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THE GRASSMANNIAN GEOMETRY OF SPECTRA

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Using bundles over Grassmannians, we construct a category of spectra which underlies Boardman's stable category, has itself a symmetric monoidal smash product, and which produces RO(G)-graded equivariant cohomology theories from group actions on spectra. May's vast array of categories of spectra embed in our single category, and a variety of adjoint functors allow us to deduce properties of our category from those of his. The construction of these functors forces us to develop a theory of parametrized spaces and spectra as well.

0. Introduction and statement of results

Despite many years of study, spectra are still not well-understood objects in topology. There is general agreement that the homotopy theory of spectra must yield Boardman's stable category [16], but there are a variety of competing geometrical constructions satisfying this criterion, with a variety of strengths and weaknesses. This paper presents a new category of spectra, $\mathscr{S}$, which we modestly claim has more strengths and fewer weaknesses than its rivals. Among its most noteworthy traits are the following three:

Theorem 0.1. *The homotopy category of* CW-*spectra in* $\mathscr{S}$ *is equivalent to Boardman's stable category.* (*We also make an infiniteness assumption that will be made precise in Section* 4.)

Theorem 0.2. *The smash product of spectra turns* $\mathscr{S}$ *into a symmetric monoidal category.*

Theorem 0.3. *For any compact group G,* RO(G)-*graded equivariant cohomology theories are represented by G-objects in* $\mathscr{S}$, *i.e., continuous homomorphisms from G to the automorphisms of a spectrum in* $\mathscr{S}$.

Our approach to spectra is inspired by Peter May's observation that spectra should be indexed by the finite-dimensional subspaces of a *universe*: a real inner product space topologized as the colimit of its finite-dimensional subspaces [12, 13]. (The usual integer indexation corresponds to the sequence of standard subspaces

$$\{0\} \subset \mathbb{R}^1 \subset \mathbb{R}^2 \subset \cdots \subset \mathbb{R}^\infty.)$$

Such indexation greatly simplifies smash products and the construction of structured ring spectra. May accordingly defines a category of spectra $\mathscr{S}\mathscr{U}$ for each universe $\mathscr{U}$. The definition of $\mathscr{S}$ (and of our category $\bar{\mathscr{S}}$ of parametrized spectra) relies on a reinterpretation of May's approach, taking seriously the natural topology on the indexing set given by Grassmann manifolds. For us, a spectrum E will be a sequence of maps $E_k \to G_k(\mathscr{U})$, where $G_k(\mathscr{U})$ is the space of k-planes in a universe $\mathscr{U}$. (The maps actually turn out to be bundles.) The systematic use of the machinery of parametrized spaces allows a larger set of maps than May's ($\mathscr{U}$ need not be held fixed) and provides the topology for the resulting morphism spaces. The larger set of maps is the key to representing RO(G)-graded equivariant cohomology theories by G-actions on spectra, and the correctness of the morphism space topology is indicated by Theorem 0.1.

May's categories $\mathscr{S}\mathscr{U}$ nevertheless play a preferred role in our theory. The category $\mathscr{S}$ is equipped with an augmentation

$$\varepsilon : \mathscr{S} \to \mathscr{U}\!\mathit{n},$$

where $\mathscr{U}\!\mathit{n}$ is the category of universes and linear isometries.

Theorem 0.4. *There is a canonical isomorphism of categories*

$$\mathscr{S}\mathscr{U} \cong \varepsilon^{-1}(\mathrm{id}_{\mathscr{U}}).$$

Given two spectra E and E', the resulting map

$$\varepsilon : \mathscr{S}(E, E') \to \mathscr{U}\!\mathit{n}(\varepsilon E, \varepsilon E')$$

will be shown in Section 4 to be 'almost' a fibration; close enough to be able to deduce Theorem 0.1 from Theorem 0.4.

Much of the work involved in the proofs of Theorems 0.1–0.3 involves a deeper connection between $\mathscr{S}$ and the various categories $\mathscr{S}\mathscr{U}$, the *twisted half-smash product*. This was introduced in [12] and exploited in [3], but gains new meaning (and a simplified construction) in our work. Specifically, let E and E' be spectra indexed on universes $\mathscr{U}$ and $\mathscr{U}'$ respectively (by this we mean $\varepsilon E = \mathscr{U}$ and $\varepsilon E' = \mathscr{U}'$). We write $\mathscr{T}/_{\mathscr{U}\!\mathit{n}(\mathscr{U}, \mathscr{U}')}$ for the category of spaces *over* $\mathscr{U}\!\mathit{n}(\mathscr{U}, \mathscr{U}')$ in the sense that, as part of their structure, they come equipped with a map into $\mathscr{U}\!\mathit{n}(\mathscr{U}, \mathscr{U}')$. The twisted half-smash product is a functor

$$\mathscr{T}/_{\mathscr{U}\!\mathit{n}(\mathscr{U}, \mathscr{U}')} \times \mathscr{S}\mathscr{U} \to \mathscr{S}\mathscr{U}'$$

with the image of (A, E) written $A \ltimes E$. We also have the *twisted function spectrum* construction, which is a functor

$$\mathscr{T}/^{\mathrm{op}}_{\mathscr{U}n(\mathscr{U},\,\mathscr{U}')} \times \mathscr{S}\mathscr{U}' \to \mathscr{S}\mathscr{U}$$

with the image of (A, E') written $F[A, E')$. One of our most important tools is

Theorem 0.5. *Let E and E' be spectra indexed on $\mathscr{U}$ and $\mathscr{U}'$ respectively, and let A be in $\mathscr{T}/_{\mathscr{U}n(\mathscr{U},\,\mathscr{U}')}$. Then there are natural isomorphisms*

$$\mathscr{T}/_{\mathscr{U}n(\mathscr{U},\,\mathscr{U}')}(A, \mathscr{S}(E,E')) \cong \mathscr{S}\mathscr{U}'(A \ltimes E, E') \cong \mathscr{S}\mathscr{U}(E, F[A, E')).$$

The most natural proof of Theorem 0.5 involves the introduction of parametrized spectra, i.e., spectra over a base space. Our view is that a parametrized spectrum should be indexed on a parametrized universe, which consists of a map of spaces $\mathscr{U} \to \mathscr{O}$ such that for each $a \in \mathscr{O}$, the preimage of a in $\mathscr{U}$ is a universe in a fashion consistent with continuous variation of a. (We also assume the fiber dimension to be constant.) We will often suppress the parameter space $\mathscr{O}$ and speak only of the parametrized universe $\mathscr{U}$. We define a *map* of parametrized universes $f\colon \mathscr{U} \to \mathscr{U}'$ to be a commutative square

$$\begin{array}{ccc} \mathscr{U} & \xrightarrow{f} & \mathscr{U}' \\ \downarrow & & \downarrow \\ \mathscr{O} & \xrightarrow{f} & \mathscr{O}' \end{array}$$

such that the map on each fiber is a linear isometry, and obtain the *category* of parametrized universes, $\bar{\mathscr{U}n}$. Our category of parametrized spectra, $\bar{\mathscr{S}}$, then augments to $\bar{\mathscr{U}n}$ just as in the unparametrized case. By analogy with Theorem 0.4, we define a preferred subcategory of parametrized spectra for each parametrized universe $\mathscr{U}$ by

$$\bar{\mathscr{S}}\mathscr{U} = \varepsilon^{-1}(\mathrm{id}_{\mathscr{U}}),$$

where ε is the augmentation functor. The following theorem gives our fundamental construction:

Theorem 0.6. *Let $f\colon \mathscr{U} \to \mathscr{U}'$ be a map of parametrized universes. Then f induces a pullback functor*

$$f^*\colon \bar{\mathscr{S}}\mathscr{U}' \to \bar{\mathscr{S}}\mathscr{U}$$

which has a left adjoint f_* and, if f is fiberwise surjective, a right adjoint $f_!$.

Assuming this theorem, it is simple to construct twisted half-smash products and twisted function spectra, and to prove Theorem 0.5. We form the diagram of parametrized universes

$$\begin{tikzcd}
\mathscr{U} \arrow[d] & A \times \mathscr{U} \arrow[l, "\pi"'] \arrow[r, "\chi"] \arrow[d] & \mathscr{U}' \arrow[d] \\
* & A \arrow[l] \arrow[r] & *
\end{tikzcd}$$

where π is the projection and χ is adjoint to the structure map $A \to \mathscr{U}n(\mathscr{U}, \mathscr{U}')$. We define

$$A \ltimes E = \chi_* \pi^* E$$

and

$$F[A, E') = \pi_! \chi^* E.$$

The second isomorphism of Theorem 0.5 is now clear, and the first follows from the next theorem, which is fairly easy to prove (we will give a more general version in Section 3).

Theorem 0.7. *With E, E', $\mathscr{U}$, $\mathscr{U}'$, and A as in Theorem* 0.5, *and π and χ as in the above diagram, there is a natural isomorphism*

$$\mathscr{T}/_{\mathscr{U}n(\mathscr{U}, \mathscr{U}')}(A, \mathscr{S}(E, E')) \cong \bar{\mathscr{S}}(A \times \mathscr{U})(\pi^* E, \chi^* E').$$

The paper is organized as follows. Since we make heavy use of the machinery of parametrized spaces throughout the paper (our definition of spectrum provides an example), we provide a summary in Section 1. Section 2 reviews May's categories of spectra and introduces our own, and then relates the two as far as the proof of Theorem 0.4. Section 3 is the technical heart of the paper; in it we introduce parametrized spectra and prove Theorem 0.6, deriving from it the appropriate generalizations of Theorems 0.5 and 0.7 to the full parametrized setting. (These generalizations will be of great use in a later paper treating structured ring and module spectra as special sorts of parametrized spectra.) Sections 4, 5, and 6 then apply this machinery to prove Theorems 0.1, 0.2, and 0.3 respectively.

1. The calculus of parametrized spaces

This section introduces some very elementary machinery that seems underused in topology, despite the efforts of James [10] and others [6, 8, 11, 15]. While it is basic to all our further work, only parts of it are necessary at a first reading. Our basic constructions are treated in Definition 1.1, Lemmas 1.2, 1.3, and Theorem 1.4; without these, most of the remainder of the paper would be incomprehensible. Many of our computations also use Lemmas 1.5 and 1.6, but Lemmas 1.7–1.9, which give generalizations, are used only occasionally. The remainder of the section introduces an external version of the theory which will come into play only when we reach Section 3.

Throughout the paper we assume we are working with k-spaces, and we generally assume the weak Hausdorff axiom as well [11, 14]. However, the reader should be cautioned that this property is not preserved by the Hom_B construction of Lemma 1.3 below, nor is it preserved in general by the construction $f_!$ (discussed in Theorem 1.4 and Lemma 1.7) which depends on Hom_B for its rigorous definition. Despite this, the weak Hausdorff property *is* preserved in all cases of interest to us, and we will not comment further on it (see [11] for a sufficient condition for preservation). We write $\mathscr{T}$ for the category of weak Hausdorff (k-) spaces, $\mathscr{T}^+$ for based spaces in $\mathscr{T}$. In the following definition, B is the parameter space for our categories of parametrized spaces.

Definition 1.1. Let B be a weak Hausdorff space. The category $\mathscr{T}/_B$ has objects all pairs (X,ξ) where X is a space and $\xi: X\to B$. The morphisms $f:(X,\xi)\to(X',\xi')$ consist of maps $f:X\to X'$ such that $\xi' f=\xi$. The category $\mathscr{T}^+/_B$ has objects all triples (X,ξ,s), where $\xi:X\to B$, $s:B\to X$, and $\xi s=\mathrm{id}_B$. The morphisms $f:(X,\xi,s)\to(X',\xi',s')$ are maps $f:X\to X'$ such that $\xi' f=\xi$ and $fs=s'$.

Our intuition is that B parametrizes the fibers of ξ, which are equipped with a basepoint if we are working in $\mathscr{T}^+/_B$. We will generally abuse notation and refer to an object in either category by its space X, leaving the *structure map* ξ_X and *section* s_X to be inferred from context.

Lemma 1.2. *The forgetful functor $\mathscr{T}^+/_B\to\mathscr{T}/_B$ has a left adjoint, whose value on X we write X^+.*

Proof. Define $X^+=X\amalg B$, with the evident structure map and section. $\square$

Both categories have a preferred product for our purposes. In $\mathscr{T}/_B$ we use the fiber product or pullback, written $X\times_B Y$, which is also the product in the category-theoretic sense. In $\mathscr{T}^+/_B$ we use the fiberwise smash product, constructed as follows. Given X and Y in $\mathscr{T}^+/_B$ we define the fiberwise wedge $X\vee_B Y$ as a pushout:

$$\begin{array}{ccc} B & \xrightarrow{s_Y} & Y \\ {\scriptstyle s_X}\downarrow & & \downarrow \\ X & \longrightarrow & X\vee_B Y \end{array}$$

This is the category-theoretic coproduct in $\mathscr{T}^+/_B$. Since $\mathscr{T}^+/_B$ has a null object (the identity map on B), there is a canonical map from $X\vee_B Y$ to the category-theoretic product, which is again $X\times_B Y$. Now the fiberwise smash product $X\wedge_B Y$ is another pushout:

$$\begin{tikzcd} X \vee_B Y \arrow[r] \arrow[d, "\xi"'] & X \times_B Y \arrow[d] \\ B \arrow[r] & X \wedge_B Y \end{tikzcd}$$

It is easy to see that

$$X^+ \wedge_B Y^+ \cong (X \times_B Y)^+.$$

We will use the following constructions repeatedly.

Lemma 1.3. *Both $\mathscr{T}/_B$ and $\mathscr{T}^+/_B$ have internal hom-functors, written* $\mathrm{Hom}_B(X, Y)$ *and $F_B(X, Y)$ respectively. For any $b \in B$, the fiber over b of* $\mathrm{Hom}_B(X, Y)$ *is $Y_b^{X_b}$ and of $F_B(X, Y)$ is $F(X_b, Y_b)$, where X_b and Y_b are the fibers over b of X and Y respectively. These enjoy the adjunction relations with their products*

$$\mathscr{T}/_B(Z \times_B X, Y) \cong \mathscr{T}/_B(Z, \mathrm{Hom}_B(X, Y))$$

and

$$\mathscr{T}^+/_B(Z \wedge_B X, Y) \cong \mathscr{T}^+/_B(Z, F_B(X, Y)).$$

Proof. $\mathrm{Hom}_B(X, Y)$ is constructed in [1]; see also [11] and [6]. We construct F_B as a pullback:

$$\begin{tikzcd} F_B(X, Y) \arrow[r] \arrow[d] & \mathrm{Hom}_B(X, Y) \arrow[d, shift left, "(s_X)^*"] \\ B = \mathrm{Hom}_B(B, B) \arrow[r, "(s_Y)_*"'] & \mathrm{Hom}_B(B, Y) \arrow[u, dashed, shift left, "\xi_X^*"] \end{tikzcd}$$

with section induced from ξ_X^*. The adjunction is now an exercise in pure category theory. □

The next theorem introduces the induced functors of a map, which appear ubiquitously throughout this paper. We will later generalize them to parametrized spectra in Theorem 0.6.

Theorem 1.4. *Any map $f: B \to A$ of weak Hausdorff spaces induces pullback functors $f^*: \mathscr{T}/_A \to \mathscr{T}/_B$ and $f^*: \mathscr{T}^+/_A \to \mathscr{T}^+/_B$. These functors have both left adjoints f_* and right adjoints $f_!$.*

Proof. Both functors f^* are trivial to construct as pullbacks (in the case with sections, the section of f^*Y is induced from id_B and s_Y by pullback). In the case without sections, f_* is given by prolongation of the structure map: $f_*(X, \xi) = (X, f\xi)$, and $f_!$ is the 'space of sections over A', formally, it is the pullback

$$\begin{tikzcd} f_!X \arrow[r] \arrow[d] & \mathrm{Hom}_A(B, f_*X) \arrow[d, "\xi_*"] \\ A \arrow[r] & \mathrm{Hom}_A(B,B) \end{tikzcd}$$

where B is in $\mathscr{T}/_A$ as (B,f) and the bottom arrow sends each point of A to the identity map in the fiber over it. In the case with sections, $f_!X$ has this same underlying space (in particular, the f_*X in the defining diagram is *not* the pushout given below) with a section induced by $(s_X)_*$. Finally, f_* in this case is given by a pushout:

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "s_X"'] & A \arrow[d] \\ X \arrow[r] & f_*X \end{tikzcd} \qquad \square$$

The following Mackey relations are among our most useful computational tools; when we point out that a square is a pullback, we generally have the next lemma in mind.

Lemma 1.5. *Given a pullback square*

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "G"'] & B \arrow[d, "g"] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

there are natural isomorphisms of functors

$$F_*G^* \cong g^*f_*$$

and

$$f^*g_! \cong G_!F^*$$

for categories of spaces either with or without section.

Proof. The first isomorphism in the case without sections is just the transitivity of pullbacks, and implies the second by the Yoneda lemma. This implies the second isomorphism in the case with sections, since the underlying spaces are the same and the section is clearly preserved. The first isomorphism now follows by another application of the Yoneda lemma. $\square$

Our next lemma records all the commutation relations between the induced functors of a map and the various products and hom-functors.

Lemma 1.6. *Let $f: B \to A$ be a map, X an object of $\mathscr{T}/_B$ or $\mathscr{T}^+/_B$, and Y and Z of $\mathscr{T}/_A$ or $\mathscr{T}^+/_A$. There are natural isomorphisms*

(a) $f^*Z \times_B f^*Y \cong f^*(Z \times_A Y)$; $\mathrm{Hom}_A(Y, f_!X) \cong f_! \,\mathrm{Hom}_B(f^*Y, X)$,

(b) $f^*Z \wedge_B f^*Y \cong f^*(Z \wedge_A Y)$; $F_A(Y, f_!X) \cong f_! F_B(f^*Y, X)$,

(c) $f_*(X \times_B f^*Y) \cong (f_*X) \times_A Y$; $\mathrm{Hom}_B(f^*Y, f^*Z) \cong f^* \mathrm{Hom}_A(Y, Z)$,

(d) $f_*(X \wedge_B f^*Y) \cong (f_*X) \wedge_A Y$; $F_B(f^*Y, f^*Z) \cong f^* F_B(Y, Z)$.

Proof. The statements in each pair are equivalent by the Yoneda lemma. The isomorphisms of products in (a) and (c) follow from transitivity of pullbacks, and the morphism statement in (d) is pulled back from that in (c). Finally, the product statement in (b) follows from (a) since f^* has both a left and a right adjoint, and therefore preserves wedges, fiber products, and pushouts. □

The following pair of lemmas generalize the adjunctions of Theorem 1.4, and will be needed in only a few places. They apply only in the case *without* sections.

Lemma 1.7. *Let $f: B \to A$, and let Z be in $\mathscr{T}/_B$, X in $\mathscr{T}/_{f_!Z}$, and Y in $\mathscr{T}/_Z$. By abuse of notation (which is the reason we consider only the case without sections) we also consider X to be in $\mathscr{T}/_A$ and Y in $\mathscr{T}/_B$, so $f_!Y$ is defined and in $\mathscr{T}/_{f_!Z}$, and f^*X is in $\mathscr{T}/_Z$ by a corresponding structure map in the isomorphism*

$$\mathscr{T}/_B(f^*X, Z) \cong \mathscr{T}/_A(X, f_!Z).$$

Then

$$\mathscr{T}/_Z(f^*X, Y) \cong \mathscr{T}/_{f_!Z}(X, f_!Y).$$

Proof. Consider the commutative square

$$\begin{tikzcd}
\mathscr{T}/_B(f^*X, Y) \arrow[d, "\xi_*"'] \arrow[r, phantom, "\cong"] & \mathscr{T}/_A(X, f_!Y) \arrow[d, "\xi_*"] \\
\mathscr{T}/_B(f^*X, Z) \arrow[r, phantom, "\cong"] & \mathscr{T}/_A(X, f_!Z)
\end{tikzcd}$$

The structure maps for f^*X and for X correspond in the bottom row, so their inverse images in the top row are isomorphic. This is precisely the claimed isomorphism. □

By a similar argument, we also have

Lemma 1.8. *Let $f: B \to A$, let Z be in $\mathscr{T}/_A$, X in $\mathscr{T}/_{f^*Z}$, and Y in $\mathscr{T}/_Z$. Then f^*Y is* *in* $\mathscr{T}/_{f^*Z}$, *and* f_*X *is in* $\mathscr{T}/_Z$ *via*

$$\mathscr{T}/_B(X, f^*Z) \cong \mathscr{T}/_A(f_*X, Z).$$

We conclude that

$$\mathscr{T}/_{f^*Z}(X, f^*Y) \cong \mathscr{T}/_Z(f_*X, Y). \qquad \square$$

Even when a square is *not* a pullback, we can salvage a bit from Lemma 1.5. The following lemma will be needed in only one spot, albeit a crucial one.

Lemma 1.9. *Given a commutative square*

$$\begin{tikzcd}
A \arrow[r, "F"] \arrow[d, "G"'] & B \arrow[d, "g"] \\
C \arrow[r, "f"] & D
\end{tikzcd}$$

there is a canonical natural transformation

$$\varphi : F_*G^* \to g^*f_*$$

which is the common map in the commutative diagram

$$\begin{tikzcd}
 & g^*g_*F_*G^* \cong g^*f_*G_*G^* \arrow[dr, "g^*f_*\varepsilon_G"] & \\
F_*G^* \arrow[ur, "\eta_g"] \arrow[dr, "F_*G^*\eta_f"'] & & g^*f_* \\
 & F_*G^*f^*f_* \cong F_*F^*g^*f_* \arrow[ur, "\varepsilon_F"'] &
\end{tikzcd}$$

where we have systematically used η *and* ε *for the unit and counit of the appropriate adjunction. This applies in the cases both with and without section.* (*There is also a statement about* $f^*g_! \to G_!F^*$ *that we will encounter in the proof, but of which we will make no other use.*)

Proof. We consider first the case without sections. Given X in $\mathscr{T}/_C$, we define φ as the induced map of pullbacks in the diagram

$$\begin{tikzcd}
G^*X \arrow[rr] \arrow[dd] \arrow[dr, dashed, "\varphi"] & & A \arrow[dr, "F"] \arrow[dd, "G" near start] & \\
 & g^*f_*X \arrow[rr] \arrow[dd] & & B \arrow[dd, "g"] \\
X \arrow[rr] \arrow[dr, "="'] & & C \arrow[dr, "f"] & \\
 & f_*X \arrow[rr] & & D
\end{tikzcd}$$

and verify that φ coincides with both ways of traversing the claimed diagram. Observe that φ is the *unique* map in $\mathscr{T}/_B$ making the square

$$\begin{tikzcd}
G^*X \arrow[r, "\varphi"] \arrow[d] & g^*f_*X \arrow[d] \\
X \arrow[r, phantom, "="] & f_*X
\end{tikzcd}$$

commute, or considering all spaces as over D, such that

$$\begin{tikzcd}
g_*F_*G^* \arrow[r, "g_*\varphi"] \arrow[d, "\cong"'] & g_*g^*f_* \arrow[d, "\varepsilon_g"] \\
f_*G_*G^* \arrow[r, "f_*\varepsilon_G"'] & f_*
\end{tikzcd}$$

commutes. But this diagram is adjoint to

$$\begin{tikzcd}
 & F_*G^* \arrow[r, "\varphi"] \arrow[dl, "\eta_g"'] & g^*f_* \arrow[dd, "="] \\
g^*g_*F_*G^* \arrow[dr, "\cong"'] & & \\
 & g^*f_*G_*G^* \arrow[r, "g^*f_*\varepsilon_G"'] & g^*f_*
\end{tikzcd}$$

which shows that φ is the top composite in the claimed diagram. If instead we pull the spaces back over A, we see that

$$\begin{tikzcd}
G^* \arrow[r, "G^*\eta_f"] \arrow[d, "\eta_F"'] & G^*f^*f_* \arrow[d, "\cong"] \\
F^*F_*G^* \arrow[r, "F^*\varphi"'] & F^*g^*f_*
\end{tikzcd}$$

must commute. But this is adjoint to

$$\begin{tikzcd}
F_*G^* \arrow[r, "F_*G^*\eta_f"] \arrow[dd, "="'] & F_*G^*f^*f_* \arrow[dr, "\cong"] & \\
 & & F_*F^*g^*f_* \arrow[dl, "\varepsilon_F"] \\
F_*G^* \arrow[r, "\varphi"'] & g^*f_* &
\end{tikzcd}$$

so φ coincides with the bottom composite as well.

To prove the case with sections, we first apply the Yoneda lemma to obtain a canonical transformation

$$\tilde{\varphi} : f^*g_! \to G_!F^*$$

which is the common map in the commutative diagram

$$\begin{tikzcd}[column sep=small] & G_!G^*f^*g_! \cong G_!F^*g^*g_! \arrow[dr, "G_!F^*\varepsilon_g"] & \\ f^*g_! \arrow[ur, "\eta_G"] \arrow[dr, "f^*g_!\eta_F"'] & & G_!F^* \\ & f^*g_!F_!F^* \cong f^*f_!G_!F^* \arrow[ur, "\varepsilon_f"'] & \end{tikzcd}$$

But here the case with sections follows from the case without sections, since the underlying spaces and maps are the same. The case with sections of the claim now follows from another application of the Yoneda lemma. $\square$

In the remainder of this section, we briefly outline the external version of the theory that we will need both in this paper and its sequel. (The most significant special case of this external version is also discussed in [2], where it is also derived from the internal theory.) We consider first the case without sections.

Definition 1.10. Let X be in $\mathscr{T}/_{B\times A}$, Y in $\mathscr{T}/_{C\times B}$. We define the *box product* $X\,\square_B\,Y$ in $\mathscr{T}/_{C\times A}$ to be the pullback

$$\begin{tikzcd} X\,\square_B\,Y \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & A \\ Y \arrow[r] \arrow[d] & B & \\ C & & \end{tikzcd}$$

with the indicated map to $C\times A$.

Corollary 1.11. *Given X in $\mathscr{T}/_{B\times A}$, Y in $\mathscr{T}/_{C\times B}$, and Z in $\mathscr{T}/_{D\times C}$, there is a natural isomorphism in $\mathscr{T}/_{D\times A}$*

$$X\,\square_B\,(Y\,\square_C\,Z) \cong (X\,\square_B\,Y)\,\square_C\,Z.$$

Proof. Transitivity of pullbacks. $\square$

Corollary 1.12. *Let* $p_1 : C\times B\times A \to B\times A$, $p_2 : C\times B\times A \to C\times A$, *and* $p_3 : C\times B\times A \to C\times B$ *be the natural projections. Then*

$$X \square_B Y \cong (p_2)_*(p_1^*X \times_{C\times B\times A} p_3^*Y).$$

Proof. All the squares in the following diagram are pullbacks:

$$\begin{tikzcd}
X \square_B Y \arrow[r] \arrow[d] & p_1^*X \arrow[r] \arrow[d] & X \arrow[d] \\
p_3^*Y \arrow[r] \arrow[d] & C\times B\times A \arrow[r] \arrow[d] & B\times A \arrow[d] \\
Y \arrow[r] & C\times B \arrow[r] & B
\end{tikzcd}$$ □

Most cases of interest to us in the following lemma will have $C=\{*\}$.

Lemma 1.13. *Let X be in* $\mathscr{T}/_{B\times A}$, *Y in* $\mathscr{T}/_{C\times B}$, *and Z in* $\mathscr{T}/_{C\times A}$. *There is a functorial construction* $\mathscr{H}_C(Y,Z)$ *in* $\mathscr{T}/_{B\times A}$ *for which*

$$\mathscr{T}/_{C\times A}(X \square_B Y, Z) \cong \mathscr{T}/_{B\times A}(X, \mathscr{H}_C(Y,Z)).$$

If $C=\{*\}$, *we just write* $\mathscr{H}(Y,Z)$.

Proof. We define

$$\mathscr{H}_C(Y,Z) = (p_1)_! \operatorname{Hom}_{C\times B\times A}(p_3^*Y, p_2^*Z)$$

and the isomorphism follows immediately from Theorem 1.4 and Corollary 1.12. □

Lemma 1.14. *Let X be in* $\mathscr{T}/_{Q\times A}$, *Y in* $\mathscr{T}/_{Q\times B}$, *and Z in* $\mathscr{T}/_{Q\times C}$. *There is an associative composition pairing*

$$\mathscr{H}_Q(Y,Z) \square_B \mathscr{H}_Q(X,Y) \to \mathscr{H}_Q(X,Z).$$

Proof. Let $\varepsilon_X : \mathscr{H}_Q(X,Y) \square_A X \to Y$ and $\varepsilon_Y : \mathscr{H}_Q(Y,Z) \square_B Y \to Z$ be counits of the adjunction in Lemma 1.13. We define the pairing to be adjoint to the composite

$$\varepsilon_Y \circ (1 \square_B \varepsilon_X) : \mathscr{H}_Q(Y,Z) \square_B \mathscr{H}_Q(X,Y) \square_A X \to Z,$$

and the proof of associativity is formally identical to the arguments proving [6, Theorem 6]. □

Although we could make analogous constructions entirely within the context of spaces with section, we will find a mixture to be more useful.

Definition 1.15. Let X be in $\mathscr{T}/_{B\times A}$, Y in $\mathscr{T}^+/_{C\times B}$, and Z in $\mathscr{T}^+/_{C\times A}$. We define

$$X \boxtimes_B Y = (p_2)_*(p_1^*X^+ \wedge_{C\times B\times A} p_3^*Y),$$

and

$$\bar{\mathscr{H}}_C(Y,Z) = (p_1)_! F_{C\times B\times A}(p_3^*Y, p_2^*Z).$$

Corollary 1.16. *With X, Y, and Z as above, there is a natural isomorphism*

$$\mathscr{T}^+/_{C\times A}(X\boxtimes_B Y, Z) \cong \mathscr{T}/_{B\times A}(X, \bar{\mathscr{H}}_C(Y,Z)).$$

Proof. This follows from Lemmas 1.2, 1.3, and Theorem 1.4. $\square$

2. Spectra and prespectra

In this section we introduce our new category of spectra $\mathscr{S}$. Since May's categories of spectra play a preferred role in ours, and our smash products arise from his, we first give a brief account of his constructions. As with May's, our spectra are embedded in a larger category of 'prespectra', which are connected to spectra by a left adjoint to the inclusion functor. We defer the construction of this left adjoint to the appendix. Having constructed $\mathscr{S}$, we relate our spectra to May's by proving Theorem 0.4.

Definition 2.1. A *universe* is a real inner product space, topologized as the colimit of its finite-dimensional subspaces. We write $\mathscr{U}\!n$ for the category of universes and linear isometries; the reason for considering only isometries is that we wish the images of orthogonal subspaces to remain orthogonal. Given two universes $\mathscr{U}$ and $\mathscr{U}'$, we give $\mathscr{U}\!n(\mathscr{U}, \mathscr{U}')$ the usual function space topology. (May writes $\mathscr{I}(\mathscr{U}, \mathscr{U}')$ for $\mathscr{U}\!n(\mathscr{U}, \mathscr{U}')$.) An *indexing set* $\mathscr{A}$ in a universe $\mathscr{U}$ is a set of finite-dimensional subspaces of $\mathscr{U}$ which is *cofinal* in the sense that any finite-dimensional subspace W of $\mathscr{U}$ is a subspace of some $V\in\mathscr{A}$. The *standard* indexing set is the set of *all* finite-dimensional subspaces of $\mathscr{U}$.

It is a fundamental insight of May that spectra should be indexed on indexing sets for the most natural treatment of smash products and structured ring spectra. The usual integer indexing corresponds to letting $\mathscr{U}=\mathbb{R}^\infty$, $\mathscr{A}=\{\mathbb{R}^n : n\geq 0\}$. May restricts his attention to countably infinite dimensional universes, as is proper for stable objects, but we will find it convenient to consider finite-dimensional universes as well. In particular, the use of the trivial universe $\{0\}$ will allow us to consider a based space as a special sort of spectrum.

Definition 2.2. Let $\mathscr{A}$ be an indexing set in a universe $\mathscr{U}$. For V and W in $\mathscr{A}$ with

$V \subset W$, we write $W - V$ for the orthogonal complement of V in W, and S^{W-V} for the one-point compactification of $W - V$. Given a based space X, we write $\Sigma^{W-V}X$ for $X \wedge S^{W-V}$ and $\Omega^{W-V}X$ for $F(S^{W-V}, X)$. An *$\mathscr{A}$-prespectrum D* consists of a based space DV for each $V \in \mathscr{A}$, together with structure maps

$$\sigma_{V,W} : \Sigma^{W-V} DV \to DW$$

whenever $V \subset W$. Given U, V, and W elements of $\mathscr{A}$ with $U \subset V \subset W$, we require the diagram

$$\begin{tikzcd}
\Sigma^{W-V}\Sigma^{V-U}DU \arrow[r, "\Sigma^{W-V}\sigma_{U,V}"] \arrow[d, "\cong"'] & \Sigma^{W-V}DV \arrow[d, "\sigma_{V,W}"] \\
\Sigma^{W-U}DU \arrow[r, "\sigma_{U,W}"] & DW
\end{tikzcd}$$

to commute. Equivalently, we may consider the adjoint structure maps

$$\bar{\sigma}_{V,W} : DV \to \Omega^{W-V} DW$$

and require the diagram

$$\begin{tikzcd}
DU \arrow[r, "\bar{\sigma}_{U,V}"] \arrow[d, "\bar{\sigma}_{U,W}"'] & \Omega^{V-U}DV \arrow[d, "\Omega^{V-U}\bar{\sigma}_{V,W}"] \\
\Omega^{W-U}DW \arrow[r, "\cong"] & \Omega^{V-U}\Omega^{W-V}DW
\end{tikzcd}$$

to commute. A *map* of $\mathscr{A}$-prespectra $f : D \to D'$ consists of maps $fV : DV \to D'V$ for each $V \in \mathscr{A}$ such that if $V \subset W$,

$$fW \circ \sigma_{V,W} = \sigma'_{V,W} \circ \Sigma^{W-V} fV,$$

or equivalently,

$$\Omega^{W-V} fW \circ \bar{\sigma}_{V,W} = \bar{\sigma}'_{V,W} \circ fV.$$

The resulting category of $\mathscr{A}$-prespectra will be written $\mathscr{P}\mathscr{A}$. An *$\mathscr{A}$-spectrum* is an $\mathscr{A}$-prespectrum for which all the adjoint structure maps $\bar{\sigma}_{V,W}$ are homeomorphisms. The category $\mathscr{S}\mathscr{A}$ of $\mathscr{A}$-spectra is the full subcategory of $\mathscr{P}\mathscr{A}$ generated by the $\mathscr{A}$-spectra. Finally, if $\mathscr{A}$ is the standard indexing set on $\mathscr{U}$, we will write $\mathscr{P}\mathscr{U}$ and $\mathscr{S}\mathscr{U}$ instead of $\mathscr{P}\mathscr{A}$ and $\mathscr{S}\mathscr{A}$.

The following basic result is due to Lewis in its correct form [12, Appendix]; it will also follow in most cases of interest from Lemma 2.7, Theorem 2.16, and Theorem A.1.

Lemma 2.3. *The inclusion functor $l : \mathscr{S}\mathscr{A} \to \mathscr{P}\mathscr{A}$ has a left adjoint $L : \mathscr{P}\mathscr{A} \to \mathscr{S}\mathscr{A}$.* □

Corollary 2.4. *The category $\mathscr{S}\mathscr{A}$ has all small limits and colimits.*

Proof. Limits and colimits in $\mathscr{P}\mathscr{A}$ are constructed spacewise, and since l has a left adjoint, limits in $\mathscr{S}\mathscr{A}$ are inherited from $\mathscr{P}\mathscr{A}$. Since l is the inclusion of a full subcategory, the colimit in $\mathscr{S}\mathscr{A}$ is provided by $L\operatorname{colim} l$-. $\square$

Once we restrict our attention to spectra, the distinction between different indexing sets is negligible.

Lemma 2.5. *Let $\mathscr{A}$ and $\mathscr{B}$ be indexing sets in a universe $\mathscr{U}$ with $\mathscr{A}\subset\mathscr{B}$. Then the natural restriction functor $\mathscr{S}\mathscr{B}\to\mathscr{S}\mathscr{A}$ is an equivalence of categories.*

Proof. We construct the inverse equivalence. Let E be a spectrum in $\mathscr{S}\mathscr{A}$; we need to extend its indexing to $\mathscr{B}$. Let $V\in\mathscr{B}$. Since $\mathscr{A}$ is indexing, there is some $W\in\mathscr{A}$ with $V\subset W$. We define $EV=\Omega^{W-V}EW$. If we choose a different $W'\in\mathscr{A}$, there is some $Z\in\mathscr{A}$ containing both W and W', and we identify $\Omega^{W-V}EW$ with $\Omega^{W'-V}EW'$ via

$$\begin{aligned}\Omega^{W-V}EW &\xrightarrow[\cong]{\Omega^{W-V}\tilde{\sigma}_{W,Z}} \Omega^{W-V}\Omega^{Z-W}EZ\cong\Omega^{Z-V}EZ\\ &\cong\Omega^{W'-V}\Omega^{Z-W'}EZ \xleftarrow[\cong]{\Omega^{W'-V}\tilde{\sigma}_{W',Z}} \Omega^{W'-V}EW'.\end{aligned}$$

The structure maps are easily provided. $\square$

As a consequence of this lemma, we will make no distinction between categories of spectra in a universe $\mathscr{U}$, writing them all indiscriminately as $\mathscr{S}\mathscr{U}$. Lemmas 2.3 and 2.5 now say that the composite

$$\mathscr{S}\mathscr{U}\to\mathscr{P}\mathscr{U}\to\mathscr{P}\mathscr{A}$$

has a left adjoint, as does the left-hand arrow. We digress briefly to answer the natural question: under what circumstances does the right-hand arrow have a left adjoint? The following condition is satisfied in all cases arising in practice:

Definition 2.6. An indexing set $\mathscr{A}$ is *complete* if it is closed under bounded sums, i.e., given $\{V_\alpha\}\subset\mathscr{A}$ such that for all α, $V_\alpha\subset W$ for some finite-dimensional W, then the sum $\sum_\alpha V_\alpha$ is in $\mathscr{A}$. (The sum is, of course, the smallest possible W.) By convention the empty sum is $\{0\}$.

Lemma 2.7. *If $\mathscr{A}$ is a complete indexing set, then the restriction functor $\mathscr{P}\mathscr{U}\to\mathscr{P}\mathscr{A}$ has a left adjoint.*

Proof. Let D be in $\mathscr{P}\mathscr{A}$; we define its image $\hat{D}$ in $\mathscr{P}\mathscr{U}$. Let W be any finite-dimensional subspace of $\mathscr{U}$, and consider the bounded set $\{V\in\mathscr{A}: V\subset W\}$. Since

$\mathscr{A}$ is complete, the sum of this collection, $\bar{W}$, is in $\mathscr{A}$, and clearly $\bar{W} \subset W$. We define

$$\hat{D}W = \Sigma^{W-\bar{W}} D\bar{W}.$$

The structure maps and unit and counit of the adjunction are easily provided. $\square$

Our flexibility in choosing indexing sets is crucial for the construction of smash products, for even if we start with standard indexing sets (or the usual indexing set $\{\mathbb{R}^n : n \geq 0\}$ in $\mathbb{R}^\infty$) we find that the smash product defined below is indexed on a more general indexing set.

Definition 2.8. Let $\mathscr{A}$ and $\mathscr{A}'$ be indexing sets on universes $\mathscr{U}$ and $\mathscr{U}'$ respectively. The indexing set $\mathscr{A} \oplus \mathscr{A}'$ on the universe $\mathscr{U} \oplus \mathscr{U}'$ consists of

$$\{V \oplus V' : V \in \mathscr{A} \text{ and } V' \in \mathscr{A}'\}.$$

The *smash product of prespectra* is a bifunctor

$$\wedge : \mathscr{P}\mathscr{A} \times \mathscr{P}\mathscr{A}' \to \mathscr{P}(\mathscr{A} \oplus \mathscr{A}')$$

with

$$(D \wedge D')(V \oplus V') = DV \wedge D'V'.$$

The *smash product of spectra* is given by the composite

$$\mathscr{S}\mathscr{U} \times \mathscr{S}\mathscr{U}' \xrightarrow{l \times l} \mathscr{P}\mathscr{A} \times \mathscr{P}\mathscr{A}' \xrightarrow{\wedge} \mathscr{P}(\mathscr{A} \oplus \mathscr{A}') \xrightarrow{L} \mathscr{S}(\mathscr{U} \oplus \mathscr{U}')$$

for any choice of indexing sets $\mathscr{A}$ and $\mathscr{A}'$. We must show that this is actually independent of the choice of $\mathscr{A}$ and $\mathscr{A}'$, which requires the following lemma:

Lemma 2.9. *Let $\mathscr{A}$ and $\mathscr{B}$ be indexing sets on $\mathscr{U}$ with $\mathscr{A} \subset \mathscr{B}$. Then the diagram*

$$\begin{tikzcd} \mathscr{P}\mathscr{B} \arrow[dd, "q"'] \arrow[dr, "L_{\mathscr{B}}"] & \\ & \mathscr{S}\mathscr{U} \\ \mathscr{P}\mathscr{A} \arrow[ur, "L_{\mathscr{A}}"'] & \end{tikzcd}$$

commutes, where q is the restriction functor.

Proof. Given D in $\mathscr{P}\mathscr{B}$ and E in $\mathscr{S}\mathscr{U}$, we show first that

$$q : \mathscr{P}\mathscr{B}(D, l_{\mathscr{B}}E) \to \mathscr{P}\mathscr{A}(qD, ql_{\mathscr{B}}E)$$

is an isomorphism by producing the inverse map. Let $f : qD \to ql_{\mathscr{B}}E$ be a map in $\mathscr{P}\mathscr{A}$ and let $V \in \mathscr{B}$. Since $\mathscr{A}$ is indexing, $V \subset W$ for some $W \in \mathscr{A}$. Then the extension of f to $\mathscr{P}\mathscr{B}$ at V is forced by the commutativity of the diagram

$$\begin{tikzcd} DV \arrow[r, "\bar{\sigma}_{V,W}"] \arrow[d, "fV"'] & \Omega^{W-V} DW \arrow[d, "\Omega^{W-V} fW"] \\ EV \arrow[r, "\bar{\sigma}_{V,W}", "\cong"'] & \Omega^{W-V} EW \end{tikzcd}$$

Now the Yoneda lemma applied to the isomorphisms

$$\begin{aligned} \mathscr{S}\mathscr{U}(L_{\mathscr{A}} qD, E) &\cong \mathscr{P}\mathscr{A}(qD, l_{\mathscr{A}} E) \\ &\cong \mathscr{P}\mathscr{A}(qD, q l_{\mathscr{B}} E) \xleftarrow[\cong]{q} \mathscr{P}\mathscr{B}(D, l_{\mathscr{B}} E) \\ &\cong \mathscr{S}\mathscr{U}(L_{\mathscr{B}} D, E) \end{aligned}$$

shows that $L_{\mathscr{A}} q \cong L_{\mathscr{B}}$. □

Corollary 2.10. *The smash product of spectra is independent of the choice of indexing sets.*

Proof. We compare an arbitrary choice of indexing sets $\mathscr{A}$ and $\mathscr{A}'$ to the standard indexing sets on $\mathscr{U}$ and $\mathscr{U}'$; write the sum of the standard indexing sets (which is *not* the standard indexing set on $\mathscr{U} \oplus \mathscr{U}'$) as $\mathscr{U} \hat{\oplus} \mathscr{U}'$. The result now follows from the commutative diagram

$$\begin{tikzcd} & \mathscr{P}\mathscr{U} \times \mathscr{P}\mathscr{U}' \arrow[r, "\wedge"] \arrow[d] & \mathscr{P}(\mathscr{U} \hat{\oplus} \mathscr{U}') \arrow[d] \arrow[dr, "L"] & \\ \mathscr{S}\mathscr{U} \times \mathscr{S}\mathscr{U}' \arrow[ur, "l"] \arrow[dr, "l"'] & & & \mathscr{S}(\mathscr{U} \oplus \mathscr{U}') \\ & \mathscr{P}\mathscr{A} \times \mathscr{P}\mathscr{A}' \arrow[r, "\wedge"] & \mathscr{P}(\mathscr{A} \oplus \mathscr{A}') \arrow[ur, "L"'] & \end{tikzcd}$$

□

This definition of the smash product is nicely symmetric monoidal, but unfortunately it is not internal to any of the categories $\mathscr{S}\mathscr{U}$ (except the trivial case $\mathscr{U} = \{0\}$). To remedy this, May chooses a linear isometric isomorphism $\mathscr{U} \oplus \mathscr{U} \xrightarrow{\cong} \mathscr{U}$ for a countably infinite dimensional $\mathscr{U}$, which cannot be done canonically, and uses the induced isomorphism of categories $\mathscr{S}(\mathscr{U} \oplus \mathscr{U}) \cong \mathscr{S}\mathscr{U}$ to internalize the smash product. Unfortunately, this destroys the symmetric monoidal structure until one passes to the stable category. Our point of view is that the categories $\mathscr{S}\mathscr{U}$ simply are not large enough: they need to be embedded in our new category $\mathscr{S}$.

Our starting point in the construction of $\mathscr{S}$ is the following trivial observation:

Lemma 2.11. *Let E be a spectrum in $\mathscr{S}\mathscr{U}$, and let V and V' be finite-dimensional subspaces of $\mathscr{U}$ with* $\dim V = \dim V'$. *Then $EV \cong EV'$.*

Proof. Embed V and V' in some finite-dimensional subspace W of $\mathscr{U}$. Then

$$EV \cong \Omega^{W-V}EW \cong \Omega^{W-V'}EW \cong EV'. \qquad \square$$

The important aspect of this proof is that the middle isomorphism is not canonical, but this is what one expects of the fibers in a bundle. The evident base space here is a Grassmannian, so we introduce the following notation:

Notation 2.12. Let $\mathscr{U}$ be a universe, k a non-negative integer. We write $G_k(\mathscr{U})$ (or just G_k if $\mathscr{U}$ is clear from the context) for the space of all k-dimensional subspaces of $\mathscr{U}$, topologized as the colimit of the Grassmann manifolds $G_k(V)$ for finite-dimensional subspaces V of $\mathscr{U}$. (This can also be expressed as the orbit space $\mathscr{U}\mathscr{n}(\mathbb{R}^k, \mathscr{U})/O_k$.) We write $\gamma_k(\mathscr{U})$ (or just γ_k) for the canonical k-plane bundle over G_k, and S^{γ_k} for its fiberwise one-point compactification, which will be considered an object of $\mathscr{T}^+/_{G_k}$. For X in $\mathscr{T}^+/_{G_k}$, we write $\Sigma^{\gamma_k}X$ for $X \wedge_{G_k} S^{\gamma_k}$ and $\Omega^{\gamma_k}X$ for $F_{G_k}(S^{\gamma_k}, X)$. We will also make use of the spaces $G_{n,k}(\mathscr{U})$ and $G_{n,j,k}(\mathscr{U})$ of ordered pairs (resp. triples) of pairwise orthogonal subspaces of dimensions (n,k) (resp. (n,j,k)) also topologized as colimits over the collection of finite-dimensional subspaces of $\mathscr{U}$. We have projection maps $p_1 : G_{n,k} \to G_n$, $p_2 : G_{n,k} \to G_k$, and a sum map $s : G_{n,k} \to G_{n+k}$. We will generally abuse notation and write γ_k indiscriminately for $p_1^*\gamma_k$, $p_2^*\gamma_k$, and $s^*\gamma_k$; this will cause no confusion in context.

The following definition gives the obvious reformulation of May's spectra and prespectra in terms of parametrized spaces.

Definition 2.13. A *prespectrum* consists of a universe $\mathscr{U}$ and a sequence D_k of objects of $\mathscr{T}^+/_{G_k(\mathscr{U})}$ for $k \geq 0$, together with structure maps in $\mathscr{T}^+/_{G_{n,k}(\mathscr{U})}$

$$\sigma_{n,k} : \Sigma^{\gamma_n} p_2^* D_k \to s^* D_{n+k}$$

or equivalently

$$\bar{\sigma}_{n,k} : p_2^* D_k \to \Omega^{\gamma_n} s^* D_{n+k}$$

for all n and k, subject to the compatibilities below. A *spectrum* is a prespectrum such that all the adjoint structure maps $\bar{\sigma}_{n,k}$ are isomorphisms. For technical reasons (discussed in the appendix) we require each D_k to be topologized as the colimit of the restrictions $D_k(V)$ of D_k to $G_k(V)$ for finite-dimensional subspaces V of $\mathscr{U}$.

To explain the compatibility relations, we examine the defining diagram

$$\begin{tikzcd}
& G_{j,k} \arrow[dl, "p_2"'] \arrow[dr, "s"] & \\
G_k & & G_{j+k} \\
& G_{n,j,k} \arrow[uu, "p"] \arrow[dl, "s_2"'] \arrow[dr, "s_1"] & \\
G_{n+j,k} \arrow[uu, "\bar{p}_2"] \arrow[dr, "\bar{s}"'] & & G_{n,j+k} \arrow[uu, "\hat{p}_2"'] \arrow[dl, "\hat{s}"] \\
& G_{n+j+k} &
\end{tikzcd}$$

and require the following diagram to commute:

$$\begin{tikzcd}
\Sigma^{\gamma_n} p^* \Sigma^{\gamma_j} p_2^* D_k \arrow[r, "\Sigma^{\gamma_n} p^* \sigma_{j,k}"] \arrow[d, "\cong"'] & \Sigma^{\gamma_n} p^* s^* D_{j+k} \arrow[d, "\cong"] \\
s_2^* \Sigma^{\gamma_{n+j}} \bar{p}_2^* D_k \arrow[d, "s_2^* \sigma_{n+j,k}"'] & s_1^* \Sigma^{\gamma_n} \hat{p}_2^* D_{j+k} \arrow[d, "s_1^* \sigma_{n,j+k}"] \\
s_2^* \bar{s}^* D_{n+j+k} \arrow[r, "\cong"] & s_1^* \hat{s}^* D_{n+j+k}
\end{tikzcd}$$

or if we abbreviate by omitting the pullback functors, simply

$$\begin{tikzcd}
\Sigma^{\gamma_n} \Sigma^{\gamma_j} D_k \arrow[r, "\Sigma^{\gamma_n} \sigma_{j,k}"] \arrow[d, "\cong"'] & \Sigma^{\gamma_n} D_{j+k} \arrow[d, "\sigma_{n,j+k}"] \\
\Sigma^{\gamma_{n+j}} D_k \arrow[r, "\sigma_{n+j,k}"] & D_{n+j+k}
\end{tikzcd}$$

We leave the corresponding diagram involving the adjoint maps $\bar{\sigma}_{n,k}$ as an exercise for the reader.

We will also occasionally use the equivalent structure maps in $\mathscr{T}^+/_{G_{n+k}}$

$$\sigma_{n,k} : s_* \Sigma^{\gamma_n} p_2^* D_k \to D_{n+k}$$

and in $\mathscr{T}^+/_{G_k}$

$$\bar{\sigma}_{n,k} : D_k \to (p_2)_! \Omega^{\gamma_n} s^* D_{n+k} .$$

May's spectra and prespectra can be thought of as determining the fibers of the spaces D_k over $G_k(\mathscr{U})$ (the fiber over V being DV) but without the topology on the total spaces. In the case of spectra, it is remarkable that May's definition *forces* the topology of the total spaces, as we will see in the proof of Theorem 0.4. Our contribution, then, is not so much in constructing new spectra, but in introducing many more morphisms, together with the correct topology on the morphism spaces (which *does* involve the topology of the total spaces).

Definition 2.14. A *map* of prespectra $f:D\to D'$ consists of a linear isometry $\theta:\mathscr{U}\to\mathscr{U}'$ of the underlying universes, which induces a pullback diagram

$$
\begin{tikzcd}
S^{\gamma_k} \arrow[r, "S^{\theta}"] \arrow[d, shift right] & S^{\gamma'_k} \arrow[d, shift right] \\
G_k \arrow[r, "G_k(\theta)"] \arrow[u, dashed, shift right] & G'_k \arrow[u, dashed, shift right]
\end{tikzcd}
$$

and a map $G_{n,k}(\theta):G_{n,k}(\mathscr{U})\to G_{n,k}(\mathscr{U}')$, together with maps $f_k:D_k\to D'_k$ for each k such that the diagrams

$$
\begin{tikzcd}
D_k \arrow[r, "f_k"] \arrow[d, shift right] & D'_k \arrow[d, shift right] \\
G_k(\mathscr{U}) \arrow[r, "G_k(\theta)"] \arrow[u, dashed, shift right] & G_k(\mathscr{U}') \arrow[u, dashed, shift right]
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
\Sigma^{\gamma_n}p_2^*D_k \arrow[rr, "\Sigma^{\theta}p_2^*f_k"] \arrow[dd, "\sigma_{n,k}"'] \arrow[dr, shift left] & & \Sigma^{\gamma'_n}p_2^*D'_k \arrow[dd, "\sigma'_{n,k}"] \arrow[dr, shift left] & \\
& G_{n,k}(\mathscr{U}) \arrow[rr, "G_{n,k}(\theta)"] \arrow[ul, dashed, shift left] \arrow[dl, dashed, shift left] & & G_{n,k}(\mathscr{U}') \arrow[ul, dashed, shift left] \arrow[dl, dashed, shift left] \\
s^*D_{n+k} \arrow[rr, "s^*f_{n+k}"'] \arrow[ur, shift left] & & s^*D'_{n+k} \arrow[ur, shift left] &
\end{tikzcd}
$$

commute (the dotted arrows indicate section maps). The space of maps is topologized as a subspace of

$$\mathscr{U}\!n(\mathscr{U},\mathscr{U}')\times\prod_{k=0}^{\infty}\mathscr{T}(D_k,D'_k).$$

The resulting category of prespectra will be written $\mathscr{P}$; $\mathscr{S}$ *is its full subcategory of spectra*. The *augmentation* functor

$$\varepsilon:\mathscr{P}\to\mathscr{U}\!n$$

sends D to its underlying universe and a map f to its underlying map θ of universes.

We will also write

$$\varepsilon : \mathscr{S} \to \mathscr{U}n$$

for the restriction of ε to $\mathscr{S}$.

We now have assembled all the necessary ingredients for the proof of Theorem 0.4, with an added bonus: when E is a spectrum, each map $E_k \to G_k(\mathscr{U})$ is a bundle. First we give the

Proof of Theorem 0.4. Given a spectrum E with underlying universe $\mathscr{U}$, we form a spectrum in $\mathscr{S}\mathscr{U}$ by letting EV be the fiber of E_k over $V \in G_k(\mathscr{U})$. It is easy to see that if $f : E \to E'$ has $\varepsilon f = \mathrm{id}_{\mathscr{U}}$, then f induces a map in $\mathscr{S}\mathscr{U}$. Conversely, suppose E is in $\mathscr{S}\mathscr{U}$. We define spaces E_k in $\mathscr{T}^+/_{G_k(\mathscr{U})}$ by specifying the restrictions $E_k(W)$ over $G_k(W)$ for finite-dimensional subspaces W of $\mathscr{U}$, and passing to the colimit. Let $\nu_k(W) \to G_k(W)$ be the normal bundle, so its fiber over $V \in G_k(W)$ is $W - V$. We define

$$E_k(W) = \Omega^{\nu_k(W)} EW,$$

where EW is the trivial bundle over $G_k(W)$ with fiber EW. If $W \subset W'$, we identify $E_k(W')|_{G_k(W)}$ with $E_k(W)$ via the isomorphisms

$$\nu_k(W')|_{G_k(W)} \cong \nu_k(W) \oplus (W' - W)$$

and

$$\Omega^{W' - W} EW' \cong EW.$$

It follows from Definition 2.2 that this is a coherent system of isomorphisms, and we define

$$E_k = \operatorname*{colim}_{W} E_k(W).$$

The structure maps are inherited from those on E. □

As a consequence of this theorem, we will consider all the categories $\mathscr{S}\mathscr{U}$ as embedded in $\mathscr{S}$. Now for the promised bonus:

Corollary 2.15. *If E is a spectrum, then $E_k \to G_k(\mathscr{U})$ is a bundle for all k.*

Proof. First, since $\nu_k(W)$ is trivial over the open set $N_V(W)$ of k-planes in W transverse to V for any $V \in G_k(W)$, $E_k(W)$ is also trivial over $N_V(W)$. Clearly

$$N_V(\mathscr{U}) \cap G_k(W) = N_V(W),$$

so $N_V(\mathscr{U})$ is open in $G_k(\mathscr{U})$ since $G_k(\mathscr{U})$ is topologized as a colimit. If $W \subset W'$, it is easy to extend the trivialization of $\nu_k(W)$ to one of $\nu_k(W')$, and therefore of $E_k(W)$ to $E_k(W')$. Since $E_k = \mathrm{colim}_W E_k(W)$, we get a trivialization of E_k over

$N_V(\mathcal{U})$, and $E_k \to G_k(\mathcal{U})$ is therefore a bundle. □

It is vain to hope for anything of this simplicity to occur for prespectra: there are just too many possible topologies for the total spaces D_k. The best we can do by way of generalizing Theorem 0.4 to prespectra is the following; I am indebted to Gaunce Lewis for providing the proof.

Theorem 2.16. *Let $\mathcal{U}$ be a universe and let $\bar{\mathscr{P}}\mathcal{U} = \varepsilon^{-1}(\mathrm{id}_{\mathcal{U}})$ where here $\varepsilon : \mathscr{P} \to \mathcal{U}n$. Then the forgetful functor $\bar{\mathscr{P}}\mathcal{U} \to \mathscr{P}\mathcal{U}$ has a left adjoint.*

Proof. Let D be in $\mathscr{P}\mathcal{U}$; we form a prespectrum in $\bar{\mathscr{P}}\mathcal{U}$ as follows. Form the disjoint union $\coprod_V DV$ over all V of dimension k; this is an object of $\mathscr{T}^+/_{G_k^\delta}$, where G_k^δ is G_k with the discrete topology. Let $\eta : G_k^\delta \to G_k$ be the identity on underlying sets. We define

$$D_k = \eta_*\Big(\coprod_V DV\Big);$$

the structure maps and proof of adjointness can now be filled in by the reader. □

We now have forgetful functors

$$\mathscr{S}\mathcal{U} \to \bar{\mathscr{P}}\mathcal{U} \to \mathscr{P}\mathcal{U}$$

and left adjoints for the composite (by Lemma 2.3) and the right-hand arrow (Theorem 2.16). Since the left-hand arrow is the inclusion of a full subcategory, a left adjoint would be useful here as well. The existence of such an adjoint follows immediately from the next theorem, which is itself a corollary of Theorem A.1, which will be proved in the appendix.

Theorem 2.17. *The inclusion functor $\mathscr{S} \to \mathscr{P}$ has a continuous left adjoint covering the identity on $\mathcal{U}n$.* □

3. Parametrized spectra

In this section we introduce our parametrized version of $\mathscr{S}$ and prove the key theorem of the whole paper, Theorem 0.6. There are a number of attractive generalizations of Theorem 0.5 (which was derived from Theorem 0.6 in the introduction) to the full parametrized context, and since these follow easily from Theorem 0.6, we give details of them first. We will also generalize Lemma 1.5 to the context of parametrized spectra, and derive from it an associativity result for twisted half-smash products that will be crucial in the applications. The proof of Theorem 0.6 itself involves a curious generalization of the construction of suspension spectra even in the non-parametrized case: the forgetful functor $\mathscr{S}\mathcal{U} \to \mathscr{T}^+/_{G_k(\mathcal{U})}$ sending

E to E_k has a left adjoint. The case $k=0$ gives suspension spectra; if we choose a particular $V \in G_k(\mathcal{U})$ and pull back to $\mathcal{T}^+/_{\{V\}} \cong \mathcal{T}^+$, the composed left adjoints give the k-fold desuspension. The general case seems to involve Thom spectra, although we will not pursue this here.

Definition 3.1. A *parametrized universe* consists of a space $\mathcal{O}$, an object $\mathcal{U}$ of $\mathcal{T}/_{\mathcal{O}}$, and maps in $\mathcal{T}/_{\mathcal{O}}$

$$+ : \mathcal{U} \times_{\mathcal{O}} \mathcal{U} \to \mathcal{U}, \qquad \cdot : \mathbb{R} \times \mathcal{U} \to \mathcal{U},$$

$$- : \mathcal{U} \to \mathcal{U}, \qquad 0 : \mathcal{O} \to \mathcal{U},$$

$$\langle\ ,\ \rangle : \mathcal{U} \times_{\mathcal{O}} \mathcal{U} \to \mathbb{R} \times \mathcal{O},$$

such that each fiber of $\mathcal{U}$ becomes a real inner product space. Further, $\mathcal{U}$ must be topologized as the colimit of its parametrized sub-vector spaces of constant finite fiber dimension. (Consequently, $\mathcal{U}$ itself must have constant fiber dimension.) We will generally write $\mathcal{U}$, $\mathcal{U}'$ for parametrized universes with base spaces $\mathcal{O}$, $\mathcal{O}'$, often omitting the base spaces. A *map* of parametrized universes $f: \mathcal{U} \to \mathcal{U}'$ is a commutative square

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{f} & \mathcal{U}' \\ \downarrow & & \downarrow \\ \mathcal{O} & \xrightarrow{f} & \mathcal{O}' \end{array}$$

such that the maps on fibers are linear isometries. The resulting category of parametrized universes will be written $\bar{\mathcal{U}\!n}$; we give $\bar{\mathcal{U}\!n}(\mathcal{U}, \mathcal{U}')$ the natural function space topology. We will usually think of $\bar{\mathcal{U}\!n}(\mathcal{U}, \mathcal{U}')$ as an object of $\mathcal{T}/_{\mathcal{T}(\mathcal{O}, \mathcal{O}')}$ via the obvious restriction map.

Of course, $\bar{\mathcal{U}\!n}$ contains $\mathcal{U}\!n$ as the full subcategory of $\mathcal{U}$'s for which $\mathcal{O} = \{*\}$. In particular, $\mathbb{R}^k$ is in $\bar{\mathcal{U}\!n}$ for all k. We use this in the following definitions:

Definition 3.2. Let $\mathcal{U}$ be a parametrized universe over $\mathcal{O}$. We write $G_k(\mathcal{U})$ (or just G_k) for the space of all k-planes in all fibers of $\mathcal{U}$, which is also parametrized over $\mathcal{O}$. We topologize $G_k(\mathcal{U})$ as the orbit space $\bar{\mathcal{U}\!n}(\mathbb{R}^k, \mathcal{U})/O_k$, which is parametrized over $\mathcal{T}(\{*\}, \mathcal{O}) \cong \mathcal{O}$ since the O_k-action is fiber-preserving. We will also need the space $G_{n,k}(\mathcal{U})$ of orthogonal pairs of subspaces in each fiber of dimensions (n,k), topologized as $\bar{\mathcal{U}\!n}(\mathbb{R}^{n+k}, \mathcal{U})/O_n \times O_k$, and similarly for $G_{n,j,k}(\mathcal{U})$. These are all spaces in $\mathcal{T}/_{\mathcal{O}}$, and we have the projection maps $p_1 : G_{n,k} \to G_n$, $p_2 : G_{n,k} \to G_k$, and the sum map $s : G_{n,k} \to G_{n+k}$ all in $\mathcal{T}/_{\mathcal{O}}$. We write $\gamma_k(\mathcal{U})$ for the fiberwise canonical bundle over $G_k(\mathcal{U})$, topologized as a subspace of $G_k(\mathcal{U}) \times_{\mathcal{O}} \mathcal{U}$. As before, we will write γ_k also for $p_1^*\gamma_k$, $p_2^*\gamma_k$, and $s^*\gamma_k$, with the meaning safely left to context.

Also as before, we write S^{γ_k} for the fiberwise (over $G_k(\mathscr{U})$) one-point compactification of $\gamma_k(\mathscr{U})$, and given X in $\mathscr{T}^+/_{G_k(\mathscr{U})}$, we write $\Sigma^{\gamma_k}X$ for $X \wedge_{G_k} S^{\gamma_k}$ and $\Omega^{\gamma_k}X$ for $F_{G_k}(S^{\gamma_k}, X)$.

Definition 3.3. A *parametrized prespectrum D* consists of a parametrized universe $\mathscr{U}$, together with spaces D_k in $\mathscr{T}^+/_{G_k(\mathscr{U})}$ for each $k \geq 0$ and structure maps in $\mathscr{T}^+/_{G_{n,k}(\mathscr{U})}$

$$\sigma_{n,k} : \Sigma^{\gamma_n} p_2^* D_k \to s^* D_{n+k}$$

or equivalently

$$\bar{\sigma}_{n,k} : p_2^* D_k \to \Omega^{\gamma_n} s^* D_{n+k},$$

satisfying exactly the same compatibility criterion as in Definition 2.13. Also as in Definition 2.13, we require D_k to be topologized as the colimit of the restrictions $D_k(V)$ to $G_k(V)$ for parametrized subspaces V of $\mathscr{U}$ of constant finite fiber dimension. A *parametrized spectrum* is a parametrized prespectrum for which all the $\bar{\sigma}_{n,k}$ are isomorphisms. A *map* of parametrized prespectra $f: D \to D'$ consists of a map $\theta: \mathscr{U} \to \mathscr{U}'$ of parametrized universes and maps $f_k : D_k \to D_k'$ satisfying exactly the same relations as in Definition 2.14. We will write $\bar{\mathscr{P}}$ for the category of parametrized prespectra, $\bar{\mathscr{S}}$ for its full subcategory of parametrized spectra, and $\varepsilon : \bar{\mathscr{P}} \to \bar{\mathscr{U}n}$ or $\varepsilon : \bar{\mathscr{S}} \to \bar{\mathscr{U}n}$ for the augmentation functor.

Definition 3.4. Given a parametrized universe $\mathscr{U}$, we write $\bar{\mathscr{P}}\mathscr{U}$ and $\bar{\mathscr{S}}\mathscr{U}$ for the categories $\varepsilon^{-1}(\mathrm{id}_{\mathscr{U}})$ in $\bar{\mathscr{P}}$ and $\bar{\mathscr{S}}$ respectively.

Parametrized spectra have also been studied by Clapp [4, 5] and Fife [7], but their categories are solely of the form $\bar{\mathscr{P}}\mathscr{U}$ and $\bar{\mathscr{S}}\mathscr{U}$.

The next theorem gives the most obvious generalization of Theorem 0.5 to the parametrized context, but we defer its proof, since it is a corollary of a deeper generalization, which itself follows from Theorem 0.6.

Theorem 3.5. *Let E and E' be parametrized spectra over $\mathscr{U}$ and $\mathscr{U}'$ respectively, and let A be in $\mathscr{T}^+/_{\bar{\mathscr{U}n}(\mathscr{U},\mathscr{U}')}$. Then there are functorial constructions of parametrized spectra*

$$A \ltimes E \quad \text{in } \bar{\mathscr{S}}\mathscr{U}'$$

and

$$F[A, E') \quad \text{in } \bar{\mathscr{S}}\mathscr{U}$$

together with natural isomorphisms

$$\mathscr{T}/_{\bar{\mathscr{U}n}(\mathscr{U},\mathscr{U}')}(A, \bar{\mathscr{S}}(E, E')) \cong \bar{\mathscr{S}}\mathscr{U}'(A \ltimes E, E') \cong \bar{\mathscr{S}}\mathscr{U}(E, F[A, E')). \qquad \square$$

In order to state the deeper versions of Theorem 0.5 from which Theorem 3.5 will follow, we need parametrized versions of $\bar{\mathscr{U}n}(\mathscr{U}, \mathscr{U}')$ and $\bar{\mathscr{S}}(E, E')$.

Definition 3.6. Let $\mathscr{U}$ and $\mathscr{U}'$ be parametrized universes over the *same* parameter space $\mathscr{O}$, and let E and E' be parametrized spectra over $\mathscr{U}$ and $\mathscr{U}'$ respectively. We define a subspace

$$\mathscr{U}\!n_{\mathscr{O}}(\mathscr{U}, \mathscr{U}') \subset \mathrm{Hom}_{\mathscr{O}}(\mathscr{U}, \mathscr{U}')$$

by requiring the maps from $\mathscr{U}_a$ to $\mathscr{U}'_a$ in each fiber to be linear isometries. $\mathscr{U}\!n_{\mathscr{O}}(\mathscr{U}, \mathscr{U}')$ is therefore the largest subspace of $\mathrm{Hom}_{\mathscr{O}}(\mathscr{U}, \mathscr{U}')$ for which the evaluation map

$$\mathrm{Hom}_{\mathscr{O}}(\mathscr{U}, \mathscr{U}') \times_{\mathscr{O}} \mathscr{U} \to \mathscr{U}'$$

induces a map of parametrized universes

$$\begin{tikzcd}
\mathscr{U}\!n_{\mathscr{O}}(\mathscr{U}, \mathscr{U}') \times_{\mathscr{O}} \mathscr{U} \arrow[r] \arrow[d] & \mathscr{U}' \arrow[d] \\
\mathscr{U}\!n_{\mathscr{O}}(\mathscr{U}, \mathscr{U}') \arrow[r] & \mathscr{O}
\end{tikzcd}$$

We also define a space $\mathscr{S}_{\mathscr{O}}(E, E')$ over $\mathscr{O}$ whose fiber over $a \in \mathscr{O}$ is $\mathscr{S}(E_a, E'_a)$, topologized as a subspace of the infinite fiber product

$$\mathscr{U}\!n_{\mathscr{O}}(\mathscr{U}, \mathscr{U}') \times_{\mathscr{O}} \prod_{k=0}^{\infty}{}_{\mathscr{O}} \mathrm{Hom}_{\mathscr{O}}(E_k, E'_k).$$

We will generally consider $\mathscr{S}_{\mathscr{O}}(E, E')$ as an object of $\mathscr{T}/_{\mathscr{U}\!n_{\mathscr{O}}(\mathscr{U}, \mathscr{U}')}$.

All of our generalizations of Theorem 0.5 depend on the following one, whose proof we defer until we have shown how all the rest arise from it:

Theorem 3.7. *Let $\mathscr{U}$, $\mathscr{U}'$, E, E', and $\mathscr{O}$ be as in Definition 3.6, and let A be an object of $\mathscr{T}/_{\mathscr{U}\!n_{\mathscr{O}}(\mathscr{U}, \mathscr{U}')}$. Then there are functorial constructions of parametrized spectra*

$$A \ltimes_{\mathscr{O}} E \ \text{ in } \bar{\mathscr{S}}\mathscr{U}', \qquad F_{\mathscr{O}}[A, E') \ \text{ in } \bar{\mathscr{S}}\mathscr{U}$$

and natural isomorphisms

$$\mathscr{T}/_{\mathscr{U}\!n_{\mathscr{O}}(\mathscr{U}, \mathscr{U}')}(A, \mathscr{S}_{\mathscr{O}}(E, E')) \cong \bar{\mathscr{S}}\mathscr{U}'(A \ltimes_{\mathscr{O}} E, E') \cong \bar{\mathscr{S}}\mathscr{U}(E, F_{\mathscr{O}}[A, E')). \quad \square$$

In order to derive Theorem 3.5 from Theorem 3.7, we need to introduce an 'external' version of the constructions of Definition 3.6, which we derive from the internal versions found there by means of Theorem 0.6. First, we must introduce the pullback functors to which Theorem 0.6 refers.

Lemma 3.8. *A map $f: \mathscr{U} \to \mathscr{U}'$ of parametrized universes induces a pullback functor $f^*: \bar{\mathscr{P}}\mathscr{U}' \to \bar{\mathscr{P}}\mathscr{U}$ which restricts on spectra to a functor $f^*: \bar{\mathscr{S}}\mathscr{U}' \to \bar{\mathscr{S}}\mathscr{U}$.*

Proof. The map f induces maps of Grassmannians $G_k(f): G_k(\mathscr{U}) \to G_k(\mathscr{U}')$; given D' in $\bar{\mathscr{P}}\mathscr{U}'$, we define

$$(f^*D')_k = (G_k f)^* D'_k.$$

We also have

$$\gamma_n(\mathscr{U}) \cong (G_n f)^* \gamma_n(\mathscr{U}');$$

referring to the commutative diagram

$$\begin{tikzcd} G_k(\mathscr{U}) \arrow[d, "f"] & G_{n,k}(\mathscr{U}) \arrow[l, "p_2"'] \arrow[r, "s"] \arrow[d, "f"] & G_{n+k}(\mathscr{U}) \arrow[d, "f"] \\ G_k(\mathscr{U}') & G_{n,k}(\mathscr{U}') \arrow[l, "p_2'"'] \arrow[r, "s'"] & G_{n+k}(\mathscr{U}') \end{tikzcd}$$

we define structure maps $\bar{\sigma}_{n,k}$ as the composites

$$\begin{aligned} p_2^* f^* D'_k &\cong f^*(p_2')^* D'_k \xrightarrow{f^*\bar{\sigma}_{n,k}} f^*\Omega^{\gamma_n'}(s')^* D'_{n+k} \\ &\cong \Omega^{f^*\gamma_n'} f^*(s')^* D'_{n+k} \cong \Omega^{\gamma_n} s^* f^* D'_{n+k}. \end{aligned}$$

Clearly, if D' is a spectrum, this composite is an isomorphism. □

Definition 3.9. Let $\mathscr{U}$ and $\mathscr{U}'$ be parametrized universes over $\mathscr{O}$ and $\mathscr{O}'$ respectively, and let $p_1: \mathscr{O} \times \mathscr{O}' \to \mathscr{O}$ and $p_2: \mathscr{O} \times \mathscr{O}' \to \mathscr{O}'$ be the projections. Then $p_1^*\mathscr{U}$ and $p_2^*\mathscr{U}'$ are both parametrized universes over $\mathscr{O} \times \mathscr{O}'$, and we write $p_1: p_1^*\mathscr{U} \to \mathscr{U}$ and $p_2: p_2^*\mathscr{U}' \to \mathscr{U}'$ for the maps of parametrized universes. We define

$$\mathscr{H}_{\mathscr{U}n}(\mathscr{U}, \mathscr{U}') = \mathscr{U}n_{\mathscr{O} \times \mathscr{O}'}(p_1^*\mathscr{U}, p_2^*\mathscr{U}').$$

Corollary 3.10. *The space of sections of $(p_1)_*\mathscr{H}_{\mathscr{U}n}(\mathscr{U}, \mathscr{U}')$ is canonically isomorphic to $\bar{\mathscr{U}n}(\mathscr{U}, \mathscr{U}')$.*

Proof. The fiber of $\mathscr{H}_{\mathscr{U}n}(\mathscr{U}, \mathscr{U}')$ over $(a, a') \in \mathscr{O} \times \mathscr{O}'$ is $\mathscr{U}n(\mathscr{U}_a, \mathscr{U}'_{a'})$, and the fiber of $(p_1)_*\mathscr{H}_{\mathscr{U}n}(\mathscr{U}, \mathscr{U}')$ over $a \in \mathscr{O}$ is then $\bar{\mathscr{U}n}(\mathscr{U}_a, \mathscr{U}')$. A section therefore assigns to each $a \in \mathscr{O}$ a fiber $\mathscr{U}'_{a'}$ of $\mathscr{U}'$ and a linear isometry $\mathscr{U}_a \to \mathscr{U}'_{a'}$, i.e., a function $\mathscr{O} \to \mathscr{O}'$ and a covering map $\mathscr{U} \to \mathscr{U}'$ which is a linear isometry on each fiber. □

Definition 3.11. Let E and E' be parametrized spectra over the parametrized universes $\mathscr{U}$ and $\mathscr{U}'$ respectively, with $p_1: p_1^*\mathscr{U} \to \mathscr{U}$ and $p_2: p_2^*\mathscr{U}' \to \mathscr{U}'$ as in Definition 3.9. We define

$$\mathscr{H}_{\mathscr{S}}(E, E') = \mathscr{S}_{\mathscr{O} \times \mathscr{O}'}(p_1^*E, p_2^*E').$$

Corollary 3.12. *The space of sections of* $(p_1)_*\mathcal{H}_{\mathscr{S}}(E,E')$ *is canonically isomorphic to* $\bar{\mathscr{S}}(E,E')$.

Proof. This is exactly as in Corollary 3.10. □

The following external version of Theorem 3.7 will allow us to derive Theorem 3.5 from Corollaries 3.10 and 3.12.

Theorem 3.13. *Let E and E' be parametrized spectra over the parametrized universes $\mathscr{U}$ and $\mathscr{U}'$ respectively, and let A be an object of $\mathscr{T}/_{\mathcal{H}_{\mathscr{U}n}(\mathscr{U},\mathscr{U}')}$. Then there are functorial constructions*

$$A \boxtimes E \quad \textit{in } \bar{\mathscr{S}}\mathscr{U}', \qquad F_{\boxtimes}[A,E') \quad \textit{in } \mathscr{S}\mathscr{U}$$

and natural isomorphisms

$$\mathscr{T}/_{\mathcal{H}_{\mathscr{U}n}(\mathscr{U},\mathscr{U}')}(A,\mathcal{H}_{\mathscr{S}}(E,E')) \cong \bar{\mathscr{S}}\mathscr{U}'(A \boxtimes E, E') \cong \mathscr{S}\mathscr{U}(E, F_{\boxtimes}[A,E')).$$

Proof. Using Theorems 3.7 and 0.6, we define

$$A \boxtimes E = (p_2)_*(A \ltimes_{\mathscr{O}\times\mathscr{O}'} p_1^*E)$$

and

$$F_{\boxtimes}[A,E') = (p_1)_! F_{\mathscr{O}\times\mathscr{O}'}[A, p_2^*E').$$

The theorem follows by applying the adjointness relations of Theorems 3.7 and 0.6 to the definition

$$\mathscr{T}/_{\mathcal{H}_{\mathscr{U}n}(\mathscr{U},\mathscr{U}')}(A,\mathcal{H}_{\mathscr{S}}(E,E'))$$
$$\cong \mathscr{T}/_{\mathscr{U}n_{\mathscr{O}\times\mathscr{O}'}(p_1^*\mathscr{U},\, p_2^*\mathscr{U}')}(A, \mathscr{S}_{\mathscr{O}\times\mathscr{O}'}(p_1^*E, p_2^*E')). \qquad \square$$

Theorem 3.5 is really a corollary of Theorem 3.13.

Proof of Theorem 3.5. Since $\bar{\mathscr{U}n}(\mathscr{U},\mathscr{U}')$ is the space of sections of $(p_1)_*\mathcal{H}_{\mathscr{U}n}(\mathscr{U},\mathscr{U}')$, we can write

$$\bar{\mathscr{U}n}(\mathscr{U},\mathscr{U}') = f_!(p_1)_*\mathcal{H}_{\mathscr{U}n}(\mathscr{U},\mathscr{U}'),$$

where $f: \mathscr{O} \to \{*\}$. As in Lemma 1.7, (which figures in the proof) we have f^*A in $\mathscr{T}/_{(p_1)_*\mathcal{H}_{\mathscr{U}n}(\mathscr{U},\mathscr{U}')} = \mathscr{T}/_{\mathcal{H}_{\mathscr{U}n}(\mathscr{U},\mathscr{U}')}$; this amounts simply to the map

$$A \times \mathscr{O} \to \mathcal{H}_{\mathscr{U}n}(\mathscr{U},\mathscr{U}')$$

which takes $(\alpha, a) \in A \times \mathscr{O}$ to the restriction to the fiber of $\mathscr{U}$ over a of the map in $\bar{\mathscr{U}n}(\mathscr{U},\mathscr{U}')$ associated to α. We now define

$$A \ltimes E = f^*A \boxtimes E, \qquad F[A,E') = F_{\boxtimes}[f^*A, E'),$$

and it follows that

$$\mathcal{T}/_{\bar{\mathcal{U}n}(\mathcal{U},\,\mathcal{U}')}(A, \bar{\mathcal{P}}(E,E'))$$
$$\cong \mathcal{T}/_{f_!(p_1)_*\mathcal{H}_{\mathcal{U}n}(\mathcal{U},\,\mathcal{U}')}(A, f_!(p_1)_*\mathcal{H}_{\mathcal{P}}(E,E'))$$
$$\cong \mathcal{T}/_{(p_1)_*\mathcal{H}_{\mathcal{U}n}(\mathcal{U},\,\mathcal{U}')}(f^*A, (p_1)_*\mathcal{H}_{\mathcal{P}}(E,E'))$$

(this is where we use Lemma 1.7)

$$\cong \mathcal{T}/_{\mathcal{H}_{\mathcal{U}n}(\mathcal{U},\,\mathcal{U}')}(f^*A, \mathcal{H}_{\mathcal{P}}(E,E'))$$

(since everything maps into $\mathcal{H}_{\mathcal{U}n}(\mathcal{U}, \mathcal{U}')$)

$$\cong \mathcal{S}\mathcal{U}'(f^*A \boxtimes E, E')$$
$$\cong \mathcal{S}\mathcal{U}(E, F_{\boxtimes}[f^*A, E')). \qquad \square$$

We can also derive a more strictly parametrized version from Theorem 3.5.

Definition 3.14. Let $\zeta: \bar{\mathcal{U}n} \to \mathcal{T}$ be the restriction functor to the parameter space, and let $\mathcal{O}$ be any space. We define the category $\mathcal{U}n/_{\mathcal{O}}$ of *universes over* $\mathcal{O}$ by

$$\mathcal{U}n/_{\mathcal{O}} = \zeta^{-1}(\mathrm{id}_{\mathcal{O}}).$$

We define the category $\mathcal{S}/_{\mathcal{O}}$ of *spectra over* $\mathcal{O}$ by the pullback diagram of categories

$$\begin{array}{ccc} \mathcal{S}/_{\mathcal{O}} & \longrightarrow & \bar{\mathcal{S}} \\ \downarrow & & \downarrow \\ \mathcal{U}n/_{\mathcal{O}} & \longrightarrow & \bar{\mathcal{U}n} \end{array}$$

Theorem 3.15. *Let $\mathcal{U}$ and $\mathcal{U}'$ be parametrized universes over the* same *space $\mathcal{O}$, let E and E' be parametrized spectra over $\mathcal{U}$ and $\mathcal{U}'$ respectively, and let A be in $\mathcal{T}/_{(\mathcal{U}n/_{\mathcal{O}})(\mathcal{U},\,\mathcal{U}')}$. Then the isomorphisms of Theorem* 3.5 *restrict to*

$$\mathcal{T}/_{(\mathcal{U}n/_{\mathcal{O}})(\mathcal{U},\,\mathcal{U}')}(A, \mathcal{S}/_{\mathcal{O}}(E,E')) \cong \mathcal{S}\mathcal{U}'(A \ltimes E, E') \cong \mathcal{S}\mathcal{U}(E, F[A, E')).$$

Proof. If we consider A as an object of $\mathcal{T}/_{\bar{\mathcal{U}n}(\mathcal{U},\,\mathcal{U}')}$ via the inclusion $\mathcal{U}n/_{\mathcal{O}} \to \bar{\mathcal{U}n}$, we have

$$\mathcal{T}/_{(\mathcal{U}n/_{\mathcal{O}})(\mathcal{U},\,\mathcal{U}')}(A, \mathcal{S}/_{\mathcal{O}}(E,E')) \cong \mathcal{T}/_{\bar{\mathcal{U}n}(\mathcal{U},\,\mathcal{U}')}(A, \bar{\mathcal{S}}(E,E'))$$

by the universal property of pullbacks, so the result follows from Theorem 3.5. $\square$

In addition to Theorem 0.6, we need the following more general version of Theorem 0.7 in order to prove Theorem 3.7 (on which all of the above generalizations of Theorem 0.5 depend).

Theorem 3.16. *Let $\mathcal{U}$, $\mathcal{U}'$, E, E', and A be as in Definition* 3.6 *and Theorem* 3.7.

The structure map $A \to \mathscr{U}n_{\mathscr{O}}(\mathscr{U}, \mathscr{U}')$ induces a diagram of parametrized universes

$$\begin{tikzcd}
\mathscr{U} \arrow[d] & A \times_{\mathscr{O}} \mathscr{U} \arrow[l, "\pi"'] \arrow[r, "\chi"] \arrow[d] & \mathscr{U}' \arrow[d] \\
\mathscr{O} & A \arrow[l] \arrow[r] & \mathscr{O}
\end{tikzcd}$$

we claim that

$$\mathscr{T}/_{\mathscr{U}n_{\mathscr{O}}(\mathscr{U}, \mathscr{U}')}(A, \mathscr{S}_{\mathscr{O}}(E, E')) \cong \bar{\mathscr{S}}(A \times_{\mathscr{O}} \mathscr{U})(\pi^* E, \chi^* E').$$

Proof. $\mathscr{S}_{\mathscr{O}}(E, E')$ is a subspace of

$$\mathscr{U}n_{\mathscr{O}}(\mathscr{U}, \mathscr{U}') \times_{\mathscr{O}} \prod_{k=0}^{\infty}{}_{\mathscr{O}} \operatorname{Hom}_{\mathscr{O}}(E_k, E_k'),$$

and consequently $\mathscr{T}/_{\mathscr{U}n_{\mathscr{O}}(\mathscr{U}, \mathscr{U}')}(A, \mathscr{S}_{\mathscr{O}}(E, E'))$ is a subspace of

$$\mathscr{T}/_{\mathscr{O}}\left(A, \mathscr{U}n_{\mathscr{O}}(\mathscr{U}, \mathscr{U}') \times_{\mathscr{O}} \prod_{k=0}^{\infty}{}_{\mathscr{O}} \operatorname{Hom}_{\mathscr{O}}(E_k, E_k')\right)$$

$$\cong \bar{\mathscr{U}n}(A \times_{\mathscr{O}} \mathscr{U}, \mathscr{U}') \times \prod_{k=0}^{\infty} \mathscr{T}/_{\mathscr{O}}(A \times_{\mathscr{O}} E_k, E_k'),$$

which also contains $\bar{\mathscr{S}}(A \times_{\mathscr{O}} \mathscr{U})(\pi^* E, \chi^* E')$. We must show that the subspaces correspond. But maps

$$A \to \mathscr{U}n_{\mathscr{O}}(\mathscr{U}, \mathscr{U}'), \qquad A \to \operatorname{Hom}_{\mathscr{O}}(E_k, E_k')$$

specify a map $A \to \mathscr{S}_{\mathscr{O}}(E, E')$ precisely when the adjoint maps

$$\chi : A \times_{\mathscr{O}} \mathscr{U} \to \mathscr{U}', \qquad f_k : A \times_{\mathscr{O}} E_k \to E_k'$$

specify a map of parametrized spectra. This means precisely that the three diagrams

$$\begin{tikzcd}
A \times_{\mathscr{O}} E_k \arrow[r, "f_k"] \arrow[d, "1 \times_{\mathscr{O}} \xi"] & E_k' \arrow[d, "\xi'"] \\
A \times_{\mathscr{O}} G_k \arrow[r, "G_k(\chi)"] & G_k'
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times_{\mathscr{O}} E_k \arrow[r, "f_k"] & E_k' \\
A \times_{\mathscr{O}} G_k \arrow[r, "G_k(\chi)"] \arrow[u, "1 \times_{\mathscr{O}} s"] & G_k' \arrow[u, "s'"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \times_{\mathscr{O}} \Sigma^{\gamma_n} p_2^* E_k = \Sigma^{\gamma_n} p_2^*(A \times_{\mathscr{O}} E_k) \arrow[r, "\Sigma^{\chi} p_2^* f_k"] \arrow[d, "1 \times_{\mathscr{O}} \sigma_{n,k}"] & \Sigma^{\gamma_n'} p_2^* E_k' \arrow[d, "\sigma_{n,k}'"] \\
A \times_{\mathscr{O}} s^* E_{n+k} = s^*(A \times_{\mathscr{O}} E_{n+k}) \arrow[r, "s^* f_{n+k}"] & s^* E_{n+k}'
\end{tikzcd}$$

commute, but these uniquely specify a map $\pi^*E \to \chi^*E'$ in $\bar{\mathscr{P}}(A \times_{\mathscr{O}} \mathscr{U})$. □

Proof of Theorem 3.7. Using the diagram in the statement of Theorem 3.16, we define

$$A \ltimes_{\mathscr{O}} E = \chi_* \pi^* E, \qquad F_{\mathscr{O}}[A, E') = \pi_! \chi^* E'.$$

Theorems 3.16 and 0.6 now easily give us the conclusion. □

The next lemma generalizes Lemma 1.5 to parametrized spectra. The proof depends on the proof of Theorem 0.6, and is therefore deferred. The succeeding corollary is one of our crucial tools for computing with twisted half-smash products.

Lemma 3.17. *If*

$$\begin{tikzcd} \mathscr{U} \arrow[r, "F"] \arrow[d, "G"'] & \mathscr{U}' \arrow[d, "g"] \\ \mathscr{V} \arrow[r, "f"'] & \mathscr{V}' \end{tikzcd}$$

is a pullback diagram in $\bar{\mathscr{U}n}$ *with G and g fiberwise surjective, then there are natural isomorphisms of functors*

$$F_* G^* \cong g^* f_* : \bar{\mathscr{P}}\mathscr{V} \to \bar{\mathscr{P}}\mathscr{U}', \qquad G_! F^* \cong f^* g_! : \bar{\mathscr{P}}\mathscr{U}' \to \bar{\mathscr{P}}\mathscr{V}. \qquad \square$$

Corollary 3.18. *Let* $\mathscr{U}$, $\mathscr{U}'$, *and* $\mathscr{U}''$ *all be parametrized universes over* $\mathscr{O}$, *let A be in* $\mathscr{T}/_{\mathscr{U}n_{\mathscr{O}}(\mathscr{U}, \mathscr{U}')}$, *B in* $\mathscr{T}/_{\mathscr{U}n_{\mathscr{O}}(\mathscr{U}', \mathscr{U}'')}$, *and E in* $\bar{\mathscr{P}}\mathscr{U}$. *We consider* $B \times_{\mathscr{O}} A$ *to be in* $\mathscr{T}/_{\mathscr{U}n_{\mathscr{O}}(\mathscr{U}, \mathscr{U}'')}$ *via the composition pairing*

$$\mathscr{U}n_{\mathscr{O}}(\mathscr{U}', \mathscr{U}'') \times_{\mathscr{O}} \mathscr{U}n_{\mathscr{O}}(\mathscr{U}, \mathscr{U}') \to \mathscr{U}n_{\mathscr{O}}(\mathscr{U}, \mathscr{U}'').$$

Then there is a natural isomorphism

$$B \ltimes_{\mathscr{O}} (A \ltimes_{\mathscr{O}} E) \cong (B \times_{\mathscr{O}} A) \ltimes_{\mathscr{O}} E.$$

Proof. We have the diagram of parametrized universes

$$\begin{tikzcd} & & \mathscr{U}'' \\ & \mathscr{B} \times_{\mathscr{O}} A \times_{\mathscr{O}} \mathscr{U} \arrow[ur, "\chi''"] \arrow[r, "1 \times_{\mathscr{O}} \chi"] \arrow[dl, "\pi''"'] \arrow[d, "\pi'"] & B \times_{\mathscr{O}} \mathscr{U}' \arrow[u, "\chi'"'] \arrow[d, "\pi'"] \\ \mathscr{U} & A \times_{\mathscr{O}} \mathscr{U} \arrow[l, "\pi"'] \arrow[r, "\chi"] & \mathscr{U}' \end{tikzcd}$$

in which the square is a pullback. Consequently,

$$\begin{aligned} B \ltimes_{\mathcal{O}} (A \ltimes_{\mathcal{O}} E) &= \chi'_*(\pi')^*\chi_*\pi^*E \\ &\cong \chi'_*(1 \times_{\mathcal{O}} \chi)_*(\pi')^*\pi^*E \\ &\cong \chi''_*(\pi'')^*E = (B \times_{\mathcal{O}} A) \ltimes_{\mathcal{O}} E. \qquad \square \end{aligned}$$

We turn now to the proof of Theorem 0.6. There are actually two versions: one for prespectra, the other for spectra. We derive the spectrum level version from that for prespectra by means of the following theorem, whose proof is given in the appendix:

Theorem A.1. *The forgetful functor* $l : \bar{\mathcal{S}} \to \bar{\mathcal{P}}$ *has a continuous left adjoint* $L : \bar{\mathcal{P}} \to \bar{\mathcal{S}}$ *covering the identity on* $\bar{\mathcal{U}}n$. $\square$

Corollary 3.19. *Theorem* 0.6 *follows from the analogous result for prespectra.*

Proof. It will follow from the construction that $f_!$ restricts to a functor on spectra (it is this point that requires the fiberwise surjectivity of f), and the left adjoint for spectra is provided by Lf_*l. $\square$

The following lemma, which is of independent interest, was already referred to in the introduction to this section. We will need it in order to construct f_*.

Lemma 3.20. *The forgetful functor*

$$\Omega_j^\infty : \bar{\mathcal{S}}\mathcal{U} \to \mathcal{T}^+/_{G_j(\mathcal{U})}$$

has a left adjoint

$$\Sigma_j^\infty : \mathcal{T}^+/_{G_j(\mathcal{U})} \to \bar{\mathcal{S}}\mathcal{U}.$$

Proof. We will actually construct a left adjoint to the forgetful functor

$$\bar{\mathcal{P}}\mathcal{U} \to \mathcal{T}^+/_{G_j(\mathcal{U})};$$

the lemma then follows by composing with the restriction of L from Theorem A.1 to $\bar{\mathcal{P}}\mathcal{U} \to \bar{\mathcal{S}}\mathcal{U}$. (In fact, it is this prespectrum version that we employ in the proof of Theorem 0.6.) Given X in $\mathcal{T}^+/_{G_j}$, we construct $\Sigma_j^\infty X$ by considering the diagram

$$G_j \xleftarrow{\bar{p}_2} G_{k-j,j} \xrightarrow{\bar{s}} G_k$$

and defining

$$(\Sigma_j^\infty X)_k = \bar{s}_*\Sigma^{\gamma_{k-j}}\bar{p}_2^*X.$$

(When $k<j$, $G_{k-j,j}=\emptyset$, and we simply have $(\Sigma_j^\infty X)_k = G_k$, the null object in $\mathcal{T}^+/_{G_k}$.) We construct structure maps

$$\sigma_{n,k} : s_*\Sigma^{\gamma_n}p_2^*(\Sigma_j^\infty X)_k \to (\Sigma_j^\infty X)_{n+k}$$

by consulting the diagram

$$
\begin{tikzcd}
 & G_{n+k-j,j} \arrow[dl, "\hat{p}"'] \arrow[dr, "\hat{s}"] & \\
G_j & & G_{n+k} \\
 & G_{n,k-j,j} \arrow[uu, "s_1"] \arrow[dl, "p"'] \arrow[dr, "s_2"] & \\
G_{k-j,j} \arrow[uu, "\bar{p}_2"] \arrow[dr, "\bar{s}"'] & & G_{n,k} \arrow[uu, "s"'] \arrow[dl, "p_2"] \\
 & G_k &
\end{tikzcd}
$$

in which the bottom square is a pullback, and defining $\sigma_{n,k}$ as the composite

$$
\begin{aligned}
s_*\Sigma^{\gamma_n}p_2^*(\Sigma_j^\infty X)_k &= s_*\Sigma^{\gamma_n}p_2^*\bar{s}_*\Sigma^{\gamma_{k-j}}\bar{p}_2^*X \\
&\cong s_*\Sigma^{\gamma_n}(s_2)_*p^*\Sigma^{\gamma_{k-j}}\bar{p}_2^*X \cong s_*(s_2)_*\Sigma^{\gamma_{n+k-j}}p^*\bar{p}_2^*X \\
&\cong \hat{s}_*(s_1)_*\Sigma^{\gamma_{n+k-j}}s_1^*\hat{p}^*X \cong \hat{s}_*(s_1)_*s_1^*\Sigma^{\gamma_{n+k-j}}\hat{p}^*X \\
&\to \hat{s}_*\Sigma^{\gamma_{n+k-j}}\hat{p}^*X = (\Sigma_j^\infty X)_{n+k},
\end{aligned}
$$

with the arrow given by the counit of the $(s_1)_* - s_1^*$ adjunction. The reader should now draw a large diagram and verify that this satisfies the requirement for a structure map.

In order to show adjointness, we display the unit and counit. The unit is the identity map

$$X = (\Sigma_j^\infty X)_j,$$

and the counit $\Sigma_j^\infty \Omega_j^\infty D = \Sigma_j^\infty D_j \to D$ is given by the structure maps for D:

$$(\Sigma_j^\infty D_j)_k = \bar{s}_*\Sigma^{\gamma_{k-j}}\bar{p}_2^*D_j \xrightarrow{\sigma_{k-j,j}} D_k. \qquad \square$$

Proof of Theorem 0.6. We prove the prespectrum version; by Corollary 3.19, this is sufficient. We begin by constructing the right adjoint $f_!: \bar{\mathscr{P}}\mathscr{U} \to \bar{\mathscr{P}}\mathscr{U}'$, given $f: \mathscr{U} \to \mathscr{U}'$. We have induced maps

$$G_k f: G_k(\mathscr{U}) \to G_k(\mathscr{U}'),$$

and we define

$$(f_! D)_k = (G_k f)_!(D_k)$$

for any prespectrum D in $\bar{\mathscr{P}}\mathscr{U}$, generally abbreviating

$$(f_! D)_k = f_!(D_k).$$

To define the structure maps, we consult the diagram

$$(*)\qquad \begin{tikzcd} G_k(\mathscr{U}) \arrow[d, "f"] & G_{n,k}(\mathscr{U}) \arrow[l, "p_2"'] \arrow[r, "s"] \arrow[d, "f"] & G_{n+k}(\mathscr{U}) \arrow[d, "f"] \\ G_k(\mathscr{U}') & G_{n,k}(\mathscr{U}') \arrow[l, "p_2'"'] \arrow[r, "s'"] & G_{n+k}(\mathscr{U}') \end{tikzcd}$$

in which both squares are pullbacks (the left one because of the fiberwise surjectivity of f), and define $\bar{\sigma}_{n,k}$ as the composite

$$\begin{aligned} (p_2')^* f_! D_k &\cong f_! p_2^* D_k \\ &\xrightarrow{f_!\bar{\sigma}_{n,k}} f_! \Omega^{\gamma_n} s^* D_{n+k} \cong f_! \Omega^{f^*\gamma_n'} s^* D_{n+k} \\ &\cong \Omega^{\gamma_n'} f_! s^* D_{n+k} \quad \text{(by Lemma 1.6(b))} \\ &\cong \Omega^{\gamma_n'} (s')^* f_! D_{n+k}. \end{aligned}$$

If D is a spectrum, this is clearly an isomorphism. We leave the check of the coherence condition to the reader. The unit and counit of the adjunction are given by the unit and counit of the $(G_k f)^* - (G_k f)_!$ adjunction for all k, which are easily seen to give maps of prespectra.

To construct the left adjoint, we also consider diagram $(*)$ above, being careful to note that the left-hand square need no longer be a pullback, since we no longer assume f to be fiberwise surjective. We construct $f_* D$ as the coequalizer of two maps in $\bar{\mathscr{P}}\mathscr{U}'$

$$\bigvee_{\substack{n\geq 0\\ k\geq 0}} \Sigma^\infty_{n+k} f_* s_* \Sigma^{\gamma_n} p_2^* D_k \to \bigvee_{k\geq 0} \Sigma^\infty_k f_* D_k.$$

The first map is induced by the structure maps $\sigma_{n,k}$ of D:

$$\Sigma^\infty_{n+k} f_* s_* \Sigma^{\gamma_n} p_2^* D_k \xrightarrow{\Sigma^\infty_{n+k} f_* \sigma_{n,k}} \Sigma^\infty_{n+k} f_* D_{n+k}.$$

The second map is induced by the natural map

$$\varphi : f_* p_2^* \to (p_2')^* f_*$$

of Lemma 1.9, and is given by the wedge of the compositions

$$\begin{aligned} \Sigma^\infty_{n+k} f_* s_* \Sigma^{\gamma_n} p_2^* D_k &\cong \Sigma^\infty_{n+k} s_*' \Sigma^{\gamma_n'} f_* p_2^* D_k \quad (\text{using } \gamma_n \cong f^*\gamma_n') \\ &\xrightarrow{\varphi_*} \Sigma^\infty_{n+k} s_*' \Sigma^{\gamma_n'} (p_2')^* f_* D_k \\ &= \Sigma^\infty_{n+k} \Omega^\infty_{n+k} \Sigma^\infty_k f_* D_k \\ &\to \Sigma^\infty_k f_* D_k, \end{aligned}$$

where the last arrow is given by the counit of the $\Sigma^\infty_{n+k} - \Omega^\infty_{n+k}$ adjunction.

To show that this gives the left adjoint, we display the counit and unit of the adjunction. The counit $f_*f^*D' \to D'$ is induced by the map on each wedge summand in $\bigvee_{k\geq 0}\Sigma_k^\infty f_*(f^*D')_k$, where we use ε generically for the counit of an adjunction:

$$\Sigma_k^\infty f_*(f^*D')_k = \Sigma_k^\infty f_*f^*(D_k') \xrightarrow{\Sigma_k^\infty \varepsilon} \Sigma_k^\infty D_k' = \Sigma_k^\infty \Omega_k^\infty D' \xrightarrow{\varepsilon} D'.$$

We leave it to the reader to verify that these maps coequalize the pair that defines $f_*(f^*D')$.

We give the unit $D \to f^*f_*D$ by its map on component spaces:

$$D_k \xrightarrow{\eta} f^*f_*D_k = f^*(\Sigma_k^\infty f_*D_k)_k \to f^*(f_*D)_k,$$

where η is the unit of the f^*-f_* adjunction on spaces, and the unlabelled arrow maps to the coequalizer. To show that this specifies a map of prespectra, we refer to the diagram in Fig. 1. The outside traces the required diagram, so we must verify the commutativity of each of the subdiagrams. The map φ_* in the upper left square is the second map of the pair coequalized by f_*D; the square commutes as a consequence of the commutativity of

$$\begin{tikzcd}
p_2^*D_k \arrow[r, "p_2^*\eta"] \arrow[d, "\eta"'] & p_2^*f^*f_*D_k \arrow[d, "\cong"] \\
f^*f_*p_2^*D_k \arrow[r, "f^*\varphi"'] & f^*(p_2')^*f_*D_k
\end{tikzcd}$$

which is adjoint to the diagram stating that φ coincides with the lower composite

$$\begin{tikzcd}
s_*\Sigma^{\gamma_n}p_2^*f^*f_*D_k \arrow[d, "\cong"] & s_*\Sigma^{\gamma_n}p_2^*D_k \arrow[l, "s_*\Sigma^{\gamma_n}p_2^*\eta"'] \arrow[r, "\sigma_{n,k}"] \arrow[d, "\eta"] & D_{n+k} \arrow[d, "\eta"] \\
f^*s_*'\Sigma^{\gamma_n'}(p_2')^*f_*D_k \arrow[d, "="] & f^*f_*s_*\Sigma^{\gamma_n}p_2^*D_k \arrow[l, "\varphi_*"'] \arrow[r, "f^*f_*\sigma_{n,k}"] & f^*f_*D_{n+k} \arrow[dd, "="] \\
f^*s_*'\Sigma^{\gamma_n'}(p_2')^*(\Sigma_k^\infty f_*D_k)_k \arrow[r, "\sigma_{n,k}"] \arrow[d] & f^*(\Sigma_k^\infty f_*D_k)_{n+k} \arrow[d] & \\
f^*s_*'\Sigma^{\gamma_n'}(p_2')^*(f_*D)_k \arrow[r, "\sigma_{n,k}"] & f^*(f_*D)_{n+k} & f^*(\Sigma_{n+k}^\infty f_*D_{n+k})_{n+k} \arrow[l]
\end{tikzcd}$$

Fig. 1.

in Lemma 1.9. (In fact, the displayed diagram is equivalent to one appearing in the proof of Lemma 1.9.) The upper right square commutes by naturality of η, and the lower left square commutes since the coequalizer map

$$\Sigma_k^\infty(f_*D_k)\to f_*D$$

is a map of prespectra. The remainder of the diagram follows from taking $n+k$th spaces in the coequalizer diagram

$$\begin{tikzcd}
\Sigma_{n+k}^\infty f_* s_* \Sigma^{\gamma_n} p_2^* D_k \arrow[r] \arrow[d] & \Sigma_{n+k}^\infty f_* D_{n+k} \arrow[d] \\
\Sigma_k^\infty f_* D_k \arrow[r] & f_*D
\end{tikzcd}$$

and applying f^*. The necessary relations between unit and counit now follow from their space level analogues, using the fact that a map from f_*D is completely determined by the induced maps from $\Sigma_k^\infty f_*D_k$ for each k. □

We can now give the promised

Proof of Lemma 3.17. The second isomorphism is true for both spectra and prespectra by restricting to component spaces and using Lemma 1.5, and the first now follows from the Yoneda lemma. □

4. The stable category

In this section we show that $\mathscr{S}$ is an appropriate category of spectra from which to build Boardman's stable category (Theorem 0.1); a more precise statement will be given below. This in turn follows from the much deeper Theorem 4.1. Although both statements refer to CW-spectra, we can derive Theorem 0.1 from Theorem 4.1 without knowing anything about them, so we do this first. The remainder of the section develops some properties of CW-spectra (we adopt May's definition) and gives the proof of Theorem 4.1.

Theorem 4.1. *Let E and E' be spectra indexed on universes $\mathscr{U}$ and $\mathscr{U}'$ respectively, with E a* CW-*spectrum. Then the restriction of the augmentation map*

$$\varepsilon:\mathscr{S}(E,E')\to \mathscr{U}n(\mathscr{U},\mathscr{U}')$$

over any compact subspace of $\mathscr{U}n(\mathscr{U},\mathscr{U}')$ is a fibration.

Given this theorem, it is quite easy to prove the precise statement of Theorem 0.1:

Theorem 0.1. *Let* CW-$\mathscr{S}_\infty$ *be the full subcategory of $\mathscr{S}$ consisting of* CW-*spectra* *indexed on countably infinite dimensional universes, and let* hCW-$\mathscr{S}_\infty$ *be its homotopy category, so*

$$\text{hCW-}\mathscr{S}_\infty(E,E') = \pi_0\mathscr{S}(E,E').$$

Then hCW-$\mathscr{S}_\infty$ *is equivalent to Boardman's stable category.*

Proof. Choose a particular countably infinite dimensional universe $\mathscr{U}$. The homotopy category of CW-spectra in $\mathscr{S}\mathscr{U}$, hCW-$\mathscr{S}\mathscr{U}$, is known to be equivalent to Boardman's stable category; we show

$$\text{hCW-}\mathscr{S}_\infty \simeq \text{hCW-}\mathscr{S}\mathscr{U}.$$

First, all countably infinite dimensional universes $\mathscr{U}'$ are isomorphic to $\mathscr{U}$, so we may choose isomorphisms $f\colon \mathscr{U}\to\mathscr{U}'$ for each. Then any E' in $\mathscr{S}\mathscr{U}'$ is isomorphic in $\mathscr{S}$ to f^*E' via the pullback maps covering f, and it will be evident that f^*E' is a CW-spectrum if and only if E' is. Therefore the restriction of CW-$\mathscr{S}_\infty$ to spectra indexed on $\mathscr{U}$ is a skeleton (in the category-theoretic sense) of all of CW-$\mathscr{S}_\infty$, and therefore equivalent to it. Next, given CW-spectra E and E' in $\mathscr{S}\mathscr{U}$, Theorem 4.1 provides us with a quasifibration

$$\mathscr{S}\mathscr{U}(E,E') \to \mathscr{S}(E,E') \to \mathscr{U}\!n(\mathscr{U},\mathscr{U}).$$

But $\mathscr{U}\!n(\mathscr{U},\mathscr{U})$ is contractible [13, I.1.3], so we have a weak equivalence

$$\mathscr{S}\mathscr{U}(E,E') \simeq \mathscr{S}(E,E').$$

In particular,

$$\pi_0\mathscr{S}\mathscr{U}(E,E') \cong \pi_0\mathscr{S}(E,E'),$$

so hCW-$\mathscr{S}\mathscr{U}$ is isomorphic to a skeleton of hCW-$\mathscr{S}_\infty$, and therefore equivalent to it. $\square$

The compactness hypothesis in Theorem 4.1 will enter its proof by means of the following lemma:

Lemma 4.2. *Let $\mathscr{U}$ be a universe, K a compact subspace of $G_k(\mathscr{U})$. Then there is a finite-dimensional subspace W of $\mathscr{U}$ such that $K\subset G_k(W)$.*

Proof. First, any compact subset C of $\mathscr{U}$ itself is contained in a finite-dimensional subspace of $\mathscr{U}$, for suppose otherwise. Then C contains an infinite linearly independent set, and there is a linear map $f\colon \mathscr{U}\to\mathscr{U}$ taking this linearly independent set into an orthonormal set, which has no accumulation point. All linear maps $f\colon\mathscr{U}\to\mathscr{U}$ are continuous, since their restrictions to finite-dimensional subspaces are, and therefore $f(C)$ is a compact subset of $\mathscr{U}$ containing an infinite set with no accumulation point, a contradiction.

Next, let $S_k(\mathscr{U})\to G_k(\mathscr{U})$ be the bundle of unit spheres in the canonical bundle.

We have the canonical map

$$S_k(\mathscr{U}) \to \mathscr{U}$$

sending a point to itself: a set K is in $G_k(W)$ if and only if the image of the restriction to K of $S_k(\mathscr{U})$ lies in W. But the restriction to K is a bundle with compact base and fiber, and therefore compact total space. The image in $\mathscr{U}$ is therefore compact, consequently contained in a finite-dimensional subspace W, so $K \subset G_k(W)$. $\square$

The major portion of the proof of Theorem 4.1 uses the definition of CW-spectrum to reduce the theorem to the case of finite-dimensional universes. If we consider a spectrum E (*not* a prespectrum) indexed on a finite-dimensional universe $\mathscr{U}$ of dimension n, it is clear that E is completely determined by $E_n = E(\mathscr{U})$, since we must have

$$E_k \cong \Omega^{\nu_k(\mathscr{U})} E_n.$$

The forgetful functor $\Omega_n^\infty : \mathscr{S}\mathscr{U} \to \mathscr{T}^+$ is therefore an equivalence of categories, with inverse equivalence Σ_n^∞. (The ∞'s here are actually spurious, due to the finite dimension.) We define sphere spectra in $\mathscr{S}$ as follows:

Definition 4.3. Let $n \geq 0$. The n-sphere S^n in $\mathscr{S}$ is the standard n-sphere in $\mathscr{S}\{0\} \cong \mathscr{T}^+$. The sphere S^{-n} is the spectrum

$$\Sigma_n^\infty S^0 \quad \text{in} \quad \mathscr{S}\mathbb{R}^n.$$

Consequently, all spheres are indexed on finite-dimensional universes. In order to obtain unambiguous spheres in $\mathscr{S}\mathscr{U}$ for all $\mathscr{U}$, we make use of the following trivial corollary of Theorem 0.6:

Corollary 4.4. *Let $f : \mathscr{U} \to \mathscr{U}'$ be a linear isometry between universes. Then the restriction functor $f^* : \mathscr{S}\mathscr{U}' \to \mathscr{S}\mathscr{U}$ has a left adjoint $f_* : \mathscr{S}\mathscr{U} \to \mathscr{S}\mathscr{U}'$.* $\square$

Lemma 4.5. *Let X be in $\mathscr{S}\mathbb{R}^n$, and let $f : \mathbb{R}^n \to \mathscr{U}$ and $g : \mathbb{R}^n \to \mathscr{U}$ be linear isometries to the universe $\mathscr{U}$. Then*

$$f_* X \cong g_* X$$

in $\mathscr{S}\mathscr{U}$.

Proof. The images of f and g are both contained in some finite-dimensional subspace W of $\mathscr{U}$, so we may factor f and g as $i \circ \bar{f}$ and $i \circ \bar{g}$, where $i : W \to \mathscr{U}$ is the inclusion. It suffices to show

$$\bar{f}_* X \cong \bar{g}_* X$$

in $\mathscr{S}W$. But letting $\dim W = k$, $V = W - \mathrm{im}(f)$, $V' = W - \mathrm{im}(g)$, we see easily that

$$\bar{f}_*X = \Sigma_k^\infty \Sigma^V X_n, \qquad \bar{g}_*X = \Sigma_k^\infty \Sigma^{V'} X_n.$$

Since $V \cong V'$, it follows that $\Sigma^V X_n \cong \Sigma^{V'} X_n$, and we are done. □

This gives substance to the following definition:

Definition 4.6. Let $\mathscr{U}$ be an infinite-dimensional universe. A *sphere* in $\mathscr{S}\mathscr{U}$ is a spectrum of the form f_*S^n for any linear isometry $f: \mathbb{R}^{-n} \to \mathscr{U}$ if $n<0$, or $f: \{0\} \to \mathscr{U}$ if $n \geq 0$. We write all of these as S^n, since they are all isomorphic in $\mathscr{S}\mathscr{U}$.

This definition is equivalent to that given in [12, I.4] in the more general equivariant context, and shows that all spheres are induced from spectra defined over finite-dimensional universes. We adopt May's definition [12, I.5] of a CW-spectrum, which is legitimate by Theorem 0.4. Our goal in proving Theorem 4.1 is to reduce to the case where E is of the form f_*X for X in $\mathscr{S}\mathbb{R}^n$, $n < \infty$. We reduce first to the case where E is a finite CW-spectrum, using the fact that a CW-spectrum is the colimit (in $\mathscr{S}\mathscr{U}$) of its finite CW-subspectra, together with the following lemma:

Lemma 4.7. *Let Λ be a connected small category, $h: \Lambda \to \mathscr{S}\mathscr{U}$ a functor and $E = \operatorname{colim} h = \operatorname{colim}_\lambda h(\lambda)$. Then*

$$\mathscr{S}(E, E') \cong \lim_\lambda \mathscr{S}(h(\lambda), E')$$

as spaces (i.e., the limit is taken in $\mathscr{T}$) for any spectrum E'.

Proof. We use Theorem 0.5 and the Yoneda lemma. Let A be an arbitrary object of $\mathscr{T}/_{\mathscr{U}n(\mathscr{U},\mathscr{U}')}$, where E' is indexed over $\mathscr{U}'$. We have

$$\begin{aligned}\mathscr{T}/_{\mathscr{U}n(\mathscr{U},\mathscr{U}')}(A, \mathscr{S}(E,E')) &\cong \mathscr{S}\mathscr{U}(E, F[A,E']) \cong \lim_\lambda \mathscr{S}\mathscr{U}(h(\lambda), F[A,E']) \\ &\cong \lim_\lambda \mathscr{T}/_{\mathscr{U}n(\mathscr{U},\mathscr{U}')}(A, \mathscr{S}(h(\lambda),E')) \cong \mathscr{T}/_{\mathscr{U}n(\mathscr{U},\mathscr{U}')}\Big(A, \lim_\lambda \mathscr{S}(h(\lambda),E')\Big).\end{aligned}$$

This last limit is taken in $\mathscr{T}/_{\mathscr{U}n(\mathscr{U},\mathscr{U}')}$, *not* in $\mathscr{T}$, but since Λ is connected, the underlying spaces are the same, and the conclusion follows. □

Corollary 4.8. *If a* CW-*spectrum E has finite* CW-*subspectra E_λ, then*

$$\mathscr{S}(E, E') \cong \lim_\lambda \mathscr{S}(E_\lambda, E'),$$

where the limit is taken over inclusions of subspectra. □

We come now to the key step in the proof of Theorem 4.1, which is to show that each map in the limit system in Corollary 4.8 is a fibration over compact subspaces of $\mathscr{U}n(\mathscr{U}, \mathscr{U}')$. This follows from inductive use of the next lemma, applied to the cofibration $S^n \to CS^n \equiv D^{n+1}$.

Lemma 4.9. *If $A \to X$ is a cofibration in $\mathscr{S}\mathbb{R}^n$, $f\colon \mathbb{R}^n \to \mathscr{U}$ is a linear isometry, and we have a pushout diagram in $\mathscr{S}\mathscr{U}$*

$$\begin{tikzcd} f_*A \arrow[r] \arrow[d] & f_*X \arrow[d] \\ E_1 \arrow[r] & E_2 \end{tikzcd}$$

then $\mathscr{S}(E_2, E') \to \mathscr{S}(E_1, E')$ is a fibration when restricted over compact subspaces of $\mathscr{U}\!n(\mathscr{U}, \mathscr{U}')$.

Proof. By a trivial application of Lemma 4.7, the diagram

$$\begin{tikzcd} \mathscr{S}(E_2, E') \arrow[r] \arrow[d] & \mathscr{S}(E_1, E') \arrow[d] \\ \mathscr{S}(f_*X, E') \arrow[r] & \mathscr{S}(f_*A, E') \end{tikzcd}$$

is a pullback, so it suffices to show that the bottom arrow is a fibration over compact subspaces of $\mathscr{U}\!n(\mathscr{U}, \mathscr{U}')$. Let

$$j\colon \mathscr{U}\!n(\mathscr{U}, \mathscr{U}') \to \mathscr{U}\!n(\mathbb{R}^n, \mathscr{U}')$$

be induced by composition with f; we claim

$$\mathscr{S}(f_*X, E') \cong j^*\mathscr{S}(X, E'),$$

and similarly with X replaced by A. To see this, let Q be any object of $\mathscr{T}/_{\mathscr{U}\!n(\mathscr{U}, \mathscr{U}')}$. We have the diagram of parametrized universes

$$\begin{tikzcd} \mathscr{U} & Q \times \mathscr{U} \arrow[l, "\pi"'] \arrow[r, "\chi"] & \mathscr{U}' \\ \mathbb{R}^n \arrow[u, "f"] & Q \times \mathbb{R}^n \arrow[l, "\pi"] \arrow[u, "1 \times f"'] & \end{tikzcd}$$

with the square a pullback, and consequently

$$\begin{aligned} &\mathscr{T}/_{\mathscr{U}\!n(\mathscr{U}, \mathscr{U}')}(Q, \mathscr{S}(f_*X, E')) \\ &\quad \cong \bar{\mathscr{S}}(Q \times \mathscr{U})(\pi^* f_*X, \chi^*E') \quad \text{(by Theorem 0.7)} \\ &\quad \cong \bar{\mathscr{S}}(Q \times \mathscr{U})((1 \times f)_*\pi^*X, \chi^*E') \quad \text{(by Lemma 3.17)} \\ &\quad \cong \bar{\mathscr{S}}(Q \times \mathbb{R}^n)(\pi^*X, (1 \times f)^*\chi^*E') \quad \text{(by Theorem 0.6)} \end{aligned}$$

$$\cong \mathscr{T}/_{\mathscr{U}\!n(\mathbb{R}^n,\, \mathscr{U}')}(j_*Q, \mathscr{S}(X,E')) \quad \text{(by Theorem 0.7)}$$

$$\cong \mathscr{T}/_{\mathscr{U}\!n(\mathscr{U},\, \mathscr{U}')}(Q, j^*\mathscr{S}(X,E')).$$

The conclusion follows by the Yoneda lemma, and similarly with X replaced by A. Since j^* is a pullback, it now suffices to show that

$$\mathscr{S}(X,E') \to \mathscr{S}(A,E')$$

is a fibration over compact subsets of $\mathscr{U}\!n(\mathbb{R}^n, \mathscr{U}')$.

We next consider the natural map

$$\mathscr{U}\!n(\mathbb{R}^n, \mathscr{U}') \to G_n(\mathscr{U}')$$

sending a map to its image, and the diagram over it

$$\begin{tikzcd}
\mathscr{S}(X,E') \arrow[rr] \arrow[dd] \arrow[dr] & & F_{G_n'}(X_n, E_n') \arrow[dd] \arrow[dr] & \\
& \mathscr{U}\!n(\mathbb{R}^n, \mathscr{U}') \arrow[rr] & & G_n(\mathscr{U}') \\
\mathscr{S}(A,E') \arrow[rr] \arrow[ur] & & F_{G_n'}(A_n, E_n') \arrow[ur] &
\end{tikzcd}$$

where X_n and A_n are trivial bundles over $G_n(\mathscr{U}')$ with fibers X_n and A_n respectively, and given $\theta \in \mathscr{U}\!n(\mathbb{R}^n, \mathscr{U}')$, the part of the arrow

$$\mathscr{S}(X,E') \to F_{G_n'}(X_n, E_n')$$

over θ sends an element of $\mathscr{S}(X,E')$ to its associated map $X_n \to E'(\theta(\mathbb{R}^n))$; similarly with X replaced by A. We see easily that all three squares in the diagram are pullbacks, since maps in $\mathscr{S}$ from X (or A) are completely determined by their restrictions to X_n (or A_n). It therefore suffices to show that

$$F_{G_n'}(X_n, E_n') \to F_{G_n'}(A_n, E_n')$$

is a fibration over compact subsets of $G_n(\mathscr{U}')$.

By Lemma 4.2, any compact subset K of $G_n(\mathscr{U}')$ is contained in $G_n(W)$ for some finite-dimensional subspace W of $\mathscr{U}'$. Since E_n' is a bundle over $G_n(\mathscr{U}')$ (Corollary 2.15) and $G_n(W)$ is a compact manifold, there is a numerable cover of $G_n(W)$, and therefore of K, trivializing E_n'. Given any set N in this cover and a point V in N, the map

$$F_{G_n'}(X_n, E_n') \to F_{G_n'}(A_n, E_n')$$

is isomorphic over N to

$$F(X_n, E'V) \times N \to F(A_n, E'V) \times N.$$

This map is a fibration since $A_n \to X_n$ is a cofibration, being the map on nth spaces of a cofibration in $\mathscr{S}\mathbb{R}^n$. Since we now have a local fibration over a numerable cover of K, Hurewicz's theorem [9, Theorem 5.1] allows us to conclude that the map is a fibration over K. $\square$

Corollary 4.10. *If $E_1 \to E_2$ is an inclusion of finite* CW*-spectra, then the induced map*

$$\mathscr{S}(E_2, E') \to \mathscr{S}(E_1, E')$$

is a fibration over compact subsets of $\mathscr{U}\!n(\mathscr{U}, \mathscr{U}')$. $\square$

Corollary 4.11. *If X is in $\mathscr{S}\mathbb{R}^n$ and $f: \mathbb{R}^n \to \mathscr{U}$ is a linear isometry, then*

$$\mathscr{S}(f_* X, E') \to \mathscr{U}\!n(\mathscr{U}, \mathscr{U}')$$

is a fibration over compact subsets of $\mathscr{U}\!n(\mathscr{U}, \mathscr{U}')$. (In fact, it is a bundle.)

Proof. Let $A = \{*\}$, and notice that $\mathscr{S}(\{*\}, E') = \mathscr{U}\!n(\mathscr{U}, \mathscr{U}')$; this works for non-degenerately based X, which is all we need. But the bundle statement follows easily from the proof of Lemma 4.9, by observing that

$$\mathscr{S}(f_* X, E') \cong j^* \mathscr{S}(X, E'),$$

which is pulled back from the bundle $F_{G_n'}(X_n, E_n')$. $\square$

Proof of Theorem 4.1. By Corollary 4.8,

$$\mathscr{S}(E, E') \cong \lim_{\lambda}(E_\lambda, E')$$

where the limit is taken over inclusions of finite CW-subspectra E_λ of E. By Corollary 4.10, each map in the limit system is a fibration over compact subsets of $\mathscr{U}\!n(\mathscr{U}, \mathscr{U}')$, and therefore so is the projection

$$\mathscr{S}(E, E') \to \mathscr{S}(E_\lambda, E')$$

for any λ. Let E_λ be the first sphere in the sequential filtration of E; by Corollary 4.11,

$$\mathscr{S}(E_\lambda, E') \to \mathscr{U}\!n(\mathscr{U}, \mathscr{U}')$$

is a fibration, and composing gives the augmentation

$$\mathscr{S}(E, E') \to \mathscr{U}\!n(\mathscr{U}, \mathscr{U}'),$$

which is therefore a fibration over compact subsets of $\mathscr{U}\!n(\mathscr{U}, \mathscr{U}')$. $\square$

5. A symmetric monoidal smash product

In this section we show that May's *external* smash product

$$\wedge : \mathscr{S}\mathscr{U} \times \mathscr{S}\mathscr{U}' \to \mathscr{S}(\mathscr{U} \oplus \mathscr{U}')$$

(Definition 2.8) extends to a symmetric monoidal smash product

$$\wedge : \mathscr{S} \times \mathscr{S} \to \mathscr{S}.$$

Since May's product gives the smash product in the stable category by composing with a map which is, for us, an isomorphism in $\mathscr{S}$, it follows from the proof of Theorem 0.1 that our extension also induces the smash product in the stable category. We could, in fact, extend to a symmetric monoidal smash product of *parametrized* spectra

$$\wedge : \bar{\mathscr{S}} \times \bar{\mathscr{S}} \to \bar{\mathscr{S}},$$

but we will leave the formulation (which requires indexing sets for parametrized universes) and proof to the interested reader. Our extension depends on the following lemma, whose proof we defer to later in the section:

Lemma 5.1. *Let $\mathscr{U}$, $\mathscr{U}'$, $\mathscr{V}$, and $\mathscr{V}'$ be universes, and let*

$$\oplus : \mathscr{U}\!n(\mathscr{U}, \mathscr{U}') \times \mathscr{U}\!n(\mathscr{V}, \mathscr{V}') \to \mathscr{U}\!n(\mathscr{U} \oplus \mathscr{V}, \mathscr{U}' \oplus \mathscr{V}')$$

be the direct sum of maps. Let A be in $\mathscr{T}/_{\mathscr{U}\!n(\mathscr{U}, \mathscr{U}')}$, B in $\mathscr{T}/_{\mathscr{U}\!n(\mathscr{V}, \mathscr{V}')}$, E in $\mathscr{S}\mathscr{U}$, H in $\mathscr{S}\mathscr{V}$. Then there is a natural isomorphism in $\mathscr{S}(\mathscr{U}' \oplus \mathscr{V}')$

$$\oplus_*(A \times B) \ltimes (E \wedge H) \cong (A \ltimes E) \wedge (B \ltimes H). \qquad \square$$

Theorem 0.2 is part (b) of the next theorem.

Theorem 5.2. *There is a smash product bifunctor*

$$\wedge : \mathscr{S} \times \mathscr{S} \to \mathscr{S}$$

with the following three properties:

(a) *It coincides with May's external smash product on the subcategories $\mathscr{S}\mathscr{U} \times \mathscr{S}\mathscr{U}'$,*

(b) *It turns $\mathscr{S}$ into a symmetric monoidal category, and*

(c) *The augmentation $\varepsilon : \mathscr{S} \to \mathscr{U}\!n$ is a map of symmetric monoidal categories. (The product on $\mathscr{U}\!n$ is given by direct sum.)*

Proof. Condition (a) determines the product on objects, and (c) tells us that to extend to morphisms, we must produce a map

$$\oplus_*[\mathscr{S}(E, E') \times \mathscr{S}(H, H')] \to \mathscr{S}(E \wedge H, E' \wedge H')$$

for spectra E, E', H, H', in $\mathscr{S}\mathscr{U}$, $\mathscr{S}\mathscr{U}'$, $\mathscr{S}\mathscr{V}$, $\mathscr{S}\mathscr{V}'$ respectively. Let

$$\varepsilon : \mathscr{S}(E, E') \ltimes E \to E'$$

be the counit of the adjunction of Theorem 0.5 between $\_ \ltimes E$ and $\mathscr{S}(E, \_)$. We define the required map as the adjoint of the composite

$$\begin{aligned}&\oplus_*[\mathcal{S}(E,E')\times\mathcal{S}(H,H')]\ltimes(E\wedge H)\\ &\quad\cong[\mathcal{S}(E,E')\ltimes E]\wedge[\mathcal{S}(H,H')\ltimes H]\quad\text{(by Lemma 5.1)}\\ &\quad\xrightarrow{\varepsilon\wedge\varepsilon}E'\wedge H'.\end{aligned}$$

The associativity isomorphism

$$(E\wedge E')\wedge E''\cong E\wedge(E'\wedge E'')$$

is the same map in $\mathcal{S}(\mathcal{U}\oplus\mathcal{U}'\oplus\mathcal{U}'')$ that gives May's external product its natural associativity. The commutativity isomorphism is the map in $\mathcal{S}$ over the interchange isomorphism

$$\tau:\mathcal{U}\oplus\mathcal{U}'\to\mathcal{U}'\oplus\mathcal{U}$$

corresponding to the isomorphism in $\mathcal{S}(\mathcal{U}\oplus\mathcal{U}')$

$$E\wedge E'\cong\tau^*(E'\wedge E),$$

and the unit is S^0 in $\mathcal{S}\{0\}\equiv\mathcal{T}^+$. The necessary diagrams for a symmetric monoidal category are all obvious for prespectra, and follow for spectra. □

The proof of Lemma 5.1 involves function spectra, which are described in May's context in [12, II.3]. They relate to smash products as follows; see [12] for details.

Theorem 5.3. *Let $\mathcal{U}$ and $\mathcal{U}'$ be universes, and suppose given spectra E in $\mathcal{S}\mathcal{U}$, E' in $\mathcal{S}\mathcal{U}'$, and X in $\mathcal{S}(\mathcal{U}\oplus\mathcal{U}')$. There is a functorial construction of a function spectrum $F(E',X)$ in $\mathcal{S}\mathcal{U}$ and a natural isomorphism*

$$\mathcal{S}(\mathcal{U}\oplus\mathcal{U}')(E\wedge E',X)\cong\mathcal{S}\mathcal{U}(E,F(E',X)).\qquad\square$$

For our purposes we need a description of $F(E',X)$ which takes into account the topology of the total spaces, but it is straightforward to check that the definition given below is equivalent to May's rather simpler one under the isomorphism of Theorem 0.4.

Definition 5.4. Let $\mathcal{U}$ and $\mathcal{U}'$ be universes and X a spectrum in $\mathcal{S}(\mathcal{U}\oplus\mathcal{U}')$. For each $n\geq 0$, we define a parametrized spectrum $X[n]$ in $\bar{\mathcal{S}}(G_n(\mathcal{U})\times\mathcal{U}')$ by considering the natural inclusion map

$$q:G_n(\mathcal{U})\times G_k(\mathcal{U}')\to G_{n+k}(\mathcal{U}\oplus\mathcal{U}'),$$

and defining

$$X[n]_k=q^*X_{n+k}.$$

Structure maps are induced from those on X. Given a spectrum E' in $\mathcal{S}\mathcal{U}'$, we may also consider the map of parametrized universes given by the projection

$$\pi_2 : G_n(\mathscr{U}) \times \mathscr{U}' \to \mathscr{U}'$$

and obtain $\pi_2^* E'$ as a spectrum in $\bar{\mathscr{S}}(G_n(\mathscr{U}) \times \mathscr{U}')$. Next, we consider the natural map

$$I : G_n(\mathscr{U}) \to \mathscr{U}\!n_{G_n(\mathscr{U})}(G_n(\mathscr{U}) \times \mathscr{U}', G_n(\mathscr{U}) \times \mathscr{U}')$$

which sends each element V of $G_n(\mathscr{U})$ to the identity on $\mathscr{U}'$ over V. We can now define $F(E', X)$ by letting

$$F(E', X)_n = I^* \mathscr{S}_{G_n(\mathscr{U})}(\pi_2^* E', X[n]).$$

This forces each element to actually be a map in $\mathscr{S}\mathscr{U}'$. Structure maps are induced by the maps of parametrized spectra

$$p_2^* X[k] \to \Omega^{\gamma_n} s^* X[n+k]$$

induced by the structure maps on X, where Ω^{γ_n} is taken componentwise on the spectrum $s^* X[n+k]$ (this makes sense since each component has a naturally defined canonical bundle γ_n on the base space).

In order to prove Lemma 5.1, we need to characterize $F(E', X)_n$ as an equalizer. Consider the diagram

$$\begin{tikzcd} G_n(\mathscr{U}) & G_n(\mathscr{U}) \times G_k(\mathscr{U}') \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "q"] & G_k(\mathscr{U}') \\ & G_{n+k}(\mathscr{U} \oplus \mathscr{U}') & \end{tikzcd}$$

If we choose a typical $V \in G_n(\mathscr{U})$, then $F(E', X)(V)$ is contained in

$$\prod_{k=0}^{\infty} F_{G_k'}(E_k', X(\{V\} \times G_k')),$$

which is the fiber over V of the infinite fiber product over $G_n(\mathscr{U})$

$$\prod_{k=0}^{\infty} {}_{G_n}(\pi_1)_! F_{G_n \times G_k'}(\pi_2^* E_k', X[n]_k).$$

We force these fiber elements to be in $F(E', X)(V)$ for all V by considering the diagram

$$G_n(\mathscr{U}) \times G_k(\mathscr{U}') \xleftarrow{1 \times p_2} G_n(\mathscr{U}) \times G_{j,k}(\mathscr{U}') \xrightarrow{1 \times s} G_n(\mathscr{U}) \times G_{j+k}(\mathscr{U}')$$

and equalizing two maps

$$\sum_{k=0}^{\infty} {}_{G_n}(\pi_1)_! F_{G_n \times G_k'}(\pi_2^* E_k', X[n]_k) \to$$

$$\to \prod_{\substack{k=0\\ j=0}}^{\infty}{}_{G_n}(\pi_1)_! F_{G_n\times G'_k}(\pi_2^* E'_k, (1\times p_2)_! \Omega^{\gamma'_j}(1\times s)^* X[n]_{j+k}).$$

The first map is simply induced by the structure maps for $X[n]$

$$\bar{\sigma}_{j,k} : X[n]_k \to (1\times p_2)_! \Omega^{\gamma'_j}(1\times s)^* X[n]_{j+k}.$$

The second map first applies the continuous functor $(1\times p_2)_! \Omega^{\gamma'_j}(1\times s)^*$ and then uses the structure map from $\pi_2^* E'$:

$$\begin{aligned} &F_{G_n\times G'_{j+k}}(\pi_2^* E'_{j+k}, X[n]_{j+k}) \\ &\quad\to F_{G_n\times G'_{j+k}}((1\times p_2)_! \Omega^{\gamma'_j}(1\times s)^* \pi_2^* E'_{j+k}, (1\times p_2)_! \Omega^{\gamma'_j}(1\times s)^* X[n]_{j+k}) \\ &\quad\to F_{G_n\times G'_{j+k}}(\pi_2^* E'_k, (1\times p_2)_! \Omega^{\gamma'_j}(1\times s)^* X[n]_{j+k}). \end{aligned}$$

These two maps correspond to the two ways of traversing the square

$$\begin{tikzcd} \pi_2^* E_k \arrow[r] \arrow[d] & (1\times p_2)_! \Omega^{\gamma'_j}(1\times s)^* \pi_2^* E_{j+k} \arrow[d] \\ X[n]_k \arrow[r] & (1\times p_2)_! \Omega^{\gamma'_j}(1\times s)^* X[n]_{j+k} \end{tikzcd}$$

defining a map in $\mathscr{S}\mathscr{U}'$, so their equalizer gives us $F(E', X)_n$. We will abuse notation slightly by observing that equalizers are a sort of limit, and write

$$F(E', X)_n = \lim_k (\pi_1)_! F_{G_n\times G'_k}(\pi_2^* E'_k, q^* X_{n+k}).$$

The next lemma is a special case of Lemma 5.1, from which the general case will follow easily.

Lemma 5.5. *Let $\mathscr{U}$, $\mathscr{U}'$, $\mathscr{V}$, $\mathscr{V}'$, A, B, E, and H be as in Lemma 5.1, let*

$$l: \mathscr{U}n(\mathscr{U}, \mathscr{U}') \to \mathscr{U}n(\mathscr{U}\oplus\mathscr{V}, \mathscr{U}'\oplus\mathscr{V})$$

be given by $l(\varphi) = \varphi\oplus 1$, and let

$$r: \mathscr{U}n(\mathscr{V}, \mathscr{V}') \to \mathscr{U}n(\mathscr{U}\oplus\mathscr{V}, \mathscr{U}\oplus\mathscr{V}')$$

be given by $r(\varphi) = 1\oplus\varphi$. Then there are natural isomorphisms

$$(l_* A)\ltimes(E\wedge H) \cong (A\ltimes E)\wedge H,$$

$$(r_* B)\ltimes(E\wedge H) \cong E\wedge(B\ltimes H).$$

Proof. We prove the adjoint form of the first isomorphism, which says that for X in $\mathscr{S}(\mathscr{U}'\oplus\mathscr{V})$, there is a natural isomorphism

$$F(H, F[l_* A, X)) \cong F[A, F(H, X)).$$

Consulting the diagram

$$
\begin{tikzcd}
 & G_n(\mathcal{U}') & & G_n(\mathcal{U}')\times G_k(\mathcal{V}) \arrow[ll, "\pi_1"'] \arrow[rr, "q"] \arrow[dd, "\bar{\pi}"] & & G_{n+k}(\mathcal{U}'\oplus\mathcal{V}) \\
A\times G_n(\mathcal{U}) \arrow[ur, "\chi"] \arrow[dd, "p"] & & A\times G_n(\mathcal{U})\times G_k(\mathcal{V}) \arrow[ll, "1\times\pi_1"'] \arrow[rr, "1\times q"] \arrow[ur, "\chi\times 1"] \arrow[dd, "p"] & & A\times G_{n+k}(\mathcal{U}\oplus\mathcal{V}) \arrow[ur, "\bar{\chi}"'] \arrow[dd, "p"] & \\
 & & & G_k(\mathcal{V}) & & \\
G_n(\mathcal{U}) & & G_n(\mathcal{U})\times G_k(\mathcal{V}) \arrow[ll, "\pi_1"'] \arrow[rr, "q"] \arrow[ur, "\pi_2"] & & G_{n+k}(\mathcal{U}\oplus\mathcal{V}) &
\end{tikzcd}
$$

we define the isomorphism on component spaces as the composite

$$
\begin{aligned}
F(H,F[l_*A,X))_n &= \lim_k (\pi_1)_! F_{G_n\times G_k}(\pi_2^* H_k, q^* p_! \bar{\chi}^* X_{n+k}) \\
&\cong \lim_k (\pi_1)_! F_{G_n\times G_k}(\pi_2^* H_k, p_!(1\times q)^* \bar{\chi}^* X_{n+k}) \\
&\cong \lim_k (\pi_1)_! p_! F_{A\times G_n\times G_k}(p^*\pi_2^* H_k, (1\times q)^* \bar{\chi}^* X_{n+k}) \\
&\cong \lim_k p_!(1\times \pi_1)_! F_{A\times G_n\times G_k}((\chi\times 1)^* \bar{\pi}^* H_k, (\chi\times 1)^* q^* X_{n+k}) \\
&\cong p_! \lim_k (1\times \pi_1)_! (\chi\times 1)^* F_{G_n'\times G_k}(\bar{\pi}^* H_k, q^* X_{n+k}) \\
&\cong p_! \lim_k \chi^* (\pi_1)_! F_{G_n'\times G_k}(\bar{\pi}^* H_k, q^* X_{n+k}) \\
&\cong p_! \chi^* \lim_k (\pi_1)_! F_{G_n'\times G_k}(\bar{\pi}^* H_k, q^* X_{n+k}) \\
&= F[A, F(H,X))_n,
\end{aligned}
$$

where we commute $p_!$ and χ^* with $\lim_k$ since they both have left adjoints, and the computation extends to the equalizer defining $\lim_k$. A similar computation shows that the structure maps are preserved.

The second isomorphism now follows formally by using the formula

$$\tau_*(E\wedge H)\cong H\wedge E$$

and applying Corollary 3.18 to the formula

$$r=\tau\circ l\circ\tau. \qquad \square$$

Proof of Lemma 5.1. This now follows from Lemma 5.5 by applying Corollary 3.18 to the formula

$$\oplus = r\circ l. \qquad \square$$

6. Group actions on spectra

In this section we define G-spectra for an arbitrary fixed topological group G; Theorem 0.3 follows directly from this definition. We then show how Theorems 0.5 and 0.6 generalize to the equivariant case. In a later paper we will give a different generalization to the case in which G is merely a small topological category; this will allow us to greatly simplify the usual description of structured ring and module spectra.

Except for Theorem 6.5, all universes and spectra in this section are parametrized.

Definition 6.1. Let G be a topological group. A *G-universe* is a universe $\mathscr{U}$ together with a continuous homomorphism

$$\Phi : G \to \bar{\mathscr{U}n}(\mathscr{U}, \mathscr{U}).$$

If $\mathscr{U}$ and $\mathscr{U}'$ are G-universes, a *map* of G-universes is an element $\theta \in \bar{\mathscr{U}n}(\mathscr{U}, \mathscr{U}')$ such that the diagram

$$\begin{tikzcd} G \arrow[r, "\Phi"] \arrow[d, "\Phi'"'] & \bar{\mathscr{U}n}(\mathscr{U}, \mathscr{U}) \arrow[d, "\theta_*"] \\ \bar{\mathscr{U}n}(\mathscr{U}', \mathscr{U}') \arrow[r, "\theta^*"] & \bar{\mathscr{U}n}(\mathscr{U}, \mathscr{U}') \end{tikzcd}$$

commutes. We write the resulting *category of G-universes* as G-$\bar{\mathscr{U}n}$; it is simply the category of functors from G to $\bar{\mathscr{U}n}$, considering G to be a category with one object and morphisms the elements of G. Similarly, a *G-spectrum* is a spectrum E together with a continuous homomorphism

$$\Phi : G \to \bar{\mathscr{S}}(E, E).$$

If E lies over the universe $\mathscr{U}$, this induces the structure of a G-universe on $\mathscr{U}$. If E and E' are G-spectra, a *map* of G-spectra is an element $f \in \bar{\mathscr{S}}(E, E')$ such that the diagram

$$\begin{tikzcd} G \arrow[r, "\Phi"] \arrow[d, "\Phi'"'] & \bar{\mathscr{S}}(E, E) \arrow[d, "f_*"] \\ \bar{\mathscr{S}}(E', E') \arrow[r, "f^*"] & \bar{\mathscr{S}}(E, E') \end{tikzcd}$$

commutes. We write the resulting *category of G-spectra* as G-$\bar{\mathscr{S}}$; it is simply the category of functors from G to $\bar{\mathscr{S}}$. The augmentation

$$\varepsilon : \bar{\mathscr{S}} \to \bar{\mathscr{U}n}$$

induces an augmentation

$$\varepsilon_G : G\text{-}\bar{\mathscr{P}} \to G\text{-}\bar{\mathscr{U}n},$$

and given a G-universe $\mathscr{U}$, we define

$$G\text{-}\bar{\mathscr{P}}\mathscr{U} = \varepsilon_G^{-1}(\mathrm{id}_{\mathscr{U}}).$$

We write $G\text{-}\mathscr{P}\mathscr{U}$ if $\mathscr{U}$ is not parametrized.

Proof of Theorem 0.3. We show that when G is a compact group and $\mathscr{U}$ is a G-universe in the sense of Lewis and May (which is a special case of our definition), then our category $G\text{-}\mathscr{P}\mathscr{U}$ is isomorphic to theirs [12, I.2]. This is sufficient, since their category provides representing objects for RO(G)-graded equivariant cohomology theories. Their G-spectra are defined exactly as in Definitions 2.1 and 2.2, except that the universe $\mathscr{U}$ has a G-action, all the indexing spaces V must be G-invariant, the component spaces EV must be G-spaces, and the structure maps and maps of G-spectra must respect these actions.

First, a G-spectrum E in our sense gives one in theirs by restricting attention to the fibers of E_k over the invariant subspaces of $\mathscr{U}$. Conversely, suppose we have a G-spectrum E in the Lewis–May sense. Then since G is compact, the invariant finite-dimensional subspaces of $\mathscr{U}$ form an indexing set, and we can form a G-spectrum in our sense by using only invariant subspaces in the proof of Theorem 0.4 (which follows Definition 2.14), and observing that the G-actions on the bundles $\Omega^{v_k(V)}EV$ agree where they overlap. □

Our next objective is to generalize Theorem 0.6 to the equivariant context. We must first introduce a G-action on the space $\bar{\mathscr{P}}(E_1, E_2)$ whenever E_1 and E_2 are G-spectra.

Lemma 6.2. *If E_1 and E_2 are G-spectra, then $\bar{\mathscr{P}}(E_1, E_2)$ has a natural G-action whose fixed point set is $G\text{-}\bar{\mathscr{P}}(E_1, E_2)$. If E_1 and E_2 are indexed on the same G-universe $\mathscr{U}$, then $\bar{\mathscr{P}}\mathscr{U}(E_1, E_2)$ is an invariant subspace of $\bar{\mathscr{P}}(E_1, E_2)$, with fixed point set $G\text{-}\bar{\mathscr{P}}\mathscr{U}(E_1, E_2)$.*

Proof. Let $\Phi_i : G \to \bar{\mathscr{P}}(E_i, E_i)$ represent the action of G on E_i for $i = 1$ or 2, and let $g \in G$, $f \in \bar{\mathscr{P}}(E_1, E_2)$. We define gf by the usual formula

$$gf = \Phi_2(g) \circ f \circ \Phi_1(g^{-1}).$$

Since the equivariance condition

$$\Phi_2(g) \circ f = f \circ \Phi_1(g)$$

is equivalent to $gf = f$, the fixed point set is $G\text{-}\bar{\mathscr{P}}(E_1, E_2)$. The invariance of $\bar{\mathscr{P}}\mathscr{U}(E_1, E_2)$ follows immediately from the functoriality of the augmentation $\varepsilon : \bar{\mathscr{P}} \to \bar{\mathscr{U}n}$. □

The following theorem gives our equivariant generalization of Theorem 0.6:

Theorem 6.3. *Let E and E' be G-spectra over the G-universes $\mathscr{U}$ and $\mathscr{U}'$ respectively, and let $f\colon \mathscr{U} \to \mathscr{U}'$ be a map of G-universes. Then there are natural G-spectrum structures on f^*E', f_*E, and, if f is fiberwise surjective, $f_!E$, such that the adjunction isomorphisms*

$$\bar{\mathscr{S}}\mathscr{U}(E, f^*E') \cong \bar{\mathscr{S}}\mathscr{U}'(f_*E, E')$$

and

$$\bar{\mathscr{S}}\mathscr{U}(f^*E', E) \cong \bar{\mathscr{S}}\mathscr{U}'(E', f_!E)$$

are equivariant maps.

The proof of this theorem will take up most of the remainder of this section; first we derive its most important consequences.

Corollary 6.4. *We may consider the pairs (f_*, f^*) and $(f^*, f_!)$ as adjoint functors between the categories of G-spectra G-$\bar{\mathscr{S}}\mathscr{U}$ and G-$\bar{\mathscr{S}}\mathscr{U}'$.*

Proof. Pass to fixed point sets in Theorem 6.3. □

The next theorem is our equivariant version of Theorem 0.5. For simplicity, we assume *in this theorem only* that G-universes and G-spectra are *not* parametrized. The parametrized versions given in Section 3 (namely, Theorems 3.5, 3.7, 3.13, and 3.15) also admit equivariant generalizations whose formulation and proof are safely left to the interested reader.

Theorem 6.5. *Let E and E' be G-spectra over the G-universes $\mathscr{U}$ and $\mathscr{U}'$ respectively, and let A be a G-space in $\mathscr{T}/_{\mathscr{U}n(\mathscr{U},\mathscr{U}')}$ with equivariant structure map. Then $A \ltimes E$ and $F[A, E')$ are naturally G-spectra, and the adjunction isomorphisms*

$$\mathscr{T}/_{\mathscr{U}n(\mathscr{U},\mathscr{U}')}(A, \mathscr{S}(E, E')) \cong \mathscr{S}\mathscr{U}'(A \ltimes E, E') \cong \mathscr{S}\mathscr{U}(E, F[A, E'))$$

are equivariant maps.

Proof. Since the structure map

$$A \to \mathscr{U}n(\mathscr{U}, \mathscr{U}')$$

is equivariant, the induced map

$$\chi : A \times \mathscr{U} \to \mathscr{U}'$$

is a map of (parametrized) G-universes. The second isomorphism is therefore equivariant by Theorem 6.3 and the definitions of $A \ltimes E$ and $F[A, E')$. For the first isomorphism, notice in the proof of Theorem 3.16 (which generalizes Theorem 0.7) that all spaces are now G-spaces, and all the isomorphisms (in particular the first one) are equivariant. The conclusion now follows easily. □

By passage to fixed point sets, we get

Corollary 6.6. *With $\mathcal{U}$, $\mathcal{U}'$, E, E', and A as in Theorem 6.5, there are natural isomorphisms*

$$G\text{-}\mathcal{T}/_{\mathcal{U}n(\mathcal{U},\mathcal{U}')}(A, \mathcal{S}(E,E')) \cong G\text{-}\mathcal{S}\mathcal{U}'(A \ltimes E, E') \cong G\text{-}\mathcal{S}\mathcal{U}(E, F[A,E']). \qquad \square$$

We now turn to the proof of Theorem 6.3. Our first step is to extend the category $\bar{\mathcal{S}}\mathcal{U}$ when $\mathcal{U}$ is a G-universe.

Definition 6.7. Let $\mathcal{U}$ be a G-universe with structure map

$$\Phi : G \to \bar{\mathcal{U}n}(\mathcal{U}, \mathcal{U}).$$

We define a category $\bar{\mathcal{S}} \downarrow \mathcal{U}$ with the same objects as $\bar{\mathcal{S}}\mathcal{U}$, but with morphisms $(g,\phi) : E_1 \to E_2$ consisting of elements $g \in G$ and $\phi \in \bar{\mathcal{S}}(E_1, E_2)$ such that $\Phi(g) = \varepsilon\phi$. (This can also be described as the pullback category $G \times_{\bar{\mathcal{U}n}} \bar{\mathcal{S}}$.) Composition is given by the formula

$$(g,\phi)\circ(g',\phi') = (gg', \phi\circ\phi').$$

Notice that $\bar{\mathcal{S}}\mathcal{U}$ is the subcategory of $\bar{\mathcal{S}} \downarrow \mathcal{U}$ consisting of those (g,ϕ) with $g = \mathrm{id}$.

Lemma 6.8. *If $F : \mathcal{U} \to \mathcal{U}'$ is a map of G-universes, then the functors $f^* : \bar{\mathcal{S}}\mathcal{U}' \to \bar{\mathcal{S}}\mathcal{U}$, $f_* : \bar{\mathcal{S}}\mathcal{U} \to \bar{\mathcal{S}}\mathcal{U}'$, and, if f is fiberwise surjective, $f_! : \bar{\mathcal{S}}\mathcal{U} \to \bar{\mathcal{S}}\mathcal{U}'$ all have functorial extensions to $\bar{\mathcal{S}} \downarrow \mathcal{U}$ and $\bar{\mathcal{S}} \downarrow \mathcal{U}'$. It will be clear from the construction that the adjunctions between f^*, f_*, and $f_!$ also extend.*

Proof. For the purposes of this lemma, it is more convenient to regard a morphism $(g,\phi) : E_1 \to E_2$ in $\bar{\mathcal{S}} \downarrow \mathcal{U}$ as an element $g \in G$ and a map *in* $\bar{\mathcal{S}}\mathcal{U}$

$$\phi : E_1 \to g^*E_2,$$

where we abbreviate g^*E_2 for $\Phi(g)^*E_2$. Using this formulation we define $f^* : \bar{\mathcal{S}} \downarrow \mathcal{U}' \to \bar{\mathcal{S}} \downarrow \mathcal{U}$ on objects to agree with $f^* : \bar{\mathcal{S}}\mathcal{U}' \to \bar{\mathcal{S}}\mathcal{U}$, and on morphisms, we send $(g,\phi') : E_1' \to E_2'$ to the composite

$$f^*E_1' \xrightarrow{f^*\phi'} f^*g^*E_2' \cong g^*f^*E_2',$$

where the isomorphism is induced by the equivariance of f, expressed in the commutative diagram

$$\begin{tikzcd} \mathcal{U} \arrow[r, "f"] \arrow[d, "\Phi(g)"'] & \mathcal{U}' \arrow[d, "\Phi'(g)"] \\ \mathcal{U} \arrow[r, "f"] & \mathcal{U}' \end{tikzcd}$$

This diagram is a pullback ($\Phi(g)$ and $\Phi'(g)$ are isomorphisms), so we can extend f_* by sending $(g,\phi): E_1 \to E_2$ to the composite

$$f_*E_1 \xrightarrow{f_*\phi} f_*g^*E_2 \cong g^*f_*E_2,$$

and similarly for $f_!$. $\square$

Corollary 6.9. *If E and E' are G-spectra over the G-universes $\mathscr{U}$ and $\mathscr{U}'$ respectively, and $f: \mathscr{U} \to \mathscr{U}'$ is a map of G-universes, then f^*E', f_*E, and, if f is fiberwise surjective, $f_!E$ all have a natural G-spectrum structure.*

Proof. The homomorphism

$$\Phi': G \to \bar{\mathscr{S}}(E', E')$$

defining E' as a G-spectrum extends canonically to a homomorphism

$$\Phi': G \to \bar{\mathscr{S}} \downarrow \mathscr{U}'(E', E');$$

we then compose with the functor

$$f^*: \bar{\mathscr{S}} \downarrow \mathscr{U}'(E', E') \to \bar{\mathscr{S}} \downarrow \mathscr{U}(f^*E', f^*E')$$

and the natural projection $\bar{\mathscr{S}} \downarrow \mathscr{U} \to \bar{\mathscr{S}}$ to get a G-spectrum structure on f^*E'. Similarly, f_*E and $f_!E$ are also G-spectra. $\square$

Corollary 6.10. *Let E_1' and E_2' be G-spectra over the G-universe $\mathscr{U}'$, let $f: \mathscr{U} \to \mathscr{U}'$ be a map of G-universes, and let G act on $\bar{\mathscr{S}} \downarrow \mathscr{U}'(E_1', E_2')$ and $\bar{\mathscr{S}} \downarrow \mathscr{U}(f^*E_1', f^*E_2')$ by conjugation (as in Lemma 6.2). Then the map*

$$f^*: \bar{\mathscr{S}} \downarrow \mathscr{U}'(E_1', E_2') \to \bar{\mathscr{S}} \downarrow \mathscr{U}(f^*E_1', f^*E_2')$$

is equivariant. Similar statements hold for f_* and $f_!$.

Proof. Let

$$\Phi_i': G \to \bar{\mathscr{S}} \downarrow \mathscr{U}'(E_i', E_i')$$

be the (extended) structure maps for E_i', $i = 1$ or 2. Given $g \in G$ and $\phi \in \bar{\mathscr{S}} \downarrow \mathscr{U}'(E_1', E_2')$, we then have

$$\begin{aligned} f^*(g\phi) &= f^*(\Phi_2'(g) \circ \phi \circ \Phi_1'(g^{-1})) \\ &= f^*\Phi_2'(g) \circ f^*\phi \circ f^*\Phi_1'(g^{-1}) = g \cdot f^*\phi, \end{aligned}$$

by the definition of the actions of G on f^*E_1' and f^*E_2'. Similar proofs hold for f_* and $f_!$. $\square$

Corollary 6.11. *With E_1', E_2', $\mathscr{U}'$, and $f: \mathscr{U} \to \mathscr{U}'$ as in Corollary 6.10, the map*

$$f^*: \bar{\mathscr{S}}\mathscr{U}'(E_1', E_2') \to \bar{\mathscr{S}}\mathscr{U}(f^*E_1', f^*E_2')$$

is equivariant. Similar statements hold for f_* and $f_!$.

Proof. This is just the restriction to an invariant subspace of the map in Corollary 6.10, and similarly for f_* and $f_!$. □

The final ingredient we need is the following elementary observation:

Lemma 6.12. *The units and counits of the adjunctions in Theorem* 6.3 *are maps of G-spectra.*

Proof. For example, consider the unit

$$\eta : E \to f^* f_* E.$$

By Lemma 6.8, this is also the unit of the $f_* - f^*$ adjunction between $\bar{\mathscr{S}} \downarrow \mathscr{U}$ and $\bar{\mathscr{S}} \downarrow \mathscr{U}'$, so by naturality, the diagram

$$\begin{tikzcd} E \arrow[r, "\eta"] \arrow[d, "\Phi(g)"'] & f^*f_*E \arrow[d, "f^*f_*\Phi(g)"] \\ E \arrow[r, "\eta"] & f^*f_*E \end{tikzcd}$$

commutes for each $g \in G$. But $f^* f_* \Phi(g)$ is the defined action of g on $f^* f_* E$, so η is equivariant. The other cases are similar. □

Proof of Theorem 6.3. This now follows easily from Corollary 6.11 and Lemma 6.12. For example, the adjunction

$$\bar{\mathscr{S}}\mathscr{U}'(f_* E, E') \cong \bar{\mathscr{S}}\mathscr{U}(E, f^* E')$$

is the composite of

$$f^* : \bar{\mathscr{S}}\mathscr{U}'(f_* E, E') \to \bar{\mathscr{S}}\mathscr{U}(f^* f_* E, f^* E')$$

and the induced map of the unit

$$\eta^* : \bar{\mathscr{S}}\mathscr{U}(f^* f_* E, f^* E') \to \bar{\mathscr{S}}\mathscr{U}(E, f^* E').$$

Since both maps are equivariant, so is their composite. □

Appendix. The construction of spectra from prespectra

This appendix is devoted to the proof of the following theorem:

Theorem A.1. *The forgetful functor $\bar{\mathscr{S}} \to \bar{\mathscr{P}}$ has a continuous left adjoint covering the identity on $\bar{\mathscr{U}}n$.*

Our construction, like Lewis and May's [12, Appendix] proceeds in two steps, with a stop at an intermediate category. Throughout this appendix, all spectra and prespectra are assumed to be *parametrized.*

Definition A.2. A prespectrum D is an *inclusion prespectrum* if all the adjoint structure maps

$$\bar{\sigma}_{n,k} : D_k \to \Omega^{\gamma_n} D_{n+k}$$

are inclusions. The full subcategory of $\bar{\mathscr{P}}$ generated by the inclusion prespectra will be written $\mathscr{Q}$.

Clearly any spectrum is an inclusion prespectrum, so the forgetful functor factors as a composite

$$\bar{\mathscr{S}} \to \mathscr{Q} \to \bar{\mathscr{P}}.$$

Theorems A.3 and A.6 construct left adjoints for each of these functors, and the composite of these provides the left adjoint for Theorem A.1.

Theorem A.3. *The forgetful functor $\bar{\mathscr{S}} \to \mathscr{Q}$ has a continuous left adjoint covering the identity on $\bar{\mathscr{U}}n$.*

The proof of Theorem A.3 is essentially a fiberwise version of May's original Ω^∞ construction in [13, II.1.4]. Since this is a colimit construction, we need a concept of fiberwise colimit.

Definition A.4. Let B be a weak Hausdorff space, and let $\mathscr{A}$ be a category internal to $\mathscr{T}/_B$ with objects $\mathscr{O}$ and morphisms $\mathscr{M}$. (We can think of $\mathscr{A}$ as a topological category parametrized over B; both $\mathscr{A}$ itself and any of the fibers over a point $b \in B$ form topological categories.) Let X be in $\mathscr{T}^+/_{\mathscr{O}}$ and let

$$\xi : \mathscr{M} \boxtimes_{\mathscr{O}} X \to X$$

be an action of $\mathscr{A}$ on X, as in [6] (although we use $\boxtimes$ instead of $\square$ to take the section of X into account). A *fiberwise colimit* for ξ is a space K in $\mathscr{T}^+/_B$ together with an action preserving map $c : X \to \pi^*K$, where $\pi : \mathscr{O} \to B$ is the structure map of $\mathscr{O}$ as an object of $\mathscr{T}/_B$, and $\mathscr{A}$ acts trivially on π^*K (which is possible since π coequalizes the source and target maps $\mathscr{M} \to \mathscr{O}$). We require (K, c) to be universal among all such pairs.

Lemma A.5. *$\mathscr{T}/_B$ has all fiberwise colimits.*

Proof. The projection map

$$p_2 : \mathscr{M} \boxtimes_{\mathscr{O}} X \to X$$

is a map over B (although not over $\mathscr{O}$); we define K to be the coequalizer in $\mathscr{T}^+/_B$ of p_2 and ξ. Given any space K' in $\mathscr{T}^+/_B$ and action preserving map $c': X \to \pi^*K'$, it is easy to see that the adjoint $\hat{c}': \pi_* X \to K'$ must factor uniquely through K, which is therefore the fiberwise colimit. $\square$

Proof of Theorem A.3. We are given an inclusion prespectrum D over the parametrized universe $\mathscr{U}$, and we must produce a spectrum E over $\mathscr{U}$ and a universal arrow $D \to E$ in $\bar{\mathscr{P}}\mathscr{U}$. We write G_k, $G_{n,k}$ etc. for $G_k(\mathscr{U})$, $G_{n,k}(\mathscr{U})$ etc. Define a category $\mathscr{A}\langle k\rangle$ internal to $\mathscr{T}/_{G_k}$ whose fiber over $V \in G_k$ consists of those finite-dimensional subspaces W of the same fiber of $\mathscr{U}$ as V such that $V \subset W$. Morphisms are inclusions of subspaces. Formally, the objects of $\mathscr{A}\langle k\rangle$ are $\coprod_{j\geq 0} G_{j,k}$ and the morphisms are $\coprod_{n\geq 0, j\geq 0} G_{n,j,k}$; a morphism (W, U, V) should be thought of as the inclusion of $U \oplus V$ in $W \oplus U \oplus V$. Source, target, and identity are given by

$$S: G_{j,n,k} \to G_{n,k},$$

$$T: G_{j,n,k} \to G_{j+n,k},$$

$$I: G_{n,k} \xrightarrow{\cong} G_{0,n,k},$$

and composition is given by

$$G_{i,j+n,k} \times_{G_{j+n,k}} G_{j,n,k} \cong G_{i,j,n,k} \to G_{i+j,n,k}.$$

We consider the parametrized space with section $D\langle k\rangle$ given by

$$\coprod_j \Omega^{\gamma_j} s^* D_{j+k} \rightleftarrows \coprod_j G_{j,k}$$

on which $\mathscr{A}\langle k\rangle$ acts (in the sense of [6]) via the prespectrum structure maps, as follows. We examine the diagram

$$\begin{array}{ccccc} G_{j,k} & \xleftarrow{S} & G_{n,j,k} & \xrightarrow{T} & G_{n+j,k} \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \bar{s}} & & \downarrow{\scriptstyle s} \\ G_{j+k} & \xleftarrow[p_2]{} & G_{n,j+k} & \xrightarrow[\underline{s}]{} & G_{n+j+k} \end{array}$$

and the action map is the coproduct of the composites

$$\begin{aligned} G^+_{n,j,k} \wedge_{G_{j,k}} \Omega^{\gamma_j} s^* D_{j+k} &\cong S^* \Omega^{\gamma_j} s^* D_{j+k} \\ &\cong \Omega^{\gamma_j} \bar{s}^* p_2^* D_{j+k} \xrightarrow{\Omega^{\gamma_j}\bar{s}^*\tilde{\sigma}_{n,j+k}} \Omega^{\gamma_j} \bar{s}^* \Omega^{\gamma_n} \underline{s}^* D_{n+j+k} \\ &\cong T^* \Omega^{\gamma_{n+j}} s^* D_{n+j+k}. \end{aligned}$$

We define E by letting E_k be the fiberwise colimit of this action, with the k-component of the universal arrow obtained by restricting to $j=0$ in the colimit map.

To complete the proof, we must describe the structure maps of E and show that they are homeomorphisms. The diagram

$$\begin{tikzcd} G_{j+n,k} \arrow[d] & G_{j,n,k} \arrow[l] \arrow[r] \arrow[d] & G_{j,n+k} \arrow[d] \\ G_k & G_{n,k} \arrow[l, "p_2"'] \arrow[r, "s"] & G_{n+k} \end{tikzcd}$$

the right-hand square of which is a pullback, induces the map on objects of a functor of internal categories

$$s^*\mathscr{A}\langle n+k\rangle \to p_2^*\mathscr{A}\langle k\rangle$$

which is cofinal (i.e., the fiberwise colimit of any action of $p_2^*\mathscr{A}\langle k\rangle$ is determined by the restriction to $s^*\mathscr{A}\langle n+k\rangle$). We also have a canonical map

$$\Omega^{\gamma_n}s^*D\langle n+k\rangle \to p_2^*D\langle k\rangle$$

given by the pullback diagram

$$\begin{tikzcd} \Omega^{\gamma_n}s^*\Omega^{\gamma_j}s^*D_{j+n+k} \arrow[r] \arrow[d] & p_2^*\Omega^{\gamma_{j+n}}s^*D_{j+n+k} \arrow[d] \\ G_{j,n,k} \arrow[r] & p_2^*G_{j+n,k} \end{tikzcd}$$

Since the base of this diagram is the map on objects of the functor defined above, the fiberwise colimits of the actions of $s^*\mathscr{A}\langle n+k\rangle$ on the two spaces are the same, by the pullback property. Structure maps are now given by

$$\begin{aligned} p_2^*E_k &= p_2^* \operatorname*{colim}_{\mathscr{A}\langle k\rangle} D\langle k\rangle \cong \operatorname*{colim}_{p_2^*\mathscr{A}\langle k\rangle} p_2^*D\langle k\rangle \\ &\cong \operatorname*{colim}_{s^*\mathscr{A}\langle n+k\rangle} p_2^*D\langle k\rangle \cong \operatorname*{colim}_{s^*\mathscr{A}\langle n+k\rangle} \Omega^{\gamma_n}s^*D\langle n+k\rangle \\ &\cong \Omega^{\gamma_n} \operatorname*{colim}_{s^*\mathscr{A}\langle n+k\rangle} s^*D\langle n+k\rangle \end{aligned}$$

(this is a fiberwise application of [12, A.2.4], which is the reason we need inclusion prespectra)

$$\cong \Omega^{\gamma_n}s^* \operatorname*{colim}_{\mathscr{A}\langle n+k\rangle} D\langle n+k\rangle \cong \Omega^{\gamma_n}s^*E_{n+k}.$$

The reader should now verify that this is a universal construction. □

The proof of Theorem A.1 is now completed by the following theorem:

Theorem A.6. *The forgetful functor $\mathscr{Q} \to \bar{\mathscr{P}}$ has a continuous left adjoint covering the identity on $\mathscr{U}\!n$.*

Proof. We mimic Lewis's original proof, and construct a functor

$$J : \bar{\mathscr{P}} \to \bar{\mathscr{P}}$$

and a natural map

$$\lambda : D \to JD$$

for all D such that

(i) each $\lambda_k : D_k \to (JD)_k$ is a surjection,

(ii) λ is an isomorphism if and only if D is an inclusion prespectrum, and

(iii) any map $f : D \to D'$, where D' is an inclusion prespectrum, factors uniquely through λ.

We will also have λ covering the identity map on the underlying universe of D and JD. We then iterate J transfinitely in the same manner as [12, A.1.4–A.1.6], so we identify all pairs of points that are going to be identified, and destroy all open sets that are going to be destroyed. This produces a universal arrow from D to $\mathscr{Q}$, and therefore a left adjoint.

We construct J and λ as follows. Let V be a parametrized subuniverse of $\mathscr{U}$ of constant finite fiber dimension N. We define a prespectrum $J_V D$ over V by setting $(J_V D)_k$ equal to the image of $\bar{\sigma}_{N-k,k}$ in the diagram

$$\begin{tikzcd}
D_k(V) \arrow[d] & p_2^* D_k(V) \arrow[l, "\cong"'] \arrow[r, "\bar{\sigma}_{N-k,k}"] \arrow[d] & \Omega^{\gamma_{N-k}} s^* D_N(V) \arrow[dl] \\
G_k(V) & G_{N-k,k}(V) \arrow[l, "\cong", "p_2"'] &
\end{tikzcd}$$

Structure maps for $J_V D$ are induced from those for D, and if $\iota : V \to \mathscr{U}$ is the inclusion, we have an obvious map of prespectra

$$\lambda_V : \iota^* D \to J_V D,$$

given on components by $\bar{\sigma}_{N-k,k}$.

Next, if $V \subset W$, we map $J_V D \to J_W D$ over this inclusion as follows. With $\dim W = M$, we have the defining diagram

$$\begin{tikzcd}
 & G_{N-k,k} \arrow[dl, "p_2"'] \arrow[dr, "s"] & \\
G_k & & G_N \\
 & G_{M-N,N-k,k} \arrow[uu, "p"] \arrow[dl, "s_2"'] \arrow[dr, "s_1"] & \\
G_{M-k,k} \arrow[uu, "\bar{p}_2"] \arrow[dr, "\bar{s}"'] & & G_{M-N,N} \arrow[uu, "\hat{p}_2"'] \arrow[dl, "\hat{s}"] \\
 & G_M &
\end{tikzcd}$$

and the commutative diagram over $G_{M-N,N-k,k}$

$$\begin{tikzcd}
p^*p_2^*D_k \arrow[r, "p^*\bar{\sigma}_{N-k,k}"] \arrow[d, "\cong"'] & p^*\Omega^{\gamma_{N-k}}s^*D_N \arrow[d, "\cong"] \\
s_2^*\bar{p}_2^*D_k \arrow[d, "s_2^*\bar{\sigma}_{M-k,k}"'] & \Omega^{\gamma_{N-k}}s_1^*\hat{p}_2^*D_N \arrow[d, "\Omega^{\gamma_{N-k}}s_1^*\bar{\sigma}_{M-N,N}"] \\
s_2^*\Omega^{\gamma_{M-k}}\bar{s}^*D_M \arrow[r, "\cong"] & \Omega^{\gamma_{N-k}}s_1^*\Omega^{\gamma_{M-N}}\hat{s}^*D_M
\end{tikzcd}$$

If we restrict this diagram along the natural inclusion

$$G_k(V)\cong\{W-V\}\times G_{N-k,k}(V)\to G_{M-N,N-k,k}$$

we obtain a map, induced from $\Omega^{\gamma_{N-k}}s_1^*\bar{\sigma}_{M-N,N}$,

$$\begin{tikzcd}
(J_VD)_k \arrow[r] \arrow[d] & (J_WD)_k \arrow[d] \\
G_k(V) \arrow[r] & G_k(W)
\end{tikzcd}$$

which forms the back square of the larger diagram

$$\begin{tikzcd}
& (J_VD)_k \arrow[rr] \arrow[dd] & & (J_WD)_k \arrow[dd] \\
D_k(V) \arrow[ur, "(\lambda_V)_k"] \arrow[rr] \arrow[dr] & & D_k(W) \arrow[ur, "(\lambda_W)_k"] \arrow[dr] & \\
& G_k(V) \arrow[rr] & & G_k(W)
\end{tikzcd}$$

We now pass to the colimit over all inclusions $V\subset W$ and define

$$(JD)_k=\operatorname*{colim}_V(J_VD)_k.$$

From our technical assumption for prespectra,

$$D_k=\operatorname*{colim}_V D_k(V),$$

so we obtain the diagram

$$\begin{tikzcd}
D_k \arrow[rr, "\lambda_k"] \arrow[dr] & & (JD)_k \arrow[dl] \\
& G_k &
\end{tikzcd}$$

It is now straightforward to verify that JD is a prespectrum and that λ is a map of prespectra with the required properties. $\square$

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