow & v_1^{-1}EO_2[0, \cdots, \infty] \end{array}$$

We will show:

Theorem 9.2. *The cohomology of Y is $H^*(Y) = A \otimes_{A(2)} \mathbb{Z}$ and the Adams spectral sequence to calculated $\pi_*(Y)$ is that given by Theorem 2.2.*

The first step is the following Lemma.

Lemma 9.3. *There is a map $g : bo \to v_1^{-1}EO_2[0, \cdots, \infty]$ such that $g_*(\iota) = \iota$, the unit in $v_1^{-1}EO_2[0, \cdots, \infty]$.*

Proof. We begin with $\iota : S^0 \to v_1^{-1}EO_2[0, \cdots, \infty]$. We recall that there is a short exact sequence

$$\Sigma^{4k-1}B(k) \to bo_k \to bo_{k+1}$$

where $B(k)$ is the integral Brown Gitler spectrum [1] and bo_k is the bo Brown Gitler spectrum. This sequence is constructed in [5]. The K theory of $B(k)$ is easily computed and it is zero in dimensions of the form $4k - 1$. Thus we can proceed by induction starting with the map ι. This constructs one copy of bo into $v_1^{-1}EO_2[0, \cdots, \infty]$. $\square$

To continue with the proof of the Theorem we next construct a map of $\Omega S^{24} \to v_1^{-1}EO_2[0, \cdots, \infty]$ which gives the polynomial algebra on v_2^4. Using the ring structure we have the desired map f of the diagram. This completes the construction of Y.

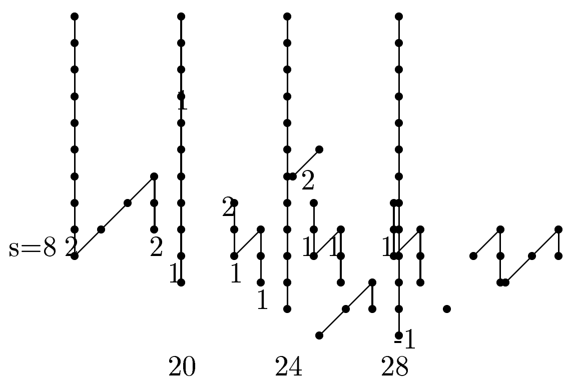
Next we want to compute the homotopy of Y. The E_2 term of the Adams spectral sequence for Y is $\mathrm{Ext}_{A(2)}(\mathbb{Z}/2, \mathbb{Z}/2)$. This has been calculated by many people. The first calculation is due to Iwai and Shimada [6]. Extensive $\mathrm{Ext}_{A(2)}(M, \mathbb{Z}/2)$ calculations are given in [2]. We refer the reader there to find the details of the calculation. The answers given there are in a compact form which is quite useful. It is based on the following definition.

Definition 9.4. *An indexed chart is a chart in which some elements are labeled with integers. A unlabeled x receives the label*

$$\max\{\text{label } (y) : x = h_0^i y \text{ or } x = h_1^i y, \text{ some } i \geq 1\}$$

or 0 if this set is empty. If C is a labeled chart then $\langle C \rangle$ denotes the chart consisting of all elements $v_1^{4i} x$ such that $i + \text{label}(x) \geq 0$.

The following is an example of an indexed chart.



Let this chart be called E_0. Then the following is proved in [2]. (Actually, the chart in [2] has a dot missing in dimension (30,6).)

Theorem 9.5. $\mathrm{Ext}_{A(2)}(\mathbb{Z}/2, \mathbb{Z}/2)$ *is free over* $\mathbb{Z}/2[v_2^8]$ *on*

$$\langle E_0 \rangle \oplus \mathbb{Z}/2[v_1, w] \cdot g_{35,7}.$$

We have the following differentials in the chart E_0. We use the notation $g_{t-s,s}$ to refer to the dot in position $(t-s, s)$.

Proposition 9.6. $d_2 g_{20,7} = g_{19,9}$.

Proof. When translated to more familiar notation this is a consequence of the following Lemma.

Lemma 9.7. *In the Adams spectral sequence of Theorem 2.1 the first differential occurs in dimension 12 and hits the class $v_1^4 h_2$.*

Proof. First we need to construct the element. Using the above formulas which are filtration preserving we see that

$$v_0 v_2^2 + v_2 h_2^2 + v_1 h_{2,0}^2$$

is a cycle and it generates an v_0 tower in the 12 stem. When we use the filtration increasing part of the differentials we see this class is not a cycle but its boundary is

$$v_1^2 v_0 h_1 v_2 + v_0^2 v_2 h_{2,0} + v_0 v_1^4 h_2 + v_1^2 h_1 h_2^2$$

We can begin to try to complete this into a cycle. The first class we would add is $v_1^3 v_2$. The boundary on this class is

$$v_1^2 v_0 h_1 v_2 + v_1^4 h_2 + v_0 v_1^3 h_{2,0} + v_1^5 h_1$$

There is nothing we can add to get rid of the $v_1^4 h_2$ class and this gives the differential of the Lemma. □

□

Using h_2 multiplications we have the following additional differentials.

$$d_2 g_{23,7} = g_{22,9}$$

$$d_2 g_{26,7} = g_{25,9}$$

$$d_2 g_{29,7} = g_{28,9}$$

$$d_2 g_{28,5} = g_{27,7}$$

We have the following d_3.

Proposition 9.8. $d_3 g_{24,6} = g_{23,9}$.

This differential implies the following in addition.

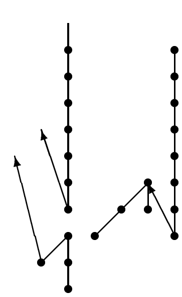
$$d_3 g_{25,8} = g_{24,11}$$

$$d_3 g_{30,6} = g_{29,9}$$

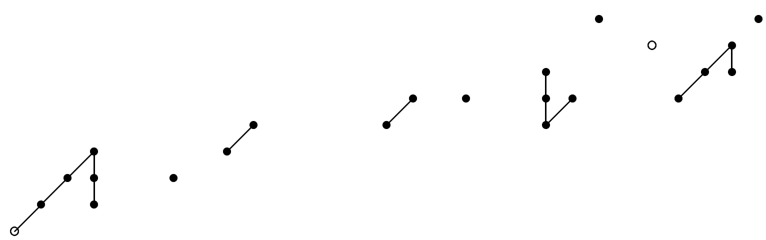
We have the following d_4.

Proposition 9.9. $d_4 g_{31,8} = v_1^4 g_{22,8}$.

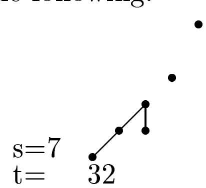
We wish to collect the result of these differentials. The pattern which is left from the upper left corner of the figure generates a copy of *bo* starting in dimension $(8, 4)$. The second pair of $\mathbb{Z}$ towers generates a *bo* in dimension $(32, 8)$. This second *bo* uses $v_1^4 g_{25,5}$ and h_1 times this class and the class in dimension $(32, 7)$ which has h_0 none zero on it. The picture looks as follows:



This picture leaves a copy of bo, after some extensions, which starts in filtration $(32, 7)$. It represents $v_1^4 v_2^4$. There is an extra dot in the above picture in filtration $(35, 10)$ which we still have to account for. The following chart lists everything which is left because the source of a differential is not present.



In addition to this part we have the polynomial algebra on the two generators and v_1^4 free on the following.



The following result gives the differentials for this part of the picture.

Proposition 9.10. *Among classes in $\mathbb{Z}/2[v_1, w]\cdot g_{35,7}$ and between this polynomial algebra and classes in the above diagram we have the following differentials:*

$$d_2 g_{35,7} = g_{36,9}$$

$$d_4 v_1 g_{35,7} = g_{38,12}$$

$$d_4 v_1 w g_{35,7} = v_1^4 g_{35,8}$$

$$d_4 v_1^2 w g_{35,7} = v_1^4 g_{35,10}$$

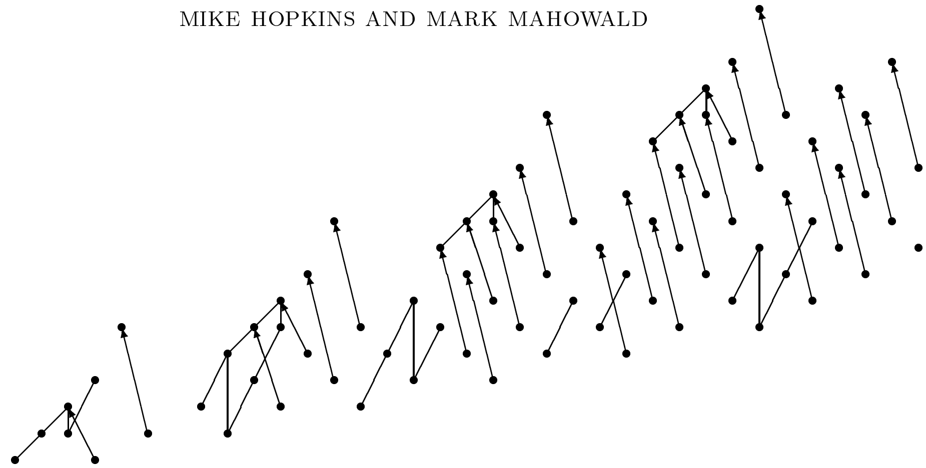
$$d_4 v_1^2 w^2 g_{35,7} = v_1^8 g_{32,7}$$

$$d_4 v_1^3 w^2 g_{35,7} = v_1^8 g_{34,8}$$

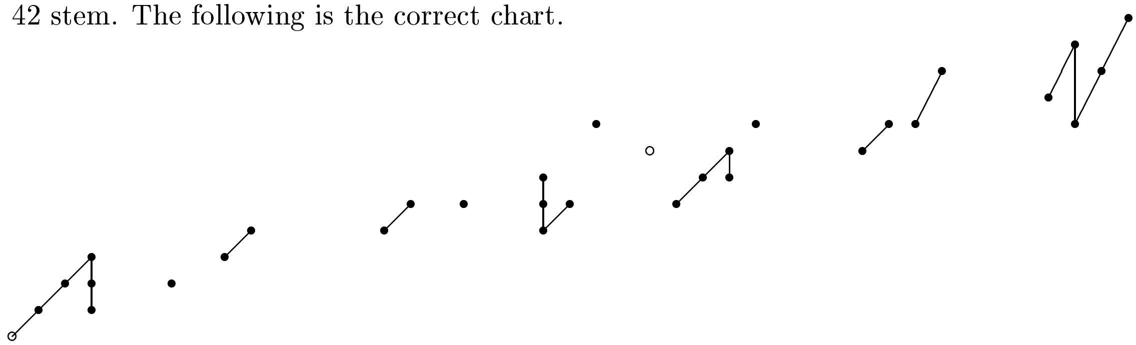
$$d_4 w^3 g_{35,7} = v_1^7 g_{35,7}$$

$$d_4 v_1^3 w^3 g_{35,7} = v_1^{10} g_{35,7}$$

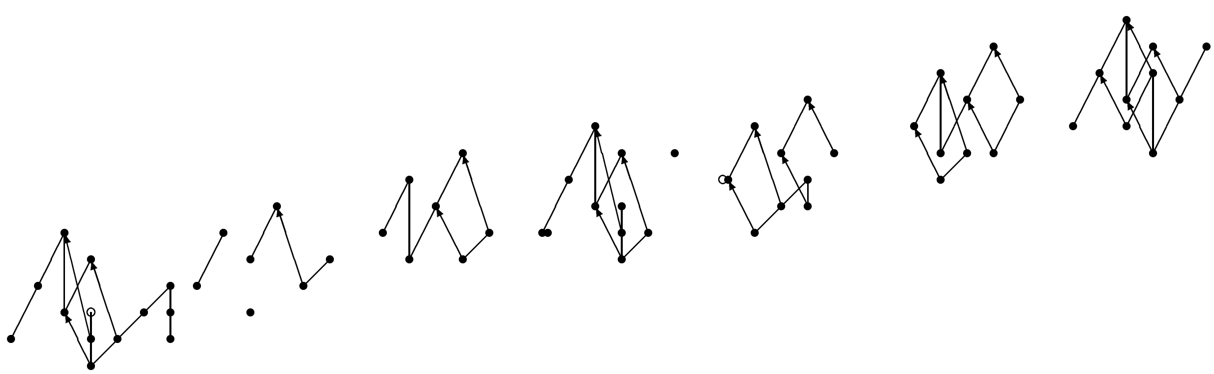
If we combine the above diagram, the polynomial algebra and the differentials above we have the following figure.



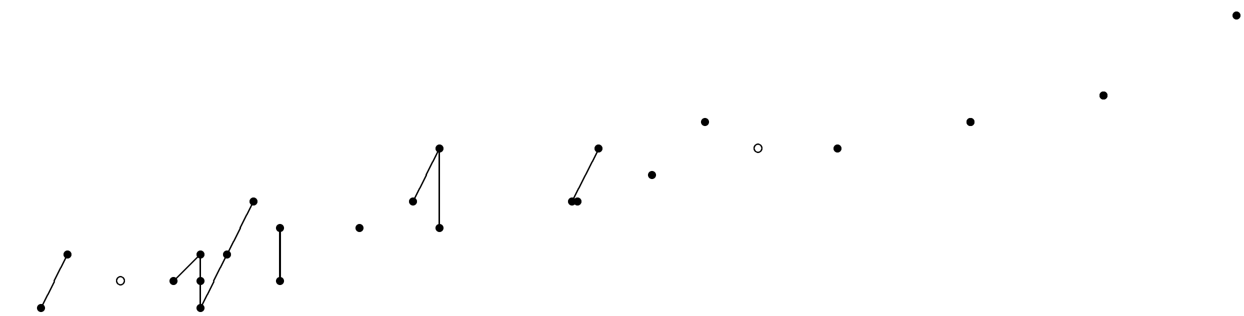
This allows us to write the v_1 torsion part of the answer out though the 42 stem. The following is the correct chart.



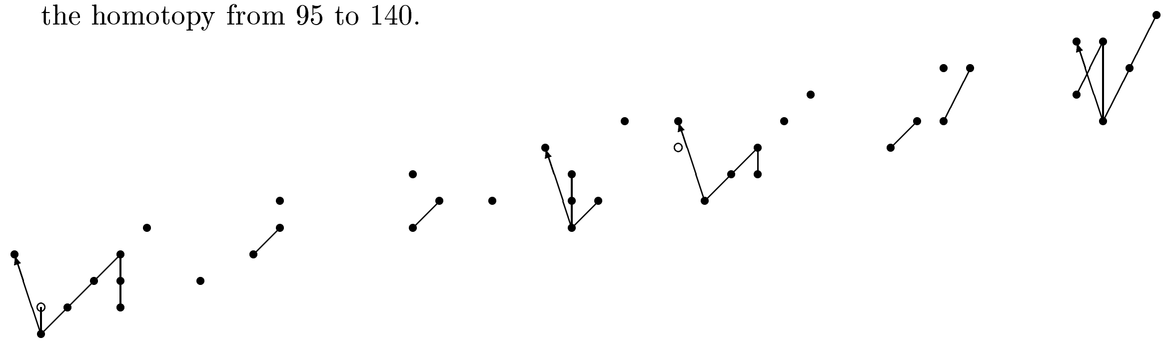
To compute the next 48 groups we need to put the earlier calculation together with the first 42 groups above multiplied by v_2^8. This gives the following chart.



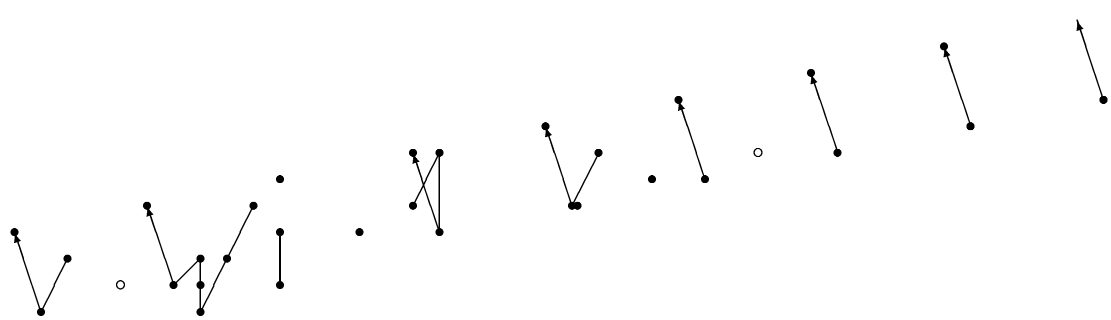
This gives the following homotopy starting in dimension 45.



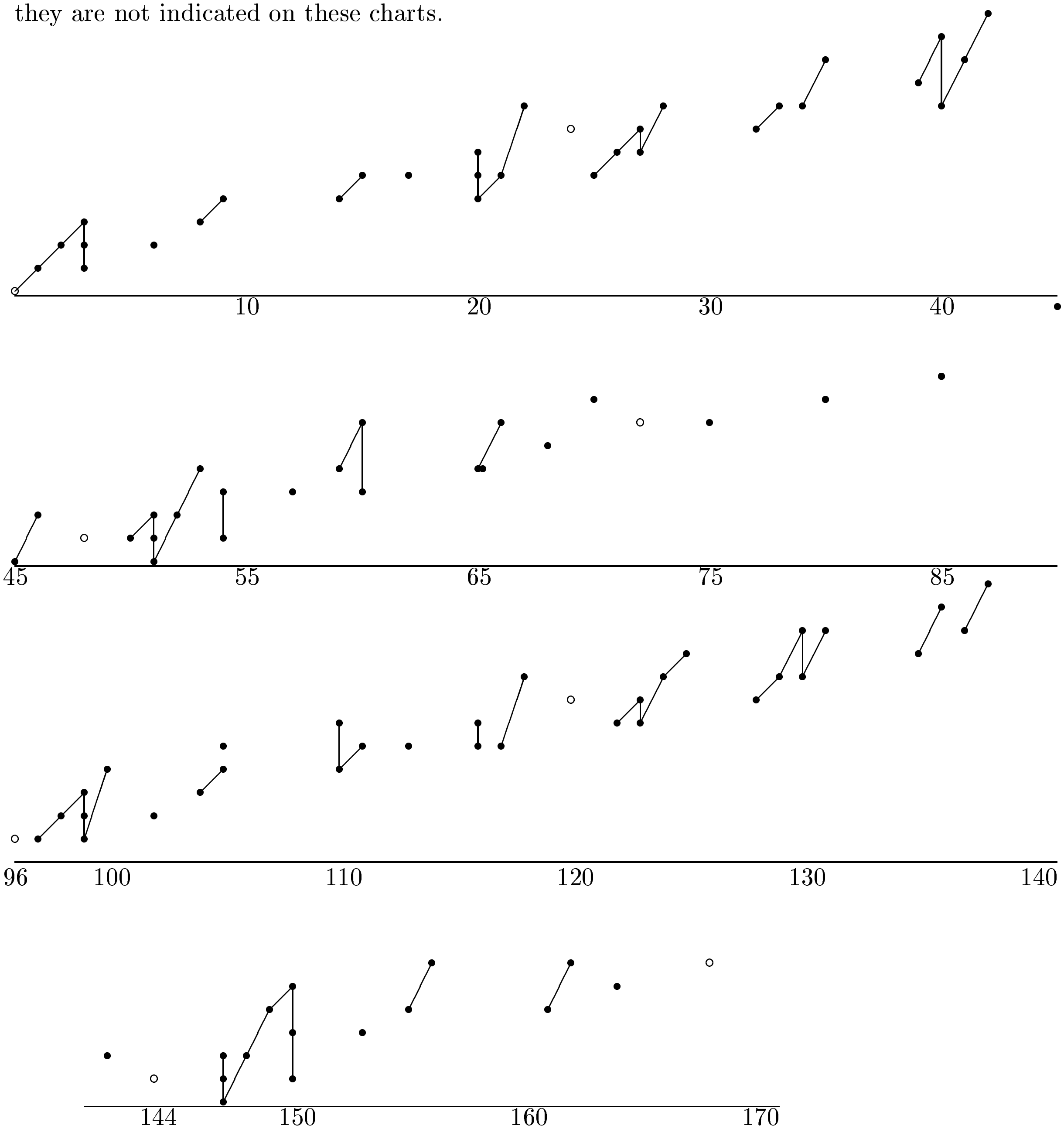
Beyond 95 this differential pattern leaves a class every 5 dimensions. Because the differential on v_2^8 is a d_2, the polynomial algebra $v_2^8\mathbb{Z}/2[v_1, w]$ is mapped monomorphically into $\mathbb{Z}/2[v_1, w]$ leaving just $w^9 g_{35,7}\mathbb{Z}/2[w]$. To complete the calculation we need to take into account v_2^{16}. We do this by putting our calculation so far together with this pattern and writing in the new differentials. This gives the following pattern. The first chart calculates the homotopy from 95 to 140.



Here is the picture for 141 to 180.



We can now collect the final charts and write in one place the v_1 torsion homotopy. Dots correspond to $\mathbb{Z}/2$'s and circles correspond to $\mathbb{Z}$'s. Vertical lines indicate multiplication by 2 and slanting lines to the right indicate multiplication by η. There are a large number of multiplications by ν but they are not indicated on these charts.



Theorem 9.11. *The homotopy of eo_2 is given by the following:*

$$v_1^4 bo[v_2^4] \oplus E[v_2^{32}]$$

where E is the homotopy described in the above charts.

10. Some self maps

Let A_1 be the suspension spectrum of one of the complexes whose cohomology is free over $A(1)$, the sub algebra of A generated by Sq^1 and Sq^2. Let $M(i_0, i_1)$ be the mapping cone of $\Sigma^{2i_1} M(2^{i_0}) \to M(2^{i_0})$ which induces an isomorphism in K-theory. In [4] it is claimed that A_1 and $M(1,4)$ admitted a self map raising dimension by 48 and inducing an isomorphism in $K(2)_*$. This result is false as the results here have shown. The argument in [4] is correct in showing the following.

Theorem 10.1. *There is a class representing $v_2^8 \in \mathrm{Ext}_A^{8,56}(A(1), A(1))$.*

Consider a resolution by Eilenberg-Mac Lane spaces constructed as follows for any suspension spectrum X with the property that there is only one class $\alpha \in \pi_*(X)$ which is non-zero in mod 2 homology. We begin with a map f_0 so that the composite

$$S^0 \xrightarrow{\alpha} X \xrightarrow{f_0} K(\mathbb{Z}/2)$$

is non-zero. Now we construct a tower inductively. Suppose we have

$$X \xrightarrow{f_s} X_s \xrightarrow{g_{s-1}} \cdots \xrightarrow{g_0} K(\mathbb{Z}/2)$$

with $g_0 \cdots g_{s-1} f_s = f_0$. Let $h_s : x_s \to K(\mathbf{ker}(H^*(f_s)))$ and let X_{s+1} be the fiber. Clearly, f_s lifts to give f_{s+1}.

Each such resolution defines a spectral sequence with

$$E_1^{s,t} = \mathbf{ker}(H^*(f_s))^{t-s+1}$$

As with the usual Adams spectral sequence we have convergence to the 2-adic completion and we can define $E_2(X, X)$ as equivalence classes of maps between these resolutions.

In this language [4] showed the following result for $M(1,4)$. A very similar argument works for $M(2,4)$.

Theorem 10.2. *For $X = M(1,4)$ or $X = M(2,4)$ there is a class*

$$v_2^8 \in E_2^{8,56}(X, X)$$

.

Using an eo_2 resolution we see that v_2^8 commutes with the possible targets of the differentials on v_2^8. Thus in each case if $d_2(v_2^8) \neq 0$, which is the case, then v_2^{32} will be a class in E_4 and for dimensional reasons, $d_4 v_2^{32} = 0$. This proves the following theorem.

Theorem 10.3. *For $X = A_1$, $X = M(1,4)$ or $X = M(2,4)$, there is a map*

$$v_2^{32} : \Sigma^{192} X \to X$$

which is detected in $K(2)_*$.

Thus the results discussed in [13], [11] and possibly other places which used the maps of [4] are established in this modified form. We will discuss some of these classes, particularly those of [11] in the next section.

11. The Hurewicz image and some homotopy constructed from EO_2

Using the results of the last section we can construct many v_2-families.

Theorem 11.1. *If $\alpha \in \pi_*(S^0)$ is represented by $a \in Ext_A(\mathbb{Z}/2, \mathbb{Z}/2)$ and under the map $Ext_A(\mathbb{Z}/2, \mathbb{Z}/2) \to Ext_{A(2)}(\mathbb{Z}/2, \mathbb{Z}/2)$ a maps to a non-zero cycle then $v_2^{32k}\alpha \neq 0$.*

Proof. This is the standard Greek letter proof. There is an $\alpha^\#$ so that the following composite is α.

$$S^{|\alpha|} \to M^{|\alpha|+10}(2,4) \xrightarrow{\alpha^\#} M^0(2,4) \to S^0$$

Then

$$\begin{tikzcd}
S^{|\alpha|+192} \arrow[r] & M^{|\alpha|+202} \arrow[r, "v"] \arrow[d, "\Sigma\alpha^\#"'] & M^{|\alpha|+10} \arrow[d, "\alpha^\#"] \arrow[dr] & & \\
& M^{192} \arrow[r, "v"'] & M^0 \arrow[r] & S^0 \arrow[r] & EO_2
\end{tikzcd}$$

is non-zero in $\pi_* EO_2$. ∎

Corollary 11.2. *We list the classes which this is known to apply to*

stem	6	8	9	14	15	17	28	32	33	34	35	39	40	41	42	45	46
name	ν^2	ϵ	$\eta\epsilon$	κ	$\eta\kappa$	$\nu\kappa$	$\epsilon\kappa$	q	ηq	e_0^2	ηe_0^2	u	$\bar\kappa^2, 2\bar\kappa$	$\eta\bar\kappa^2$	$\eta^2\kappa^2$	w	ηw

stem	52	53	59	60	65	66	80	85
name	$\bar\kappa q$	$\bar\kappa\eta q$	$\bar\kappa u$	$\bar\kappa^3, 2\bar\kappa^3$	$\bar\kappa w$	$\eta\bar\kappa w$	$\bar\kappa^4$	$\bar\kappa^2 w$

Next consider $\bar\kappa$. The only problem is that it has order 8 and so the argument does not quite work.

Proposition 11.3. *$\bar\kappa$ is v_2^{32}-periodic.*

Proof. We look at map $\bar\kappa^\#$ which makes this diagram commute.

$$\begin{tikzcd}
M^{29}(2,4) \arrow[r, "\bar\kappa^\#"] & M^0(2,4) \arrow[d] \\
M^{20}(2) \arrow[u] \arrow[r, "\bar\kappa\wedge 1"'] & M^0(2)
\end{tikzcd}$$

We do the Greek letter construction and get

$$\begin{array}{ccccccc}
M^{20+192}(2,4) & \longrightarrow & M^{29+192}(2,4) & \longrightarrow & M^{29}(2,4) & \longrightarrow & M^0(2) \\
 & & \downarrow & & \downarrow & & \downarrow \\
* & & M^{192} & \longrightarrow & M^0(2,4) & \longrightarrow & M^0(2)\wedge EO_2
\end{array}$$

This composite is non-zero. We can break apart the calculation and get that $\bar{\kappa}$ as an element of order 8 is v_2^{32}-periodic. Of course we get easily $4\bar{\kappa} = v^2\kappa$ is v_2^{32}-periodic. $\square$

The next problem class is $\nu\bar{\kappa}$.

Proposition 11.4. *$\nu\bar{\kappa}$ is v_2^{32} periodic.*

Proof. Suppose not, that is, suppose $\nu\bar{\kappa}(v_2^{32}) = 0$. Then we have a map f

$$\begin{tikzcd} S^{20+192} \cup e^{24+192} \arrow[r, "f"] \arrow[dd] & S^0 \arrow[d] \\ & EO_2 \arrow[d] \\ S^{24+192} \arrow[r, "g"'] & E_2 \end{tikzcd}$$

in the diagram, the map g is v_2^{36}. Thus f would have ∞ order. This contradiction completes the proof. $\square$

Next we have classes in EO_2 which do not come from the sphere. Since there should be nothing extra in homotopy we use these to detect something also.

Proposition 11.5. *The classes η_4, $\eta\eta_4$, $\eta^2\eta_4$ and $\frac{1}{2}\eta^2\eta_4$ are v_2-periodic.*

Proof. It is an easy *Ext* calculation to show that in the following diagram

$$\begin{tikzcd} & M^{26}(1,4) \arrow[r, "\eta_4^{\#}"] \arrow[dd] & M^0(1,4) \arrow[d] \\ & & S^0 \arrow[d] \\ S^{25} \arrow[r] & M^{26}(1) \arrow[r] & EO_2 \end{tikzcd}$$

the composite along the bottom row is non-zero. $M(1,4)$ also has a v_2^{32} self map and this gives the diagram

$$\begin{tikzcd} S^{208} \arrow[r] & M^{26+192}(1,4) \arrow[r] \arrow[d] & M^{26}(1,4) \arrow[d] & \\ & M^{192}(1,4) \arrow[r] & S^0 \arrow[r] & EO_2 \end{tikzcd}$$

The map $M^{26+192} \to EO_2$ is essential. Suppose the composite $S^{16+192} \to M^{26+192} \to M^{26}(1,4) \to S^0$ was zero. Then M^{26+192}/S^{16+192} factors through S^0 giving $M^{26+192}/S^{16+192} \to S^0 \to EO_2$. This map is v_1-periodic and this contradicts the v_1-structure of S^0. Thus $v_2^{32}\eta_4$ is essential. $\eta\eta_4$ works the same way and also $\eta^2\eta_4/2$. $\square$

It is interesting that each extra class in EO_2 detected something in the sphere.

The next place to study is the 47 stem. Here the differential on v_2^8 eliminates homotopy classes which are in the sphere. Similar to Proposition 11.4 we have

Proposition 11.6. *The classes $\{e_0 r\}$ and $\eta\{e_0 r\}$ are v_2-periodic.*

Proof. We have established that $2v_2^{32}\bar{\kappa}^2 = \eta v_2^{32}\{u\}$ by Theorem 11.1. Thus we can form the bracket $\langle v_2^{32}\bar{\kappa}^2, 2\nu, \nu\rangle$. Suppose this bracket contains zero. Then

$$\begin{tikzcd} S^{40+192} \cup e^{45+192} \cup e^{48+192} \arrow[r] \arrow[d] & S^0 \arrow[d] \\ S^{48+192} \arrow[r, "v_2^{40}"'] & BP\langle 2\rangle \end{tikzcd}$$

is a commutative diagram. This gives a class of ∞-order in $I\pi_*S^0$, a contradiction. Thus the bracket is essential and defines $v_2^{32}\{e_0 r\}$. Now $\eta\{e_0 r\} = \nu\{\bar{w}\}$. Suppose $v_2^{32}\nu w = 0$. Then we could form the bracket $\langle v_2^{32}\{w\}, \nu, \eta\rangle \in \pi_{192+49}(S^0)$ and this class maps to $v_2^{40}\eta^2$ in EO_2, which is a v_1 periodic class. This contradiction establishes the result. $\square$

We also get some mileage out of the failure of the v_2^8 self map.

Proposition 11.7. *There is a family of classes of order 4 in $38+k192$ stem detected by v_2^{8+32k} in BP.*

Proof. Here we use the centrality of our self map. We have established $\{e_0 r\}$ is v_2^{32} periodic and the failure of the proof of v_2^8 self map shows that the composite $M^{47}(2,4) \xrightarrow{g} M^{47}(4) \to S^0$ is null. This gives the following diagram

$$\begin{tikzcd} M^{47+192}(2,4) \arrow[r] \arrow[d] & M^{58+192}(2,4) \arrow[d] & \\ M^{47}(2,4) \arrow[r] & M^{58}(2,4) \arrow[r] & M^0(2) \end{tikzcd}$$

Going around the top is the same, by centrality, as going around bottom. The top represents $v_2^{32}g$ and so is null which implies that there is a class β such that $M^{47+192}(2) \xrightarrow{v_1^4} M^{39+192}(2) \to S^0$ is $v_2^{32}\{e_0 r\}$. As before, β must be on the bottom class or we would get a v_1-periodic class. $\square$

Following closely the proof of Proposition 11.5 we get,

Proposition 11.8. *$v_1^4\eta_5$, $\mu\eta_5$, $\eta\mu n_5/4$ are v_2^{32} periodic. The class θ_4 is v_2 to some power periodic but we don't know the power.*

Proof. First we look at $\eta_5\mu$. We have $M^{51}(1,4) \to S^0$ extending $\eta_5\mu$. The map is detected in EO_2 by $\{v_2^8 h_1^2\}$. Thus we have an essential composite

$$M^{51+192}(1,4) \to S^0 \to EO_2$$

detected by $v_2^{40}h_1^2$.

Suppose the composite $S^{41+192} \to M^{51+192}(1,4) \to S^0$ is zero. Then it factors through $M^{51+192}(1,4)/S^{41+192} \to S^0 \to EO_2$. This map would be v_1 periodic. This completes the proof. The argument is similar for the other cases. We look at θ_4. We have the following

$$M^{47}(1) \xrightarrow{v_1^8} M^{31}(1) \xrightarrow{\theta_4} S^0 \text{ is essential and detected by } \{e_0 r\}.$$

Thus we can consider the composite $M^{47}(1,8) \to M^{56}(1,4) \to S^0$. This is null. We have

$$\begin{tikzcd}
M^{47+2^i\cdot 6}(1,8) \arrow[r] \arrow[d] & M^{56+2^i\cdot 6}(1,4) \arrow[d] & \\
M^{47}(1,8) \arrow[r] & M^{56}(1,4) \arrow[r] & S^0
\end{tikzcd}$$

The map from $M^{47+2^i\cdot 6}(1,8) \to S^0$ is null so the map $M^{47+2^i\cdot 6}(1)$ factors through $M^{31+2^i\cdot 6}$. As before, it must be on the bottom cell or we get a v_1-periodic class. $\square$

Some of these homotopy classes were discussed in [1]. The proofs there are valid for $32k$ replacing $8k$. In that note $\rho_k\eta_j$ was also studied and shown to be non-zero. We first discuss $\eta_4\sigma$. In the sphere we are looking at the classes $\{h_4h_2\}$, h_3^3, h_3c_0, $h_3h_1c_0$ and $\nu^2\bar{\kappa}$.

Proposition 11.9. *The classes $\{h_4h_3\}$, h_3^3, h_3c_0,$h_3h_1c_0$ and $\nu^2\bar{\kappa}$ are v_2^{32} periodic.*

Proof. We begin with $\nu^2\bar{\kappa}$. The $M^{27}(1) \xrightarrow{\nu^2\bar{\kappa}^\#} S^0 \to EO_2$ is detected by $M^{27}(1) \to S^{27} \to EO_2$. Thus the map of $M^{27}(1) \to S^0$ is v_2-periodic. Suppose $S^{26+192} \to M^{27+192}(1) \to S^0$ is null. Then $S^{27+192} \to EO_2$ factors through S^0. This class has finite order and so extends to $M^{28+192}(2^i) \to S^0 \to EO_2$. The composite is v_1-periodic, a contradiction. Next note that the composite $M^{26}(1,4) \to M^{26}(1) \to S^{26} \xrightarrow{\nu^1\kappa} S^0$ is null since $M^{26}(1) \to S^0$ factors through $M^{26}(1) \to M^{18}(1) \to S^{18} \xrightarrow{\{h_2h_4\}} S^0$. By centrality $M^{26+192}(1,4) \to M^{26+192}(1) \xrightarrow{\nu} S^{26+192} \xrightarrow{v_2^{32}\nu^2\bar{\kappa}} S^0$ is null. This gives $v_2^{32}\{h_2h_4\}$. $\nu \circ \{h_4h_2\} = \sigma^3$. This gives $S^{26+192} \xrightarrow{n^2\bar{n}} M^{22+192}(1) \to S^0$ is $\nu^2\bar{\kappa}v_2^{32}$ and so all the in between classes are non-zero too. $\square$

We start with $\{\rho^1 h_2 h_5\}$. There is an extension of $M^{42}(1) \to S^{42} \xrightarrow{\rho^1 h_1 h_5} S^0$ to $M^{51}(1,4) \to S^0$ and the composite is detected by $v_2^8\nu$ and so this gives a v_2^{32} family. As before, this class must live in the $S^{42+k192}$ stem. Composing with ν to get $M^{55}(1,4) \xrightarrow{\{h_2\rho^1 h_2 h_5\}} S^0$ and this is detected by $\nu^2 v_2^8$. This gives $v_2 \cdot \{\rho^1 h_2 h_5\}$, ρn_5, $\rho \wedge n_5$, as v_2^{32k} periodic classes. The same family of arguments works for n_5 and related classes. The approach in [1] is different since EO_2 was not available but complimentary.

The remaining task is to show all the classes in EO_2 come from the sphere in the above sense. This requires constructing new homotopy classes. These classes are covered by the following propositions.

Proposition 11.10. *The class of order 4 represented by $\{h_5 h_0 i\}$ in the 54 stem is v_2^{32} periodic and $2\{h_5 h_0 i\} = \bar{\kappa}\{e_0^2\}$.*

Proof. We first note that $\bar{\kappa}\{e_0^2\}$ is $\{e_0^2 g\}$ and this class fits 11.1. This completes the proof since $\{h_5 h_0 i\}$ must map essentially to EO_{2*}. □

Proposition 11.11. *The class corresponding to $\{P h_5 h_1 e_0\}$ is v_2^{32} periodic.*

Proof. In EO_{2*} we have $\nu\{h_5 h_0 i\} \neq 0$. Since $h_2 h_5 h_0 i = h_1 h_5 P e_0$ in Ext we are done. □

Proposition 11.12. *The class in the 65 stem with Adams spectral sequence name $P h_5 j$ maps to a non zero class in EO_{2*}. Thus it represents a homotopy class which is v_2^{32} periodic.*

We remark that this is the first class beyond Kochman's calculations that was not covered by 11.1.

Proof. In EO_{2*} the Toda bracket $\langle \bar{\kappa}^3, \eta, \nu \rangle \neq 0$. we can form the bracket in the sphere too and so it must be non-zero there. Since $d_4 PG = gz$ and $\bar{\kappa}^2\eta = \{z\}$ we see that $\langle \bar{\kappa}^3, \eta, \nu \rangle \in \{h_2 PG\} = 0$. Thus it has filtration ≥ 12. The only other class of higher filtration not in J_* is $\bar{\kappa}\{w\}$ and this is also present in EO_{2*}. This gives the result. □

Corollary 11.13. *The class $P h_5 h_0 k$ in the 63 stem maps to a class in EO_{2*} and thus is v_2^{32} periodic.*

Proof. In EO_{2*} we have $\nu\langle \bar{\kappa}^3, \eta, \nu \rangle \neq 0$. We also have $h_2 P h_5 j = P h_5 h_0 k$ in Ext. This completes the proof. □

We remark that in the E_∞ term of the May spectral sequence the class $P h_5 h_0 k$ is divided by only h_0. Brunner's calculation of the Ext shows that it is actually divided by h_0^5. In particular, we have $h_0^5 G_{21} = P h_5 h_0 k$.

Proposition 11.14. *The bracket $\langle \epsilon, \bar{\kappa}^3, \eta \rangle$ is detected in EO_{2*}. In the Adams spectral sequence it is in the coset $\{P^2 G\}$. In addition η on this class is divisible by 4 which is represented by the bracket $\langle \bar{\kappa}^3, \nu, 2\nu, \nu \rangle$ and is in the coset $\{Q_5\}$. We also have $\eta^2\langle \bar{\kappa}^3, \nu, 2\nu, \nu \rangle \neq 0$.*

Proof. In EO_{2*} it is easy to see that $\langle \epsilon, \bar{\kappa}^3, \eta \rangle \neq 0$. It is also straightforward to see that it is in $\{P^2G\}$. In Ext we have $h_1 P^2 G = h_0^2 Q_5$. This is verified by Brunner's calculations. Now Q_5 is a permanent cycle because there is nothing of higher filtration for it to hit. It has order eight for the same reason. It is in the four fold bracket $\langle \bar{\kappa}^3, \nu, 2\nu, \nu \rangle$ by construction. The other claims follow by easy arguments except we need to show that Q_5 is non-zero. To this end consider the following diagram.

$$\begin{tikzcd}
S^{72} \arrow[rr, "f"] & & EO_2 \\
M^{72}(8) \arrow[u] \arrow[rr, "g"] & & S^0 \arrow[u] \\
& S^{71} \arrow[ul, "i"] \arrow[ur, "\{Q_5\}"'] &
\end{tikzcd}$$

Since $\eta\langle \epsilon, \bar{\kappa}^3, \eta \rangle = 4\{Q_5\}$, $f_*(i) = \{h_0^3 v_2^{12}$, the generator of Z_2 in the the 72 stem. Thus Q_5 must be a non-zero cycle. ☐

Proposition 11.15. *The composition $\nu\{Q_5\}$ is a class of order 4 with the generator having Adams spectral sequence name $\{PD'\}$. This class is also $\bar{\kappa}\{Ph_5 i h_0\}$.*

Proof.

$$\begin{tikzcd}
S^{75} \arrow[rr, "f"] & & EO_2 \\
M^{75}(2) \arrow[u] \arrow[rr, "g"] & & S^0 \arrow[u] \\
& S^{71} \arrow[ul, "i"] \arrow[ur, "\kappa\bar{\kappa}^3"'] &
\end{tikzcd}$$

The map f is $v_2^8 \eta^2 b$ where b is the EO_{2*} class of order 2 in dimension 27. Thus the map g can not be null but f can not factor through S^0 since it would have Adams filtration at least 16 and there is nothing there. Thus $\kappa\bar{\kappa}^3 \neq 0$ in $\pi_*(S^0)$ and we have $\bar{\kappa}2\ \{h_1 P h_5 e_0\} = \kappa\bar{\kappa}^3$ by 11.11. Thus $\bar{\kappa}\{Ph_5 i h_0\} \neq 0$ and generates a $Z/4$. We need to show that it is $\nu\{Q_5\}$. This follows from the bracket constructed for $\{Q_5\}$ in 11.14. ☐

Proposition 11.16. *The remaining classes through the 95 stem detected by EO_{2*} are $\bar{\kappa}^4$, $\bar{\kappa}^3\{w\}$ and $\langle \bar{\kappa}^3\{w\}, \nu, \eta \rangle = \{v_2^8 P^1 d_0 g\}$.*

References

[1] F. Cohen, D. M. Davis, P. Goerss, and M. Mahowald, *Integral Brown-Gitler spectra*, Proc. AMS 103 (1988), 262-266.
[2] D. M. Davis and M. E. Mahowald, *Ext over the subalgebra A_2 of the Steenrod Algebra for stunted projective spaces* Canad. Math. Soc. Conf. Proc. Vol. 2, Part 1 (1982), 297–342.
[3] D. M. Davis and M. E. Mahowald, *The non realizability of the quotient $A//A_2$ of the Steenrod algebra,* Amer. J. of Math. 104 (1982) 1211-1216.
[4] D. M. Davis and M. E. Mahowald, *v_1 and v_2 periodicity in homotopy theory,* Amer. J. of Math. 103 (1981) 615-659.
[5] Paul G. Goerss, John D. S. Jones, and Mark Mahowald, *Some generalized Brown-Gitler spectra,* Trans. Amer. Math. Soc. 294 (1986) 113-132.
[6] A. Iwai and N. Shimada, *On the cohomology of some Hopf algebras,* Nagoya Mat. Jour. 30 (1967) 103-111.
[7] M. J. Hopkins and H. Miller, Title to be announced.
[8] S. O. Kochman, *Stable homotopy groups of spheres, a computer assisted approach,* Lecture Notes in Math. 1423, Springer Verlag, Berlin 1990.
[9] S. O. Kochman and M. Mahowald, *On the computation of the stable stems,* Contemporary Mathematics #181 (1993)299-316.
[10] W. Lellmann and M. Mahowald, *The bo-Adams spectral sequence,* Trans. AMS 300 (1987), 593-623.
[11] Mark Mahowald, The primary v_2 family, *Math. Z. 177(1981) 381-393.*
[12] Mark Mahowald, *Ring spectra which are Thom spectra,* Duke Journal. 46 (1979) 549-559.
[13] D. C. Ravenel, Complex Cobordism and Stable Homotopy Groups of Spheres, Academic Press, 1986.
[14] J. Silverman, The Arithmetic of Elliptic Curves, Springer-Verlag (1992).
[15] M. C. Tangora, *Computing the homology of the lambda algebra,* Memoirs of the AMS Vol 58 Number 337 (1985).

Department of Mathematics, MIT, Cambridge MA 02139
E-mail address: `mjh@@math.mit.edu`

Department of Mathematics, Northwestern University, Evanston, Illinois 60208
E-mail address: `mark@@math.nwu.edu`