Homotopy theory and classifying spaces: Happy Birthday!

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Outline

I

Homotopy theories and model categories

What is a homotopy theory $\mathbb{T}$?

Equivalent answers

- $\mathbb{T}_{\text{pair}}$ category $\mathcal{C}$ with a subcategory $\mathcal{E}$ of *equivalences*
- $\mathbb{T}_{\text{en}}$ category $\mathcal{R}$ enriched over spaces (simplicial sets)
- $\mathbb{T}_{\text{sc}}$ Segal category
- $\mathbb{T}_{\text{css}}$ complete Segal space
- $\mathbb{T}_{\text{qc}}$ quasi-category
- $\mathbb{T}_{\infty}$ ∞-category (or $(\infty, 1)$ category)

Just like categories

Internal function objects $\mathbb{T}_3 = \text{Cat}^h(\mathbb{T}_1, \mathbb{T}_2) = \mathbb{T}_2^{h\mathbb{T}_1}$

Examples of homotopy theories ($\mathbb{T}_{\text{pair}}$ presentation)

Geometry

Notation	$\mathcal{C}$	$\mathcal{E}$
$\mathbf{Top}_{\text{he}}$	**Top**	homotopy equivalences
Top	**Top**	weak homotopy equivalences
$\mathbf{Top}_R$	**Top**	R-homology isomorphisms
$\mathbf{Top}_{\leq n}$	**Top**	iso on π_i for $i \leq n$

Algebra

Sp	simplicial sets	$\lvert f \rvert$ an equivalence in **Top**
sGrp	simplicial groups	$\lvert f \rvert$ an equivalence in **Top**
	simp. rings, Lie alg.,etc	(same)
$\mathbf{Ch}_R$	chain complexes over R	homology isomorphisms
	DG algebras	homology isomorphisms

The homotopy category Ho($\mathbb{T}$)

The most visible invariant of a homotopy theory

$\mathbb{T}_{\text{pair}}$ $\text{Ho}(\mathcal{C}, \mathcal{E}) = \mathcal{E}^{-1}\mathcal{C}$

$\mathbb{T}_{\text{en}}$ $\text{Ho}(\mathcal{R}) = \pi_0\mathcal{R}$ (i.e. π_0(morphism spaces))

Pluses and minuses

- Elegant (but only a small part of the structure)
- Can be hard to compute in the $\mathbb{T}_{\text{pair}}$ case.

Examples of homotopy categories

Geometry

$(\mathcal{C},\mathcal{E})$	Maps $X \to Y$ in $\mathrm{Ho}(\mathcal{C},\mathcal{E})$
$\mathbf{Top}_{\mathrm{he}}$	homotopy classes $X \to Y$
$\mathbf{Top}$	homotopy classes $\mathrm{CW}(X) \to Y$
$\mathbf{Top}_R$	homotopy classes $\mathrm{CW}(X) \to L_R(Y)$
$\mathbf{Top}_{\leq n}$	homotopy classes $\mathrm{CW}(X) \to P_n(Y)$

Algebra

$\mathbf{Sp}$	homotopy classes $\lvert X\rvert \to \lvert Y\rvert$
$\mathbf{sGrp}$	pointed homotopy classes $B\lvert X\rvert \to B\lvert Y\rvert$
$\mathbf{Ch}_R$	chain homotopy classes Proj. Res.$(X) \to Y$

Model category: $(\mathcal{C}, \mathcal{E})$ with extras

Routine axioms

- **MC0** *equivalences* ($\sim$), *cofibrations* ($\hookrightarrow$), *fibrations* ($\twoheadrightarrow$)
- **MC1-3** composites, retracts, 2 out of 3, limits, colimits

Lifting

- **MC4**

$$\begin{array}{ccc} A & \longrightarrow & X \\ f \big\downarrow & \overset{h}{\nearrow} & \big\downarrow g \\ B & \longrightarrow & Y \end{array} \qquad \exists h \text{ if } f \text{ or } g \text{ is } \sim$$

Factorization

- **MC5** any map factors $\overset{\sim}{\hookrightarrow} \cdot \twoheadrightarrow$ and $\hookrightarrow \cdot \overset{\sim}{\twoheadrightarrow}$

Examples of model categories $(\mathcal{C}, \mathcal{E})$

Geometry

- $\mathbf{Top}_{\mathrm{he}}$, Hurewicz fibrations, closed NDR-pair inclusions
- **Top**, Serre fibrations, retracts of relative cell inclusions
- $\mathbf{Top}^{\mathcal{D}}$, objectwise Serre fibrations, retracts of relative diagram cell inclusions.

Algebra

- **Sp**, Kan fibrations, monomorphisms
- $\mathbf{Ch}_R^+$, surjections in degrees > 0, monomorphisms such that the cokernel in each degree is projective

Calculate

- Ho($\mathcal{C}$) (or even $\mathbb{T}_{\text{en}}(\mathcal{C}, \mathcal{E})$) from $(\mathcal{C}, \mathcal{E})$
- Diagrams: $\mathbb{T}$heory of $(\mathcal{C}, \mathcal{E})^{(\mathcal{D}, \mathcal{F})} \quad \sim \quad \mathbb{T}(\mathcal{C}, \mathcal{E})^{\mathrm{h}\mathbb{T}(\mathcal{D}, \mathcal{F})}$

Construct

- Derived functors
- Homotopy limits & colimits

Identify

- Equivalences $\mathbb{T}(\mathcal{C}, \mathcal{E}) \sim \mathbb{T}(\mathcal{C}', \mathcal{E}')$

Equivalences between homotopy theories

Paradox


There is a homotopy theory of homotopy theories.

Equivalences vary with context

($\mathbb{T}_{\text{en}}$) $F : \mathcal{R} \to \mathcal{R}'$ is an equivalence if $\mathrm{Ho}(F)$ is an equivalence of categories and $\mathrm{Hom}_{\mathcal{R}}(x, y) \sim \mathrm{Hom}_{\mathcal{R}'}(Fx, Fy)$

($\mathbb{T}_{\text{css}}$) $F : X_* \to Y_*$ is an equivalence if $X_n \sim Y_n$, $n \geq 0$

($\mathbb{T}_{\text{pair}}$) $F : (\mathcal{C}, \mathcal{E}) \to (\mathcal{C}', \mathcal{E}')$ is an equivalence if (??)

Special $\mathbb{T}_{\text{pair}}$ case

Filling in (??) easier for model categories

Conditions on adjoint functors $F : \mathcal{C} \leftrightarrow \mathcal{C}' : G$

1. $F(\hookrightarrow) = (\hookrightarrow)$ and $G(\twoheadrightarrow) = (\twoheadrightarrow)$
2. $f : A^{\mathrm{c}} \xrightarrow{\sim} G(B^{\mathrm{f}}) \quad \Leftrightarrow \quad f^{\flat} : F(A^{\mathrm{c}}) \xrightarrow{\sim} B^{\mathrm{f}}$

Definition

(1) = *Quillen pair*, (1) + (2) = *Quillen equivalence*

Theorem

A Quillen equivalence (F, G) *induces* $\mathbb{T}(\mathcal{C}, \mathcal{E}) \sim \mathbb{T}(\mathcal{C}', \mathcal{E}')$

Examples of equivalences between homotopy theories

Examples

- **Top** and **Sp**
- $\mathbf{Sp}_*$ and **sGrp**
- Simplicial algebras and DG_+ algebras
- Chain complexes and $H\mathbb{Z}$-module spectra

Meta-examples

- $\mathbb{T}_{\text{en}}$, $\mathbb{T}_{\text{sc}}$, $\mathbb{T}_{\text{css}}$, and $\mathbb{T}_{\text{qc}}$ ($\mathbb{T}_{\text{pair}}$ belong here?)

II

Homotopy limits and colimits

Colimits and related constructions

Colimit for $X : \mathcal{C} \to \mathcal{S}$

$$\mathrm{colim} : \mathcal{S}^{\mathcal{C}} \leftrightarrow \mathcal{S} \ : \Delta$$

$$\mathrm{Hom}_{\mathcal{S}^{\mathcal{C}}}(X, \Delta(Y)) \cong \mathrm{Hom}_{\mathcal{S}}(\mathrm{colim}\, X, Y)$$

LKan_F for $X : \mathcal{C} \to \mathcal{S}$ and $F : \mathcal{C} \to \mathcal{D}$

$$\mathrm{LKan}_F : \mathcal{S}^{\mathcal{C}} \leftrightarrow \mathcal{S}^{\mathcal{D}} \ : F^*$$

$$\mathrm{Hom}_{\mathcal{S}^{\mathcal{C}}}(X, F^* Y) \cong \mathrm{Hom}_{\mathcal{S}^{\mathcal{D}}}(\mathrm{LKan}_F\, X, Y)$$

Coend for $X : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$

$$A(\mathcal{C}) = \begin{array}{ccc} a & \xrightarrow{f} & b \\ \uparrow & & \downarrow \\ a' & \longrightarrow & b' \end{array} \qquad \mathrm{coend}\, X \cong \mathrm{colim}_{A(\mathcal{C})}(f \mapsto X(a, b))$$

Aside: notation for special coends

Monoidal category $\mathcal{S}$

- Bifunctor $\otimes : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$
- Usually associative, unital (commutative) up to ...

$\otimes$ over $\mathcal{C}$ of functors $\mathcal{C} \to \mathcal{S}$

$$X : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}, \quad Y : \mathcal{C} \to \mathcal{S}$$

$$X \otimes_{\mathcal{C}} Y := \text{coend of } (a, b) \mapsto X(a) \otimes Y(b)$$

Usually $\otimes = \times$

$$\mathcal{S} = \mathbf{Sp}, \quad \otimes = \times$$

$$\mathrm{Map}_{\mathbf{Sp}}(X \times_{\mathcal{C}} Y, Z) \cong \mathrm{Hom}_{\mathbf{Sp}^{\mathcal{C}}}(X, \mathrm{Map}(Y, Z))$$

Homotopy colimit for $X : \mathcal{C} \to \mathcal{S}$

$$\mathrm{hocolim} : \mathcal{S}^{\mathcal{C}} \leftrightarrow \mathcal{S} : \Delta$$

$$\mathrm{Hom}^{\mathrm{h}}_{\mathcal{S}^{\mathcal{C}}}(X, \Delta(Y)) \sim \mathrm{Hom}^{\mathrm{h}}_{\mathcal{S}}(\mathrm{hocolim}\, X, Y)$$

$\mathrm{LKan}^{\mathrm{h}}_F$ for $X : \mathcal{C} \to \mathcal{S}$ and $F : \mathcal{C} \to \mathcal{D}$

$$\mathrm{LKan}^{\mathrm{h}}_F : \mathcal{S}^{\mathcal{C}} \leftrightarrow \mathcal{S}^{\mathcal{D}} : F^*$$

$$\mathrm{Hom}^{\mathrm{h}}_{\mathcal{S}^{\mathcal{C}}}(X, F^* Y) \cong \mathrm{Hom}^{\mathrm{h}}_{\mathcal{S}^{\mathcal{D}}}(\mathrm{LKan}^{\mathrm{h}}_F X, Y)$$

Homotopy coend for $X : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$

$$\mathrm{hocoend}\, X = \mathrm{hocolim}_{A(\mathcal{C})} \text{ of } (a \to b) \mapsto X(a, b)$$

$$U \otimes^{\mathrm{h}}_{\mathcal{C}} V = \mathrm{hocolim}_{A(\mathcal{C})} \text{ of } (a \to b) \mapsto U(a) \otimes V(b)$$

Model category dividend

Theorem

If $\mathcal{S}$ admits a model category structure:

Existence

hocolim, $\mathrm{LKan}^{\mathrm{h}}_{F}$, hocoend exist for target $\mathcal{S}$.

Realizability

- hocolim $\sim$ functor $\mathcal{S}^{\mathcal{C}} \to \mathcal{S}$,
- $\mathrm{LKan}^{\mathrm{h}}_{F}$ $\sim$ functor $\mathcal{S}^{\mathcal{C}} \to \mathcal{S}^{\mathcal{D}}$, and
- hocoend $\sim$ functor $\mathcal{S}^{\mathcal{C}^{\mathrm{op}} \times \mathcal{C}} \to \mathcal{S}$.

Construction of hocolim, version I

Assumptions

- $\mathcal{S}$ admits a model structure
- $\mathcal{S}^{\mathcal{C}}$ admits the projective model structure (fibrations are objectwise)

The assumptions hold if $\mathcal{S}$ is **Sp**, or **Top**.

Conclusion

$$\operatorname{hocolim} X \sim \operatorname{colim} X^{\mathsf{c}}, \quad X \in \mathcal{S}^{\mathcal{C}}$$

(A) $|Y|$ for $Y : \Delta^{\mathrm{op}} \longrightarrow \{\textbf{Top} \text{ or } \textbf{Sp}\}$

$$|Y| = \begin{cases} \Delta_* \times_{\Delta^{\mathrm{op}}} Y & \text{for } \textbf{Top} \\ \Delta[*] \times_{\Delta^{\mathrm{op}}} Y & \text{for } \textbf{Sp} \end{cases}$$

(B) $\mathrm{Repl}_*(X) : \Delta^{\mathrm{op}} \longrightarrow \mathcal{S}$ for $X : \mathcal{C} \longrightarrow \mathcal{S}$

$$\mathrm{Repl}_*(X)(\mathbf{n}) = \coprod_{f:\,\{0\to 1\to\cdots\to n\}\to\mathcal{C}} f(0)$$

hocolim $=$ **(A)** $+$ **(B)**

$$\mathrm{hocolim}\, X \sim |\mathrm{Repl}_*(X)|$$

Homotopy limit for $X : \mathcal{C} \to \mathcal{S}$

$$\Delta : \mathcal{S} \leftrightarrow \mathcal{S}^{\mathcal{C}} : \text{holim}$$

$$\text{Hom}^{\text{h}}_{\mathcal{S}^{\mathcal{C}}}(\Delta(Y), X) \sim \text{Hom}^{\text{h}}_{\mathcal{S}}(Y, \text{holim}\, X)$$

RKan^{h}_F for $X : \mathcal{C} \to \mathcal{S}$ and $F : \mathcal{C} \to \mathcal{D}$

$$F^* : \mathcal{S}^{\mathcal{D}} \leftrightarrow \mathcal{S}^{\mathcal{C}} : \text{RKan}^{\text{h}}_F$$

$$\text{Hom}^{\text{h}}_{\mathcal{S}^{\mathcal{C}}}(F^*Y, X) \cong \text{Hom}^{\text{h}}_{\mathcal{S}^{\mathcal{D}}}(Y, \text{RKan}^{\text{h}}_F\, X)$$

Homotopy end for $X : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{S}$

$$\text{hoend}\, X = \text{holim}_{A(\mathcal{C})} \text{ of } (a \to b) \mapsto X(a, b)$$

Another model category dividend

Theorem

If $\mathcal{S}$ admits a model category structure:

Existence

holim, $\mathrm{RKan}_F^{\mathrm{h}}$, hoend exist for target $\mathcal{S}$.

Realizability

- holim $\sim$ functor $\mathcal{S}^{\mathcal{C}} \to \mathcal{S}$,
- $\mathrm{RKan}_F^{\mathrm{h}} \sim$ functor $\mathcal{S}^{\mathcal{C}} \to \mathcal{S}^{\mathcal{D}}$, and
- hoend $\sim$ functor $\mathcal{S}^{\mathcal{C}^{\mathrm{op}} \times \mathcal{C}} \to \mathcal{S}$.

Construction of holim, version I

Assumptions

- $\mathcal{S}$ admits a model structure
- $\mathcal{S}^{\mathcal{C}}$ admits the injective model structure (cofibrations are objectwise)

The assumptions hold if $\mathcal{S}$ is **Sp** (**Top**?)

Conclusion

$$\operatorname{holim} X \sim \lim X^{\mathrm{f}}, \quad X \in \mathcal{S}^{\mathcal{C}}$$

(A) Tot Y for $Y : \Delta \to \{\textbf{Top} \text{ or } \textbf{Sp}\}$

$$\mathrm{Tot}\, Y = \begin{cases} \mathrm{Map}_{\textbf{Top}^{\Delta}}(\Delta_*, Y) & \text{for } \textbf{Top} \\ \mathrm{Map}_{\textbf{Sp}^{\Delta}}(\Delta[*], Y) & \text{for } \textbf{Sp} \end{cases}$$

(B) $\mathrm{Repl}^*(X) : \Delta \to \mathcal{S}$ for $X : \mathcal{C} \to \mathcal{S}$

$$\mathrm{Repl}^*(X)(\mathbf{n}) = \prod_{f:\, \{0 \to 1 \to \cdots \to n\} \to \mathcal{C}} f(n)$$

holim $=$ **(A)** $+$ **(B)**

$$\mathrm{holim}\, X \sim \mathrm{Tot}\, \mathrm{Repl}^*(c \mapsto X(c)^{\mathrm{f}})$$

Universal maps

$$\operatorname{hocolim} X \to \operatorname{colim} X$$
$$\lim X \to \operatorname{holim} X$$

Homotopy invariance

$$X \xrightarrow{\simeq} Y \quad \Longrightarrow \begin{cases} \operatorname{hocolim} X \xrightarrow{\simeq} \operatorname{hocolim} Y \\ \operatorname{holim} X \xrightarrow{\simeq} \operatorname{holim} Y \end{cases}$$

Mapping adjointness

$$\operatorname{Hom}^{\mathrm{h}}_{\mathcal{S}}(\operatorname{hocolim} X, Y) \sim \operatorname{holim}_{\mathcal{C}^{\mathrm{op}}} \text{ of } c \mapsto \operatorname{Hom}^{\mathrm{h}}(X(c), Y)$$

$$\operatorname{Hom}^{\mathrm{h}}_{\mathcal{S}}(Y, \operatorname{holim} X) \sim \operatorname{holim}_{\mathcal{C}} \text{ of } c \mapsto \operatorname{Hom}^{\mathrm{h}}(Y, X(c))$$

Homotopy colimit

Suppose that X takes on cofibrant values.

$$\begin{array}{lcccc} \operatorname{hocolim} X & \sim & * & \times^{\mathrm{h}}_{\mathcal{C}} & X \\ \text{(I)} & \sim & * & \times_{\mathcal{C}} & X^{\mathrm{c}}_{\mathrm{proj}} \\ \text{(II)} & \sim & *^{\mathrm{c}}_{\mathrm{proj}} & \times_{\mathcal{C}} & X \end{array}$$

Homotopy limit

Suppose that X takes on fibrant values.

$$\begin{array}{lcccc} \operatorname{holim} X & \sim & \mathrm{Hom}^{\mathrm{h}}(& *, & X) \\ \text{(I)} & \sim & \mathrm{Map}(& *, & X^{\mathrm{f}}_{\mathrm{inj}}) \\ \text{(II)} & \sim & \mathrm{Map}(& *^{\mathrm{c}}_{\mathrm{proj}}, & X) \end{array}$$

III

Spaces from categories

Geometrization

$$\mathrm{N} : \mathbf{Cat} \to \mathbf{Sp}$$

Properties

- $F : \mathcal{C} \to \mathcal{D} \quad \mapsto \quad \mathrm{N}F : \mathrm{N}\mathcal{C} \to \mathrm{N}\mathcal{D}$
- $\tau : F \stackrel{\cdot}{\to} G \quad \mapsto \quad H : \mathrm{N}\mathcal{C} \times \Delta[1] \to \mathcal{D}$

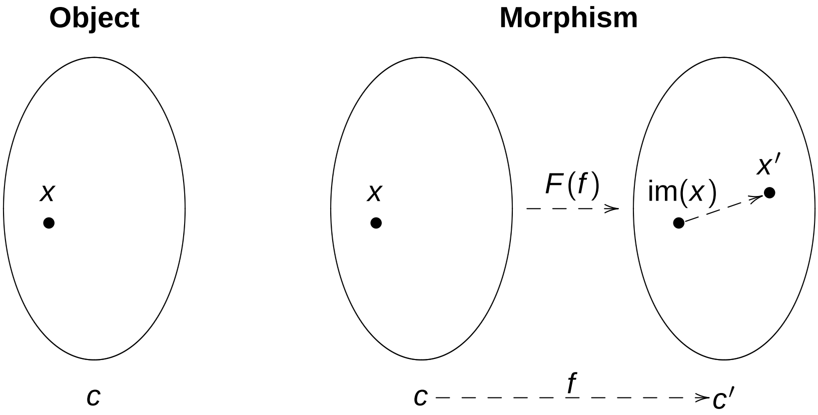
Advantages

- Categories more visible than spaces.
- Natural transformations more accessible than homotopies.

Can homotopy colimits (coends) be built in?

$$F : \mathcal{C} \to \mathbf{Cat} \implies \operatorname{hocolim}_{\mathcal{C}} \mathrm{N}(F) \quad \sim \quad \mathrm{N}(?)$$

$F : \mathcal{C} \to \mathbf{Cat} \quad \Longrightarrow \quad \text{category } \mathcal{C} \ltimes F$



Theorem

$$\mathrm{N}(\mathcal{C} \ltimes F) \quad \sim \quad \operatorname{hocolim}_{\mathcal{C}} \mathrm{N}(F)$$

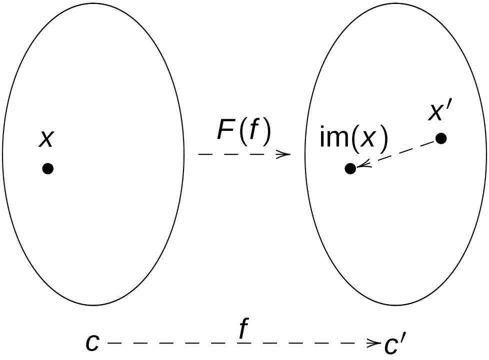
Example

$F : \mathcal{C} \to \mathbf{Set}$, $\mathcal{C} \ltimes F =$ Transport Category

- Object: $(c, x),\ c \in \mathcal{C},\ x \in F(c)$
- Morphism: $f : c \to c'$ with $f(x) = x'$

N(Transport Category) $\sim$ hocolim F

$\mathcal{C}\ltimes(F^{\mathrm{op}})$



(c, x) (c', x')

Also have $(\mathcal{C}\ltimes F)^{\mathrm{op}}$ and $(\mathcal{C}\ltimes(F^{\mathrm{op}}))^{\mathrm{op}}$!

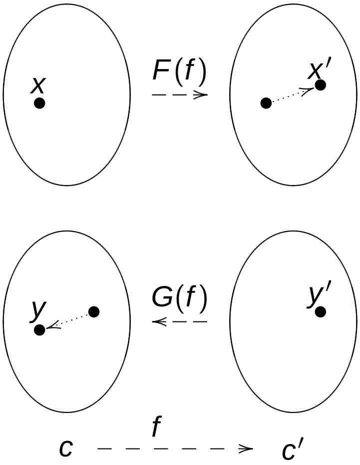
Extension to homotopy coends

$F : \mathcal{C} \to \mathbf{Cat}, \; G : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$

$\mathrm{N}(G \rtimes \mathcal{C} \ltimes F) \sim \mathrm{N}(G) \times^{\mathrm{h}}_{\mathcal{C}} \mathrm{N}(F)$

$G \rtimes \mathcal{C} \ltimes F = \{(c, x, y)\}$

Morphism



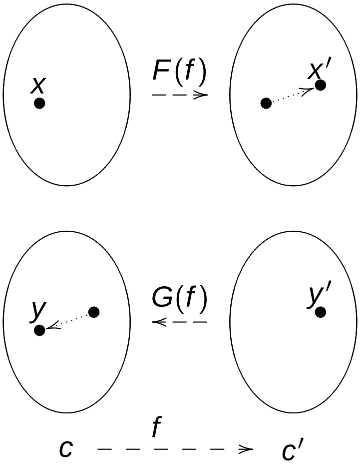
Extension to homotopy coends

$F : \mathcal{C} \to \textbf{Cat}, \; G : \mathcal{C}^{\mathrm{op}} \to \textbf{Cat}$

$\mathrm{N}(G \rtimes \mathcal{C} \ltimes F) \sim \mathrm{N}(G) \times^{\mathrm{h}}_{\mathcal{C}} \mathrm{N}(F)$

$$G \rtimes \mathcal{C} \ltimes F = \{(c, x, y)\}$$

Morphism



Setup

• $R \to S$ aug. k-algebras	$F : \mathcal{C} \to \mathcal{D}$ categories
• ${}_R M$, ${}_S N$	${}_{\mathcal{C}} X$, ${}_{\mathcal{D}} Y$

Dictionary

• $k \otimes_R M$	$\text{hocolim}_{\mathcal{C}} X \sim * \times^{\text{h}}_{\mathcal{C}} X$
• $\text{Hom}_R(k, M)$	$\text{holim}_{\mathcal{C}} X \sim \text{Hom}^{\text{h}}_{\mathcal{C}}(*, X)$
• $S \otimes_R M$	$\text{LKan}^{\text{h}}_F(X) \sim \mathcal{D} \times^{\text{h}}_{\mathcal{C}} X$
• $\text{Hom}_R(S, M)$	$\text{RKan}^{\text{h}}_F(X) \sim \text{Hom}^{\text{h}}_{\mathcal{C}}(\mathcal{D}, X)$

Transitivity (pushing forward over functors)

$$M, R \longrightarrow S \longrightarrow T \qquad X, \mathcal{C} \longrightarrow \mathcal{D} \longrightarrow \mathcal{S}$$

On the left

Algebra $T \otimes_R M \cong T \otimes_S S \otimes_R M$

Topology $\mathcal{S} \times^{\mathrm{h}}_{\mathcal{C}} X \sim \mathcal{S} \times^{\mathrm{h}}_{\mathcal{D}} \mathcal{D} \times^{\mathrm{h}}_{\mathcal{C}} X$

On the right

Algebra $\mathrm{Hom}_R(T, M) \cong \mathrm{Hom}_S(T, \mathrm{Hom}_R(S, M))$

Topology $\mathrm{Hom}^{\mathrm{h}}_{\mathcal{C}}(\mathcal{S}, X) \sim \mathrm{Hom}^{\mathrm{h}}_{\mathcal{D}}(\mathcal{S}, \mathrm{Hom}^{\mathrm{h}}_{\mathcal{C}}(\mathcal{D}, X))$

Does pulling back preserve hocolim?

Cofinality (pulling back)

$R \longrightarrow S, \quad {}_S N$ $\qquad\qquad$ $\mathcal{C} \longrightarrow \mathcal{D}, \quad {}_{\mathcal{D}} Y$

On the left

Algebra	$\forall N:$	$k \otimes_R N$	$\cong$	$k \otimes_S N$?
Topology	$\forall Y:$	$\operatorname{hocolim} F^*(Y)$	$\sim$	$\operatorname{hocolim} Y$?

Does pulling back preserve hocolim?

Cofinality (pulling back)

$R \longrightarrow S, \quad {}_S N$ $\qquad\qquad$ $\mathcal{C} \longrightarrow \mathcal{D}, \quad {}_{\mathcal{D}} Y$

On the left

Algebra	$\forall N:$	$k \otimes_R N$	$\cong$	$k \otimes_S N$	?
Topology	$\forall Y:$	$* \times^{\mathrm{h}}_{\mathcal{C}} Y$	$\sim$	$* \times^{\mathrm{h}}_{\mathcal{D}} Y$	?

Does pulling back preserve hocolim?

Cofinality (pulling back)

$R \to S, \quad {}_S N$ $\qquad\qquad$ $\mathcal{C} \to \mathcal{D}, \quad {}_{\mathcal{D}} Y$

On the left

Algebra	$\forall N :$	$k \otimes_R N$	$\cong$	$k \otimes_S N$	?
Topology	$\forall Y :$	$* \times^{h}_{\mathcal{C}} Y$	$\sim$	$* \times^{h}_{\mathcal{D}} Y$	?

Solution

Algebra $\quad k \otimes_R N \cong (k \otimes_R S) \otimes_S N, \quad$ want $\boxed{k \otimes_R S \cong k}$

Topology $\quad * \times^{h}_{\mathcal{C}} Y \sim (* \times^{h}_{\mathcal{C}} \mathcal{D}) \times^{h}_{\mathcal{D}} Y \quad$ want $\boxed{* \times^{h}_{\mathcal{C}} \mathcal{D} \sim *}$

Definition

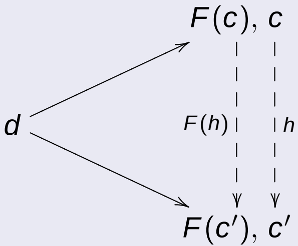
$F : \mathcal{C} \to \mathcal{D}$ *terminal* if $* \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{D} \sim *$.

Theorem

$F : \mathcal{C} \to \mathcal{D}$ terminal, ${}_{\mathcal{D}}Y \implies \mathrm{hocolim}_{\mathcal{C}}\, Y \sim \mathrm{hocolim}_{\mathcal{D}}\, Y$

Interpretation (Grothendieck construction!)

$$(* \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{D})(d) = \mathrm{N}(d{\downarrow}F)$$



Does pulling back preserve holim?

Cofinality (pulling back)

$R \to S, \quad {}_S N \qquad\qquad \mathcal{C} \to \mathcal{D}, \quad {}_{\mathcal{D}} Y$

On the right

Algebra	$\forall N$:	$\mathrm{Hom}_R(k, N)$	$\cong$	$\mathrm{Hom}_S(k, N)$?
Topology	$\forall Y$:	$\mathrm{holim}\, F^*(Y) \sim$		$\mathrm{holim}\, Y$?

Does pulling back preserve holim?

Cofinality (pulling back)

$R \to S, \quad {}_S N$ $\qquad\qquad \mathcal{C} \to \mathcal{D}, \quad {}_{\mathcal{D}} Y$

On the right

Algebra	$\forall N :$	$\mathrm{Hom}_R(k, N)$	$\cong$	$\mathrm{Hom}_S(k, N)$?
Topology	$\forall Y :$	$\mathrm{Hom}^{\mathrm{h}}_{\mathcal{SC}}(*, Y)$	$\sim$	$\mathrm{Hom}^{\mathrm{h}}_{\mathcal{SD}}(*, Y)$?

Does pulling back preserve holim?

Cofinality (pulling back)

$R \to S, \quad {}_S N \qquad\qquad \mathcal{C} \to \mathcal{D}, \quad {}_{\mathcal{D}} Y$

On the right

Algebra	$\forall N:$	$\mathrm{Hom}_R(k, N)$	$\cong$	$\mathrm{Hom}_S(k, N)$?
Topology	$\forall Y:$	$\mathrm{Hom}^{\mathrm{h}}_{\mathcal{S}\mathcal{C}}(*, Y)$	$\sim$	$\mathrm{Hom}^{\mathrm{h}}_{\mathcal{S}\mathcal{D}}(*, Y)$?

Solution

Algebra $\mathrm{Hom}_R(k, N) \cong \mathrm{Hom}_S(S \otimes_R k, N)$ $\boxed{S \otimes_R k \cong k}$

Topology $\mathrm{Hom}^{\mathrm{h}}_{\mathcal{S}\mathcal{C}}(*.Y) \cong \mathrm{Hom}^{\mathrm{h}}_{\mathcal{S}\mathcal{D}}(\mathcal{D} \times^{\mathrm{h}}_{\mathcal{C}} *, Y)$ $\boxed{\mathcal{D} \times^{\mathrm{h}}_{\mathcal{C}} * \sim *}$

Initial functors

Definition

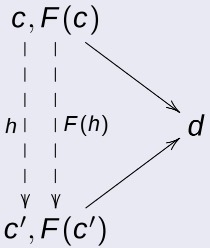
$F : \mathcal{C} \to \mathcal{D}$ *initial* if $\mathcal{D} \times^{\mathrm{h}}_{\mathcal{C}} * \sim *$.

Theorem

$F : \mathcal{C} \to \mathcal{D}$ initial, ${}_{\mathcal{D}}Y \implies \operatorname{holim}_{\mathcal{C}} Y \sim \operatorname{holim}_{\mathcal{D}} Y$

Interpretation (Grothendieck construction again!)

$$(\mathcal{D} \times^{\mathrm{h}}_{\mathcal{C}} *)(d) = \mathrm{N}(F{\downarrow}d)$$



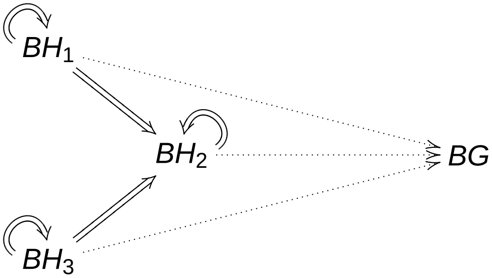
IV

Homology decompositions

Approximation data for BG

Approximating BG by B(subgroups)

- $F : \mathcal{D} \to \mathbf{Sp}$
- $\forall d, F(d) \sim BH_d, \quad H_d \subset G$
- Approximation: $\operatorname{hocolim} F \to BG$



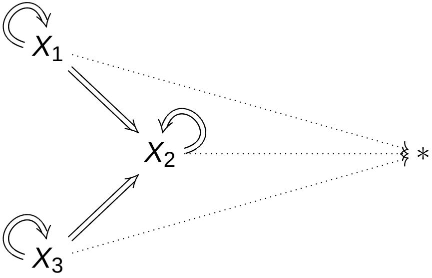
Homology decomposition $\Leftrightarrow$ $\operatorname{hocolim} F \sim_p BG$

Approximation data from G-orbits

G-orbit $\Leftrightarrow$ B(subgroup)

$$(X \in \mathcal{O}_G) \quad \Leftrightarrow \quad (X_{hG} \sim BG_x,\, x \in X)$$

$$S : \mathcal{D} \to \mathcal{O}_G \quad \Longrightarrow \quad S_{hG} \text{ is approximation data}$$



Obtaining alternative approximation data

Collection C: set of subgroups of G closed under conjugation

Subgroup diagram of C $\quad (H_i \in C)$

$$\mathcal{I}_C \quad = \quad \begin{array}{c} G/H_1 \\ \vdots \\ \downarrow \\ G/H_2 \end{array}$$

$$I(G/H) = G/H$$

$$I_{hG}(G/H) \sim BH$$

Centralizer diagram of C $\quad (\mathrm{im}(H_i) \in C)$

$$\mathcal{J}_C \quad = \quad \begin{array}{ccc} H_1 \subset & \overset{\Sigma_1}{\rightsquigarrow} & \\ \vdots & & G \\ \downarrow & & \\ H_2 \subset & \underset{\Sigma_2}{\rightsquigarrow} & \end{array}$$

$$J(H, \Sigma) = \Sigma$$

$$J_{hG}(H, \Sigma) \sim BZ_G(\mathrm{im}\, H)$$

Six $\mathbb{Z}/p$-homology decompositions

Collections $(p \mid \#(G))$

- $C_1 = \{\text{non-trivial } p\text{-subgroups}\}$
- $C_2 = \{\text{non-trivial elementary abelian } p\text{-subgroups}\}$
- $C_3 = \{V \in C_2 \mid V = {}_pZ(Z_G(V))\}$

$\mathbb{Z}/p$-homology decompositions?

C	**Subgroup decomposition**	**Centralizer decomposition**
C_1	Yes	Yes
C_2	Yes	Yes
C_3	Yes	Yes

How to obtain the six decompositions

$\mathcal{K}_C = \{$poset C under inclusion$\}$, $K_C = \mathrm{N}(\mathcal{K}_C)$

Identify hocolims

$I: \mathcal{I}_C \to \mathbf{Sp}$	$\operatorname{hocolim} I_{hG} \sim (K_C)_{hG}$	(subgroup diagram)
$J: \mathcal{J}_C \to \mathbf{Sp}$	$\operatorname{hocolim} J_{hG} \sim (K_C)_{hG}$	(cent'lizer diagram)

Relate posets

$$K_{C_1} \sim K_{C_2} \sim K_{C_3} \qquad \text{(via } G\text{-maps)}$$

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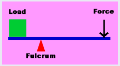
Relate posets

$K_{C_1} \sim K_{C_2} \sim K_{C_3}$ (via G-maps)

Start here

$(K_{C_1})_{hG} \sim_p BG$



Identify hocolim for subgroup diagram

$$I_{hG} : \mathcal{I}_C \to \mathbf{Sp} \qquad I(G/H) = G/H$$

Reduction to hocolim I

Want:	$\operatorname{hocolim}(I_{hG})$	$\sim$	$(K_C)_{hG}$
Have:	$\operatorname{hocolim}(I_{hG})$	$\sim$	$(\operatorname{hocolim} I)_{hG}$
Need:	$\operatorname{hocolim} I$	$\sim$	K_C

Grothendieck construction

$$\mathcal{I}_C \ltimes I \quad = \quad \begin{array}{ccc} (x_1 \in G/H_1) & \longmapsto & G_{x_1} \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ (x_2 \in G/H_2) & \longmapsto & G_{x_2} \end{array} \quad = \quad \mathcal{K}_C$$

Equivalence of categories, G-equivariant

Identify hocolim for subgroup diagram

$$I_{hG} : \mathcal{I}_C \to \mathbf{Sp} \qquad I(G/H) = G/H$$

Reduction to hocolim I

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Equivalence of categories, G-equivariant

Identify hocolim for centralizer diagram

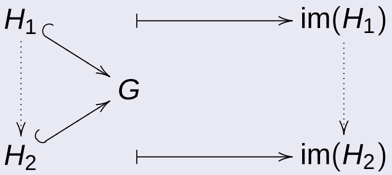
$$J_{hG} : \mathcal{J}_C \to \mathbf{Sp} \qquad J(H, \Sigma) = \Sigma$$

Reduction to hocolim J

Want:	$\operatorname{hocolim}(J_{hG})$	$\sim$	$(K_C)_{hG}$
Have:	$\operatorname{hocolim}(J_{hG})$	$\sim$	$(\operatorname{hocolim} J)_{hG}$
Need:	$\operatorname{hocolim} J$	$\sim$	K_C

Grothendieck construction

$\mathcal{J}_C \ltimes J \quad =$



$= \quad \mathcal{K}_C$

Equivalence of categories, G-equivariant

Identify hocolim for centralizer diagram

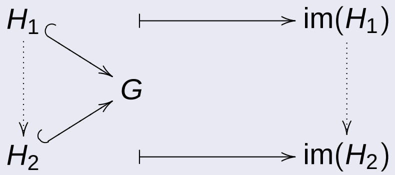
$$J_{hG} : \mathcal{J}_C \to \mathbf{Sp} \qquad J(H, \Sigma) = \Sigma$$

Reduction to hocolim J

Want:	$\operatorname{hocolim}(J_{hG})$	$\sim$	$(K_C)_{hG}$
Have:	$\operatorname{hocolim}(J_{hG})$	$\sim$	$(\operatorname{hocolim} J)_{hG}$
Need:	$\operatorname{hocolim} J$	$\sim$	K_C

Grothendieck construction

$\mathcal{J}_C \ltimes J \quad =$



$= \quad \mathcal{K}_C$

Equivalence of categories, G-equivariant

Identify hocolim for centralizer diagram

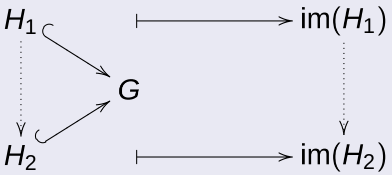
$$J_{hG} : \mathcal{J}_C \to \mathbf{Sp} \qquad J(H, \Sigma) = \Sigma$$

Reduction to hocolim J

Want:	$\operatorname{hocolim}(J_{hG})$	$\sim$	$(K_C)_{hG}$
Have:	$\operatorname{hocolim}(J_{hG})$	$\sim$	$(\operatorname{hocolim} J)_{hG}$
Need:	$\operatorname{hocolim} J$	$\sim$	K_C

Grothendieck construction

$\mathcal{J}_C \ltimes J \quad =$



$= \quad \mathcal{K}_C$

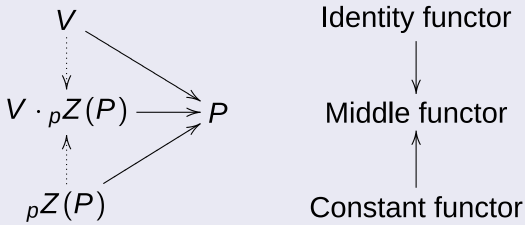
Equivalence of categories, G-equivariant

Relate posets $K_{C_2} \sim K_{C_1}$

$\mathcal{K}_{C_1} = \{\text{nontrivial } p\text{-subgroups}\}$
$\mathcal{K}_{C_2} = \{\text{nontrivial elementary abelian } p\text{-subgroups}\}$

An initial functor $\iota\colon \mathcal{K}_{C_2} \to \mathcal{K}_{C_1}$

$(\iota{\downarrow}P) \sim *$



$$\mathcal{K}_{C_2} \to \mathcal{K}_{C_1} \text{ initial } \implies K_{C_2} \sim K_{C_1}$$

$\mathcal{K}_{C_2} = \{$nontrivial elementary abelian p-subgroups$\}$
$\mathcal{K}_{C_3} = \{V \in C_2 \mid V = {}_pZ(Z_G(V))\}$

Adjoint functors $\beta : \mathcal{K}_{C_2} \leftrightarrow \mathcal{K}_{C_3} : \iota$

$$(\beta\iota \to \mathrm{id}, \quad \mathrm{id} \to \iota\beta)$$

$$W \xleftarrow{\ \iota\ } W$$

$$V \xmapsto{\ \beta\ } {}_pZ(Z_G(V))$$

β and ι give inverse $\sim$ on nerves.

V

Localizations

Left Bousfield localization

Setup

$(\mathcal{C}, \mathcal{E})$ a homotopy theory, $\mathcal{E} \subset \mathcal{F} \subset \mathcal{C}$ $\qquad (\mathcal{E} = \sim, \ \mathcal{F} = \approx)$

Definition

$I\colon (\mathcal{C}, \mathcal{E}) \to (\mathcal{C}, \mathcal{F})$ a *left Bousfield localization* if $\exists J$

$$I\colon (\mathcal{C}, \mathcal{E}) \leftrightarrow (\mathcal{C}, \mathcal{F}) \ : J \quad J \text{ full \& faithful}$$

Properties

- $L = JI$ (localization functor), X *local* $\Leftrightarrow X \in \mathrm{im}(L)$
- $L^2 \sim L$
- $X \overset{\approx}{\to} L(X)$ terminal among $X \overset{\approx}{\to} Y$
- $X \overset{\approx}{\to} L(X)$ initial among $X \to$ (local)

Example with discrete categories

Context

- $\mathcal{C} = \mathbf{Ab}$
- $R = \mathbb{Z}[1/p]$
- $\mathcal{E} =$ (isos), $\mathcal{F} = \{f \mid R \otimes f \text{ an iso}\}$
- $(\mathcal{C}, \mathcal{E}) \sim \mathbf{Mod}_{\mathbb{Z}}, (\mathcal{C}, \mathcal{F}) \sim \mathbf{Mod}_R$

Properties

- $L = R \otimes (-)$, X *local* $\Leftrightarrow X$ an R-module
- $L^2 \sim L$
- $X \xrightarrow{\approx} L(X)$ terminal among $X \xrightarrow{\approx} Y$
- $X \xrightarrow{\approx} L(X)$ initial among $X \to$ (local)

Definition (f a morphism of $\mathcal{C}$, X an object)

- $f \perp X$ if $\mathrm{Hom}^{\mathrm{h}}_{(\mathcal{C},\mathcal{E})}(f, X)$ is an equivalence in **Sp**.
- $f^{<} = \{g : f \perp X \implies g \perp X\}$.

Assume

- $(\mathcal{C}, \mathcal{E})$ is a model category^{++}
- there is a map f such that $\mathcal{F} = f^{<}$

Theorem

1. $(\mathcal{C}, \mathcal{E}) \to (\mathcal{C}, \mathcal{F})$ is a left Bousfield localization.
2. $(\mathcal{C}, \mathcal{F})$ is a model category with $\mathcal{C}^{\mathrm{c}}_{\mathcal{F}} = \mathcal{C}^{\mathrm{c}}_{\mathcal{E}}$.
3. L is the fibrant replacement functor in $(\mathcal{C}, \mathcal{F})$.

Examples of model category localizations (I)

Localization with respect to a map

Pick your favorite map f and let $\mathcal{F} = f^{<}$

$\mathcal{C} = \mathbf{Sp}$, $f = (S^{n+1} \to *)$

- $f^{<} = \{g : \pi_i(g) \text{ iso for } i \leq n\}$
- $L = P_n$ (n'th Postnikov stage)
- (local) $=$ (π_i vanishes for $i > n$)

$\mathcal{C} = \mathbf{Ch}_{\mathbb{Z}}$, $f = \oplus_i(\Sigma^i\mathbb{Z}[1/p] \to 0)$

- $f^{<} = \{g : \mathbb{Z}/p \otimes^{\mathrm{h}} g \text{ is an equivalence}\}$
- $L =$ derived p-completion
- (local) $=$ homology groups are Ext-p-complete

Examples of model category localizations (II)

Localization with respect to a homology theory E_*

Pick your favorite E_*, let $\mathcal{F} = \{E_*\text{-isos}\}$, and follow Bousfield.

Theorem (Bousfield)

There exists a magic morphism f with $f^< = \mathcal{F}$.

Widely applicable

- Spaces
- Spectra
- $\mathbf{Ch}_R$ (e.g, $E_*(X) = H_*(A \otimes^h_R X)$)
- simplicial universal algebras

Localization with respect to R-homology, $R \subset \mathbb{Q}$

Properties

L_R = localization in **Sp** with respect to $H_*(-;R)$

- L_R preserves components, connectivity, nilpotency.
- X 1-connected $\implies \pi_i L_R(X) \cong R \otimes \pi_i(X)$
- X 1-connected $\implies H_i(L_R X;\mathbb{Z}) \cong R \otimes H_i(X;\mathbb{Z})$
- L_R preserves fibrations of connected nilpotent spaces.

Algebraization if $R = \mathbb{Q}$

$\mathbf{Sp}_1$ = 1-connected spaces, $\mathcal{F} = \{H_*(-;\mathbb{Q})$ isos$\}$

$\mathbf{Lie}_0 = \{$0-connected DG Lie alg$/\mathbb{Q}\}$, $\mathcal{H} = \{H_*$ isos$\}$.

$$\boxed{(\mathbf{Sp}_1, \mathcal{F}) \quad \sim \quad (\mathbf{Lie}_0, \mathcal{H})}$$

Localization with respect to $\mathbb{Z}/p$-homology

Properties (Completion?)

$L_p =$ localization in **Sp** with respect to $H_*(-; \mathbb{Z}/p)$

- L_p preserves components, connectivity, nilpotency.
- X 1-connected, fin. type $\implies \pi_i L_p(X) \cong \mathbb{Z}_p \otimes \pi_i(X)$
- X 1-connected $\implies H_i(L_p X; \mathbb{Z}) \cong$??
- L_p preserves fibrations of connected nilpotent spaces.

General formula for homotopy groups of $L_p X$

X 1-connected $\implies$

$$0 \to \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_{i-1}X) \to \pi_i L_p X \to \mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_i X) \to 0$$

Mixing $L_{\mathbb{Q}}(X)$ with the $L_p(X)$'s to recover X

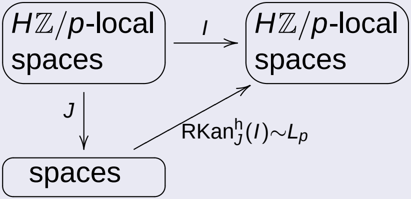
The Arithmetic Square (Sullivan)

X nilpotent $\implies$ homotopy fibre square

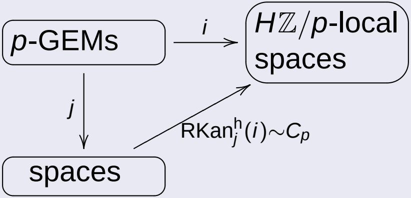
$$\begin{array}{ccc} X & \longrightarrow & \prod_p L_p(X) \\ \downarrow & & \downarrow \\ L_{\mathbb{Q}}X & \longrightarrow & L_{\mathbb{Q}}(\prod_p L_p(X)) \end{array}$$

An approximation to L_p

$H\mathbb{Z}/p$-localization functor L_p



Bousfield-Kan p-completion functor C_p ($L_p \to C_p$)



Properties

- C_p preserves disjoint unions, connectivity
- C_p preserves $H\mathbb{Z}/p_*$-nilpotent fibrations
- $C_p(\sim_p) = \sim$
- X nilpotent $\implies L_p(X) \sim C_p(X)$.

Computability

Unstable Adams spectral sequence $\Rightarrow \pi_* C_p(X)$

Building $C_p X$ if $H_* X$ has finite type

$X \to \{X_s\}$, X_s p-finite, $H^* X \cong \operatorname{colim} H^* X_s$:

$$C_p(X) \sim \operatorname{holim} X_s$$

The Bousfield-Kan p-completion C_p: good & bad

Definition

- $L_pX \sim C_pX = X$ p-good
- $L_pX \not\sim C_pX = X$ p-bad

p-good examples

- X nilpotent (e.g. $\pi_1 X$ trivial)
- $\pi_1 X$ finite

p-bad examples

- $S^1 \vee S^1$
- $S^1 \vee S^n$

News flash

C_p(p-bad X) sighted in the wild!

VI

Cohomology of function spaces

The Steenrod algebra A_p

p = fixed prime, $H^* = H^*(-;\mathbb{Z}/p)$

p odd	Structure of A_p	$p=2$
$\{\beta, \mathcal{P}^i, i \geq 0\}$		$\{\mathrm{Sq}^i, i \geq 0\}$
$\mathcal{P}^0 = 1$		$\mathrm{Sq}^0 = 1, \mathrm{Sq}^1 = \beta$
Adem relations $\mathcal{P}^i\mathcal{P}^j = \cdots$		$\mathrm{Sq}^i\,\mathrm{Sq}^j = \cdots$
$A_p \to A_p \otimes A_p$		$\mathrm{Sq}^n \mapsto \sum_{i+j=n} \mathrm{Sq}^i \otimes \mathrm{Sq}^j$
$\lvert\beta\rvert = 1$, $\lvert\mathcal{P}^i\rvert = 2i(p-1)$		$\lvert\mathrm{Sq}^i\rvert = i$

All in all

A_p = cocommutative Hopf algebra over $\mathbb{F}_p$

Modules and algebras over A_p

Modules over A_p

- *Graded* modules $\quad \mathrm{Sq}^i : M^k \to M^{k+i}$
- $\mathcal{U}$ = unstable modules $\quad \mathrm{Sq}^i x = 0,\ i > |x|$
- M, N (unstable) modules $\implies$ $M \otimes N$ (unstable) module

Algebra over A_p

- Graded algebras, $\mu\colon M \otimes M \to M$ respects A_p
- $\mathcal{K}$ = unstable algebras $\quad \mathrm{Sq}^{|x|} x = x^2$
- R, S (unstable) algebras $\implies$ $R \otimes S$ (unstable) algebra

Geometry

- H^*(suspension spectrum) $\in \mathcal{U}$
- H^*(space) $\in \mathcal{K}$

The functor T

$V = V_1 = \mathbb{Z}/p$, $\mathbb{H} = H^*BV$

Left adjoint(s) to $\mathbb{H} \otimes -$

$T_{\mathcal{U}} : \mathcal{U} \leftrightarrow \mathcal{U} : (\mathbb{H} \otimes -)$ $\boxed{T_{\mathcal{U}} = T_{\mathcal{K}} \; (= T)}$ $T_{\mathcal{K}} : \mathcal{K} \leftrightarrow \mathcal{K} : (\mathbb{H} \otimes -)$

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow[T_{\mathcal{K}}]{} & \mathcal{K} \\ {\scriptstyle \text{forget}} \downarrow & & \downarrow {\scriptstyle \text{forget}} \\ \mathcal{U} & \xrightarrow{T_{\mathcal{U}}} & \mathcal{U} \end{array}$$

Crucial properties

- T is exact
- T preserves $\otimes$
- Same properties with $V_1 \leftrightarrow V_n = (\mathbb{Z}/p)^n$

$X \in$ **Sp**, want to understand $\mathrm{Hom}^{\mathrm{h}}(BV, X)$.

Consequence of adjointness

$$X \leftarrow BV \times \mathrm{Hom}^{\mathrm{h}}(BV, X)$$

$$H^*X \to \mathbb{H} \otimes H^* \mathrm{Hom}^{\mathrm{h}}(BV, X)$$

$$\lambda_X : T(H^*X) \to H^* \mathrm{Hom}^{\mathrm{h}}(BV, X)$$

Question

How often is λ_X an isomorphism?

Assume: H^*X of finite type, $T(H^*X)$ of finite type.

Lannes works his magic

Tweak of the question

$$\lambda_X^{\mathrm{c}} : T(H^*X) \to H^* \operatorname{Hom}^{\mathrm{h}}(BV, C_pX)$$

N-conditions on $(T(H^*X), \operatorname{Hom}^{\mathrm{h}}(BV, C_pX))$

1. $T(H^*X)^1 = 0$
2. $\operatorname{Hom}^{\mathrm{h}}(BV, C_pX)$ is an H-space, is nilpotent, is p-complete
3. $\beta : T(H^*X)^1 \to T(H^*X)^2$ is injective
4. $\pi_1 \operatorname{Hom}^{\mathrm{h}}(BV, C_pX)$ is finite

Lannes Theorem

$$T(H^*X)^0 \cong H^0 \operatorname{Hom}^{\mathrm{h}}(BV, C_pX).$$

λ_X^{c} is an isomorphism if any **N**-condition holds.

Outlining the proof

X p-finite $\leftrightarrow$

- $\pi_0 X$ finite
- $\pi_i X$ finite p-group
- $\pi_i X = 0$ for $i \gg 0$.

One step at a time

1. λ_X an iso if $X = K(\mathbb{Z}/p, n)$
2. λ_X an iso if X is p-finite
3. A tour of towers
4. λ_X^c exists, iso if any **N**-condition holds

The case $X = K = K(\mathbb{Z}/p, n)$

This slide: $Y^* = H^*Y, \quad Y^*$ of finite type

$T_Y = $ left adjoint to $(Y^* \otimes -)$ on $\mathcal{K}$

$M = \mathrm{Hom}^{\mathrm{h}}(Y, K) \sim \prod K(\mathbb{Z}/p, m_i)$

Theorem

$$T_Y(K^*) \cong H^* \mathrm{Hom}^{\mathrm{h}}(Y, K)$$

Proof (Yoneda, Hom $= \mathrm{Hom}_{\mathcal{K}}$)

(1) $\mathrm{Hom}(T_Y(K^*), Z^*) \cong \mathrm{Hom}(K^*, Y^* \otimes Z^*)$

(2) $\mathrm{Hom}(M^*, Z^*) \cong [Z, M] \cong [Y \times Z, K]$

$\cong \mathrm{Hom}(K^*, Y^* \otimes Z^*)$

$$T(H^*K) \cong H^* \mathrm{Hom}^{\mathrm{h}}(BV, K)$$

The case in which X is p-finite.

$$X \to Y \to K = K(\mathbb{Z}/p, n)$$
$$H^*X \Leftarrow \mathrm{Tor}^{H^*K}(H^*Y, \mathbb{F}_p)$$
$$\lambda_Y \text{ an iso}$$

EMSS

Mapping side (EMSS + $\mathrm{Hom}^{\mathrm{h}}$ preserves fibrations)

$$H^* \mathrm{Hom}^{\mathrm{h}}(BV, X) \Leftarrow \mathrm{Tor}^{TH^*K}(TH^*Y, \mathbb{F}_p)$$

Space side (T is nice)

$$TH^*X \Leftarrow \mathrm{Tor}^{TH^*K}(TH^*Y, \mathbb{F}_p)$$

$$T(H^*X) \cong H^* \mathrm{Hom}^{\mathrm{h}}(BV, X)$$

The case in which X is p-finite.

$$X \to Y \to K = K(\mathbb{Z}/p, n)$$

$$H^*X \Leftarrow \mathrm{Tor}^{H^*K}(H^*Y, \mathbb{F}_p)$$ EMSS

λ_Y an iso (Ex. 6.15)

Mapping side (EMSS + $\mathrm{Hom}^{\mathrm{h}}$ preserves fibrations)

$$H^* \mathrm{Hom}^{\mathrm{h}}(BV, X) \Leftarrow \mathrm{Tor}^{TH^*K}(TH^*Y, \mathbb{F}_p)$$

Space side (T is nice)

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The case in which X is p-finite.

$$X \to Y \to K = K(\mathbb{Z}/p, n)$$

$$H^*X \Leftarrow \mathrm{Tor}^{H^*K}(H^*Y, \mathbb{F}_p) \qquad \text{EMSS}$$

λ_Y an iso

Mapping side (EMSS + $\mathrm{Hom}^{\mathrm{h}}$ preserves fibrations)

$$H^* \mathrm{Hom}^{\mathrm{h}}(BV, X) \Leftarrow \mathrm{Tor}^{TH^*K}(TH^*Y, \mathbb{F}_p)$$

Space side (T is nice)

$$TH^*X \Leftarrow \mathrm{Tor}^{TH^*K}(TH^*Y, \mathbb{F}_p)$$

$$T(H^*X) \cong H^* \mathrm{Hom}^{\mathrm{h}}(BV, X)$$

Tower $\{U_s\} = \{\cdots \to U_n \to U_{n-1} \to \cdots \to U_0\}$

$$\operatorname{colim} H^* U_s \stackrel{?}{=} H^* \operatorname{holim} U_s$$

Assume: each U_n p-finite, $\operatorname{colim} H^* U_s$ of finite type

N-conditions on $(H^*_\infty = \operatorname{colim} H^* U_s,\ U_\infty = \operatorname{holim} U_s)$

1. $H^1_\infty = 0$
2. U_∞ is an H-space, is nilpotent, is p-complete
3. $\beta : H^1_\infty \to H^2_\infty$ is injective
4. $\pi_1 U_\infty$ is finite

Tower Theorem

$H^0_\infty \cong H^0 U_\infty$, always.
$H^*_\infty \cong H^* U_\infty$ if any **N**-condition holds.

H^*X, $T(H^*X)$ of finite type

BK: $C_pX \sim \text{holim}\{X_s\}$, X_s p-finite, $\text{colim}\, H^*X_s \cong H^*X$

Proof of Lannes Theorem

$U_s = \text{Hom}^{\text{h}}(BV, X_s)$

1. $U_\infty \sim \text{Hom}^{\text{h}}(BV, C_pX)$
2. $H^*U_s \cong T(H^*X_s)$
3. $\text{colim}\, T(H^*X_s) = \text{colim}\, T(H^*X_s) \cong TH^*X$
4. Tower Theorem

$$T(H^*X)^0 \cong H^0U_\infty$$

$T(H^*X) \cong H^*U_\infty$ if any **N**-condition holds.

In the presence of a volunteer. . .

Tower version $\{U_s\}$ as before

V-condition: $\exists Y \to \{U_x\}$ inducing $H^*Y \cong \operatorname{colim} H^*U_s$.

Theorem: $\operatorname{holim} U_s \sim C_p Y$

***T*-version** $H^*X, T(H^*X)$ as before

V-condition: $\exists BV \times Y \to X$ inducing $H^*Y \cong T(H^*X)$.

Theorem: $\mathrm{Hom}^{\mathrm{h}}(BV, C_p X) \sim C_p Y$

Y is a (cohomological) *volunteer*.

VII

Maps between classifying spaces

Cohomology of homotopy fixed point sets

$V = \mathbb{Z}/p$, acts on X

X^{hV} and a close relative

$$\begin{array}{ccc} & & X_{hV} \\ & \nearrow & \downarrow \\ BV & \longrightarrow & BV \end{array} \qquad \boxed{X^{hV}} \qquad \begin{array}{ccc} X^{hV} & \longrightarrow & \mathrm{Hom}^{h}(BV, X_{hV}) \\ \downarrow & & \downarrow \\ * & \xrightarrow{\text{id}} & \mathrm{Hom}^{h}(BV, BV) \end{array}$$

$$\begin{array}{ccc} & & X_{hV} \\ & \nearrow & \downarrow \\ BV & \rightsquigarrow & BV \end{array} \qquad \boxed{X^{\mathcal{H}V}} \qquad \begin{array}{ccc} X^{\mathcal{H}V} & \longrightarrow & \mathrm{Hom}^{h}(BV, X_{hV}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}^{h}(BV, BV)_1 & \xrightarrow{\text{id}} & \mathrm{Hom}^{h}(BV, BV) \end{array}$$

Definition & question ($\mathbb{H} \cong$ identity piece of $T\mathbb{H}$)

$$\mathcal{F}\mathrm{ix}(X, V) \equiv \mathbb{H} \otimes_{T\mathbb{H}} T(X_{hV}) \rightarrow H^{*}X^{\mathcal{H}V} \qquad \text{(iso?)}$$

$$\mathcal{F}\text{ix}(X, V) \equiv \mathbb{H} \otimes_{T\mathbb{H}} T(H^* X_{\text{h}V})$$

M an $\mathbb{H}$-module: $\mathcal{F}\text{ix}(M) \equiv \mathbb{H} \otimes_{T\mathbb{H}} T(M)$

$\mathcal{F}\text{ix}(M) \cong 0$ if $M = H^* S^n$

$$T\mathbb{H} \to TM = H^*(\,\text{Hom}^{\text{h}}(BV, BV) \leftarrow \text{Hom}^{\text{h}}(BV, S^n)\,)$$

$\mathcal{F}\text{ix}(M) \cong 0$ if M is finite

Exactness.

$\mathcal{F}\text{ix}(X, V) \cong \mathbb{H} \otimes H^*(X^V)$ (Smith Theory)

$$(\text{finite}) \leftarrow H^*(BV \times X^V) \leftarrow H^*(X_{\text{h}V}) \leftarrow (\text{finite})$$

$$\mathcal{F}\mathrm{ix}(X, V) \equiv \mathbb{H} \otimes_{T\mathbb{H}} T(H^* X_{\mathrm{h}V})$$
$$\mathcal{F}\mathrm{ix}(M) \equiv \mathbb{H} \otimes_{T\mathbb{H}} T(M)$$

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Exactness.

$\mathcal{F}\mathrm{ix}(X, V) \cong \mathbb{H} \otimes H^*(X^V)$ (Smith Theory)

$$(\text{finite}) \leftarrow H^*(BV \times X^V) \leftarrow H^*(X_{\mathrm{h}V}) \leftarrow (\text{finite})$$

$BV \times X^V$ is a volunteer for $(C_pX)^{\mathcal{H}V}$!

Theorem

$$(C_pX)^{\mathcal{H}V} \sim BV \times C_p(X^V)$$
$$(C_pX)^{hV} \sim C_p(X^V)$$

Corollary (T-free, V-free)

Q a finite p-group, X a finite Q-complex $\implies$

$$(C_pX)^{hQ} \sim C_p(X^Q)$$

Aside: sighting of C_p(p-bad space)

- $Y = S^1 \vee S^1$
- $X = Y * S^1 * S^1 * \cdots * S^1$
- $Q = \mathbb{Z}/p$
- Q-actions: trivial on Y, rotation on S^1, diagonal on X

Y has problems ($C_pY \neq L_pY$, $C_pC_pY \neq C_pY \ldots$)

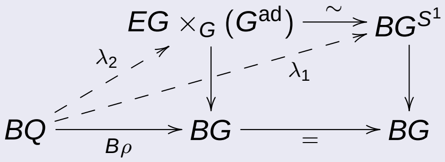
p-bad news

$$(C_pX)^{hQ} \sim_p C_pY$$

You can run but you can't hide.

- G a connected compact Lie group,
- Q a finite p-group, $\rho : Q \to G$
- $Z(B\rho) = \mathrm{Hom}^{\mathrm{h}}(BQ, BG)_{B\rho}$

Adjoint action and the free loop space



$$\Omega Z(B\rho) \sim \{\lambda_1\text{'s}\} \sim \{\lambda_2\text{'s}\} \sim G^{hQ}$$

Theorem (C_p-free)

$$Z(B\rho) \;\sim_p\; B(Z_G(\rho Q))$$

Maps $BQ \to BG$: how many components?

G, Q as before

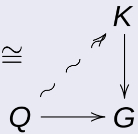
New ingredient (Q acts on X)

$$H^*X < \infty \quad \& \quad \chi(X) \neq 0 \mod p \quad \Longrightarrow \quad (C_pX)^{hQ} \neq \emptyset$$

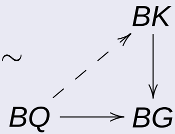
Theorem: $[BQ, BG] \cong \mathrm{Hom}(Q, G)/G$ (Sketch)

- OK if G is p-toral
- $\exists K \subset G$ with K p-toral, $\chi(G/K) \neq 0 \mod p$

$(G/K)^Q \cong$



$(G/K)^{hQ} \sim$



$$\coprod_{\langle \rho \rangle} BZ_G(\rho) \sim_p \mathrm{Hom}^h(BQ, BG) \quad \Leftrightarrow \quad \mathcal{C}_G^{\mathcal{C}_Q} \sim_p \mathcal{C}_G^{h\mathcal{C}_Q}$$

Maps into C_pBG for G finite

Q a finite p-group, G compact Lie

$$G \text{ connected} \implies \mathrm{Hom}^h(BQ, BG) \sim_p \mathrm{Hom}^h(BQ, C_p(BG))$$

Theorem

$$G \text{ finite} \implies \mathrm{Hom}^h(BQ, BG) \sim_p \mathrm{Hom}^h(BQ, C_p(BG))$$

1. find a faithful $\rho \colon G \to U(n) = K$
2. $$\begin{array}{ccccc} K/G & \longrightarrow & BG & \longrightarrow & BK \\ \downarrow & & \downarrow & & \downarrow \\ C_p(K/G) & \longrightarrow & C_p(BG) & \longrightarrow & C_p(BK) \end{array}$$
3. Apply $\mathrm{Hom}^h(BQ, -)$, compare bases and fibres.

$\mathcal{P} = \{\text{finite } p\text{-groups}\}$
$\bar{\mathcal{P}} = \{\text{finite } p\text{-groups}\} + \{\textit{monomorphisms}\}$ $\qquad I : \bar{\mathcal{P}} \to \mathcal{P}$

Fusion functor (X a space)

$$\Phi_X : \mathcal{P}^{\mathrm{op}} \to \mathbf{Set}, \qquad \Phi_X(Q) = [BQ, X]$$

$$\Phi_{BG} \sim \Phi_{C_p BG}$$

Does Φ_{BG} determine $C_p BG$?

Frugal fusion functor

$$\Psi_G : \bar{\mathcal{P}}^{\mathrm{op}} \to \mathbf{Set}, \quad \Psi_G(Q) = \{Q \hookrightarrow G\}$$

$$\Phi_{BG} \cong \mathrm{LKan}_{I^{\mathrm{op}}} \Psi_G$$

Does Ψ_G determine $C_p BG$?

$\mathcal{P} = \{\text{finite } p\text{-groups}\}$
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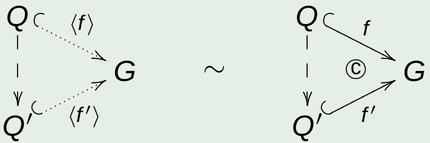
Does Ψ_G determine $C_p BG$?

$\Psi_G \rtimes \bar{\mathcal{P}} \ltimes *$ $=$ 

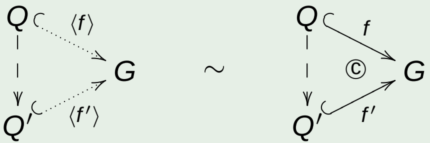
Fusion system $\mathcal{F} = \mathcal{F}_p(G)$

- $P \subset G$ a Sylow p-sugroup
- $\mathcal{F} = \{Q \subset P\}$
- $Q \to_{\mathcal{F}} Q' \;=\; f : Q \to Q$ s.t. $\exists g : f = g(-)g^{-1}$

$$\mathcal{F} \quad \sim \quad \Psi_G \rtimes \bar{\mathcal{P}} \ltimes *$$

Does $\mathcal{F}$ determine C_pBG?

$$\Psi_G \rtimes \bar{\mathcal{P}} \ltimes * \quad = \quad \cdots \quad \sim \quad \cdots$$



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- $Q \to_{\mathcal{F}} Q' \ = \ f : Q \to Q$ s.t. $\exists g : f = g(-)g^{-1}$

$$\mathcal{F} \quad \sim \quad \Psi_G \rtimes \bar{\mathcal{P}} \ltimes *$$

Does $\mathcal{F}$ determine C_pBG?

VIII

Linking systems and p-local classifying spaces

$$\mathcal{F} \sim \{Q \overset{\langle f \rangle}{\hookrightarrow} G\}$$

Centralizer diagram $\Longrightarrow$ something missing $(C = \{Q \subset G\})$

$C_pBG \sim_p \operatorname{hocolim}_{\mathcal{F}} J \qquad J : \langle f \rangle \mapsto BZ_G(fQ)$

Troubling questions

1. Need $\mathcal{F}$ **and** J (better, C_pJ) to get C_pBG?
2. J too fancy? $C_pBZ_G(fQ)$ not "algebraic"
3. Circular scam? $J(1 \to G) = BG \sim_p C_pBG$

Solution to (2) and (3): find better C

Does $\mathcal{F}$ determine C_pBG?

G a finite group, $\mathcal{F} = \mathcal{F}_p(G)$

$$\mathcal{F} \sim \{ Q \overset{\langle f \rangle}{\hookrightarrow} G \}$$

Centralizer diagram $\implies$ something missing $(\mathcal{C} = \{Q \subset G\})$

$C_pBG \sim_p \operatorname{hocolim}_{\mathcal{F}} J \qquad J : \langle f \rangle \mapsto BZ_G(fQ)$

Troubling questions

1. Need $\mathcal{F}$ **and** J (better, C_pJ) to get C_pBG?
2. J too fancy? $\quad C_pBZ_G(fQ)$ not "algebraic"
3. Circular scam? $\quad J(1 \to G) = BG \sim_p C_pBG$

Solution to (2) and (3): **find better** $\mathcal{C}$

Switch to the p-centric collection

A more economical approach

Definition: $Q \subset G$ p-centric (Q a p-group)

$$Z(Q) = \text{Sylow } p\text{-subgroup in } Z_G(Q)$$

Theorem $C = \{p\text{-centric } Q \subset G\}$

$$\begin{aligned} BG &\sim_p (K_C)_{hG} \\ &\sim_p \operatorname{hocolim} I \qquad I(-) \sim BQ \\ &\sim_p \operatorname{hocolim} J \qquad J(-) \sim BZ_G(Q) \end{aligned}$$

Fewer subgroups, smaller(?) centralizers, rarely $G = Z_G(Q)\ldots$

$$\mathcal{F}_c \sim \{Q \overset{\langle f \rangle}{\hookrightarrow} G,\ f(Q) \subset G\ p\text{-centric}\}$$

Centralizer decomposition

$$C_p BG \sim_p \operatorname{hocolim} J \qquad J : \langle f \rangle \mapsto BZ_G(fQ)$$

Centralizers: simplify at p to centers

$$Q\ p\text{-centric} \implies Z_G(Q) \cong Z(Q) \times\ p'\text{-group}$$

$$\implies C_p BZ_G(Q) \sim BZ(Q)$$

$$C_p BG \sim \operatorname{hocolim} BZ(fQ)$$

Centers: algebraic, and easily extracted from $C_p BG$

$$BZ(fQ) \sim_p \operatorname{Map}(BQ, C_p BG)_f$$

The categorical p-centric centralizer model

$\mathcal{F}_c = \{\langle f\rangle : f\colon Q \hookrightarrow G \text{ } p\text{-centric}\}$

$$\text{Functor } \mathcal{Z}\colon \mathcal{F}_c \to \mathbf{Grpd}, \qquad \mathcal{Z}(\langle f\rangle) \sim Z(Q)$$

Grothendieck construction

$$\begin{aligned} C_pBG &\sim_p \operatorname{hocolim} B\mathcal{Z} \\ &\sim_p \mathrm{N}(\mathcal{Z} \rtimes \mathcal{F}_c) \end{aligned}$$

$\mathcal{F}$ vs. $\mathcal{Z} \rtimes \mathcal{F}_c$

$$\mathcal{F} \sim \begin{array}{ccc} Q & \hookrightarrow & G \\ \downarrow & & \downarrow {\scriptstyle gZ_G(Q)} \\ Q' & \hookrightarrow & G \end{array} \qquad \mathcal{Z} \rtimes \mathcal{F}_c \sim \begin{array}{ccc} Q & \xrightarrow{p.\text{-c}} & G \\ \downarrow & & \downarrow {\scriptstyle gZ'_G(Q)} \\ Q' & \xrightarrow{p.\text{-c}} & G \end{array}$$

$$Z_G(Q) \cong Z(Q) \times Z'_G(Q)$$

The linking system

Linking system $\mathcal{L}_c := \mathcal{Z} \rtimes \mathcal{F}_c$

Recovering $\mathcal{L}_c$ from C_pBG $\quad X = C_pBG$

$$\mathcal{F}_c \quad \sim \quad \begin{array}{ccc} Q, & BQ & \xrightarrow{\langle f \rangle\text{:p.c.}} \\ h \downarrow & Bh \downarrow & \qquad X \\ Q', & BQ' & \xrightarrow{\langle f' \rangle\text{:p.c.}} \end{array} \qquad \mathcal{Z}\langle f \rangle = \Pi_1 \operatorname{Hom}^h(BQ, X)_f$$

Now just form $\mathcal{Z} \rtimes \mathcal{F}_c$

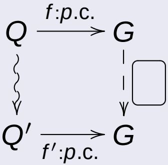
Theorem

$$C_pBG \sim C_pBG' \iff \mathcal{L}_c(G) \sim \mathcal{L}_c(G')$$

$\mathcal{L}_c$ vs. $\mathcal{F}_c$ – the orbit picture

$Z_G(Q) \cong Z(Q) \times \mathbb{Z}'_G(Q)$

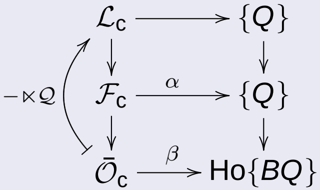
Three categories



$\mathcal{L}_c$: $\boxed{gZ'_G(Q)}$

$\mathcal{F}_c$: $\boxed{gZ_G(Q)}$

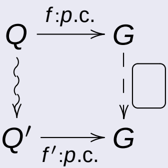
$\bar{\mathcal{O}}_c$: $\boxed{Q'gZ_G(Q)}$



Three categories $X = BG, C_pBG$

Category	Object	Morphism
$\mathcal{L}_c$	$Q + f_{p.c.} : BQ \to X$	$h : Q \to Q'_* + \omega$
$\mathcal{F}_c$	$Q + \langle f_{p.c.} \rangle \in [BQ, X]$	$h : Q \to Q'$
$\bar{\mathcal{O}}_c$	$Q + \langle f_{p.c.} \rangle \in [BQ, X]$	$\langle h \rangle \in [BQ, BQ']$

Three categories



$\mathcal{L}_{\mathrm{c}}$: $\boxed{gZ'_G(Q)}$

$\mathcal{F}_{\mathrm{c}}$: $\boxed{gZ_G(Q)}$

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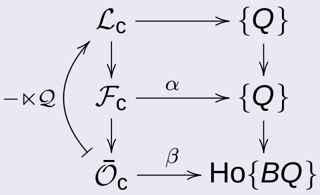
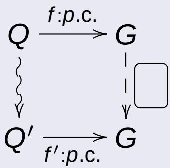


$\mathcal{L}_c$ vs. $\mathcal{F}_c$ – the orbit picture

$Z_G(Q) \cong Z(Q) \times \mathbb{Z}'_G(Q)$

Three categories



$\mathcal{L}_c$: $\boxed{gZ'_G(Q)}$

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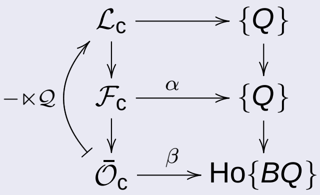
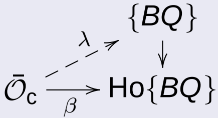


Conclusions

$\mathcal{Q} =$



$$\mathcal{F}_c + \alpha = \bar{\mathcal{O}}_c + \beta$$

$$\mathcal{L}_c = \mathcal{F}_c + u + \lambda$$

Lifting from Ho(**Sp**) to **Sp**

An example $\qquad$ (K a group)

$$\begin{array}{ccc} & & \{BQ\} \\ & \overset{\lambda}{\nearrow} & \downarrow \\ \mathcal{C}_K & \longrightarrow & \mathrm{Ho}\{BQ\} \end{array} \quad = \quad \begin{array}{ccc} & & B\,\mathrm{Aut}^{\mathrm{h}}(BQ) \\ & \overset{\lambda}{\nearrow} & \downarrow \\ BK & \longrightarrow & B\,\mathrm{Out}(Q) \end{array}$$

Obstruction $\in H^3(BK; Z(Q))$, $\qquad \langle\lambda\rangle \leftrightarrow H^2(BK, Z(Q))$

A generalization

$$\begin{array}{ccc} & & \{BQ\} \\ & \overset{\lambda}{\nearrow} & \downarrow \\ \beta\colon \mathcal{D} & \longrightarrow & \mathrm{Ho}\{BQ\}_{p\text{.-c}} \end{array}$$

Obstruction $\in \lim^3_{\mathcal{D}^{\mathrm{op}}} \mathcal{Z}$, $\quad \langle\lambda\rangle \leftrightarrow \lim^2_{\mathcal{D}^{\mathrm{op}}} \mathcal{Z}$, $\quad \mathcal{Z}(d) = Z(\pi_1\beta(d))$

G = finite group

$\mathcal{Z} : \bar{\mathcal{O}}_c^{op} \to \mathrm{Ho}(\mathbf{Sp})$, $\mathcal{Z}(Q \to G) = Z(Q)$

Oliver's Theorem

$$\lim{}^2 \mathcal{Z} = 0$$

Consequences

There is only one way to enrich $\mathcal{F}_c$ to $\mathcal{L}_c$.

$$\mathcal{F}_c(G) \sim_{\{Q\}} \mathcal{F}_c(G') \quad \Leftrightarrow \quad C_p(G) \sim C_p(G')$$

p-local finite group X

I. Fusion data (many versions) (+ Axioms. . .)

- (Frugal) fusion functor (Ψ) Φ, fusion category $\bar{\mathcal{P}} \ltimes \Psi$
- Fusion system $\mathcal{F}$ based on "Sylow" group P

II. p-centric fusion data (two versions)

- p-centric subcategory $\mathcal{F}_c \subset \mathcal{F}$
- p-centric orbit category $\bar{\mathcal{O}}_c$, $\quad \beta : \bar{\mathcal{O}}_c \to \mathrm{Ho}\{BQ\}$

III. Linking system

- $\mathcal{L}_c = \Pi_1 \lambda \rtimes \bar{\mathcal{O}}_c$, $\lambda \colon \bar{\mathcal{O}}_c \to \{BQ\}$ a lift of β

$C_p BG$ sans G!

$$BX \sim C_p\, \mathrm{N}(\mathcal{L}_c) \sim C_p(\mathrm{hocolim}\,\lambda)$$

IX

p-compact groups

Dictionary

Compact Lie group	p-compact group
group G	loop space X
compact, smooth	$H^*X < \infty, \quad BX \sim C_pBX$
$\rho\colon G \to H$	$B\rho : BX \to BY$
$\ker \rho = \{1\}$	$H^*\text{fibre}(BX \to BY) < \infty$
H/G	$Y/X =$fibre $BX \to BY$
$Z_H(\rho G)$	$Z_Y(\rho X) = \Omega\,\text{Hom}^{\text{h}}(BX, BY)_{B\rho}$
abelian	$Z_X(X) \sim X$
torus $T = (S^1)^r$	p-complete torus $\hat{T} = C_pT$
...	...

$$Z_X(\rho V) \to X \qquad \text{(eval at } *\text{)}$$

Theorem: centralizer is a p-compact group.

$Z_X(\rho V)$ is a p-compact group.

Theorem: centralizer is a “subgroup” of X.

$Z_X(\rho V) \to X$ is a monomorphism.

Theorem: there exist nontrivial ρ’s.

$$X \not\simeq * \implies \exists \text{ nontrivial } \rho\colon V \to X$$

More cohomology of homotopy fixed point sets

$V = \mathbb{Z}/p$, acts on X, $H^*X < \infty$, $X \sim C_pX$

$$H^*X_{hV} \cong \text{finitely generated module over } \mathbb{H} = H^*BV$$

Algebraic Smith theory (no need for X^V)

- $(\text{finite}) \leftarrow \mathbb{H} \otimes (\text{finite}) \leftarrow H^*X_{hV} \leftarrow (\text{finite})$
- $\mathcal{F}\text{ix}(X, V) \cong \mathbb{H} \otimes (\text{finite})$

Consequences

If any **N**-condition holds

- $H^*X^{hV} < \infty$
- $H^*(X, X^{hV})_{hV} < \infty$
- $\chi(X^{hV}) = \chi(X) \mod p$

The centralizer of V in X is a p-compact group.

$V = \mathbb{Z}/p$, $\rho : V \to X$

Formula for $Z_X(\rho V)$ $\qquad \Lambda BX = \mathrm{Hom}^{\mathrm{h}}(S^1, BX)$

$$Z_X(\rho V) \sim \begin{array}{ccc} & & \Lambda BX \\ & \nearrow & \downarrow \\ BV & \xrightarrow[B\rho]{} & BX \end{array} \sim \begin{array}{c} B\rho^*\Lambda BX \\ \downarrow \uparrow \\ BV \end{array} \sim X^{\mathrm{h}V}$$

Algebraic Smith theory $\qquad$ **N**-condition: $X^{\mathrm{h}V}$ is an H-space

$$H^*X^{\mathrm{h}V} < \infty$$

End game $\qquad Z = Z_X(\rho V)$

BX p-complete	$\Longrightarrow$	BX p-local $\Longrightarrow$ BZ p-local
$+\ \pi_0 Z$ finite	$\Longrightarrow$	$\pi_1 BZ$ a finite p-group
	$\Longrightarrow$	BZ p-good, p-complete

The centralizer of V in X is a subgroup

$V = \mathbb{Z}/p$, $\rho : V \to X$, $Z = Z_X(\rho V)$

Construction of X/Z: V acts on BX^p by permutation.

$$\begin{array}{ccccc}
X^p/X & \longrightarrow & BX & \xrightarrow{\text{diag}} & BX^p \\
\downarrow{\scriptstyle (-)^{hV}} & & \downarrow{\scriptstyle (-)^{hV}} & & \downarrow{\scriptstyle (-)^{hV}} \\
\coprod_\rho X/Z\rho \sim (X^p/X)^{hV} & \longrightarrow & \coprod_\rho BZ\rho & \longrightarrow & BX
\end{array}$$

$H^*(\coprod_\rho X/Z\rho) < \infty$ by algebraic Smith theory

N-condition:

$$X/Z\rho \longrightarrow BZ\rho \longrightarrow BX \quad \Longrightarrow \quad \coprod_\rho X/Z\rho \text{ is } p\text{-complete}$$

$$\coprod_\rho X/Z\rho \sim (X^p/X)^{hV}$$

Existence of a nontrivial ρ ($\rho \neq 0$)

Statement	Reason
$\chi(X) = 0 \mod p$	Milnor-Moore
$\chi(X^p/X) = 0 \mod p$	$X^p/X \sim X^{p-1}$
$\chi(\coprod_\rho X/Z\rho) = 0 \mod p$	algebraic Smith theory
$\chi(X/Z_0) = 1$	Miller's theorem

$$\therefore \exists \rho \neq 0 \text{ such that } \chi(X/Z\rho) \neq 0 \mod p$$

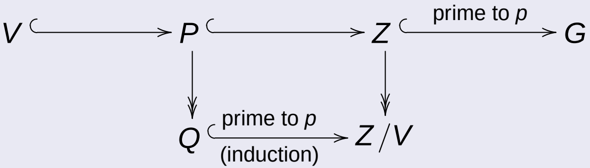
Existence of a Sylow p-subgroup for finite G

G finite, $V = \mathbb{Z}/p$

Seek $P \subset G$ such that $p \nmid \#(G/P)$

Induction on $\#(G)$ $(Z = Z_G(V))$

Find $V \subset G$ with $p \nmid \#(G/Z)$.



Existence of a maximal torus in X

Seek $\hat{T} \subset X$ such that $\chi(X/\hat{T}) \neq 0$

Induction on size of X $\quad (Z = Z_X(V))$

Find $V \subset X$ with $\chi(X/Z) \neq 0$.

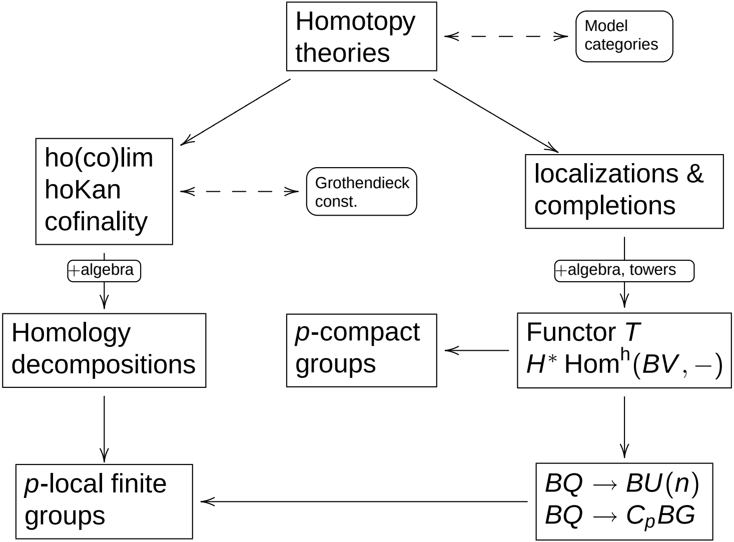
$$\begin{array}{ccccccc}
V & \hookrightarrow & \hat{T} & \hookrightarrow & Z & \xrightarrow[]{\chi\neq 0} & X \\
 & & \downarrow & & \downarrow & & \\
 & & \hat{S} & \underset{\text{(induction)}}{\xrightarrow{\chi\neq 0}} & Z/V & &
\end{array}$$

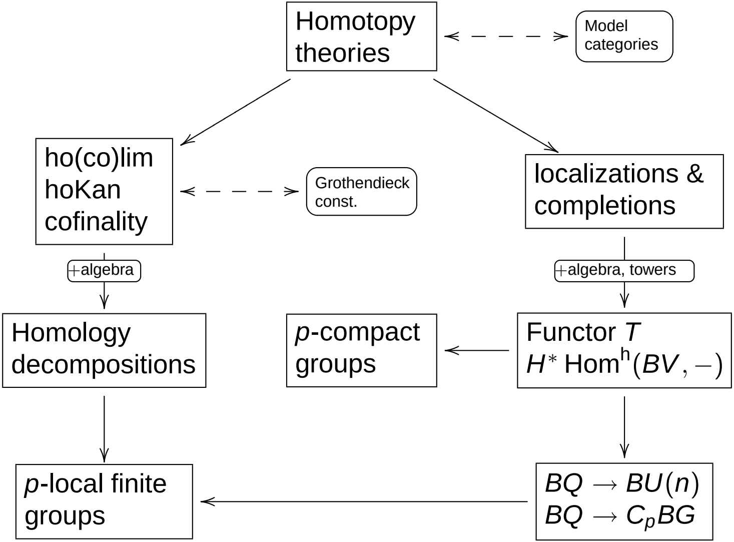
More or less OK if Z/V “smaller” than X

X

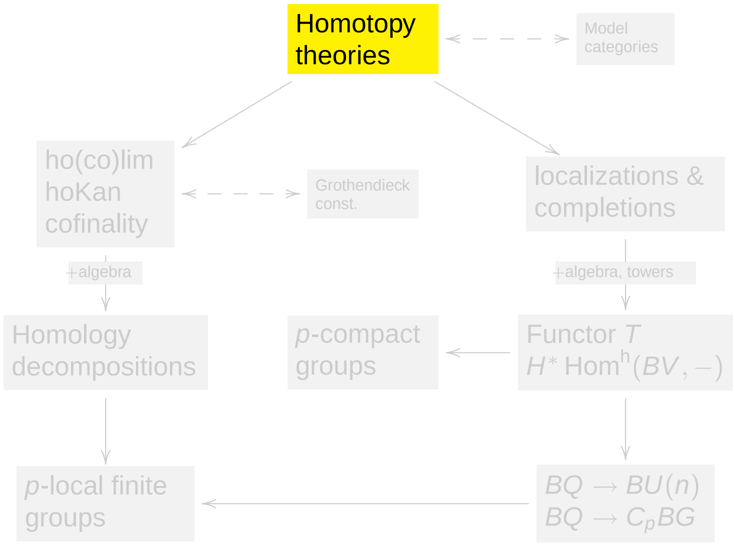
Wrapping up

Another commutative diagram?





Homotopy theories
Model categories
ho(co)lim hoKan cofinality
Grothendieck const.
localizations & completions
+algebra
+algebra, towers
Homology decompositions
p-compact groups
Functor T
$H^* \text{Hom}^h(BV, -)$
p-local finite groups
$BQ \to BU(n)$
$BQ \to C_pBG$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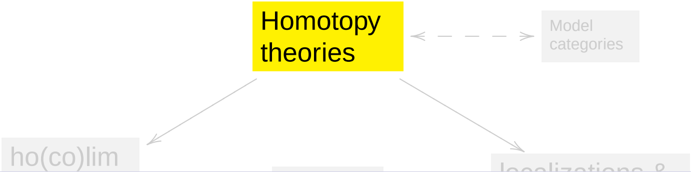
Homotopy theories

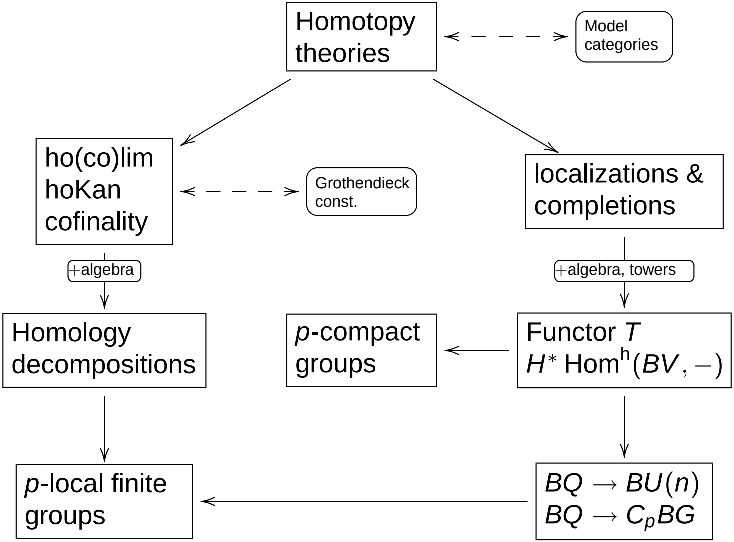


Higher category theory

- Homotopy theory $\mathbb{T} = (\infty, 1)$-category
- Category of $\mathbb{T}$'s $= (\infty, 2)$-category(?)
- Conformal field theories $\leftrightarrow (\infty, n)$-categories(?)

Enriched morphisms

- Spectra, chain complexes, . . .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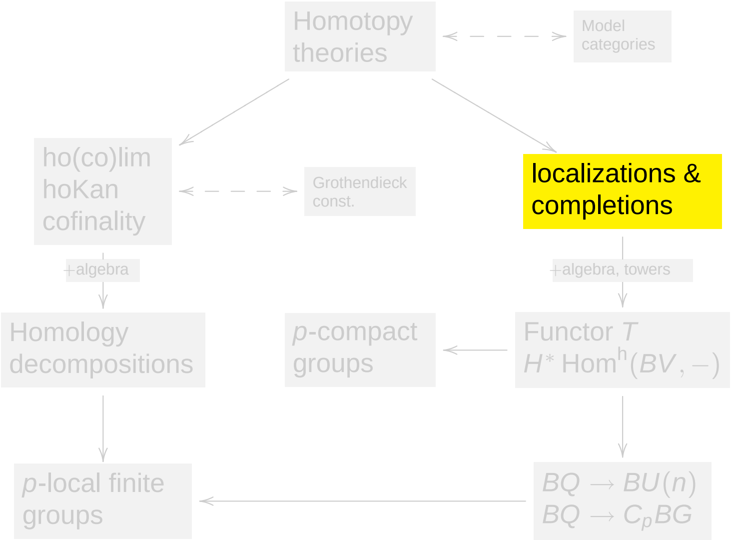
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Right Bousfield localization

$I\colon (\mathcal{C},\mathcal{E}) \to (\mathcal{C},\mathcal{F})$ a *right Bousfield localization* if $\exists J$

$$J\colon (\mathcal{C},\mathcal{F}) \leftrightarrow (\mathcal{C},\mathcal{E}) \ : I \quad J \text{ full \& faithful}$$

ho(co)lim
hoKan
cofinality

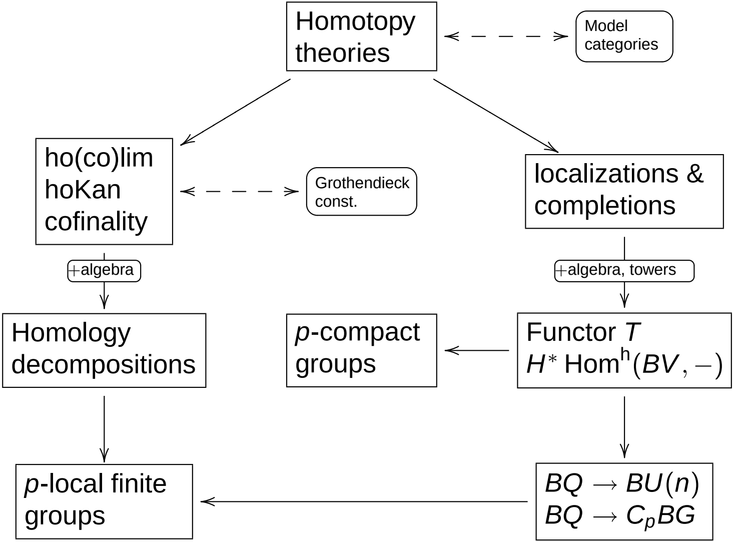
Grothendieck const.

localizations & completions

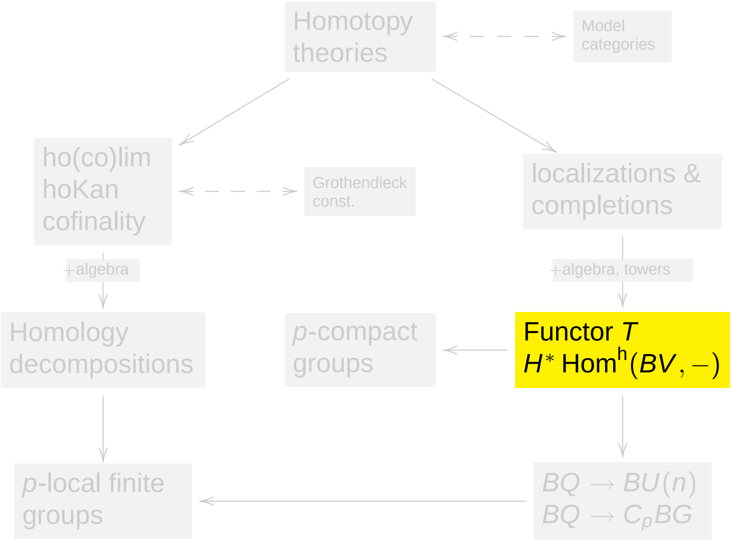
Examples

- Cellularization
- Local cohomology ($\mathcal{C} = \mathbf{Ch}_R$)
- Fibre of $X \to L_n^f(X)$
- Homology approximations?

The functor T and $H^* \operatorname{Hom}^h(BV, -)$



The functor T and $H^* \operatorname{Hom}^{\mathrm{h}}(BV, -)$



Homotopy

Model

Applications of T

- Structure of $\mathcal{U}$ and $\mathcal{K}$
- Realization of unstable modules
- H^*(arithmetic groups)
- Cohomological uniqueness

Homology decompositions

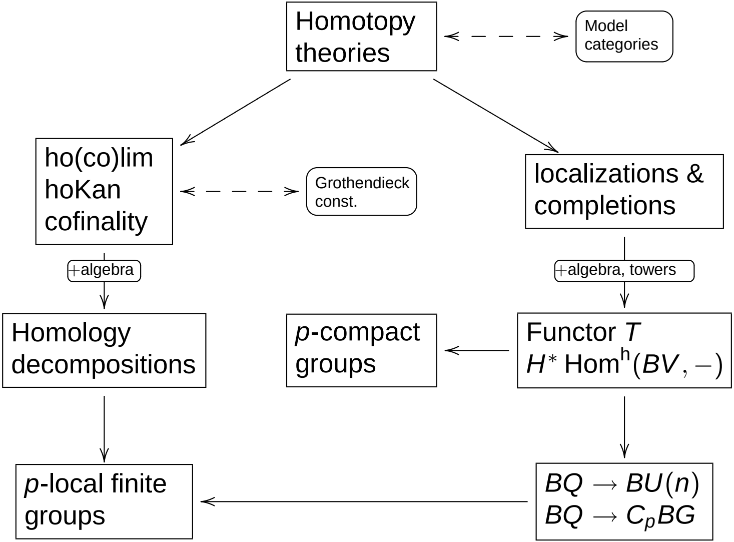
p-compact groups

Functor T
$H^* \operatorname{Hom}^{h}(BV, -)$

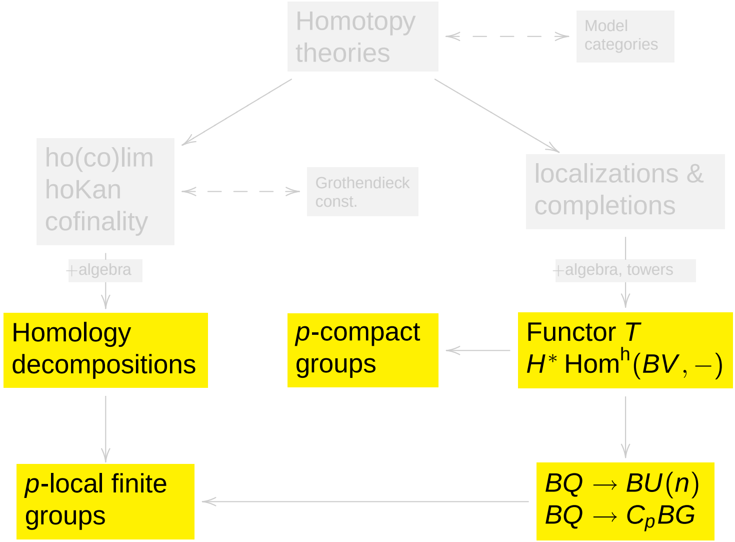
Out with BV!

- $BV \leftrightarrow B$(abelian) via $H^* \leftrightarrow MU_*$

Homology decompositions and maps to BG



Homology decompositions and maps to BG



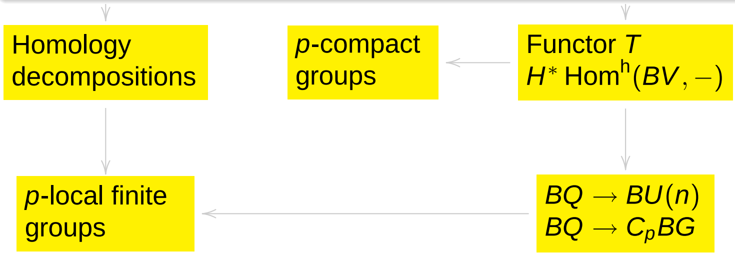
Homology decompositions and maps to BG

Revealed

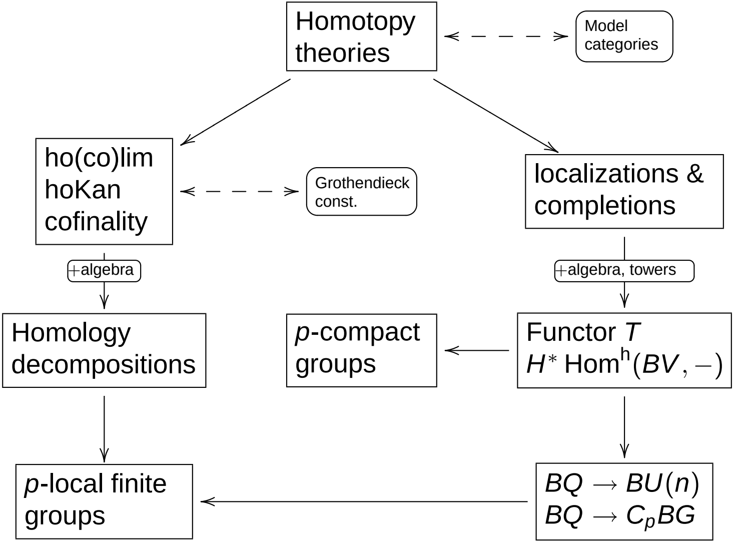
- Decompositions of BG (compact Lie), BX, $B\mathcal{F}$
- $\mathrm{Aut}^{\mathrm{h}}(BG)$, $\mathrm{Aut}^{\mathrm{h}}(BX)$, $\mathrm{Aut}^{\mathrm{h}}(B\mathcal{F})$

Hidden

- $\mathrm{Hom}^{\mathrm{h}}(BG, BK)$, $\mathrm{Hom}^{\mathrm{h}}(BX, BY)$, $\mathrm{Hom}^{\mathrm{h}}(B\mathcal{F}, B\mathcal{F}')$



p-compact groups and p-local finite groups



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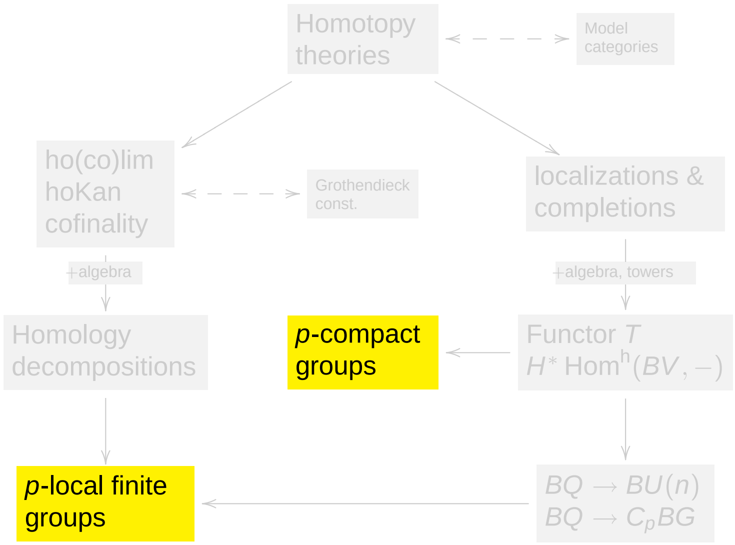

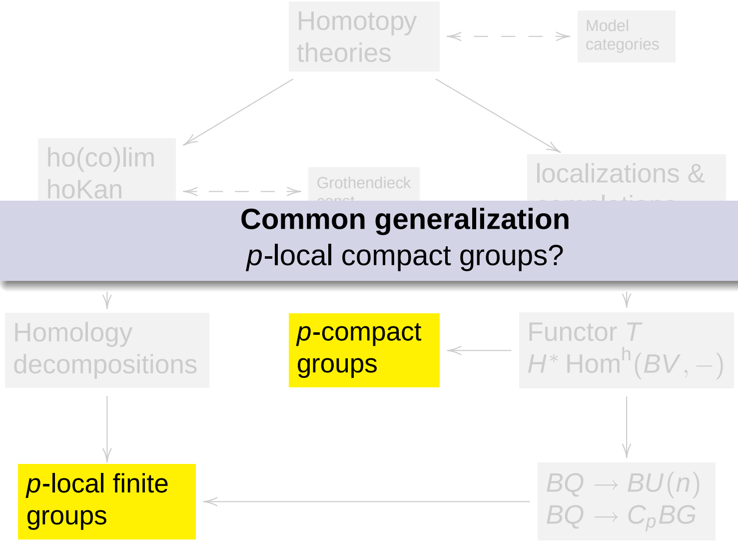

Common generalization

p-local compact groups?

Many other directions

- Realization of polynomial algebras (Steenrod's problem)
- Finite loop spaces
- etc. etc., . . .

- `ArXiv`
- `hopf.math.purdue.edu`
- MathSciNet (review and reference crosslinks)

Happy Surfing!