THE SINGULAR HOMOLOGY OF THE HAWAIIAN EARRING

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1. *Introduction*

The singular homology groups of compact CW-complexes are finitely generated, but the groups of compact metric spaces in general are very easy to generate infinitely and our understanding of these groups is far from complete even for the following compact subset of the plane, called the *Hawaiian earring*:

$$\mathbb{H} = \bigcup_{m=1}^{\infty} C_m, \quad \text{where } C_m = \left\{(x,y)\in \mathbf{R}^2 \mid \left(x-\frac{1}{m}\right)^2 + y^2 = \frac{1}{m^2}\right\}.$$

Griffiths [**11**] gave a presentation of the fundamental group of $\mathbb{H}$ and the proof was completed by Morgan and Morrison [**15**]. The same group is presented as the free σ-product $\circledast^{\sigma}_{\mathbb{N}}\mathbb{Z}$ of integers $\mathbb{Z}$ in [**4**, Appendix]. Hence the first integral singular homology group $\mathrm{H}_1(\mathbb{H})$ is the abelianization of the group $\circledast^{\sigma}_{\mathbb{N}}\mathbb{Z}$. These results have been generalized to non-metrizable counterparts $\mathbb{H}_I$ of $\mathbb{H}$ (see Section 3).

In Section 2 we prove that $\mathrm{H}_1(X)$ is torsion-free and $\mathrm{H}_i(X) = 0$ for each one-dimensional normal space X and for each $i \geqslant 2$. The result for $i \geqslant 2$ is a slight generalization of [**2**, Theorem 5]. In Section 3 we provide an explicit presentation of $\mathrm{H}_1(\mathbb{H})$ and also $\mathrm{H}_1(\mathbb{H}_I)$ by using results of [**4**].

Throughout this paper, a *continuum* means a compact connected metric space and all maps are assumed to be continuous. All homology groups have the integers $\mathbb{Z}$ as the coefficients. The bouquet with n circles $\bigcup_{j=1}^{n} C_j$ is denoted by B_n. The base point $(0,0)$ of B_n is denoted by o for simplicity.

2. *The singular homology groups of one-dimensional continua*

The purpose of this section is to prove the following theorem.

THEOREM 2.1. *For each one-dimensional normal space X,* $\mathrm{H}_1(X)$ *is torsion-free and* $\mathrm{H}_i(X) = 0$ *for each $i \geqslant 2$.*

COROLLARY 2.2. $\mathrm{H}_1(\mathbb{H})$ *is torsion-free and* $\mathrm{H}_i(\mathbb{H}) = 0$ *for each $i \geqslant 2$.*

The second statement of Theorem 2.1 has been proved by Curtis and Fort for one-dimensional separable metrizable spaces [**2**, Theorem 5]. Their proof makes use of some results on the universal Menger curve (see [**2**, Sections 3 and 4]). Theorem 2.1 slightly generalizes their result with a somewhat direct proof. The idea of using the monotone–light factorization theorem (Theorem 2.5) and making use of the one-dimensionality of X comes from [**1**].

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We give a unified proof of Theorem 2.1 based on the following well-known theorem, the proof of which will be outlined below for completeness. The authors are grateful to the referee for suggesting the unified proof. For a (singular) cycle z of a space X, $[z]$ denotes the homology class represented by z.

THEOREM 2.3. *Let X be a space and let z be an i-cycle of X. There exists a compact CW-complex K, a map $f: K \longrightarrow X$, and an i-cycle w such that $f_*([w]) = [z]$. In addition, if $n[z] = 0 \in H_i(X)$ for some integer n, then K and w can be chosen so that $n[w] = 0 \in$ $\mathrm{H}_i(K)$.*

Proof (*sketch*). Let $\varphi: W \longrightarrow X$ be a map of a CW-complex W which induces isomorphisms between all homotopy groups (see, for example, [**17**, Chapter V, Theorem 3.2]). The map φ induces isomorphisms on all homology groups as well. Let w be an i-cycle of W such that $\varphi_*([w]) = [z]$. Take a compact CW-complex K which supports the cycle w. Then for the restriction $f = \varphi \mid K$, we have $f_*([w]) = [z]$. This completes the proof of the first statement.

To prove the second, let us assume that $n \cdot [z] = 0$. Then for the above cycle w in W, we have $\varphi_*(n \cdot [w]) = n \cdot [z] = 0$ and hence $n \cdot [w] = 0 \in \mathrm{H}_i(W)$. Let v be an $(i+1)$-chain which is bounded by $n \cdot w$ and take a CW-complex K' which supports the chains v and $n \cdot w$. Clearly $[n \cdot w] = 0 \in \mathrm{H}_i(K')$ and the homomorphism induced by $f' = \varphi \mid K'$ maps $[w]$ to $[z]$. This completes the proof. □

A couple of auxiliary results are necessary for the proof of Theorem 2.1.

PROPOSITION 2.4. *Let X be a locally connected one-dimensional continuum with a finitely generated first Čech cohomology group. Then X has the homotopy type of the bouquet B_n for some n.*

Proof. Since $\dim X = 1$ and $\check{\mathrm{H}}^1(X)$ is finitely generated, X contains only a finite number of simple closed curves, that is, X is a local dendrite [**13**, Section 52]. By [**13**, Theorem 8, p. 306], X has the homotopy type of a compact connected one-dimensional polyhedron which is homotopy equivalent to B_n for some n. □

A map $f: X \longrightarrow Y$ between continua is said to be *monotone* (respectively *light*) if each fibre of f is connected (respectively zero-dimensional). The factorization described in the next theorem is called the *monotone–light factorization* of a map.

THEOREM 2.5 [**18**, p. 141]. *For any map $f: X \longrightarrow Y$ between continua, there exist a continuum Z, a monotone map $m: X \longrightarrow Z$ and a light map $l: Z \longrightarrow Y$ such that $f = l \cdot m$.*

The next theorem follows from [**14**].

THEOREM 2.6. *If $f: X \longrightarrow Y$ is a monotone map, then it induces a monomorphism $f^*: \check{\mathrm{H}}^1(Y) \longrightarrow \check{\mathrm{H}}^1(X)$.*

Proof of Theorem 2.1. Since $\mathrm{H}_i(X)$ is the direct sum of the ith homologies of path components of X, we may assume that X is connected.

Case 1: $\mathrm{H}_1(X)$ is torsion-free.

Let z be a 1-cycle of X and assume that $n\cdot[z]=0\in \mathrm{H}_1(X)$. Apply Theorem 2.3 to obtain a compact CW-complex K, a map $f\colon K\longrightarrow X$ and a cycle w in K such that $f_*([w])=[z]$ and $n\cdot[w]=0\in \mathrm{H}_1(K)$. Notice that K is a compact metric space.

By the monotone–light factorization theorem (Theorem 2.5), the map f is factorized as $h=v\cdot u$, where $u\colon K\longrightarrow G$ is a monotone map onto a continuum G and $v\colon G\longrightarrow X$ is a light map. Since $v(G)$ is a compact metrizable space (because so is K), by [**8**, Theorem 3.3.10] we see that

$$1\leqslant \dim G\leqslant \dim v(G)+\sup\{\dim v^{-1}(x)\mid x\in X\}\leqslant \dim X\leqslant 1$$

and G is locally connected as a continuous image of a locally connected continuum K (the Hahn–Mazurkiewicz theorem). Applying Theorem 2.6 to the monotone map u, we see that $\check{\mathrm{H}}^1(G)$ is isomorphic to a subgroup of $\check{\mathrm{H}}^1(K)$ and hence is finitely generated. Proposition 2.4 implies that G has the same homotopy type as B_m for some m, and in particular, $\mathrm{H}_1(G)$ is torsion-free.

Since $n\cdot u_*([w])=u_*(n\cdot[w])=0$, we have $u_*([w])=0$. Thus $[z]=f_*([w])=v_*(u_*([w]))=0$. This completes the proof.

Case 2: $\mathrm{H}_i(X)=0$ for each $i\geqslant 2$.

Take an element $[z]\in \mathrm{H}_i(X)$ for $i\geqslant 2$. By Theorem 2.3, there exists a compact CW-complex K and a map $f\colon K\longrightarrow X$ such that $[z]=f_*([w])$ for some $[w]\in H_i(K)$. Apply Theorem 2.5 to obtain a monotone map $u\colon K\longrightarrow G$ and a light map $v\colon G\longrightarrow X$ such that $f=v\cdot u$ and G is a locally connected continuum. By the same argument as in case (1), we see that G has the same homotopy type of B_m for some m and hence $\mathrm{H}_i(G)=0$. Therefore $[z]=f_*([w])=v_*\cdot v_*([w])=0$. □

REMARK 2.7. Since $H_1(\mathbb{H})$ is isomorphic to the abelianization of $\circledast_{\mathbb{N}}\mathbb{Z}$ by Theorem 3.2, Corollary 2.2 answers the first half of [**4**, Question 4.20] affirmatively. Though the question itself is formulated algebraically, we have not succeeded in finding a purely algebraic proof of it.

3. *An explicit presentation of the first singular homology group of the Hawaiian earring*

The goal of this section is to prove Theorem 3.1. For an index set I, $\mathbb{H}_I$ denotes the compact Hausdorff space which is, as a set, obtained from I-many copies, denoted by C_i ($i\in I$), of the circle $\{(x,y):(x-1)^2+y^2=1\}$ by identifying all points corresponding to $(0,0)$ to a single point o. The topology is described by specifying neighbourhood bases. The neighbourhood base of each point $x\neq o$ is the same as the standard base of the circle, while each member of the neighbourhood base of o is of the form $\bigcup_{i\in F}U_i\cup\bigcup_{j\in I\setminus F}C_i$, where F is a finite subset of I and U_i is a neighbourhood of $(0,0)$ in C_i with respect to the standard topology of the circle.

The subgroup of the direct product $\prod_I\mathbb{Z}$ consisting of all functions with countable support is denoted by $\prod_I^{\sigma}\mathbb{Z}$.

THEOREM 3.1. *Let I be an index of infinite cardinality and let κ be the cardinality of $I^{\mathbb{N}}$. Then, $\mathrm{H}_1(\mathbb{H}_I)$ is isomorphic to*

$$\textstyle\prod_I^{\sigma}\mathbb{Z}\oplus\prod_{p:\mathrm{prime}}A_p\oplus\bigoplus_{\kappa}\mathbb{Q},$$

where A_p is isomorphic to the p-adic completion of $\bigoplus_{\kappa}\mathbb{J}_p$ and $\mathbb{J}_p$ is the group of p-adic integers.

For the proof of this theorem, we recall some algebraic notions and results from [**9**, VII] and [**4**]. Undefined notions are standard and are found in [**9**] or [**12**]. For groups G_i $(i \in I)$, the free product is denoted by $*_{i\in I} G_i$. The notation $X \Subset Y$ means that 'X is a finite subset of Y'. For $F, G \Subset I$ with $F \subset G$, let $p_{FG}: *_{i\in G} G_i \longrightarrow *_{i\in F} G_i$ be the canonical projection. Next we define the free σ-product of G_i, assuming that $G_i \cap G_j = \{e\}$ for distinct $i, j \in I$. For each $F \Subset I$, the canonical projection from the inverse limit $\lim_{\leftarrow}(*_{i\in F} G_i, p_{FG}: F \subset G \Subset I)$ to $*_{i\in F} G_i$ is denoted by p_F. The free σ-product $※^{\sigma}_{i\in I} G_i$ is the subgroup of $\lim_{\leftarrow}(*_{i\in F} G_i, p_{FG}: F \subset G \Subset I)$ consisting of all elements x with the following property:

> There exists a countable linearly ordered set $\bar{W}$ and a map $W: \bar{W} \longrightarrow \bigcup\{G_i : i \in I\}$ (called the word of countable length that corresponds to x) such that $W^{-1}(G_i)$ is finite for each $i \in I$ and $p_F(x)$ is equal to the word obtained by restricting W to the set $\bigcup\{G_i : i \in F\}$ for each $F \Subset I$ [**4**].

We refer the reader to [**4**, **5**] for more information on free σ-products. (In [**4**], the free complete product $※_{i\in I} G_i$ of G_i is defined. When I is countable, we have $※_{i\in I} G_i = ※^{\sigma}_{i\in I} G_i$.) For a group G, the commutator subgroup of G is denoted by G'. The abelianization G/G' of G is denoted by $\mathrm{Ab}(G)$. Let $\prod^{\sigma}_{i\in I} G_i$ be the subgroup of the direct product $\prod_{i\in I} G_i$ consisting of $x \in \prod_{i\in I} G_i$ such that $\{i \in I : x(i) \neq e\}$ is countable. There exists a canonical surjection $※^{\sigma}_{i\in I} G_i$ onto $\prod^{\sigma}_{i\in I} G_i$ which induces a canonical surjection $\sigma: \mathrm{Ab}(※^{\alpha}_{i\in I} G_i) \longrightarrow \prod^{\sigma}_{i\in I} G_i$ if G_i is abelian for each i.

For an abelian group A, the Ulm subgroup is $U(A) = \{a \in A : n \mid a \text{ for every } n \in \mathbb{N}\}$. When A is torsion-free, $U(A)$ is the maximal divisible subgroup of A. An abelian group A is said to be complete modulo the Ulm subgroup, if for each sequence $x_n \in A$ $(n \in \mathbb{N})$ with $n! \mid x_{n+1} - x_n$ there exists $x \in A$ such that $n! \mid x - x_n$ for all $n \in N$, that is, if $A/U(A)$ is complete [**9**]. It is known that A is algebraically compact if and only if $UU(A) = U(A)$ and A is complete modulo the Ulm subgroup [**3**]. If A is torsion-free, then $UU(A) = U(A)$ always holds. Consequently, each torsion-free, complete modulo the Ulm subgroup, abelian group is algebraically compact.

THEOREM 3.2 (H. B. Griffiths, J. W. Morgan and I. Morrison [**4**, Theorem A.1]). *The fundamental group of $\mathbb{H}_I$ is isomorphic to the free σ-product $※^{\sigma}_{i\in I} \mathbb{Z}$. In particular, the fundamental group of the Hawaiian earring $\mathbb{H}$ is isomorphic to $※_{\mathbb{N}} \mathbb{Z}$.*

From the above, the first homology group $\mathrm{H}_1(\mathbb{H}_I)$ is the abelianization $\mathrm{Ab}(※^{\sigma}_{i\in I} \mathbb{Z})$ on which there exists a canonical homomorphism $\sigma: \mathrm{Ab}(※^{\sigma}_{i\in I} \mathbb{Z}) \longrightarrow \prod^{\sigma}_{i\in I} \mathbb{Z}$. Theorem 3.1 is proved by studying this homomorphism. In the remaining part of the paper, σ stands for the homomorphism above.

LEMMA 3.3. *Let $\sigma: \mathrm{Ab}(※^{\sigma}_{i\in I} \mathbb{Z}) \longrightarrow \prod^{\sigma}_{i\in I} \mathbb{Z}$ be the canonical surjection. Then,* $\mathrm{Ker}(\sigma)$ *is complete modulo the Ulm subgroup and consequently is a torsion-free, algebraically compact abelian group.*

Proof. Let $\varphi: ※^{\sigma}_{i\in I} \mathbb{Z} \longrightarrow \mathrm{Ab}(※^{\sigma}_{i\in I} \mathbb{Z})$ be the canonical homomorphism. Then $\varphi^{-1}(\mathrm{Ker}\,\sigma) = \mathrm{Ker}(\sigma\varphi) = \{x \in ※^{\sigma}_{i\in I} \mathbb{Z} \mid p_i(x) = 0 \text{ for each } i\} = (※^{\sigma}_{i\in I} \mathbb{Z})^{\sigma'}$ by [**4**, Theorem 4.5]. Also $\mathrm{Ker}(\varphi) = (※^{\sigma}_{i\in I} \mathbb{Z})' \subset \varphi^{-1}(\mathrm{Ker}\,\sigma)$. Thus $\mathrm{Ker}(\sigma) = (※^{\sigma}_{i\in I} \mathbb{Z})^{\sigma'}/(※^{\sigma}_{i\in I} \mathbb{Z})'$ which is complete modulo the Ulm subgroup by [**4**, Theorem 4.7]. By Theorem 3.2, $\mathrm{Ab}(※^{\sigma}_{i\in I} \mathbb{Z})$ is isomorphic to $\mathrm{H}_1(\mathbb{H}_I)$, which is torsion-free by Theorem 2.1. Therefore, $\mathrm{Ker}(\sigma)$ is torsion-free as well and hence is algebraically compact. □

LEMMA 3.4. *The maximal divisible subgroup of* $\mathrm{Ker}(\sigma)$ *is isomorphic to* $\bigoplus_\kappa \mathbb{Q}$.

Proof. In [**4**, Theorem 4.14], it is proved that $\mathrm{Ker}(\sigma)$ contains the direct sum of $2^{\aleph_0}$-many copies of $\mathbb{Q}$. For the proof we used the existence of a family $\mathscr{F}$ consisting of countably infinite subsets of I such that $X \cap Y$ is finite for each distinct pair $X, Y \in \mathscr{F}$. Since I is infinite, it is easy to see the existence of such a family of cardinality $\kappa = |I^{\mathbb{N}}|$. This allows us to perform the same proof as that of [**4**, Theorem 4.14] to conclude that $\mathrm{Ker}(\sigma)$ contains $\bigoplus_\kappa \mathbb{Q}$. Since $\mathrm{Ker}(\sigma)$ is torsion-free and of cardinality κ, the maximal divisible subgroup of $\mathrm{Ker}(\sigma)$ is isomorphic to $\bigoplus_\kappa \mathbb{Q}$. □

We generalize [**4**, Lemma 4.12] as follows.

LEMMA 3.5. $\mathrm{Ker}(\sigma)$ *contains the free abelian group* $\bigoplus_I \mathbb{Z}$ *as a pure subgroup.*

Proof. The generator of the ith component $\mathbb{Z}$ of $※^\sigma_{i\in I} \mathbb{Z}$ is denoted by δ_i. First, we recall the proof of the following claim [**4**, Lemma 4.12]: $\mathrm{Ker}(\sigma)$ contains $\mathbb{Z}$ as a pure subgroup if $I = \mathbb{N}$.

Let $\varphi\colon ※_{n\in\mathbb{N}} \mathbb{Z} \longrightarrow \mathrm{Ab}(※_{n\in\mathbb{N}} \mathbb{Z})$ be the canonical homomorphism and let a be the element of $※_{n\in\mathbb{N}} \mathbb{Z}$ presented by $\delta_1 \ldots \delta_n \ldots \delta_1^{-1} \ldots \delta_n^{-1} \ldots$. Then, $\varphi(a) \in \mathrm{Ker}(\sigma)$ and it is shown that $\varphi(a)$ generates a pure subgroup, which is isomorphic to $\mathbb{Z}$.

To generalize the above argument for the proof of the present lemma, we take the family $\mathscr{F} = \{X_\alpha : \alpha < \kappa\}$ of cardinality κ as in the proof of Lemma 3.4. Let $\{k_{\alpha n} : n < \omega\}$ be an enumeration of X_α without repetition. Let a_α be the element of $※^\sigma_{i\in I} \mathbb{Z}$ presented by $\delta_{k_{\alpha 1}} \ldots \delta_{k_{\alpha n}} \ldots \delta^{-1}_{k_{\alpha 1}} \ldots \delta^{-1}_{k_{\alpha n}} \ldots$. We claim that $\bigoplus_{\alpha<\kappa} \langle \varphi(a_\alpha) \rangle$ is the required subgroup. To see the purity of the subgroup $\langle \varphi(a_\alpha) : \alpha < \kappa \rangle$, let $m \mid \sum_{\alpha\in F} m_\alpha \varphi(a_\alpha)$ in $\mathrm{Ab}(※^\sigma_{i\in I} \mathbb{Z})$ for $F \Subset I$ and $0 \neq m$, $m_\alpha \in \mathbb{Z}$. We show that $m \mid m_\alpha$ for each $\alpha \in F$, which immediately implies that $m \mid \sum_{\alpha\in F} m_\alpha \varphi(a_\alpha)$ in $\langle \varphi(a_\alpha) : \alpha < \kappa \rangle$. For $\alpha \in F$, there exists an infinite subset X of X_α such that $X \cap X_\beta = \emptyset$ for $\beta \in F$ with $\beta \neq \alpha$. Let $p_X\colon ※^\sigma_{i\in I} \mathbb{Z} \longrightarrow ※_{i\in X} \mathbb{Z}$ be the projection and $\bar{p}_X\colon \mathrm{Ab}(※^\sigma_{i\in I} \mathbb{Z}) \longrightarrow \mathrm{Ab}(※_{i\in X} \mathbb{Z})$ be the induced homomorphism satisfying $\varphi \cdot p_X = \bar{p}_X \cdot \varphi$. Since we have

$$\begin{aligned} \bar{p}_X\left(\sum_{\alpha\in F} m_\alpha \varphi(a_\alpha)\right) &= \sum_{\alpha\in F} m_\alpha \varphi \cdot p_X(a_\alpha) \\ &= m_\alpha \varphi \cdot p_X(a_\alpha), \end{aligned}$$

$m \mid m_\alpha \varphi \cdot p_X(a_\alpha)$ holds in $\mathrm{Ab}(※_{i\in X} \mathbb{Z})$. By [**4**, Lemma 4.11], there exists a canonical m-form $U_1 \ldots U_k$ such that $U_1 \ldots U_k = p_X(a_\alpha)^{m_\alpha}$. The reduced word for $p_X(a_\alpha)$ is of the form such that the letters are well-ordered from the left to the right. Therefore, as in the proof of [**4**, Lemma 4.12] we conclude that $m \mid m_\alpha$. Since $\alpha \in F$ is arbitrary, this proves the purity of $\langle \varphi(a_\alpha) : \alpha < \kappa \rangle$. The linear independence (over $\mathbb{Z}$) of the elements $\varphi(a_\alpha) (\alpha < \kappa)$ can be proved in a similar way to the one of the purity of $\langle \varphi(a_\alpha) : \alpha < \kappa \rangle$ above, and we omit the proof. Now, $\langle \varphi(a_\alpha) : \alpha < \kappa \rangle \simeq \bigoplus_\kappa \mathbb{Z}$ holds. □

We omit a straightforward proof of the following lemma, where $\hat{\mathbb{Z}}$ is the $\mathbb{Z}$-completion of $\mathbb{Z}$ and is isomorphic to $\prod_{p:\mathrm{prime}} \mathbb{J}_p$.

LEMMA 3.6. *Let A be a torsion-free algebraically compact group. If A contains a free abelian group $\bigoplus_I \mathbb{Z}$ as a pure subgroup, then $A/U(A)$ contains $\bigoplus_I \hat{\mathbb{Z}}$ as a pure subgroup.*

Proof of Theorem 3.1. Recall again that $H_1(\mathbb{H}_I) = \mathrm{Ab}(\circledast^{\sigma}_{i\in I}\mathbb{Z})$ and consider the canonical homomorphism $\sigma : H_1(\mathbb{H}_I) = \mathrm{Ab}(\circledast^{\sigma}_{i\in I}\mathbb{Z}) \longrightarrow \prod^{\sigma}_I \mathbb{Z}$. Since $\mathrm{Ab}(\circledast^{\sigma}_{i\in I}\mathbb{Z})/\mathrm{Ker}(\sigma) \simeq \prod^{\sigma}_I \mathbb{Z}$, which is torsion-free, $\mathrm{Ker}(\sigma)$ is a pure subgroup of $\mathrm{Ab}(\circledast^{\sigma}_{i\in I}\mathbb{Z})$. Therefore, $\mathrm{Ker}(\sigma)$ is a direct summand of $\mathrm{Ab}(\circledast^{\sigma}_{i\in I}\mathbb{Z})$ by Lemma 3.3 and consequently $\mathrm{Ab}(\circledast^{\sigma}_{i\in I}\mathbb{Z})$ is isomorphic to $\prod^{\sigma}_I \mathbb{Z} \oplus \mathrm{Ker}(\sigma)$. Since $\mathrm{Ker}(\sigma)$ is of cardinality κ, by Lemma 3.4 we conclude that $\mathrm{Ker}(\sigma)$ is isomorphic to $A \oplus \bigoplus_{\kappa}\mathbb{Q}$, where A is reduced, torsion-free and algebraically compact. By [**9**, Proposition 40.1], A is the direct product $\prod_{p:\mathrm{prime}} A_p$, where $A_p = \bigcap\{q^k A : q \text{ is prime}, q \neq p, k \geqslant 0\}$ is isomorphic to the p-adic completion of a direct sum of copies of $\mathbb{J}_p$. By Lemmas 3.5 and 3.6 and also the fact that $|A| = \kappa$, A_p is isomorphic to the p-adic completion of $\bigoplus_{\kappa}\mathbb{J}_p$. Now, $H_1(\mathbb{H}_I) = \mathrm{Ab}(\circledast^{\sigma}_{i\in I}\mathbb{Z})$ is isomorphic to

$$\prod_I^{\sigma} \mathbb{Z} \oplus \prod_{p:\mathrm{prime}} A_p \oplus \bigoplus_{\kappa} \mathbb{Q},$$

which completes the proof. □

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