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ON THE ASPHERICITY OF ONE-POINT UNIONS OF CONES

KATSUYA EDA AND KAZUHIRO KAWAMURA

Abstract. We prove that the one-point union of two copies of the cone over the Hawaiian earring is aspherical.

1. Introduction and definitions

The one-point union $C\mathbb{H} \vee C\mathbb{H}$ of two copies of the cone over the Hawaiian earring $\mathbb{H}$ is not simply connected [9]. This is a well-known example of a non-contractible one-point union of two contractible spaces [15, p. 59]. The non-triviality of its fundamental group follows from the presentation of the group given by H. B. Griffiths in [10], a flaw in which was remedied in [13]. Another proof was suggested by R. H. Fox in his review of [9] and is proved in detail in [3, Theorem 2].

On the other hand, the Hawaiian earring and, more generally, every planar or one-dimensional space are aspherical in the sense that all homotopy groups of dimension at least 2 is trivial [17], [2],

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and [1]. In [4], the authors constructed a 2-dimensional, simply-connected, cell-like Peano continuum $SC(\mathbb{S}^1)$ such that the second homotopy group $\pi_2(SC(\mathbb{S}^1))$ is non-trivial. In [5], the authors demonstrated variants of $SC(\mathbb{S}^1)$- construction which, on one hand, produces a space homotopy equivalent to $SC(\mathbb{S}^1)$ [5, Theorem 4.3(2)] and, on the other hand, produces a space homotopy equivalent to $C\mathbb{H} \vee C\mathbb{H}$ [5, Theorem 4.3(3)]. This leads to a question of whether the space $C\mathbb{H} \vee C\mathbb{H}$ is aspherical. The present paper answers this question in the affirmative.

For a Hausdorff space X, CX denotes the cone over X

$$CX = X \times [0,1]/X \times \{1\},$$

with the quotient topology. The *peak point* of CX is the point represented by $X \times \{1\}$, and is denoted by p. The space X is identified with the subspace $X \times \{0\}$. Let X_0 and X_1 be two Hausdorff spaces with two points $o_0 \in X_0$ and $o_1 \in X_1$. For $i = 0, 1$, the peak point of CX_i is denoted by p_i. The one-point union $CX_0 \vee CX_1$ is the space obtained from the topological sum $CX_0 \oplus CX_1$ with the points o_0 and o_1 being identified with a point o.

Theorem 1.1. *Let X_0 and X_1 be one-dimensional compact metric spaces. Then $\pi_n(CX_0 \vee CX_1)$ is trivial for each $n \geq 2$.*

Consequently, we have an answer to the question above.

Corollary 1.2. *Let $\mathbb{H}_0$ and $\mathbb{H}_1$ be copies of the Hawaiian earring $\mathbb{H}$. Then $\pi_n(C\mathbb{H}_0 \vee C\mathbb{H}_1)$ is trivial for each $n \geq 2$.*

Since the cone construction makes a space contractible, it does not seem that "the coning" adds any complexity to one-point unions.

Question 1.3. Let X_0 and X_1 be path-connected (Hausdorff) spaces such that the n-th homotopy group $\pi_n(X_0 \vee X_1)$ is trivial. Then is the group $\pi_n(CX_0 \vee CX_1)$ also trivial?

At the time of this writing, we can answer the above question only for $n = 2$.

Theorem 1.4. *Let X_0 and X_1 be path-connected Hausdorff spaces such that the second homotopy group $\pi_2(X_0 \vee X_1)$ is trivial. Then the group $\pi_2(CX_0 \vee CX_1)$ is also trivial.*

All spaces are assumed to be Hausdorff and all maps are assumed to be continuous unless otherwise stated. The word "components" means "path-connected components." The reader is referred to [15] for undefined notions.

2. Proofs of theorems 1.1 and 1.4

Let K be a polyhedron with a triangulation $\mathcal{T}$. By abuse of notation, the subcomplex of $\mathcal{T}$ that defines a subpolyhedron L of K is denoted by the same symbol L. For an n-dimensional PL submanifold Q of $\mathbb{S}^n$ (with the standard triangulation), the manifold boundary of Q coincides with the topological boundary of Q in $\mathbb{S}^n$ and is denoted by ∂Q. Also, $\mathrm{Int}Q = Q \setminus \partial Q$.

The following result seems to be well known and a proof is provided for completeness of the argument. Let n be an integer such that $n \geq 2$. Note that, for $n = 2$, we make no assumption on the space X other than its path-connectivity.

Lemma 2.1. *Let X be a path-connected space with base point o such that $\pi_i(X, o) = 0$ for each $i = 2, \cdots, n-1$. Let P be a compact n-dimensional PL submanifold of $\mathbb{S}^n$ and let $f : P \to X$ be a map such that*

(1) *for each map $g : \mathbb{S}^1 \to \partial P$, the composition $f \circ g : \mathbb{S}^1 \to X$ is null homotopic.*

Then the map f admits an extension to a map $\bar{f} : \mathbb{S}^n \to X$.

Proof: Let $P_0, \cdots, P_k$ be the components of P and let $\{C_{ij} | j = 0, \cdots l_i\}$ be the components of ∂P_i. We take a sufficiently fine triangulation $\mathcal{T}$ of $\mathbb{S}^n$ such that

(2) each P_i is a subpolyhedron with respect to $\mathcal{T}$, and
(3) no 1-simplex of $\mathcal{T}$ connects distinct components C_{ij} and $C_{i'j'}$.

We define an extension $\bar{f}$ of f by an induction on the skeleton $\mathcal{T}^{(m)}$. At the outset, we fix a maximal tree $T_{ij} \subseteq C_{ij} \subseteq (\partial P_i)^{(1)}$ and a vertex $v_{ij} \in T_{ij}$ for each C_{ij}. Additionally, we choose and fix a path p_{ij} from $f(v_{ij})$ to o. For a 1-simplex with vertices u and v, (u, v) denotes the 1-simplex endowed with the orientation from u toward v.

Define $\bar{f}(v) = f(v)$ for each vertex $v \in P$ and $f(v) = o$ for $v \notin P$. For a 1-simplex $\sigma \notin P$ with vertices v_0 and v_1, we define $\bar{f}$ on σ as follows:

(1.1) if $\sigma \cap P = \emptyset$, then let $\bar{f}$ on σ be the constant map c_o to the point o, and

(1.2) if $v_0 \in C_{ij}$ and $v_1 \notin P$, take the unique path q_{v_0} in T_{ij} from v_0 to v_{ij} and let $\bar{f}|(v_0, v_1)$ be a map defined by the concatenation $(f \circ q_{v_0}) * p_{ij}$ of the paths $f \circ q_{v_0}$ (from $f(v_0)$ to $f(v_{ij})$) and p_{ij} (from $f(v_{ij})$ to o). Notice that $\bar{f}(v_0) = f(v_0)$ and $\bar{f}(v_1) = o$.

Next, we take a 2-simplex σ with vertices v_0, v_1, and v_2. If $\sigma \cap P = \emptyset$, then let $\bar{f}|\sigma$ be the constant map c_o. Assume that σ intersects with P.

(2.1) If $v_0, v_1 \notin P$ and $v_2 \in C_{ij}$, then the restriction $\bar{f}|\partial\sigma = f|(v_0, v_1, v_2)$ is null homotopic because it is represented by the concatenation $c_o * (f \circ q_{v_2} * p_{ij})^{-1} * (f \circ q_{v_2} * p_{ij})$. Thus, $\bar{f}|\partial\sigma$ admits an extension on σ.

(2.2) If $v_0 \notin P$ and $v_1, v_2 \in C_{ij}$, then let $g : \partial\sigma \to C_{ij} \subset P_i \subset P$ be a map defined by the loop $q_{v_1}^{-1} * (v_1, v_2) * q_{v_2}$ at v_{ij}. Then $f|\partial\sigma$ is a map defined by the path $p_{ij}^{-1} * (f \circ q_{v_1})^{-1} * f|(v_1, v_2) * (f \circ q_{v_2}) * p_{ij}$ which is freely homotopic to the map $f \circ (q_{v_1}^{-1} * (v_1, v_2) * q_{v_2}) \simeq f \circ g \simeq 0$ by the hypothesis (1). Hence, $f|\partial\sigma$ is null homotopic and it extends to a map on σ.

The above completes an extension procedure of f to the 2-skeleton $\mathcal{T}^{(2)}$ and thus completes the proof for $n = 2$. For $n > 2$, we can make use of the triviality of $\pi_i(X, o)$ to continue the extension process and, at the n-th step, obtain the desired extension $\bar{f}$ on $\mathbb{S}^n$. $\square$

The proof of Theorem 1.1 relies on the following lemma. The idea of using the monotone-light factorization theorem is due to M. L. Curtis and M. K. Fort, Jr. [2] and was applied in [6]. A *local dendrite* (a *dendrite*, respectively) is a one-dimensional locally connected compact connected metric space containing at most finitely many (no, respectively) simple closed curves. A map $h : S \to T$ between compact metric spaces is said to be *monotone* (*light*, respectively) if every point inverse of h is connected (zero-dimensional, respectively).

Lemma 2.2. *Let $f : N \to X$ be a map of a compact polyhedron N to a compact metric space X such that* $\dim X \leq 1$. *Then there exist* *a compact metric space G and maps $m : N \to G$ and $l : G \to X$ such that*

1. $f = l \circ m$,
2. *the map m is monotone and the map l is light, and*
3. *the space G has finitely many components, each of which is a local dendrite or a singleton.*

Proof: Applying the monotone-light factorization [16, Chap. VIII, section 4] to the map f, we find a monotone map $m : N \to G$ and a light map $l : G \to X$ satisfying conditions (1) and (2). We show that the space G satisfies condition (3). Since l is a light map, by [8, Theorem 3.3.10] and the hypothesis, we see $\dim G \leq \dim X + 0 = 1$. By the monotonicity of m, every component of N is of the form $m^{-1}(S)$ where S is a component of G. The space N has finitely many components and so does G. Enumerate the components of G as $\{G_j\}$ and let $N_j = m^{-1}(G_j)$. Each N_j is a component of N and the restriction $m|N_j : N_j \to G_j$ is monotone. By the Hahn-Mazurkiewicz Theorem, G_j, as a continuous image of a locally connected compact connected metric space N_j, is locally connected. Furthermore, by the monotonicity of $m|N_j$, the induced homomorphisms $(m|N_j)^* : \check{\mathrm{H}}^1(G_j; \mathbb{Z}) \to \check{\mathrm{H}}^1(N_j; \mathbb{Z})$ is a monomorphism [12] to a finitely generated abelian group. Hence, $\check{\mathrm{H}}^1(G_j; \mathbb{Z})$ is finitely generated. By [11, section 52], every one-dimensional locally connected compact connected metric space with finitely generated first Čech cohomology is a local dendrite. Hence, we obtain the desired conclusion (3). □

Proof of Theorem 1.1: Fix an integer $n \geq 2$ and take a map $f : \mathbb{S}^n \to CX_0 \vee CX_1$. Notice that the set $f^{-1}(CX_0 \vee CX_1 \setminus \{o\})$ consists of at most countably many connected components, each of which is open in $\mathbb{S}^n$. Among these components, at most finitely many of them meet $f^{-1}(\{p_0, p_1\})$ and neither of them intersects both of $f^{-1}(p_0)$ and $f^{-1}(p_1)$.

We construct a map $g : \mathbb{S}^n \to CX_0 \vee CX_1$ such that

1. g is homotopic to f, and
2. $g(\mathbb{S}^n) \subset CX_0 \vee CX_1 \setminus \{p_0, p_1\}$.

Let us assume, for a moment, that we have the above map g. Since $X_0 \vee X_1$ is a strong deformation retract of $CX_0 \vee CX_1 \setminus \{p_0, p_1\}$, g is homotopic to a map from $\mathbb{S}^n$ to $X_0 \vee X_1$. Since $\dim(X_0 \vee X_1) = 1$,

$\pi_n(X_0 \vee X_1)$ is trivial by [2] and [1] and hence, g is null homotopic. Consequently, we conclude that f is null homotopic, as desired.

The map g and the homotopy between f and g are defined on each component of $f^{-1}(CX_0 \vee CX_1 \setminus \{o\})$. If a component O does not meet $f^{-1}(\{p_0, p_1\})$, then $g|O = f|O$ and the homotopy $H_O : O \times [0,1] \to CX_0 \vee CX_1$ is given by $H(x,t) = f(x)$ for each point $(x,t) \in O \times [0,1]$.

Next, take a component O of $f^{-1}(CX_0 \vee CX_1 \setminus \{o\})$ such that $O \cap f^{-1}(\{p_0, p_1\}) \neq \emptyset$. Without loss of generality, we may assume that $O \cap f^{-1}(p_0) \neq \emptyset = O \cap f^{-1}(p_1)$. Take a compact PL submanifold N of $\mathbb{S}^n$ such that $\mathbb{S}^n \setminus O \subset \mathrm{Int} N$ and $N \cap (f^{-1}(\{p_0\}) \cap O) = \emptyset$. Define a map $f_O : \mathbb{S}^n \to CX_0$ by $f_O(x) = f(x)$ for $x \in O$ and $f_O(x) = o$ otherwise. Let $r : CX_0 \vee CX_1 \setminus \{p_0, p_1\} \to X_0 \vee X_1$ be the standard retraction which is a homotopy equivalence.

Applying Lemma 2.2 to the composition $r \circ f_O|N : N \to X_0 \vee X_1$, we obtain a compact metric space G, a monotone map $m : N \to G$, and a light map $l : G \to X_0 \vee X_1$ such that $r \circ f_O|N = l \circ m$ and

(3) the space G has finitely many components G_j, each of which is a local dendrite or a singleton.

Let $C_j = l^{-1}(\{o\}) \cap G_j$ and note $l^{-1}(o) = \bigcup_j C_j$. Since $\dim C_j = 0$, the above condition (3) implies that there exists a closed neighborhood D_j of C_j such that D_j is the disjoint union of finitely many dendrites, each of which intersects with C_j. In particular, D_j contains no simple closed curve and hence,

(4) the inclusion $i_j : D_j \to G_j$ is null homotopic.

Observe that $\mathbb{S}^n \setminus O \subseteq (f_O|N)^{-1}(\{o\}) \subseteq (r \circ f_O|N)^{-1}(\{o\}) = (l \circ m)^{-1}(\{o\})$. There exists a compact PL submanifold P of N with the components $P_0, \cdots, P_k$ such that $\overline{\mathbb{S}^n \setminus P}$ is a PL submanifold and also

(5) $\mathbb{S}^n \setminus O \subseteq (l \circ m)^{-1}(\{o\}) \subseteq \mathrm{Int} P$, $(l \circ m)^{-1}(\{o\}) \cap P_i \neq \emptyset$ for each $i = 0, \cdots, k$, and each $m(P_i)$ is a subset of some D_j.

Then, for each $h : \mathbb{S}^1 \to P_i$, the composition $r \circ (f_O|P) \circ h = l \circ m \circ h$ is null homotopic by (4). Since r is a homotopy equivalence, the map $(f_O|P) \circ h$ is null homotopic as well. By Lemma 2.1, $f_O|P$ extends to a map $g_0 : \mathbb{S}^n \to CX_0 \setminus \{p_0\}$. Define $g_1 : \mathbb{S}^n \to CX_0 \vee CX_1$ by $g_1(x) = g_0(x)$ for $x \in O$ and $g_1(x) = f(x)$ otherwise. Then $g_1 \,|\, P = f \,|\, P$. Since P and $\overline{\mathbb{S}^n \setminus P}$ are compact PL-submanifolds and CX_0 is contractible, we see that f and g_1 are homotopic relative to P and $g_1(\bar{O}) \cap \{p_0, p_1\} = \emptyset$.

We iterate this procedure for every component O of $f^{-1}(CX_0 \vee CX_1 \setminus \{o\})$. The continuity of f implies that there are at most finitely many such components. Carrying out all these procedures, we obtain the desired map $g : \mathbb{S}^n \to CX_0 \vee CX_1 \setminus \{p_0, p_1\}$, satisfying conditions (1) and (2). □

The next lemma is for the proof of Theorem 1.4.

Lemma 2.3. *Let K_0, K_1 be disjoint closed subsets of $\mathbb{S}^2$ and X be a path-connected space with a point $o \in X$ specified. There exists a compact surface $P \subset \mathbb{S}^2$ with boundary such that*

(1) $K_0 \subset \mathrm{Int}P$ *and* $K_1 \cap P = \emptyset$,
(2) *each component of the boundary ∂P is a polygonal simple closed curve, and*
(3) *for each map $f : P \to X$ with $f(K_0) = \{o\}$, the restriction $f|\partial P : \partial P \to X$ is null homotopic.*

Proof: Take a compact surface P satisfying (1) and (2) above and let $P_0, \cdots, P_k$ be the components of P. We may assume that

(4) each component of $\mathbb{S}^2 \setminus (K_0 \cap P_i)$ contains at most one component of $\mathbb{S}^2 \setminus P_i$ for every i.

Indeed, if a component of $\mathbb{S}^2 \setminus (K_0 \cap P_i)$ contains two components of $\mathbb{S}^2 \setminus P_i$, then by cutting open P_i along an arc connecting these components, we have a smaller neighborhood $P_i' \subset P_i$ of $P_i \cap K_0$ so that these components are contained in a single component of $\mathbb{S}^2 \setminus P_i'$. Iterating this procedure, we can make P satisfy condition (4).

For a map $f : P \to X$ satisfying the hypothesis of (3), we show that $f|\partial P_i$ is null homotopic for each component P_i, which follows from

$$f|C \text{ is null homotopic for each component } C \text{ of } \partial P_i.$$

Let O be a component of $\mathbb{S}^2 \setminus K_0$ which intersects with the component of $\mathbb{S}^2 \setminus P_i$ whose boundary is equal to C. The curve C divides $\mathbb{S}^2$ into two components. Let U be the component of $\mathbb{S}^2 \setminus C$ containing $\mathrm{Int}(P_i)$.

The closure $\overline{U} = U \cup C$ is the closed disk such that $\overline{U} \supset P_i \cap O$. Define $g : \overline{U} \to X$ by

$$g(u) = \begin{cases} f(u) & \text{for } u \in P_i \cap O, \\ o & \text{for } u \in U \setminus O. \end{cases}$$

By (4), g is actually defined on $\overline{U}$ and is a continuous extension of $f|C$ and hence, $f|C$ is null homotopic. □

Proof of Theorem 1.4: Let $f : \mathbb{S}^2 \to CX_0 \vee CX_1$ be a map. As in the proof of Theorem 1.1, we construct a map $g : \mathbb{S}^2 \to CX_0 \vee CX_1$ such that

(i) $g(\mathbb{S}^2) \subset CX_0 \vee CX_1 \setminus \{p_0, p_1\}$, and
(ii) $g \simeq f : \mathbb{S}^2 \to CX_0 \vee CX_1$.

Having constructed such a map g, the proof is completed as follows: Let $r : CX_0 \vee CX_1 \setminus \{p_0, p_1\} \to X_0 \vee X_1$ be the standard retraction. Then the hypothesis $\pi_2(X_0 \vee X_1) = 0$, together with (ii), implies $f \simeq g \simeq r \circ g \simeq 0$.

Choose a component O of $f^{-1}(CX_0 \vee CX_1 \setminus \{o\})$ such that $O \cap f^{-1}(\{p_0, p_1\}) \neq \emptyset$ and assume, without loss of generality, $O \cap f^{-1}(\{p_0\}) \neq \emptyset$. We construct a map $g_1 : \mathbb{S}^2 \to CX_0 \vee CX_1$ such that

(1) g_1 is homotopic to f, and
(2) $g_1|\mathbb{S}^2 \setminus O = f|\mathbb{S}^2 \setminus O$ and $g_1(\overline{O}) \cap \{p_0, p_1\} = \emptyset$.

First we apply Lemma 2.3 to $K_0 = \mathbb{S}^2 \setminus O, K_1 = f^{-1}(\{p_0\}) \cap O$ and obtain a compact surface P with polygonal boundary such that $\mathbb{S}^2 \setminus O \subset \mathrm{Int}P$ and

(3) for each map $\varphi : P \to CX_0 \setminus \{p_0\}$ with $\varphi(\mathbb{S}^2 \setminus O) = \{o\}$, the restriction $\varphi|\partial P : \partial P \to X$ is null homotopic.

Define $f_O : P \to CX_0$ by

$$f_O(x) = \begin{cases} f(x) & \text{for } x \in P \cap O, \\ o & \text{for } x \in \mathbb{S}^2 \setminus O. \end{cases}$$

Condition (3) above guarantees that $f_O : P \to CX_0 \setminus \{p_0\}$ satisfies hypotheses (1) and (2) of Lemma 2.1 and hence admits an extension $g_0 : \mathbb{S}^2 \to CX_0 \setminus \{p_0\}$. Define $g_1 : \mathbb{S}^2 \to CX_0 \vee CX_1$ by $g_1(x) = g_0(x)$ for $x \in O$ and $g_1(x) = f(x)$ otherwise. Then $g_1 \,|\, P = f \,|\, P$. Since P and $\overline{\mathbb{S}^2 \setminus P}$ are compact surfaces and CX_0 is contractible, we see that f and g_1 are homotopic relative to P. By the definition, we also have $g_1(\bar{O}) \cap \{p_0, p_1\} = \emptyset$.

As in the proof of Theorem 1.1, we obtain the desired map $g : \mathbb{S}^2 \to CX_0 \vee CX_1 \setminus \{p_0, p_1\}$ by iterating the above procedure at most finitely many times on each component of $f^{-1}(CX_0 \vee CX_1 \setminus \{o\})$ such that $O \cap f^{-1}(\{p_0, p_1\}) \neq \emptyset$. □

For more information on homology and homotopy groups on one-point unions of cones, see [3] and [7].

3. Remark on Lemma 2.3

The proof of Lemma 2.3 shows the following: for each compact subset K_0 of $\mathbb{R}^2$ and for each neighborhood U of K_0, there exists a compact surface P such that

(1) $K_0 \subset \mathrm{Int} P \subseteq U$, and
(2) for each map $f : P \to X$ with $f(K_0) = \{o\}$, the restriction $f|\partial P : \partial P \to X$ is null homotopic.

The following result illustrates that the 3-dimensional analogue of the above result does not hold. This is the main technical obstacle to answering Question 1.3 in its full generality.

Proposition 3.1. *Let ST be a solid torus in $\mathbb{R}^3$ which contains Antoine's necklace K_0 in its interior in the standard way* [14, pp. 71-72]. *Let P be a compact 3-manifold-neighborhood of K_0 in ST and Y_0 be the quotient space P/K_0 with the quotient map $q : P \to Y_0$. Then, the restriction $q|\partial P : \partial P \to Y_0$ is not null-homotopic.*

To prove the above, it is convenient to make the following lemma.

Lemma 3.2. *Let X be a simply-connected PL manifold and Y be a connected PL submanifold of X. Then for each component Z of $X \setminus Y$, the topological boundary of Z is path-connected.*

Proof: It suffices to verify the conclusion when $\dim Y = \dim X$. Suppose the topological boundary ∂Z of Z is not path-connected. Then we have two points p and q in ∂Z which are not joined by arcs in ∂Z. We have, on one hand, an arc A in Y connecting p and q and, on the other hand, an arc B in $\overline{Z}$ connecting p and q such that $A \cap B = \{p, q\}$. The union $A \cup B$ is a simple closed curve in X which is not null homotopic. This contradicts the assumption. □

Proof of Proposition 3.1: For simplicity, $ST \setminus K_0$ is regarded as a subspace of Y_0 via the homeomorphism $q|ST \setminus K_0$. Let y_0 be the point with $\{y_0\} = q(K_0)$.

Suppose that there exists a homotopy $H : \partial P \times \mathbb{I} \to Y_0$ such that

$$H(x,0) = x \quad \text{and} \quad H(x,1) = y_0 \quad \text{for } x \in \partial P.$$

Let S be a component of ∂P. By making use of the homotopy $H|S \times \mathbb{I}$ between the inclusion $S \to Y_0$ and the constant map to y_0, we show that

($*$) there exists a homotopy $\overline{H} : S \times \mathbb{I} \to P$ between the inclusion $S \to P$ to a constant map.

Take the component O of $(H|S \times \mathbb{I})^{-1}(Y_0 \setminus \{y_0\})$ which contains $S \times \{0\}$. Define

$$H_0(x,t) = H(x,t) \text{ if } (x,t) \in O, \quad H_0(x,t) = y_0, \text{ otherwise.}$$

Then H_0 is also a homotopy from the inclusion $S \to Y_0$ to the constant map. Hence, we assume that $O := (H|S \times \mathbb{I})^{-1}(Y_0 \setminus \{y_0\})$ is connected and let $C = S \times \mathbb{I} \setminus O = (H|S \times \mathbb{I})^{-1}(\{y_0\})$. Then we have $S \times \{0\} \subseteq O$ and $S \times \{1\} \subseteq C$. In the next lemma, $H|S \times \mathbb{I}$ is abbreviated to H.

Lemma 3.3. *Let C_0 be a component of C. Then there exists a unique $u \in K_0$ such that $H_1 : O \cup C_0 \to P$ defined by $H_1|O = H|O$ and $H_1(x,t) = u$ for $(x,t) \in C_0$ is continuous.*

Proof: We show that there exists a unique point $u \in K_0$ such that for each sequence $\{p_n\} \subset O$ with $\lim_{n\to\infty} p_n \in \partial C_0$, the sequence $\{H(p_n)\}$ accumulates to u. It is easily seen that u is the desired point.

To show this by contradiction, suppose there exist two points $a, b \in \partial C_0$ and sequences $\{a_n\}, \{b_n\} \subset O$ such that $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$, and further, $\lim_{n\to\infty} H(a_n)$ and $\lim_{n\to\infty} H(b_n)$ are distinct points of K_0. Since K_0 is 0-dimensional, we have open sets U and V in P such that $\lim_{n\to\infty} H(a_n) \in U$, $\lim_{n\to\infty} H(b_n) \in V$, $U \cap V = \emptyset$, and $K_0 \subseteq U \cup V$. There exists a PL-manifold-neighborhood P_1 of C such that $H(P_1) \subseteq \{y_0\} \cup (U \setminus K_0) \cup (V \setminus K_0)$. Let P_2 be the component of P_1 containing C_0. Choose a_n and b_n so that $a_n, b_n \in P_2$. Notice that

($\sharp$) there is no arc connecting a_n and b_n in $P_2 \cap O$,

because $H(P_2 \cap O) \subseteq (U \cup V) \setminus K_0$. In other words, C separates the connected manifold P_2.

As S is a surface in ST, $S \times \mathbb{I}$ is naturally embedded in $\mathbb{R}^3$ by "thickening S." Under this embedding, the topological boundary of $S \times \mathbb{I}$ in $\mathbb{R}^3$ is $S \times \{0,1\}$. We apply Lemma 3.2 to $X = \mathbb{R}^3$ and $Y = P_2$. If P_2 does not intersect with $S \times \{1\}$, then the topological boundary of P_2 in $\mathbb{R}^3$ is contained in O. If P_2 meets $\partial P \times \{1\}$, then it contains $S \times \{1\}$ and the topological boundary of P_2 in $\mathbb{R}^3$ is contained in the disjoint union of $S \times \{1\}$ and O. Hence, we have the following remark:

> $S \times \mathbb{I} \setminus P_2$ consists of finitely many components and the topological boundary of each component in $S \times \mathbb{I}$ is path-connected and is contained in O.

We have a polygonal arc A in O which connects a_n and b_n. There exist finitely many pairwise disjoint subarcs $B_1, \ldots, B_r$ of A such that the endpoints of each B_j belong to ∂P_2, each B_j is contained in the union of P_2 and a unique component of the complement of P_2, and $A \setminus \cup B_j \subset P_2$. By the preceding remark, for each B_j, we have an arc on the boundary of P_2 which connects the endpoints of B_j. Hence, we obtain an arc in $P_2 \cap O$ connecting a_n and b_n, which contradicts ($\sharp$). □

Proof of Proposition 3.1 (continued): Applying the above lemma to each component of C, we have a map $\overline{H}_S : S \times \mathbb{I} \to P$ such that $\overline{H}_S | O \cup C_0$ is continuous for each component C_0 of C. To see the continuity of $\overline{H}_S$ on $S \times \mathbb{I}$, it suffices to show the following.

(∗∗) Let $\{p_n\}$ be a sequence of C such that $\lim_{n\to\infty} p_n = p \in C$ and let C_n (C_0, respectively) be the component of C containing p_n (p, respectively). Take the unique points u_n for C_n and u for C_0 as in the previous lemma. Then $\lim_{n\to\infty} u_n = u$.

To show the above, we may assume that $p_n \in \partial C_n$ and $p \in \partial C$. By the definition of $\overline{H}_S$, $\overline{H}_S(p_n) = u_n$ and $\overline{H}_S(p) = u$. By the continuity of $\overline{H}_S | O \cup C_n$, we may take $a_n \in O$ so close to p_n that $\lim_{n\to\infty} a_n = p$ and $\lim_{n\to\infty} \overline{H}_S(a_n) = \lim_{n\to\infty} \overline{H}_S(p_n)$. Then, by the uniqueness of u, we obtain $\lim_{n\to\infty} \overline{H}_S(a_n) = u$. Thus, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \overline{H}_S(p_n) = \lim_{n\to\infty} \overline{H}_S(a_n) = u$. This proves (∗∗) and hence completes the proof of (∗).

Taking the union $\overline{H} := \cup \overline{H}_S$ over all components S of ∂P, we have a homotopy $\overline{H} : \partial P \times \mathbb{I} \to P$ such that $\overline{H}(x, 0) = x, x \in \partial P$, $\overline{H}(\partial P \times \{1\}) \subseteq K_0$. It is easy to see that for each component P_0 of P, we have $\overline{H}(\partial P_0 \times \mathbb{I}) \subseteq P_0$.

By the construction of Antoine's necklace, there exists a component P_0 of P such that the inclusion $\partial P_0 \to P_0$ is not null-homotopic. Then there exists a component S of ∂P_0 such that the inclusion $S \to P_0$ is not null-homotopic. However, the restriction $\overline{H}|S \times \mathbb{I}$ provides a homotopy between the inclusion $S \to P_0$ and a constant map because $\overline{H}(S \times \{1\})$, as a connected set of the zero-dimensional K_0, is a singleton. This contradiction completes the proof of the proposition. □

For the tame Cantor set K in $\mathbb{R}^3$, there exists an arbitrarily small neighborhood P of K which is the disjoint union of 3-balls. For the quotient map $q : P \to P/K$, the restriction $q|\partial P : \partial P \to P/K$ is null-homotopic, since the restriction $q|\partial P_0$ is easily seen to be null-homotopic for each component P_0 of P.

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