Combinatorial Model Categories Have Presentations

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We show that every combinatorial model category is Quillen equivalent to a localization of a diagram category (where ''diagram category'' means diagrams of simplicial sets). This says that every combinatorial model category can be built from a category of ''generators'' and a set of ''relations.''   © 2001 Elsevier Science

1. INTRODUCTION

In the companion paper [D2] we introduced a technique for constructing model categories via generators and relations. The two main points were as follows:

(1)   From a small category $\mathscr{C}$ one can construct a model category $U\mathscr{C}$ which is in some sense the free model category generated by $\mathscr{C}$. This $U\mathscr{C}$ is simply the category of diagrams $s\mathscr{S}et^{\mathscr{C}^{op}}$ with an appropriate model structure.

(2)   Given a set of maps S in $U\mathscr{C}$, one can form the *localization* $U\mathscr{C}/S$—this is the closest model category to $U\mathscr{C}$ in which the maps from S have been added to the weak equivalences. We regard this process of localization as ''imposing relations'' into the model category $U\mathscr{C}$.

When a model category can be built from generators and relations we say that it has a *small presentation*. More precisely, a presentation for a model category $\mathscr{M}$ consists of (1) a small category $\mathscr{C}$, (2) a set of maps S in the diagram category $U\mathscr{C}$, and (3) a specified Quillen equivalence $L: U\mathscr{C}/S \rightleftarrows \mathscr{M} : R$. As in [D2] we will denote a Quillen pair as a map of model categories in the direction of the left adjoint, so that our presentation takes on the form $U\mathscr{C}/S \rightsquigarrow \mathscr{M}$.

There are certainly model categories which cannot be given presentations, but the majority of those one encounters can indeed be built up in this way.

In this paper we will deal with a very broad class called the *combinatorial* model categories, which were introduced by J. Smith [Sm]. These include essentially any model category of algebraic origin, as well as anything constructed in some way out of simplicial sets. Our aim is to prove the following result, announced in [D2]:

THEOREM 1.1. *Every combinatorial model category has a small presentation.*

This has the following corollary.

COROLLARY 1.2. *Every combinatorial model category is Quillen equivalent to one which is simplicial and left proper and (this is slightly harder) in which every object is cofibrant.*

In [D1] it was proven, using very different methods, that every left-proper, combinatorial model category is Quillen equivalent to a simplicial model category. The above corollary offers a slight improvement on this, in that it eliminates the left-properness assumption.

We close this Introduction with a word about the proof of Theorem 1.1. To establish some intuition, consider the problem of giving a presentation for an abelian group A. The first thing one does is to find a surjection $\mathbb{Z}^r \twoheadrightarrow A$, and if R denotes the kernel then one automatically has the presentation $\mathbb{Z}^r/R \cong A$. Following this analogy our approach will be to define what it means for a map of model categories $\mathscr{M} \to \mathscr{N}$ to be a "surjection" (3.1), and we'll see that once one has a surjection $U\mathscr{C} \to \mathscr{M}$ then a presentation follows automatically (3.2). So getting the surjection will be the tricky part, and this depends on carefully choosing the category $\mathscr{C}$.

Since we think of $\mathscr{C}$ as a category of "generators," it's natural to try and choose it as a subcategory of $\mathscr{M}$. Intuitively, $\mathscr{C}$ should be big enough so that every object $X \in \mathscr{M}$ can be built up as a certain homotopy colimit of objects from $\mathscr{C}$, but this turns out to be more delicate than it sounds (see Section 4 for a precise statement). In the case when $\mathscr{M}$ is a *simplicial* model category, though, this plan can indeed be carried out.

When $\mathscr{M}$ is not simplicial we have to be more clever. Instead of choosing $\mathscr{C}$ as a subcategory of $\mathscr{M}$, we are forced to fatten $\mathscr{M}$ a little by looking at the cosimplicial objects $c\mathscr{M}$. By choosing an appropriate subcategory $\mathscr{C}$ of $c\mathscr{M}$ (Section 6) we are able to get our surjective map $U\mathscr{C} \to \mathscr{M}$.

1.3. *Overview.* Section 2 contains a brief discussion of combinatorial model categories, especially several key properties that will appear throughout the paper. In Section 3 we give a definition of "homotopically surjective" maps, and we prove both Theorem 1.1 and its corollary by assuming the existence of a surjective map $U\mathscr{C} \to \mathscr{M}$. The remainder of the paper is the quest for this surjective map.

Section 4 deals with "canonical homotopy colimits." Given a map $\gamma: \mathscr{C} \to \mathscr{M}$ and a fibrant object $X \in \mathscr{M}$ the canonical homotopy colimit hocolim($\mathscr{C} \times \varDelta \downarrow X$) is an object built out of all the information contained in the homotopy function complexes $\underline{\mathscr{M}}(\gamma c, X)$, for all $c \in \mathscr{C}$—it is a kind of homotopical approximation to X based on $\mathscr{C}$. We show in (4.4) that finding a surjective map $U\mathscr{C} \to \mathscr{M}$ is equivalent to finding a functor $\gamma: \mathscr{C} \to \mathscr{M}$ such that these homotopical approximations to X always give back X itself (up to weak equivalence, of course).

In Section 5 we produce the required surjective map in the case when $\mathscr{M}$ is a *simplicial* model category. This case is fairly simple based on our work so far. Section 6 handles the more general case—the ideas are similar to those from Section 5, but with one more level of complication.

Sections 7 and 8 contain some of the auxiliary proofs postponed from previous sections. Finally, since much of the paper is spent working with various homotopy colimits, we have for convenience enclosed an appendix recalling some of the basic properties we need. In particular, there are several instances in the paper where we have to identify two homotopy colimits over different indexing categories, and the key result letting us do this is Proposition A.4.

1.4. *Notation.* This paper is intended as a companion to [D2], and we assume a general familiarity with the notation and results of Sections 2, 3, and 5 of that paper. We deal quite a bit with overcategories here, so recall that if $F: \mathscr{C} \to \mathscr{D}$ is a functor and $X \in \mathscr{D}$ then $(F \downarrow X)$—often written $(\mathscr{C} \downarrow X)$ by abuse of notation—is the category whose objects are pairs $[c, Fc \to X]$ where $c \in \mathscr{C}$ and $Fc \to X$ is a map in $\mathscr{D}$. The morphisms of $(\mathscr{C} \downarrow X)$ are the obvious candidates.

2. COMBINATORIAL MODEL CATEGORIES

In this section we review the definition of combinatorial model categories, due to Smith [Sm], together with several of their important properties. The main theorem of this paper (1.1) is the homotopy-theoretic analog of a standard result about locally presentable categories, recalled in (2.4).

DEFINITION 2.1. A model category $\mathscr{M}$ is called *combinatorial* if it is cofibrantly-generated and the underlying category is locally presentable.

This definition is surprisingly powerful considering how simple it is. We'd better recall what all the words mean, though. The notion of a *cofibrantly-generated* model category is standard by now and may be found

in [Ho, Definition 2.1.17]—it requires that there are basic sets of cofibrations and trivial cofibrations which one can use to do the small object argument. The notion of a category being *locally presentable* is less familiar to homotopy theorists, so here's the definition.

DEFINITION 2.2. A category $\mathscr{C}$ is *locally presentable* if it is co-complete, and if there is a regular cardinal λ and a set of objects $\mathscr{A}$ in $\mathscr{C}$ such that:

(i) Every object in $\mathscr{A}$ is small with respect to λ-filtered colimits.

(ii) Every object of $\mathscr{C}$ can be expressed as a λ-filtered colimit of elements of $\mathscr{A}$.

For background on locally presentable categories one may consult [AR, Section 1.B] or [B]. An indexing category I is called λ-*filtered* if any subcategory with fewer than λ morphisms has a compatible cocone in I. The condition that an object A be small with respect to λ-filtered colimits is called λ-*presentable* in [AR], but we will follow Smith and call it λ-*small*. This means that for any diagram $X: I \to \mathscr{C}$ in which I is λ-filtered, one has $\mathscr{C}(A, \operatorname{colim} X_i) = \operatorname{colim} \mathscr{C}(A, X_i)$.

Locally presentable categories have the following important characteristics:

(1) For every object A, there exists a regular cardinal λ such that A is λ-small.

(2) For each regular cardinal λ, the λ-small objects in $\mathscr{C}$ have a *set* of representatives with respect to isomorphism—we'll use $\mathscr{C}_\lambda$ to denote the full subcategory determined by any such set.

The following proposition brings together the properties of combinatorial model categories we will need in this paper. Most of these statements are due to Smith and should one day appear in [Sm]. For the reader's convenience we provide proofs (or sketches of proofs, when we are lazy) in Section 7.

PROPOSITION 2.3. *Let $\mathscr{M}$ be a combinatorial model category.*

(i) *There exist cofibrant- and fibrant-replacement functors which preserve sufficiently large filtered colimits.*

(ii) *Sufficiently large filtered colimits of weak equivalences are again weak equivalences: if λ is a sufficiently large regular cardinal, I is a λ-filtered indexing category, and if $D_1, D_2: I \to \mathscr{C}$ are diagrams with a natural weak equivalence $D_1 \to D_2$, then $\operatorname{colim} D_1 \to \operatorname{colim} D_2$ is also a weak equivalence.*

(iii) *There exist functorial factorizations of maps $X \to Y$ as $X \overset{\sim}{\rightarrowtail} \tilde{X} \twoheadrightarrow Y$ and $X \rightarrowtail \tilde{Y} \overset{\sim}{\twoheadrightarrow} Y$ with the following property: for sufficiently large regular cardinals μ, if $X \to Y$ is a map between μ-small objects then both $\tilde{X}$ and $\tilde{Y}$ are μ-small as well.*

Note that the above properties are well-known for the model category of simplicial sets. They in some sense say that for a combinatorial model category the interesting part of the homotopy theory is all concentrated within some small subcategory—beyond sufficiently large cardinals the homotopy theory is somehow "formal." Model categories of the form $U\mathscr{C}/S$ certainly have this property (as they are combinatorial), and this observation explains why not every model category can have a small presentation.

2.4. *Locally Presentable Categories and Diagram Categories.*   The proof of Theorem 1.1 is somewhat involved, and it will help for us to establish a little background. Like many of the results in [D2], the theorem is once again a homotopy-theoretic analog of a standard result in category theory.

Suppose that $\mathscr{C}$ is a category and $F: \mathscr{A} \to \mathscr{C}$ is a functor (in many cases this will be the inclusion of a subcategory). For any $x \in \mathscr{C}$ consider the overcategory $(\mathscr{A} \downarrow x)$ together with the canonical diagram $(\mathscr{A} \downarrow x) \to \mathscr{C}$ which sends $[a, Fa \to x]$ to Fa. The colimit of this diagram (when it exists) is called the *canonical colimit of x with respect to $\mathscr{A}$* and we'll denote it $\mathrm{colim}(\mathscr{A} \downarrow x)$.

Remark 2.5.   In a locally presentable category $\mathscr{C}$ the following is true: For large enough regular cardinals λ, every x in $\mathscr{C}$ has the property that $(\mathscr{C}_\lambda \downarrow x)$ is λ-filtered and the natural map $\mathrm{colim}(\mathscr{C}_\lambda \downarrow x) \to x$ is an isomorphism (see [AR, Proposition 1.22]). We say that $\mathscr{C}_\lambda$ is *dense* in $\mathscr{C}$. So choosing a large enough cardinal gives us a way of canonically expressing any object of $\mathscr{C}$ as a λ-filtered colimit of λ-small objects.

When $\mathscr{C}$ is locally presentable it is co-complete, and so the inclusion $\mathscr{C}_\lambda \hookrightarrow \mathscr{C}$ extends to an adjoint pair

$$Re : Pre(\mathscr{C}_\lambda) \rightleftarrows \mathscr{C} : Sing$$

as in [D2, Proposition 2.1]. It's not hard to check that $Re(Sing\, x)$ is precisely the canonical colimit of x with respect to $\mathscr{C}_\lambda$, and so the map $Re(Sing\, x) \to x$ is an isomorphism (again, for large enough λ). A standard result in the theory of locally presentable categories roughly says that there is a "localization functor" $L: Pre(\mathscr{C}_\lambda) \to Pre(\mathscr{C}_\lambda)$ such that the above adjoint pair restricts to the image of L and becomes an equivalence of categories—in other words, $\mathscr{C}$ is equivalent to a full, reflective subcategory of the diagram category $Pre(\mathscr{C}_\lambda)$ [AR, Proposition 1.46].

Theorem 1.1 above is a direct homotopy-theoretic analog of this result. In Section 4 we define the notion of canonical *homotopy* colimits, and we will see in Section 5 that combinatorial model categories $\mathscr{M}$ which are simplicial have the following property: for sufficiently large regular cardinals

λ the subcategory $\mathcal{M}_\lambda$ is "homotopically dense," in that every x in $\mathcal{M}$ is weakly equivalent to its canonical homotopy colimit with respect to $\mathcal{M}_\lambda$. This condition will allow us to get a Quillen equivalence $U(\mathcal{M}_\lambda)/S \rightleftarrows \mathcal{M}$ (actually, we will replace $\mathcal{M}_\lambda$ with its subcategory of cofibrant objects, for technical convenience).

For combinatorial model categories which are not simplicial the story is slightly more complex, but the above ideas are still the central points. We refer the reader to the discussion which begins Section 6 for more about this.

3. HOMOTOPICALLY SURJECTIVE MAPS

In this section we will define what it means for a map of model categories $\mathcal{M} \to \mathcal{N}$ to be "surjective" (3.1), and we'll see that a surjection from a universal model category automatically yields a presentation (3.2). At the end of the section we show how Theorem 1.1 and Corollary 1.2 will follow as soon as one has found a surjective map of the form $U\mathcal{C} \to \mathcal{M}$.

DEFINITION 3.1.   A map of model categories $\mathcal{M} \xrightarrow{L} \mathcal{N}$ is *homotopically surjective* if it has the following property: for every fibrant object X in $\mathcal{N}$ and for every cofibrant replacement $[RX]^{cof} \xrightarrow{\sim} RX$ for RX, the induced map $L([RX]^{cof}) \to X$ is a weak equivalence in $\mathcal{N}$. We often omit the word "homotopically" for brevity.

Equivalently, the definition says that on the level of homotopy categories the derived functors $\mathbb{L}: \operatorname{Ho} \mathcal{M} \rightleftarrows \operatorname{Ho} \mathcal{N} : \mathbb{R}$ are such that $\mathbb{L} \circ \mathbb{R}$ is naturally isomorphic to the identity.

The following result says that for combinatorial model categories any homotopically surjective map may be localized so as to become a Quillen equivalence. (Note that the left-properness assumption on $\mathcal{M}$ is there so that we may form the localization $\mathcal{M}/S$; otherwise it is unimportant.)

PROPOSITION 3.2.   *Let $\mathcal{M}$ and $\mathcal{N}$ be combinatorial model categories, where $\mathcal{M}$ is left proper. Suppose that $L: \mathcal{M} \to \mathcal{N}$ is a surjective map. Then there is a set of maps S in $\mathcal{M}$ which become weak equivalences under L^{cof} and such that the induced map $\mathcal{M}/S \to \mathcal{N}$ is a Quillen equivalence.*

(Recall that L^{cof} (which we call the left-derived functor of L) denotes the result of precomposing L with some cofibrant-replacement functor in $\mathcal{M}$.)

Proof.   Choose a regular cardinal λ which is large enough so that the following are true:

(1) $\mathcal{M}_\lambda$ is dense in $\mathcal{M}$ (cf. Remark 2.5),

(2) λ-filtered colimits of weak equivalences in $\mathcal{M}$ are again weak equivalences,

(3) $\mathcal{M}$ has a cofibrant replacement functor $A \to A^{cof}$ which preserves λ-filtered colimits,

(4) $\mathcal{N}$ has a fibrant replacement functor $X \to X^{fib}$ which preserves λ-filtered colimits, and

(5)   the right adjoint R to L preserves λ-filtered colimits (see [AR, Proposition 1.66]).

Let S be the set consisting of all the natural maps

$$A^{cof} \to R([LA^{cof}]^{fib}),$$

where $A \in \mathcal{M}_\lambda$. The condition that $\mathcal{M} \to \mathcal{N}$ is homotopically surjective shows that the derived functor of L takes maps in S to weak equivalences, and so L descends to a map $\mathcal{M}/S \to \mathcal{N}$. It is readily checked that this new map is also homotopically surjective. To check that this is a Quillen equivalence one must verify that for every object X in $\mathcal{M}$, the composite $X^{cof} \to R([LX^{cof}]^{fib})$ is a weak equivalence in $\mathcal{M}/S$. But any X in $\mathcal{M}$ is a λ-filtered colimit of objects in $\mathcal{M}_\lambda$ by the assumption (1), and all the functors in sight commute with such colimits by the assumptions (3)–(5). So the map in question is a λ-filtered colimit of maps in S, which are weak equivalences in $\mathcal{M}/S$. Finally, the assumption (2) says that λ-filtered colimits preserve weak equivalences in $\mathcal{M}$, and it's easy to check that this property is inherited by any localization of $\mathcal{M}$. This completes the proof.   ∎

The following proposition will be our focus in the rest of the paper. Granting it for the moment, we can prove Theorem 1.1 and its corollary.

PROPOSITION 3.3.   *If $\mathcal{M}$ is a combinatorial model category then there exists a small category $\mathcal{C}$ and a homotopically surjective map $U\mathcal{C} \to \mathcal{M}$.*

Proof of Theorem 1.1.   Let $\mathcal{C}$ be the category guaranteed by the above proposition. The model category $U\mathcal{C}$ is left proper and combinatorial, so by Proposition 3.2 we can find a set of maps S in $U\mathcal{C}$ which become weak equivalences in $\mathcal{M}$ and such that $U\mathcal{C}/S \to \mathcal{M}$ is a Quillen equivalence.   ∎

Proof of Corollary 1.2.   If $\mathcal{M}$ is a combinatorial model category then the theorem gives us a Quillen equivalence $U\mathcal{C}/S \simeq \mathcal{M}$ for some $\mathcal{C}$ and S. The point is that the universal model category $U\mathcal{C}$ is simplicial and left proper, and these properties are inherited by the localization $U\mathcal{C}/S$.

We must work a little harder to show that $\mathcal{M}$ is Quillen equivalent to a model category in which every object is cofibrant. Recall that the diagram category $s\mathscr{S}et^{\mathscr{C}^{op}}$ has a *Heller model structure* [He] in which a map $D \to E$ is a weak equivalence (resp. cofibration) if $D(c) \to E(c)$ is a weak equivalence (resp. cofibration) for every $c \in \mathscr{C}$. The Heller model structure is related to $U\mathscr{C}$ by a Quillen equivalence $U\mathscr{C} \rightleftarrows s\mathscr{S}et_H^{\mathscr{C}^{op}}$ (where the "H" is for "Heller"). This map will still be a Quillen equivalence when we localize, so that we get a zigzag of Quillen equivalences

$$s\mathscr{S}et_H^{\mathscr{C}^{op}}/S \rightleftarrows U\mathscr{C}/S \rightleftarrows \mathcal{M}.$$

But now the point is that the Heller model structure is simplicial, left proper, and has the property that every object is cofibrant; these properties all pass to the localization. ∎

The application of replacing combinatorial model categories with ones in which everything is cofibrant was suggested to me by Jeff Smith.

4. CANONICAL HOMOTOPY COLIMITS

In this section we introduce a homotopical generalization of canonical colimits, which were discussed in (2.4). When $\mathscr{C}$ is a subcategory of $\mathcal{M}$ then the *canonical homotopy colimit* of a fibrant object X with respect to $\mathscr{C}$ is a certain "approximation" to X based on $\mathscr{C}$: one takes all the information from the homotopy function complexes $\underline{\mathcal{M}}(c, X)$ as $c \in \mathscr{C}$ varies, and from this data one constructs the canonical homotopy colimit. (In the general case X need not be fibrant, and $\mathscr{C}$ need not be a subcategory.) The importance for us is Corollary 4.4, which says that a map $U\mathscr{C} \to \mathcal{M}$ is homotopically surjective precisely if taking canonical homotopy colimits with respect to $\mathscr{C}$ always gives back the original object up to weak equivalence.

Consider a functor $\gamma: \mathscr{C} \to \mathcal{M}$ together with a cosimplicial resolution $\Gamma: \mathscr{C} \to c\mathcal{M}$ (see [D2, Definition 3.2]). If $c \in \mathscr{C}$ then we'll use $\Gamma^n c$ to denote the component of $\Gamma(c)$ lying in dimension n. The cosimplicial resolution induces a functor $\mathscr{C} \times \varDelta \to \mathcal{M}$ sending $(c, [n])$ to $\Gamma^n c$. For each X in $\mathcal{M}$ one can form the over-category $(\mathscr{C} \times \varDelta \downarrow X)$, together with the canonical functor $(\mathscr{C} \times \varDelta \downarrow X) \to \mathcal{M}$.

DEFINITION 4.1. The homotopy colimit of this functor is called the *canonical homotopy colimit* of X with respect to Γ (or with respect to $\mathscr{C}$, if we are lazy), and it will be denoted $\text{hocolim}(\mathscr{C} \times \varDelta \downarrow X)$.

As usual, talking about "the" canonical homotopy colimit is somewhat inappropriate since the actual object depends on the framing used in calculating the homotopy colimit—we will ignore this point of etiquette, though. Note that there are canonical maps

$$\text{hocolim}(\mathscr{C} \times \varDelta \downarrow X) \to \text{colim}(\mathscr{C} \times \varDelta \downarrow X) \to X.$$

We will be very interested in the composite.

Everyone knows at least one example of a canonical homotopy colimit: For the model category $\mathscr{T}op$, consider the inclusion of the one-point category $pt \hookrightarrow \mathscr{T}op$ and its standard cosimplicial resolution given by the topological simplices $\varDelta^n$. The canonical homotopy colimit of a space X with respect to this subcategory turns out to be essentially the same as the realization of the singular complex of X.

In general, $\text{hocolim}(\mathscr{C} \times \varDelta \downarrow X)$ is a kind of "homotopical approximation" to X based on the functor $\gamma \colon \mathscr{C} \to \mathscr{M}$. We look at all ways of mapping n-fold homotopies $\varGamma^n c$ into X, and from this data we concoct some strange object which is like a phantom image of X as seen through the eyes of γ. In the above example from topological spaces the natural maps $\text{hocolim}(\mathscr{C} \times \varDelta \downarrow X) \to X$ are all weak equivalences, but this will typically be far from true.

From the above definition it is not immediately clear to what extent the homotopy type of $\text{hocolim}(\mathscr{C} \times \varDelta \downarrow X)$ depends on the cosimplicial resolution $\varGamma$, which contributed to the indexing category $(\mathscr{C} \times \varDelta \downarrow X)$. We will show in a moment (4.3i) that if X is fibrant then choosing a different cosimplicial resolution yields a weakly equivalent object. We will also show (4.3ii) that if $X \to Y$ is a weak equivalence between fibrant objects, then the induced map of canonical homotopy colimits is again a weak equivalence. The key to proving these statements is the following result, which says that canonical homotopy colimits can always be interpreted as certain realizations of singular complexes (compare (2.4)):

PROPOSITION 4.2.   *Let* $Re : U\mathscr{C} \rightleftarrows \mathscr{M} : Sing$ *be the Quillen pair induced by* $\varGamma$. *Then* $Re \, ^{cof} Sing \, X$ *is weakly equivalent to* $\text{hocolim}(\mathscr{C} \times \varDelta \downarrow X)$.

Proof.   Consider the Yoneda embedding $\mathscr{C} \hookrightarrow U\mathscr{C}$, together with its canonical cosimplicial resolution induced by the simplicial structure on $U\mathscr{C}$. In [D2, Proposition 2.9] we showed that for any $F \in U\mathscr{C}$ the natural map $\text{hocolim}(\mathscr{C} \times \varDelta \downarrow F) \to F$ gives a cofibrant-approximation to F. We aim to apply this in the case where F is $Sing \, X$.

It's easy to see using adjointness that the overcategory $(\mathscr{C} \times \varDelta \downarrow Sing \, X)$ is isomorphic to the overcategory $(\mathscr{C} \times \varDelta \downarrow X)$. It's then clear that applying the realization Re to $\text{hocolim}(\mathscr{C} \times \varDelta \downarrow Sing \, X)$ gives precisely the object $\text{hocolim}(\mathscr{C} \times \varDelta \downarrow X)$.

So we've recovered hocolim($\mathscr{C} \times \Delta \downarrow X$) by starting with *Sing X*, taking a certain cofibrant-approximation in $U\mathscr{C}$, and then applying the realization *Re*. This is precisely what we needed to prove. ∎

COROLLARY 4.3. (i) *If $X \to Y$ is a weak equivalence between fibrant objects then the induced map* hocolim($\mathscr{C} \times \Delta \downarrow X$) $\to$ hocolim($\mathscr{C} \times \Delta \downarrow Y$) *is also a weak equivalence.*

(ii) *Suppose that Γ' is another resolution for γ. Then the canonical homotopy colimits* hocolim$_\Gamma(\mathscr{C} \times \Delta \downarrow X)$ *and* hocolim$_{\Gamma'}(\mathscr{C} \times \Delta \downarrow X)$ *are weakly equivalent, provided X is fibrant.*

Proof. Part (i) follows directly from the fact that (Re, $Sing$) is a Quillen pair: the weak equivalence between fibrant objects $X \to Y$ yields a weak equivalence $Sing\,X \to Sing\,Y$, and therefore the map $Re^{\,cof}Sing\,X \to Re^{\,cof}Sing\,Y$ is also a weak equivalence.

For part (ii) recall that any two cosimplicial resolutions of γ can be connected by a zigzag of weak equivalences. So it suffices to prove the result in the case where there is a weak equivalence $\Gamma \to \Gamma'$.

We will use (Re', $Sing'$) for the Quillen pair corresponding to Γ'. It is easy to see (using the formulas of [D2, Section 9.5], for instance) that there are natural transformations $Re \to Re'$ and $Sing' \to Sing$ induced by $\Gamma \to \Gamma'$, and these have the properties that $Re\,A \to Re'\,A$ and $Sing'\,X \to Sing\,X$ are weak equivalences when A is cofibrant and X is fibrant. (In the language of [D2, Definition 5.9], this is a *Quillen homotopy* from Re to Re'.)

If X is fibrant we have a weak equivalence $Sing'\,X \to Sing\,X$ and hence a weak equivalence $QSing'\,X \to QSing\,X$ where Q denotes any cofibrant-replacement functor in $U\mathscr{C}$. Consider the square

$$
\begin{array}{ccc}
Re\,QSing'\,X & \longrightarrow & Re\,QSing\,X \\
\downarrow & & \downarrow \\
Re'\,QSing'\,X & \longrightarrow & Re'\,QSing\,X.
\end{array}
$$

All the maps in the square are readily seen to be weak equivalences, and so we've shown that $Re\,QSing\,X$ and $Re'\,QSing'\,X$ are weakly equivalent via a zigzag. By the above proposition, this is what we wanted. ∎

COROLLARY 4.4. *Let $\gamma: \mathscr{C} \to \mathscr{M}$ be a functor and $\Gamma: \mathscr{C} \to c\mathscr{M}$ be a cosimplicial resolution of γ. Then the induced map $U\mathscr{C} \to \mathscr{M}$ is homotopically surjective if and only if the natural maps* hocolim($\mathscr{C} \times \Delta \downarrow X$) $\to X$ *are weak equivalences for every fibrant object X.*

Proof. In light of the above proposition, this is just a restatement of the definitions. ∎

4.5. *Results about Canonical Homotopy Colimits.* In contrast to the canonical homotopy colimit defined above, we can also consider a more naive construction where we take the homotopy colimit of the canonical diagram $(\mathscr{C} \downarrow X) \to \mathscr{M}$: this new object will be denoted hocolim$(\mathscr{C} \downarrow X)$. Note the difference between "hocolim$(\mathscr{C} \downarrow X)$" and "hocolim$(\mathscr{C} \times \varDelta \downarrow X)$" —the former was constructed only out of maps $\gamma c \to X$, whereas the latter used all "higher homotopies" $\varGamma^n c \to X$. The problem with hocolim$(\mathscr{C} \downarrow X)$ is that it is usually not a homotopy invariant construction—replacing X with another weakly equivalent object, even if they are both fibrant, may change the homotopy type of hocolim$(\mathscr{C} \downarrow X)$. On the other hand, there are many instances in this paper where we get our hands on the canonical homotopy colimit precisely by showing it agrees with the more naive construction; this works because we will have chosen $\mathscr{C}$ to be so large that it already contains all the information from the "higher homotopies."

Assume now that the image of $\gamma \colon \mathscr{C} \to \mathscr{M}$ is contained in the cofibrant objects. In this case we may choose a cosimplicial resolution $\varGamma$ such that the 0th object of $\varGamma c$ is γc itself (rather than an arbitrary cofibrant replacement). There is an obvious map of categories $i \colon (\mathscr{C} \downarrow X) \to (\mathscr{C} \times \varDelta \downarrow X)$ which sends $[c, \gamma c \to X]$ to the object $[c \times [0], \varGamma^0 c \to \gamma c \to X]$, and this induces a map of homotopy colimits $i_* \colon$ hocolim$(\mathscr{C} \downarrow X) \to$ hocolim$(\mathscr{C} \times \varDelta \downarrow X)$. Our concern will be conditions for this map to be a weak equivalence.

We have need for one final piece of notation: Let $(\mathscr{C}^n \downarrow X)$ denote the overcategory of $\varGamma^n \colon \mathscr{C} \to \mathscr{M}$, whose objects are pairs $[c, \varGamma^n c \to X]$. Consider the functor $(\mathscr{C}^n \downarrow X) \to \mathscr{M}$ which sends $[c, \varGamma^n c \to X]$ to γc (this is *not* the canonical functor). The homotopy colimit of this functor will be denoted hocolim$(\mathscr{C}^n \downarrow X)$. There is an obvious map $j \colon (\mathscr{C}^0 \downarrow X) \to (\mathscr{C}^n \downarrow X)$ sending $[c, \varGamma^0 c \to X]$ to $[c, \varGamma^n c \to \varGamma^0 c \to X]$, and this induces a map $j_* \colon$ hocolim$(\mathscr{C} \downarrow X) \to$ hocolim$(\mathscr{C}^n \downarrow X)$.

PROPOSITION 4.6. *Assume as above that the image of $\gamma \colon \mathscr{C} \to \mathscr{M}$ lies in the cofibrant objects and that the cosimplicial resolution has been chosen so that $\varGamma^0 c = \gamma c$. If the maps $j_* \colon$ hocolim$(\mathscr{C}^0 \downarrow X) \to$ hocolim$(\mathscr{C}^n \downarrow X)$ are weak equivalences for all n, then the map $i_* \colon$ hocolim$(\mathscr{C} \downarrow X) \to$ hocolim$(\mathscr{C} \times \varDelta \downarrow X)$ is also a weak equivalence.*

Proof.   Postponed until Section 8.   ∎

The following proposition will be our starting point for obtaining presentations for combinatorial model categories. As you will see, it only concerns the naive construction hocolim$(\mathscr{C} \downarrow X)$. Much of the work in the rest of the paper involves bootstrapping ourselves up to a result about canonical homotopy colimits. (It's useful to once again compare the result below to our discussion in Subsection 2.4.)

PROPOSITION 4.7. *Let $\mathcal{M}$ be a combinatorial model category. Then for sufficiently large regular cardinals λ one has the following:*

(i)   *For every $X \in \mathcal{M}$ the canonical map* $\mathrm{hocolim}(\mathcal{M}_\lambda \downarrow X) \to X$ *is a weak equivalence.*

(ii)   *The same is true for* $\mathrm{hocolim}(\mathcal{M}_\lambda^{cof} \downarrow X) \to X$, *where $\mathcal{M}_\lambda^{cof}$ denotes the full subcategory of $\mathcal{M}_\lambda$ consisting of the cofibrant objects.*

Proof.   The proof of (i) is very easy: Since the underlying category of $\mathcal{M}$ is locally presentable, for sufficiently large regular cardinals λ the maps $\mathrm{colim}(\mathcal{M}_\lambda \downarrow X) \to X$ are isomorphisms for all X. On the other hand, the indexing categories $(\mathcal{M}_\lambda \downarrow X)$ are λ-filtered, and in combinatorial model categories one has that λ-filtered colimits are the same as λ-filtered homotopy colimits for large enough λ (cf. the assumption 2.3ii). So by picking λ large enough we may ensure both that the map $\mathrm{hocolim}(\mathcal{M}_\lambda \downarrow X) \to \mathrm{colim}(\mathcal{M}_\lambda \downarrow X)$ is a weak equivalence and that the map $\mathrm{colim}(\mathcal{M}_\lambda \downarrow X) \to X$ is an isomorphism. This finishes (i).

For (ii) we must be more careful. Choose λ large enough so that (i) is satisfied, but also so that $\mathcal{M}$ has a cofibrant-replacement functor which maps $\mathcal{M}_\lambda$ to itself (2.3(iii)). Let $F \colon \mathcal{M}_\lambda \to \mathcal{M}_\lambda$ denote this functor, $I = (\mathcal{M}_\lambda \downarrow X)$, and $J = (\mathcal{M}_\lambda^{cof} \downarrow X)$.

Observe that one has maps $I \xrightarrow{f} J \xrightarrow{g} I$, where g is the obvious inclusion and f is the functor sending $[c, c \to X]$ to $[Fc, Fc \to c \to X]$. These functors come to us with natural transformations $gf \to id$ and $fg \to id$ induced by the natural transformation $Fc \to c$. Let $D \colon I \to \mathcal{M}$ be the canonical diagram sending an object $[c, c \to X]$ to c.

The criteria of Proposition A.4 are readily checked, and so we can conclude that $\mathrm{hocolim}_J g^*D \to \mathrm{hocolim}_I D$ is a weak equivalence. But this is precisely the natural map $\mathrm{hocolim}(\mathcal{M}_\lambda^{cof} \downarrow X) \to \mathrm{hocolim}(\mathcal{M}_\lambda \downarrow X)$. By Part (i) the codomain is weakly equivalent to X, so we are done.   ∎

5. THE PROOF FOR SIMPLICIAL MODEL CATEGORIES

Our goal in this section is to prove Proposition 3.3 for the special case where $\mathcal{M}$ is a combinatorial model category which is also a *simplicial* model category. The proof for arbitrary $\mathcal{M}$ will be given in the next section. This special case is presented separately because it is quite a bit easier, yet the steps are very similar to what we will do for the general case.

Choose a regular cardinal λ for which Proposition 4.7(ii) holds; that is, choose a λ so that the natural map $\mathrm{hocolim}(\mathcal{M}_\lambda^{cof} \downarrow X) \to X$ is a weak equivalence for all X in $\mathcal{M}$. Let $\mathscr{C}$ denote $\mathcal{M}_\lambda^{cof}$, for brevity. Since $\mathcal{M}$ is

simplicial there is a canonical cosimplicial resolution for the inclusion $\mathscr{C} \hookrightarrow \mathscr{M}$, and this gives a map $U\mathscr{C} \to \mathscr{M}$. The goal will be to show that this map is homotopically surjective.

LEMMA 5.1.   *The maps* $\mathrm{hocolim}(\mathscr{C}^0 \downarrow X) \to \mathrm{hocolim}(\mathscr{C}^n \downarrow X)$ *are weak equivalences provided that X is fibrant.*

Proof.   The objects of $(\mathscr{C}^n \downarrow X)$ are pairs $[c, c \otimes \varDelta^n \to X]$ where c is an object of $\mathscr{C}$ and $c \otimes \varDelta^n \to X$ is some map in $\mathscr{M}$. Since $\mathscr{M}$ is simplicial, this map has an adjoint $c \to X^{\varDelta^n}$. In this way we see that the category $(\mathscr{C}^n \downarrow X)$ is isomorphic to $(\mathscr{C}^0 \downarrow X^{\varDelta^n})$. The map in which we are interested is isomorphic to the map

$$\mathrm{hocolim}(\mathscr{C}^0 \downarrow X) \to \mathrm{hocolim}(\mathscr{C}^0 \downarrow X^{\varDelta^n})$$

induced by $X \to X^{\varDelta^n}$.

Now by our choice of $\mathscr{C}$ we know that $\mathrm{hocolim}(\mathscr{C}^0 \downarrow Z)$ is naturally weakly equivalent to Z, for any Z. So the above map is weakly equivalent to $X \to X^{\varDelta^n}$, which of course is a weak equivalence because X was fibrant.   ∎

Proof of Proposition 3.3, *Simplicial Case.*   By Corollary 4.4 we must show that for any fibrant X in $\mathscr{M}$, the natural map $\mathrm{hocolim}(\mathscr{C} \times \varDelta \downarrow X) \to X$ is a weak equivalence. Consider the diagram

$$\mathrm{hocolim}(\mathscr{C} \downarrow X) \longrightarrow \mathrm{hocolim}(\mathscr{C} \times \varDelta \downarrow X)$$

$$\searrow \qquad \qquad \downarrow$$

$$X.$$

The above lemma, together with Proposition 4.6, shows that the horizontal map is a weak equivalence. The diagonal map is a weak equivalence by our choice of $\mathscr{C}$ (Proposition 4.7). Therefore the vertical map is also a weak equivalence, which is what we needed to prove.   ∎

6. THE PROOF FOR NONSIMPLICIAL MODEL CATEGORIES

In this section we prove Proposition 3.3 for arbitrary model categories. The main surprise is that the category of "generators" $\mathscr{C}$ is not chosen to be a subcategory of $\mathscr{M}$—instead we have to chose something bigger.

6.1. *The Plan.*   We begin with some general remarks about our approach. The first hope would be to take $\mathscr{C}$ to be the category $\mathscr{M}_\lambda^{cof}$ for a sufficiently large cardinal λ, just as we did for simplicial model categories.

The difficulty is that we don't know how to prove Lemma 5.1 in this generality. Our second hope might be to find a method for somehow reducing to the simplicial case, which we've already handled.

The category of cosimplicial objects $c\mathcal{M}$ has a natural simplicial action on it: given an $A \in c\mathcal{M}$ and a $K \in s\mathcal{S}et$ one can form new objects $A \otimes K$ and A^K (see the Appendix). There is also an adjoint pair $ev_0 : c\mathcal{M} \rightleftarrows \mathcal{M} : c^*$ where $ev_0(A) = A^0$ and c^*X is the cosimplicial object consisting of X in every dimension. In good cases one can find a model structure on $c\mathcal{M}$ for which (1) this adjoint pair is a Quillen equivalence, (2) $c\mathcal{M}$ is a *simplicial* model category, and (3) the cofibrant objects of $c\mathcal{M}$ are precisely the cosimplicial resolutions. (This model structure is dual to the one constructed in [D1].) Recall that a cosimplicial resolution is a Reedy cofibrant object of $c\mathcal{M}$ with the property that all coface and codegeneracy maps are weak equivalences.

Now if we did have such a model structure on $c\mathcal{M}$ then we could apply the result from the previous section to get a presentation for $c\mathcal{M}$, and this would also yield a presentation for $\mathcal{M}$ (using the Quillen equivalence $c\mathcal{M} \simeq \mathcal{M}$). Our $\mathcal{C}$ would be a certain subcategory of the cofibrant objects in $c\mathcal{M}$, which are the cosimplicial resolutions. Essentially what we do in this section is unravel this plan in such a way that we never have to actually use the existence of the model structure on $c\mathcal{M}$.

6.2. *The Proof.* Choose a regular cardinal λ which is large enough so that Proposition 4.7(ii) holds and so that the condition of Proposition 2.3(iii) is satisfied. Let $\mathcal{C}R$ denote the full subcategory of $c\mathcal{M}$ consisting of all cosimplicial resolutions A^* with the property that $A^n \in \mathcal{M}_\lambda$ for all n. There is an obvious functor $\gamma : \mathcal{C}R \to \mathcal{M}$ sending A^* to A^0, and this comes equipped with a natural cosimplicial resolution $\Gamma : \mathcal{C}R \to c\mathcal{M}$ which is just the inclusion of $\mathcal{C}R$ as a subcategory. These induce a map $U(\mathcal{C}R) \to \mathcal{M}$, and the goal will be to show that this is homotopically surjective.

Let $\mathcal{C}$ denote the category $\mathcal{M}_\lambda^{cof}$ of cofibrant objects in $\mathcal{M}_\lambda$, and let $f : \mathcal{C}R \to \mathcal{C}$ be the functor sending A^* to A^0. For any $X \in \mathcal{M}$ there is an induced map on overcategories $(\mathcal{C}R \downarrow X) \to (\mathcal{C} \downarrow X)$ sending $[A^*, A^0 \to X]$ to $[A^0, A^0 \to X]$.

LEMMA 6.3. *The map* $f_* : \operatorname{hocolim}(\mathcal{C}R \downarrow X) \to \operatorname{hocolim}(\mathcal{C} \downarrow X)$ *is a weak equivalence, for any* $X \in \mathcal{M}$.

Proof. Let $\mathcal{E}$ denote the subcategory of $c\mathcal{M}$ consisting of the objects A^* for which A^n is in $\mathcal{C}$ for all n (unlike for $\mathcal{C}R$, we are not requiring A^* to be a cosimplicial resolution). It is possible to find a functor $\mathcal{R} : \mathcal{E} \to c\mathcal{M}$ with the following properties:

    (i)   Each $\mathscr{R}(A)$ is a Reedy cofibrant object contained in $\mathscr{E}$.

    (ii)   There is a natural weak equivalence $\eta\colon \mathscr{R}(A) \to A$.

    (iii)   The object of $\mathscr{R}(A)$ in dimension 0 is equal to A^0, and the map $\eta\colon \mathscr{R}(A)^0 \to A^0$ is the identity.

The map $\mathscr{R}$ is just a certain Reedy cofibrant-replacement functor defined on a subcategory of $c\mathcal{M}$. In order to construct Reedy cofibrant replacements, one starts with the 0th object and first makes that cofibrant in $\mathcal{M}$. For $A^* \in \mathscr{E}$ the 0th object is already cofibrant, so we can just let it be. Next one moves inductively up the cosimplicial object and factors the latching maps as cofibrations followed by trivial cofibrations (see [Ho, Chapter 5]). By (2.3iii) and our choice of λ, there are factorization functors which will never take us outside the category $\mathcal{M}_\lambda$—this is all that we wanted.

If $A \in \mathscr{C}$ then we will write $\mathscr{R}(A)$ for the result of applying $\mathscr{R}$ to the constant cosimplicial object c^*A consisting of A in every dimension.

Consider the maps $(\mathscr{C} \downarrow X) \to (\mathscr{C}R \downarrow X)$ and $(\mathscr{C}R \downarrow X) \to (\mathscr{C} \downarrow X)$ induced by $\mathscr{R}$ and f, respectively. These maps have the following behavior:

$$\mathscr{R}\colon [c, c \to X] \mapsto [\mathscr{R}(c), \mathscr{R}(c)^0 \xrightarrow{id} c \to X]$$

$$f\colon [A^*, A^0 \to X] \mapsto [A^0, A^0 \to X].$$

The composite $f\mathscr{R}\colon (\mathscr{C} \downarrow X) \to (\mathscr{C} \downarrow X)$ is the identity by property (iii) of $\mathscr{R}$. We will show that the other composite $\mathscr{R}f$ can be connected to the identity by a zigzag of natural transformations, and then we'll apply Proposition A.4.

The composite $\mathscr{R}f$ sends an object $[A^*, A^0 \to X]$ of $(\mathscr{C}R \downarrow X)$ to the object $[\mathscr{R}(A^0), A^0 \to X]$. Consider the map $H\colon (\mathscr{C}R \downarrow X) \to (\mathscr{C}R \downarrow X)$ which maps $[A^*, A^0 \to X]$ to $[\mathscr{R}A^*, [\mathscr{R}A]^0 \to A^0 \to X]$. The transformation η from property (ii) gives a natural transformation $H \to Id$. On the other hand, for any cosimplicial object A^* there is a natural map $A^* \to c^*(A^0)$ and therefore a map $\mathscr{R}(A^*) \to \mathscr{R}(A^0)$. This gives a natural transformation $H \to \mathscr{R}f$. It is easy to check that these transformations satisfy the conditions of Proposition A.4 (see Remark A.5 as well). So we conclude that $f_*\colon \mathrm{hocolim}(\mathscr{C} \downarrow X) \to \mathrm{hocolim}(\mathscr{C}R \downarrow X)$ is a weak equivalence, and the same is true for $\mathscr{R}_*$ going in the other direction. ∎

LEMMA 6.4. *The canonical map* $\mathrm{hocolim}(\mathscr{C}R^0 \downarrow X) \to \mathrm{hocolim}(\mathscr{C}R^n \downarrow X)$ *is a weak equivalence.*

Proof. Let $j\colon \Delta^0 \to \Delta^n$ denote the map of simplicial sets which includes Δ^0 as the last vertex of Δ^n. For any cosimplicial object A^* there is a corresponding map $j\colon A^0 \to A^n$; from this we can define a functor $j\colon (\mathscr{C}R^n \downarrow X) \to (\mathscr{C}R^0 \downarrow X)$ sending the object $[A^*, A^n \to X]$ to the object $[A^*, A^0 \to A^n \to X]$.

Let $i: (\mathscr{C}R^0 \downarrow X) \to (\mathscr{C}R^n \downarrow X)$ denote the functor sending $[A^*, A^0 \to X]$ to $[A^*, A^n \to A^0 \to X]$. The map we are concerned with in the statement of the lemma is i_*. Note that the composite ji is the identity. We will show that the other composite ij can be related to the identity by a zigzag of natural transformations, and then we'll apply Proposition A.4.

There is a map of simplicial sets $H: \Delta^n \times \Delta^1 \to \Delta^n$ such that H restricts to the identity map on $\Delta^n \times \{0\}$ and H restricts to the map $j\pi: \Delta^n \to \Delta^0 \to \Delta^n$ on $\Delta^n \times \{1\}$ (left to the reader). Recall that if A^* is a cosimplicial object and K is a simplicial set then one gets a new cosimplicial object $A \otimes K$ in a natural way (see the Appendix). So H induces a map $H: A^* \otimes (\Delta^n \times \Delta^1) \to A^* \otimes \Delta^n$, and by looking at the objects in dimension 0 we get a map $H: (A^* \otimes \Delta^1)^n \to A^n$ which is natural in A^*.

Consider the functor $H: (\mathscr{C}R^n \downarrow X) \to (\mathscr{C}R^n \downarrow X)$ defined by

$$[A^*, A^n \to X] \mapsto [A^* \otimes \Delta^1, (A^* \otimes \Delta^1)^n \xrightarrow{H} A^n \to X].$$

The two inclusions $\Delta^0 \to \Delta^1$ are readily seen to induce natural transformations $Id \to H$ and $ij \to H$. The hypotheses of Proposition A.4 are checked to hold, and so we may conclude that $i_*: \operatorname{hocolim}(\mathscr{C}R^0 \downarrow X) \to \operatorname{hocolim}(\mathscr{C}R^n \downarrow X)$ and j_* going in the other direction are both weak equivalences. $\blacksquare$

We can now close out the main proof.

Proof of Proposition 3.3, *General Case.* Again, by Corollary 4.4 we must show that for any fibrant object X in $\mathscr{M}$ the natural map $\operatorname{hocolim}(\mathscr{C}R \times \Delta \downarrow X) \to X$ is a weak equivalence.

Consider the commutative diagram

$$\operatorname{hocolim}(\mathscr{C} \downarrow X) \xleftarrow{f_*} \operatorname{hocolim}(\mathscr{C}R \downarrow X) \xrightarrow{i_*} \operatorname{hocolim}(\mathscr{C}R \times \Delta \downarrow X)$$

with maps a, p going down to X.

Lemma 6.3 says that f_* is a weak equivalence. Lemma 6.4, together with Proposition 4.6, implies the same about i_*. Finally, our assumption on λ guarantees that a is a weak equivalence (Proposition 4.7). We therefore conclude that p is also a weak equivalence, which is what we wanted. $\blacksquare$

7. MORE ABOUT COMBINATORIAL MODEL CATEGORIES

At this point we have finished with the main ideas of the paper. In this section and the next we have only to complete the proofs for some of the auxiliary results. This section fills in some of the details behind the properties

of combinatorial model categories singled out in Proposition 2.3. The authoritative reference for results like these will eventually be [Sm].

PROPOSITION 7.1 (Smith).   *In a combinatorial model category $\mathcal{M}$ there are functorial factorizations of a map into a trivial cofibration followed by a fibration which preserve λ-filtered colimits for sufficiently large regular cardinals λ. The same is true for the factorizations as a cofibration followed by a trivial fibration.*

Proof.   The usual factorizations provided by the small object argument will have the required properties, as long as we use the transfinite version of the small object argument for a sufficiently large ordinal. See [Sm].   ∎

Proposition 2.3(i) is a special case of the above. Part (iii) of the proposition is the following.

PROPOSITION 7.2 (Smith).   *The factorizations guaranteed by the above proposition have the following property: For sufficiently large regular cardinals μ, if $X \to Y$ is a map between μ-small objects then the factorizations produce maps $X \overset{\sim}{\rightarrowtail} \tilde{X} \twoheadrightarrow Y$ and $X \rightarrowtail \tilde{Y} \overset{\sim}{\twoheadrightarrow} Y$, where both $\tilde{X}$ and $\tilde{Y}$ are also μ-small.*

In a locally presentable category one can define the *size* of an object X to be the smallest regular cardinal λ for which X is λ-small. The proposition says that past a certain point the factorizations don't increase size anymore.

Proof.   Pick a regular cardinal λ large enough to satisfy the previous proposition and also large enough so that $\mathcal{M}_\lambda$ is dense in $\mathcal{M}$ (using local presentability). The category $\mathcal{M}_\lambda$ is small, so applying our given factorizations to maps $X \to Y$ in $\mathcal{M}_\lambda$ only produces a set of new objects. Therefore there exists a regular cardinal ν such that applying our factorizations to maps between λ-small objects always produces ν-small objects.

Let μ be any regular cardinal larger than both λ and ν, and let $X \to Y$ be a map between μ-small objects. It follows from [MP, Proposition 2.3.11] that we can write $X \to Y$ as a colimit of maps $X_\alpha \to Y_\alpha$ where X_α, Y_α are λ-small and where the indexing category is both μ-small and λ-filtered. Applying our factorization produces maps $X \overset{\sim}{\rightarrowtail} \tilde{X} \twoheadrightarrow Y$ which are isomorphic to the colimit of the maps $X_\alpha \overset{\sim}{\rightarrowtail} \tilde{X}_\alpha \to Y_\alpha$. Each $\tilde{X}_\alpha$ is ν-small (hence μ-small) by our choice of ν, and so $\tilde{X}$ is a μ-small colimit of μ-small objects, hence is itself μ-small [AR, Proposition 1.16]. This completes the proof.   ∎

PROPOSITION 7.3.   *Let $\mathcal{M}$ be a combinatorial model category. Then for sufficiently large regular cardinals λ, λ-filtered colimits of weak equivalences are again weak equivalences.*

Proof.   Let λ be a regular cardinal large enough so that there are functorial factorizations preserving λ-filtered colimits and so that the model category has a set of generating cofibrations whose domains and codomains are λ-small. Let I be a λ-filtered indexing category, and let $D_1, D_2 : I \to \mathcal{M}$ be two diagrams. We suppose that $\eta : D_1 \to D_2$ is a map of diagrams such that $D_1(i) \to D_2(i)$ is a weak equivalence for every $i \in I$, and we'll show that colim $D_1 \to$ colim D_2 must also be a weak equivalence.

Start by factoring the map colim $D_1 \xrightarrow{\eta}$ colim D_2 into a trivial cofibration followed by a fibration, using our preferred functorial factorization

$$\text{colim } D_1 \xrightarrow{\sim} X \twoheadrightarrow \text{colim } D_2.$$

These maps are the colimits of the maps obtained by applying the factorization to each spot in the diagram,

$$D_1(i) \xrightarrow{\sim} X(i) \twoheadrightarrow D_2(i).$$

Now the maps $D_1(i) \to D_2(i)$ were assumed to be weak equivalences, and therefore the maps $X(i) \to D_2(i)$ are actually *trivial* fibrations. If we can show that λ-filtered colimits of trivial fibrations are again trivial fibrations then we will be done: the map $X \to$ colim D_2 will be a trivial fibration, and so colim $D_1 \to$ colim D_2 will be a weak equivalence.

But we can test if a map is a trivial fibration by checking the lifting property with respect to our generating cofibrations. Since the domains and codomains of these generating cofibrations are λ-small, they will factor through some stage of the λ-filtered colimit and we will get our lift. ∎

8. A LEFTOVER PROOF

Our goal is to prove Proposition 4.6. Recall the scene: $\gamma : \mathcal{C} \to \mathcal{M}$ is a functor taking its values in the cofibrant objects and $\Gamma : \mathcal{C} \to c\mathcal{M}$ is a cosimplicial resolution of γ with $\Gamma^0 c = \gamma c$. There is a canonical map   hocolim$(\mathcal{C} \downarrow X) \to$ hocolim$(\mathcal{C} \times \Delta \downarrow X)$ for each $X \in \mathcal{M}$, and Proposition 4.6 gives sufficient conditions for this map to be a weak equivalence. The key ingredient in our proof is the cofibrant-replacement functor Q for $U\mathcal{C}$, written down in [D2, Section 2.6].

The cosimplicial resolution Γ induces a Quillen pair $Re : U\mathcal{C} \rightleftarrows \mathcal{M} : Sing$. Let $Sing_n X$ denote the presheaf which forms the degree n part of

Sing X—as usual, we will implicitly identify *Sing $_n$X* with the corresponding discrete simplicial presheaf in $U\mathscr{C}$.

In terms of the above Quillen pair, we know by Proposition 4.2 that hocolim($\mathscr{C} \times \varDelta \downarrow X$) is weakly equivalent to *Re ^{cof}Sing X*. We claim that hocolim($\mathscr{C} \downarrow X$) is weakly equivalent to *Re ^{cof}Sing $_0$X*. The way to see this is to make use of the functor *Q* mentioned above. It's easy to check that *QSing $_0$X* can be identified with the simplicial replacement of the canonical diagram ($\mathscr{C} \downarrow X) \rightarrow \mathscr{M}$, and then *Re QSing $_0$X* gives the usual geometric realization—the resulting object is precisely hocolim($\mathscr{C} \downarrow X$). Moreover, there is an obvious map *Sing $_0$X* → *Sing X* obtained by including the 0-simplices, and the induced map *Re QSing $_0$X* → *Re QSing X* will be weakly equivalent to the map hocolim($\mathscr{C} \downarrow X$) → hocolim($\mathscr{C} \times \varDelta \downarrow X$) we're interested in.

The first thing we will show is that the object *Re ^{cof}Sing X* can be built up as a homotopy colimit of the objects *Re ^{cof}Sing $_n$X*. The hypotheses of the proposition translate into saying that all the *Re ^{cof}Sing $_n$X* have the same homotopy type (as *n* varies), and so we will be able to collapse the homotopy colimit down to the $n = 0$ piece.

We start out with a couple of technical lemmas. Suppose given a simplicial diagram $\mathscr{F}: \varDelta^{op} \rightarrow U\mathscr{C}$. Since $U\mathscr{C}$ is a simplicial model category we may form the geometric realization $|\mathscr{F}|$. On the other hand, we may also form the homotopy colimit using the formula in [BK], and we will denote this object "badhocolim $\mathscr{F}$." The "bad-" prefix is to remind us that this is not *a priori* a homotopy invariant construction, because the objects in the diagram $\mathscr{F}$ need not be cofibrant. Note that there is a *Bousfield–Kan map* badhocolim$\mathscr{F} \rightarrow |\mathscr{F}|$, just as one has in any simplicial category.

LEMMA 8.1. *For any diagram $\mathscr{F}$ as above, the Bousfield–Kan map*

$$\text{badhocolim } \mathscr{F} \rightarrow |\mathscr{F}|$$

is a weak equivalence in $U\mathscr{C}$.

Proof. The point is that the homotopy theory in $U\mathscr{C}$ comes from simplicial sets: the weak equivalences are the objectwise weak equivalences, and the simplicial structure is the objectwise structure. So the lemma is immediately reduced to the corresponding fact for simplicial sets, which is well-known. ∎

We will need one other fact about the Bousfield–Kan map in this context: there are of course natural maps $\mathscr{F}_0 \rightarrow |\mathscr{F}|$ and $\mathscr{F}_0 \rightarrow$ badhocolim $\mathscr{F}$, and it is easy to check that the triangle

$$\text{badhocolim } \mathscr{F} \longrightarrow |\mathscr{F}|$$

$$\mathscr{F}_0$$

is commutative.

PROPOSITION 8.2. *Let $\mathscr{F}: \Delta^{op} \to \mathscr{M}$ be the diagram given by $[n] \mapsto$ Re QSing $_n X$. There is a commutative triangle in $\mathscr{M}$*

$$\text{hocolim } \mathscr{F} \xrightarrow{\;\sim\;} Re \; QSing \; X$$

$$\beta \Big\uparrow \qquad \qquad \alpha$$

$$Re \; QSing \,_0 X$$

in which the horizontal map is a weak equivalence.

Proof. Consider the diagram $\mathscr{E}: \Delta^{op} \to U\mathscr{C}$ given by $[n] \mapsto QSing \,_n X$. The geometric realization $|\mathscr{E}|$ is isomorphic to $QSing \; X$ (using the definition of Q, together with the fact that for bisimplicial sets the realization is isomorphic to the diagonal). Our above discussion therefore gives a commutative triangle

$$\text{badhocolim } \mathscr{E} \longrightarrow QSing \; X$$

$$QSing \,_0 X \qquad \qquad ,$$

where the horizontal arrow is the Bousfield–Kan map and therefore a weak equivalence. Note that every object of $\mathscr{E}$ is cofibrant, being in the image of Q—therefore badhocolim $\mathscr{E}$ actually has the correct homotopy type, and we may drop the "bad-" prefix. Moreover, badhocolim $\mathscr{E}$ is known to be cofibrant in this case.

Now we apply the realization functor to the above triangle to get

$$\text{hocolim}(Re \; QSing \,_* X) \xrightarrow{\;\sim\;} Re \; QSing \; X$$

$$\beta \Big\uparrow \qquad \qquad \alpha$$

$$Re \; QSing \,_0 X \qquad \qquad .$$

The horizontal map is still a weak equivalence because we applied Re to a weak equivalence between cofibrant objects. ∎

Proof of Proposition 4.6.   Our assumption is that the maps

$$j_* \colon \operatorname{hocolim}(\mathscr{C}^0 \downarrow X) \to \operatorname{hocolim}(\mathscr{C}^n \downarrow X)$$

are weak equivalences, for every n. We claim that $\operatorname{hocolim}(\mathscr{C}^0 \downarrow X)$ and $\operatorname{hocolim}(\mathscr{C}^n \downarrow X)$ may be identified with $Re\, QSing_0 X$ and $Re\, QSing_n X$, and under this identification the maps j_* correspond to the iterated degeneracies $Re\, QSing_0 X \to Re\, QSing_n X$ in the simplicial object $Re\, QSing_* X$.

To see this, note that $Sing_n X$ is the presheaf whose value on c is $\mathscr{M}(\Gamma^n c, X)$. Therefore $QSing_n X$ is the simplicial object which in spot k is the coproduct

$$\coprod_{\substack{c_0 \to \cdots \to c_k \\ \Gamma^n(c_k) \to X}} c_0.$$

Here the coproduct is indexed by all sequences of maps $c_0 \to \cdots \to c_k$ in $\mathscr{C}$ together with a map $\Gamma^n(c_k) \to X$ in $\mathscr{M}$, and the "c_0" inside the coproduct denotes the representable presheaf in $U\mathscr{C}$. One observes that if we had $\gamma(c_0)$ instead of c_0 inside the coproduct, then this would be exactly the simplicial replacement of the diagram $D \colon (\mathscr{C}^n \downarrow X) \to \mathscr{M}$ which sends $[c, \Gamma^n c \to X]$ to γc. Applying the realization functor to get $Re\, QSing_n X$ is the same as tensoring this simplicial object with the framing Γ, and this is one description of how to compute $\operatorname{hocolim} D$, or what we have been calling $\operatorname{hocolim}(\mathscr{C}^n \downarrow X)$. The same reasoning justifies our claims about $\operatorname{hocolim} (\mathscr{C}^0 \downarrow X)$ and the map j_*.

So our assumption translates into saying that the iterated degneracies in $Re\, QSing_* X$ are weak equivalences. It then readily follows that *every* map in this simplicial object is a weak equivalence, and therefore the natural map $\beta \colon Re\, QSing_0 X \to \operatorname{hocolim}[Re\, QSing_* X]$ must also be a weak equivalence (see [D1, Proposition 5.4], for instance).

At this point we look at the triangle from the above proposition and conclude by the two-out-of-three property that $\alpha \colon Re\, QSing_0 X \to Re\, QSing X$ is a weak equivalence. This is what we wanted. ∎

APPENDIX A: HOMOTOPY COLIMITS

This section has two main goals. We recall that if $\mathscr{M}$ is a model category then the category of cosimplicial objects $c\mathscr{M}$ has a natural simplicial structure: if $A \in c\mathscr{M}$ and $K \in s\mathscr{S}et$ then one can form objects $A \otimes K$ and A^K (in fact, all this needs is that $\mathscr{M}$ is complete and cocomplete). The tensoring operation is used in the proof of Lemma 6.4, and it's also the basis for the

way homotopy colimits are defined in [DHK] and [H]. We briefly recall this definition of homotopy colimits, and we list some basic properties which aren't always stressed. These properties are used to prove Proposition A.4, which is a technique for identifying two homotopy colimits over different indexing categories. This technique is needed several times in the course of the paper.

A.1. *The Simplicial Structure on $c\mathcal{M}$.* If S is a set and $W \in \mathcal{M}$ let $W \cdot S$ denote a coproduct of copies of W, one for each element of S. Similarly, let $W^{\cdot S}$ denote a product of copies of W indexed by the set S.

If $K \in s\mathcal{S}et$ and $A \in c\mathcal{M}$ then we define $A \otimes_\Delta K \in \mathcal{M}$ as the coend

$$A \otimes_\Delta K = \text{coeq}\left[\coprod_{[k] \to [m]} A_k \cdot K_m \rightrightarrows \coprod_n A_n \cdot K_n \right].$$

The object $A \otimes K \in c\mathcal{M}$ is then defined to be the cosimplicial object

$$[n] \mapsto A \otimes_\Delta (K \times \Delta^n).$$

The exponential $A^K \in c\mathcal{M}$ can be defined in a more straightforward way: It is the cosimplicial object

$$[n] \mapsto A_n^{\cdot K_n}$$

with the obvious coface and codegeneracy operators. It is routine work to check that these definitions give a simplicial structure on the category $c\mathcal{M}$—it is exactly dual to the standard simplicial structure on $s\mathcal{M}$ (written down in [D1], for instance).

A.2. *Homotopy Colimits.* Suppose that $X: I \to \mathcal{M}$ is a diagram in a model category, and let $\Gamma: I \to c\mathcal{M}$ denote a cosimplicial resolution for X. The *homotopy colimit* of X is defined to be the object

$$\underset{I}{\text{hocolim}}\ X = \text{coeq}\left[\coprod_{i \to j} \Gamma(i) \otimes_\Delta B(j \downarrow I)^{op} \rightrightarrows \coprod_i \Gamma(i) \otimes_\Delta B(i \downarrow I)^{op} \right].$$

Here $B(i \downarrow I)^{op}$ denotes the classifying space of the category $(i \downarrow I)^{op}$. Note that technically speaking the object $\text{hocolim}_I X$ depends on the cosimplicial resolution Γ, although the homotopy type of $\text{hocolim}_I X$ does not.

This construction of homotopy colimits is essentially the one given in [DHK] and [H]. The only difference is that those sources only require Γ to be a cosimplicial *framing* rather than a full resolution, which has the effect of giving a construction which is only homotopy invariant for diagrams of cofibrant objects. This distinction is a minor one.

Remark A.3.   The above construction can be seen to have the following properties:

(i)   If $X_1 \to X_2$ is a map of I-diagrams which is an objectwise weak equivalence and if $\Gamma_1 \to \Gamma_2$ is a corresponding map of resolutions, then the induced map $\operatorname{hocolim}_I X_1 \to \operatorname{hocolim}_I X_2$ is also a weak equivalence.

(ii)   Given a functor $f: I_1 \to I_2$ and a diagram $X: I_2 \to \mathcal{M}$ one can define a pullback diagram $f^*X: I_1 \to \mathcal{M}$ by $f^*X(i) = X(fi)$. A cosimplicial resolution of X pulls back to a cosimplicial resolution of f^*X, and there is an induced map $f_*: \operatorname{hocolim}_{I_1} f^*X \to \operatorname{hocolim}_{I_2} X$.

(iii)   If $I_1 \xrightarrow{f} I_2 \xrightarrow{g} I_3$ are functors and X is an I_3-diagram (with a chosen cosimplicial resolution) then the following triangle commutes:

$$
\begin{array}{ccc}
\operatorname{hocolim}_{I_1}(gf)^* X & \xrightarrow{\ f_* \ } & \operatorname{hocolim}_{I_2} g^*X \\
 & \searrow^{(gf)_*} & \downarrow{\scriptstyle g_*} \\
 & & \operatorname{hocolim}_{I_3} X.
\end{array}
$$

(iv)   If $f, g: I_1 \to I_2$ are functors, X is an I_2-diagram, and $\eta: f \to g$ is a natural transformation, then η induces a map of diagrams $\eta_*: f^*X \to g^*X$. The triangle

$$
\begin{array}{ccc}
\operatorname{hocolim}_{I_1} f^*X & \xrightarrow{\ f_* \ } & \operatorname{hocolim}_{I_2} X \\
\downarrow{\scriptstyle \eta_*} & \nearrow_{g_*} & \\
\operatorname{hocolim}_{I_1} g^*X & &
\end{array}
$$

commutes in the homotopy category.

The following result and its generalizations (see Remark A.5 below) are used several times in the body of the paper. They are our main tool for identifying two homotopy colimits over different indexing categories.

PROPOSITION A.4.   *Let I and J be small categories with a functor $g: J \to I$, and let $X: I \to \mathcal{M}$ be a diagram. Suppose that there is also a functor $f: I \to J$, together with natural transformations $\eta: gf \to Id_I$ and $\theta: fg \to Id_J$, such that the following hold:*

(i)   *Applying X to the maps $\eta(i): gf(i) \to i$ yields weak equivalences.*

(ii)   *Applying X to the maps $g(\theta j): g(fg(j)) \to g(j)$ also yields weak equivalences.*

Under these hypotheses, the map $g_*: \operatorname{hocolim}_J(g^*X) \to \operatorname{hocolim}_I X$ is a weak equivalence.

The easiest example to which the result applies is when f and g are actually an equivalence of categories—in this case conditions (i) and (ii) are automatic. In general the conditions are saying that f and g look like an equivalence as far as X is concerned.

Proof.   Consider the triangle

$$\text{hocolim}_I (gf)^* X \xrightarrow{f_*} \text{hocolim}_J g^* X \xrightarrow{g_*} \text{hocolim}_I X$$

$$\eta_* \downarrow \qquad \qquad \qquad \nearrow \, (id)_*$$

$$\text{hocolim}_I (id)^* X$$

which commutes up to homotopy. The slanted map is the identity, and the vertical map is a weak equivalence because of assumption (i) on X (which says that $(gf)^* X \to X$ is an objectwise weak equivalence). So it follows that the composite across the top row $(gf)_*$ is a weak equivalence as well.

Likewise, in the triangle

$$\text{hocolim}_J (gfg)^* X \xrightarrow{g_*} \text{hocolim}_I (gf)^* X \xrightarrow{f_*} \text{hocolim}_J g^* X$$

$$(g\theta)_* \downarrow \qquad \qquad \qquad \nearrow \, (id)_*$$

$$\text{hocolim}_J g^* X$$

assumption (ii) on X again shows that the composite across the top is a weak equivalence. The notation is a little confusing because the maps labeled g_* in the two diagrams are not exactly the same, although they are both induced by g. But the maps labeled f_* *are* the same and this is all we need. If

$$A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} D$$

are maps in some category such that ba and cb are both isomorphisms, then each of a, b, and c is also an isomorphism. Applying this to our situation shows that f_* and the two g_*'s are isomorphisms in the homotopy category of $\mathscr{M}$, hence they are weak equivalences.   ∎

Remark A.5.   In most of the cases where we want to apply this result we actually don't have a simple natural transformation from the composites gf and fg to their respective identities. Rather, usually what we have is a zigzag of natural transformations. This is fine, though, because the same line of reasoning applied to each of the steps in the zigzag still shows that the required maps are weak equivalences. Rather than give a messy formulation of some general result along these lines, we will leave that to the reader's imagination.

ACKNOWLEDGMENT

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