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Connective Versions of $TMF(3)$

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Abstract

We study three connective versions of the spectrum for topological modular forms of level 3. All three were described briefly by Mahowald and Rezk in [10], but we add much detail to their discussion. Letting tmf(3) denote our connective model which is a ring spectrum, we compute $\text{tmf}(3)_*(RP^\infty)$.

Keywords: Topological modular forms, Adams spectral sequence.

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1 Introduction

In [10], the second author and Rezk discuss the periodic spectrum $\mathrm{TMF}(\Gamma_0(3))$, abbreviated here as $\mathrm{TMF}(3)$, associated to topological modular forms of level 3. In Section 7 of [10], they discuss briefly three connective models of $\mathrm{TMF}(3)$. The main purpose of this paper is to clarify and fill in details for these connective models.

The first model is $X \wedge \mathrm{tmf}$, where X is a certain 10-cell complex, and tmf is the connective 2-primary spectrum discussed in [1]. The spectrum $X \wedge \mathrm{tmf}$ was first introduced by the second author and Gorbounov in their study of $MO[8]$ in [7]. It is probably the best of our three models because it is a ring spectrum. In Section 2, we define it and compute its homotopy groups. Our description and method of computation differ somewhat from that of [7].

In [10], another connective model for $\mathrm{TMF}(3)$ is discussed, which is $Z \wedge \mathrm{tmf}$, where Z is a certain 8-cell complex. Although $Z \wedge \mathrm{tmf}$ is not a ring spectrum, its importance is primarily that the dimensions of the cells of Z allow one to easily construct a map $Z \to \mathrm{TMF}(3)$ thanks to certain homotopy groups of $\mathrm{TMF}(3)$ being 0. The other models are then related to $\mathrm{TMF}(3)$ via the Z-model. In Section 3, we provide some additional details to the sketch given in [10].

In Section 4, we consider a third model which was also introduced in [10]. This one is closely related to consideration of a splitting of $\mathrm{tmf} \wedge \mathrm{tmf}$. There is a Brown-Gitler-type splitting of the A-module $H^*(\mathrm{tmf} \wedge \mathrm{tmf})$, and we show that it is not realized by a spectrum splitting. Again we add some clarity and detail to the description in [10] of this model and its homotopy groups.

All three of our models are equivalent after inversion of v_2, but as connective models they are different. The homotopy groups of the second and third models are subsets of those of the first, obtained by omitting certain initial portions. One nice feature of our approach is to relate the Ext calculations for the second and third models directly to that of the first, even though the constructions of the spectra are very different.

In Section 5 we compute $\pi_*(P_1 \wedge X \wedge \mathrm{tmf})$, where $P_1 = RP^\infty$. If we think of $X \wedge \mathrm{tmf}$ as our best model of $\mathrm{tmf}(3)$, then this is $\mathrm{tmf}(3)_*(P_1)$. Our original goal in undertaking this study was to use $\mathrm{TMF}(3)$ in obstruction theory, and this computation would be a first step toward doing that.

2 The Model of Gorbounov and Mahowald

In their study of $\pi_*(MO[8])$ in [7], the second author and Gorbounov introduced a new spectrum, which turns out to be the best model for a connective version of $\mathrm{TMF}(3)$. Certain aspects of the construction in [7] were unclear to the first author, and so we have prepared this alternative account. In Theorem 2.1 we define the spectrum, and in Theorems 2.3 and 2.4 we determine its homotopy groups. In Section 3, we will establish its relationship with $\mathrm{TMF}(3)$.

Theorem 2.1. *(a) There is a 9-cell CW complex Y with one cell of each dimension* 0, 2, 3, 4, 6, 7, 8, 9, *and* 10, *in which the following Steenrod operations are nonzero on the bottom class g:*

$$\mathrm{Sq}^2,\ \mathrm{Sq}^3,\ \mathrm{Sq}^4,\ \mathrm{Sq}^4\,\mathrm{Sq}^2,\ \mathrm{Sq}^5\,\mathrm{Sq}^2,\ \mathrm{Sq}^6\,\mathrm{Sq}^2 = \mathrm{Sq}^8,\ \mathrm{Sq}^6\,\mathrm{Sq}^3,\ \mathrm{Sq}^7\,\mathrm{Sq}^3\,. \tag{2.1}$$

This together with $\mathrm{Sq}^6\,g = 0$ completely describes $H^*(Y)$ as an A-module.

(b) There is a map $\Sigma^3 Y \xrightarrow{\alpha} S^0$ extending 2ν.

(c) Let X denote the mapping cone of α. There is a map $X \xrightarrow{f} bo$ which is the identity on the bottom cell.

(d) Let $\widetilde{f}$ denote the composite

$$X \wedge \mathrm{tmf} \xrightarrow{f \wedge 1} bo \wedge \mathrm{tmf} \xrightarrow{\mu} bo,$$

where μ gives the tmf-module structure of bo described in Remark 2.2. Let C denote the mapping cone of $\widetilde{f}$. There is an isomorphism of A-modules

$$H^*(C) \approx \Sigma^4 A/A(\mathrm{Sq}^4, \mathrm{Sq}^5\,\mathrm{Sq}^1).$$

(e) $X \wedge \mathrm{tmf}$ is a ring spectrum.

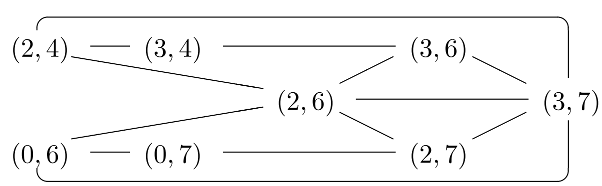
Remark 2.1. This $X \wedge \mathrm{tmf}$ will be our preferred model for the connective $\mathrm{tmf}(3)$, because it is a ring spectrum. The spectrum $\Sigma^{16}X \wedge \mathrm{tmf}$ is apparently a subspace of $MO[8]/\,\mathrm{tmf}$, but this will not enter into our argument. This was the motivation for the initial discussion of $X \wedge \mathrm{tmf}$ in [7].

Remark 2.2. Mark Behrens explained to the authors in an e-mail the following argument that bo is a tmf-module, which fact was used in the preceding theorem. In [8, §4.3], there is an argument involving derived stacks, which, upon taking global sections, gives an E_∞-map $\widehat{\mathrm{tmf}} \to KO[\![q]\!]$, where $KO[\![q]\!] = KO \wedge \mathbb{N}_+$ (i.e. smash KO with the suspension spectrum of the monoid given by the natural numbers.) Here, $\widehat{\mathrm{tmf}}$ is a certain non-connective version of tmf whose connective cover is tmf. It is similar to, but slightly different than, the $E(2)$-localization of the connective spectrum tmf. Taking connective covers of the composite $\widehat{\mathrm{tmf}} \to KO[\![q]\!] \to KO$ gives an E_∞-map $\mathrm{tmf} \to bo$, leading to the desired module structure.

Throughout the paper, A_n denotes the subalgebra of the mod 2 Steenrod algebra A generated by Sq^i for $i \le 2^n$. Also η and ν denote the (class of the) Hopf maps in the 1- and 3-stems. All cohomology groups have coefficients in $\mathbb{Z}_2 = \mathbb{Z}/2$. Our spectra are localized at 2. We make frequent use of the isomorphisms $H^*(\mathrm{tmf}) \approx A/\!/A_2$ and $H^*(bo) \approx A/\!/A_1$, and the fact that if M is an A-module, then $A/\!/A_n \otimes M \approx A \otimes_{A_n} M$.

Proof. (a.) Let $X_3 = S^0 \cup_\eta e^2 \cup_2 e^3$ and $X_7 = S^0 \cup_\nu e^4 \cup_\eta e^6 \cup_2 e^7$. Let Q denote the quotient of $X_3 \wedge X_7$ by its 4-skeleton. The Steenrod algebra structure, or equivalently the cell structure, of Q is depicted in Diagram 2.1. Here a symbol (i, j) is the product class or cell of an i-cell of X_3 and a j-cell of X_7. We indicate both Sq^1 and Sq^2 by straight lines, and Sq^4 by a curved line.

Diagram 2.1. Cell structure of quotient of $X_3 \wedge X_7$



There is a map g from this Q to $S^6 \cup_2 e^7 \cup_\eta e^9$ which sends the cells $(2, 4)$, $(3, 4)$, and $(3, 6)$ by degree 1, and the cells $(0, 6)$, $(0, 7)$, and $(2, 7)$ by degree -1. This map extends over the $(2, 6)$ and then $(3, 7)$ cells because the image of the attaching map of each is 0. The fiber of the composite

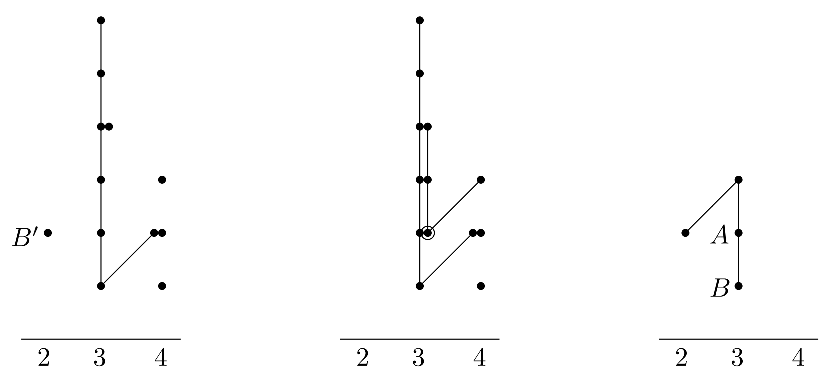
$$X_3 \wedge X_7 \xrightarrow{\mathrm{coll}} Q \xrightarrow{g} S^6 \cup_2 e^7 \cup_\eta e^9$$

is the desired complex Y. The Steenrod operations in Y can be determined from the Cartan formula together with the fact that Sq^2 and Sq^3 are nonzero in X_3, and Sq^4, Sq^6, and Sq^7 are nonzero in X_7. For example, $\mathrm{Sq}^4\,\mathrm{Sq}^2$ on the bottom class is $(2,4)$, while Sq^6 is $(2,4)+(0,6)$, which is $g^*(x_6)$ and hence is 0 in the fiber.

(b.) Let DY denote the S-dual of Y, with cells of dimensions the negative of those of Y. Thus the top cell of DY has dimension 0. Note that $\mathrm{Sq}^8 = 0$ in $H^*(DY)$, since it is dual to $\chi\,\mathrm{Sq}^8$, which is 0 in $H^*(Y)$. Let $(DY)^{(-1)}$ denote the (-1)-skeleton of DY. We will now use the Adams spectral sequence (ASS) to show that 2ν is in the image of $\pi_3(DY) \xrightarrow{c_*} \pi_3(S^0)$, where c collapses $(DY)^{(-1)}$. We use Bruner's software ([3]) to compute $\mathrm{Ext}_A^{s,t}(H^*(DY))$ for $2 \le t-s \le 4$ to be as in Diagram 2.2. Here and throughout, we omit writing $\mathbb{Z}_2$ as the second argument of our Ext groups.

Diagram 2.2. Ext groups for $2 \le t-s \le 4$

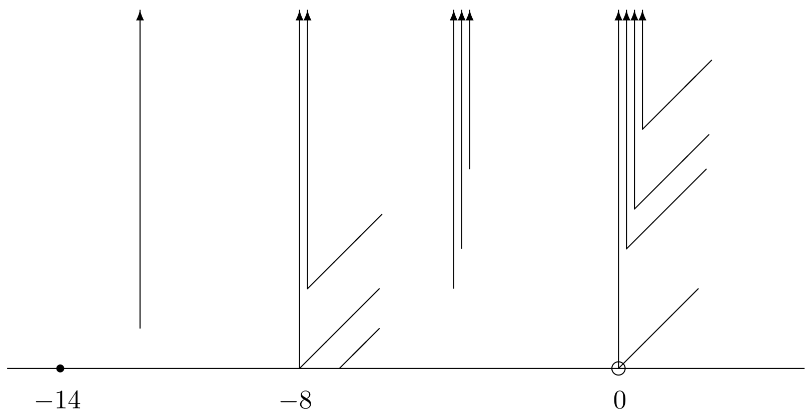
$$\mathrm{Ext}_A(H^*((DY)^{(-1)})) \longrightarrow \mathrm{Ext}_A(H^*DY) \longrightarrow \mathrm{Ext}_A(H^*S^0) \xrightarrow{\delta}$$



The desired class 2ν is denoted by A in the diagram, and is the image of the circled class. The class ν, indicated by B, maps to B' in the exact sequence.

(c.) Let DX denote the S-dual of X, with 10 cells, in dimensions -14 up to 0. Then $[\Sigma^i X, bo] \approx \pi_i(DX \wedge bo)$, and this can be computed by the ASS with $E_2 = \mathrm{Ext}_{A_1}(H^*DX)$. The A_1-structure of $H^*(DX)$ is easily seen, and the Ext_{A_1}-calculation easily made, giving the result in Diagram 2.3 in dimension < 4. There are clearly no possible differentials, and our desired map is detected in filtration 0 by the circled element.

Diagram 2.3. $\mathrm{Ext}_{A_1}(H^*DX)$ in $t-s<4$



(d.) There is a commutative diagram in which horizontal and vertical sequences are fiber sequences.

$$\begin{tikzcd}
S^0 \wedge \mathrm{tmf} \arrow[r, "="] \arrow[d] & \mathrm{tmf} \arrow[d] & & \\
X \wedge \mathrm{tmf} \arrow[r, "\widetilde{f}"] \arrow[d] & bo \arrow[r] \arrow[d] & C \arrow[d, "="'] & \\
(X/S^0) \wedge \mathrm{tmf} \arrow[r, "\widehat{f}"] & bo/\,\mathrm{tmf} \arrow[r] & C \arrow[r] & \Sigma(X/S^0) \wedge \mathrm{tmf}
\end{tikzcd}$$

The restriction of $\widetilde{f}$ to the 4-skeleton is $S^0 \cup_{2\nu} e^4 \to S^0 \cup_\nu e^4$ of degree 1 on the bottom cell. Thus $\widehat{f}$ has degree 2 on its bottom 4-cell. Let $\overline{A_2//A_1}$ denote the kernel of the augmentation of $A_2//A_1$. The A-module $H^*(bo/\,\mathrm{tmf})$ is isomorphic to $A \otimes_{A_2} \overline{A_2//A_1}$, and the A_2-module $\overline{A_2//A_1}$ has basis

$$\{g_4,\ \mathrm{Sq}^2 g_4,\ \mathrm{Sq}^3 g_4,\ \mathrm{Sq}^4 \mathrm{Sq}^2 g_4,\ \mathrm{Sq}^4 \mathrm{Sq}^3 g_4,\ \mathrm{Sq}^6 \mathrm{Sq}^3 g_4,\ \mathrm{Sq}^4 \mathrm{Sq}^6 \mathrm{Sq}^3 g_4\}. \tag{2.2}$$

Thus $(\widehat{f})^* = 0$, and, since $X/S^0 = \Sigma^4 Y$, there is a short exact sequence of A-modules

$$0 \to H^*(\Sigma^5 Y \wedge \mathrm{tmf}) \to H^*(C) \to A \otimes_{A_2} \overline{A_2//A_1} \to 0,$$

and $\mathrm{Sq}^1 g_4 \neq 0$ in H^*C. We conclude that $H^*(C)$ is an extended cyclic A_2-module on a 4-dimensional generator, with nonzero operations being those in (2.2) and Sq^1 and the operations listed in (2.1) applied to Sq^1. One easily checks that this A_2-module equals $A_2/(\mathrm{Sq}^4, \mathrm{Sq}^5 \mathrm{Sq}^1)$, and so the A-module $H^*(C)$ is as claimed.

(e.) We will prove there is a map $m' : X \wedge X \to bo$ extending the inclusion of the bottom cell and that when followed by the map $bo \to C$ of part (d), the composite is trivial. Thus by the definition of C, m' factors through a map $m : X \wedge X \to X \wedge \mathrm{tmf}$ extending the inclusion of the bottom cell. Smashing this twice with tmf and following by two multiplications of tmf yield the desired product on $X \wedge \mathrm{tmf}$.

We construct the dual of m', an element of $\pi_0(DX \wedge DX \wedge bo)$. The E_2-term of the ASS converging to $\pi_*(DX \wedge DX \wedge bo)$ is $\mathrm{Ext}_{A_1}(H^*(DX \wedge DX))$. The A_1-structure of $H^*(DX)$ is easily seen and Ext_{A_1} of tensor products of the summands is easily computed, as, for example, in [4]. We obtain that in the vicinity of $t-s=0$, the chart has a copy of bo_* beginning in position (0,0) and 15 additional copies of bo_* beginning in positions $(0,s)$ for $3 \le s \le 12$. The groups in $t-s=-1$, i.e. corresponding to π_{-1}, are all 0. Thus there are no possible differentials from $t-s=0$ in the ASS, and we deduce the existence of our map $S^0 \to DX \wedge DX \wedge bo$, whose dual is m'.

Next we compute the ASS for $\pi_*(DX \wedge DX \wedge C)$. Let Y be as in part (a). Then $DX = \Sigma^{-4}DY \cup_{2\nu} e^0$, and so $H^*(DX) \approx H^*(\Sigma^{-4}DY) \oplus H^*(S^0)$ as A-modules. Thus the ASS converging to $\pi_*(DX \wedge DX \wedge C)$ has

$$\begin{aligned} E_2 \quad \approx \quad & \mathrm{Ext}_A(H^*(\Sigma^{-4}DY \wedge DY) \otimes A/(\mathrm{Sq}^4, \mathrm{Sq}^{5,1})) \\ & \oplus \mathrm{Ext}_A(H^*(DY) \otimes A/(\mathrm{Sq}^4, \mathrm{Sq}^{5,1})) \\ & \oplus \mathrm{Ext}_A(H^*(DY) \otimes A/(\mathrm{Sq}^4, \mathrm{Sq}^{5,1})) \oplus \mathrm{Ext}_A(\Sigma^4 A/(\mathrm{Sq}^4, \mathrm{Sq}^{5,1})). \end{aligned}$$

Note that the bottom class of DY is in grading -10. We can use Bruner's software to see that each of these Ext groups is 0 in $t-s=0$. For example,

$$\mathrm{Ext}_A(H^*(\Sigma^{-4}DY \wedge DY) \otimes A/(\mathrm{Sq}^4, \mathrm{Sq}^{5,1}))$$

has 15 $\mathbb{Z}$-towers in the (-3)-stem, beginning in filtrations 2 through 8. It is 0 in stems -2, -1, and 0, and then in the 1-stem has 21 $\mathbb{Z}$-towers, on each of which η and η^2 are nonzero.

Thus $\pi_0(DX \wedge DX \wedge C) = 0$ and hence $[X \wedge X, C] = 0$. Therefore the map $X \wedge X \xrightarrow{m'} bo \to C$ is trivial, implying the result by the argument of the first paragraph of the proof.

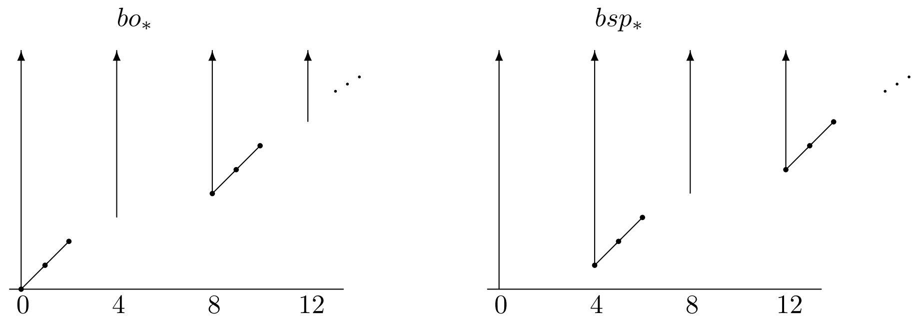
We must also show that the unital property of a ring spectrum is satisfied; i.e., that the composite

$$S^0 \wedge X \wedge \mathrm{tmf} \xrightarrow{\iota \wedge 1} X \wedge \mathrm{tmf} \wedge X \wedge \mathrm{tmf} \xrightarrow{\mu} X \wedge \mathrm{tmf}$$

is homotopic to the identity. This follows because $H^*(X \wedge \mathrm{tmf}) \approx A \otimes_{A_2} (H^*(S^0) \oplus H^*(\Sigma^4 Y))$, and a map sending the bottom cell of $X \wedge \mathrm{tmf}$ by degree 1 does the same for the bottom cell of Y, since it is attached by 2ν. Here we also use that H^*Y is a cyclic A_2-module. $\square$

The main step toward describing $\pi_*(X \wedge \mathrm{tmf})$ is, because of 2.1(d), the Ext calculation in Theorem 2.2. This calculation was first made in [7], but our approach will be somewhat different. Our approach will be useful in performing other related Ext calculations. The description is in terms of bo_* and bsp_*, which are depicted in Diagram 2.4.

Diagram 2.4. bo_* and bsp_*



We will denote by $a_{x,y}$ an element of $\mathrm{Ext}^{y,x+y}$. This corresponds to the usual (x, y) components in an ASS. There are standard elements h_1, h_2, and v_2^4 of (x, y)-grading $(1, 1)$, $(3, 1)$, and $(24, 4)$, respectively. Here and throughout, $R[a]\langle b_1, \ldots, b_r\rangle$ denotes a free module over a polynomial algebra $R[a]$ with basis $\{b_1, \ldots, b_r\}$.

Theorem 2.2. *As a bigraded abelian group,* $\mathrm{Ext}_A^{*,*}(A/A(\mathrm{Sq}^4, \mathrm{Sq}^5\,\mathrm{Sq}^1), \mathbb{Z}_2)$ *is isomorphic to*

$$\begin{array}{rl} & \mathbb{Z}_2[v_2^8]\langle a_{0,0},\ h_2a_{0,0},\ a_{14,2},\ h_1a_{14,2},\ h_2a_{14,2},\ a_{31,5},\ h_2a_{31,5},\ a_{39,7}\rangle \\ \oplus & \ker(bo_*[v_2^4]\langle a_{5,1}, a_{21,3}\rangle \to \mathbb{Z}_2[v_2^8]\langle a_{21,3}\rangle) \\ \oplus & bsp_*[v_2^4]\langle a_{9,2}, a_{17,4}\rangle. \end{array}$$

Proof. By the Change-of-Rings Theorem, it is equivalent to compute

$$\mathrm{Ext}_{A_2}(A_2/A_2(\mathrm{Sq}^4, \mathrm{Sq}^5\,\mathrm{Sq}^1), \mathbb{Z}_2).$$

One can verify that there is an exact sequence of A_2-modules:

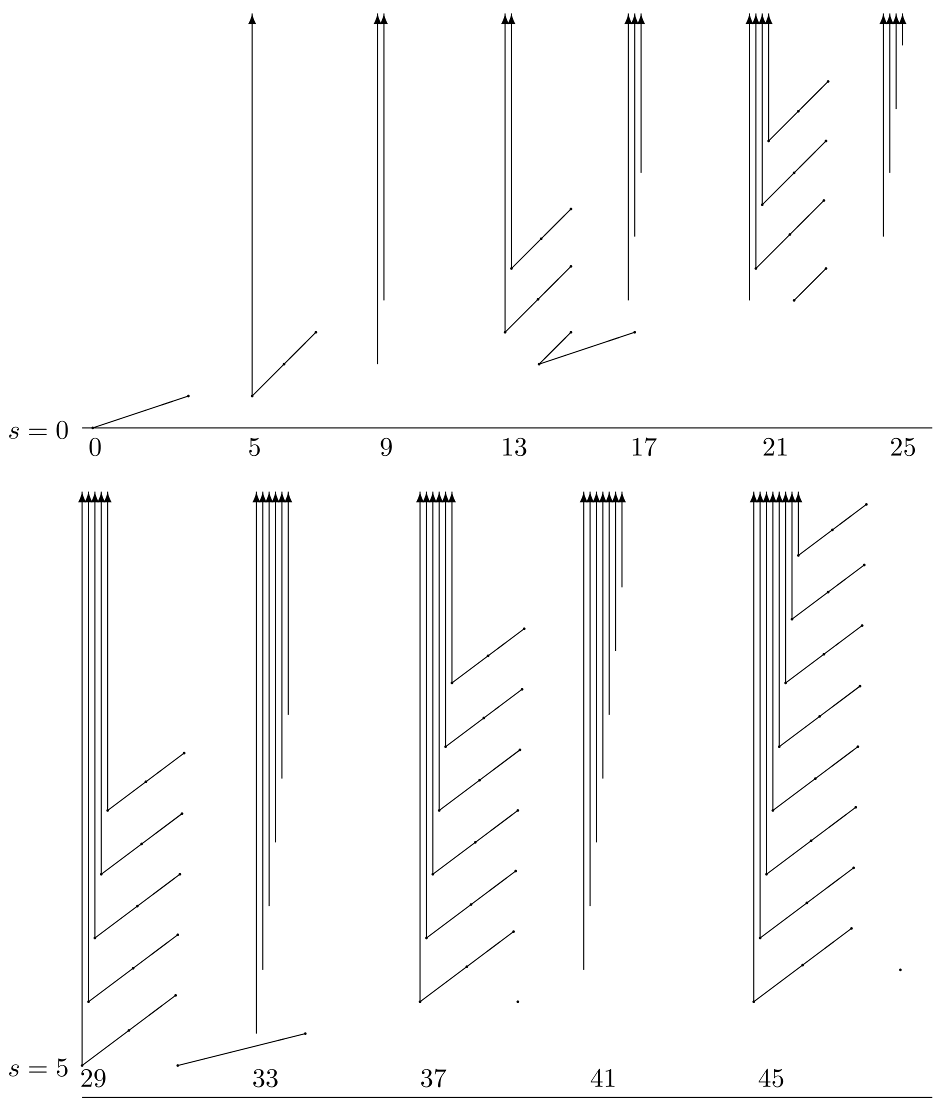
$$\begin{array}{lll} 0 & \leftarrow & A_2/(\mathrm{Sq}^4, \mathrm{Sq}^{5,1}) \xleftarrow{d_0} A_2 \xleftarrow{d_1} \Sigma^4 A_2 \oplus \Sigma^6 A_2//A_1 \\ & \xleftarrow{d_2} & \Sigma^{11} A_2/(\mathrm{Sq}^1, \mathrm{Sq}^5) \oplus \Sigma^{16} A_2 \\ & \xleftarrow{d_3} & \Sigma^{18} A_2/(\mathrm{Sq}^3) \oplus \Sigma^{20} A_2 \xleftarrow{d_4} (\Sigma^{25} A_2 \oplus \Sigma^{26} A_2)/(\mathrm{Sq}^1\, I_{25}, \mathrm{Sq}^3\, I_{25} + \mathrm{Sq}^2\, I_{26}) \\ & \xleftarrow{d_5} & \Sigma^{34} A_2//A_1 \oplus \Sigma^{36} A_2/(\mathrm{Sq}^3) \xleftarrow{d_6} \Sigma^{40} A_2 \\ & \xleftarrow{d_7} & \Sigma^{46} A_2/(\mathrm{Sq}^3) \oplus \Sigma^{52} A_2//A_1 \xleftarrow{d_8} \Sigma^{56} A_2/(\mathrm{Sq}^4, \mathrm{Sq}^{5,1}) \leftarrow 0 \end{array} \qquad (2.3)$$

with

$$\begin{array}{rcl} d_1(I_4) & = & \mathrm{Sq}^4 \\ d_1(I_6) & = & \mathrm{Sq}^5\,\mathrm{Sq}^1 \\ d_2(I_{11}) & = & \mathrm{Sq}^7\, I_4 \\ d_2(I_{16}) & = & (\mathrm{Sq}^{6,6} + \mathrm{Sq}^{7,5}) I_4 + \mathrm{Sq}^{4,6}\, I_6 \\ d_3(I_{18}) & = & \mathrm{Sq}^2\, I_{16} + \mathrm{Sq}^7\, I_{11} \\ d_3(I_{20}) & = & \mathrm{Sq}^4\, I_{16} + \mathrm{Sq}^{6,3}\, I_{11} \end{array}$$

$$
\begin{aligned}
d_4(I_{25}) &= \mathrm{Sq}^7\, I_{18} + \mathrm{Sq}^5\, I_{20}\\
d_4(I_{26}) &= \mathrm{Sq}^{7,1}\, I_{18} + \mathrm{Sq}^6\, I_{20}\\
d_5(I_{34}) &= \mathrm{Sq}^{2,7}\, I_{25}\\
d_5(I_{36}) &= (\mathrm{Sq}^{5,6} + \mathrm{Sq}^{6,5}) I_{25} + \mathrm{Sq}^{4,6}\, I_{26}\\
d_6(I_{40}) &= \mathrm{Sq}^4\, I_{36} + \mathrm{Sq}^6\, I_{34}\\
d_7(I_{46}) &= \mathrm{Sq}^6\, I_{40}\\
d_7(I_{52}) &= \mathrm{Sq}^{7,5}\, I_{40}\\
d_8(I_{56}) &= \mathrm{Sq}^4\, I_{52} + (\mathrm{Sq}^{4,6} + \mathrm{Sq}^{6,3,1}) I_{46}.
\end{aligned}
$$

Diagram 2.5. $\mathrm{Ext}_A(A/(\mathrm{Sq}^4, \mathrm{Sq}^{5,1}))$ through degree 48



For $0 \le i \le 7$, let C_i denote the A_2-module which is the domain of d_i. Because the domain of d_8 is Σ^{56} of the beginning module, the exact sequence could be continued periodically with the $\Sigma^{56}A_2/(\mathrm{Sq}^4, \mathrm{Sq}^{5,1})$ removed, and $C_{i+8} \approx \Sigma^{56}C_i$. There is a spectral sequence building $\mathrm{Ext}(A_2/(\mathrm{Sq}^4, \mathrm{Sq}^{5,1}))$ from $\bigoplus_{i\ge 0} \phi^i \,\mathrm{Ext}(\Sigma^{-i}C_i)$, where ϕ^i increases filtration by i. In this proof, Ext means Ext_{A_2}.

Of the modules that appear in C_i, $\mathrm{Ext}(A_2)$ is just $\mathbb{Z}_2$ in $(0,0)$, $\mathrm{Ext}(A_2/\!/A_1)$ is bo_*, $\mathrm{Ext}(A_2/(\mathrm{Sq}^1, \mathrm{Sq}^5))$ is bsp_*, $\mathrm{Ext}(A_2/(\mathrm{Sq}^3))$ is $\mathrm{Ext}(A_2) \oplus \phi\,\mathrm{Ext}(\Sigma^2 bsp_*)$, and $\mathrm{Ext}((A_2 \oplus \Sigma^1 A_2)/(\mathrm{Sq}^1 I_0, \mathrm{Sq}^3 I_0 + \mathrm{Sq}^2 I_1))$ is $\phi^{-1}(\ker(bo_* \to \mathbb{Z}_2))$. When these are put together, one obtains exactly the claim of the theorem. There can be no differentials because differentials are h_i-natural. The differentials would go from position (x,y) of $\phi^i\,\mathrm{Ext}(\Sigma^{-i}C_i)$ to position $(x-1, y+1)$ of $\phi^{i+r}\,\mathrm{Ext}(\Sigma^{-(i+r)}C_{i+r})$. In Diagram 2.5, we depict this chart for $x \le 48$, to show the impossibility of differentials in both this SS converging to Ext, and in an ASS to be considered later. Note that the $\mathbb{Z}_2$ in the 48-stem is v_2^8 times the initial $\mathbb{Z}_2$. □

The following result is an easy consequence of Theorems 2.1 and 2.2.

Theorem 2.3. *There is an isomorphism of graded abelian groups*

$$\begin{aligned}\pi_*(X \wedge \mathrm{tmf}) \quad \approx \quad & bo_*[v_2^4]\langle v_1 v_2\rangle \oplus \ker(bo_*[v_2^4] \to \mathbb{Z}_2[v_2^8]\langle v_2^4\rangle) \\ & \oplus bsp_*[v_2^4]\langle 2v_2^2, 2v_1v_2^3\rangle \\ & \oplus \mathbb{Z}_2[v_2^8]\langle \nu,\, \nu^2,\, x,\, \eta x,\, \nu x,\, x^2,\, \eta x^2,\, y\rangle,\end{aligned}$$

where the (homotopy group, Adams filtration) of elements is $(2,1)$ *for* v_1, $(6,1)$ *for* v_2, $(17,3)$ *for* x, *and* $(42,8)$ *for* y.

Proof. We use the exact sequence

$$\to \pi_*(\Sigma^{-1}C) \to \pi_*(X \wedge \mathrm{tmf}) \to \pi_*(bo) \to \tag{2.4}$$

from Theorem 2.1(d). The morphism $H^*(C) \to H^*(bo)$ sends the bottom class of $H^*(C)$ to $\mathrm{Sq}^4 \iota_0$, and hence $\mathrm{Sq}^4 \iota_0 = 0$ in $H^*(X \wedge \mathrm{tmf})$. Thus the class in $(0,0)$ in Diagram 2.5 should be placed in position $(3,1)$ in a chart for a $\pi_*(X \wedge \mathrm{tmf})$, and **a chart for a spectral sequence converging to** $\pi_*(X \wedge \mathrm{tmf})$ **can be formed from** bo_* **of Diagram 2.4 together with Diagram 2.5 shifted by** $(3,1)$ **units**. We emphasize this because $\pi_*(X \wedge \mathrm{tmf})$ is our main item of interest, but we don't want to draw the chart again. By h_i-naturality there are no differentials or extensions, and so the chart depicts $\pi_*(X \wedge \mathrm{tmf})$. Equivalently, the sequence (2.4) is short exact.

The classes x and y of the theorem correspond to the lowest classes in the 14- and 39-stems in Diagram 2.5. The names v_1v_2, $2v_2^2$, and $2v_1v_2^3$ which we give to certain generators are, at least at this point, meant to only describe stem and filtration. □

Our next result simplifies the bo_*-bsp_*-part of this description and also incorporates as much as we can say about the ring structure from our approach. Our limitation is that our approach can only give the ring structure of $\pi_*(X \wedge \mathrm{tmf})$ up to elements of higher filtration in the Adams-type spectral sequence we have been using. Note that we say "Adams-type" because we have elevated the filtrations of the part from C by 1 compared to an actual ASS. The reason that we can't say any better than "up to higher filtration" is, first of all the usual limitation of an ASS, and secondly that our multiplication of $X \wedge \mathrm{tmf}$ is only defined up to maps of higher filtration. It seems that such deviations would change the product structure in $\pi_*(X \wedge \mathrm{tmf})$. For example, the product of classes that we call $2v_2^2$ and $2v_1^4v_2^2$ (so-called because of their image in BP_*; note that these classes are generators—the elements without the factor 2 are not present in $\pi_*(X \wedge \mathrm{tmf})$) would naturally be $4v_1^4v_2^4$, an element which would be divisible by 4 in $\pi_*(X \wedge \mathrm{tmf})$. However, we cannot assert that this product of generators is divisible by 4; it might equal, for example, $4v_1^4v_2^4 + v_1^{16}$.

Theorem 2.4. *There is an isomorphism of graded abelian groups*

$$\pi_*(X \wedge \mathrm{tmf}) \approx K \oplus \mathbb{Z}_2[v_2^8]\langle \nu,\, \nu^2,\, x,\, \eta x,\, \nu x,\, x^2,\, \nu x^2,\, v_1v_2x^2\rangle,$$

where

$$K = \ker(R \to \mathbb{Z}_2[v_2^8]\langle v_2^4\rangle)$$

with R the subring of $\mathbb{Z}[v_1, v_2, \eta]/(2\eta, \eta^3)$ generated by $2v_1^2$, v_1^4, v_1v_2, $2v_2^2$, and v_2^4. The isomorphism is, up to elements of higher filtration, an isomorphism of rings, with the additional relations $v_1^4x = \eta v_1^3v_2^3$, $v_1v_2x = \eta v_2^4$, $x^3 = \nu v_2^8$, and $x^7 = 0$.

Stems of elements are as in Theorem 2.3. Note that $x^7 = 0$, not just up to elements of higher filtration, as it lies in a zero group.

Proof. It is not difficult to check that this description is consistent as an Adams-filtered graded abelian group with the description in Theorem 2.3. We must establish various product formulas.

First we show that x^2 is nonzero, corresponding to $a_{31,5}$ in Theorem 2.2. Note that

$$\mathrm{im}(\pi_*(\Sigma^{-1}C) \to \pi_*(X \wedge \mathrm{tmf})) = h_2 \cdot \mathrm{Ext}_{A_2}(A_2/(\mathrm{Sq}^4, \mathrm{Sq}^{5,1})).$$

Thus the product in $\pi_*(X \wedge \mathrm{tmf})$ on elements in the image from $\pi_*(\Sigma^{-1}C)$ can be considered as

$$\begin{aligned} &\mathrm{Ext}_{A_2}^{s,t}(A_2/(\mathrm{Sq}^4, \mathrm{Sq}^{5,1})) \otimes \mathrm{Ext}_{A_2}^{s',t'}(A_2/(\mathrm{Sq}^4, \mathrm{Sq}^{5,1})) \\ \to\quad &\mathrm{Ext}_{A_2}^{s+s'+1,t+t'+4}(A_2/(\mathrm{Sq}^4, \mathrm{Sq}^{5,1})) \end{aligned} \tag{2.5}$$

with $\alpha \otimes \beta \mapsto h_2\alpha\beta$. With $x \in \mathrm{Ext}_{A_2}^{2,16}(A_2/(\mathrm{Sq}^4, \mathrm{Sq}^{5,1}))$, the image of $x \otimes x$ is in $\mathrm{Ext}_{A_2}^{5,36}(A_2/(\mathrm{Sq}^4, \mathrm{Sq}^{5,1}))$. We wish to show it is nonzero. Thus we want the Yoneda product of h_2x with x.

Using the minimal "resolution" (2.3), we consider the following diagram:

$$\begin{tikzcd}
C_3 \arrow[d, "h_2 x"'] & C_4 \arrow[l] \arrow[d, "f_4"'] & C_5 \arrow[l] \arrow[d, "f_5"'] \\
\Sigma^{20} C_0 & \Sigma^{20} C_1 \arrow[l] & \Sigma^{20} C_2 \arrow[l] \arrow[d, "x"'] \\
 & & \Sigma^{36} \mathbb{Z}_2 .
\end{tikzcd}$$

Although the modules in (2.3) are not projective, we can still find enough preimages to compute many Yoneda products. Since $h_2 x \in \mathrm{Ext}_{A_2}^{3,20}(A_2/(\mathrm{Sq}^4, \mathrm{Sq}^{5,1}))$, the relevant parts are

$$\begin{tikzcd}
\Sigma^{20} A_2 \arrow[d, "1"'] & (\Sigma^{25} A_2 \oplus \Sigma^{26} A_2)/R \arrow[l, "{\mathrm{Sq}^5, \mathrm{Sq}^6}"'] \arrow[d, "f_4"'] & \Sigma^{36} A_2 \arrow[l, "{\mathrm{Sq}^{5,6} + \mathrm{Sq}^{6,5}, \mathrm{Sq}^{4,6}}"'] \arrow[d, "f_5"'] \\
\Sigma^{20} A_2 & \Sigma^{24} A_2 \oplus \Sigma^{26} A_2 /\!/ A_1 \arrow[l, "{\mathrm{Sq}^4, \mathrm{Sq}^{5,1}}"'] & \Sigma^{36} A_2 \arrow[l, "{\mathrm{Sq}^{6,6} + \mathrm{Sq}^{7,5}, \mathrm{Sq}^{4,6}}"'] \arrow[d] \\
 & & \Sigma^{36} \mathbb{Z}_2 .
\end{tikzcd}$$

We find that $f_4(\iota_{25}) = \mathrm{Sq}^1 \iota_{24}$ and $f_4(\iota_{26}) = \mathrm{Sq}^2 \iota_{24} + \iota_{26}$, and then that f_5 is the identity.

A similar argument works to show $x^3 = \nu v_2^8$. Relations for x^4, x^5, and x^6 can be deduced from the stated relations.

The elements ηx, x^2, and y generate the three occurrences of $A_2/(\mathrm{Sq}^3)$ in the resolution in the proof of Theorem 2.2. The bsp_*'s on $a_{17,4}$, $v_2^4 a_{9,2}$, and $v_2^4 a_{17,4}$ in Theorem 2.3 are obtained from Ext_{A_2} of these three $A_2/(\mathrm{Sq}^3)$'s by omitting the initial $\mathbb{Z}_2$. This implies that $v_1^4 \eta x = \eta^2 v_1^3 v_2^3$, $v_1^4 x^2 = \eta^2 v_1^2 v_2^6$, and $v_1^4 y = \eta^2 v_1^3 v_2^7$. One of our relations is obtained by dividing the first of these by η, while the latter two imply that $y = v_1 v_2 x^2$. This is valid because η and v_1^4 act injectively in the relevant stems.

The elements which we call $v_1^{4i} v_2^{8j} \cdot (2v_2^2)^e$ with $1 \le e \le 3$ in $E_2^{*,*}(H^* X)$ are in the image of the ring map from $\mathrm{Ext}_{A_2}(\mathbb{Z}_2)$, and so products among them are as we claim because of the ring structure of $\mathrm{Ext}_{A_2}(\mathbb{Z}_2)$. That the products of $(v_1 v_2)^i$ with $2v_2^2$ are as claimed can be proved by a Yoneda product argument with the element $2v_2^2$ of $\mathrm{Ext}_{A_2}^{3,15}(\mathbb{Z}_2)$. To verify this using a minimal resolution of $A/(\mathrm{Sq}^4, \mathrm{Sq}^{5,1})$, one should expand the efficient resolution used in the proof of Theorem 2.2 to use only A_2 and $A_2/(\mathrm{Sq}^1)$ (and not the more efficient $A_2/\!/A_1$ and $A/(\mathrm{Sq}^3)$). This produces some additional $\mathrm{Sq}^4 \mathrm{Sq}^6$ terms in the resolution. The following not-quite-commutative diagram of not-quite-exact sequences shows the most relevant terms in the morphism from a portion of the resolution built on $(v_1 v_2)^i$ to the most relevant terms of the resolution of $\mathbb{Z}_2$, and can be used to establish that the Yoneda product of the element that we call $(v_1 v_2)^i$ followed by the element that we call

$2v_2^2$ equals the element that we call $2v_1^i v_2^{i+2}$.

$$\begin{array}{ccccccc}
A_2/(\mathrm{Sq}^1) & \xleftarrow{\mathrm{Sq}^{4,6}} & \Sigma^{10}A_2 & \xleftarrow{\mathrm{Sq}^2} & \Sigma^{12}A_2 & \xleftarrow{\mathrm{Sq}^3} & \Sigma^{15}A_2 \\
{\scriptstyle 1}\big\downarrow & & {\scriptstyle \mathrm{Sq}^{4,2}}\big\downarrow & & {\scriptstyle \mathrm{Sq}^4}\big\downarrow & & {\scriptstyle 1}\big\downarrow \\
A_2/(\mathrm{Sq}^1) & \xleftarrow{\mathrm{Sq}^4} & \Sigma^4 A_2 & \xleftarrow{\mathrm{Sq}^4} & \Sigma^8 A_2 & \xleftarrow{\mathrm{Sq}^7} & \Sigma^{15}A_2.
\end{array}$$

For example, when $i = 2$, the top row of this diagram corresponds to elements in $(13, 3)$, $(22, 4)$, $(23, 5)$, and $(25, 6)$ in Diagram 2.5.

To see that the elements that we call $v_1^i v_2^i$ multiply by one another as the notation suggests, we consider the morphism of minimal resolutions inducing (2.5). Let

$$\mathbf{C} : C_0 \leftarrow C_1 \leftarrow \cdots \qquad (\text{resp. } \mathbf{D} : D_0 \leftarrow D_1 \leftarrow \cdots)$$

be a minimal A_2-resolution of $\Sigma^3 A_2/(\mathrm{Sq}^4, \mathrm{Sq}^{5,1})$ (resp. $\Sigma^2 \ker(A_2 \to A_2/(\mathrm{Sq}^4, \mathrm{Sq}^{5,1}))$). Then (2.5) is induced by a morphism $\mathbf{D} \xrightarrow{\psi} \mathbf{C} \otimes \mathbf{C}$. The class which we call $v_1^i v_2^i$ is dual to a generator $\alpha_i \in (C_{2i-1})_{10i-1}$ and to a generator $\beta_i \in (D_{2i-2})_{10i-2}$.

First we show that the square of our $v_1 v_2$ class equals our class called $v_1^2 v_2^2$. The relevant parts are that $C_1 \xrightarrow{d_1} C_0$ has $C_0 = \Sigma^3 A_2$, $C_1 = \Sigma^7 A_2 \oplus \Sigma^9 A_2//A_0$, with $d_1(\iota_7) = \mathrm{Sq}^4\, \iota_3$ and $d_1(\iota_9) = \mathrm{Sq}^{5,1}\, \iota_3$, while the relevant part of $\mathbf{D}$ is

$$\Sigma^{18}A_2 \xrightarrow{d_2} \Sigma^{13}A_2//A_0 \xrightarrow{d_1} \Sigma^6 A_2$$

with $d_2(\iota_{18}) = \mathrm{Sq}^5\, \iota_{13}$ and $d_1(\iota_{13}) = \mathrm{Sq}^7\, \iota_6$. In the commutative diagram of exact sequences

$$\begin{array}{ccccc}
D_2 & \xrightarrow{d_2} & D_1 & \xrightarrow{d_1} & D_0 \\
{\scriptstyle f_2}\big\downarrow & & {\scriptstyle f_1}\big\downarrow & & {\scriptstyle f_0}\big\downarrow \\
C_1 \otimes C_1 & \longrightarrow & C_0 \otimes C_1 \oplus C_1 \otimes C_0 & \longrightarrow & C_0 \otimes C_0
\end{array}$$

we must have

$$\begin{aligned}
f_0(\iota_6) &= \iota_3 \otimes \iota_3 \\
f_1(\iota_{13}) &= (1+T)(\iota_3 \otimes \mathrm{Sq}^3\, \iota_7 + \mathrm{Sq}^1\, \iota_3 \otimes (\mathrm{Sq}^2\, \iota_7 + \iota_9) \\
&\quad + \mathrm{Sq}^2\, \iota_3 \otimes \mathrm{Sq}^1\, \iota_7 + \mathrm{Sq}^3\, \iota_3 \otimes \iota_7) \\
f_2(\iota_{18}) &= \iota_9 \otimes \iota_9,
\end{aligned}$$

implying the result. Here $T(x \otimes y) = y \otimes x$. Note that the important term here was the ι_9, which occurred because of the difference between Sq^6 and $\mathrm{Sq}^2\,\mathrm{Sq}^4$.

Now we show that the class which we call $v_1^2 v_2^2$ times the class which we call $v_1^i v_2^i$ equals the class that we call $v_1^{i+2} v_2^{i+2}$. This, with the result of the preceding paragraph, implies that all powers of $v_1 v_2$ are as claimed.

The class which we call $2v_1^i v_2^{i+2}$ is dual to a generator $\gamma_{i+1} \in (C_{2i+2})_{10i+14}$ and to a generator $\delta_{i+1} \in (D_{2i+1})_{10i+13}$. In the resolutions, $d(\alpha_{i+1}) \equiv \mathrm{Sq}^5 \gamma_i$ and $d(\beta_{i+1}) \equiv \mathrm{Sq}^5 \delta_i$ mod other terms, where α_{i+1} and β_{i+1} are dual to $v_1^{i+1} v_2^{i+1}$ as above. Because the product of our $2v_2^2$ class and our $v_1^i v_2^i$ class equals our $2v_1^i v_2^{i+2}$ class, as was shown earlier, we conclude that in $\mathbf{D} \xrightarrow{\psi} \mathbf{C} \otimes \mathbf{C}$, $\psi(\delta_{i+1}) = \gamma_1 \otimes \alpha_i$ plus other terms. Thus, modulo other terms, we have

$$d(\psi(\beta_{i+2})) = \psi(d(\beta_{i+2})) \equiv \mathrm{Sq}^5 \gamma_1 \otimes \alpha_i$$

and

$$d(\alpha_2 \otimes \alpha_i) \equiv \mathrm{Sq}^5 \gamma_1 \otimes \alpha_i,$$

from which we conclude $\psi(\beta_{i+2}) = \alpha_2 \otimes \alpha_i$, which is equivalent to our claim.

Now that we know that the classes which we have named by monomials in v_1 and v_2 multiply consistently with these names, we can deduce the final relation $v_1 v_2 x = \eta v_2^4$ from $v_1^4 x = \eta v_1^3 v_2^3$ by multiplying the latter by $v_1 v_2$ and then dividing by v_1^4. $\square$

3 An 8-Cell Model Related to $\mathrm{TMF}(3)$

In [10, §7], another connective model for $\mathrm{TMF}(3)$ is discussed, which is $Z \wedge \mathrm{tmf}$, where Z is a certain 8-cell complex. Although $Z \wedge \mathrm{tmf}$ is not a ring spectrum, it is still true that $v_2^{-1} Z \wedge \mathrm{tmf} \simeq \mathrm{TMF}(3)$. The importance of this model is primarily that the dimensions of the cells of Z allow one to construct a map $Z \to \mathrm{TMF}(3)$ thanks to certain homotopy groups of $\mathrm{TMF}(3)$ being 0. The other models are then related to $\mathrm{TMF}(3)$ via the Z-model. In this section, we provide some additional details to the sketch given in [10].

Let $X_7 = S^0 \cup_\nu e^4 \cup_\eta e^6 \cup_2 e^7$ be as in the proof of Theorem 2.1.a, and let $X_{421} = \Sigma^7 D X_7 = S^0 \cup_2 e^1 \cup_\eta e^3 \cup_\nu e^7$. The following lemma was proved in [10], using the ASS of X_{421} through dimension 13.

Lemma 3.1. *The map $S^6 \xrightarrow{\nu^2} S^0 \hookrightarrow X_{421}$ extends to a map $\Sigma^6 X_7 \to X_{421}$.*

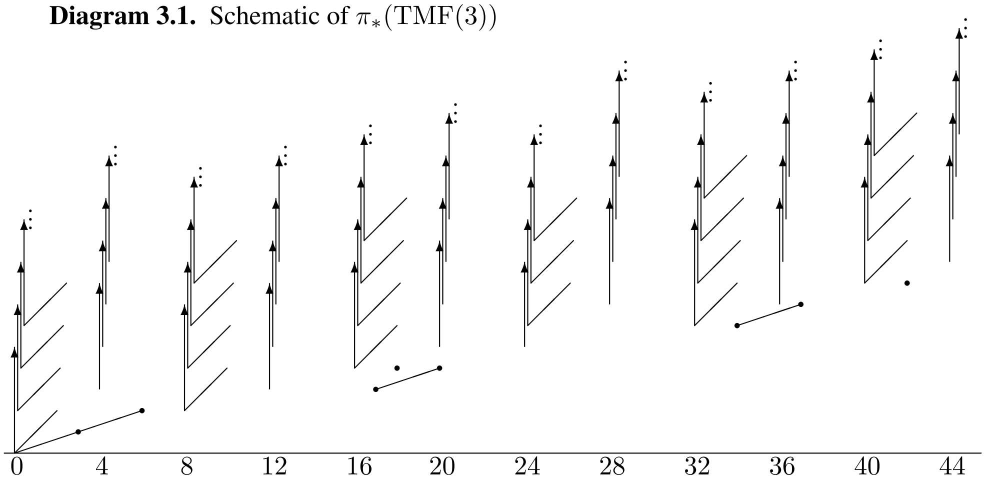
Definition 3.1. Let Z denote the mapping cone of the map $\Sigma^{23} X_7 \to \Sigma^{17} X_{421}$ obtained from Lemma 3.1.

Proposition 3.1. *There is an element $x \in \pi_{17}(\mathrm{TMF}(3))$ of order 2 which is not divisible by η, and a map $Z \to \mathrm{TMF}(3)$ which extends this map x.*

Proof. This is where we need input from the theory of topological modular forms. In [10], a 48-periodic ring spectrum $\mathrm{TMF}(3)$ (called there $\mathrm{TMF}(\Gamma_0(3))$) is defined and its homotopy groups calculated, using a spectral sequence defined using results about elliptic curves. Their result ([10, 4.1]), localized at 2, is a v_2^8-inverted version of our Theorem 2.4, but with their ring structure being precise, not just up to elements of higher filtration. We emphasize that our Theorem 2.4 and [10, 4.1] are totally independent calculations. Our 2.4 uses only homotopy theory (and the existence of a ring spectrum tmf with $H^*(\mathrm{tmf}) \approx A//A_2$), while [10, 4.1] uses the Weierstrass curve. We will realize this isomorphism of homotopy groups by a map of spectra later in Corollary 3.1 and Theorem 3.2, but for now we mean just to refer to the result of [10, 4.1] without actually stating it.

A schematic of $\pi_*(\mathrm{TMF}(3))$ from the ASS viewpoint is given in Diagram 3.1. Each collection of four closely-spaced towers represents infinitely many such towers in the same stem. If the lowest of these begins in filtration s, then there are such towers in filtration $s+2i$ for all $i \geq 0$, with a slight exception in dimension 24. The names of the bottom generators are 1, $2v_1^2$, v_1v_2, $2v_2^2$, $v_1^2v_2^2$, $2v_1v_2^3$, $2v_2^4$, $2v_1^2v_2^4$, $v_1v_2^5$, $2v_2^6$, $v_1^2v_2^6$, and $2v_1v_2^7$. The name of the generator in filtration $s+2i$ is $v_1^{3i}v_2^{-i}$ times that of the bottom generator, except that in dimension 24, we have $2v_2^4$ and $v_1^{3i}v_2^{4-i}$ for all $i > 0$. The eight $\mathbb{Z}_2$'s along the bottom, indicated by a solid dot, occur only once, in the indicated filtration. Because of period 48, Diagram 3.1 is a complete depiction of $\pi_*(\mathrm{TMF}(3))$.

Diagram 3.1. Schematic of $\pi_*(\mathrm{TMF}(3))$



The extension of x over Z occurs because $2x = 0$ and $\pi_i(\mathrm{TMF}(3)) = 0$ for $i = 19$, 23, 27, 29, and 30, showing that the obstructions to extending over the remaining cells are all 0. □

We illustrate the relationship between Diagrams 2.5 and 3.1 by considering the towers in the 0-stem in 3.1. Because of the fact that $\pi_*(TMF(3)) \approx v_2^{-1}\pi_*(X \wedge \mathrm{tmf})$, alluded to above, this corresponds to the direct limit of

$$\pi_0(X \wedge \mathrm{tmf}) \xrightarrow{v_2^8} \pi_{48}(X \wedge \mathrm{tmf}) \xrightarrow{v_2^8} \pi_{96}(X \wedge \mathrm{tmf}) \xrightarrow{v_2^8} \cdots .$$

We have $\pi_0(X \wedge \mathrm{tmf}) \approx \pi_0(bo) \approx \mathbb{Z}_{(2)}$. Next, $\pi_{48}(X \wedge \mathrm{tmf})$ is the sum of nine $\mathbb{Z}_{(2)}$'s, corresponding to the eight in the 45-stem in Diagram 2.5 plus one from bo, which will be in filtration 2 higher than the top one pictured. The lowest of the nine towers is v_2^8 times the one in $\pi_0(X \wedge \mathrm{tmf})$ and so they are identified in the direct limit. These will correspond to the first nine towers in $\pi_0(TMF(3))$ in Diagram 3.1. Similarly, the first seventeen towers in $\pi_0(TMF(3))$ can be seen in $\pi_{96}(X \wedge \mathrm{tmf})$, of which the lowest nine are divisible by v_2^8 and hence identified with those from $\pi_{48}(X \wedge \mathrm{tmf})$ just described.

The spectrum $Z \wedge \mathrm{tmf}$ will be one of our connective models of $\mathrm{TMF}(3)$. The following result gives its homotopy groups, which are closely related to those of $X \wedge \mathrm{tmf}$.

Theorem 3.1. *There is an isomorphism of graded abelian groups*

$$\pi_*(Z \wedge \mathrm{tmf}) \approx \widetilde{K} \oplus \mathbb{Z}_2[v_2^8]\langle x,\, \eta x,\, \nu x,\, x^2,\, \nu x^2,\, v_1 v_2 x^2,\, v_2^8\nu,\, v_2^8\nu^2\rangle,$$

where

$$\widetilde{K} = \ker(\widetilde{R} \to \mathbb{Z}_2[v_2^8]\langle v_2^4\rangle)$$

with $\widetilde{R}$ the subgroup of the ring R of Theorem 2.4 spanned by all elements divisible by v_2^3.

In dimension ≤ 51, $\pi_*(Z \wedge \mathrm{tmf})$ may be seen in Diagram 2.5 by removing the first two $\mathbb{Z}_2$'s, and the bo_* starting in the 5-stem, and the bsp_* starting in the 9-stem, and increasing stems of all elements by 3. Thus the first element would be the $\mathbb{Z}_2$ class x, which appears in 2.5 in position $(14, 2)$, and is in the 17-stem for $Z \wedge \mathrm{tmf}$. For the ASS-type chart that we will describe in our proof, filtrations should be decreased by 2, so that x appears in filtration 0.

Proof. Let $M_7 = H^*(X_7)$ be the A-module (or A_2-module) whose only nonzero groups are $\mathbb{Z}_2$ in dimensions 0, 4, 6, and 7 with $\mathrm{Sq}^7 \neq 0$, and let M_{421} be the A-module or A_2-module whose only nonzero groups are $\mathbb{Z}_2$ in dimensions 0, 1, 3, and 7 with $\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1 \neq 0$. There is an exact sequence

$$\begin{aligned} &\to \quad \mathrm{Ext}_{A_2}^{s-2,t-1}(\Sigma^{24}M_7) \to \mathrm{Ext}_{A_2}^{s,t}(\Sigma^{17}M_{421}) \to E_2^{s,t}(Z \wedge \mathrm{tmf}) \\ &\to \quad \mathrm{Ext}_{A_2}^{s-1,t-1}(\Sigma^{24}M_7) \xrightarrow{d} \mathrm{Ext}_{A_2}^{s+1,t}(\Sigma^{17}M_{421}) \to, \end{aligned} \tag{3.1}$$

with $d(\iota_{24}) = h_2^2\iota_{17}$. Here $E_2(Z \wedge \mathrm{tmf})$ is the E_2-term of a spectral sequence converging to $\pi_*(Z \wedge \mathrm{tmf})$. We could compute $E_2(Z \wedge \mathrm{tmf})$ by first computing $\mathrm{Ext}_{A_2}(M_7)$ and $\mathrm{Ext}_{A_2}(M_{421})$ (and these have been computed in [5] and [7]), but we prefer the following method which relates it directly to $E_2(X \wedge \mathrm{tmf})$.

Let $P = \ker(d_1)$ in the resolution in the proof of Theorem 2.2. One easily verifies that there is an exact sequence of A_2-modules

$$0 \to \Sigma^{24}M_7 \xrightarrow{i} \Sigma^{11}A_2/(\mathrm{Sq}^1, \mathrm{Sq}^5) \xrightarrow{d_2} P \xrightarrow{q} \Sigma^{16}M_{421} \to 0$$

with $d_2(\iota_{11}) = \mathrm{Sq}^7 I_4$, $q(\mathrm{Sq}^{6,6+7,5} I_4 + \mathrm{Sq}^{4,6} I_6) = \mathrm{gen}_{16}$, and $i(\iota_{24}) = \mathrm{Sq}^{6,7+4,6,3}\,\iota_{11}$.

Note that $\mathrm{Ext}_{A_2}(P)$ consists of a shifted version of Diagram 2.5 minus the first two $\mathbb{Z}_2$'s and the first bo_*. It is shifted so that the (now) initial tower, which did begin in $(9,2)$, now begins in $(11,0)$. Note also that

$$\mathrm{Ext}_{A_2}(P) \xrightarrow{d_2^*} \mathrm{Ext}_{A_2}(\Sigma^{11}A_2/(\mathrm{Sq}^1,\mathrm{Sq}^5)) \tag{3.2}$$

is surjective, because of the bsp_* in 2.5 beginning in $(9,2)$.

Let $K = \mathrm{im}(d_2) = \ker(q)$. There is a commutative diagram of exact horizontal and vertical sequences, with $\mathrm{Ext} = \mathrm{Ext}_{A_2}$ and all Ext groups having the same second superscript t,

$$\begin{tikzcd}
0 \arrow[d] & & & 0 \arrow[d] \\
\mathrm{Ext}^{s-2}(\Sigma^{24}M_7) \arrow[d] \arrow[dr, "\delta"] & & & \mathrm{Ext}^{s-1}(\Sigma^{24}M_7) \arrow[d] \\
\mathrm{Ext}^{s-1}(K) \arrow[r] \arrow[d] & \mathrm{Ext}^s(\Sigma^{16}M_{421}) \arrow[r] & \mathrm{Ext}^s(P) \arrow[r, "j"] \arrow[dr, "d_2^s"'] & \mathrm{Ext}^s(K) \arrow[d] \\
\mathrm{Ext}^{s-1}(\Sigma^{11}A_2/(\mathrm{Sq}^1,\mathrm{Sq}^5)) \arrow[d] & & & \mathrm{Ext}^s(\Sigma^{11}A_2/(\mathrm{Sq}^1,\mathrm{Sq}^5)) \arrow[d] \\
0 & & & 0
\end{tikzcd}$$

in which d_2^* is surjective. By a diagram chase, this implies exactness of

$$\to \mathrm{Ext}^{s-2}(\Sigma^{24}M_7) \xrightarrow{\delta} \mathrm{Ext}^s(\Sigma^{16}M_{421}) \to \ker(d_2^s) \to \mathrm{Ext}^{s-1}(\Sigma^{24}M_7) \to . \tag{3.3}$$

This δ must send ι_{24} to $h_2^2\iota_{16}$ since $\mathrm{Ext}^{2,24}(P) = 0$. Thus it must agree totally with d of (3.1), and so the exact sequences (3.1) and (3.3) are identical. Therefore, $E_2(Z \wedge \mathrm{tmf}) \approx \ker(d_2^*)$, and this is the chart obtained from Diagram 2.5, extended indefinitely, by removing the first two dots, the initial bo_*, and the bsp_* starting in $(9,2)$, and regrading so that the $\mathbb{Z}_2$ in $(14,2)$ in Diagram 2.5 is now in $(17,0)$. $\square$

Corollary 3.1. *There is a map $Z \wedge \mathrm{tmf} \to X \wedge \mathrm{tmf}$ such that the induced map $v_2^{-1}Z \wedge \mathrm{tmf} \to v_2^{-1}X \wedge \mathrm{tmf}$ is an equivalence.*

Proof. There is a map $Z \to X \wedge \mathrm{tmf}$ extending x for the same reason as in the proof of Proposition 3.1, namely 0 obstructions. Smashing with tmf and following by the multiplication of tmf yields the desired map. The proof of Theorem 3.1 identified $\pi_*(Z \wedge \mathrm{tmf})$ with the kernel of (3.2), which is contained in $\pi_*(X \wedge \mathrm{tmf})$. Thus $\pi_*(Z \wedge \mathrm{tmf})$ injects into all of $\pi_*(X \wedge \mathrm{tmf})$ except ν, ν^2, and the integer multiples of $v_2^i v_1^j$ for $i \le 2$. These latter classes are, for $i = 0$ the bo_* which is $\mathrm{coker}(\pi_*(\Sigma^{-1}C) \to \pi_*(X \wedge \mathrm{tmf}))$, for $i = 1$ the initial bo_* in 2.5, and for $i = 2$ the bsp_* which appears in 2.5 to begin in $(9, 2)$. Since v_2^8 times these classes are in the image from $\pi_*(Z \wedge \mathrm{tmf})$, we deduce the claim that it is an equivalence after v_2^8 is inverted. □

Theorem 3.2. *The map $Z \to \mathrm{TMF}(3)$ of Proposition 3.1 induces an equivalence*

$$v_2^{-1} Z \wedge \mathrm{tmf} \to \mathrm{TMF}(3).$$

Proof. We need a fact from topological modular forms that there is a map

$$\mathrm{tmf} \wedge \mathrm{TMF}(3) \to \mathrm{TMF}(3)$$

making $\mathrm{TMF}(3)$ a tmf-module. Using this, the map $Z \to \mathrm{TMF}(3)$, and the product in $TMF(3)$, we obtain a map $Z \wedge \mathrm{tmf} \to \mathrm{TMF}(3)$. We will show it sends $\pi_*(Z \wedge \mathrm{tmf})$ to elements of $\pi_*(\mathrm{TMF}(3))$ with the same names (as those of Theorem 3.1). In the proof of Proposition 3.1, we discussed how [10, 4.1] can be interpreted to give $\pi_*(\mathrm{TMF}(3))$ as a v_2-inverted version of our Theorem 2.4. Then the same argument as was used in the proof of Corollary 3.1 gives the asserted equivalence.

The class x maps across by construction. We must deduce from this, by various types of naturality, that all other classes map across. Our map is one of tmf_*-modules. The relation $v_1^4 x = \eta v_1^3 v_2^3$ is present in both $\pi_*(Z \wedge \mathrm{tmf})$ and $\pi_*(\mathrm{TMF}(3))$ (by Theorem 2.4 and [10, 4.1], resp.), and hence $\eta v_1^3 v_2^3$ maps across, and then so also does $v_1^3 v_2^3$. Since $16 v_2^2$ is in tmf_*, we deduce that all $v_1^i v_2^j$ with $i \equiv 3$ mod 4 and j odd map across. By the Toda bracket formula $2 v_1^5 v_2^3 = \langle \eta^2 v_1^3 v_2^3, \eta, 2\rangle$, which is valid in both $Z \wedge \mathrm{tmf}$ and $\mathrm{TMF}(3)$, we now have that all $v_1^i v_2^j$ with i and j odd map across.

In [10, 4.1], it is noted that $\pi_{20}(S^0) \to \pi_{20}(\mathrm{TMF}(3))$ sends $\overline{\kappa}$ to νx. One can show, for example using Yoneda products, that $\overline{\kappa}$ acting on $x \in \pi_{17}(Z \wedge \mathrm{tmf})$ yields the class that we call νx^2. Thus νx^2 maps across, and hence so does x^2. There is a bracket formula $2 v_2^6 = \langle x^2, \eta, 2\rangle$ in both spectra, and so v_2^6 maps across. Arguing as before, we deduce that all $v_1^i v_2^j$ with i and j even map across. Knowing that v_2^8 maps across implies the same for νv_2^8 and $\nu^2 v_2^8$. We have now accounted for all of $\pi_*(Z \wedge \mathrm{tmf})$. □

The following corollary is immediate from Corollary 3.1 and Theorem 3.2.

Corollary 3.2. *There is an equivalence $v_2^{-1} X \wedge \mathrm{tmf} \to \mathrm{TMF}(3)$.*

Thus both $X \wedge \mathrm{tmf}$ and $Z \wedge \mathrm{tmf}$ can serve as connective models of $\mathrm{TMF}(3)$. We prefer $X \wedge \mathrm{tmf}$ because it is a ring spectrum and gives a better approximation to $\pi_*(\mathrm{TMF}(3))$ prior to inverting v_2, but $Z \wedge \mathrm{tmf}$ was useful because it was so easy to get a map from it into $\mathrm{TMF}(3)$.

4 A Model Related to $\mathrm{tmf} \wedge \mathrm{tmf}$

In this section we study a third model of $\mathrm{tmf}(3)$ introduced in [10]. This one is closely related to $\mathrm{tmf} \wedge \mathrm{tmf}$, and we provide a proof that a plausible splitting of $\mathrm{tmf} \wedge \mathrm{tmf}$ does not occur. We clarify some aspects of the construction in [10] and compute the homotopy groups.

Let $A^* = \mathbb{Z}_2[\zeta_1, \zeta_2, \ldots]$ denote the dual of the mod 2 Steenrod algebra. Here $\zeta_i = \chi(\xi_i)$, the conjugates of the usual generators. Assign a weight wt on A^* by $\mathrm{wt}(\zeta_i) = 2^{i-1}$ and $\mathrm{wt}(ab) = \mathrm{wt}(a) + \mathrm{wt}(b)$. It is well-known and easily verified that

$$(A//A_2)^* = \mathbb{Z}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \zeta_5, \ldots]$$

and there is a splitting as A_2-modules

$$(A//A_2)^* \approx \bigoplus_{n \geq 0} M_n,$$

where M_n is spanned by all monomials in $(A//A_2)^*$ of weight $8n$. The A-action is given by $\zeta_i(\chi\,\mathrm{Sq}) = \zeta_i + \zeta_{i-1}^2$. Note that $H_*(\mathrm{tmf}) \approx (A//A_2)^*$.

Similarly $H_*(bo) = (A//A_1)^*$ is isomorphic to a polynomial algebra on ζ_1^4, ζ_2^2, and ζ_i for $i \geq 3$. There are bo-Brown-Gitler spectra bo_n satisfying that $H_*(bo_n)$ is the span of all monomials in $H_*(bo)$ with weight $\leq 4n$.([6]) One easily verifies that there is an isomorphism of A_2-modules

$$\bigoplus \phi_n : \bigoplus_{n \geq 0} H_*(\Sigma^{8n} bo_n) \to H_*(\mathrm{tmf})$$

defined by $\phi_n(\sigma^{8n} \zeta_1^{i_1} \zeta_2^{i_2} \cdots) = \zeta_1^{8n - \sum 2^j i_j} \zeta_2^{i_1} \zeta_3^{i_2} \cdots$. The image of ϕ_n is M_n, the span of monomials of weight $8n$. One might ask if this isomorphism is induced by an equivalence of the spectra $\mathrm{tmf} \wedge \mathrm{tmf}$ and $\bigvee \Sigma^{8n} bo_n \wedge \mathrm{tmf}$. An analogous equivalence $bo \wedge bo \simeq \bigvee \Sigma^{4n} \overline{B}_n \wedge bo$ was proved in [9]. In that case $\overline{B}_n$ was an integral Brown-Gitler spectrum.

We answer this question and prepare for a new $\mathrm{TMF}(3)$ model by proving the following result.

Theorem 4.1. *The spectra* $\mathrm{tmf} \wedge \mathrm{tmf}$ *and* $\bigvee_{n \geq 0} \Sigma^{8n} bo_n \wedge \mathrm{tmf}$ *are not homotopy equivalent. Indeed, in the ASS converging to* $\pi_*(\mathrm{tmf} \wedge \mathrm{tmf})$*, which has*

$$E_2 \approx \bigoplus_{n \geq 0} \mathrm{Ext}_{A_2}(H^*(\Sigma^{8n} bo_n)),$$

there is a class $g \in \mathrm{Ext}_{A_2}^{0,24}(H^*(\Sigma^{16} bo_2))$ *and an element* $w \in \mathrm{Ext}_{A_2}^{3,26}(H^*(\Sigma^{8} bo_1))$ *such that* $d_3(g) = w$.

Proof. Let $\overline{\mathrm{tmf}}$ denote the cofiber of $S^0 \to \mathrm{tmf}$. Since tmf is a ring spectrum, there is a splitting

$$\mathrm{tmf} \wedge \mathrm{tmf} \simeq (S^0 \wedge \mathrm{tmf}) \vee (\overline{\mathrm{tmf}} \wedge \mathrm{tmf}).$$

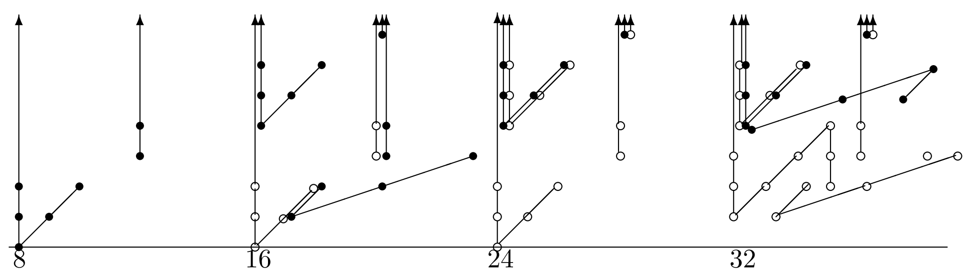
We will use the cofibration

$$\overline{\mathrm{tmf}} \wedge S^0 \to \overline{\mathrm{tmf}} \wedge \mathrm{tmf} \to \overline{\mathrm{tmf}} \wedge \overline{\mathrm{tmf}} \tag{4.1}$$

and a differential in the ASS of $\overline{\mathrm{tmf}}$ to deduce the claimed differential in the ASS of $\overline{\mathrm{tmf}} \wedge \mathrm{tmf}$.

In Diagram 4.1, we depict $\mathrm{Ext}_{A_2}^{s,t}(H^*(\Sigma^8 bo_1 \vee \Sigma^{16} bo_2))$ for $s < 8$, $t - s < 40$. Elements suggested by solid dots come from the first summand, and those with open circles (or connected to open circles by lines) come from the second summand.

Diagram 4.1. $\mathrm{Ext}_{A_2}^{s,t}(H^*(\Sigma^8 bo_1 \vee \Sigma^{16} bo_2))$ in a range



The cofibration which defines $\overline{\mathrm{tmf}}$ induces an exact sequence

$$\begin{aligned} &\to \mathrm{Ext}_A^{s,t}(H^*(\mathrm{tmf})) \to \mathrm{Ext}_A^{s,t}(H^*(\overline{\mathrm{tmf}})) \\ &\to \mathrm{Ext}_A^{s+1,t}(H^*(S^0)) \to \mathrm{Ext}_A^{s+1,t}(H^*(\mathrm{tmf})) \to . \end{aligned}$$

There is a lower vanishing line in $\mathrm{Ext}_A(H^*(\mathrm{tmf})) \approx \mathrm{Ext}_{A_2}(\mathbb{Z}_2)$ (e.g. [5, 2.6]) which implies that $\mathrm{Ext}_A^{s,t}(H^*(\overline{\mathrm{tmf}})) \approx \mathrm{Ext}_A^{s+1,t}(H^*(S^0))$ if $s \le 6$ and $t - s \ge 31$. In [2], it was shown that in the ASS of S^0 there are nonzero elements $e_1 \in \mathrm{Ext}_A^{4,42}(H^*(S^0))$ and $h_1 t \in \mathrm{Ext}_A^{7,44}(H^*(S^0))$ satisfying $d_3(e_1) = h_1 t$. These elements are in the range of our asserted isomorphism, and so there must be corresponding elements $\overline{e_1} \in \mathrm{Ext}_A^{3,42}(H^*(\overline{\mathrm{tmf}}))$ and $\overline{h_1 t} \in \mathrm{Ext}_A^{6,44}(H^*(\overline{\mathrm{tmf}}))$ related by a d_3-differential.

Now we consider the exact sequences in $\mathrm{Ext}_A(-)$ and $\pi_*(-)$ induced by (4.1). Using Bruner's software, we see that $\mathrm{Ext}_A^{s,t}(H^*(\overline{\mathrm{tmf}} \wedge \overline{\mathrm{tmf}})) = 0$ if $t - s = 39$ and $s > 3$. Thus neither of the elements $\overline{e_1}$ or $\overline{h_1 t}$ can be in the image from $\mathrm{Ext}_A(H^*(\overline{\mathrm{tmf}} \wedge \overline{\mathrm{tmf}}))$, the second since there is nothing to hit it, and the first since a class which hits it would have to support a differential, but there is nothing for it to hit. Thus the elements $\overline{e_1}$ and $\overline{h_1 t}$ related by the d_3 in the ASS of $\overline{\mathrm{tmf}}$ map nontrivially to $\mathrm{Ext}_A(H^*(\overline{\mathrm{tmf}} \wedge \mathrm{tmf}))$. One easily checks that $\mathrm{Ext}_{A_2}(H^*(\Sigma^{24} bo_3) \oplus H^*(\Sigma^{32} bo_4))$ is 0 in these bigradings. Thus the elements $\overline{e_1}$ and

$\overline{h_1 t}$ must map nontrivially to classes in $\mathrm{Ext}_{A_2}(H^*(\Sigma^8 bo_1) \oplus H^*(\Sigma^{16} bo_2))$ involved in a d_3-differential. These must be the two classes at the extreme right end of Diagram 4.1, one in filtration 6 from $\Sigma^8 bo_1$ and the other in filtration 3 from $\Sigma^{16} bo_2$.

This already implies the first conclusion of the theorem, that $\mathrm{tmf} \wedge \mathrm{tmf}$ does not split as $\bigvee_{n \geq 0} \Sigma^{8n} bo_n \wedge \mathrm{tmf}$. We would like to infer from this differential the claimed nontrivial d_3 on the class g in position $(24, 0)$. Clearly the h_2-action and the nonzero d_3 from $(39, 3)$ imply that d_3 is nonzero on the class in $(33, 1)$. Let $X_7 = S^0 \cup_\nu e^4 \cup_\eta e^6 \cup_2 e^7$ as before. If $d_3(g) = 0$, then the homotopy class g would extend to a map $\Sigma^{24} X_7 \to \overline{\mathrm{tmf}} \wedge \mathrm{tmf}$, since Diagram 4.1 shows that there are no obstructions to the extension. Smashing with tmf and following by the multiplication of tmf would yield a map $\Sigma^{24} X_7 \wedge \mathrm{tmf} \to \overline{\mathrm{tmf}} \wedge \mathrm{tmf}$ extending g. Since $X_7 = bo_1$, the ASS of $\Sigma^8 X_7 \wedge \mathrm{tmf}$ is just the black elements in Diagram 4.1. The 16-suspension of the element in $(17, 1)$ in that diagram does not support a differential in $\Sigma^{24} X_7 \wedge \mathrm{tmf}$ but would map to the class in $(33, 1)$ in $\overline{\mathrm{tmf}} \wedge \mathrm{tmf}$ which we showed does support a differential. This contradicts the assumption that $d_3(g) = 0$.

$\square$

Now we begin working toward the construction of our third connective model of $\mathrm{TMF}(3)$.

Proposition 4.1. *There is a subcomplex W_1 of $\overline{\mathrm{tmf}}$ such that there is a cofibration*

$$\Sigma^8 bo_1 \to W_1 \to \Sigma^{16} bo_2$$

which has a short exact sequence in mod-2 cohomology.

Proof. We use the description of $H_*(\overline{\mathrm{tmf}})$ given in the second paragraph of this section. All elements of weight ≤ 16 are in dimension ≤ 31, and the first few elements of weight greater than 16 are ζ_1^{24}, $\zeta_1^{16}\zeta_2^4$, $\zeta_1^{16}\zeta_3^2$, and $\zeta_1^{16}\zeta_4$. The A-module structure of $H^*(\overline{\mathrm{tmf}}^{(31)}/\overline{\mathrm{tmf}}^{(23)})$ is

$$\langle \mathrm{Sq}^0, \mathrm{Sq}^2, \mathrm{Sq}^3, \mathrm{Sq}^4, \mathrm{Sq}^5, \mathrm{Sq}^6, \mathrm{Sq}^7 \rangle \widehat{\zeta_2^8} \oplus \langle \mathrm{Sq}^0, \mathrm{Sq}^4, \mathrm{Sq}^6, \mathrm{Sq}^7 \rangle \widehat{\zeta_1^{24}}, \tag{4.2}$$

with the first (resp. second) summand dual to monomials of weight 16 (resp. 24). Here the $\widehat{(\quad)}$ represents duality. Bruner's software shows that there is a map

$$\overline{\mathrm{tmf}}^{(31)}/\overline{\mathrm{tmf}}^{(23)} \to \Sigma^{24} X_7$$

which induces the identity homomorphism from the second summand of (4.2) and 0 from the first. This is done by computing Ext_A of the tensor product of the dual of the module in (4.2) with M_7, and seeing that there are no possible differentials from the obvious filtration-0 class. The desired complex W_1 is the fiber of the composite

$$\overline{\mathrm{tmf}}^{(31)} \to \overline{\mathrm{tmf}}^{(31)}/\overline{\mathrm{tmf}}^{(23)} \to \Sigma^{24} X_7,$$

where the second map is the one just noted. $\square$

The E_2-term of the ASS for $W_1 \wedge \mathrm{tmf}$ in dimension less than 40 is given in Diagram 4.1, and, as established in Theorem 4.1, there are d_3-differentials on the classes in positions $(24,0)$, $(33,1)$, $(36,2)$, and $(39,3)$. Let $f : S^{32} \to W_1 \wedge \mathrm{tmf}$ be a nontrivial map of Adams filtration 1, which exists by Diagram 4.1. Smash with tmf and follow by the multiplication of tmf, obtaining a map $S^{32} \wedge \mathrm{tmf} \to W_1 \wedge \mathrm{tmf}$.

Definition 4.1. Define W to be the cofiber of this map $S^{32} \wedge \mathrm{tmf} \to W_1 \wedge \mathrm{tmf}$.

This W will be our third connective model of $\mathrm{TMF}(3)$. Note that, unlike the first two, it is not obtained as the smash product of a finite complex with tmf, since the above map f does not factor through W_1 itself.

Similarly, let $S^{16} \to bo_2 \wedge \mathrm{tmf}$ correspond to essentially the same class as f, as the open circles in Diagram 4.1 depict the ASS of $\Sigma^{16} bo_2$. Extend this to a map $S^{16} \wedge \mathrm{tmf} \to bo_2 \wedge \mathrm{tmf}$, and let $\widetilde{bo_2}$ denote the cofiber of this. There is a cofiber sequence

$$\Sigma^8 bo_1 \wedge \mathrm{tmf} \to W \to \Sigma^{16} \widetilde{bo_2}. \tag{4.3}$$

The short exact sequence of A-modules

$$0 \to \Sigma^{17} A//A_2 \to H^*(\widetilde{bo_2}) \to A \otimes_{A_2} H^*(bo_2) \to 0$$

induces an exact sequence in Ext_A which implies that $\mathrm{Ext}_A(H^*(\widetilde{bo_2}))$ begins as the 16-desuspension of the open circles in Diagram 4.1 with the portion connected to the element in $(32,1)$ removed. It contains no unpictured elements in filtration 0 or 1. Therefore, $H^*(\widetilde{bo_2}) = A \otimes_{A_2} B$, where B sits in a short exact sequence of A_2-modules

$$0 \to \Sigma^{17} \mathbb{Z}_2 \to B \to H^*(bo_2) \to 0, \tag{4.4}$$

with the new class in B equal to $\mathrm{Sq}^4 \mathrm{Sq}^6 \mathrm{Sq}^7 \iota_0$, or equivalently $\mathrm{Sq}^4 \mathrm{Sq}^2 \mathrm{Sq}^3 \iota_8$. It also equals Sq^2 of the top class of $H^*(bo_2)$. The A_2-module B cannot be given the structure of A-module, as the Adem relation $\mathrm{Sq}^2 \mathrm{Sq}^{15} = \mathrm{Sq}^1 \mathrm{Sq}^{16} + \mathrm{Sq}^{16} \mathrm{Sq}^1$ would be violated.

Our next result gives a direct relationship among $\mathrm{Ext}_{A_2}(A_2/(\mathrm{Sq}^4, \mathrm{Sq}^{5,1}))$, which was depicted through degree 48 in Diagram 2.5 and is very closely related to the homotopy groups described in Theorem 2.4, and $\mathrm{Ext}_{A_2}(B)$ and $\mathrm{Ext}_{A_2}(H^*(X_7))$, which two together are related to the ASS of W. After stating and proving this result, we will use it to determine $\pi_*(W)$ and see that $v_2^{-1} W$ is another model for $\mathrm{TMF}(3)$.

We begin by noting that $\mathrm{Ext}_{A_2}^{s,t}(A_2(\mathrm{Sq}^4, \mathrm{Sq}^{5,1})) \approx \mathrm{Ext}_{A_2}^{s+1,t}(A_2/(\mathrm{Sq}^4, \mathrm{Sq}^{5,1}))$.

Theorem 4.2. *Let* $\widetilde{\mathrm{Ext}}_{A_2}(A_2(\mathrm{Sq}^4, \mathrm{Sq}^{5,1}))$ *denote* $\mathrm{Ext}_{A_2}(A_2(\mathrm{Sq}^4, \mathrm{Sq}^{5,1}))$ *without the* $\mathbb{Z}_2$ *in* $\mathrm{Ext}^{0,4}$ *or the tower beginning in* $\mathrm{Ext}^{1,11}$*. There is an exact sequence*

$$\to \mathrm{Ext}_{A_2}^{s+2,t}(\Sigma^6 M_7) \to \widetilde{\mathrm{Ext}}_{A_2}^{s+2,t}(A_2(\mathrm{Sq}^4, \mathrm{Sq}^{5,1})) \to \mathrm{Ext}_{A_2}^{s,t}(\Sigma^{16} B) \to \mathrm{Ext}_{A_2}^{s+3,t}(\Sigma^6 M_7).$$

Proof. One can verify that there is an exact sequence of A_2-modules

$$0 \to K \xrightarrow{i} \Sigma^4 A_2 \to A_2(\mathrm{Sq}^4, \mathrm{Sq}^{5,1}) \xrightarrow{\phi} \Sigma^6 M_7 \to 0,$$

where $\Sigma^6 M_7$ is generated by $\phi(\mathrm{Sq}^{5,1})$, and $i(K)$ is the submodule of $\Sigma^4 A_2$ generated by $\mathrm{Sq}^7\,\iota_4$, and that there is a short exact sequence of A_2-modules

$$0 \to \Sigma^{16} B \to \Sigma^{11} A_2//A_0 \to K \to 0$$

with B as above, and the A_2-generators of $\Sigma^{16}B$ mapping to $\mathrm{Sq}^5\,\iota_{11}$ and $\mathrm{Sq}^{4,6,3}\,\iota_{11}$.

Let $R = \mathrm{coker}(i) = \ker(\phi)$. Except for the classes omitted in forming $\widetilde{\mathrm{Ext}}$, we have isomorphisms

$$\mathrm{Ext}^s_{A_2}(\Sigma^{16}B) \xrightarrow{\approx} \mathrm{Ext}^{s+1}_{A_2}(K) \xrightarrow{\approx} \mathrm{Ext}^{s+2}_{A_2}(R)$$

and an exact sequence

$$\to \mathrm{Ext}^{s+2}_{A_2}(\Sigma^6 M_7) \to \mathrm{Ext}^{s+2}_{A_2}(A_2(\mathrm{Sq}^4, \mathrm{Sq}^{5,1})) \to \mathrm{Ext}^{s+2}_{A_2}(R) \to \mathrm{Ext}^{s+3}_{A_2}(\Sigma^6 M_7),$$

from which the result follows. □

Similarly to Theorem 3.1, we can now deduce the following result without using complete information about $\mathrm{Ext}_{A_2}(B)$.

Theorem 4.3. *There is an isomorphism of graded abelian groups*

$$\pi_*(W) \approx K' \oplus \mathbb{Z}_2[v_2^8]\langle\, x,\, \eta x,\, \nu x,\, x^2,\, \nu x^2,\, v_1 v_2 x^2,\, v_2^8\nu,\, v_2^8\nu^2\rangle,$$

where

$$K' = \ker(R' \to \mathbb{Z}_2[v_2^8]\langle v_2^4\rangle)$$

with R' the subgroup of the ring R of 2.4 spanned by all elements divisible by v_2 but not including the cyclic group generated by $2v_2^2$.

Proof. The map $\Sigma^{15}\widetilde{bo_2} \to \Sigma^8 bo_1 \wedge \mathrm{tmf}$ whose cofiber is W has Adams filtration 3 since $H^i(\Sigma^{15}\widetilde{bo_2}) = 0$ for $i < 15$ and for $i = 17$, 18, and 20, the values of i for which $\pi_i(\Sigma^8 bo_1 \wedge \mathrm{tmf})$ has nonzero classes in filtration less than 3. We obtain a homomorphism

$$\mathrm{Ext}^{s,t}_{A_2}(\Sigma^{15}B) \to \mathrm{Ext}^{s+3,t+3}_{A_2}(\Sigma^8 M_7).$$

We show in the next paragraph that this is the same homomorphism as the one at the end of the exact sequence in Theorem 4.2.

Both homomorphisms are nontrivial on the class in $\mathrm{Ext}^{0,24}_{A_2}(\Sigma^{16}B))$, the first by Theorem 4.1 and the second since Diagram 2.5 is 0 in position $(21, 3)$. Let $\mathbf{C}$ (resp. $\mathbf{D}$) be a minimal A_2-resolution of $\Sigma^8 M_7$ (resp. $\Sigma^{15}B$). There is a morphism $C_3 \xrightarrow{\phi} \Sigma^{15}B$ which lifts to a morphism $C_3 \to D_0$ and then to $C_{s+3} \to D_s$ for all s. Since $B_5 = 0$, ϕ must be

0 on the generators in 8, 12, and 20, and it must send the generator in 23 nontrivially to get the correct Ext morphism. This completely determines the entire Ext morphism. The same is true of the Ext morphism at the end of the sequence of Theorem 4.2. Thus the two Ext morphisms are equal.

We obtain that $E_2^{s,t}(W) \approx \widetilde{\mathrm{Ext}}_{A_2}^{s,t-2}(A_2(\mathrm{Sq}^4, \mathrm{Sq}^{5,1}))$. We have already seen that there are no possible differentials in an ASS with $E_2 \approx \widetilde{\mathrm{Ext}}_{A_2}(A_2(\mathrm{Sq}^4, \mathrm{Sq}^{5,1}))$. Thus $\pi_*(W)$ is like the groups described in Theorem 2.4 without the initial bo_*, ν, ν^2, or the $2v_2^2$-tower. $\square$

Similarly to Corollary 3.2, we obtain the following result, giving a third connective model for $\mathrm{TMF}(3)$. The significance of this one is its close relationship to $\mathrm{tmf} \wedge \mathrm{tmf}$.

Corollary 4.1. *There is an equivalence* $v_2^{-1}W \to \mathrm{TMF}(3)$.

Proof. Similarly to Corollary 3.1, we construct a map $Z \to W$, then use the tmf-module structure of W to extend to a map $Z \wedge \mathrm{tmf} \to W$. This becomes an equivalence after inverting v_2. Then we use Theorem 3.2. $\square$

5 $\mathrm{tmf}(3)$-Homology of Real Projective Space

In this section, we compute $\pi_*(X \wedge \mathrm{tmf} \wedge P_1)$, where X is as in Theorem 2.1 and $P_1 = RP^\infty$. Because $X \wedge \mathrm{tmf}$ is probably the best connective model for $\mathrm{TMF}(3)$, this could be considered as $\mathrm{tmf}(3)_*(P_1)$. More work will be required to deduce results for P_n or P_1^m from this, but this should provide a model. One possible application of this calculation would be to obstruction theory, which was an initial motivation for this project.

It is convenient to state and prove the result for ΣP_1. Some of the tmf_*-module structure is included in the result. We now state the main theorem of this section. Although it is not exactly an ASS, we describe the groups in an ASS-like way, with bigrading (i, s) for an element of $\pi_i(X \wedge \mathrm{tmf} \wedge \Sigma P_1)$ of filtration s. Many elements are expressed as $a^{e_1} v_2^{e_2}$ of bigrading $(2e_1 + 6e_2, e_2)$. Thus a (resp. v_2) is thought of as having bigrading $(2, 0)$ (resp. $(6, 1)$), although a and v_2 themselves are not actually elements of $\pi_*(X \wedge \mathrm{tmf} \wedge \Sigma P_1)$. Certain powers of v_2 can be thought of as being part of the tmf_*-module structure. Note that the elements $a^{e_1} v_2^{e_2}$ are not really products, since $X \wedge \mathrm{tmf} \wedge \Sigma P_1$ is not a ring spectrum. The element a roughly corresponds to $v_1/2$.

Theorem 5.1. *For each pair* (e_1, e_2) *such that* $e_1 > 0$*,* $e_2 \geq 0$*, and* $e_1 \equiv e_2$ (2)*,* $\pi_*(X \wedge \mathrm{tmf} \wedge \Sigma P_1)$ *has a summand* $\mathbb{Z}/2^{e_1}$ *generated by*

$$\begin{cases} a^{e_1} v_2^{e_2} & \text{if } e_1 \equiv e_2 \ (4) \\ 2a^{e_1} v_2^{e_2} & \text{if } e_1 \equiv e_2 + 2 \ (4), \end{cases} \tag{5.1}$$

with the following two variations:

- *if* $e_1 = 2$ *and* $e_2 \equiv 0\ (8)$, *it is* $\mathbb{Z}/8$ *generated by* $a^2 v_2^{e_2}$;
- *if* $e_1 = 1$ *and* $e_2 \equiv 1$ *or* $3\ (8)$, *it is* $\mathbb{Z}/4$ *generated by* $a v_2^{e_2}$.

If $e_1 \geq 5$ *and* $e_1 \equiv e_2\ (4)$, *or if* $(e_1, (e_2 \bmod 8)) = (4,0)$ *or* $(3,3)$, *then* $\eta^2 a^{e_1} v_2^{e_2} \neq 0$. *If* $(e_1, (e_2 \bmod 8)) = (1,1)$, $(4,4)$, $(2,6)$, *or* $(3,7)$, *then* $\eta a^{e_1} v_2^{e_2} \neq 0$.

If $e_1 \geq 3$ *and* $e_1 \equiv e_2 + 2\ (4)$, *or* $e_1 = 2$ *and* $e_2 \equiv 0\ (8)$, *then there exists* b_{e_1,e_2} *of bigrading* $(e_1 + e_2 - 2, 2e_1 + 6e_2 - 2)$ *and order 2 satisfying* $\eta^2 b_{e_1,e_2} = 2^{e_1} a^{e_1} v_2^{e_2}$. *If* $(e_1, (e_2 \bmod 8)) = (1,3)$ *or* $(2,4)$, *there exists* b'_{e_1,e_2} *of bigrading* $(e_1 + e_2 - 1, 2e_1 + 6e_2 - 1)$ *and order 2 satisfying* $\eta b'_{e_1,e_2} = 2^{e_1} a^{e_1} v_2^{e_2}$.

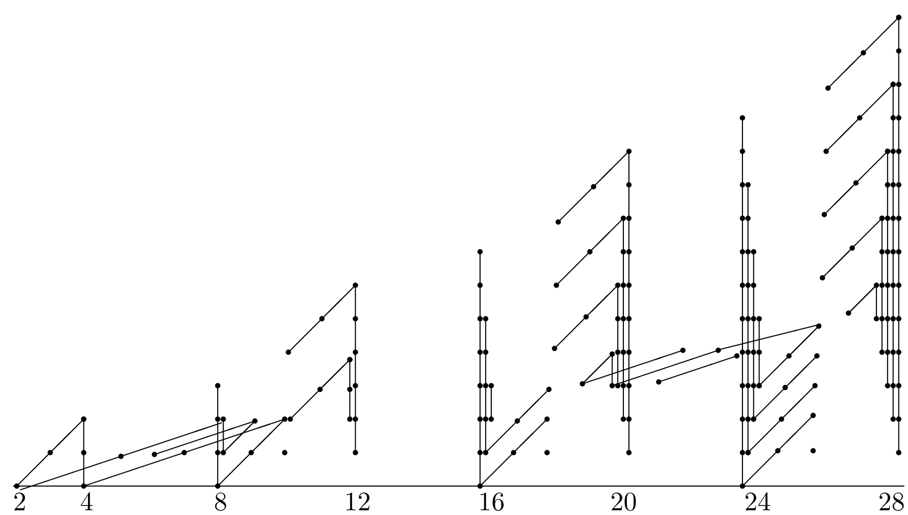
In addition, there are the following $\mathbb{Z}_2$ *classes* $x_{i,s}$ *of bigrading* (i,s).[1]

- $x_{8i+2,1}$ *for* $i \geq 1$.

  All the rest are acted on freely by v_2^8.
- $x_{5,1} = \nu b_{2,0}$, $x_{7,1} = \nu a^2$;
- $x_{6,1}$ *satisfying* $\nu x_{6,1} = \eta a v_2$;
- $x_{21,3}$ *and* $\nu x_{21,3}$;
- $x_{22,4} = \nu b'_{1,3}$, $x_{23,4} = \nu a v_2^3$;
- $x_{36,6}$ *and* $\nu x_{36,6}$, $x_{37,6}$ *and* $\nu x_{37,6}$;
- $x_{38,6}$ *satisfying* $\nu x_{38,6} = \eta a^2 v_2^6$.

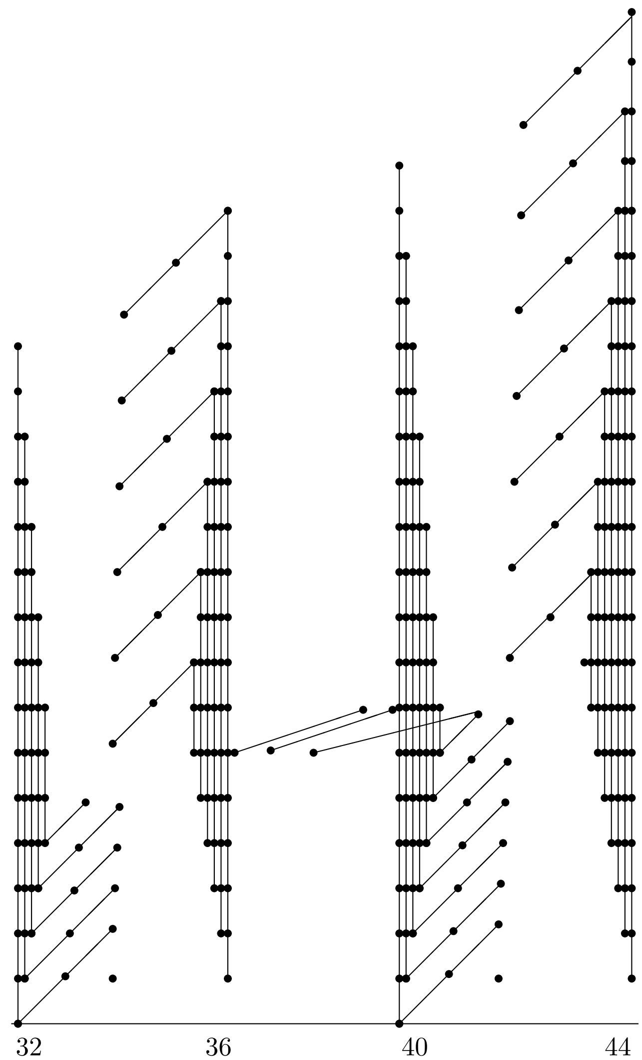
In Diagrams 5.1 and 5.2 we depict the groups of Theorem 5.1. All elements except those in position $(8i+2,1)$ for $i \geq 1$ in Diagram 5.1 are acted on freely by v_2^8.

Diagram 5.1. $\pi_*(X \wedge \mathrm{tmf} \wedge \Sigma P_1)$ in $* < 32$



[1] Note that the subscripts of x refer to bigrading, while the subscripts of b and b' do not.

Diagram 5.2. $\pi_*(X \wedge \mathrm{tmf} \wedge \Sigma P_1)$, $32 \leq * < 48$



The remainder of this section is devoted to the proof of Theorem 5.1. By Theorem 2.1, there is an exact sequence

$$bo_*(P_1) \xrightarrow{g_*} \pi_*(C \wedge P_1) \xrightarrow{\delta_*} \pi_*(X \wedge \mathrm{tmf} \wedge \Sigma P_1) \xrightarrow{\widetilde{f}_*} bo_*(\Sigma P_1). \tag{5.2}$$

As is well-known, $bo_*(P_1)$ can be computed from $\mathrm{Ext}_{A_1}(H^*(P_1))$, and from Theorem 2.1(d), $\pi_*(C \wedge P_1)$ can be computed from

$$\mathrm{Ext}_{A_2}(\Sigma^4 A_2/(\mathrm{Sq}^4, \mathrm{Sq}^{5,1}) \otimes H^*(P_1)). \tag{5.3}$$

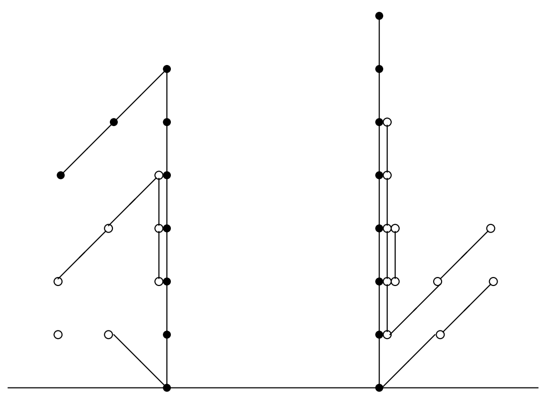
We can use Bruner's software to compute (5.3) through a large range of dimensions, enough to see patterns. In order to prove that these patterns continue, v_2^8-periodicity, which follows from the resolution in the proof of 2.2, is very helpful, but we still need to prove what happens in filtration less than 8 beyond dimension 48. Most of our analysis will go into computing (5.3), but we begin by analyzing (5.2).

It is convenient to use (5.2) to form a chart for $\pi_*(X \wedge \mathrm{tmf} \wedge \Sigma P_1)$ from

$$\phi\,\mathrm{Ext}_{A_2}(\Sigma^4 A_2/(\mathrm{Sq}^4, \mathrm{Sq}^{5,1}) \otimes H^*(P_1)) \oplus \mathrm{Ext}_{A_1}(H^*(\Sigma P_1)).$$

Recall that ϕ increases filtration by 1. The behavior for $10 \le i \le 18$ is typical, and is depicted in Diagram 5.3, in which black dots are from $\mathrm{Ext}_{A_1}(H^*(\Sigma P_1))$ and $\circ$'s are from $\phi\,\mathrm{Ext}_{A_2}(\Sigma^4 A_2/(\mathrm{Sq}^4, \mathrm{Sq}^{5,1}) \otimes H^*(P_1))$.

Diagram 5.3. Forming $\pi_*(X \wedge \mathrm{tmf} \wedge \Sigma P_1)$, $10 \le * < 18$



The content in this chart is the d_1-differential from $(12, 0)$ and the η-extension from $(16, 0)$. These are generalized and proved in Theorem 5.2.

Theorem 5.2. *In (5.2),*

- $bo_{8i+3}(P_1) \xrightarrow{g_*} \pi_{8i+3}(C \wedge P_1)$ *is nontrivial.*
- *There is an element* $\gamma_{8i} \in \pi_{8i}(X \wedge \mathrm{tmf} \wedge \Sigma P_1)$ *such that* $\widetilde{f}_*(\gamma_{8i})$ *has Adams filtration 0, and* $\eta\gamma_{8i} = \delta_*(y_{8i}) \neq 0$ *with* y_{8i} *of Adams filtration 0 in* $\pi_{8i+1}(C \wedge P_1)$.

Proof. We will see in Theorem 5.3 that $\mathrm{Ext}_A^{0,8i+3}(H^*(C \wedge P_1)) \approx \mathbb{Z}_2$ with nonzero class $\iota_4 \otimes x_{8i-1}$. The morphism g_* is induced by

$$\begin{array}{ccccc} \Sigma^4 A/(\mathrm{Sq}^4, \mathrm{Sq}^{5,1}) \otimes H^* P_1 & \to & A//A_1 \otimes H^* P_1 & \approx & A \otimes_{A_1} H^* P_1 \\ \iota_4 \otimes x_{8i-1} & \mapsto & \mathrm{Sq}^4 \otimes x_{8i-1} & \leftrightarrow & \mathrm{Sq}^4(1 \otimes x_{8i-1}) + 1 \otimes x_{8i+3}, \end{array}$$

which proves the first statement. The η-extension follows similarly from

$$\iota_4 \otimes x_{8i-3} \mapsto \mathrm{Sq}^4 \otimes x_{8i-3} \leftrightarrow \mathrm{Sq}^4(1 \otimes x_{8i-3}) + \mathrm{Sq}^2(1 \otimes x_{8i-1}).$$

To know that the class γ_{8i} is nonzero in $\pi_{8i}(-)$, we use Theorem 5.3 to see that, unless $i \equiv 5 \bmod 8$, the only possible target of a differential from γ_{8i} is ruled out by h_2-naturality.

If $i \equiv 5 \bmod 8$, the differential, if nonzero in the ASS of ΣP_1, would have to also be nonzero in the ASS of the cofiber R of the Kahn-Priddy map $\lambda : P_1 \to S^0$, but it is ruled out there by h_2-naturality. $\square$

Let $L = A_2/(\mathrm{Sq}^4, \mathrm{Sq}^{5,1})$. A good way to obtain $\mathrm{Ext}_{A_2}(L \otimes H^*(P_1))$ begins by computing $\mathrm{Ext}_{A_2}(L \otimes Q)$, where Q is the A_2-module which has as its only nonzero classes x_i for $i \geq 1$ and $i \in \{-9, -5, -3, -2, -1\}$ with $\mathrm{Sq}^j\, x_i = \binom{i}{j} x_{i+j}$. Then Q is an extension of copies of $\Sigma^{8i-1} A_2//A_1$ for $i \geq -1$. See [5, p.299]. Thus there is a spectral sequence converging to $\mathrm{Ext}_{A_2}(L \otimes Q)$ with

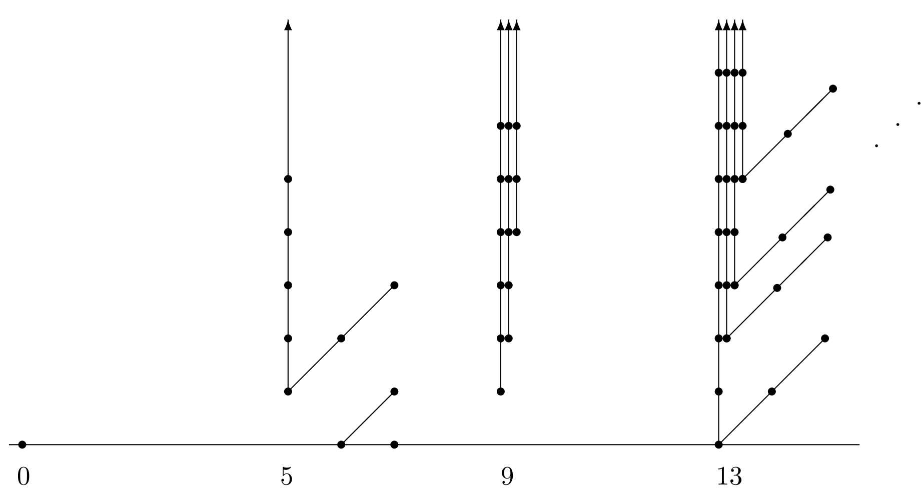
$$E_2^{*,*} = \bigoplus_{i \geq -1} \mathrm{Ext}_{A_1}^{*,*}(\Sigma^{8i-1} L).$$

One easily computes $\mathrm{Ext}_{A_1}(L)$ to be as in Diagram 5.4, from which it is immediate that the spectral sequence collapses and

$$\mathrm{Ext}_{A_2}(L \otimes Q) \approx \bigoplus_{i \geq -1} \mathrm{Ext}_{A_1}(\Sigma^{8i-1} L). \tag{5.4}$$

We obtain that, in grading $8i - 4$, $\mathrm{Ext}_{A_2}(L \otimes Q)$ has a tower beginning in filtration s for all nonnegative $s \leq 4i + 1$ except $s = 4i$. This will explain the low-filtration form of Diagrams 5.1 and 5.2.

Diagram 5.4. $\mathrm{Ext}_{A_1}(L)$



There is a short exact sequence of A_2-modules

$$0 \to H^* P_1 \to Q \to \Sigma^{-9} M_7 \oplus \Sigma^{-1} \mathbb{Z}_2 \to 0,$$

and also after tensoring with L. Thus there is an exact sequence

$$\begin{aligned} &\operatorname{Ext}^s_{A_2}(L\otimes\Sigma^{-9}M_7)\oplus\operatorname{Ext}^s_{A_2}(\Sigma^{-1}L)\to\operatorname{Ext}^s_{A_2}(L\otimes Q) \\ \to\quad &\operatorname{Ext}^s_{A_2}(L\otimes H^*P_1)\to\operatorname{Ext}^{s+1}_{A_2}(L\otimes\Sigma^{-9}M_7)\oplus\operatorname{Ext}^{s+1}_{A_2}(\Sigma^{-1}L). \end{aligned}\tag{5.5}$$

In Theorem 2.2 and Diagram 2.5, we computed and displayed $\operatorname{Ext}_{A_2}(L)$. A nice computation of $\operatorname{Ext}_{A_2}(L\otimes M_7)$ can be obtained by tensoring the exact sequence at the beginning of the proof of Theorem 2.2 with M_7. This yields a spectral sequence computing $\operatorname{Ext}_{A_2}(L\otimes M_7)$ from things such as $\operatorname{Ext}_{A_2}(M_7\otimes A_2)$, which is just four $\mathbb{Z}_2$'s, and $\operatorname{Ext}_{A_2}(M_7\otimes A_2//A_1)\approx\operatorname{Ext}_{A_1}(M_7)$, which is $bo_*\oplus\Sigma^4 bsp_*$. The resulting spectral sequence has only a very few possible differentials, which are most easily settled using Bruner's software, although they can be settled without it. Both $\operatorname{Ext}_{A_2}(L)$ and $\operatorname{Ext}_{A_2}(L\otimes M_7)$ have lower vanishing lines. From these and the exact sequence, we obtain that

$$\operatorname{Ext}^{s,t}_{A_2}(L\otimes Q)\to\operatorname{Ext}^{s,t}_{A_2}(L\otimes H^*P_1)$$

is an isomorphism if $s\le 8$ and $t-s\ge 53$.

Thus a Bruner calculation of $\operatorname{Ext}^{s,t}_{A_2}(L\otimes H^*P_1)$ for $t-s\le 53$, which is easily done and is consistent with Theorem 5.3, together with the complete description of $\operatorname{Ext}_{A_2}(L\otimes Q)$ in (5.4) and Diagram 5.4 and v_2^8-periodicity, gives a complete determination of the groups $\operatorname{Ext}^{s,t}_{A_2}(L\otimes H^*P_1)$. Note that the Bruner software is not absolutely necessary for the calculation in $t-s\le 53$. First of all, it is just a finite calculation, and secondly there are rather simple patterns for the boundary homomorphism in (5.5), which could be determined directly.

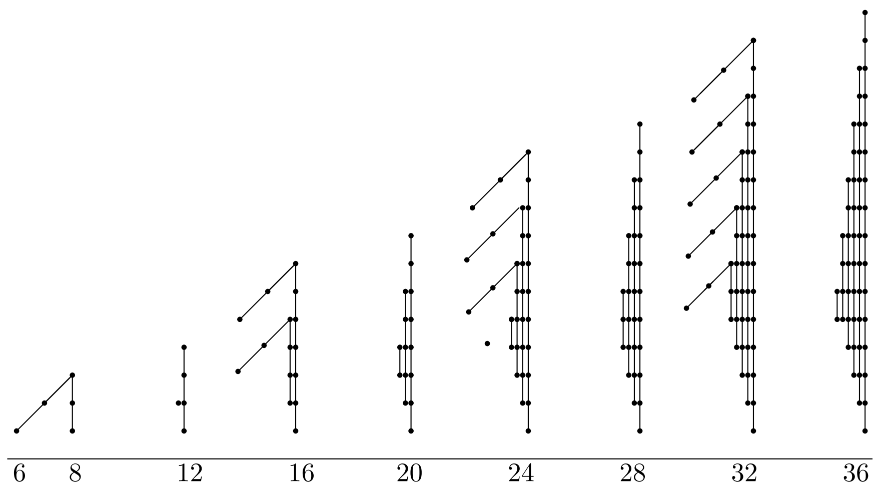
There is one more thing required in order to determine the chart for $\operatorname{Ext}^{s,t}_{A_2}(L\otimes H^*P_1)$, and the resulting $\pi_*(C\wedge P_1)$. In dimensions greater than 53 and congruent to 0 mod 4, we know from the determination of $\operatorname{Ext}_{A_2}(L\otimes Q)$ that in filtration ≤ 8 $\operatorname{Ext}^{s,t}_{A_2}(L\otimes H^*P_1)$ has h_0-towers beginning in each filtration (>0 in dimension 0 mod 8), and we know from the Bruner calculation and periodicity that in high filtration it has towers which end in every second filtration coming down from a certain maximum filtration. But how do we know the way these match up? We must show that, as suggested in Diagrams 5.1 and 5.2, the lowest bottoms match up with the highest tops.

One way to do this is to use the spectral sequence which builds $\operatorname{Ext}_{A_2}(L\otimes H^*P_1)$ from

$$\bigoplus_{s\ge 0}\phi^s\operatorname{Ext}^{*,*}_{A_2}(\Sigma^{-s}C_s\otimes H^*P_1),\tag{5.6}$$

where C_s are the A_2-modules in the resolution of L at the beginning of the proof of Theorem 2.2. The s-summand provides a bunch of $\mathbb{Z}_2$'s at height s in the resulting chart (coming from $\phi^s\operatorname{Ext}^0(-)$) together with the portion of Diagram 5.5 consisting of towers beginning at height s. Note that there are no such towers when $s=0$.

Diagram 5.5. Portion of spectral sequence building $\mathrm{Ext}_{A_2}(L \otimes H^*P_1)$



The desired form for the bottoms of the towers, as obtained from the complete description of $\mathrm{Ext}_{A_2}(L \otimes Q)$ in (5.4) and Diagram 5.4, differs slightly from this, in that in dimensions congruent to 4 mod 8 most of the towers should begin one filtration lower. This can only be accounted for by an extension from a $\mathbb{Z}_2$ from the next smaller s-value.

For example, in dimension 28, Diagram 5.5 shows towers beginning at height 1, 2, 3, and 4, coming from summands $s = 1$, 2, 3, and 4 in (5.6) with tops at height 12, 10, 8, and 6, respectively. These correspond to $\pi_{32}(C \wedge P_1)$, which, according to Theorem 5.3, corresponds to $\pi_{32}(X \wedge \mathrm{tmf} \wedge \Sigma P_1)$ in Diagram 5.2 with its largest tower removed and filtrations decreased by 1; hence, towers beginning at height 0, 1, 2, and 3 ending at height 12, 10, 8, and 6. Then, for example, the tower in $\pi_{32}(C \wedge P_1)$ (corresponding to $\mathrm{Ext}_{A_2}^{*,*+28}(L \otimes H^*P_1)$) going from filtration 0 to 12 can only come, in the spectral sequence of (5.6), from the $s = 1$ tower with an extension from a $\mathbb{Z}_2$ from $s = 0$.

The main thing that was obtained from using Q which was not easily obtained from (5.6) is the η^2-hooks on the bottom of towers. In (5.6) these come about from the filtration-0 $\mathbb{Z}_2$'s in the various s-summands in a complicated way, but they are clear in Diagram 5.4. The above remarks imply the following result, the computation of (5.3), since there are no possible differentials in the ASS.

Theorem 5.3. *The ASS converging to* $\pi_*(C \wedge P_1)$ *has* $E_2^{s,t} = \mathrm{Ext}_{A_2}^{s,t}(\Sigma^4 L \otimes H^*(P_1))$ *and collapses. The description of* $\pi_*(C \wedge P_1)$ *can be obtained from that of* $\pi_*(X \wedge \mathrm{tmf} \wedge \Sigma P_1)$ *in Theorem 5.1 by making the following changes:*

- *Remove summands in (5.1) for which* $e_2 = 0$, *(but do not remove* ηa^{e_1} *and* $\eta^2 a^{e_1}$ *when* $e_1 \equiv 0$ *mod 4);*
- *Remove* $b_{e_1,0}$ *and* $\eta b_{e_1,0}$ *with* $e_1 \equiv 2$ *mod 4;*
- *Add elements* $c_{8i+3,0}$ *of order 2 for* $i \geq 1$*;*
- *Decrease filtrations by 1.*

The proof of Theorem 5.1 is now immediate from the exact sequence (5.2), Theorem 5.3, and Theorem 5.2, which describes the only possible differentials and extensions in (5.2).

Remark 5.1. The way that we have chosen to describe these things is reversed from the way they are derived. We first compute the groups in 5.3 and then use them to determine the groups in 5.1. However, we are mostly interested in 5.1, and so we felt that it should be stated up front. It seemed like overkill to state the whole thing again for $\pi_*(C \wedge P_1)$, since it is so similar.

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References

[1] T. Bauer, *Computation of the homotopy of the spectrum* tmf, Geom. Topol. Monogr. **13** (2008) 11–40.
[2] R. R. Bruner, *A new differential in the Adams spectral sequence*, Topology **23** (1984) 271–276.
[3] R. R. Bruner, *Ext in the nineties*, Contemp. Math. **146** (1993) 71–90.
[4] D. M. Davis, *Generalized homology and the generalized vector field problem*, Quart. J. Math. Oxford **25** (1974) 169–193.
[5] D. M. Davis and M. Mahowald, *Ext over the subalgebra A_2 of the Steenrod algebra for stunted projective spaces*, Can. Math. Soc. Conf. Proc. **2** (1982) 297–342.
[6] P. G. Goerss, J. D. S. Jones, and M. Mahowald, *Some generalized Brown-Gitler spectra*, Trans. Amer. Math. Soc. **294** (1986) 113–132.
[7] V. Gorbounov and M. Mahowald, *Some homotopy of the cobordism spectrum $MO\langle 8\rangle$*, Contemp. Math. **188** (1995) 105–119.
[8] J. Lurie, *A survey of elliptic cohomology*, Algebraic topology, Abel Symp. **4**, Springer, (2009) 219–277.
[9] M. Mahowald, *bo-resolutions*, Pacific J. Math. **92** (1982) 365–383.
[10] M. Mahowald and C. Rezk, *Topological modular forms of level 3*, Pure Appl. Math. Quar. **5** (2009) 853–872.

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