COMPUTATIONS OF HEIGHT 2 HIGHER REAL K-THEORY SPECTRA AT PRIME 2

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Abstract. We completely compute the G-homotopy fixed point spectral sequences at prime 2 for the height 2 Lubin–Tate theory E_2, in the case of finite subgroups G of the Morava stabilizer group for $G = Q_8, SD_{16}, G_{24}$, and G_{48}. Our computation uses recently developed equivariant techniques since Hill–Hopkins–Ravenel. We also compute the $(* - \sigma_i)$-graded Q_8- and SD_{16}-homotopy fixed point spectral sequences where σ_i is a non-trivial one dimensional Q_8-representation.

Contents

1. Introduction and main results

1.1. **Motivation and main results.** Chromatic homotopy theory studies large scale phenomena in the stable homotopy category using the algebraic geometry of smooth 1-parameter formal groups [Qui69, Mor85]. The moduli stack of formal groups has a stratification by heights, which in the stable homotopy category corresponds to localizations with respect to the Lubin–Tate theories E_n for heights $n \geq 0$.

We fix a prime p. Let Γ_n be the p-typical height n Honda formal group law over $\mathbb{F}_p$, and let $\mathbb{S}_n$ be the automorphism group of Γ_n (extended to $\mathbb{F}_{p^n}$). Let $\mathbb{G}_n = \mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ be the (extended) Morava stabilizer group. Goerss–Hopkins–Miller showed that the continuous action of $\mathbb{G}_n$ on $\pi_* E_n$ can be refined to a unique $\mathbb{E}_\infty$-action of $\mathbb{G}_n$ on E_n [Rez98, GH04, Lur18].

At a prime p, theoretically one can assemble the information of E_n with the $\mathbb{G}_n$ action for all heights $n \geqslant 0$ to recover the p-local sphere. More precisely, the chromatic convergence theorem due to Hopkins and Ravenel [Rav92] exhibits the p-local sphere spectrum $S^0_{(p)}$ as the homotopy inverse limit of the E_n-local spheres (in the sense of Bousfield [Bou79])

$$\cdots \longrightarrow L_{E_n} S^0 \longrightarrow \cdots \longrightarrow L_{E_1} S^0 \longrightarrow L_{E_0} S^0.$$

Furthermore, these localizations can be built inductively via the following homotopy pullback square (the chromatic fracture square)

$$\begin{array}{ccc} L_{E_n}S^0 & \longrightarrow & L_{K(n)}S^0 \\ \downarrow & & \downarrow \\ L_{E_{n-1}}S^0 & \longrightarrow & L_{E_{n-1}}L_{K(n)}S^0, \end{array}$$

where $L_{K(n)}$ denotes the localization functor with respect to $K(n)$, the nth Morava K-theory. From this point of view, the $K(n)$-local sphere $L_{K(n)}S^0$ is the building block of the p-local stable homotopy category. Devinatz and Hopkins showed that $L_{K(n)}S^0$ is equivalent to the homotopy fixed point $E_n^{h\mathbb{G}_n}$ [DH04].

Let G denote a finite subgroup of the Morava stabilizer group. From the finite resolution point of view, the spectrum E_n^{hG} is the building block of the $K(n)$-local stable homotopy category [Hen07, GHMR05]. In particular, its homotopy groups $\pi_*E_n^{hG}$ detect important families of classes in the stable homotopy groups of spheres [HHR16, LSWX19, BMQ20]. Therefore, the computation of E_n^{hG} is a central topic in chromatic homotopy theory and extremely challenging in general.

From now on, we focus on the prime $p = 2$. Hewett classified all the finite subgroups of $\mathbb{S}_n$ [Hew95] (see also [Buj12]). If $n = 2^{m-1}\ell$ where ℓ is odd, then when $m \neq 2$, the maximal finite 2-subgroups of $\mathbb{G}_n$ are isomorphic to C_{2^m}, the cyclic group of order 2^m; when $m = 2$, n is of the form $4k + 2$, and the maximal finite 2-subgroups are isomorphic to Q_8, the quaternion group.

There are breakthroughs of computations of E_n^{hG} when G is cyclic due to the recent development of the equivariant methods [HHR17, HSWX18, BBHS20, HS20]. These computations are done by a new tool called the slice spectral sequence. The slice spectral sequence computations of the norm of real cobordism theories induce computations of E_n^{hG} at prime 2 for the case $G = C_{2^m}$. As far as the authors are aware, there are no such computations for the case $G = Q_8$ due to the lack of the slice information.

At height 2, the group Q_8 first appears as a subgroup of the (small) Morava stabilizer group $\mathbb{S}_2$. Maximal finite subgroups of $\mathbb{S}_2$ are isomorphic to $G_{24} = Q_8 \rtimes C_3$. Similarly, in the (extended) Morava stabilizer group $\mathbb{G}_2$, there are subgroups isomorphic to SD_{16} and G_{48}. Homotopy fixed points of E_2 with respect to the above subgroups appear in the finite resolution of $E_2^{h\mathbb{G}_2}$, the $K(2)$-local sphere at prime 2, as building blocks [Bea15, BG18]. Moreover, they also appear in the interplay between chromatic layer 2 and the theory of elliptic curve (see for example [Hop02, DFHH14, HM14, BO16, HL16]). Important examples like tmf are related to computations of $E_2^{hG_{48}}$.

In this paper, we use equivariant methods and a new method, which we called "the vanishing line method", to compute the G-homotopy fixed point spectral sequence (G-HFPSS) of the height 2 Lubin–Tate theory E_2 at the prime 2 for $G = Q_8, SD_{16}, G_{24}$ and G_{48}.

Let σ_i (resp. σ_j, σ_k) be the one dimensional non-trivial Q_8 representation that $i \in Q_8$ (resp. $j, k \in Q_8$) acts trivially. We compute the integer-graded as well as $(* - \sigma_i)$-graded G-HFPSS for E_2. By symmetry, this gives the $(* - \sigma_j)$-graded and $(* - \sigma_k)$-graded G-HFPSS for E_2.

Theorem A. *(1) The integer-graded Q_8-HFPSS for E_2 has differentials as listed in Table 8 (also see Figs. 5 to 8). The E_∞-page with all 2 extensions is presented in Fig. 9. Furthermore, we have*

$$SD_{16}\text{-}HFPSS(E_2) \otimes_{\mathbb{Z}_2} \mathbb{W}(\mathbb{F}_4) = Q_8\text{-}HFPSS(E_2),$$

where the tensor products happen on E_r and d_r for every $2 \leq r \leq \infty$.

(2) The $(*-\sigma_i)$-graded Q_8-HFPSS for E_2 has differentials in Table 9 (also see Figs. 11 to 14) and the E_∞-page with 2 extensions is presented as Fig. 15.
Furthermore, we have

$$SD_{16}\text{-}HFPSS(E_2) \otimes_{\mathbb{Z}_2} \mathbb{W}(\mathbb{F}_4) = Q_8\text{-}HFPSS(E_2),$$

where the tensor products happen on E_r and d_r for every $2 \le r \le \infty$.

Theorem B. *The integer-graded G_{24}-HFPSS for E_2 is a subobject of the integer-graded Q_8-HFPSS for E_2 which consists of classes with D^m where $3 \mid m$, and the differentials are exactly the same. The E_∞-page with all 2 extensions is presented as in Fig. 10. Furthermore, we have*

$$G_{48}\text{-}HFPSS(E_2) \otimes_{\mathbb{Z}_2} \mathbb{W}(\mathbb{F}_4) = G_{24}\text{-}HFPSS(E_2),$$

where the tensor products happen on E_r and d_r for every $2 \le r \le \infty$.

In Theorem B, we only compute the integer-graded part because σ_i cannot be lifted to a G_{24}-representation.

Theorem A gives the complete computation of the integer-graded G_{48}-HFPSS for E_2. Though the result is known to experts and can be deduced from the tmf computation [Bau08], as far as the authors are aware, it is not written down in literature before. The $(*-\sigma_i)$-graded computation in Theorem A is new. Moreover, our methods for Q_8-HFPSS computations are independent of previous computations and can potentially work for higher heights. The first method is the recently developed equivariant method which uses the restriction, transfer and norm structures of the spectral sequence to deduce differentials and hidden extensions. More precisely, we deduce differentials and hidden extensions in the Q_8-HFPSS for E_2 from differentials in the C_4-HFPSS for E_2 (computed in [HHR17, BBHS20]) via restrictions, transfers and norms. For example the restriction functor from Q_8 to C_4 implies a hidden 2 extension from a class at $(54, 2)$ to a class at $(54, 10)$ in the Q_8-HFPSS for E_2 (See Lemma 4.23) which is crucial to deduce the d_{13}-differential proved in Proposition 4.25. This exempts us from using the Toda bracket shuffling method as in [Bau08, Proposition 8.5 (3)]. $RO(G)$-gradings have been proved to be helpful in computations [HHR17, BBHS20]. For example, for groups $H \subset G$, the norm map on the E_2-page of a G-HFPSS is only defined after extending to $RO(G)$-gradings [Ull13, HHR17, MSZ20]. Norm maps allow us to pull back and push forward known differentials for new differential information. In our computation, $(* - \sigma_i)$-graded G-HFPSS for E_2 gives an alternative proof of a d_9-differential by the norm map (See Proposition 4.43).

We also introduce a new method: “the vanishing line method”. The vanishing result [DLS22] shows that all permanent cycles in filtration at least 25 must be hit, which forces differentials to happen in many cases. For example, in Proposition 4.14 the vanishing line method forces three differentials, including the longest d_{23}-differentials, just from the E_2-page information.

Along the way, we prove properties that help the computation and work for general heights. In particular, we improve the vanishing result in [DLS22, Theorem 6.1] for the Q_8 case to which is sharp for all known cases.

Theorem C (Theorem 4.8)**.** *The Q_8-HFPSS for E_{4k+2} admits a strong vanishing line of filtration $2^{4k+5} - 9$.*

Recall that having a strong horizontal vanishing line of filtration f means that the spectral sequence collapses after the E_f-page, and any element of filtration greater than or equal to f supports a differential or is hit.

Equivariant methods and the vanishing line method work for general heights. However, the computation of the E_2-page of HFPSS(E_{4k+2}) is not known due to the lack of the explicit Q_8-action on E_{4k+2} for $k > 1$.

Question. *How to describe explicitly the Q_8-action on the Lubin-Tate theory E_{4k+2} for $k > 1$?*

1.2. **Summary of the contents.** This paper is organized as follows. Section 2 provides a necessary background for the computational tools for the $RO(G)$-graded homotopy fixed point spectral sequence, and the input for the computation of the Q_8-HFPSS for E_2. In particular, we review the norm structure in $RO(G)$-graded homotopy fixed point spectral sequences (Theorem 2.8) and the interplay between the homotopy fixed point spectral sequences and the Tate spectral sequences in general (Lemma 2.1). We briefly review the Q_8-action on $\pi_*(E_2)$ (Eq. (2.3)) and the computation of $RO(C_4)$-graded Mackey-functor-valued C_4-HFPSS for E_2 (Section 2.4). We take these as the input for the Q_8-HFPSS for E_2. In Section 3 we compute the E_2-page of the integer-graded and $(* - \sigma_i)$-graded Q_8-HFPSS(E_2) by Bockstein spectral sequences.

In Section 4, we derive all differentials in the integer-graded Q_8-HFPSS for E_2 via equivariant methods and the vanishing line method (Theorem 4.8). In Section 4.1, we prove the properties of the Q_8-HFPSS for E_2 that we need for our computation. The vanishing line (Theorem 4.8) works for general heights and is of its own interests. In Section 4.2, we give a complete computation of all differentials in the logical order. The vanishing line method gives some difficult differentials (for example Proposition 4.14). In Section 4.3, we solve all 2 extensions. In Section 4.4, we present alternative proofs for those differentials that can be proved by more than one way.

In Section 5, we also apply equivariant methods and the vanishing line method to compute the $(* - \sigma_i)$-graded Q_8-HFPSS for E_2. In particular, this computation gives an alternative proof of a d_9-differential in the integer-graded part. In Section 6, we list figures that present our computation. In Appendix A, we explain algebraic computations of the Q_8 group cohomology. In addition, we explain how the Hurewicz image of $E_2^{hC_4}$ helps to compute the restriction map from Q_8-HFPSS to C_4-HFPSS.

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2. Preliminaries

2.1. $RO(G)$-graded homotopy fixed point spectral sequences and Tate spectral sequences.

Let X be a G-spectrum, and let $P^\bullet X$ be the slice tower of X. We have a diagram of towers:

$$\begin{array}{ccccc} EG_+ \wedge P^\bullet X & \longrightarrow & P^\bullet X & \longrightarrow & \tilde{E}G \wedge P^\bullet X \\ \downarrow \simeq & & \downarrow & & \downarrow \\ EG_+ \wedge F(EG_+, P^\bullet X) & \longrightarrow & F(EG_+, P^\bullet X) & \longrightarrow & \tilde{E}G \wedge F(EG_+, P^\bullet X). \end{array}$$

This diagram of towers further induces a Tate diagram of spectral sequences

$$\begin{array}{ccccc} \mathrm{HOSS(X)} & \longrightarrow & \mathrm{SliceSS}(X) & \longrightarrow & \mathrm{LSliceSS}(X) \\ \downarrow = & & \downarrow & & \downarrow \\ \mathrm{HOSS}(X) & \longrightarrow & \mathrm{HFPSS}(X) & \longrightarrow & \mathrm{TateSS}(X). \end{array} \tag{2.1}$$

We briefly explain the above notations as follows. We use $*$ to denote an integer and $\bigstar$ to denote an $RO(G)$-grading. We denote the underling homotopy group $\pi_0^e(X \wedge S^{-\bigstar})$ as a G-module by $\pi_\bigstar(X)$.

- The spectral sequence $\mathrm{HOSS}(X)$ of the tower $EG_+ \wedge P^\bullet X$ is the $RO(G)$-graded homotopy fixed point spectral sequence of X with the E_2-page as
$$\underline{H}_*(G, \pi_\bigstar(X))$$
which converges to $\pi_{\bigstar+*}(EG_+ \wedge X)$.
- The spectral sequence $\mathrm{SliceSS}(X)$ of the tower $P^\bullet X$ is the slice spectral sequence of X with the E_2-page as
$$\underline{\pi}_{\bigstar-*}(P^{|\bigstar|}_{|\bigstar|}X)$$
which converges to $\underline{\pi}_{\bigstar-*}(X)$. Where $P^\bullet_\bullet X$ is the fiber of $P^\bullet X \to P^{\bullet-1}X$ and $|\bigstar|$ is the underlying dimension of $\bigstar$.
- The spectral sequence $\mathrm{HFPSS}(X)$ of the tower $F(EG_+, P^\bullet X)$ is the $RO(G)$-graded homotopy fixed point spectral sequence of X with the E_2-page as
$$\underline{H}^*(G, \pi_\bigstar(X))$$
which converges to $\underline{\pi}_{\bigstar-*}(F(EG_+, X))$.
- The spectral sequence $\tilde{E}G \wedge P^\bullet X$ of the tower is the $RO(G)$-graded localized slice spectral sequence. It follows the treatment of a forthcoming paper by Meier-Shi-Zeng.
- The spectral sequence $\mathrm{TateSS}(X)$ of the tower $\tilde{E}G \wedge F(EG_+, P^\bullet X)$ is the $RO(G)$-graded Tate spectral sequence of X with the E_2-page as
$$\widehat{\underline{H}}^*(G, \pi_\bigstar(X))$$
which converges to $\underline{\pi}_{\bigstar-*}(\tilde{E}G \wedge F(EG_+, X))$.

The following result is first proven in [Ull13] for the integer gradings and extended to the $RO(G)$-gradings in[DLS22, Theorem 3.3]. It shows that the natural map from $\mathrm{SliceSS}(X)$ to $\mathrm{HFPSS}(X)$ is an isomorphism in a certain range.

Lemma 2.1 ([Ull13], [DLS22])**.** *The map from the $RO(G)$-graded slice spectral sequence to the $RO(G)$-graded homotopy fixed point spectral sequence*

$$\begin{array}{ccc} \pi^G_{V-s} P^{|V|}_{|V|} X & \longrightarrow & \pi^G_{V-s} F(EG_+, P^{|V|}_{|V|} X) \\ \Downarrow & & \Downarrow \\ \pi^G_{V-s} X & \longrightarrow & \pi^G_{V-s} F(EG_+, X) \end{array}$$

induces an isomorphism on the E_2-page in the region defined by the inequality
$$\tau(V - s - 1) > |V|, \quad \tau(V) := \min_{\{e\} \subsetneq H \subset G} |H| \cdot \dim V^H.$$
Furthermore, the map induces a one-to-one correspondence between the differentials in this isomorphism region.

We recall two kinds of distinguished classes in the $RO(G)$-graded homotopy groups that are useful for naming the relevant classes on the E_2-page of the slice spectral sequence (see [HHR16, Section 3.4] and [HSWX18, Section 2.2]) and the homotopy fixed point spectral sequence.

Definition 2.2. Let V be a G-representation. We denote the inclusion of the fixed points $S^0 \to S^V$ by a_V. This is a class in $\pi^G_{-V} S^0$. Moreover, for a ring spectrum X with G-action, we abuse notation to denote the image of a_V by a_V under the map $S^0 \to X$. We will also denote the class on the E_2-page of the G-HFPSS(S^0) or the G-HFPSS(X) that detects the image of a_V by a_V.

By construction, we have the following property.

Proposition 2.3. *With the above notation, the class a_V on the E_2-page of the G-HFPSS(X) is a permanent cycle.*

If the representation V has non-trivial fixed points (i.e. $V^G \neq \{0\}$), then $a_V = 0$. Moreover, for any two G-representations V and W, we have the relation $a_{V\oplus W} = a_V a_W$ in $\pi^G_{-V-W}(S^0)$. Moreover, a_V-class is always a torsion class, according to [HHR17, Lemma 3.6]

$$|G/G_V|a_V = 0$$

where G_V is the isotropy subgroup of V.

For an orientable G-representation V, a choice of orientation for V gives an isomorphism $H^G_{|V|}(S^V;\underline{\mathbb{Z}}) \cong \mathbb{Z}$. In particular, the restriction map

$$H^G_{|V|}(S^V,\underline{\mathbb{Z}}) \longrightarrow H_{|V|}(S^{|V|},\mathbb{Z}) \tag{2.2}$$

is an isomorphism.

Definition 2.4. Let V be an orientable G-representation. We define the orientation class of V $u_V \in H^G_{|V|}(S^V;\underline{\mathbb{Z}})$ to be the generator that maps to 1 under the above restriction isomorphism 2.2.

The orientation class u_V is stable in V in the sense that if 1 is the trivial representation, then $u_{V\oplus 1} = u_V$. Moreover, if V and W are two orientable G-representations, then $V\oplus W$ is also orientable with the direct sum orientation, and $u_{V\oplus W} = u_V u_W$.

Norms of a_V classes and u_V classes are given as follows.

Proposition 2.5. ([HHR16, Lemma 3.13]) *Let $H \subset G$ be a subgroup and V is a G-representation*

$$N^G_H(a_V) = a_{\mathrm{Ind}\,V};$$
$$u_{\mathrm{Ind}\,|V|}N^G_H(u_V) = u_{\mathrm{Ind}\,V}$$

where Ind *means* Ind^G_H.

Given a G-oriented representation V and a G-equivariant commutative ring spectrum X, by [HHR16, Corollary 4.54] and the unit map $S^0 \to X$, Hill–Hopkins–Ravenel defines the u_V classes on the E_2-page of the slice spectral sequence for X via the following map on 0-th slices

$$H\underline{\mathbb{Z}} = P^0_0 S^0 \to P^0_0 X.$$

With Lemma 2.1, we can define u_V classes in the $RO(G)$-graded HFPSS for X.

The computation of the TateSS and the HFPSS are closely related. In any $RO(G)$-graded page the natural map from HFPSS(X) to TateSS(X) is isomorphic in positive filtration ([DLS22, Theorem 3.6], see also [BM94, Lemma 2.12]).

Lemma 2.6. *The map from the $RO(G)$-graded homotopy fixed point spectral sequence to the $RO(G)$-graded Tate spectral sequence induces an isomorphism on the E_2-page for classes in filtration $s > 0$, and a surjection for classes in filtration $s = 0$. Furthermore, there is a one-to-one correspondence between differentials whose source is in non-negative filtrations.*

One advantage of considering Tate spectral sequences is that they are whole plane spectral sequences with more invertible classes. This feature makes the calculations more accessible.

If V is a G-representation such that its fixed point set V^H is trivial for any non-trivial subgroup H of G, then $S^{\infty V}$ is a geometric model for $\tilde{E}G$. If X is a G-spectrum, we have

$$\tilde{E}G \wedge X \simeq S^{\infty V} \wedge X = a_V^{-1} X$$

This implies that for such representation V, the class a_V is invertible in the Tate spectral sequence.

Method 2.7. When it is multiplicative, the TateSS is extremely useful for proving permanent cycles in the HFPSS. Assume the TateSS of a G-spectrum X is multiplicative. Then we can find permanent cycles in the G-HFPSS for X as follows. Assume that we find a differential $d_r(a) = b$ in the HFPSS, then there is a corresponding differential $d_r(a') = b'$ in the TateSS by Lemma 2.6. We can move this differential by some r-cycle c' in the TateSS such that $d_r(c'a') = c'b'$ is a differential with the source $c'a'$ in a negative filtration and the target $c'b'$ in a non-negative filtration. (One can choose $c' = a_V^{-k}$ for proper integer k where a_V is an invertible class as above.) Then $c'b'$ is a permanent cycle in the TateSS and hence the corresponding class of $c'b'$ in the HFPSS is also a permanent cycle by Lemma 2.6. This method allows us to identify permanent cycles at E_r-page for $r < \infty$.

Now we focus on $G = Q_8$ and its subgroups. We will use the following notations for representations of C_2, C_4 and Q_8.

- When $G = C_2$, $RO(C_2) = \mathbb{Z}\{1, \sigma_2\}$ where σ_2 is the sign representation.
- When $G = C_4$, $RO(C_4) = \mathbb{Z}\{1, \sigma, \lambda\}$. The representation σ is the sign representation and λ is the 2-dimensional representation by rotating the plane $\mathbb{R}^2$ by degree $\frac{\pi}{2}$.
- When $G = Q_8$, $RO(Q_8) = \mathbb{Z}\{1, \sigma_i, \sigma_j, \sigma_k, \mathbb{H}\}$. The representations σ_i, σ_j, and σ_k are one-dimensional representations whose kernels are $C_4\langle i\rangle$, $C_4\langle j\rangle$, and $C_4\langle k\rangle$ i.e, the three C_4 subgroups generated by i, j and k, respectively. The representation $\mathbb{H}$ is a four-dimensional irreducible representation, obtained by the action of Q_8 on the quaternion algebra $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ by left multiplication.

By the above discussion, $S^{\infty\mathbb{H}}$ is a model of $\tilde{E}Q_8$. Therefore, the class $a_{\mathbb{H}}$ is invertible in any Q_8-Tate spectral sequence.

2.2. **Norm differentials and strong vanishing lines in spectral sequences.** The Hill-Hopkins-Ravenel norm structure holds in nice equivariant spectral sequences. Let $H \subset G$ be a subgroup. Consider the following diagram of G-spectra

$$\cdots \to P^{n+1} \to P^n \to P^{n-1} \to \cdots$$

Recall that P_n^m denotes the fiber of $P^m \to P^{n-1}$ and $P_n = P_n^\infty$.

We denote the spectral sequence associated to this tower by $\{E_r^{n,\bigstar}, d_r\}$, where n denotes the filtration and the second grading denotes the $RO(G)$-graded stem. We say the spectral sequence has a norm structure if there are two types of maps $N_H^G P_n \to P_{|G/H|}$ and $N_H^G P_n^n \to P_{|G/H|}^{|G/H|}$ such that the following two diagrams commute up to homotopy.

$$\begin{array}{ccc} N_H^G P_n & \longrightarrow & P_{|G/H|} \\ \downarrow & & \downarrow \\ N_H^G P_{n-1} & \longrightarrow & P_{|G/H|(n-1)} \end{array} \qquad \begin{array}{ccc} N_H^G P_n & \longrightarrow & P_{|G/H|} \\ \downarrow & & \downarrow \\ N_H^G P_n^n & \longrightarrow & P_{|G/H|n}^{|G/H|n} \end{array}$$

The norm structure induces a map between towers

$$\begin{array}{ccccccccc} \cdots \to N_H^G P_n & & \longrightarrow & & & & N_H^G P_{n-1} & \longrightarrow \cdots \\ \downarrow & & & & & & \downarrow & \\ \cdots \to P_{|G/H|} & \to & P_{|G/H|-1} & \to & \cdots & \to P_{(n-1)|G/H|+1} & \to P_{(n-1)|G/H|} & \to \cdots \end{array}$$

which induces a map from the E_2-page of the H-level spectral sequence H-$E_2^{*,\bigstar}$ to the E_2-page of the G-level spectral sequence G-$E_2^{*,\bigstar}$ as follows

$$N_H^G : H\text{-}E_2^{n,V+n} \longrightarrow G\text{-}E_2^{|G/H|n,\mathrm{Ind}_H^G V+|G/H|n}.$$

It is proved in [MSZ20] that if X is a commutative G-ring spectrum then its slice spectral sequence, homotopy fixed point spectral sequence, and Tate spectral sequence (at least for $H \neq e$) have a norm structure.

One consequence of having a norm structure is that we can predict differentials in the G-level from differentials in the H-level.

Theorem 2.8. ([Ull13, Proposition I.5.17][HHR17, Theorem 4.7]) *In a spectral sequence with norm structures, if we have a differential $d_r(x) = y$ in the spectral sequence of a H-spectrum X. Then in the spectral sequence for $Y = N_H^G(X)$ there is a predicted differential*

$$d_{|G/H|(r-1)+1}(a_{\bar{\rho}} N_H^G(x)) = N_H^G(y)$$

where $\rho = \mathrm{Ind}_H^G(1)$ and $\bar{\rho}$ is the reduced representation of ρ.

In [DLS22] the authors use the norm structures to show that every class in G-TateSS(E_n) is hit before a specific page depending on n and G.

Theorem 2.9. ([DLS22, Theorem 5.1]) *At the prime 2, for any height n and any $G \subset \mathbb{G}_n$ a finite subgroup, let H be a Sylow 2-subgroup of G. All the classes in the $RO(G)$-graded Tate spectral sequence of E_n vanish after the $E_{N_{n,H}}$-page. Here $N_{n,H}$ is a positive integer defined as follows:*

- *when $(n, H) = (2^{m-1}\ell, C_{2^m})$, $N_{n,H} = 2^{n+m} - 2^m + 1$;*
- *when $(n, H) = (4k + 2, Q_8)$, $N_{n,H} = 2^{n+3} - 7$.*

The isomorphism range of the natural map G-HFPSS$(E_n) \to G$-TateSS(E_n) implies there is a strong horizontal vanishing line in E_∞-page of G-HFPSS(E_n).

Theorem 2.10. ([DLS22, Theorem 6.1]) *At the prime 2, for any height n and any $G \subset \mathbb{G}_n$ a finite subgroup, let H be a Sylow 2-subgroup of G. There is a strong horizontal vanishing line of filtration $N_{n,H}$ in the $RO(G)$-graded homotopy fixed point spectral sequence of E_n.*

It turns out that the existence of such horizontal vanishing lines is extremely helpful for determining higher differentials in homotopy fixed point spectral sequences. In particular, for our computation in Q_8-HFPSS(E_2), the vanishing line gives an independent proof of several higher differentials in the integer-gradings. Moreover, this vanishing line plays a crucial role in the computation of $(* - \sigma_i)$-graded Q_8-HFPSS for E_2.

2.3. **Lubin-Tate Theory E_2 with G_{24}-action.** We fix a pair $(\mathbb{F}_{p^n}, \Gamma_n)$ where Γ_n is the height n Honda formal group law over $\mathbb{F}_p$ extended to $\mathbb{F}_{p^n}$. Then Lubin–Tate [LT65] shows that there is a universal deformation F_n defined over a complete local ring

$$\mathbb{W}(\mathbb{F}_{p^n})[\![u_1, \ldots, u_{n-1}]\!][u^{\pm 1}]$$

where $\mathbb{W}(\mathbb{F}_{p^n})$ is the p-typical Witt vector of $\mathbb{F}_{p^n}$ and $|u_i| = 0$, $|u^{-1}| = 2$. The Landweber exactness theorem shows that this ring can be realized by a complex oriented ring spectrum E_n.

Let $\mathbb{S}_n$ be the automorphism group of Γ_n, namely the small n-th Morava stabilizer group. Let $\mathbb{G}_n = \mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ be the automorphism group of $(\mathbb{F}_{p^n}, \Gamma_n)$, namely the (extended) n-th Morava stabilizer group. By universality, $\pi_* E_n$ admits a $\mathbb{G}_n$-action. The Goerss–Hopkins–Miller theorem [Rez98, GH04, Lur18] lifts this action uniquely to an $\mathbb{E}_\infty$-action on E_n.

We are interested in computing $\pi_* E_n^{hG}$ for G a finite subgroup of $\mathbb{G}_n$ via G-homotopy fixed point spectral sequences. For these computations, the action of the Galois group $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ will not change the differential pattern. More precisely, we review the following result.

Lemma 2.11. ([BG18, Lemma 1.32][BGH22, Lemma 2.2.6, Lemma 2.2.7]) *Let $F \subset \mathbb{G}_n$ be a closed subgroup and let $F_0 = F \cap \mathbb{S}_n$. Suppose the canonical map*

$$F/F_0 \to \mathbb{G}_n/\mathbb{S}_n \cong \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$$

is an isomorphism. Then there is a commutative diagram of homotopy fixed point spectral sequences

$$\begin{array}{ccc} \mathbb{W}(\mathbb{F}_{p^n}) \otimes_{\mathbb{Z}_p} H^*(F, \pi_* E_n) & \Longrightarrow & \mathbb{W}(\mathbb{F}_{p^n}) \otimes_{\mathbb{Z}_p} \pi_*(E_n^{hF}) \\ \downarrow \cong & & \downarrow \cong \\ H^*(F_0, \pi_* E_n) & \Longrightarrow & \pi_*(E_n^{hF_0}). \end{array}$$

In this paper, we will focus on the case $p = 2$ and $n = 2$. The Galois group $\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$ is isomorphic to C_2 and we write $\mathbb{W}$ for the Witt vector $\mathbb{W}(\mathbb{F}_4)$. There are finite subgroups Q_8 and $G_{24} \cong Q_8 \rtimes C_3$ in the small Morava stabilizer group $\mathbb{S}_2$ and $SD_8 = Q_8 \rtimes \mathrm{Gal}$ and $G_{48} \cong G_{24} \rtimes \mathrm{Gal}$ in the extended Morava stabilizer group $\mathbb{G}_2$. The subgroups Q_8, G_{24} are unique up to conjugacy in $\mathbb{S}_2$ [Buj12] (see also [BGH22, Remark 2.4.5]). Therefore, there is no ambiguity of the notation $\pi_* E_2^{hQ_8}$ or $\pi_* E_2^{hG_{24}}$. The subgroup Q_8 and complex orientation coordinates can be chosen specifically from the theory of elliptic curves at prime 2 so that the action has explicit formulas as follows (See [Bea17, Section 2] for more details).

We recall the action of G_{24} on $\pi_* E_2$ [Bea17, Lemma A.1]. The coefficient ring is a complete local ring $\pi_* E_2 = \mathbb{W}[\![u_1]\!][u^{\pm 1}]$ with a maximal ideal $I = (2, u_1)$. Denote $u_1 u^{-1}$ by v_1, the generator of the quaternion group Q_8 by i, j, k and the generator of C_3 by ω. We regard the third root of unit ζ as a class in the Witt vector $\mathbb{W}$. The G_{24}-actions on u^{-1} and v_1 are

$$\begin{aligned} \omega_*(v_1) &= v_1, & \omega_*(u^{-1}) &= \zeta^2 u^{-1}, \\ i_*(u^{-1}) &= \frac{v_1 - u^{-1}}{\zeta^2 - \zeta}, & i_*(v_1) &= \frac{v_1 + 2u^{-1}}{\zeta^2 - \zeta}, \\ j_*(u^{-1}) &= \frac{\zeta v_1 - u^{-1}}{\zeta^2 - \zeta}, & j_*(v_1) &= \frac{v_1 + 2\zeta^2 u^{-1}}{\zeta^2 - \zeta}, \\ k_*(u^{-1}) &= \frac{\zeta^2 v_1 - u^{-1}}{\zeta^2 - \zeta}, & k_*(v_1) &= \frac{v_1 + 2\zeta u^{-1}}{\zeta^2 - \zeta}. \end{aligned} \tag{2.3}$$

We define D to be $\prod_{g \in Q_8/C_2} g_*(u^{-1})$ which is Q_8-invariant. Then $(E_2)_*$ could be expressed as

$$\pi_* E_2 \cong (\mathbb{W}[v_1, u^{-1}][D^{-1}])^{\wedge}_I,$$

which is more convenient for the Q_8-cohomology computation.

Lemma 2.12. *There is an isomorphism*

$$H^*(Q_8, \pi_* E_2) \cong (H^*(Q_8, \mathbb{W}[v_1, u^{-1}])[D^{-1}])^{\wedge}_I.$$

Proof. Because D is Q_8-invariant, we have

$$H^*(Q_8, \mathbb{W}[v_1, u^{-1}][D^{-1}]) \cong H^*(Q_8, \mathbb{W}[v_1, u^{-1}])[D^{-1}].$$

Note that $\mathbb{W}[v_1, u^{-1}][D^{-1}]$ is finitely generated as a $\mathbb{W}$-algebra. Therefore, the completion is an exact functor [AM16, Theorem 10.12] [HS99, Theorem A.1] and we have

$$H^*(Q_8, \pi_* E_2) \cong (H^*(Q_8, \mathbb{W}[v_1, u^{-1}])[D^{-1}])^{\wedge}_I.$$

□

2.4. **Mackey functor C_4-homotopy fixed point spectral sequence for E_2.** In this subsection, we recall some results on the Mackey-functor-valued C_4-HFPSS for E_2 in [BBHS20]. See also the slice spectral sequence computation of the truncated C_4-normed Real Brown–Petersen spectrum $BP^{((C_4))}\langle 1\rangle$ [HHR17][HSWX18].

Proposition 2.13. ([BBHS20, Proposition 5.6]) *There is an isomorphism*

$$H^*(C_2, \pi_\bigstar E_2) \cong \mathbb{W}[\![\mu_0]\!][\bar{r}_1^{\pm 1}, a_{\sigma_2}, u_{2\sigma_2}^{\pm 1}]/(2a_{\sigma_2}),$$

where the $(\bigstar - *, *)$*-degree of the classes is given by* $|\mu_0| = (0,0)$, $|\bar{r}_1| = (\rho_2, 0)$, $|a_{\sigma_2}| = (-\sigma_2, 1)$, *and* $|u_{2\sigma_2}| = (2-2\sigma_2, 0)$.

We partially rewrite the names of classes on the E_2-page of C_4-HFPSS(E_2) in [BBHS20, Proposition 5.10] with slice names. For slice names, see [HHR17, HSWX18] for details. One advantage of using slice names is that it is better to organize differentials by the slice differential theorem [HHR16, Theorem 9.9].

Proposition 2.14. ([BBHS20, Proposition 5.10]) *There is an isomorphism*

$$H^*(C_4, \pi_\bigstar E_2) \cong \mathbb{W}[\![\mu]\!][T_2, \eta, \eta', a_\lambda, a_\sigma][\bar{\mathfrak{d}}_1^{\pm 1}, u_\lambda^{\pm 1}, u_{2\sigma}^{\pm 1}]/\sim$$

where $\mu = \mathrm{tr}_{C_2}^{C_4}(\mu_0)$, $T_2 = \bar{s}_1^2 u_{2\sigma_2} = \mathrm{tr}_{C_2}^{C_4}(\bar{r}_1^2 u_{2\sigma_2})$, $\eta = \bar{s}_1 a_{\sigma_2} = \mathrm{tr}_{C_2}^{C_4}(\bar{r}_1 a_{\sigma_2})$ *and* $\eta' = \bar{s}_1 u_\sigma a_{\sigma_2} = \mathrm{tr}_{C_2}^{C_4}(\bar{r}_1 a_{\sigma_2} u_\sigma)$. *Although* σ *is not an oriented* C_4*-representation, we apply* u_σ *here indicating that* η' *is transfered from* $\bar{r}_1 a_{\sigma_2}$ *from integer-graded part in* C_2*-level to* $(1-\sigma)$*-page in* C_4*-level. And the relation* $\sim$ *is the ideal generated by the following relation*

$$\begin{aligned} 2\eta = 2\eta' = 2a_\sigma = 4a_\lambda &= 0, & T_2^2 &= \Delta_1((\mu-2)^2+4),\\ \eta^2 u_{2\sigma} = \eta'^2 &= T_2 u_\lambda^{-1} u_{2\sigma} a_\lambda, & T_2\eta' &= \bar{\mathfrak{d}}_1 \mu\eta u_\lambda u_{2\sigma},\\ T_2\eta &= \bar{\mathfrak{d}}_1 \mu \eta' u_\lambda, & \eta\eta' &= \mu u_{2\sigma} a_\lambda,\\ u_\lambda a_{2\sigma} &= 2a_\lambda u_{2\sigma}, & \mu a_\sigma = \eta a_\sigma = \eta' a_\sigma &= T_2 a_\sigma = 0. \end{aligned}$$

Here $\Delta_1 = \bar{\mathfrak{d}}_1^2 u_{2\lambda} u_{2\sigma}$ *at* $(8,0)$ *is an invertible class in* $\pi_* E_2^{hC_4}$.

Remark 2.15. Proposition 2.13 and Proposition 2.14 give a full description of the Mackey functor $\underline{H}^*(C_4, \pi_\bigstar E_2)$ by the Frobenius relation [BBHS20, Remark 5.17] and the multiplicative property of restriction.

Remark 2.16. A warning is that one needs to be careful about the isomorphism range(See Lemma 2.1) to translate between the slice spectral sequence and the homotopy spectral sequence. For example, in the C_4-SliceSS($BP^{((C_4))}\langle 1\rangle$), the class $u_{2\sigma}$ supports a non-trivial d_5-differential [HSWX18, Theorem 3.4], while in the corresponding C_4-HFPSS(E_2), the class $u_{2\sigma}$ actually supports a non-trivial d_7-differential [BBHS20, Remark 5.23].

The computation of the Mackey-functor-valued C_4-homotopy fixed point spectral sequence for E_2 is explained in detail in [BBHS20, Section 5] and presented by [BBHS20, Figure 5.8] and [BBHS20, Figure 5.14].

The $RO(G)$-graded Mackey functor computation is useful even if one only cares about the computation of the integer-graded part $\pi_* E_n^{hG}$. The following discussion of hidden extensions is a good example. We can use exotic operations (exotic transfers, exotic restrictions, and so on) in Mackey-functor-valued spectral sequences to deduce differentials and hidden extensions inside the spectral sequences. For more detailed definitions and properties of such phenomena, one could refer to [MSZ20, Section 3.3].

In [HHR17, Lemma 4.2], the authors introduce a useful trick to determine exotic restrictions and transfers on the E_∞-page of Mackey-functor-valued G-HFPSS.

Lemma 2.17. ([HHR17, Lemma 4.2]) *Let G be a cyclic 2-group and G' be its index 2 subgroup then in $\underline{\pi}_\bigstar(F(EG_+, X))$ we have*

- $\ker(\mathrm{res}^G_{G'}) = im(a_\sigma)$
- $im(\mathrm{tr}^G_{G'}) = \ker(a_\sigma)$

where σ is the sign representation of G.

The following hidden 2 extension in stem 22 is a good example showing that equivariant structures provide extra integer-graded information (see a similar 2 extension in stem 2 in [MSZ20, Remark 5.15]). In [HHR17, Figure 15] and [BBHS20, Figure 5.6], they drew all exotic restrictions and transfers in the E_∞-page of the Mackey functor valued C_4-HFPSS(E_2). The 2 extension follows from an exotic transfer and an exotic restriction in 22 stem. We spell out the details in Lemma 2.18.

Lemma 2.18. *In the Mackey-functor-valued C_4-HFPSS for E_2, there is an exotic restriction in stem 22 from $\bar{\mathfrak{d}}_1^6 u_{6\lambda} u_{4\sigma} a_{2\sigma}$ to $\bar{\mathfrak{d}}_1^{10} \bar{r}_1^6 u_{8\sigma_2} a_{6\sigma_2}$ and there is an exotic transfer in stem 22 from $\bar{\mathfrak{d}}_1^4 \bar{r}_1^6 u_{8\sigma_2} a_{6\sigma_2}$ to $\bar{\mathfrak{d}}_1^8 u_{4\lambda} u_{6\sigma} a_{4\lambda} a_{2\sigma}$. As a consequence, there is a hidden 2 extension from $\bar{\mathfrak{d}}_1^6 u_{6\lambda} u_{4\sigma} a_{2\sigma}$ to $\bar{\mathfrak{d}}_1^8 u_{4\lambda} u_{6\sigma} a_{4\lambda} a_{2\sigma}$.*

Proof. According to the computations in [HHR17][BBHS20], in stem 22 there are only three classes survives: $\bar{\mathfrak{d}}_1^6 u_{6\lambda} u_{4\sigma} a_{2\sigma}$ and $\bar{\mathfrak{d}}_1^8 u_{4\lambda} u_{6\sigma} a_{4\lambda} a_{2\sigma}$ in C_4-level and $\bar{\mathfrak{d}}_1^4 \bar{r}_1^6 u_{8\sigma_2} a_{6\sigma_2}$ in C_2-level. We first claim the class $\bar{\mathfrak{d}}_1^6 u_{6\lambda} u_{4\sigma} a_{2\sigma}$ is not in the image of multiplication by a_σ. If there is some x such that $a_\sigma x$ is $\bar{\mathfrak{d}}_1^6 u_{6\lambda} u_{4\sigma} a_{2\sigma}$, then x is detected by classes at $(22+\sigma, 1)$ or $(22+\sigma, 0)$. There is only one class at $(22+\sigma, 1)$ which is $\bar{\mathfrak{d}}_1^4 u_{4\lambda} u_{4\sigma} a_\sigma$ on E_2-page. According to [HSWX18, Theorem 3.11], this class supports a d_{13}-differential

$$d_{13}(\bar{\mathfrak{d}}_1^4 u_{4\lambda} u_{4\sigma} a_\sigma) = \bar{\mathfrak{d}}_1^4 u_{4\sigma} d_{13}(u_{4\lambda} a_\sigma) = \bar{\mathfrak{d}}_1^7 u_{8\sigma} a_{7\lambda}$$

And moreover there is no non-trivial class at $(22+\sigma, 0)$. Therefore, in homotopy level there is no class such that its multiplication by a_σ hits the class $\bar{\mathfrak{d}}_1^6 u_{6\lambda} u_{4\sigma} a_{2\sigma}$. Then according to Lemma 2.17, this class must have a non-trivial restriction in homotopy level, and the desired exotic restriction follows from degree reasons.

On the other hand by the gold relation $u_\lambda a_{2\sigma} = 2u_{2\sigma} a_\sigma$ and $2a_\sigma = 0$ we know on E_2-page

$$\bar{\mathfrak{d}}_1^8 u_{4\lambda} u_{6\sigma} a_{4\lambda} a_{2\sigma} \cdot a_\sigma = 0$$

Moreover, according to the computation on $* - \sigma$-page of C_4-HFPSS(E_2) [BBHS20], there is no hidden a_σ-extension from $\bar{\mathfrak{d}}_1^8 u_{4\lambda} u_{6\sigma} a_{4\lambda} a_{2\sigma}$ by degree reasons. Since we have $\mathrm{im}(\mathrm{tr}^G_{G'}) = \ker(a_\sigma)$, the class $\bar{\mathfrak{d}}_1^8 u_{4\lambda} u_{6\sigma} a_{4\lambda} a_{2\sigma}$ must be a transfer of a class from C_2-level. Then the desired exotic transfer follows from degree reasons. □

Remark 2.19. For degree reasons, the class $\bar{\mathfrak{d}}_1^6 u_{6\lambda} u_{4\sigma} a_{2\sigma}$ cannot be in the image of the transfer from C_2. However, by the gold relation, the product of this class and a_σ is zero on the E_2-page. Therefore, this class must have a hidden a_σ-extension.

Remark 2.20. The hidden 2 extension in Lemma 2.18 will play a crucial rule in deducing several higher differentials in Q_8-HFPSS(E_2) (see Lemma 4.23, Proposition 4.25). A similar 2 extension can also be seen in the homotopy groups of tmf in stem 54. The proof of this hidden 2 extension in [Bau08, Proposition 8.5 (3)] uses shuffling arguments of 4-fold Toda brackets. In our Q_8-HFPSS(E_2) computation, the corresponding hidden 2 extension follows directly from the C_4-computation by restriction (see Lemma 4.23).

2.5. $RO(G)$**-graded periodicity.** When computing HFPSS, another advantage of expanding to $RO(G)$-gradings is having more periodicities. These periodicities have their own theoretic importance. They can also move integer-graded calculations to certain $RO(G)$-gradings where the calculations might be simpler. In either the slice spectral sequence for $BP^{(\!(C_4)\!)}\langle 1\rangle$ [HSWX18] or the C_4-homotopy fixed point spectral sequence for E_2 [HHR17][BBHS20], we have the following periodicities in the $RO(G)$-gradings.

Lemma 2.21. *The following permanent cycles in C_4-HFPSS(E_2) [HHR17][BBHS20] are periodic classes.*

- *The class $\bar{\mathfrak{d}}_1$ gives $(1+\sigma+\lambda)$-periodicity.*
- *The class $u_{8\lambda}$ gives $(16-8\lambda)$-periodicity.*
- *The class $u_{4\sigma}$ gives $(4-4\sigma)$-periodicity.*
- *The class $u_{4\lambda}u_{2\sigma}$ gives $(10-4\lambda-2\sigma)$-periodicity.*

Since the norm functor is symmetric monoidal, we can apply it to the above three invertible permanent cycles, which gives some $RO(Q_8)$-periodicities in Q_8-HFPSS(E_2). The quaternion group Q_8 has three C_4 subgroups $C_4\langle i\rangle$, $C_4\langle j\rangle$ and $C_4\langle k\rangle$ generated by i, j and k respectively. For each C_4 copy we have the associated C_4-periodicities and their norms give $RO(Q_8)$-periodicities as follows.

Corollary 2.22. *We have the following $RO(Q_8)$-periodicities in Q_8-HFPSS(E_2).*

- $N_{C_4}^{Q_8}(\bar{\mathfrak{d}}_1)$:
$$1+\sigma_i+\sigma_j+\sigma_k+\mathbb{H}$$
- $N_{C_4}^{Q_8}(u_{4\sigma})$:
$$4+4\sigma_i-4\sigma_j-4\sigma_k$$
$$4+4\sigma_j-4\sigma_i-4\sigma_k$$
$$4+4\sigma_k-4\sigma_i-4\sigma_j$$
- $N_{C_4}^{Q_8}(u_{4\lambda}u_{2\sigma})$:
$$10+10\sigma_i-2\sigma_j-2\sigma_k-4\mathbb{H}$$
$$10+10\sigma_j-2\sigma_i-2\sigma_k-4\mathbb{H}$$
$$10+10\sigma_k-2\sigma_j-2\sigma_i-4\mathbb{H}$$
- $N_{C_4}^{Q_8}(u_{8\lambda})$:
$$16+16\sigma_i-8\mathbb{H}$$
$$16+16\sigma_j-8\mathbb{H}$$
$$16+16\sigma_k-8\mathbb{H}$$

Corollary 2.23. *There are periodicities of $4-4\sigma_i, 4-4\sigma_j$ and $4-4\sigma_k$ in Q_8-HFPSS(E_2).*

Proof. It suffices to show that $4-4\sigma_i$ is a periodicity. This periodicity is given by the following product:
$$N_{C_4\langle j\rangle}^{Q_8}(u_{4\lambda}u_{2\sigma})N_{C_4\langle k\rangle}^{Q_8}(u_{4\lambda}u_{2\sigma})N_{C_4\langle i\rangle}^{Q_8}(u_{8\lambda})^{-1}N_{C_4\langle i\rangle}^{Q_8}(u_{2\sigma})^2N_{C_4\langle j\rangle}^{Q_8}(u_{2\sigma})^{-1}N_{C_4\langle k\rangle}^{Q_8}(u_{2\sigma})^{-1}.$$
□

3. E_2-page of the Q_8-HFPSS(E_2)

In this section, we recollect the computation of the E_2-page of the integer-graded Q_8-HFPSS for E_2 by the 2-Bockstein spectral sequence (2-BSS) from [Bea17, Bau08]. Then we compute the E_2-page of the $(*-\sigma_i)$-graded part by the same method. By Lemma 2.12 we can compute $H^*(Q_8, \pi_*E_2)$, the E_2-page of the Q_8-HFPSS for E_2, by first computing $H^*(Q_8, \mathbb{W}[v_1, u^{-1}])$.

3.1. **2-BSS, integer-graded.** The integer-graded 2-Bockstein spectral sequence for $H^*(Q_8, \mathbb{W}[v_1, u^{-1}])$ is

$$H^*(Q_8, \mathbb{F}_4[v_1, u^{-1}])[h_0] \implies H^*(Q_8, \mathbb{W}[v_1, u^{-1}])$$

where h_0 detects 2. The computation of the E_1-page, $H^*(Q_8, \mathbb{F}_4[v_1, u^{-1}])$, is from [Bea17, Appendix A]. We follow the notation in [Bea17], except that we use h_1 for η and h_2 for ν. The differentials of this 2-BSS are essentially from [Bau08, Section 7] and we list them in Table 1.

Proposition 3.1. *The bigradings of generators of $H^*(Q_8, \mathbb{F}_4[v_1, u^{-1}])$ are:*

$$|v_1| = (2,0), \quad |D| = (8,0), \quad |k| = (-4,4), \quad |h_1| = (1,1),$$
$$|h_2| = (3,1), \quad |x| = (-1,1), \quad |y| = (-1,1), \quad |D^{-1}h_2^2 y| = (-3,3).$$

The relation ($\sim$) is generated by:

(1) in filtration 1:
$$v_1 h_2, \quad v_1^2 x, \quad v_1 y;$$
(2) in filtration 2:
$$h_1 h_2, \quad h_2 x - v_1 h_1 x, \quad h_1 y - v_1 x^2, \quad xy, \quad Dy^2 - h_2^2;$$
(3) in filtration 3:
$$h_1^2 Dx - h_2^3, \quad Dx^3 - h_2^2 y, \quad D(D^{-1}h_2^2 y) - h_2^2 y;$$
(4) in filtration 4:
$$h_1^4 - v_1^4 k.$$

Proof. Note that the composition

$$H^*(G_{24}, \mathbb{F}_4[v_1, u^{-1}]) \xrightarrow{\text{res}} H^*(Q_8, \mathbb{F}_4[v_1, u^{-1}]) \xrightarrow{\text{tr}} H^*(G_{24}, \mathbb{F}_4[v_1, u^{-1}])$$

is multiplication by $|G_{24}/Q_8| = 3$ which is a unit in the coefficient $\mathbb{F}_4[v_1, u^{-1}]$. This implies that $H^*(Q_8, \mathbb{F}_4[v_1, u^{-1}])$ is just 3 copies of $H^*(G_{24}, \mathbb{F}_4[v_1, u^{-1}])$. The result follows from the computation of the cohomology $H^*(G_{24}, \mathbb{F}_4[v_1, u^{-1}])$ in [Bea17, Thm. A.20]. □

The differentials in the integer-graded 2-BSS for the cohomology $H^*(Q_8, \mathbb{W}[v_1, u^{-1}])$ are essentially from [Bau08, Section 7] which are determined by the ones in Table 1 and the multiplicative structure.

TABLE 1. 2-BSS differentials, integer-graded

(s,f)	x	r	$d_r(x)$
$(4k+2,0)$	v_1^{2k+1}	1	$2v_1^{2k}h_1$
$(7,1)$	Dx	1	$2h_2^2$
$(-1,1)$	x	1	$2y^2$
$(-1,1)$	y	1	$2x^2$
$(4,0)$	v_1^2	2	$4h_2$
$(5,3)$	yh_2^2	3	$8kD$

The 2-Bockstein computation gives the following result (see also [Bau08, Section 7]).

Theorem 3.2. *Table 2 and Table 3 present $H^*(Q_8, \mathbb{W}[v_1, u^{-1}])$, the E_∞-page of the integer-graded 2-Bockstein spectral sequence (also see Fig. 2 and Fig. 3).*

Remark 3.3. We note that in $H^*(Q_8, \mathbb{W}[v_1, u^{-1}])$, there is an exotic h_2-extension

$$h_2 \cdot x^2 h_2 = 4kD$$

by [Bau08, Equation (7.13)] which is useful in later computations.

Table 2. E_∞-page, multiplicative generators, integer-graded

(s,f)	x	2-torsion
$(-4,4)$	k	$\mathbb{Z}/8$
$(-2,2)$	x^2	$\mathbb{Z}/2$
$(-2,2)$	y^2	$\mathbb{Z}/2$
$(0,2)$	xh_1	$\mathbb{Z}/2$
$(1,1)$	h_1	$\mathbb{Z}/2$
$(3,1)$	h_2	$\mathbb{Z}/4$
$(5,1)$	$v_1^2h_1$	$\mathbb{Z}/2$
$(8,0)$	D	$\mathbb{Z}$
$(8,0)$	v_1^4	$\mathbb{Z}$

Table 3. E_∞-page, relation generators, integer-graded

f	relation generators
1	$v_1^4h_2$
2	h_1h_2, $v_1^2h_1\cdot h_2$, $Dy^2-h_2^2$, $xh_1\cdot v_1^4$, $x^2\cdot v_1^4$, $y^2\cdot v_1^4$
3	$xh_1\cdot h_2$, $xh_1\cdot v_1^2h_1$, $x^2\cdot v_1^2h_1$, y^2h_1, $y^2\cdot v_1^2h_1$, $D\cdot xh_1\cdot h_1-h_2^3$
4	$h_1^4-v_1^4k$, $(xh_1)^2$, $(x^2)^2$,$(y^2)^2$, h_2^4, $x^2\cdot xh_1$, $y^2\cdot xh_1$, $x^2\cdot y^2$, $xh_1\cdot h_1^2$, $x_2\cdot h_1^2$, $y^2\cdot h_1^2$,$h_2^2\cdot x^2-4kD$, $y^2\cdot h_2^2$, $xh_1\cdot h_2^2$

We refer readers to §6 for charts of the E_1-page and the E_∞-page.

3.2. 2-BSS, $(*-\sigma_i)$-graded.

We discuss the $RO(G)$-graded case and restrict it to the $(*-\sigma_i)$-graded case. A variation of Lemma 2.12 still holds in this case. Thus we can compute $H^*(Q_8,\pi_{*-\sigma_i}E_2)$ by first computing the $(*-\sigma_i)$-graded 2-BSS, and then inverting D and taking the completion. Note that after modulo 2, the representation σ_i is oriented and the orientation class u_{σ_i} gives an isomorphism between $\pi_*E_2/2$ and $\pi_{*+1-\sigma_i}E_2/2$ as Q_8-modules. Therefore, the E_1-page of the $(*-\sigma_i)$-graded 2-BSS is abstractly isomorphic to that of the integer-graded part. We denote the E_1-page by

$$H^*(Q_8,\mathbb{F}_4[v_1,u^{-1}])\{u_{\sigma_i}\}$$

where u_{σ_i} denote a generator of the class at $(1-\sigma_i,0)$.

Proposition 3.4. *In the 2-BSS, there is a differential*

$$d_1(u_{\sigma_i})=2xu_{\sigma_i}+2yu_{\sigma_i}.$$

Proof. The group cohomology computation shows that $H^1(Q_8,\pi_{1-\sigma_i}(E_2))$ is 2-torsion according to Proposition A.7. Hence in the 2-BSS, there must be a d_1-differential hit the bigrading $(-\sigma_i,1)$. Then u_{σ_i} in the 2-BSS must support a non-trivial d_1-differential by degree reasons. Assume that $d_1(u_{\sigma_i})=2axu_{\sigma_i}+2byu_{\sigma_i}$ where $a,b\in\mathbb{F}_4$. By the Leibniz rule, we have $d_1(v_1u_{\sigma_i})=2h_1u_{\sigma_i}+2axv_1u_{\sigma_i}$. Since h_1 is a permanent cycle, the Leibniz rule implies that $h_1u_{\sigma_i}$ also supports a non-trivial d_1-differential. Therefore, the d_1-target of $v_1u_{\sigma_i}$ cannot be $2h_1u_{\sigma_i}$. We deduce that $a\neq 0$.

Similarly, by considering $d_1(h_2^2u_{\sigma_i})$ and $d_1(yu_{\sigma_i})$, we deduce that $b\neq 0$.

□

The remaining $(* - \sigma_i)$-graded 2-BSS d_1-differentials can be determined by the Leibniz rule and the differential on u_{σ_i} in Proposition 3.4.

Proposition 3.5. *There is a 2-BSS differential*

$$d_2(x\eta^2 u_{\sigma_i}) = 4kv_1^2 u_{\sigma_i}.$$

Proof. By Example A.5, the class at $(1 - \sigma_i, 4)$ is 4-torsion in the E_∞-page. This forces the desired d_2-differential.

□

We list non-trivial differentials on classes of the form $\{\text{multiplicative generators}\}u_{\sigma_i}$ in the table below.

TABLE 4. 2-BSS differentials, $(* - \sigma_i)$-graded

(s, f)	x	r	$d_r(x)$
$(1 - \sigma_i, 0)$	u_{σ_i}	1	$2xu_{\sigma_i} + 2yu_{\sigma_i}$
$(-\sigma_i, 1)$	xu_{σ_i}	1	$2x^2u_{\sigma_i} + 2y^2u_{\sigma_i}$
$(3 - \sigma_i, 0)$	$v_1u_{\sigma_i}$	1	$2\eta u_{\sigma_i} + 2xv_1u_{\sigma_i}$
$(4 - \sigma_i, 1)$	$h_2u_{\sigma_i}$	1	$2xh_2u_{\sigma_i} + 2yh_2u_{\sigma_i}$
$(2 - \sigma_i, 3)$	$x\eta^2u_{\sigma_i}$	2	$4kv_1^2u_{\sigma_i}$

Theorem 3.6. *Table 5 and Table 6 present $H^*(Q_8, \mathbb{W}[v_1, u^{-1}])$, the E_∞-page of the $(* - \sigma_i)$-graded 2-Bockstein spectral sequence.*

Proof. The result follows from the 2-BSS computation. □

TABLE 5. E_∞ page, module generators, $(* - \sigma_i)$-graded

(s, f)	x	2-torsion
$(-2, 2)$	$\{x^2 + y^2\}u_{\sigma_i}$	$\mathbb{Z}/2$
$(-1, 1)$	$\{x + y\}u_{\sigma_i}$	$\mathbb{Z}/2$
$(1, 1)$	$\{h_1 + xv_1\}u_{\sigma_i}$	$\mathbb{Z}/2$
$(0, 2)$	$v_1^2u_{\sigma_i}$	$\mathbb{Z}$

TABLE 6. E_∞-page, relation generators, $(* - \sigma_i)$-graded

f	relation generators
1	$\{h_1 + xv_1\}u_{\sigma_i} \cdot v_1^4 - v_1^2u_{\sigma_i} \cdot v_1^2h_1$, $v_1^2u_{\sigma_i} \cdot h_2$, $\{x + y\}u_{\sigma_i} \cdot v_1^4$
2	$\{h_1 + xv_1\}u_{\sigma_i} \cdot h_2$, $\{h_1 + xv_1\}u_{\sigma_i} \cdot v_1^2h_1 - v_1^2u_{\sigma_i} \cdot h_1^2$,
	$v_1^2u_{\sigma_i} \cdot x^2$, $v_1^2u_{\sigma_i} \cdot y^2$, $v_1^2u_{\sigma_i} \cdot xh_1$, $v_1^2u_{\sigma_i} \cdot h_2^2$, $\{x + y\}u_{\sigma_i} \cdot v_1^2h_1$, $\{x^2 + y^2\}u_{\sigma_i} \cdot v_1^4$
3	$\{h_1 + xv_1\}u_{\sigma_i} \cdot x^2 - \{x^2 + y^2\}u_{\sigma_i} \cdot h_1$, $\{h_1 + xv_1\}u_{\sigma_i} \cdot y^2$, $\{x^2 + y^2\}u_{\sigma_i} \cdot v_1^2h_1$
	$\{x + y\}u_{\sigma_i} \cdot h_1^2 - \{h_1 + xv_1\}u_{\sigma_i} \cdot xh_1$, $\{x + y\}u_{\sigma_i} \cdot x^2 - \{x + y\}u_{\sigma_i} \cdot y^2$
4	$\{x^2 + y^2\}u_{\sigma_i} \cdot h_1^2$, $\{x^2 + y^2\}u_{\sigma_i} \cdot h_2^2$, $\{x^2 + y^2\}u_{\sigma_i} \cdot x^2$, $\{x^2 + y^2\}u_{\sigma_i} \cdot y^2$, $\{x^2 + y^2\}u_{\sigma_i} \cdot xh_1$
	$\{h_1 + xv_1\}u_{\sigma_i} \cdot h_1^3 - v_1^2u_{\sigma_i}$, $4v_1^2u_{\sigma_i} \cdot k$

We refer the readers to §6 for charts of the E_1-page and the E_∞-page.

By Lemma 2.12, in both the integer-graded and the $(* - \sigma_i)$-graded case, the E_2-page of Q_8-HFPSS(E_2) follows from Theorem 3.2 and Theorem 3.6.

Remark 3.7. The E_2-page of TateSS(E_2) follows by further inverting the class k from that of Q_8-HFPSS(E_2), and then replacing the 0-line with the cokernel of the norm map.

4. Computation of the integer-graded Q_8-HFPSS(E_2)

In this section, we derive all differentials in the integer-graded Q_8-HFPSS for E_2 via the following two methods.

1. Equivariant methods: apply the restrictions, transfers, and norms to deduce differentials in the Q_8-HFPSS for E_2 from the C_4-HFPSS for E_2;
2. The vanishing line method: use the fact that the Q_8-HFPSS for E_2 admits a strong vanishing line of filtration 23 (Theorem 4.8, for general cases, see [DLS22, Theorem 6.1]) to force differentials.

We also solve all hidden 2 extensions via equivariant methods and investigation of the Tate spectral sequence.

We will rename several classes on the E_2-page of the Q_8-HFPSS for E_2 as follows. The advantage is that these names are compatible with the tmf computation and the Hurewicz images in $E_2^{hQ_8}$ (see [Bau08], also compare to [Isa18]). For example, we rename the class kD^3 by g, which is compatible with [Bau08] and suggests that this class detects the Hurewicz image of $\bar{\kappa}$ (see 4.9).

Table 7. Distinguished classes

Classes	Bauer's notation	Bigrading
Dxh_1	c	$(8,2)$
D^2x^2	d	$(14,2)$
kD^3	g	$(20,4)$

When we talk about the restriction map from Q_8 to C_4, the subgroup C_4 usually indicates the subgroup $C_4\langle i\rangle$ generated by i if there is no further specification. Some of the arguments in the proofs of this section are easier to see when accompanied by charts in §6.

4.1. General properties of the Q_8-HFPSS for E_{4k+2}.

It is a result of Shi–Wang–Xu, using the Slice Differential Theorem and the norm functor of Hill–Hopkins–Ravenel [HHR16], that the homotopy fixed point spectrum $E_{4k+2}^{hQ_8}$ is 2^{4k+6}-periodic.

The periodicity of $E_2^{hQ_8}$ is known by computation to be 64 classically. Here we give a proof that $E_2^{hQ_8}$ is 64-periodic before compute it using Q_8-HFPSS.

Proposition 4.1. *The homotopy groups of the spectrum $E_2^{hQ_8}$ is 64-periodic and the periodicity class can be given by the class D^8.*

Proof. The product

$$N_{C_4}^{Q_8}(\bar{\mathfrak{d}}_1)^8 N_{C_4\langle i\rangle}^{Q_8}(u_{4\sigma})^2 N_{C_4\langle i\rangle}^{Q_8}(u_{8\lambda}) N_{C_4\langle j\rangle}^{Q_8}(u_{4\sigma})^4 N_{C_4\langle k\rangle}^{Q_8}(u_{4\sigma})^4$$

gives the 64 periodicity of $E_2^{hQ_8}$. This product is in bigrading $(64,0)$ and is invertible. On the other hand, the generator D^8 of $\pi_{64}(E_2)$ is Q_8-invariant and invertible. Therefore, this periodicity class is D^8 up to a unit. □

From now on we can simply view D^8 as a periodicity class of $E_2^{hQ_8}$. In the following property, we show that the Q_8-HFPSS for E_2 can split into three parts such that there are no differentials across different parts.

Note that the universal space EG_{24} can be viewed as a model for EQ_8. The transfer and the restriction of the genuine spectrum $F(EG_{24}, E_2)$ give a sequence $E_2^{hG_{24}} \xrightarrow{\text{res}} E_2^{hQ_8} \xrightarrow{\text{tr}} E_2^{hG_{24}}$, which is compatible with the filtration of the HFPSS.

Proposition 4.2. *The composition*

$$E_2^{hG_{24}} \xrightarrow{res} E_2^{hQ_8} \xrightarrow{tr} E_2^{hG_{24}}$$

is an equivalence. In particular, the G_{24}-HFPSS for E_2 splits as a summand of the Q_8-HFPSS for E_2.

Proof. The composition $\text{tr} \circ \text{res}$ is multiplication by $|G_{24}|/|Q_8| = 3$. All spectra are 2-local and 3 is coprime to 2 so this composition is an equivalence. □

We identify the E_2-page of Q_8-HFPSS(E_2) as a free module over the E_2-page of G_{24}-HFPSS(E_2) generated by $\{1, D, D^2\}$.

Corollary 4.3. *Let a, b be two classes on the E_2-page of G_{24}-HFPSS(E_2). View a, b as classes in Q_8-HFPSS(E_2) and consider classes aD^{k_a}, bD^{k_b} where $k_a, k_b \in \{0, 1, 2\}$. Then there is a differential $d_r aD^{k_a} = bD^{k_b}$ in the Q_8-HFPSS(E_2) iff there is a differential $d_r a = b$ in the G_{24}-HFPSS(E_2) and $k_a = k_b$.*

Proof. When $k_a = 0$, this follows from Proposition 4.2. For $k_a = 1$, note that the Q_8-HFPSS for E_2 is D^8-periodic by Proposition 4.1. The two differentials

$$(1)\ d_r aD = bD^{k_b} \text{ and } (2)\ d_r(aD^9) = bD^{k_b+8}$$

imply each other. We observe that the class aD^9 is a class in G_{24}-HFPSS(E_2). Then by the case $k_a = 0$, the differential (2) happens in G_{24}-HFPSS(E_2). This implies the desired result. The case $k_a = 2$ is similar. □

As a consequence, the computation of the Q_8-HFPSS for E_2 splits into three copies with the same differential patterns and there are no differentials across different copies. In particular, the G_{24}-HFPSS for E_2 is 192-periodic.

Remark 4.4. A similar statement holds for general heights $4k + 2$. A maximal finite subgroup in $\mathbb{S}_{4k+2}$ is $Q_8 \rtimes C_{3(2^{2k+1}-1)} \cong G_{24} \times C_{(2^{2k+1}-1)}$ [Hew95][Buj12, Section 4.3]. The computation of the Q_8-HFPSS for E_{4k+2} also splits into copies of the computation of the $Q_8 \rtimes C_{3(2^{2k+1}-1)}$-HFPSS for E_{4k+2}.

Remark 4.5. The G_{24}-HFPSS for E_2 computation is essentially the same as the 2-local tmf computation [Bau08]. However, our computation only relies on the C_4 computation of E_2 and hence is an independent computation of the classical tmf computations.

In Theorem 4.8, we will improve the horizontal vanishing line result of the Q_8-HFPSS for E_{4k+2} in Theorem 2.10. In the case of the Q_8-HFPSS for E_2, the improved vanishing line of filtration 23 turns out to be sharp by computation. We start with the following fact.

Proposition 4.6. *Let $H\underline{\mathbb{Z}}$ be the Eilenberg-Mac Lane spectrum with trivial Q_8-action. Then on the E_2-page of Q_8-HFPSS($H\underline{\mathbb{Z}}$), the product $a_{\sigma_i} a_{\sigma_j} a_{\sigma_k}$ is trivial.*

Proof. We prove a stronger statement that the whole group $H^3(Q_8, \pi_{3-\sigma_i-\sigma_j-\sigma_k}(H\underline{\mathbb{Z}}))$, where the class $a_{\sigma_i} a_{\sigma_j} a_{\sigma_k}$ lies in, is trivial. According to Proposition A.7, the group $H^3(Q_8, \mathbb{Z})$ is trivial. We observe that the homotopy group $\pi_{3-\sigma_i-\sigma_j-\sigma_k}(H\underline{\mathbb{Z}})$ as a Q_8-module is a copy of $\mathbb{Z}$ with trivial Q_8-action ($\sigma_i \otimes \sigma_j \otimes \sigma_k$ is a trivial Q_8-representation). Then we have

$$H^0(Q_8, \pi_{3-\sigma_i-\sigma_j-\sigma_k}(H\underline{\mathbb{Z}})) = (\pi_{3-\sigma_i-\sigma_j-\sigma_k}(H\underline{\mathbb{Z}}))^{Q_8} \cong \mathbb{Z}.$$

Similarly we also have

$$H^0(Q_8, \pi_{-3+\sigma_i+\sigma_i+\sigma_k}(H\underline{\mathbb{Z}})) = (\pi_{-3+\sigma_i+\sigma_i+\sigma_k}(H\underline{\mathbb{Z}}))^{Q_8} \cong \mathbb{Z}.$$

Let u be a generator of $H^0(Q_8, \pi_{3-\sigma_i-\sigma_j-\sigma_k}(H\underline{\mathbb{Z}}))$. Then the class u is invertible on the E_2-page of HFPSS for $H\underline{\mathbb{Z}}$ by the following paring

$$\pi_{3-\sigma_i-\sigma_j-\sigma_k}(H\underline{\mathbb{Z}}) \otimes \pi_{-3+\sigma_i+\sigma_i+\sigma_k}(H\underline{\mathbb{Z}}) \cong \mathbb{Z}.$$

Therefore, the class u induces an isomorphism $H^3(Q_8, \pi_{3-\sigma_i-\sigma_j-\sigma_k}(H\underline{\mathbb{Z}})) \simeq H^3(Q_8, \mathbb{Z})$, the latter of which is trivial.

□

Remark 4.7. We thanked Guillou for confirming and explaining Proposition 4.6. This proposition also follows from Guillou and Slone's computation of quaternionic Eilenberg–Mac Lane spectra [GS22].

Theorem 4.8. *The $RO(Q_8)$-graded Q_8-TateSS for E_{4k+2} vanishes after $E_{2^{4k+5}-9}$-page. And the $RO(Q_8)$-graded Q_8-HFPSS for E_{4k+2} admits a strong vanishing line of filtration $2^{4k+5}-9$.*

Proof. Denote the height $4k+2$ by h. We briefly review the proof of the vanishing line of filtration $2^{h+3}-7$ in [DLS22, Theorem 6.1] and explain the filtration improvement by 2. By Theorem 2.8, in the Q_8-TateSS(E_h), there is a predicted differential

$$d_{2^{h+3}-7}(N_{C_2}^{Q_8}(\bar{v}_h^{-1}u_{2\sigma_2}^{2^h-1}a_{\sigma_2}^{1-2^{h+1}})a_{\bar{\rho}}) = 1. \tag{4.1}$$

By naturality, the unit 1 has to be hit by a differential d_r with $r \leq 2^{h+3}-7$. Note that since 1 is hit, the spectral sequence vanishes at E_r-page.

The ring map $\mathbb{Z} \to \pi_*(E_h)$ induces a map between E_2-pages of the Q_8-HFPSS for $H\underline{\mathbb{Z}}$ and E_h. Then the naturality forces the source of Eq. (4.1) is trivial since $a_{\bar{\rho}} = a_{\sigma_i}a_{\sigma_j}a_{\sigma_k} = 0$ by Proposition 4.6. For degree reasons, we conclude $r \leq 2^{h+3}-9$. So every class in the Q_8-TateSS(E_h) will disappear on or before the $E_{2^{h+3}-9}$-page. Finally by Lemma 2.6 there is a strong vanishing line of filtration $2^{4k+5}-9$. □

Lemma 4.9. *In the Q_8-HFPSS for E_2, the class h_1, h_2, g are permanent cycles.*

Proof. Consider the following maps

$$S^0 \xrightarrow{unit} E_2^{hQ_8} \xrightarrow{res} E_2^{hC_2}.$$

By [LSWX19, Theorem 1.8], the class $\bar{\kappa} \in \pi_{20}S^0$ maps to a non-trivial class in $E_2^{hC_2}$ in filtration 4 in the C_2-HFPSS for E_2. Thus the image of $\bar{\kappa}$ in $\pi_*(E_2^{hQ_8})$ is non-trivial. For degree reasons, it is detected by the class g in Q_8-HFPSS(E_2). The proofs for h_1, h_2 are similar. □

We only use the Hurewicz image of $E_2^{hC_2}$ as the input. This has been systematically studied in [LSWX19]. Our method does not assume the knowledge of the Hurewicz image of $E_2^{hC_4}$.

4.2. Differentials in the integer-graded pages.

We suggest readers refer to the charts while reading the proofs in this section.

All statements about differentials in this subsection are differentials in integer-graded Q_8-HFPSS(E_2) if there is no specification.

Proposition 4.10. *The class v_1^6 in $(12, 0)$ supports a d_3-differential*

$$d_3(v_1^6) = v_1^4h_1^3.$$

Proof. By construction, we have $\mathrm{res}^{Q_8}_{C_4}(v_1^6) = T_2^3, \mathrm{res}^{Q_8}_{C_4}(h_1) = \eta$. In C_4-HFPSS(E_2), [BBHS20, Proposition 5.21] implies that we have

$$d_3(T_2^3) = T_2^2\eta^3.$$

The result follows by naturality.

□

Corollary 4.11. *The class $v_1^2h_1$ at $(5,1)$ supports a d_3-differential*

$$d_3(v_1^2h_1) = h_1^4.$$

Proof. By Proposition 4.10, we have $d_3(v_1^6h_1) = v_1^4h_1^4$. Note that v_1^4 is a 3-cycle. This forces the desired d_3-differential. □

Proposition 4.10 produces a family of d_3-differentials by the Leibniz rule:

$$d_3(D^mg^sv_1^{4l+2}h_1^n) = D^mg^sv_1^{4l}h_1^{n+3}, \text{ and } d_3(D^mg^sv_1^2h_1^n) = D^mg^sh_1^{n+3}$$

for any $(m,s,l,n) \in \mathbb{Z}\times\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 1}\times\mathbb{Z}_{\geq 1}$.

For degree reasons (and the following proposition), these are all the non-trivial d_3-differentials.

Proposition 4.12. *The following classes survive to the E_∞-page.*

$$2D^mv_1^{4l+2}, D^mv_1^{4l}, D^mv_1^{4l}h_1, D^mv_1^{4l}h_1^2\ ,\ (m,l)\in\mathbb{Z}\times\mathbb{Z}_{\geq 1}.$$

Proof. The classes $D^mv_1^{4l}, D^mv_1^{4l}h_1, D^mv_1^{4l}h_1^2$ cannot be hit by degree reasons. They are permanent cycles by Lemma 2.6 and the Q_8-TateSS(E_2) d_3-differentials

$$d_3(D^{m+3}g^{-1}v_1^{4l-2}h_1^{n+4}) = D^mv_1^{4l}h_1^{n+3},\ m,l,n\in\mathbb{Z},\ l\neq 0.$$

As for the classes $2D^mv_1^{4l+2}$, we consider the additive norm map

$$H_0(Q_8,(E_2)_*) \xrightarrow{N} H^0(Q_8,(E_2)_*)$$

where $N(x) = \sum\limits_{g\in Q_8} g(x)$. By the Q_8-action formulas (Eq. (2.3)), we have

$$\begin{aligned}
&N(v_1^{2l+1}(u^{-1})^{2l+1}) = \sum_{g\in Q_8} g(v_1^{2l+1}(u^{-1})^{2l+1})\\
&= 2v_1^{2l+1}(u^{-1})^{2l+1} + 2\left(\frac{v_1+2u^{-1}}{\zeta^2-\zeta}\right)^{2l+1}\left(\frac{v_1-u^{-1}}{\zeta^2-\zeta}\right)^{2l+1}\\
&+2\left(\frac{v_1+2\zeta^2u^{-1}}{\zeta^2-\zeta}\right)^{2l+1}\left(\frac{\zeta v_1-u^{-1}}{\zeta^2-\zeta}\right)^{2l+1} + 2\left(\frac{v_1+2\zeta u^{-1}}{\zeta^2-\zeta}\right)^{2l+1}\left(\frac{\zeta^2v_1-u^{-1}}{\zeta^2-\zeta}\right)^{2l+1}.
\end{aligned}$$

The leading term of the above formula on the E_∞-page of the 2-BSS for $H^*(Q_8,\mathbb{W}[v_1,u^{-1}])$ is $2v_1^{4l+2}$ for $l\geq 1$. Then we have

$$N(D^mv_1^{2l+1}(u^{-1})^{2l+1}) = D^m\sum_{g\in Q_8} g(v_1^{2l+1}(u^{-1})^{2l+1}) = 2D^mv_1^{4l+2}$$

since D is Q_8-invariant. As the additive norm map is the d_1-differential on E_1-page of the Q_8-TateSS for E_2, we have the classes $2D^mv_1^{4l+2}$ are permanent cycles who survive to the E_∞-page by Lemma 2.6. □

Remark 4.13. All the classes supporting or receiving non-trivial d_3-differentials and all classes in Proposition 4.12 are sometimes referred to as the *bo*-pattern. They match the pattern of (many copies of) π_*KO, the homotopy groups of the real K-theory. See [BG18, Definition 2.1] for more details.

The following result is the first example of the strong vanishing line method (Theorem 4.8). The method gives differentials of three lengths (including the longest d_{23}-differential) all at once (see Fig. 1).

Proposition 4.14. *There are differentials*

(1) $d_5(D^{-13}g^5dh_2) = 4D^{-16}g^7$;
(2) $d_{13}(D^{-7}g^3ch_1) = 2D^{-16}g^7$;
(3) $d_{23}(D^{-1}gh_1) = D^{-16}g^7$.

Proof. We suggest readers comparing the arguments with Fig. 1. The class $D^{-16}g^7$ is a permanent cycle in filtration $28 \geqslant 23$. By Theorem 4.8, the classes $D^{-16}g^7$, $2D^{-16}g^7$ and $4D^{-16}g^7$ must receive differentials. According to Corollary 4.3, Q_8-HFPSS(E_2) splits into three parts. On the E_2-page, these three parts are modules over the E_2-page of G_{24}-HFPSS(E_2) and all differentials do not cross different copies. In Fig. 1, we highlight the relevant copy. By inspection, we obtain the desired d_5, d_{13} and d_{23}-differentials. □

Corollary 4.15. *The class* D *at* $(8, 0)$ *supports a* d_5*-differential*

$$d_5(D) = D^{-2}gh_2.$$

Proof. Note that D^8 is an invertible permanent cycle (Proposition 4.1), and g^5 is a permanent cycle (Lemma 4.9). By Proposition 4.14(1) and the Leibniz rule, there is a d_5-differential

$$d_5(D^3dh_2) = 4g^2. \tag{4.2}$$

The relation $dh_2^2 = 4g$ (see Remark 3.3 under 2BSS names) forces the following d_5-differential

$$d_5(D^3d) = gdh_2. \tag{4.3}$$

With Eq. (4.3), it suffices to show D^2d is a 5-cycle. In fact, the only possible d_5 target of D^2d supports a differential

$$d_5(D^{-1}gdh_2) = 4D^{-4}g^3.$$

by multiplying $D^{-4}gh_2$ with Eq. (4.3). Note that D^{-4} is a 5-cycle since D is a 3-cycle.

□

All the remaining d_5-differentials follow from the Leibniz rule. There are no more d_5-differentials by degree reasons and Corollary 4.3.

We also get a d_9-differential from the d_{13}-differential in Proposition 4.14(2).

Corollary 4.16. *The class* Dc *at* $(16, 2)$ *supports a* d_9*-differential*

$$d_9(Dc) = D^{-5}g^2dh_1.$$

Proof. We observe that in Q_8-HFPSS(E_2) there is an h_1-extension from Dc to Dch_1. We proof by contradiction. Suppose that Dc does not support the claimed d_9-differential. Then for degree reasons, Dc becomes a 13-cycle. However, this contradicts Proposition 4.14 since Dch_1 supports a non-trivial d_{13}-differential. □

Proposition 4.17. *The classes* $4D$ *and* $2D^2$ *at* $(16, 0)$ *support the following* d_7*-differentials*

(1) $d_7(4D) = D^{-2}gh_1^3$;
(2) $d_7(2D^2) = D^{-1}gh_1^3$.

Proof. By Corollary 4.15 and the hidden 2 extension from $2h_2$ to h_1^3 (see [Tod62]), $D^{-2}gh_1^3$ has to be hit by a differential. For degree reasons and Corollary 4.3, the only possible source is either $4D$. The second d_7-differential follows similarly from $d_5(D^2) = 2D^{-1}gh_2$. □

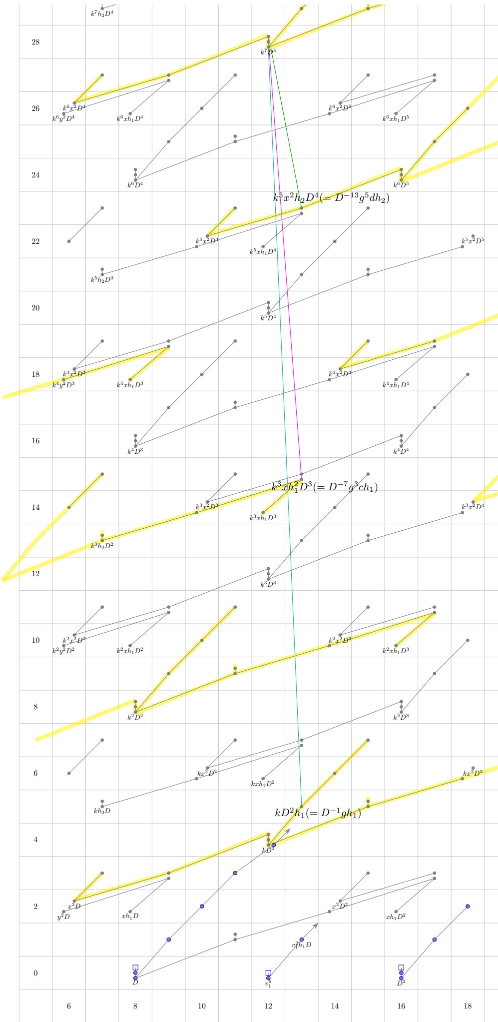

FIGURE 1. d_5, d_{13}, d_{23}-differentials

The d_7-differential on D^4 (which we prove in Proposition 4.28) turns out to be a hard one, as it does not follow from primary relations like the Leibniz rule or (hidden) extensions. We will first prove several d_9, d_{13}-differentials, and then the d_7-differential follows from the vanishing line method.

Proposition 4.18. *The class D^5ch_1 at $(49,3)$ and the class D^5c at $(48,2)$ supports the following differentials.*

(1) $d_{13}(D^5ch_1) = 2D^{-4}g^4$;
(2) $d_9(D^5c) = D^{-1}g^2dh_1$.

Proof. By a similar argument as in Corollary 4.16, it is enough to show the (1). We first observe that the class $2D^{-4}g^4$ is in the image of the transfer map from C_4-HFPSS(E_2) since

$$\mathrm{tr} \circ \mathrm{res}(D^{-4}g^4) = [Q_8 : C_4]D^{-4}g^4 = 2D^{-4}g^4.$$

According to [BBHS20, Proposition 5.28], the class $\mathrm{res}(D^{-4}g^4)$ receives a d_{13}-differential in C_4-HFPSS(E_2). The naturality forces that $2D^{-4}g^4$ dies on or before the the E_{13}-page in Q_8-HFPSS(E_2). The only possibility is the desired d_{13}-differential by Corollary 4.3 and degree reasons. □

Remark 4.19. Since C_4-HFPSS(E_2) is 32-periodic with the periodicity class $\Delta_1^4 = \bar{\mathfrak{d}}_1^8 u_{8\lambda}u_{8\sigma}$ [HHR17][BBHS20], the same argument in the proof of Proposition 4.18 gives an alternative proof of Proposition 4.14(2) and Corollary 4.16.

Lemma 4.20. *The class D^3h_1 is a permanent cycle.*

Proof. By Corollary 4.3, it suffices to show D^3h_1 is a permanent cycle in G_{24}-HFPSS(E_2). For degree reasons, D^3h_1 can only possibly hit $D^{-3}gc$ or $2D^{-12}g^6$ in G_{24}-HFPSS(E_2). Because D^{-8}, g are permanent cycles, Proposition 4.18 implies

$$d_{13}(D^{-3}g^2ch_1) = 2D^{-12}g^6 \quad \text{and} \quad d_9(D^{-3}gc) = D^{-9}g^3dh_1.$$

Therefore, the class D^3h_1 has to be a permanent cycle. □

Remark 4.21. It turns out that D^3h_1 is hit by a d_{23}-differential in the Tate spectral sequence by Corollary 4.22.

Corollary 4.22. *There are non-trivial d_{23}-differentials*

(1) $d_{23}(D^2h_1^2) = D^{-13}g^6h_1$;
(2) $d_{23}(D^5h_1^3) = D^{-10}g^6h_1^2$.

Proof. The claimed d_{23}-differentials follow from Proposition 4.14(3) and Lemma 4.20 □

We write $m \doteq n$ if $m = ln$ for some $l \in \mathbb{W}(F_4)^\times$.

Lemma 4.23. *There is a hidden 2 extension from $D^6h_2^2$ to g^2d.*

Proof. According to Lemma 2.18, there is a hidden 2 extension in stem 54 from $\Delta_1^4\bar{\mathfrak{d}}_1^6u_{6\lambda}u_{4\sigma}a_{2\sigma}$ to $\Delta_1^4\bar{\mathfrak{d}}_1^8u_{4\lambda}u_{6\sigma}a_{4\lambda}a_{2\sigma}$ in the C_4-HFPSS(E_2) since it is Δ_1^4-periodic. Note that the restriction of D to the E_2-page of the C_4-HFPSS(E_2) is invertible then it equals Δ_1 up to a unit, i.e., $\mathrm{res}_{C_4}^{Q_8}(D) \doteq \Delta_1$. In Appendix A we show that the restriction of the classes h_2, d and g are non-trivial. Then in stem 54 of Q_8-HFPSS(E_2), we have the following two restrictions

$$\mathrm{res}_{C_4}^{Q_8}(D^6h_2^2) \doteq \Delta_1^4\bar{\mathfrak{d}}_1^6u_{6\lambda}u_{4\sigma}a_{2\sigma}$$
$$\mathrm{res}_{C_4}^{Q_8}(g^2d) \doteq \Delta_1^4\bar{\mathfrak{d}}_1^8u_{4\lambda}u_{6\sigma}a_{4\lambda}a_{2\sigma}.$$

Note that in G_{24}-HFPSS(E_2), there are no other classes between these two filtrations. Then the naturality forces a hidden 2 extension from $D^6h_2^2$ to g^2d in G_{24}-HFPSS(E_2). This hidden 2 extension also happens in Q_8-HFPSS(E_2) by Corollary 4.3. □

As the C_4-HFPSS for E_2 is 32-periodic, a similar proof gives the following hidden 2 extension in stem 22 in the Q_8-HFPSS for E_2.

Corollary 4.24. *There is a hidden 2 extension from $D^2h_2^2$ to $D^{-4}g^2d$.*

Proposition 4.25. *The classes $2Dh_2$ at $(11,1)$ and $2D^5h_2$ at $(43,1)$ support d_{13}-differentials*

(1) $d_{13}(2Dh_2) = D^{-8}g^3d$;
(2) $d_{13}(2D^5h_2) = D^{-4}g^3d$.

Proof. (1) By Lemma 4.23 and the E_∞-page class g, there is a hidden 2 extension from $D^{-2}gh_2^2$ to $D^{-8}g^3d$ in stem 10 of the Q_8-HFPSS for E_2. By Corollary 4.15, we have

$$d_5(Dh_2) = D^{-2}gh_2^2.$$

Then the hidden 2 extension forces the desired differential.
(2) It follows similarly from the hidden 2 extension from $D^2g^2h_2^2$ to $D^{-4}g^3d$ by Corollary 4.24. □

Remark 4.26. In Bauer's computation for tmf [Bau08], the hidden 2 extension in Lemma 4.23 is proved using four-fold Toda brackets. In our approach, the hidden 2 extension follows from the restriction and the C_4-HFPSS hidden 2 extension, which again is forced by the exotic restrictions and transfers in Lemma 2.18.

Lemma 4.27. *The class Dh_1^3 is a permanent cycle.*

Proof. The class Dh_1^3 is a 5-cycle. By Corollary 4.3 and degree reasons, Dh_1^3 can only possibly hit $D^{-8}g^3d$ and $D^{-14}g^6h_1^2$, of which the former is hit by a d_{13}-differential by Proposition 4.25 and the latter supports a d_{23}-differential by Corollary 4.22 ($D^{-14}g^6h_1^2 = D^{-16}g^6D^2h_1^2$). The result thereby follows. □

Proposition 4.28. *The class D^4 at $(32, 0)$ supports a d_7-differential*

$$d_7(D^4) = Dgh_1^3.$$

Proof. Note that g and D^{-8} are permanent cycles. Then by Lemma 4.27 the class $D^{-15}h_1^3g^6$ at $(3, 27)$ is also a permanent cycle. This class has to be hit by a differential via the vanishing line method (Theorem 4.8). By Corollary 4.3, the potential source is either $D^{-3}gc$ or $D^{-12}g^5$. The former supports a d_9 by Proposition 4.18. Therefore, the only possibility is the d_7-differential

$$d_7(D^{-12}g^5) = D^{-15}h_1^3g^6.$$

Since $D^{-8}g^5$ is a permanent cycle, the result follows. □

All d_7-differentials follow from Proposition 4.17, Proposition 4.28 and the Leibniz rule.

Before proving the next two d_9-differentials in Corollary 4.32, we need to first prove a permanent cycle in Lemma 4.29 and two d_{11}-differentials in Proposition 4.30.

Lemma 4.29. *The class D^3dh_1 is a permanent cycle.*

Proof. By Corollary 4.16 in the Q_8-TateSS for E_2, we have a d_9-differential

$$d_9(D^9g^{-2}c) = D^3dh_1.$$

Then D^3dh_1 is a permanent cycle in the Q_8-TateSS. By Lemma 2.6 it is also a permanent cycle in Q_8-HFPSS(E_2). □

Proposition 4.30. *The classes D^2d at $(30, 2)$ and D^6d at $(62, 2)$ support d_{11}-differentials*

(1) $d_{11}(D^2d) = D^{-4}g^3h_1$;
(2) $d_{11}(D^6d) = g^3h_1$.

Proof. According to Proposition A.7, the restriction of the class d from Q_8-HFPSS(E_2) to C_4-HFPSS(E_2) is non-trivial, and supports a non-trivial d_{13}-differential by [BBHS20, Proposition 5.28]. This implies the class D^2d supports a non-trivial differential with a length at most 13. The desired differential in (1) follows by degree reasons. The proof for (2) is similar since C_4-HFPSS(E_2) is 32-periodic. □

Corollary 4.31. *The classes D^2dh_1 at $(31, 3)$ and D^6dh_1 at $(63, 3)$ support d_{11}-differentials*

(1) $d_{11}(D^2dh_1) = D^{-4}g^3h_1^2$;
(2) $d_{11}(D^6dh_1) = g^3h_1^2$.

Corollary 4.32. *The classes Dh_1 at $(9,1)$ and D^5h_1 at $(41,1)$ support d_9-differentials*

(1) $d_9(Dh_1) = D^{-5}g^2c$;
(2) $d_9(D^5h_1) = D^{-1}g^2c$.

Proof. By Corollary 4.3 and degree reasons, the class Dh_1 either supports a non-trivial d_9-differential or is an 11-cycle. We show it is the first case.

If Dh_1 were a 11-cycle then by Proposition 4.30 and the Leibniz rule, there would be a d_{11}-differential

$$d_{11}(D^3dh_1) = D^{-3}g^3h_1^2.$$

This contradicts Lemma 4.29. Therefore, we have the desired d_9-differential in (1). The proof for (2) is similar. □

Proposition 4.33. *The class $D^{-1}h_1$ is a 13-cycle.*

Proof. Since D^8 is the periodic class, it suffices to prove that D^7h_1 is a 13-cycle. The D^7h_1 is a 7-cycle from our computation of E_9-page. It cannot support a d_9-differential since the possible target g^2Dc supports a differential by Corollary 4.16. Then for degree reasons, D^7h_1 is a 13-cycle. □

Corollary 4.34. *The classes D^2c at $(24,2)$ and D^6c at $(56,2)$ support d_9-differentials*

(1) $d_9(D^2c) = D^{-4}g^2dh_1$;
(2) $d_9(D^6c) = g^2dh_1$.

Proof. Suppose D^2c doesn't support a non-trivial d_9-differential. Then for degree reasons, it is a 13-cycle. However, since $D^{-1}h_1$ is also a 13-cycle, the Leibniz rule show that Dh_1c is also a 13-cycle. This contradicts Proposition 4.14 and proves the d_9-differential in (1). The d_9-differential in (2) follows similarly by Proposition 4.18. □

Corollary 4.35. *The classes Ddh_1 at $(23,3)$ and D^5dh_1 at $(55,3)$ support d_{11}-differentials*

(1) $d_{11}(Ddh_1) = D^{-5}g^3h_1^2$;
(2) $d_{11}(D^5dh_1) = D^{-1}g^3h_1^2$.

Proof. According to Proposition 4.33, the class $D^{-1}h_1$ is a 13-cycle. Then these two d_{11}-differentials follow by Proposition 4.30 and the Leibniz rule. □

Lemma 4.36. *The class d is a permanent cycle.*

Proof. Proposition 4.25 shows d is hit by a d_{13}-differential from $2D^9g^{-3}h_2$ in Q_8-TateSS(E_2). By Lemma 2.6 d is a permanent cycle. □

Remark 4.37. In fact, the class d is in the image of the Hurewicz map $S^0 \to E_2^{hQ_8}$. This follows from the Hurewicz image of $E_2^{hC_4}$ [HSWX18, Figure 12] (see Proposition A.8).

Proposition 4.38. *The classes D^2h_1 at $(17,1)$ and D^6h_1 at $(49,1)$ support d_9-differentials*

(1) $d_9(D^2h_1) = g^2D^{-4}c$;
(2) $d_9(D^6h_1) = g^2c$.

Proof. We prove by contradiction. Assume D^2h_1 does not support the desired differential. Then it is a 11-cycle by degree reasons. The Leibniz rule forces the class Dh_1 to support a non-trivial d_{11}-differential but this contradicts Lemma 4.36. The proof of (2) is similar. □

Table 8 lists the differentials we have computed so far. They generate differentials via the Leibniz rule. By inspection, these are all non-trivial differentials since the remaining classes are permanent cycles by Method 2.7.

4.3. **Extension problem.** Now we solve all the 2-extensions on the E_∞-page.

Theorem 4.39. *All the hidden 2 extensions in the integer-graded G_{24}-HFPSS(E_2) are displayed in Fig. 10 by gray vertical lines.*

Proof. Since the G_{24}-HFPSS for E_2 is 192-periodic, it suffices to consider the stem range from 0 to 192. We divide these 2 extensions into three types by their proofs. The first type follows from the fact that in homotopy groups of spheres $4\nu = \eta^3$ and h_1 detects η, h_2 detects ν (Lemma 4.9). This type of hidden 2 extensions happens in stem $3, 27, 51, 99, 123$ and 147 in the period from 0 to 192.

The second type consists of the 2 extensions in stem 54 and 150. The proof of the first is in Lemma 4.23, and proof of the second is similar using the 32-periodicity of C_4-HFPSS(E_2) and Lemma 2.18.

The third type consists of three hidden 2 extensions in the first period. The first one is in stem 110 from $D^{12}d$ to $D^6g^3h_1^2$. The other two in stem 130 and 150 (from filtration 10 to 22) follow from the first one by multiplying g and g^2 respectively.

In G_{24}-TateSS(E_2) we have the following two differentials by Proposition 4.25 and Corollary 4.22.

$$d_{13}(2D^{21}g^{-3}h_2) = D^{12}d,$$
$$d_{23}(D^{21}g^{-3}h_1^3) = D^6g^3h_1^2.$$

Now consider the cofibration

$$(E_2)_{hG_{24}} \to E_2^{hG_{24}} \to E_2^{tG_{24}}$$

In the negative filtrations in G_{24}-TateSS(E_2), there is a hidden 2 extension from $2D^{21}g^{-3}h_2$ to $D^{21}g^{-3}h_1^3$, then this hidden 2 extension under the additive norm map gives a 2 extension relation in $\pi_*E_2^{hG_{24}}$ from an element detected by $D^{12}d$ to some element detected by $D^6g^3h_1^2$. This forces a hidden 2 extension from $D^{12}d$ to $D^6g^3h_1^2$ in G_{24}-HFPSS(E_2).

We claim there are no further 2 extensions in G_{24}-HFPSS(E_2). By degree reasons, the other possible hidden 2 extensions either have sources that are h_1 divisible or have targets that support h_1 extensions. Therefore, the hidden 2 extensions cannot happen in these cases. □

Corollary 4.40. *All the hidden 2 extensions in the integer-graded Q_8-HFPSS(E_2) are displayed in Fig. 9 by gray vertical lines.*

Proof. This follows from Theorem 4.39 and Proposition 4.2. □

Our result of 2 extensions via the equivariant and the Tate methods matches the tmf computation in [Bau08]. In [Bau08], because his arguments for proving differentials rely on (hidden) η and ν extensions, almost all these hidden extension are also computed (there are another ν extension from $D^{15}h_1^2$ at $(122, 2)$ and its $\bar{\kappa}$ multiples [Isa09, Lemma 5.3]). Here our new methods only use hidden 2 extensions and the h_1, h_2 multiplications on the E_2-page. Therefore, we do not need to work out hidden η and ν extensions and in our figures we only draw h_1, h_2 multiplications.

4.4. **Differentials: alternative methods.** In this subsection, we revisit several differentials in the integer-graded part via different approaches.

Proposition 4.41. *The class D at $(8, 0)$ supports a d_5-differential*

$$d_5(D) = D^{-2}gh_2.$$

Proof. The restriction of D to the C_4-HFPSS for E_2 is Δ_1, which supports a non-trivial d_5-differential according to [BBHS20, Proposition 5.24]. By naturality, D must support a non-trivial differential with length ≤ 5. Then by Corollary 4.3 and degree reasons, it has to be $d_5(D) = D^{-2}gh_2$. □

Moreover, given all d_5, d_7-differentials, then the vanishing line forces the d_{11}-differential in Proposition 4.30.

Proposition 4.42. *The class D^6d at $(62, 2)$ supports a d_{11}-differential*

$$d_{11}(D^6d) = g^3h_1.$$

Proof. It is enough to prove the d_{11}-differential

$$d_{11}(D^6g^5dh_1) = g^8h_1^2$$

since g is invertible in the Q_8-TateSS for E_2. The target $g^8h_1^2$ is a permanent cycle in filtration $34 \geqslant 23$. By Theorem 4.8 and Theorem 2.9 it has to be hit by a differential. Since $D^6g^5dh_1$ is a 7-cycle, the only possibility is the desired d_{11}-differential. □

We here present another proof of the d_9-differential in Proposition 4.38 which combines the partial calculations in $(* - \sigma_i)$-gradings by the norm method (see Proposition 5.16).

Proposition 4.43. *The class D^2h_1 at $(17, 1)$ supports a d_9-differential*

$$d_9(D^2h_1) = D^{-4}g^2c.$$

Proof. Suppose the claimed d_9-differential doesn't happen, then D^2h_1 is a 9-cycle. According to Lemma 5.6, the class $\{x+y\}D^4u_{\sigma_i}$ is a 9-cycle. Then the Leibniz rule implies that $\{x+y\}D^6h_1u_{\sigma_i}$ is also a 9-cycle. This contradicts the fact that $\{x+y\}D^6h_1u_{\sigma_i}$ supports a non-trivial d_9-differential in Proposition 5.16. □

4.5. **Summary of differentials.** We summarize differentials in Table 8. All differentials follow from this list by the Leibniz rule.

TABLE 8. HPFSS differentials, integer page

toprule (s, f)	x	r	$d_r(x)$	Proof
$(12, 0)$	v_1^6	3	$v_1^4h_1^3$	Proposition 4.10 (restriction)
$(8, 0)$	D	5	$D^{-2}gh_2$	Corollary 4.15 (vanishing line) or Proposition 4.41 (restriction)
$(8, 0)$	$4D$	7	$D^{-2}gh_1^3$	Proposition 4.17 ($8\nu = \eta^3$)
$(16, 0)$	$2D^2$	7	$D^{-1}gh_1^3$	Proposition 4.17
$(32, 0)$	D^4	7	Dgh_1^3	Proposition 4.28 (vanishing line)
$(9, 1)$	Dh_1	9	$D^{-5}g^2c$	Corollary 4.32
$(41, 1)$	D^5h_1	9	$D^{-1}g^2c$	Corollary 4.32
$(16, 2)$	Dc	9	$D^{-5}g^2dh_1$	Corollary 4.16
$(48, 2)$	D^5c	9	$D^{-1}g^2dh_1$	Proposition 4.18
$(17, 1)$	D^2h_1	9	$D^{-4}g^2c$	Proposition 4.38
$(49, 1)$	D^6h_1	9	g^2c	Proposition 4.38
$(24, 2)$	D^2c	9	$D^{-4}g^2dh_1$	Corollary 4.34
$(56, 2)$	D^6c	9	g^2dh_1	Corollary 4.34
$(30, 2)$	D^2d	11	$D^{-4}g^3h_1$	Proposition 4.30 (restriction)
$(62, 2)$	D^6d	11	g^3h_1	Proposition 4.30 (restriction)

TABLE 8. HPFSS differentials, integer page

toprule (s, f)	x	r	$d_r(x)$	Proof
				or Proposition 4.42 (vanishing line)
$(23, 3)$	Ddh_1	11	$D^{-5}g^3h_1^2$	Corollary 4.35
$(55, 3)$	D^5dh_1	11	$D^{-1}g^3h_1^2$	Corollary 4.35
$(17, 3)$	Dch_1	13	$2D^{-8}g^4$	Proposition 4.14 (vanishing line)
$(49, 3)$	D^5ch_1	13	$2D^{-4}g^4$	Proposition 4.18 (transfer)
$(11, 1)$	$2Dh_2$	13	$D^{-8}g^3d$	Proposition 4.25 (hidden 2 extension)
$(43, 1)$	$2D^5h_2$	13	$D^{-4}g^3d$	Proposition 4.25
$(-7, 1)$	$D^{-1}h_1$	23	$D^{-16}g^6$	Proposition 4.14 (vanishing line)
$(18, 2)$	$D^2h_1^2$	23	$D^{-13}g^6h_1$	Corollary 4.22
$(43, 3)$	$D^5h_1^3$	23	$D^{-10}g^6h_1^2$	Corollary 4.22

5. The $(* - \sigma_i)$-graded computation

In this section, we compute the $(* - \sigma_i)$-graded Q_8-HFPSS for E_2. We adapt the following convention: a class at $(n - \sigma_i, m)$ will be denoted as in degree $(n - 1, m)$. Since the Q_8-representation σ_i cannot be lifted to G_{24}, in this section, we only consider the groups Q_8 and SD_{16}. We name classes by their names in the 2-BSS in Table 5, and also use 2-BSS names for the integer-graded classes as it makes the multiplication relation clearer.

Proposition 5.1. *The class $v_1^2 u_{\sigma_i}$ at $(4, 0)$ supports a d_3-differential*

$$d_3(v_1^2 u_{\sigma_i}) = h_1^3 u_{\sigma_i}.$$

Proof. We consider the restriction map from $(* - \sigma_i)$-graded Q_8-HFPSS(E_2) to the integer-graded C_4-HFPSS(E_2). Note that the C_4-invariant element $T_2 \in H^0(C_4, \pi_4 E_2)$ equals v_1^2 modulo 2. This implies $\mathrm{res}_{C_4}^{Q_8}(v_1^2 u_{\sigma_i}) = T_2$. Recall that in the C_4-HFPSS for E_2, the class T_2 supports a non-trivial d_3-differential ([BBHS20, Proposition 5.21]). The class $v_1^2 u_{\sigma_i}$ must support a non-trivial differential of length ≤ 3. By degree reasons, we have

$$d_3(v_1^2 u_{\sigma_i}) = h_1^3 u_{\sigma_i}.$$

□

Since the $(* - \sigma_i)$-graded part is a module over the integer-graded part, this d_3-differential implies a family of d_3-differentials as follows:

$$d_3(k^s D^m v_1^{4l+2} h_1^n u_{\sigma_i}) = k^s D^m v_1^{4l} h_1^{n+3} u_{\sigma_i}$$

where $k, m, l, n \in \mathbb{Z}$ and $l, n \geq 0$. By taking out these d_3-differentials, an argument similar to the proof in Proposition 4.12 shows that the following classes are permanent cycles

$$2D^m v_1^{4l-2}, D^m v_1^{4l}, D^m v_1^{4l} h_1, D^m v_1^{4l} h_1^2$$

where $l \geq 1$ and $m \geq 0$. All the classes above either support non-trivial d_3-differentials or are permanent cycles. Similar to the *bo*-pattern in the integer graded part, we do not need to consider this part in later computations of higher differentials.

However, this is not the only kind of d_3-differentials in $(* - \sigma_i)$-graded part. In order to derive the second kind of d_3-differentials, we first need to show the d_5-differential pattern and several other facts.

Lemma 5.2. *The class $\{x + y\} u_{\sigma_i}$ is a permanent cycle.*

Proof. For degree reasons, this class is a_{σ_i} on the E_2-page defined in Definition 2.2. By Proposition 2.3, this class is a permanent cycle. □

Corollary 5.3. *The class $\{x+y\}Du_{\sigma_i}$ at $(7,1)$ supports a d_5-differential*

$$d_5(\{x+y\}Du_{\sigma_i}) = k\{yh_2 + xh_1v_1\}Du_{\sigma_i}.$$

Proof. Since the $(*-\sigma_i)$-graded part is a module over the integer-graded part, the claimed differential follows from Lemma 5.2, Corollary 4.15 and the Leibniz rule. □

Corollary 5.3 generates the first kind of d_5-differentials via the Leibniz rule.

Lemma 5.4. *The class $\{x^2+y^2\}Du_{\sigma_i}$ is a permanent cycle.*

Proof. According to Proposition 4.38, there is a d_9-differential $d_9(D^6h_1) = k^2D^7xh_1$. Then the Leibniz rule implies that the class $k^2x^2h_1D^7u_{\sigma_i} = k^2D^7xh_1 \cdot \{x+y\}u_{\sigma_i}$ is hit by a differential of length ≤ 9. For degree reasons, it is hit by either a d_9-differential or a d_7-differential. In either case, the degree reasons force $k^2\{x^2+y^2\}D^7u_{\sigma_i}$ to be hit on or before the E_9-page. Then the class $\{x^2+y^2\}Du_{\sigma_i}$ must be a permanent cycle; otherwise the class $k^2\{x^2+y^2\}D^7u_{\sigma_i}$ would support a non-trivial differential since

$$k^2\{x^2+y^2\}D^7u_{\sigma_i} = \{x^2+y^2\}Du_{\sigma_i} \cdot k^2D^6$$

where $k^2D^6 = g^2$ is a permanent cycle that survives to E_∞-page in the integer-graded part. □

Corollary 5.5. *The class $\{x^2+y^2\}u_{\sigma_i}$ at $(-2,2)$ supports a d_5-differential*

$$d_5(\{x^2+y^2\}u_{\sigma_i}) = k\{x+y\}h_1^2u_{\sigma_i}.$$

All d_5-differentials in $(*-\sigma_i)$-graded part follows from Corollary 5.3 and Corollary 5.5 by the Leibniz rule.

Lemma 5.6. *The class $\{x+y\}D^4u_{\sigma_i}$ is a 11-cycle.*

Proof. According to [BBHS20, Remark 5.23], the class $u_{2\sigma}$ is a 5-cycle in the C_4-HFPSS for E_2. Therefore, Theorem 2.8 implies that $N_{C_4}^{Q_8}(u_{2\sigma})a_{\sigma_i}$ is a 9-cycle. Because the norm functor is symmetric monoidal [HHR16] and $u_{2\sigma}$ is an invertible class on the E_2-page of the C_4-HFPSS for E_2, the class $N_{C_4}^{Q_8}(u_{2\sigma})$ in the Q_8-HFPSS for E_2 is also invertible on the E_2-page. Hence $N_{C_4}^{Q_8}(u_{2\sigma}) \cdot a_{\sigma_i}$ is non-trivial on the E_2-page. By multiplying $N_{C_4}^{Q_8}(u_{2\sigma}) \cdot a_{\sigma_i}$ with the periodicity classes in Corollary 2.22, we get a non-trivial class at $(31,1)$. For degree reasons, this class must be $\{x+y\}D^4u_{\sigma_i}$ (up to a unit). This implies that $\{x+y\}D^4u_{\sigma_i}$ is also a 9-cycle. For degree reasons, $\{x+y\}D^4u_{\sigma_i}$ is a 11-cycle . □

Remark 5.7. We will show in Proposition 5.20 that the above class supports a non-trivial d_{13}-differential.

Corollary 5.8. *The classes $x^3u_{\sigma_i}$ at $(-3,3)$ and $x^3D^4u_{\sigma_i}$ at $(29,3)$ support d_{11}-differentials*

(1) $d_{11}(x^3u_{\sigma_i}) = k^3\{x+y\}Dh_1u_{\sigma_i}$;
(2) $d_{11}(x^3D^4u_{\sigma_i}) = k^3\{x+y\}D^5h_1u_{\sigma_i}$.

Proof. According to Proposition 4.30, there is a d_{11}-differential in the integer-gradings

$$d_{11}(x^2) = k^3Dh_1.$$

Note that $\{x+y\}u_{\sigma_i}$ and $\{x+y\}D^4u_{\sigma_i}$ are both 11-cycles. By the Leibniz rule, we have

$$d_{11}(x^3u_{\sigma_i}) = \{x+y\}u_{\sigma_i}d_{11}(x^2) = k^3\{x+y\}Dh_1u_{\sigma_i}.$$

The proof of the second d_{11}-differential is similar.

□

Proposition 5.9. *The class $\{h_1 + xv_1\}u_{\sigma_i}$ at $(1, 1)$ supports a d_3-differential*

$$d_3(\{h_1 + xv_1\}u_{\sigma_i}) = 2kv_1^2 u_{\sigma_i}.$$

Proof. We argue by contradiction. Suppose this differential doesn't happen. By Corollary 5.8 the class $k^3\{x + y\}D^5 h_1 u_{\sigma_i}$ must be hit by a differential of length ≤ 11. Because $kD^3 = g$ is a permanent cycle, the class $k^2\{x + y\}D^2 u_{\sigma_i}$ has to be hit by a differential of length ≤ 11 in the Q_8-TateSS for E_2. So is the class $k^2\{x + y\}D^2 h_1^2 u_{\sigma_i}$. For degree reasons, the class $k^2\{x + y\}D^2 h_1^2 u_{\sigma_i}$ has to be hit by the following d_9-differential

$$d_9(\{h_1^2 + xh_1v_1\}Du_{\sigma_i}) = k^2\{x + y\}D^2 h_1^2 u_{\sigma_i}. \tag{5.1}$$

By Lemma 2.6 this d_9-differential also happens in the Q_8-HFPSS for E_2 . This forces the following d_9-differential

$$d_9(\{h_1 + xv_1\}Du_{\sigma_i}) = k^2\{x + y\}D^2 h_1 u_{\sigma_i}.$$

Since $D^{-1}h_1$ is a 9-cycle according to Proposition 4.33, there is a non-trivial d_9-differential

$$d_9(\{h_1^2 + xh_1v_1\}u_{\sigma_i}) = k^2\{x + y\}Dh_1^2 u_{\sigma_i}.$$

By the assumption that $\{h_1 + xv_1\}u_{\sigma_i}$ is a 3-cycle and degree reasons, this class survives to the E_9-page. Then the above d_9-differential forces

$$d_9(\{h_1 + xv_1\}u_{\sigma_i}) = k^2\{x + y\}Dh_1 u_{\sigma_i}.$$

Recall that Dh_1 supports a non-trivial d_9-differential by Proposition 4.33. Then by the Leibniz rule, we have

$$d_9(\{h_1^2 + xh_1v_1\}Du_{\sigma_i}) = d_9(Dh_1) \cdot \{h_1 + xv_1\}u_{\sigma_i} + Dh_1 \cdot d_9(\{h_1 + xv_1\}u_{\sigma_i}) = 0.$$

This contradicts Eq. (5.1).

Therefore, the claimed d_3-differential must happen. □

Remark 5.10. Proposition 5.9 shows that $2kv_1^2 u_{\sigma_i}$ is hit by a d_3-differential. Recall that the class $kv_1^2 u_{\sigma_i}$ itself supports a non-trivial d_3-differential by Proposition 5.1.

By the above discussion and by inspection, all d_3-differentials in the $(* - \sigma_i)$-graded part follows from Proposition 5.1, Proposition 5.9 and the Leibniz rule.

Proposition 5.11. *The classes $\{h_1^2 + xh_1v_1\}Du_{\sigma_i}$ at $(10, 2)$ and $\{h_1^2 + xh_1v_1\}D^5 u_{\sigma_i}$ at $(42, 2)$ support d_9-differentials*

(1) $d_9(\{h_1^2 + xh_1v_1\}Du_{\sigma_i}) = k^2\{x + y\}h_1^2 D^2 u_{\sigma_i}$;
(2) $d_9(\{h_1^2 + xh_1v_1\}D^5 u_{\sigma_i}) = k^2\{x + y\}h_1^2 D^6 u_{\sigma_i}$.

Proof. Because $kD^3 = g$ is an invertible permanent cycle in Q_8-TateSS(E_2), the d_{11}-differential in Corollary 5.8

$$d_{11}(x^3 D^4 u_{\sigma_i}) = k^3\{x + y\}h_1 D^5 u_{\sigma_i}$$

implies that in the $(* - \sigma_i)$-graded Q_8-TateSS(E_2) we have

$$d_{11}(k^{-1}x^3 Du_{\sigma_i}) = (kD^3)^{-1} d_{11}(x^3 D^4 u_{\sigma_i}) = k^2\{x + y\}h_1 D^2 u_{\sigma_i}.$$

Since $k^2\{x + y\}h_1 D^2 u_{\sigma_i}$ is hit by a d_{11}-differential in Q_8-TateSS(E_2), its h_1 extension, $k^2\{x + y\}h_1^2 D^2 u_{\sigma_i}$, has to be hit on or before the E_{11}-page. For degree reasons, this class $k^2\{x+y\}D^2 h_1^2 u_{\sigma_i}$ must be hit by the claimed d_9-differential in Q_8-TateSS(E_2). By Lemma 2.6, the first claimed d_9-differential also happens in Q_8-HFPSS(E_2). The second d_9-differential follows similarly. □

We have the following d_9-differentials by the Leibniz rule and integer-graded d_9-differentials.

Proposition 5.12. *We have the following d_9-differentials*

(1) $d_9(\{x + y\}h_1 Du_{\sigma_i}) = k^2 x^2 h_1 D^2 u_{\sigma_i}$;

(2) $d_9(\{x+y\}h_1D^2u_{\sigma_i}) = k^2x^2h_1D^3u_{\sigma_i}$;
(3) $d_9(\{x+y\}h_1D^5u_{\sigma_i}) = k^2x^2h_1D^6u_{\sigma_i}$;
(4) $d_9(\{x+y\}h_1D^6u_{\sigma_i}) = k^2x^2h_1D^7u_{\sigma_i}$.

Proof. We prove the first differential, and the proofs of the rest three differentials are similar. According to Corollary 4.32, in the integer-graded Q_8-HFPSS(E_2) we have

$$d_9(Dh_1) = k^2xh_1.$$

Note that the class $\{x+y\}u_{\sigma_i}$ is a permanent cycle by Lemma 5.2. Then the Leibniz rule implies

$$d_9(\{x+y\}h_1Du_{\sigma_i}) = \{x+y\}u_{\sigma_i}d_9(Dh_1) = k^2x^2h_1D^2u_{\sigma_i}.$$

□

Corollary 5.13. *The classes $\{x+y\}D^2u_{\sigma_i}$ at $(15,1)$ and $\{x+y\}D^6u_{\sigma_i}$ at $(47,1)$ support d_9-differentials*
(1) $d_9(\{x+y\}D^2u_{\sigma_i}) = k^2\{x^2+y^2\}D^3u_{\sigma_i}$;
(2) $d_9(\{x+y\}D^6u_{\sigma_i}) = k^2\{x^2+y^2\}D^7u_{\sigma_i}$.

Proof. By Proposition 4.33, the class $D^{-1}h_1$ is a 9-cycle. These two d_9-differentials hold since otherwise the classes $(\{x+y\}Dh_1u_{\sigma_i}$ and $\{x+y\}D^5h_1u_{\sigma_i}$ would be 9-cycles by the Leibniz rule, which contradicts Proposition 5.12. □

In order to derive the last type of d_9-differential, we first need to show the following d_{17}-differential in the $(*-\sigma_i)$-graded part.

Proposition 5.14. *The class $\{h_1^2+xh_1v_1\}u_{\sigma_i}$ at $(2,2)$ supports a d_{17}-differential*

$$d_{17}(\{h_1^2+xh_1v_1\}u_{\sigma_i}) = k^4\{x+y\}h_1^2D^2u_{\sigma_i}.$$

Proof. Consider the class $k^6\{h_1^2+xh_1v_1\}D^{10}u_{\sigma_i}$ in filtration 26. By Theorem 4.8 this class cannot survive to the E_∞-page.

After the E_5-page, all the potential sources that could support a differential hitting the class $k^6\{h_1^2+xh_1v_1\}D^{10}u_{\sigma_i}$ are $k^3x^2h_1D^9u_{\sigma_i}$, $k^3\{x+y\}D^9u_{\sigma_i}$ and $kx^2h_1D^8u_{\sigma_i}$. We ruin out all three possibilities one by one. The class $k^3x^2h_1D^9u_{\sigma_i}$ is hit by the following d_9-differential in Proposition 5.12

$$d_9(kxh_1D^9u_{\sigma_i}) = kD^3d_9(\{x+y\}h_1D^6u_{\sigma_i}) = k^3x^2h_1D^9u_{\sigma_i}.$$

The class $k^3\{x+y\}D^9u_{\sigma_i}$ is a permanent cycle since $\{x+y\}u_{\sigma_i}$ is a permanent cycle by Lemma 5.2. The class $kx^2h_1D^8$ is also a permanent cycle since it is hit by a known d_9-differential in the Q_8-TateSS for E_2 according to Proposition 5.12

$$d_9(k^{-1}\{x+y\}D^7h_1u_{\sigma_i}) = kx^2h_1D^8u_{\sigma_i}.$$

Therefore, the class $k^6\{h_1^2+xh_1v_1\}D^{10}u_{\sigma_i}$ must support a non-trivial differential. Since $kD^3 = g$ is an invertible permanent cycle in TateSS, the class $\{h_1^2+xh_1v_1\}u_{\sigma_i} = D^8(kD^3)^{-6}k^6\{h_1^2+xh_1v_1\}D^{10}u_{\sigma_i}$ also has to support a non-trivial differential. Then for degree reasons, since $k^6\{h_1^2+xh_1v_1\}D^{10}u_{\sigma_i}$ is 5-cycle, the only potential targets are $k^4xh_1^2D^2u_{\sigma_i}$ and $k^5x^3D^3u_{\sigma_i}$. However, the class $k^5x^3D^3u_{\sigma_i}$ supports the following d_{11}-differential by Corollary 5.8:

$$d_{11}(k^5x^3D^3u_{\sigma_i}) = (kD^3)^5D^{-16}d_{11}(x^3D^4u_{\sigma_i}) = k^8D^4\{x+y\}h_1u_{\sigma_i}.$$

Therefore, the class $\{h_1^2+xh_1v_1\}u_{\sigma_i}$ supports the desired d_{17}-differentials

$$d_{17}(\{h_1^2+xh_1v_1\}u_{\sigma_i}) = k^4\{x+y\}h_1^2D^2u_{\sigma_i}.$$

□

It turns out that this is the only d_{17}-differential in one period of the $(*-\sigma_i)$-graded part of Q_8-HFPSS(E_2).

Proposition 5.15. *The classes* $\{x^2+y^2\}D^3u_{\sigma_i}$ *at* $(22,2)$ *and* $\{x^2+y^2\}D^7u_{\sigma_i}$ *at* $(54,2)$ *support* d_9*-differentials*

(1) $d_9(\{x^2+y^2\}D^3u_{\sigma_i}) = k^2x^3D^4u_{\sigma_i}$;
(2) $d_9(\{x^2+y^2\}D^7u_{\sigma_i}) = k^2x^3D^8u_{\sigma_i}$.

Proof. According to Proposition A.7, the restriction of $\{x^2+y^2\}u_{\sigma_i}$ to the integer-graded C_4-HFPSS for E_2 is non-trivial. It implies the following restriction by degree reasons

$$\mathrm{res}_{C_4}^{Q_8}(\{x^2+y^2\}D^3u_{\sigma_i}) \doteq \bar{\mathfrak{d}}_1^6 u_{6\lambda}u_{4\sigma}a_{2\sigma}.$$

We now prove that the class $\{x^2+y^2\}D^3u_{\sigma_i}$ supports a non-trivial differential by contradiction. Suppose that the class $\{x^2+y^2\}D^3u_{\sigma_i}$ is a permanent cycle that survives to the E_∞-page. Note that its C_4-restriction $\bar{\mathfrak{d}}_1^6 u_{6\lambda}u_{4\sigma}a_{2\sigma}$ has a hidden 2 extension in $\pi_*(E_2^{hC_4})$ by Lemma 2.18. Then $\{x^2+y^2\}D^3u_{\sigma_i}$ also has a hidden 2 extension in the E_∞-page. However, since hidden extensions and natural maps between spectral sequences will not decrease filtration, the potential target of the hidden 2 extension from the class $\{x^2+y^2\}D^3u_{\sigma_i}$ can only be $k^2\{x^2+y^2\}D^4u_{\sigma_i}$, $k\{yh_2+xh_1v_1\}D^3u_{\sigma_i}$ and $k\{h_1^2+xh_1v_1\}D^3$ by degree reasons. However, the first class $k^2\{x^2+y^2\}D^4u_{\sigma_i}$ supports a non-trivial d_5-differential by Corollary 5.5

$$d_5(k^2\{x^2+y^2\}D^4u_{\sigma_i}) = k^3xh_1^2D^4u_{\sigma_i}.$$

The second class $k\{yh_2+xh_1v_1\}D^3u_{\sigma_i}$ is hit by a d_5-differential by Corollary 5.3

$$d_5(\{x+y\}D^3u_{\sigma_i}) = k\{yh_2+xh_1v_1\}D^3u_{\sigma_i}.$$

The third class $k\{h_1^2+xh_1v_1\}D^3u_{\sigma_i}$ supports a d_{17}-differential by Proposition 5.14

$$d_{17}(k\{h_1^2+xh_1v_1\}D^3u_{\sigma_i}) = k^5xh_1^2D^5u_{\sigma_i}.$$

Therefore, all the potential targets of the hidden 2 extension from the class $\{x^2+y^2\}D^3u_{\sigma_i}$ will not survive to the E_∞-page. This is a contradiction. Hence the class $\{x^2+y^2\}D^3u_{\sigma_i}$ must support a non-trivial differential.

After the E_5-page, the only two potential targets are $k^2x^3D^4u_{\sigma_i}$ and $k^5\{x+y\}h_1^2D^5u_{\sigma_i}$ by degree reasons. However, the class $k^5\{x+y\}h_1^2D^5u_{\sigma_i}$ is hit by the following d_{17}-differential by Proposition 5.14 and the Leibniz rule

$$d_{17}(k\{h_1^2+xh_1v_1\}D^3u_{\sigma_i}) = kD^3d_{17}(\{h_1^2+xh_1v_1\}u_{\sigma_i}) = k^5\{x+y\}h_1^2D^5u_{\sigma_i}.$$

Then the first desired d_9-differential follows. The proof of the second d_9-differential in the statement is similar since the C_4-HFPSS for E_2 is 32-periodic. □

Actually, we can apply the norm method to get a d_9-differential directly (after the calculation of E_3-page) which is independent of the d_9 information in the integer-graded part.

Proposition 5.16. *There is a normed* d_9*-differential in* $(*-\sigma_i)$*-page*

$$d_9(\{x+y\}D^6u_{\sigma_i}) = k^2\{x^2+y^2\}D^7u_{\sigma_i}.$$

Proof. According to [HHR17, Theorem 11.13], the class $u_{2\lambda}$ supports a non-trivial d_5-differential in C_4-HFPSS(E_2)

$$d_5(u_{2\lambda}) = \bar{\mathfrak{d}}_1u_\lambda a_{2\lambda}a_\sigma.$$

Then Theorem 2.8 implies there is a predicted d_9-differential in Q_8-HFPSS(E_2)

$$d_9(N_{C_4}^{Q_8}(u_{2\lambda})a_{\sigma_i}) = N_{C_4}^{Q_8}(\bar{\mathfrak{d}}_1)N_{C_4}^{Q_8}(u_\lambda)a_{2\mathbb{H}}a_{\sigma_j}a_{\sigma_k}.$$

We claim the target of this predicted d_9-differential is non-trivial on the E_2-page. It suffices to show that the class $a_{\sigma_j}a_{\sigma_k}$ is non-trivial since $N_{C_4}^{Q_8}(u_\lambda)a_{2\mathbb{H}}$ is invertible in TateSS(E_2). We observe that

$$\mathrm{res}_{C_4}^{Q_8}(a_{\sigma_j}a_{\sigma_k}) = a_{2\sigma}$$

where $a_{2\sigma}$ is non-trivial in C_4-HFPSS(E_2). This implies that $a_{\sigma_j}a_{\sigma_k}$ is also non-trivial. Therefore, the non-trivial class on the E_2-page $N_{C_4}^{Q_8}(\bar{\mathfrak{d}}_1)N_{C_4}^{Q_8}(u_\lambda)a_{2\mathbb{H}}a_{\sigma_j}a_{\sigma_j}$ must be hit on or before the E_9-page. By multiplying this class with the periodicity classes in Corollary 2.22, we get a non-trivial class at $(46, 10)$, which has to be the class $k^2\{x^2+y^2\}D^7u_{\sigma_i}$ (up to a unit) by degree reasons. Therefore, the class $k^2\{x^2+y^2\}D^7u_{\sigma_i}$ has to be hit on or before the E_9-page too. For degree reasons, the desired d_9-differential happens in the $(*-\sigma_i)$-graded part. □

All d_9-differentials follow from Proposition 5.11, Proposition 5.12, Corollary 5.13, Proposition 5.15 and the Leibniz rule.

Proposition 5.17. *The classes $\{x+y\}h_1^2D^2u_{\sigma_i}$ at $(17, 3)$ and $\{x+y\}h_1D^7u_{\sigma_i}$ at $(56, 2)$ support d_{23}-differentials*

(1) $d_{23}(\{x+y\}h_1^2D^2u_{\sigma_i}) = k^6\{x+y\}h_1D^5u_{\sigma_i}$;
(2) $d_{23}(\{x+y\}h_1D^7u_{\sigma_i}) = k^6\{x+y\}D^{10}u_{\sigma_i}$.

Proof. By Corollary 4.22 we have the following two d_{23}-differentials in the integer-graded part

$$d_{23}(D^2h_1^2) = k^6h_1D^5, \text{ and } d_{23}(D^7h_1) = k^6D^{10}.$$

Note that the class $\{x+y\}u_{\sigma_i}$ is a permanent cycle by Lemma 5.2. Then the desired two differentials follow from these d_{23}-differentials and the Leibniz rule. □

All d_{23}-differentials follow from Proposition 5.17 and the Leibniz rule.

Lemma 5.18. *The classes $\{h_1^2+xh_1v_1\}D^7u_{\sigma_i}$ and $\{h_1^2+xh_1v_1\}D^4u_{\sigma_i}$ are both permanent cycles.*

Proof. After the E_5-page, the potential targets of $\{h_1^2+xh_1v_1\}D^7u_{\sigma_i}$ are the classes $k^2\{x+y\}h_1^2D^8u_{\sigma_i}$ and $k^3x^3D^9u_{\sigma_i}$, since lengths of differentials in the $RO(Q_8)$-graded Q_8-HFPSS(E_2) are less than or equal to 23 by Theorem 4.8. However, the class $k^2\{x+y\}h_1^2D^8u_{\sigma_i}$ supports a non-trivial d_{23}-differential by Proposition 5.17 and the class $k^3x^3D^9u_{\sigma_i}$ supports a non-trivial d_{11}-differential by Corollary 5.8. By similar reasons, the class $\{h_1^2+xh_1v_1\}D^4u_{\sigma_i}$ is also a permanent cycle. □

Proposition 5.19. *There are four non-trivial d_{11}-differentials*

(1) $d_{11}(x^2h_1D^2u_{\sigma_i}) = k^3\{h_1^2+xh_1v_1\}D^3u_{\sigma_i}$;
(2) $d_{11}(x^2h_1D^3u_{\sigma_i}) = k^3\{h_1^2+xh_1v_1\}D^4u_{\sigma_i}$;
(3) $d_{11}(x^2h_1D^6u_{\sigma_i}) = k^3\{h_1^2+xh_1v_1\}D^7u_{\sigma_i}$;
(4) $d_{11}(x^2h_1D^7u_{\sigma_i}) = k^3\{h_1^2+xh_1v_1\}D^8u_{\sigma_i}$.

Proof. We prove the last d_{11}-differential, and the proofs for the other differentials are similar. The class $\{h_1^2+xh_1v_1\}D^7u_{\sigma_i}$ is a permanent cycle by Lemma 5.18. Then $k^6\{h_1^2+xh_1v_1\}D^9u_{\sigma_i} = D^{-16}\{h_1^2+xh_1v_1\}D^7u_{\sigma_i}(kD^3)^6$ is also a permanent cycle. Since the filtration of this class is 26 which is greater than 23, Theorem 4.8 implies that the class $k^5\{h_1^2+xh_1v_1\}D^7u_{\sigma_i}$ must be hit by a differential. For degree reasons, we have the d_{11}-differentials

$$d_{11}(k^3x^2h_1D^8u_{\sigma_i}) = k^6\{h_1^2+xh_1v_1\}D^9u_{\sigma_i}.$$

Note that the class $(k^3D)^{-1} = (k^3D^9D^{-8})^{-1}$ is an invertible permanent cycle in Q_8-TateSS(E_2). Then the Leibniz rule implies the desired d_{11}-differential in Q_8-TateSS(E_2). By Lemma 2.6 this d_{11}-differential also happens in Q_8-HFPSS(E_2). □

All d_{11}-differentials follow from Corollary 5.8, Proposition 5.19 and the Leibniz rule.

Proposition 5.20. *The class* $\{x+y\}D^4u_{\sigma_i}$ *at* $(31,1)$ *supports a* d_{13}*-differential*

$$d_{13}(\{x+y\}D^4u_{\sigma_i}) = k^3\{h_1^2 + xh_1v_1\}D^5u_{\sigma_i}.$$

Proof. We first claim the class $k^3\{h_1^2+xh_1v_1\}D^5u_{\sigma_i}$ is a permanent cycle. In the Q_8-TateSS for E_2, by multiplying it with $k^{-3}D^{-9}\cdot D^8$, we obtain $\{h_1^2+xh_1v_1\}D^4u_{\sigma_i}$, which is a permanent cycle by Lemma 5.18. So $k^3\{h_1^2+xh_1v_1\}D^5u_{\sigma_i}$ is also a permanent cycle in the Q_8-HFPSS for E_2.

Next we consider the class $k^6\{h_1^2+xh_1v_1\}D^6u_{\sigma_i} = k^3\{h_1^2+xh_1v_1\}D^5u_{\sigma_i}\cdot k^3D^9\cdot D^{-8}$ above the vanishing line. By Theorem 4.8 it must be hit by a differential since it is a permanent cycle. Then for degree reasons, the only two possible sources are $\{x+y\}D^4u_{\sigma_i}$ and $x^2h_1D^4u_{\sigma_i}$. Note that the class $x^2h_1D^4$ is a permanent cycle since it is hit by a d_9-differential in Q_8-TateSS(E_2)

$$d_9(k^{-2}xh_1D^3u_{\sigma_i}) = x^2h_1D^4u_{\sigma_i}.$$

Therefore, the claimed d_{13}-differential must happen. □

This d_{13}-differential can also be deduced via the norm method.

Second proof of Proposition 5.20. According to [HHR17, Theorem 11.13][HSWX18, Corollary 3.14], there is a d_7-differential in the C_4-HFPSS for E_2

$$d_7(u_{4\lambda}) = \bar{\mathfrak{d}}_1\eta' u_{2\lambda}a_{3\lambda}.$$

Then Theorem 2.8 shows that there is a predicted d_{13}-differential

$$d_{13}(N_{C_4}^{Q_8}(u_{4\lambda})a_{\sigma_i}) = N_{C_4}^{Q_8}(\bar{\mathfrak{d}}_1)N_{C_4}^{Q_8}(\eta')N_{C_4}^{Q_8}(u_{2\lambda})a_{3\mathbb{H}}.$$

According to [Sch11, Proposition 10.4 (viii)], $\mathrm{res}_{C_4}^{Q_8} N_{C_4}^{Q_8}(\eta') = \eta'^2$ is non-trivial. Then $N_{C_4}^{Q_8}(\eta')$ is non-trivial on the E_2-page and so is the class $N_{C_4}^{Q_8}(\bar{\mathfrak{d}}_1)N_{C_4}^{Q_8}(\eta')N_{C_4}^{Q_8}(u_{2\lambda})a_{3\mathbb{H}}$. By multiplying the non-trivial class $N_{C_4}^{Q_8}(\bar{\mathfrak{d}}_1)N_{C_4}^{Q_8}(\eta')N_{C_4}^{Q_8}(u_{2\lambda})a_{3\mathbb{H}}$ with the periodicity classes in Corollary 2.22, we get a non-trivial class at $(30,14)$ on the E_2-page, which has to be the class $k^3\{h_1^2+xh_1v_1\}D^5u_{\sigma_i}$ by degree reasons. Therefore, the class $k^3\{h_1^2+xh_1v_1\}D^5u_{\sigma_i}$ must be hit on or before the E_{13}-page. For degree reasons, the desired d_{13}-differential follows. □

All d_{13}-differentials follow from Proposition 5.20 and the Leibniz rule.

Table 9 lists the differentials we have computed so far. They generate differentials via the Leibniz rule. By inspection, these are all non-trivial differentials since the remaining classes are permanent cycles by Method 2.7.

Theorem 5.21. *There are no hidden 2 extensions on the* E_∞*-page of* $(*-\sigma_i)$*-graded* Q_8*-HFPSS*(E_2).

Proof. Since $a_{\sigma_i} = \{x+y\}u_{\sigma_i}$ already lives in the homotopy group $\pi_{-\sigma_i}(E_2^{hQ_8})$ and it is 2-torsion. Then in $(*-\sigma_i)$-graded Q_8-HFPSS(E_2) all a_{σ_i} multiples that survive to E_∞-page can not support hidden 2 extensions. Then for degree reasons, the only possible 2 extensions are from $\{x^2+y^2\}D^{4k+1}u_{\sigma_i}$ to $k\{h_1^2+xh_1v_1\}D^{4k+1}u_{\sigma_i}$, where $k\in\mathbb{Z}$. We now show there is actually no hidden 2 extension on the class $\{x^2+y^2\}Du_{\sigma_i}$; and the rest are similar. We observe that in $(*-\sigma_i)$-graded Q_8-TateSS(E_2), there is a differential

$$d_9(k^{-2}\{x+y\}u_{\sigma_i}) = \{x^2+y^2\}Du_{\sigma_i}.$$

By sparseness of the Q_8-TateSS for E_2, the above differential implies that under the homotopy group map induced by the additive norm map

$$N\colon (E_2)_{hQ_8} \to E_2^{hQ_8},$$

the elements detected by $k^{-2}\{x+y\}u_{\sigma_i}$ maps to elements detected by $\{x^2+y^2\}Du_{\sigma_i}$. However, in $(*-\sigma_i)$-graded Q_8-HOSS(E_2)

$$2k^{-2}\{x+y\}u_{\sigma_i} = 0,$$

then it forces

$$2\{x^2+y^2\}Du_{\sigma_i} = 0.$$

□

The result is presented in Fig. 15. We only draw h_1, h_2 extensions from the E_2-page.

5.1. **Summary of differentials.** Differentials in $(*-\sigma)$-graded part are given by Table 9. All differentials follow from this list by multiplying permanent cycles and the Leibniz rule.

TABLE 9. HPFSS differentials, $(*-\sigma_i)$-page

(s,f)	x	r	$d_r(x)$	Proof
$(1,1)$	$\{h_1+xv_1\}u_{\sigma_i}$	3	$2kv_1^2u_{\sigma_i}$	Proposition 5.9
$(4,0)$	$v_1^2u_{\sigma_i}$	3	$h_1^3u_{\sigma_i}$	Proposition 5.1 (restriction)
$(7,1)$	$\{x+y\}Du_{\sigma_i}$	5	$k\{yh_2+xh_1v_1\}Du_{\sigma_i}$	Corollary 5.3 (module structure)
$(14,2)$	$\{x^2+y^2\}D^2u_{\sigma_i}$	5	$kxh_1^2D^2u_{\sigma_i}$	Corollary 5.5 (module structure)
$(10,2)$	$\{h_1^2+xh_1v_1\}Du_{\sigma_i}$	9	$k^2\{x+y\}h_1^2D^2u_{\sigma_i}$	Proposition 5.11
$(42,2)$	$\{h_1^2+xh_1v_1\}D^5u_{\sigma_i}$	9	$k^2\{x+y\}h_1^2D^6u_{\sigma_i}$	Proposition 5.11
$(8,2)$	$\{x+y\}h_1Du_{\sigma_i}$	9	$k^2x^2h_1D^2u_{\sigma_i}$	Proposition 5.12 (module structure)
$(40,2)$	$\{x+y\}h_1D^5u_{\sigma_i}$	9	$k^2x^2h_1D^6u_{\sigma_i}$	Proposition 5.12
$(15,1)$	$\{x+y\}D^2u_{\sigma_i}$	9	$k^2\{x^2+y^2\}D^3u_{\sigma_i}$	Corollary 5.13
$(47,1)$	$\{x+y\}D^6u_{\sigma_i}$	9	$k^2\{x^2+y^2\}D^7u_{\sigma_i}$	Corollary 5.13
$(22,2)$	$\{x^2+y^2\}D^3u_{\sigma_i}$	9	$k^2x^3D^4u_{\sigma_i}$	Proposition 5.15 (hidden 2 extension)
$(54,2)$	$\{x^2+y^2\}D^7u_{\sigma_i}$	9	$k^2x^3D^8u_{\sigma_i}$	Proposition 5.15
$(15,3)$	$x^2h_1D^2u_{\sigma_i}$	11	$k^3\{h_1^2+xh_1v_1\}D^3u_{\sigma_i}$	Proposition 5.19 (vanishing line)
$(47,3)$	$x^2h_1D^6u_{\sigma_i}$	11	$k^3\{h_1^2+xh_1v_1\}D^7u_{\sigma_i}$	Proposition 5.19
$(23,3)$	$x^2h_1D^3u_{\sigma_i}$	11	$k^3\{h_1^2+xh_1v_1\}D^4u_{\sigma_i}$	Proposition 5.19
$(55,3)$	$x^2h_1D^7u_{\sigma_i}$	11	$k^3\{h_1^2+xh_1v_1\}D^8u_{\sigma_i}$	Proposition 5.19
$(29,3)$	$x^3D^4u_{\sigma_i}$	11	$k^3xh_1D^5u_{\sigma_i}$	Corollary 5.8 (module structure)
$(61,3)$	$x^3D^8u_{\sigma_i}$	11	$k^3xh_1D^9u_{\sigma_i}$	Corollary 5.8
$(31,1)$	$\{x+y\}D^4u_{\sigma_i}$	13	$k^3\{h_1^2+xh_1v_1\}D^5u_{\sigma_i}$	Proposition 5.20 (vanishing line or norm differential)
$(2,2)$	$\{h_1^2+xh_1v_1\}u_{\sigma_i}$	17	$k^4\{x+y\}h_1^2D^2u_{\sigma_i}$	Proposition 5.14 (vanishing line)
$(17,3)$	$\{x+y\}h_1^2D^2u_{\sigma_i}$	23	$k^6\{x+y\}h_1D^5$	Proposition 5.17 (module structure)
$(56,2)$	$\{x+y\}h_1D^7u_{\sigma_i}$	23	$k^6\{x+y\}D^{10}u_{\sigma_i}$	Proposition 5.17

6. Charts and Tables

6.1. **Keys for the charts.** In all charts, a gray line denotes a multiplication. See the following table for the keys.

Table 10. keys for multiplications

line	meanings
vertical	2 multiplication
slope 1	h_1 multiplication
slope 1/3	h_2 multiplication
dashed (only in 2BSS)	hidden extension

The colored lines denote the differentials. We use different colors to distinguish different lengths.

Table 11. keys for classes

class	meaning
dot	k
blue dot	$k[\![j]\!]$
red dot	$k[\![j]\!]\{j\}$
square	$\mathbb{W}(k)$

Here k is $\mathbb{F}_2$ for $G = SD_{16}$ or G_{48}, and is $\mathbb{F}_4$ for $G = Q_8$ or G_{24}; j is $v_1^{12}D^{-3}$ for G_{24} or G_{48}, and $v_1^4D^{-1}$ otherwise.

Remark 6.1. We elaborate more on boxes and dots connected by vertical lines in the same bidegree. Such pattern is a 2-adic presentation of a class. Namely, the bottom dot is generated by the generator and represents a 2-torsion copy, the dot or box just above is generated by twice the generator, and so on.

For example, on the E_∞-page of the integral degrees (Fig. 9), in bigrading $(32, 0)$ the bottom red dot represents the class $\mathbb{W}/2[\![v_1^4D^{-1}]\!]\{v_1^4D^3\}$ and the blue box above represents $\mathbb{W}[\![v_1^4D^{-1}]\!]\{2D^4\}$; Note that there is a 2 extension. Thus the class at $(32, 0)$ is $\mathbb{W}[\![v_1^4D^{-1}]\!]\{v_1^4D^3\} \oplus \mathbb{W}\{2D^4\}$.

Such presentations help to demonstrate where the differentials or extensions come from. For example, in Fig. 5 in bigrading $(12, 0)$, only the generator v_1^6 supports a non-trivial d_3-differential and $2v_1^6$ survives. This convention is due to Dan Isaksen.

Remark 6.2. We comment on the extensions between dots of different colors. For example, in the bidegree $(24, 0)$ and $(25, 1)$ in Fig. 9, there is an h_1 multiplication connecting a red and a blue dot. The red dot represents the class $\mathbb{W}/2[\![v_1^4D^{-1}]\!]\{v_1^4D^2\}$ and the blue dot represents the class $\mathbb{W}/2[\![v_1^4D^{-1}]\!]\{h_1D^3\}$. The h_1 multiplication happens whenever it is indicated by the class names. Note that the class $\mathbb{W}/2\{h_1D^3\}$ is not h_1-divisible in this case since the source is missing.

6.1.1. *2-BSS.*

Fig. 2 - Fig. 4 are charts for the 2-Bockstein spectral sequences. All three charts have $(8, 0)$ periodicity by multiplying D and $(-4, 4)$ periodicity by multiplying k (except the v_1 local classes in low filtration). We only depict part of the spectral sequence here, which contains a full periodic range.

In Fig. 2, a blue line indicates the multiplication by x, while an orange line indicates the multiplication by y.

Recall the $(* - \sigma_i)$-graded part and the integer-graded part have isomorphic E_1-pages. When interpret the chart as the $(* - \sigma_i)$-graded part, the name of a class at (s, f) is its label multiplied by u_{σ_i}, and its degree is $(s + 1 - \sigma_i, f)$. For example, the class 1 at $(0, 0)$, when interpreted as an $(* - \sigma_i)$-graded part class, denotes u_{σ_i} at $(1 - \sigma_i, 0)$ in the 2BSS.

Fig. 3 and Fig. 4 show the E_∞-page of 2BSS, for the integer-graded part and $(* - \sigma_i)$-graded part respectively.

6.1.2. *HFPSS.*

Fig. 5–Fig. 9 depict the integer degree calculation of the integer-graded G-HFPSS(E_2) for $G = Q_8$ or SD_{16}, and Fig. 11–Fig. 15 depict the $(* - \sigma_i)$-graded calculation. Both E_2-pages are $(8, 0)$ periodic by multiplying D, and other pages are $(64, 0)$ periodic by multiplying D^8. All charts are $(20, 4)$ periodic by multiplying kD^3 (except the v_1 local classes in low filtration). The differentials are denoted by the colored lines with their length classified by the color. When the target or the source of the differential is out of range, we replace the line with an arrow. There are horizontal vanishing lines in filtration 23 on E_∞-pages.

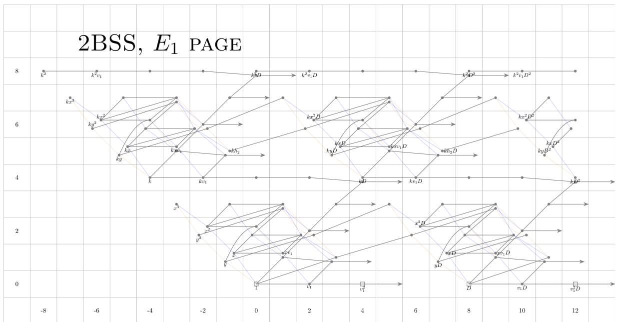

FIGURE 2. The E_1-page of the integer/$(* - \sigma_i)$-graded 2BSS.

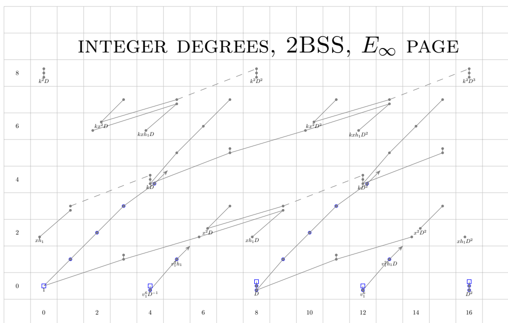

FIGURE 3. The E_∞-page of the integer-graded 2BSS.

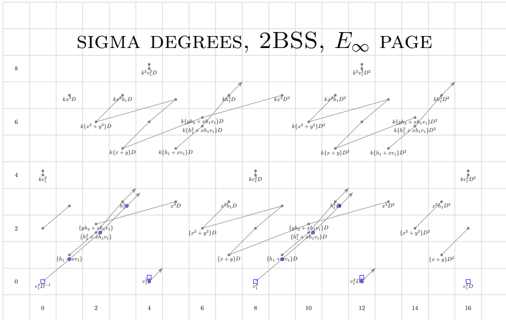

FIGURE 4. The E_∞-page of the $(* - \sigma_i)$-graded 2BSS.

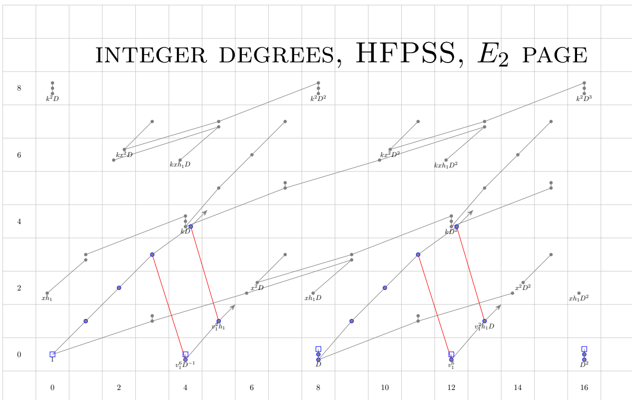

FIGURE 5. The E_3-page of the integer-graded Q_8-HFPSS(E_2). The red lines are d_3-differentials.

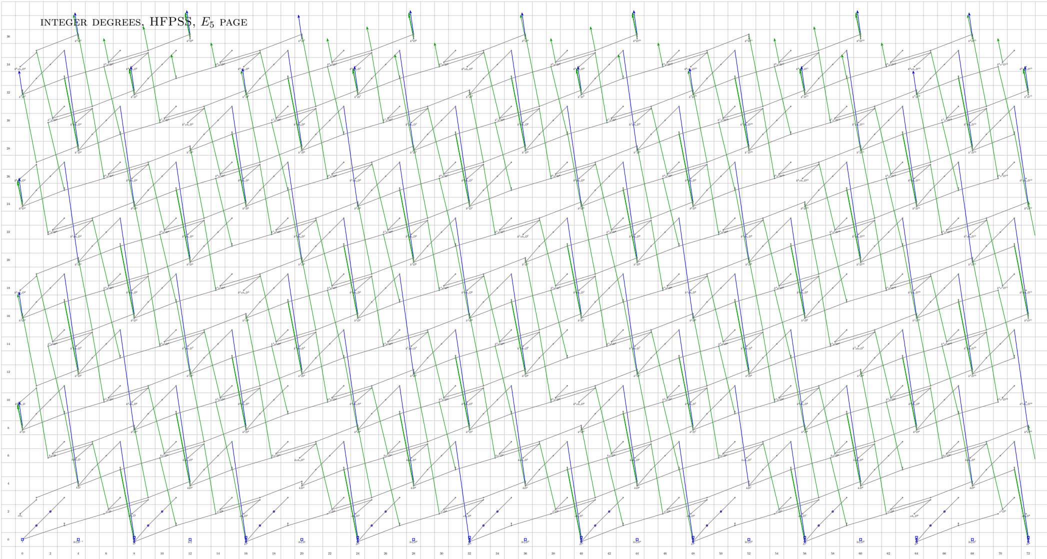

FIGURE 6. The E_5-page of the integer-graded Q_8-HFPSS(E_2). The green lines are d_5-differentials. The blue lines are d_7-differentials.

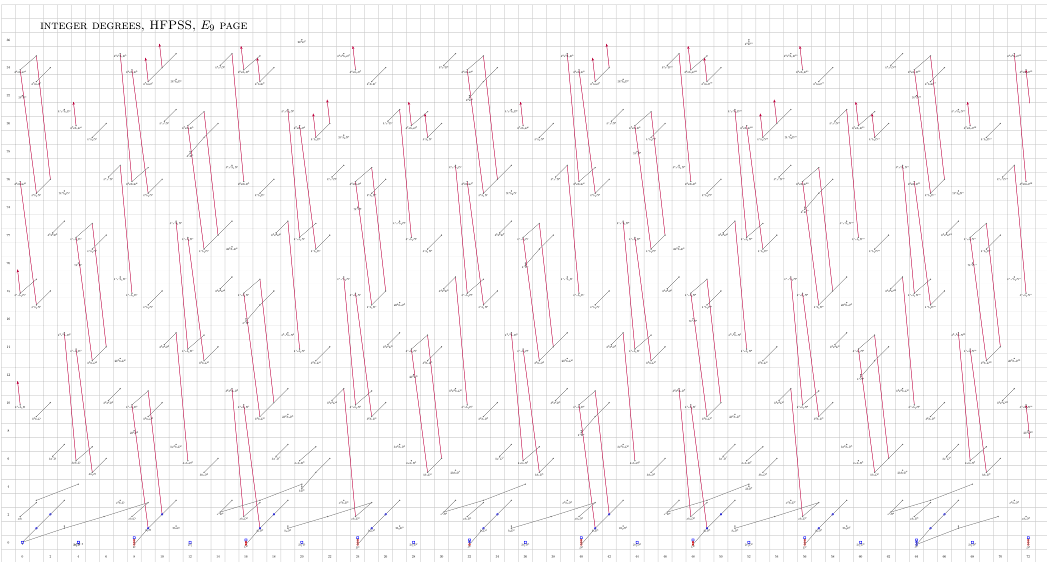

FIGURE 7. The E_9-page of the integer-graded Q_8-HFPSS(E_2). The purple lines are d_9-differentials.

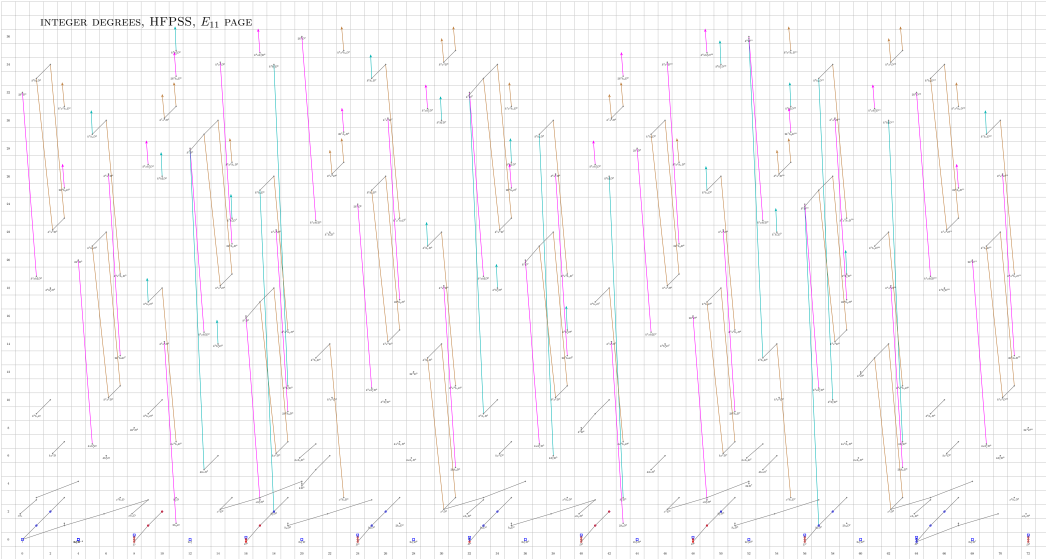

FIGURE 8. The E_{11}-page of the integer-graded Q_8-HFPSS(E_2). The brown lines are d_{11}-differentials. The magenta lines are d_{13}-differentials. The green lines are d_{13}-differentials.

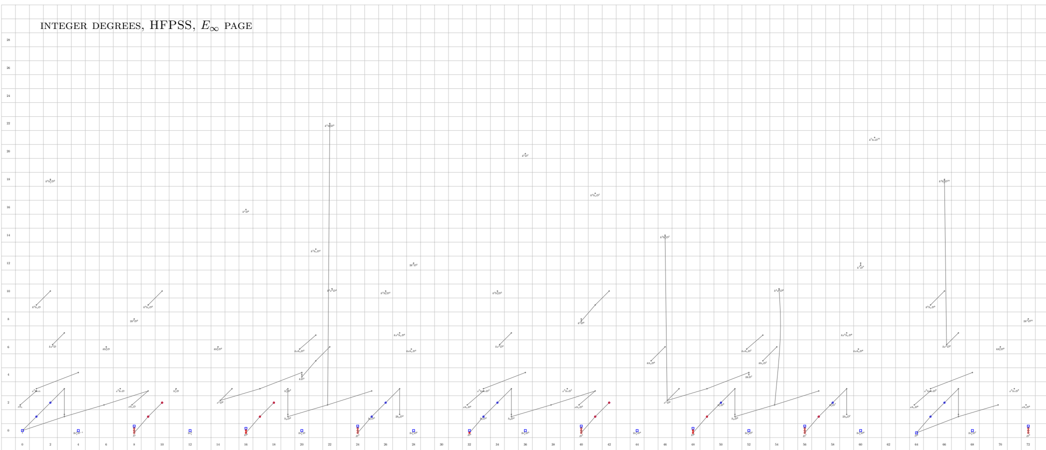

FIGURE 9. The E_∞-page of the integer-graded Q_8/SD_{16}-HFPSS(E_2).

FIGURE 10. The E_∞-page of the integer-graded G_{24}/G_{48}-HFPSS(E_2).

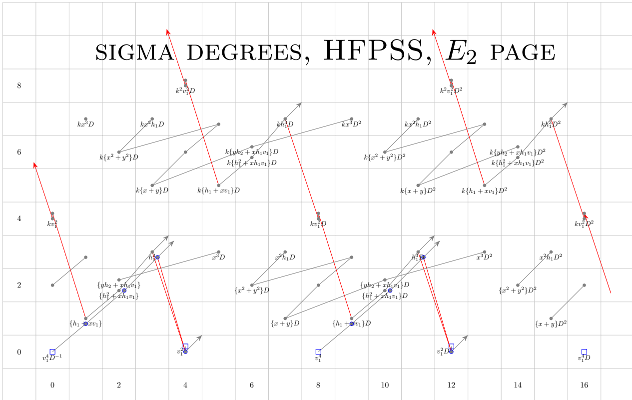

FIGURE 11. The E_2-page of the $(* - \sigma_i)$-graded Q_8-HFPSS(E_2). The red lines are d_3-differentials.

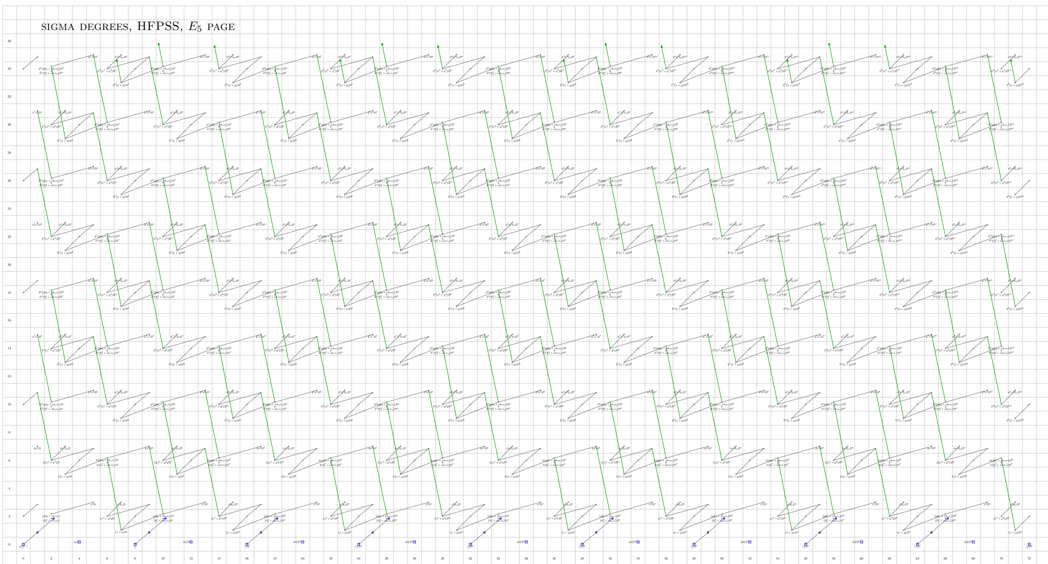

FIGURE 12. The E_5-page of the $(* - \sigma_i)$-graded Q_8-HFPSS(E_2). The green lines are d_5-differentials.

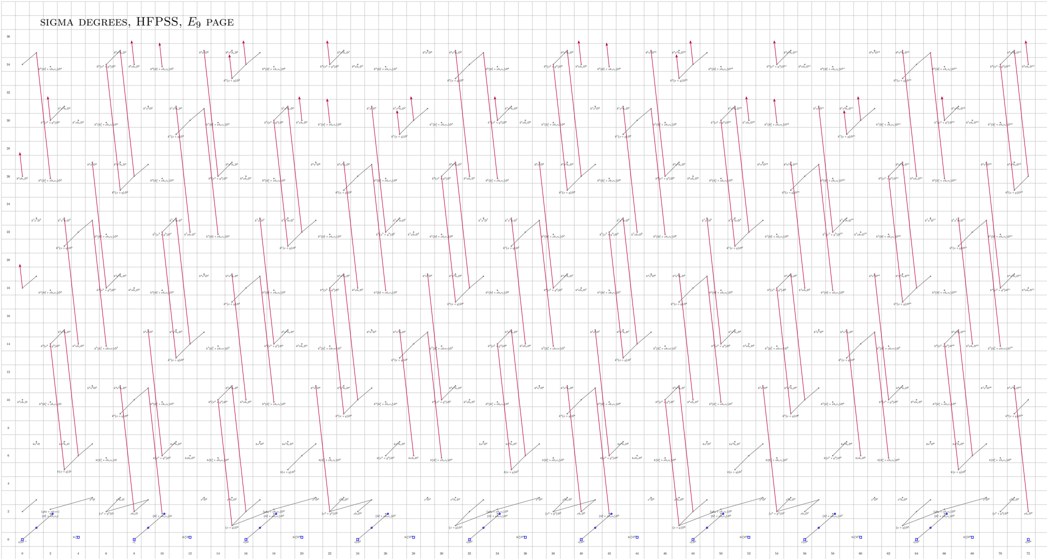

FIGURE 13. The E_9-page of the $(*-\sigma_i)$-graded Q_8-HFPSS(E_2). The purple lines are d_9-differentials.

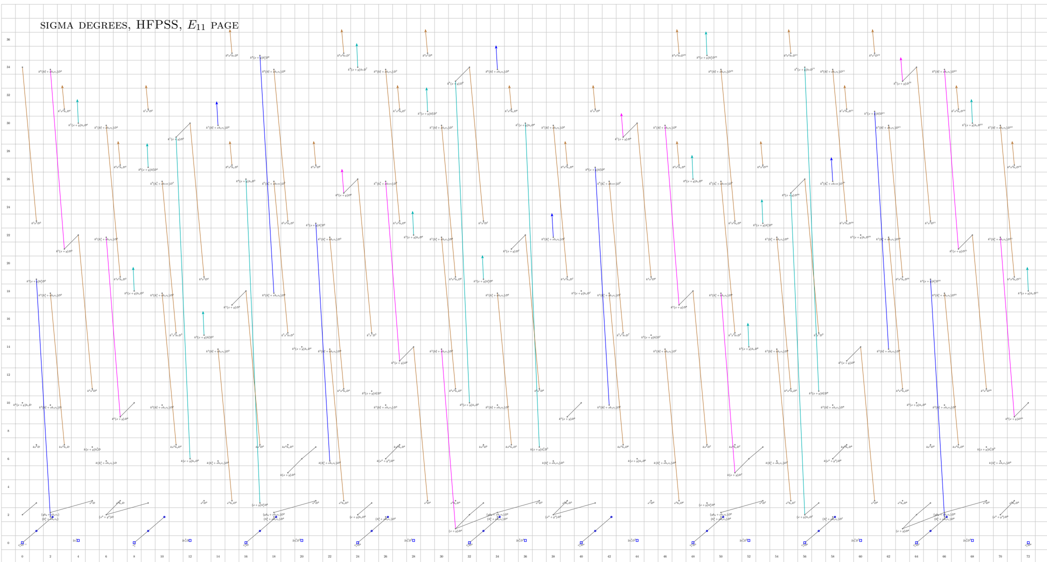

FIGURE 14. The E_{11}-page of the $(*-\sigma_i)$-graded Q_8-HFPSS(E_2). The brown lines are d_{11}-differentials. The magenta lines are d_{13}-differentials. The blue lines are d_{17}-differentials. The green lines are d_{13}-differentials.

Figure 15. The E_∞-page of the $(* - \sigma_i)$-graded Q_8/SD_{16}-HFPSS(E_2).

Appendix A. Group Cohomology

In this appendix, we collect and present examples of computations of group cohomology. There are two main applications: one is to calculate it as the input for the E_2-page of the integer- and $(* - \sigma_i)$-graded homotopy fixed points spectral sequences for E_2, the other is to utilize restrictions, transfers, and norm maps for proofs of differentials. All the rests needed for our computation of the Q_8-HFPSS for E_2 are listed in Proposition A.7.

Let Q_8 be presented as

$$Q_8 = \langle i, j \,|\, i^4, j^2, ijij^{-1} \rangle$$

with its real representation ring $RO(Q_8) = \mathbb{Z}\{1, \sigma_i, \sigma_j, \sigma_k, \mathbb{H}\}$. To calculate $H^*(Q_8, A)$ we will use the following 4-periodic free $\mathbb{Z}[Q_8]$-resolution:

$$0 \leftarrow \mathbb{Z} \xleftarrow{\nabla} X_0 \xleftarrow{d_0} X_1 \xleftarrow{d_1} X_2 \xleftarrow{d_2} ...,$$

where $X_0 = \mathbb{Z}[Q_8]\{a_0\}$, $\nabla(a_0) = 1$, and for $k \geq 0$,

$$\begin{aligned}
X_{4k+1} &= \mathbb{Z}[Q_8]\{b_{k,1}, b_{k,2}\}, & d(b_{k,1}) &= (i-1)a_k, \\
& & d(b_{k,2}) &= (j-1)a_k, \\
X_{4k+2} &= \mathbb{Z}[Q_8]\{c_{k,1}, c_{k,2}\}, & d(c_{k,1}) &= (1+i)b_{k,1} - (1+j)b_{k,2}, \\
& & d(c_{k,2}) &= (1+ij)b_{k,1} + (i-1)b_{k,2}, \\
X_{4k+3} &= \mathbb{Z}[Q_8]\{e_k\}, & d(e_k) &= (i-1)c_{k,1} - (ij-1)c_{k,2}, \\
X_{4k+4} &= \mathbb{Z}[Q_8]\{a_{k+1}\}, & d(a_{k+1}) &= \textstyle\sum_{g \in Q_8} g \cdot e_k
\end{aligned}$$

Suppose that A is a Q_8-module, then $H^*(Q_8; A)$ is the cohomology of the cochain complex

$$A \xrightarrow{d_0} A \oplus A \xrightarrow{d_1} A \oplus A \xrightarrow{d_2} A \xrightarrow{d_3} A \to ...$$

where the differentials (by abuse of notation) are given by the following matrices

$$d_{4k} = \begin{pmatrix} i-1 \\ j-1 \end{pmatrix}, \quad d_{4k+1} = \begin{pmatrix} 1+i & -1-j \\ 1+ij & -1+i \end{pmatrix}, \quad d_{4k+2} = \begin{pmatrix} -1+i & 1-ij \end{pmatrix},$$

and $d_{4k+3} = \sum_{g \in Q_8} g$.

We record here the group cohomology of Q_8 with trivial $\mathbb{Z}$ coefficients

$$H^{4k+2}(Q_8,\mathbb{Z}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2,$$
$$H^{4k+4}(Q_8,\mathbb{Z}) = \mathbb{Z}/8,$$
$$H^{2q+1}(Q_8,\mathbb{Z}) = 0,$$

where $k \geq 0, q \geq 0$, and the generator of $H^4(Q_8,\mathbb{Z})$ gives the 4-periodicity.

In addition to the integer-graded Q_8-HFPSS for E_2, we also compute the $(* - \sigma_i)$-graded part. For this purpose, we study the structure of $\pi_* E_2 \otimes \sigma_i$ as a Q_8-module, which is given by the following analog of [HM17, Lemma 4.6] :

Lemma A.1. *Let E be a Q_8-spectrum. Then*

$$\pi^e_*(E \wedge S^{1-\sigma_i}) \cong \pi^e_* E \otimes \sigma_i$$

as Q_8-modules.

Recall that we define $v_1 = u_1 u^{-1}$ and its Q_8 action is given in Eq. (2.3). By Lemma 2.12, we may first compute $H^*(Q_8, \mathbb{W}[u^{-1}, v_1])$, and then invert D and complete at $I = (2, u_1)$.

Remark A.2. If we define $s = i_*(u^{-1})$ and denote u^{-1} by t, then the actions of Q_8 on s, t are given by

$$\begin{aligned} i_*(s) &= -t, & i_*(t) &= s \\ j_*(s) &= -\zeta^2 s + \zeta t, & j_*(t) &= \zeta s + \zeta^2 t \\ k_*(s) &= \zeta s + \zeta^2 t, & k_*(t) &= \zeta^2 s - \zeta t \end{aligned}$$

For computational purposes, it is equivalent to replace generators u^{-1}, v_1 by s, t, and the form of the action turns out to be more compact.

We first calculate the 0-th cohomology ring. Behrens and Ormsby [BO16] have determined the $C_4\langle i\rangle$-invariants:

Proposition A.3. *Let $b_2 = s^2 + t^2$, $b_4 = s^3 t - s t^3$ and $\delta = s^2 t^2$, then*

$$H^0(C_4, \mathbb{W}[u^{-1}, v_1]) = \mathbb{W}[b_2, b_4, \delta]/(b_4^2 - b_2^2\delta + 4\delta^2).$$

The j-actions on b_2, b_4, δ are the following:

$$\begin{aligned} j_*(b_2) &= -b_2, \\ j_*(b_4) &= -(2\zeta+1)b_2^2 + 7b_4 + 8(2\zeta+1)\delta, \\ j_*(\delta) &= b_2^2 + 2(2\zeta+1)b_4 - 7\delta. \end{aligned}$$

Proposition A.4. *We have the 0-th cohomology ring*

$$H^0(Q_8, \mathbb{W}[u^{-1}, v_1]) = \mathbb{W}[s_1, s_2, s_3]/(s_1^3 = 4(2\zeta+1)s_1^2 s_2 + 16 s_1 s_2^2)$$

where $s_1 = b_2^2$, $s_2 = b_4 + (2\zeta+1)\delta$, and $s_3 = b_2^3 + 2(2\zeta+1)b_4 b_2 - 8 b_2 \delta$.

Proof. Since $\pi_* E_2$ is 16-periodic, it suffices to compute the j-invariants of $H^0(C_4, \mathbb{W}[u^{-1}, v_1])$ in low degrees. The result follows by direct computation. □

In the main computations, we sometimes need to rely on explicit group cohomology results. The following is an example.

Example A.5. The calculation of $H^4(Q_8, \pi_4 E_2 \otimes \sigma_i) \cong \mathbb{W}/4$.

The cochain complex at degree 4 looks like

$$\mathbb{W}\{s^2, st, t^2\} \xrightarrow{d_3} \mathbb{W}\{s^2, st, t^2\} \xrightarrow{d_4} \mathbb{W}\{s^2, st, t^2\}^2$$

By Lemma A.1, the actions are

$$i_*(s^2) = t^2, \qquad i_*(st) = -st, \qquad i_*(t^2) = s^2$$
$$j_*(s) = -\zeta s^2 + 2st - \zeta^2 t^2, \quad j_*(t) = \zeta s + \zeta^2 t, \quad j_*(t^2) = \zeta^2 s^2 + 2st + \zeta t^2.$$

Therefore, $\ker d_4 = \ker(i-1) = \mathbb{W}\{s^2+t^2\}$.

Meanwhile, since we have

$$d_3(s^2) = 4(s^2+t^2),$$
$$d_3(st) = 0,$$
$$d_3(t^2) = 4(s^2+t^2),$$

we conclude that $H^4(Q_8, \pi_4 E_2 \otimes \sigma_i) \cong \mathbb{W}/4$.

We also calculate a couple of restriction maps in group cohomology. In the case of the integer-graded part, most calculations are easy. By Proposition A.8 we deduce that the generators η, ν, c, d, g have to restrict non-trivially to their C_4-counterparts, which lie in the Hurewicz image. For the $(* - \sigma_i)$-graded part, some chain level calculations seem to be inevitable.

Example A.6. In the integer-graded part, calculate $\operatorname{res}^{Q_8}_{C_4\langle i\rangle} D^{-2}d \neq 0$. This is used in the proof of Proposition 4.30.

The class $D^{-2}d$ lies in bigrading $(-2,2)$. We are looking at the degree 0 part of $\mathbb{W}[u^{-1}, v_1]$. The generator of $H^2(Q_8, \mathbb{W}\{1\})$ is given by the cochain

$$\alpha : \mathbb{Z}[Q_8]\{c_{0,1}, c_{0,2}\} \to \mathbb{W}\{1\},$$
$$c_{0,1} \mapsto 1, \quad c_{0,2} \mapsto 0.$$

Restricting to $C_4\langle i\rangle$, we rewrite $X_2 = \mathbb{Z}[Q_8]\{c_{0,1}, c_{0,2}\}$ as $\mathbb{Z}[C_4\langle i\rangle]\{c_{0,1}, jc_{0,1}, c_{0,2}, jc_{0,2}\}$, and similarly for X_1. Then α restricts to the cochain

$$\alpha : \mathbb{Z}[Q_8]\{c_{0,1}, jc_{0,1}, c_{0,2}, jc_{0,2}\} \to \mathbb{W}\{1\},$$
$$c_{0,1},\ jc_{0,1} \mapsto 1, \quad c_{0,2},\ jc_{0,2} \mapsto 0.$$

Now we check the image of d_1. Let $\beta_1, \beta_2, \beta_3, \beta_4$ be the dual basis of $b_{0,1}, jb_{0,1}, b_{0,2}, jb_{0,2}$ in $\operatorname{Hom}_{C_4\langle i\rangle}(X_1, \mathbb{W}\{1\})$. The image of β_1 is calculated by evaluating $\beta_1 \circ d_1$ at the $C_4\langle i\rangle$-basis of X_2. As an example, we have

$$(\beta_1 \circ d_1)(c_{1,0}) = \beta_1((1+i)b_{0,1} - b_{0,2} - jb_{0,2}) = 2.$$

Similarly, we verify that the restriction of α does not lie in the coboundary; hence the restriction is non-trivial.

Sometimes the restriction to $C_4\langle i\rangle$ is trivial, but it becomes non-trivial when restricted to $C_4\langle j\rangle$ or $C_4\langle k\rangle$. By similar calculations we have $\operatorname{res}^{Q_8}_{\langle j\rangle}(x+y)u_{\sigma_j} = 0$, while $\operatorname{res}^{Q_8}_{\langle j\rangle}(x+y)u_{\sigma_j} \neq 0$.

Finally, we present the collection of calculated results.

Proposition A.7. ***Summary of calculated group cohomology***

- $H^3(Q_8, \mathbb{Z}) = 0$.
- $H^4(Q_8, \pi_4 E_2 \otimes \sigma_i) = \mathbb{W}/4$.
- $H^3(Q_8, \pi_4 E_2 \otimes \sigma_i) = \mathbb{W}/2$.
- $H^2(Q_8, \pi_4 E_2 \otimes \sigma_i) = \mathbb{W}/2 \oplus \mathbb{W}/2$.
- $H^1(Q_8, \pi_0 E_2 \otimes \sigma_i) = \mathbb{W}/2$.

Summary of calculated restrictions

- $\operatorname{res}^{Q_8}_{\langle i\rangle} h_1 \neq 0$.
- $\operatorname{res}^{Q_8}_{\langle i\rangle} h_2 \neq 0$.

- $\mathrm{res}^{Q_8}_{\langle i\rangle} d \neq 0$.
- $\mathrm{res}^{Q_8}_{\langle i\rangle} g \neq 0$.
- $\mathrm{res}^{Q_8}_{\langle i\rangle}\{x^2+y^2\}u_{\sigma_i} \neq 0$.

In fact, the restriction map from $H^*(Q_8, \pi_*E_2)$ to $H^*(C_4, \pi_*E_2)$ is determined by the Hurewicz image of $E_2^{hC_4}$. The direct algebraic computation we gave above could potentially adapt to computations at higher heights.

We recall the known result of the Hurewicz image result of $E_2^{hC_4}$. We follow names introduced in Proposition 2.14.

Proposition A.8. (*see* [HSWX18, Figure 12]) *The following classes on the E_∞-page of the C_4-HFPSS for E_2 detects images of the Hurewicz map: $S^0 \to E_2^{hC_4}$:*

- $\bar{s}_1 a_{\sigma_2}$ *at* $(1,1)$ *detects the image of* $\eta \in \pi_1 S^0$,
- $\bar{\mathfrak{d}}_1 u_\lambda a_\sigma$ *at* $(3,1)$ *detects the image of* $\nu \in \pi_3 S^0$,
- $\bar{\mathfrak{d}}_1^4 u_{4\sigma} a_{4\lambda}$ *at* $(8,8)$ *detects the image of* $\epsilon \in \pi_8 S^0$,
- $\bar{\mathfrak{d}}_1^4 u_{4\lambda} u_{2\sigma} a_{2\sigma}$ *at* $(14,2)$ *detects the image of* $\kappa \in \pi_{14} S^0$,
- $\bar{\mathfrak{d}}_1^6 u_{4\lambda} u_{6\sigma} a_{2\lambda}$ *at* $(20,4)$ *detects the image of* $\bar{\kappa} \in \pi_{20} S^0$.

The unit map $S^0 \to E_2^{hC_4}$ factors as

$$S^0 \xrightarrow{\text{unit}} E_2^{hQ_8} \xrightarrow{\text{res}} E_2^{hC_4}.$$

There is a map of spectral sequences from the Adams–Novikov spectral sequence of the sphere to the C_4-HFPSS for E_2, and it factors through the Q_8-HFPSS for E_2. By comparing the Adams–Novikov spectral sequence of the sphere (e.g., see [Rav78, Table 2]) and the C_4-HFPSS for E_2, we see that the classes detecting η, ν, g, d do not jump filtrations under this map. Hence in the Q_8-HFPSS for E_2, these classes are detected by classes h_1, h_2, d, g, and the C_4-restriction of these classes are non-trivial as follows.

Proposition A.9. *The restriction map from the E_2-page of the Q_8-HFPSS for E_2 to the E_2-page of the C_4-HPFSS for E_2 is determined by the following and the multiplicative structure.*

$$\begin{aligned} &\mathrm{res}^{Q_8}_{C_4}(h_1) = \eta, && \mathrm{res}^{Q_8}_{C_4}(h_2) = \nu,\\ &\mathrm{res}^{Q_8}_{C_4}(c) = 0, && \mathrm{res}^{Q_8}_{C_4}(d) = \bar{\mathfrak{d}}_1^4 u_{4\sigma} a_{4\lambda},\\ &\mathrm{res}^{Q_8}_{C_4}(g) = \bar{\mathfrak{d}}_1^6 u_{4\lambda} u_{6\sigma} a_{2\lambda}. \end{aligned}$$

The element $\epsilon \in \pi_8 S^0$ is detected by a class at filtration 2 in the Adams–Novikov spectral sequence of the sphere. However, the image of ϵ in $\pi_8 E_2^{hC_4}$ is detected by $\bar{\mathfrak{d}}_1^4 u_{4\sigma} a_{4\lambda}$ at filtration 8 in the C_4-HFPSS for E_2. There is a jump of filtration by 6. By degree reasons, in Q_8-HFPSS(E_2), the image of ϵ could be potentially detected by a class of filtration $2 \leq f \leq 8$. By the fact that the unit map $S^0 \to E_2^{hQ_8}$ further factors through $S^0 \xrightarrow{unit} E_2^{hG_{24}}$, the image of ϵ is detected by the class c at $(8,2)$ (up to a unit) in Q_8-HFPSS(E_2). Therefore, there is an exotic restriction in HFPSS from Q_8 to C_4 that maps the class c to the class $\bar{\mathfrak{d}}_1 u_{4\sigma} a_{4\lambda}$.

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