THE ACTION OF THE MORAVA STABILIZER GROUP ON THE LUBIN-TATE MODULI SPACE OF LIFTS

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Introduction. The purpose of this paper is indicated by its title. Namely, for each prime number p, there is a certain canonical one-dimensional (commutative) height n formal group law Γ_n defined over $\mathbb{F}_p$ and hence over $\mathbb{F}_{p^n}$, the field with p^n elements. The automorphism group of this formal group law over $\mathbb{F}_{p^n}$ is defined to be the n^{th} Morava stabilizer group S_n. Now suppose A is a noetherian local ring complete with respect to its maximal ideal $\mathfrak{m}$ and such that the residue ring $A/\mathfrak{m}$ is an $\mathbb{F}_{p^n}$-algebra. Then, according to Lubin and Tate [15], the set $\text{lifts}_n^\star(A)$ of $\star$-isomorphism classes of lifts of Γ_n to A is in natural bijective correspondence with continuous $W\mathbb{F}_{p^n}$-algebra homomorphisms from $E_n^\wedge = W\mathbb{F}_{p^n}[[u_1, \ldots, u_{n-1}]]$ to A. ($W\mathbb{F}_{p^n}$ denotes the ring of Witt vectors with coefficients in $\mathbb{F}_{p^n}$.) S_n acts naturally on $\text{lifts}_n^\star(A)$; hence it acts on $E_n^\wedge$ by $W\mathbb{F}_{p^n}$-algebra homomorphisms. There is also an equivalence relation of $\star$-isomorphism on the set of pairs (F, v), where F is a lift of Γ_n to A and v is a "1-form on F." The set of equivalence classes is denoted $(\text{lifts}_n^\star)_*(A)$ and is also acted on by S_n. Since $(\text{lifts}_n^\star)_*$ is "corepresented" by $E_{n*}^\wedge \equiv W\mathbb{F}_{p^n}[[u_1, \ldots, u_{n-1}]][u, u^{-1}]$, S_n acts on $E_{n*}^\wedge$ as well. This action is gradation preserving, where $|u| = -2$ and $|u_i| = 0$, $1 \leq i \leq n-1$. In this paper, we give explicit formulas for the action. We also provide an application of our computation to Brown-Comenetz duality.

Using more sophisticated methods, Gross and Hopkins ([8], [9]) have found a framework for understanding the action of S_n on $E_{n*}^\wedge$ which allowed them to complete our understanding of the Brown-Comenetz dual of the $E(n)_*$-localization of an $E(n-1)_*$-acyclic finite spectrum (for p sufficiently large compared to n). However, it is our hope that the present paper will provide topologists with an accessible account of the genesis of the ideas in their work.

Let us indicate the homotopy-theoretic context of this action of S_n on $E_{n*}^\wedge$. Recall that for each prime p, there is a p-local spectrum BP with coefficient ring $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_k, \ldots]$, where v_i is the i^{th} Hazewinkel generator and has degree $2(p^i - 1)$. (BP_*, BP_*BP) is a Hopf algebroid, and, if X is any spectrum, BP_*X is a comodule over this Hopf algebroid, also expressed by saying that

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BP_*X is a BP_*BP-comodule. (See [18] for generalities about BP.) $E^{\wedge}_{n*}$ becomes a BP_*-algebra via the map r given by

$$r(v_i) = \begin{cases} u_i u^{1-p^i} & i < n \\ u^{1-p^n} & i = n \\ 0 & ow. \end{cases} \tag{0.1}$$

(The reader should be aware that the map r does *not* classify the universal lift of Theorem 1.1; in fact, the formal group law classified by r will not be used in this paper.) Now let M be a BP_*BP-comodule which is finitely generated as a BP_*-module. Suppose also that $v_i^{-1}M = 0$ for $0 \le i < n$. (By convention, $v_0 = p$.) Then Morava has shown ([16], see also [7]) that $E^{\wedge}_{n*} \otimes_{BP_*} M$ is naturally a continuous Galois equivariant twisted $S_n - E^{\wedge}_{n*}$ module. We recall the definition of such an object here; again, for more details, the reader is referred to [16] or [7]. First of all, S_n is canonically a profinite group, and Gal $\equiv \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$, the Galois group of automorphisms of $\mathbb{F}_{p^n}$, acts continuously in an evident way on S_n. Gal also acts on $W\mathbb{F}_{p^n}$; letting Gal act trivially on the u_i's and u produces an action of this Galois group on $E^{\wedge}_{n*}$. Now let N be a discrete continuous (graded) $E^{\wedge}_{n*}$-module. (Discreteness is here included only for convenience; note also that N is discrete if and only if $u_i^{-1}N = 0$ for all $0 \le i \le n-1$, where, once again, $u_0 = p$.) Suppose also that N is a Gal-module and a continuous S_n-module. Both of these actions are to be gradation preserving. Then N is said to be a continuous Galois equivariant twisted $S_n - E^{\wedge}_{n*}$ module if the following relations are satisfied for all $n \in N$, $e \in E^{\wedge}_{n*}$, $g \in S_n$, and $\sigma \in$ Gal:

$$\begin{array}{lrcl} \text{i.} & g(en) & = & g(e)g(n) \\ \text{ii.} & \sigma(en) & = & \sigma(e)\sigma(n) \\ \text{iii.} & \sigma(gn) & = & \sigma(g)\sigma(n) \end{array} \tag{0.2}$$

Alternatively, one could call N a continuous twisted $(S_n \rtimes \mathrm{Gal}) - E^{\wedge}_{n*}$ module. The Galois equivariance of the action of S_n on N implies that Gal acts on $H^*_c(S_n; N)$, the continuous cohomology of S_n with coefficients in N. On H^0, this action is just the given action on N.

Now Let M be as before. Then the map

$$v_n^k : \Sigma^{2k(p^n-1)}M \longrightarrow M,$$

given by multiplication by v_n^k, is a BP_*BP-comodule map for some $k > 0$. Hence, $v_n^{-1}M$ is also a BP_*BP-comodule. Morava has proved (again see [16] or [7]) that there is a natural isomorphism

$$\mathrm{Ext}^{*\,*}_{BP_*BP}(BP_*, v_n^{-1}M) \xrightarrow{\approx} H_c^{**}(S_n; E^{\wedge}_{n*} \otimes_{BP_*} M)^{\mathrm{Gal}}.$$

The significance of Morava's change-of-rings theorem is this: Let L_n denote the $E(n)_*$-localization functor. (Recall that $E(n)$ is a spectrum with coefficient ring $\mathbb{Z}_{(p)}[v_1, \ldots, v_{n-1}, v_n, v_n^{-1}]$.) Stable homotopy is "built out of" these localizations as n varies; one may think of L_n as detecting periodicity of order less than or equal to n. If X is any spectrum, Hopkins and Ravenel [20; 8.2-3] have shown that the Adams spectral sequence

$$\mathrm{Ext}^{**}_{BP_*BP}(BP_*, BP_*L_nX) \Rightarrow \pi_*L_nX$$

is strongly convergent. If X is a finite complex with $v_i^{-1}BP_*X = 0$ for all $i < n$; i.e., X is $E(n-1)_*$-acyclic, then

$$BP_*L_nX = v_n^{-1}BP_*X \qquad \text{(see [17;6] and [19])}$$

and so the Adams spectral sequence becomes

$$H_c^{**}(S_n, E_{n*}^{\wedge} \otimes_{BP_*} BP_*X)^{\mathrm{Gal}} \Rightarrow \pi_*L_nX. \tag{0.3}$$

The $E(n)_*$-localizations of complexes of this sort are the constituents of the "n^{th} monochromatic piece" of stable homotopy (cf. §5); to understand their homotopy using the Adams spectral sequence, one must understand the action of S_n on $E_{n*}^{\wedge}$.

Indeed, if $p-1 \nmid n$, S_n is essentially a Poincaré group of dimension n^2 (see Proposition 5.10); hence

$$H_c^i(S_n; E_{n*}^{\wedge} \otimes_{BP_*} BP_*X) = 0, \quad i > n^2,$$

for any $E(n-1)_*$-acyclic finite spectrum X. It further follows by sparseness that, if $BP_iX = 0$ whenever $2(p-1) \nmid i$, and if $2p > n^2+1$, there can be no differentials or extensions in the Adams spectral sequence 0.3 and thus

$$H_c^{s,t}(S_n; E_{n*}^{\wedge} \otimes_{BP_*} BP_*X)^{\mathrm{Gal}} = \begin{cases} \pi_{t-s}L_nX & 0 \le s \le n^2,\ 2(p-1)|t \\ 0 & \text{o w.} \end{cases} \tag{0.4}$$

One might therefore expect in this situation a more direct relationship between $E_{n*}^{\wedge} \otimes_{BP_*} BP_*X$ and L_nX. Such is the case; in §5 we show that (with certain restrictions on X), L_nX is Brown-Comenetz self-dual if and only if $E_{n*}^{\wedge} \otimes_{BP_*} BP_*X$ is Pontryagin self-dual as a continuous Galois equivariant twisted $S_n - E_{n*}^{\wedge}$ module. Understanding Brown-Comenetz duality in the $E(n)_*$-local category is crucial in our approach to the generating hypothesis. As an application of this reduction to algebra, we prove (Theorem 5.3), using our formulas for the action of S_n on $E_{n*}^{\wedge}$, that, at least for $n = 2$, L_nX is not Brown-Comenetz self-dual if $pBP_*X \neq 0$. The converse of this statement (for all n) is proved by Gross and Hopkins.

Our method of determining the action of S_n on $E_{n*}^{\wedge}$ involves a comparison of two approaches to studying lifts of a given formal group (law) Γ of finite height over a characteristic p perfect field k. In §1, we recall the (graded) Lubin-Tate solution to the lifting problem and the definition of the action of S_n on $E_{n*}^{\wedge}$. The coordinates u_i in $E_{n*}^{\wedge}$ are well suited for most of the needs of algebraic topologists—unfortunately, as our formulas will make clear, it would be rather complicated to explicitly determine the action of S_n on u and the u_i's directly from the definition. On the other hand, Cartier has developed a theory for determining $(\mathrm{lifts}_{\Gamma}^{\star})_*(A)$ which applies when A is a local ring with divided power structure whose residue field is k. This theory, together with the requisite formal group theory background, is summarized in §2; an appendix develops what is needed concerning divided power structures and envelopes. Specializing to the case $\Gamma = \Gamma_n$, Cartier's theory allows us to determine certain canonical coordinates w_i, $1 \leq i \leq n-1$, and wu^{-1} in the divided power envelope of $(E_n^{\wedge})_0$ whose transformation under the action of S_n can be easily described. We then determine the universal lift of Γ_n to this divided power envelope in terms of the canonical coordinates. These steps are carried out in §2 and §3. In §4, we compare the universal Lubin-Tate lift of Γ_n, described in terms of the u_i's, to the above universal lift to express the w_i's and w in terms of the u_i's and u.

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1. The Lubin-Tate theory of lifts. In this section we recall the Lubin-Tate theory of lifts and its graded extension. Only the most familiar (to algebraic topologists) parts of the theory of formal group laws are needed here; a good reference is [18; Appendix 2]. In particular, all formal group laws are assumed to be commutative and one-dimensional.

Let k be a perfect field of characteristic p, let n be a positive integer, and let Γ be a height n p-typical formal group law over k. If A is a complete local ring with maximal ideal $\mathfrak{m}$ and residue field a k-algebra, say that a formal group law F is a lift of Γ if $F \equiv \Gamma \bmod \mathfrak{m}$. Two lifts F and G of Γ to A are said to be $\star$-isomorphic if there exists an isomorphism (not necessarily strict) $\varphi : F \longrightarrow G$ such that $\varphi(x) \equiv x \bmod \mathfrak{m}$. If F and G are $\star$-isomorphic, then the $\star$-isomorphism between them is unique [16; Lemma 1.1.2]. Denote by $\mathrm{lifts}_{\Gamma}^{\star}(A)$ the set of $\star$-isomorphism classes of lifts of Γ to A. Note that every lift F of Γ is $\star$-isomorphic to a p-typical one—the canonical strict isomorphism from F to a p-typical formal group law [18; A2.1.18] is a $\star$-isomorphism since Γ is already p-typical. Therefore, we need only consider p-typical lifts, if we wish.

We can now state the main result of [15]. As usual, Wk denotes the ring of Witt vectors with coefficients in k [23; Chapter II §5, §6].

THEOREM 1.1. [15; 3.1] *There exists a lift $\widehat{F}$ of Γ to $Wk[[u_1, \ldots, u_{n-1}]]$ with the following property: If A is a complete Noetherian local ring whose residue field* *is a k-algebra, and if F is a lift of Γ to A, then there exists a unique continuous Wk-algebra homomorphism*

$$\iota_F : Wk[[u_1, \ldots, u_{n-1}]] \longrightarrow A$$

such that $\iota_{F*}\widehat{F}$ is $\star$-isomorphic to F. In other words

$$lifts^{\star}_{\Gamma}(A) = \mathrm{Hom}^{c}_{Wk-alg}(Wk[[u_1, \ldots, u_{n-1}]], A).$$

Remark 1.2. The hypotheses on A can be relaxed considerably. This is discussed in [7], where it proves useful to do so.

For our purposes, Γ will be taken to be the p-typical formal group law Γ_n over $\mathbb{F}_p$ (and hence over k) whose p-series is $[p]_{\Gamma_n}(x) = x^{p^n}$. (Although the notation $[p]_{\Gamma}(x)$ is often used for the p-series, it is not really good notation, since [ ] will mean something quite different in §2.) Then $\widehat{F}$ in Theorem 1.1 can be taken to be the formal group law classified by $\theta : BP_* \to Wk[[u_1, \ldots, u_{n-1}]]$, where

$$\theta(v_i) = \begin{cases} u_i & i < n \\ 1 & i = n \\ 0 & i > n \end{cases} \qquad \text{(cf. [15; 1.1])}. \tag{1.3}$$

We write $\mathrm{lifts}^{\star}_n$ for $\mathrm{lifts}^{\star}_{\Gamma_n}$.

For the "graded" Lubin-Tate theory, we consider pairs (F, u), where F is a lift of Γ to A and u is a unit in A. (F, u) is $\star$-isomorphic to (G, v) provided that F is $\star$-isomorphic to G and the (unique) $\star$-isomorphism $\varphi : F \longrightarrow G$ satisfies $\varphi'(0) \cdot v = u$. Define $(\mathrm{lifts}^{\star}_{\Gamma})_*(A)$ to be the set of equivalence classes of such pairs. Then with the hypotheses of Theorem 1.1,

$$(\mathrm{lifts}^{\star}_{\Gamma})_*(A) = \mathrm{Hom}^{c}_{Wk-alg}(Wk[[u_1, \ldots, u_{n-1}]][u, u^{-1}], A). \tag{1.4}$$

(A continuous homomorphism here is one that maps each u_i into $\mathfrak{m}$.) $E^{\wedge}_{\Gamma *} \equiv Wk[[u_1, \ldots, u_{n-1}]][u, u^{-1}]$ is graded by setting $|u_i| = 0$ and $|u| = -2$; for the meaning of this grading, see [7]. Write $E^{\wedge}_{n*} = E^{\wedge}_{\Gamma_n *} = W\mathbb{F}_{p^n}[[u_1, \ldots, u_{n-1}]][u, u^{-1}]$.

$\mathrm{Aut}_{\Gamma}(k)$, the group of automorphisms of Γ (over k), acts on $(\mathrm{lifts}^{\star}_{\Gamma})_*(A)$ on the right as follows: Suppose $g \in k[[x]]$ is an element of $\mathrm{Aut}_{\Gamma}(k)$. Let $\hat{g} \in Wk[[x]]$ be any power series with $\hat{g}(0) = 0$ and $\hat{g} \equiv g \bmod (p)$. Then define

$$(F, u)g = (\hat{g}^{-1}(F),\ \hat{g}'(0)u),$$

where $\hat{g}^{-1}(F)(x, y) = \hat{g}^{-1}[F(\hat{g}(x), \hat{g}(y))]$. It is easy to check that this definition passes to $\star$-isomorphism classes and is there independent of the choice of lift of g, giving us the desired action. Now a right action on a space yields a left action on the space of functions, which in this case is $E_{\Gamma *}^{\wedge}$. Explicitly, the Wk-algebra homomorphism

$$i(g) : E_{\Gamma *}^{\wedge} \longrightarrow E_{\Gamma *}^{\wedge}$$

given by the action of $g \in \mathrm{Aut}_{\Gamma}(k)$ on $E_{\Gamma *}^{\wedge}$ is defined on $(E_{\Gamma}^{\wedge})_0$ to be the endomorphism of $(E_{\Gamma}^{\wedge})_0$ classifying the lift $\hat{g}^{-1}(\widehat{F})$, where $\hat{g}$ is a lift of g to $(E_{\Gamma}^{\wedge})_0[[x]]$ as above. Let

$$\varphi_{\hat{g}} : i(g)_*\widehat{F} \longrightarrow \hat{g}^{-1}(\widehat{F})$$

be the $\star$-isomorphism. Then

$$i(g)u = \hat{g}'(0) \cdot (\varphi_{\hat{g}})'(0) \cdot u.$$

Note that this action preserves the grading.

In particular, the above construction gives an action of $S_n = \mathrm{Aut}_{\Gamma_n}(\overline{\mathbb{F}}_p) = \mathrm{Aut}_{\Gamma_n}(\mathbb{F}_{p^n})$ ([18; A2.2.20]) on $E_{n*}^{\wedge}$. Here $\overline{\mathbb{F}}_p$ denotes the algebraic closure of $\mathbb{F}_p$.

2. The Cartier theory of lifts. For this section we require some more sophisticated aspects of the theory of (commutative) formal groups. Our basic reference is [14]. Begin by defining a nilalgebra over a basic ring R to be a commutative R-algebra without unit, all of whose elements are nilpotent. For each indexing set I, one has a functor $D^{(I)}$ from the category of nilalgebras over R to the category of pointed sets, defined by

$$D^{(I)}(B) = \bigoplus_I B = B^{(I)}$$

for each nilalgebra B. A formal variety over R is a functor from the category of nilalgebras over R to the category of pointed sets, which is isomorphic to the functor $D^{(I)}$ for some I. Finally, a formal group over R is defined to be a commutative group object in the category of formal varieties over R.

A formal group law over R is a formal group over R together with a choice of coordinate system. Indeed, let $\mathbb{N}^{(I)}$ denote the set of I-multi-indices whose entries are nonnegative integers, only finitely many of which are not zero. Then the power series

$$f(x) = \sum_{\alpha \in \mathbb{N}^{(I)}} c_\alpha x^\alpha,$$

where $x = (x_i)_{i \in I}$, $c_\alpha \in R^{(J)}$ for all $\alpha \in \mathbb{N}^{(I)}$, and $c_0 = 0$, yields a natural transformation $D^{(I)} \to D^{(J)}$. There is a bijective correspondence between such power series and such natural transformations [14; I 3.2 and 5.2]. Thus, if G is a formal group, an isomorphism $D^{(I)} \xrightarrow{\simeq} G$ yields a multiplication $D^{(I)} \times D^{(I)} \to$

$D^{(I)}$ represented by a power series satisfying the axioms for an $|I|$-dimensional formal group law. Different isomorphisms $D^{(I)} \xrightarrow{\cong} G$ lead to isomorphic formal group laws.

Now let Γ be a p-typical height n formal group law over the perfect field k of characteristic p, and let A be a complete local ring whose residue field is a k-algebra K. Let $\pi : A \to K$ be the quotient map. Γ may also be regarded as a formal group over k; take Γ to be the formal variety D with multiplication $D \times D \to D$ given by the power series defining the formal group law Γ. Note finally that Γ may be considered as a formal group over K; in general, if G is a formal group over R and $f : R \to S$ is a ring homomorphism, then f_*G, defined in the evident way, is a formal group over S. With these notations we have the following easy result.

PROPOSITION 2.1. *$lifts^{\star}_{\Gamma}(A)$ may be identified with equivalence classes of couples (G, δ), where G is a formal group over A and $\delta : \Gamma \to \pi_* G$ is an isomorphism of formal groups over K. (G, δ) is equivalent to (G', δ') provided there exists an isomorphism (of formal groups) $f : G \to G'$ such that $\pi_* f \circ \delta = \delta'$. If such an f exists, it is unique.*

We are, however, interested in interpreting $(\mathrm{lifts}^{\star}_{\Gamma})_*(A)$. This requires consideration of $\mathfrak{T}G$, the tangent space of G.

Let V be a formal variety over R, and let CV denote the collection of curves in V; that is, the set of natural transformations $\gamma : D \to V$. Let

$$\mathrm{nil}(R, n) = \text{collection of nilalgebras over } R \text{ such that every product of } n+1 \text{ elements is } 0.$$

Then define $\mathfrak{T}V$ to be the collection of equivalence classes of curves in V, where two curves γ_0, γ_1 are equivalent if $\gamma_0 | \mathrm{nil}\,(R,1) = \gamma_1 | \mathrm{nil}\,(R,1)$. If one chooses an isomorphism $V \xrightarrow{\cong} D^{(I)}$ so that γ_0 and γ_1 are represented by power series, then γ_0 is equivalent to γ_1 if and only if they have the same terms in degree 1. Since a choice of coordinate system induces an isomorphism $\mathfrak{T}V \xrightarrow{\cong} \mathfrak{T}D^{(I)} = R^{(I)}$, $\mathfrak{T}V$ can be given the structure of an R-module. This R-module structure is in fact independent of the choice of coordinate system (cf. [14; I 6.6]), and $\mathfrak{T}$ is a functor from the category of formal varieties to the category of (free) R-modules. Furthermore, a map f of formal varieties is an isomorphism if and only if $\mathfrak{T}f$ is an isomorphism of R-modules [14; I 8.1].

PROPOSITION 2.2. *$(lifts^{\star}_{\Gamma})_*(A)$ may be identified with equivalence classes of triples (G, δ, ε), where G, δ are as in Proposition 2.1 and $\varepsilon : \mathfrak{T}G \to A$ is an isomorphism. (G, δ, ε) is equivalent to $(G', \delta', \varepsilon')$ provided there exists a formal* *group isomorphism* $f : G \to G'$ *such that* $\pi_* f \circ \delta = \delta'$ *and* $\varepsilon' \circ \mathfrak{T} f = \varepsilon$. *The action of* $\mathrm{Aut}_\Gamma(k)$ *on* $(\mathit{lifts}^\star_\Gamma)_*(A)$ *is given by*

$$(G, \delta, \varepsilon)g = (G, \delta \circ g, \varepsilon), \quad g \in \mathrm{Aut}_\Gamma(k).$$

Proof. Let $(F, u) \in (\mathrm{lifts}^\star_\Gamma)_*(A)$. As before, F may be regarded as the formal group G_F whose underlying formal variety is D with multiplication given by the power series F. Let $\varepsilon_u : \mathfrak{T} G_F = \mathfrak{T} D = A \to A$ be defined by $\varepsilon_u(1) = u$. Then let (G_F, id, ε_u) correspond to (F, u). It is easy to see that this correspondence defines a bijection between $(\mathrm{lifts}^\star_\Gamma)_*(A)$ and equivalence classes of triples (G, δ, ε).

As for the description of the group action, chase down the identifications.

Cartier's solution to the lifting problem utilizes the equivalence between the category of formal groups over R and a certain category of modules over a ring $\mathrm{Cart}\, R$. This equivalence is given by the Dieudonné module of a formal group. In more detail, consider the curves functor C from the category of formal groups over R to the category of abelian groups. (If G is a formal group, the abelian group structure on CG arises in the usual way from the structure of G as a commutative group object in the category of formal varieties over R.) Let $\mathrm{Cart}\, R$ be the ring of natural transformations $C \to C$. $\mathrm{Cart}\, R$ is in fact a topological ring and CG is a topological $\mathrm{Cart}\, R$-module [14; III 5–7]. As a ring, $\mathrm{Cart}\, R$ is generated by the homotheties $[c]$ with $c \in R$ [14; I 10], the verschiebungen V_n for each positive integer n [14; I 10], and the frobenius operators F_n for each positive integer n [14; III 3]. The functor C is fully faithful upon restricting its target to the subcategory of continuous $\mathrm{Cart}\, R$-modules (and continuous $\mathrm{Cart}\, R$-module maps) [14; III 8.4]; moreover, if the target is further restricted to the full subcategory of *reduced* $\mathrm{Cart}\, R$-modules [14; III 7.15], C becomes an equivalence [14; III 10, 11].

Unfortunately, $\mathrm{Cart}\, R$ is a very large and unmanageable ring. However, if R is p-local, the study of formal groups over R reduces to the study of modules over a smaller ring $\mathrm{Cart}_p R$. Here is a summary of the situation: If G is a formal group over (the p-local ring) R, let $C_p G$ denote the group of p-typical curves of G; i.e., those curves $\gamma \in CG$ such that $F_q \gamma = 0$ for all primes $q \neq p$. $\mathfrak{T} G$ has a basis consisting of p-typical curves ([14; IV, 8.10]); in different language, this says that any formal group law over a p-local ring is strictly isomorphic to a p-typical one. Moreover, in Lazard's theory, this fact is crucial in proving that G is a p-typical formal group in a unique way [14; IV, 7–8]. Now $C_p G$ is a topological left $\mathrm{Cart}_p R$ module; the ring $\mathrm{Cart}_p R$ is a quotient ring of a subring of $\mathrm{Cart}\, R$ ([14; IV, 6.3]) and is generated by the homotheties, the verschiebung $V_p \equiv V$, and the frobenius $F_p \equiv F$. For the reader's convenience, we summarize the relations determining the structure of $\mathrm{Cart}_p R$.

DEFINITION THEOREM 2.3. ([14; IV, 2–4; VI, 1])

i. *Every element x of* $\mathrm{Cart}_p R$ *has a unique representation as*

$$x = \sum_{m,n \geq 0} V^m [x_{m,n}] F^n.$$

The sum may be infinite, but for each m, only finitely many $x_{m,n}$ may be nonzero.

ii.

$$\begin{aligned} [a]V &= V[a^p] \\ F[a] &= [a^p]F \\ [a][b] &= [ab] \\ FV &= p \cdot 1_{\mathrm{Cart}_p R} = p \cdot [1] \end{aligned}$$

iii. *Let $\{x_k\}$ be a sequence of indeterminates indexed on the nonnegative integers, and define*

$$w_n(\{x_k\}) = \sum_{i=0}^{n} p^i x_i^{p^{n-i}}$$

for each $n \geq 0$. Then, for all sequences $\{a_k\}$, $\{b_k\}$ in R, we have

$$\sum_{n \geq 0} V^n [a_n] F^n + \sum_{n \geq 0} V^n [b_n] F^n = \sum_{n \geq 0} V^n [c_n] F^n,$$

where

$$w_n(\{a_k\}) + w_n(\{b_k\}) = w_n(\{c_k\})$$

for all $n \geq 0$. In particular,

$$[a] + [b] = [a + b] \bmod V \,\mathrm{Cart}_p R.$$

iv. *The right ideals, V^n* $\mathrm{Cart}_p R$, *define a complete filtration on* $\mathrm{Cart}_p R$. $\mathrm{Cart}_p R$ *is a topological ring with the topology given by this filtration.*

Remark 2.4. $\mathrm{Cart}_p R$ is the ring of natural transformations from C_p to C_p (cf. [14; IV, 7.20]).

Remark 2.5. Let WR denote the ring of Witt vectors with coefficients in R ([23; II, §6]). Then the map

$$WR \xrightarrow{op} \mathrm{Cart}_p R$$

defined by

$$\{a_k\} \mapsto \sum_{n\geq 0} V^n[a_n]F^n$$

is a ring isomorphism onto a subring of $\mathrm{Cart}_p R$ ([14; IV, 4]).

DEFINITION 2.6. *A left* $\mathrm{Cart}_p R$*-module* M *is reduced if:*

i. $M = \varprojlim_n M/V^nM$.

ii. $V^n : M/VM \longrightarrow V^nM/V^{n+1}M$ *is an isomorphism for all* n.

iii. M/VM *is a free* R*-module, the* R*-module structure being given by* $am \equiv [a]m$ *for* $m \in M/VM$.

Remark 2.7. Let M be reduced, and give M the topology obtained from the filtration V^nM, $n \in \mathbb{N}$. Then M is a topological $\mathrm{Cart}_p R$-module.

C_pG is a reduced $\mathrm{Cart}_p R$-module [14; IV, 7.9]. Indeed, all reduced $\mathrm{Cart}_p R$-modules arise in this way—the functor C_p from formal groups over R to reduced $\mathrm{Cart}_p R$-modules is an equivalence [14; IV, 7.12]. We also note for future use that

$$C_pG/VC_pG = \mathfrak{T}G.$$

Let us call C_pG, regarded as a topological $\mathrm{Cart}_p R$-module, the Cartier module of G.

Although $\mathrm{Cart}_p R$ is still in general unwieldly, the theory simplifies greatly when R is a perfect field k of characteristic p. Recall that in this case, Wk is a discrete valuation ring with uniformizing parameter p and residue field k. It also has a frobenius automorphism σ defined by

$$\sigma(a_0, a_1, a_2, \ldots) = (a_0^p, a_1^p, a_2^p, \ldots). \tag{2.8}$$

Regard Wk as a subring of $\mathrm{Cart}_p k$ via the map in 2.5. Then $\mathrm{Cart}_p k$ is obtained by appropriately adjoining V and F to Wk, subject to the following relations:

$$\begin{aligned} &\text{i.} && VF = p = FV \\ &\text{ii.} && aV = Va^\sigma \text{ for all } a \in Wk \\ &\text{iii.} && Fa = a^\sigma F \text{ for all } a \in Wk \end{aligned} \tag{2.9}$$

Properties ii and iii follow from i and 2.3. A proof that $VF = p$ may be found in [14; IV, 4].

Let G be a formal group over k. Of course, multiplication by V is always a monomorphism on C_pG; however, by 2.9i, multiplication by F is a monomorphism if and only if C_pG has no p-torsion. If G is finite dimensional, this is the case if and only if C_pG is a free Wk-module of finite rank [14; VI, 7]. This rank is defined to be the height of G; it agrees in the one-dimensional case with the definition of height given in [18].

Now let Γ be a height n (one-dimensional) formal group over k, and let M denote its Cartier module. The next result is Cartier's solution to the lifting problem.

THEOREM 2.10. ([4], [14; VII]). *Let $A = Wk$ with maximal ideal $\mathfrak{m}$. $(lifts^\star_\Gamma)_*(A)$ is in bijective correspondence with the set of Wk-linear maps $M \xrightarrow{t} A$ for which there exists an isomorphism $A/\mathfrak{m} \to M/VM$ of k-vector spaces such that the composite*

$$M \xrightarrow{t} A \to A/\mathfrak{m} \to M/VM$$

is the projection.

Remark 2.11. In [4], Cartier claims this result to be valid for A any complete local ring with a divided power structure and with residue field k. He, however, does not give a complete proof; that is done in [14; VII] but only for the ring Wk. For our purposes this is enough.

We will need to understand the construction of the map $t : M \to Wk$ from an element of $(\text{lifts}^\star_n)_*(Wk)$. This construction will be used to determine certain canonical coordinates w_i, w in the divided power envelope of the ring $W\mathbb{F}_{p^n}[[u_1, \dots, u_{n-1}]][u, u^{-1}]$ of functions on the moduli space of $\star$-isomorphism classes of lifts of Γ_n. The action of S_n on these canonical coordinates will be easy to understand. We will also use this construction to express the universal lift of Γ_n in terms of the w_i's. This will in turn allow us to express one set of coordinates in terms of the other, thus giving us the action of S_n on u and the u_i's.

Following Lazard, we proceed to outline the construction of this t. First, however, some preliminaries are needed.

PROPOSITION 2.12. ([14; VII, 6.3]). *Write $\Lambda = Wk$. There exists a unique ring homomorphism $\Delta : \Lambda \to W\Lambda$ satisfying*

$$w_n(\Delta(\xi)) = \xi^{\sigma^n}$$

for all $n \in \mathbb{N}$ and $\xi \in \Lambda$. (See 2.3iii and 2.8 for the notation.)

DEFINITION 2.13. *Let M' be a* Cart_p *k-module, and let C' be a* Cart_p *Wk-module. An additive map $f' : M' \to C'$ is (W, F)-linear if*

$$f(\xi\gamma) = \Delta(\xi)f(\gamma)$$
$$f(F\gamma) = Ff(\gamma)$$

for all $\xi \in Wk$ and $\gamma \in M'$.

THEOREM 2.14. ([14; VII, 6.14]). *Let $(G, \delta) \in \mathit{lifts}^\star_\Gamma(Wk)$, and let π_* be the composite*

$$C_p(G) \to C_p(\pi_* G) \xrightarrow{C_p(\delta^{-1})} M \equiv C_p(\Gamma).$$

Then there exists a unique (W, F)-linear section $s : M \to C_pG$ of π_*.

Now, if $(G, \delta, \varepsilon) \in (\mathrm{lifts}^\star_\Gamma)_*(Wk)$, let s be as in 2.14, and consider the composite

$$t : M \xrightarrow{s} C_p(G) \to C_pG/VC_pG = \mathfrak{T}G \xrightarrow{\varepsilon} Wk. \tag{2.15}$$

t is the map corresponding to (G, δ, ε) in Theorem 2.10.

COROLLARY 2.16. *Let (G, δ, ε) and t be as above, and let $g \in \mathrm{Aut}_\Gamma(k)$. Then*

$$M \xrightarrow{g} M \xrightarrow{t} Wk$$

is the map corresponding to $(G, \delta, \varepsilon)g \in (\mathit{lifts}^\star_\Gamma)_*(Wk)$. Here g also denotes the map induced by g on the Cartier module of Γ.

Proof. Use Proposition 2.2 and the definition of the correspondence in Theorem 2.10.

Remark 2.17. Although not needed for this paper, the following more conceptual account of the bijection in 2.10 may help the reader understand [14; VII] and [4]. One starts ([14; VII, 6.17–6.23]) by constructing a certain (n-dimensional) formal group G^* over Wk with tangent space M together with a canonical (W, F)-linear section

$$\tau : M \to C_p(G^*)$$

of the projection

$$C_p(G^*) \to C_p(G^*)/VC_p(G^*) = M.$$

(Despite the notation, G^* is constructed independently of any lift G of Γ.) G^* has the following universal property: For each reduced Cart_p Wk-module C' and

(W, F)-linear map $s : M \longrightarrow C'$, there exists a unique map $\alpha : C_p(G^*) \longrightarrow C'$ of $\mathrm{Cart}_p\, Wk$-modules satisfying $s = \alpha \circ \tau$. In particular, with the notation of Theorem 2.14, there exists a unique homomorphism $f : G^* \longrightarrow G$ of formal groups such that

$$\begin{tikzcd} M \arrow[r, "s"] \arrow[d, "\tau"'] & C_pG \\ C_p(G^*) \arrow[ur, "f"'] & \end{tikzcd}$$

commutes. The map t of 2.15 is then just the map $\mathfrak{T}f : \mathfrak{T}G^* \longrightarrow \mathfrak{T}G$. Moreover, f has additional significance. Not only is it surjective as a map from $C_p(G^*) \longrightarrow C_pG$, but its kernel is the Cartier module of a maximal embedded additive formal group A_G ([14; VII, 7.5–7.10]). The extension

$$0 \longrightarrow A_G \longrightarrow G^* \longrightarrow G \longrightarrow 0$$

turns out to be the "universal extension of G with additive kernel" ([14; VII, 8.5]).

Finally, one uses G^* to construct a lift of Γ from a map $t' : M \longrightarrow Wk$. Indeed, consider the kernel L of t'. If t' satisfies the hypotheses of Theorem 2.10, then L is a direct summand of M such that $L + pM = VM$. This implies ([14; VII, 7]) that L is the tangent space of a unique maximal embedded additive subgroup A of G^*. Furthermore, there is a unique isomorphism

$$\delta : \Gamma \longrightarrow \pi_*(G^*/A)$$

such that the diagram

$$\begin{tikzcd} C_p(G^*) \arrow[r] \arrow[d] & C_p(G^*)/C_p(A) = C_p(G^*/A) \arrow[d] \\ C_p(G^*)/VC_p(G^*) = M \arrow[r, "C_p(\delta)"] & C_p(\pi_*(G^*/A)) \end{tikzcd}$$

commutes. Let ϵ be the map

$$\mathfrak{T}(G^*/A) = M/L \xrightarrow{t'} Wk,$$

where we have also written t' for its factorization through M/L. Then $(G^*/A, \delta, \epsilon)$

is an element of $(\mathrm{lifts}^\star_\Gamma)_*(Wk)$, and one can show that the correspondence

$$t' \mapsto (G^*/A, \delta, \epsilon)$$

is the inverse to the correspondence described earlier.

For the remainder of this section, we specialize to the case $k = \mathbb{F}_{p^n}$ and $\Gamma = \Gamma_n$. We first explicitly describe $M = C_p(\Gamma_n)$ and S_n as the group of automorphisms of M. By Theorem 2.10 and Corollary 2.16, this gives a very good description of $(\mathrm{lifts}^\star_n)_*(W\mathbb{F}_{p^n})$, which will allow us to understand the action of S_n on the canonical coordinates.

PROPOSITION 2.18. *Let $\gamma \in M$ be the curve $\gamma(x) = x$. Then M is the free $W\mathbb{F}_{p^n}$-module with generators $\gamma, V\gamma, \ldots, V^{n-1}\gamma$. The action of the frobenius operator is given by $F\gamma = V^{n-1}\gamma$.*

Proof. By the definition of Γ_n and γ, together with 2.9i, we have

$$VF\gamma = p\gamma = V^n\gamma.$$

Hence $F\gamma = V^{n-1}\gamma$. Now the image of γ in $\mathfrak{T}\Gamma_n$ is a generator; therefore, every curve $\zeta \in M$ is written uniquely as

$$\zeta = \sum_{i\geq 0} V^i[a_i]\gamma, \quad a_i \in \mathbb{F}_{p^n}.$$

But this is equivalent to

$$\zeta = b_0\gamma + b_1 V\gamma + \cdots + b_{n-1}V^{n-1}\gamma$$

where

$$b_i = \sum_{k\geq 0} p^k[a_{nk+i}^{p^{-i}}] \in W\mathbb{F}_{p^n},$$

proving that $\gamma, V\gamma, \ldots, V^{n-1}\gamma$ is a basis for M.

PROPOSITION 2.19. *For each $\zeta \in M$, there exists a unique endomorphism g of Γ_n such that $g\gamma = \zeta$.*

Proof. By the preceding proposition, it is clear that if such a g exists, it is unique. To prove existence, write

$$\zeta = \sum_{i=o}^{n-1} a_i^{\sigma^{-i}} V^i\gamma = \sum_{i=0}^{n-1} V^i a_i \gamma, \quad a_i \in W\mathbb{F}_{p^n},$$

and define a $W\mathbb{F}_{p^n}$-module map $g : M \to M$ by

$$g(V^j\gamma) = \sum_{i=0}^{n-1} V^{i+j}a_i\gamma, \quad 0 \le j \le n-1.$$

Certainly $g\gamma = \zeta$ and g commutes with V. Moreover,

$$\begin{aligned} g(F\gamma) &= g(V^{n-1}\gamma) = \sum_{i=0}^{n-1} V^{n-1+i}a_i\gamma \\ &= \sum_{i=0}^{n-1} V^i a_i^{\sigma^{-(n-1)}} V^{n-1}\gamma \\ &= \sum_{i=0}^{n-1} V^i a_i^{\sigma^{-(n-1)}} F\gamma \\ &= \sum_{i=0}^{n-1} V^i F a_i^{\sigma^{-n}}\gamma \\ &= F\sum_{i=0}^{n-1} V^i a_i\gamma = Fg(\gamma), \end{aligned}$$

where 2.9 has been used several times. Thus g is a map of $\mathrm{Cart}_p\,\mathbb{F}_{p^n}$-modules, completing the proof.

Continue with the notation of the last proposition and its proof. Then, with respect to the ordered basis $(\gamma, V\gamma, \ldots, V^{n-1}\gamma)$, g is represented by the matrix

$$\begin{bmatrix} a_0 & pa_{n-1} & pa_{n-2} & & pa_1 \\ a_1^{\sigma^{-1}} & a_0^{\sigma^{-1}} & pa_{n-1}^{\sigma^{-1}} & & pa_2^{\sigma^{-1}} \\ \cdot & \cdot & \cdot & & \vdots \\ \cdot & \cdot & \cdot & \cdots & \\ \cdot & \cdot & \cdot & & pa_{n-1}^{\sigma^{-(n-2)}} \\ a_{n-1}^{\sigma^{-(n-1)}} & a_{n-2}^{\sigma^{-(n-1)}} & a_{n-3}^{\sigma^{-(n-1)}} & & a_0^{\sigma^{-(n-1)}} \end{bmatrix} \tag{2.20}$$

Note also that g is an automorphism if and only if the image of ζ in $\mathfrak{T}\Gamma_n$ is a generator; i.e., a_0 is a unit in $W\mathbb{F}_{p^n}$ (cf. discussion of $\mathfrak{T}$ preceding Proposition 2.2). 2.20 therefore gives us a convenient matrix representation of S_n.

Remark 2.21. From the power series point of view, S_n consists of all power series of the form

$$g = \sum_{i\geq 0}{}^{\Gamma} b_i x^{p^i}, \quad b_i \in \mathbb{F}_{p^n}, \ b_0 \neq 0.$$

Then

$$g\gamma = \sum_{i\geq 0}{}^{\Gamma} V^i[b_i]\gamma = \sum_{i=0}^{n-1} V^i a_i \gamma$$

where

$$a_i = \sum_{k\geq 0} p^k [b_{nk+i}].$$

Recall also that in [18; A2.2.17], S_n is identified with the group of units of the ring obtained by adjoining an indeterminate S to $W\mathbb{F}_{p^n}$ with relations $S^n = p$ and $Sw = w^\sigma S$ for all $w \in W\mathbb{F}_{p^n}$. The element g then corresponds to $\sum_{i=0}^{n-1} a_i S^i$.

3. The universal lift in Cartier's theory. In this section we use the theory of §2 to describe the action of S_n on, and the universal lift of Γ_n in terms of, the previously advertised canonical coordinates. Write $W \equiv W\mathbb{F}_{p^n}$. Our first result provides the definition of these coordinates.

PROPOSITION 3.1. *Let $(G, \delta, \varepsilon) \in (lifts_n^\star)_*(W)$ and let $t : M \to W$ be the map corresponding to (G, δ, ε) in the bijection of Theorem 2.10. Let $\gamma \in M = C_p(\Gamma_n)$ be the curve $\gamma(x) = x$. Then the map*

$$\mu : (lifts_n^\star)_*(W) \to \mathrm{Hom}^c_{W-alg}(W[[w_1, \ldots, w_{n-1}]][w, w^{-1}], W)$$

defined by $\mu(G, \delta, \varepsilon) = h_t$, where

$$\begin{aligned} h_t(ww_i) &= t(V^{n-i}\gamma), \quad 1 \leq i \leq n-1 \\ h_t(w) &= t(\gamma), \end{aligned}$$

is an isomorphism. (Once again, a W-algebra homomorphism

$$h : W[[w_1, \ldots, w_{n-1}]][w, w^{-1}] \to W$$

is said to be continuous if it maps the maximal ideal of $W[[w_1, \ldots, w_{n-1}]]$ into the maximal ideal of W.)

Proof. Clear.

Remark 3.2. If c is a unit of W, and $t' : M \to W$ corresponds to $(G, \delta, c\varepsilon)$, then $h_{t'} = ch_t$, so that

$$\begin{aligned} h_{t'}(w_i) &= h_t(w_i), \ 1 \le i \le n-1 \\ h_{t'}(w) &= ch_t(w). \end{aligned}$$

This means that each w_i has gradation 0 and w has gradation -2 (see discussion of gradings in [7]; also compare Theorem 3.7ii).

PROPOSITION 3.3. *The (right) action of S_n on $(lifts_n^\star)_*(W)$ is induced by a left action of S_n on $W[[w_1, \dots, w_{n-1}]][w, w^{-1}]$ which is linear on the coordinates $ww_1, \dots, ww_{n-1}, w$. If $g \in S_n$ is given by*

$$g\gamma = \sum_{i=0}^{n-1} V^i a_i \gamma, \quad a_i \in W \ (cf.\ Prop.\ 2.19),$$

then

$$g(w) = a_0 w + \sum_{j=1}^{n-1} a_{n-j}^{\sigma^j} ww_j$$

and

$$\begin{aligned} g(ww_i) &= pa_i w + pa_{i+1}^{\sigma^{n-1}} ww_{n-1} + \cdots + pa_{n-1}^{\sigma^{i+1}} ww_{i+1} \\ &\quad + a_0^{\sigma^i} ww_i + \cdots + a_{i-1}^{\sigma} ww_1. \end{aligned}$$

Proof. Combine 2.16, 2.19, 2.20, and 3.1.

It follows from Remark 3.2 that μ of Proposition 3.1 induces an isomorphism

$$(3.4) \qquad \mu_0 = (\mathrm{lifts}_n^\star)(W) \to \mathrm{Hom}^c_{W-alg}(W[[w_1, \dots, w_{n-1}]], W).$$

One might ask whether there exists a lift $\widetilde{F}$ of Γ_n to $W[[w_1, \dots, w_{n-1}]]$ such that, for every $(G, \delta) \in \mathrm{lifts}_n^\star(W)$, $\mu_0(G, \delta)_* \widetilde{F}$ is $\star$-isomorphic to (G, δ). This is not the case[1]; however, observe that since W has a divided power structure, we have isomorphisms

$$(3.5) \qquad \begin{aligned} \mu : (\mathrm{lifts}_n^\star)_*(W) &\to \mathrm{Hom}^{P.D.}_{W-alg}(W\langle\langle w_1, \dots, w_{n-1}\rangle\rangle[w, w^{-1}], W) \\ \mu_0 : (\mathrm{lifts}_n^\star)(W) &\to \mathrm{Hom}^{P.D.}_{W-alg}(W\langle\langle w_1, \dots, w_{n-1}\rangle\rangle, W) \end{aligned}$$

[1]Indeed, if such an $\widetilde{F}$ existed, it would follow that the image of the map ι of §4 would lie in $W[[w_1, \dots, w_{n-1}]][w, w^{-1}]$. But Theorem 4.4 shows that this does not happen.

and that the action of S_n on $W[[w_1, \dots, w_{n-1}]][w, w^{-1}]$ passes to an action on $W\langle\langle w_1, \dots, w_{n-1}\rangle\rangle[w, w^{-1}]$. Here $W\langle\langle w_1, \dots, w_{n-1}\rangle\rangle$ is the divided power envelope of the W-algebra $W[[w_1, \dots, w_{n-1}]]$ and once again, the target of μ is to be interpreted as those W-algebra homomorphisms whose restriction to $W\langle\langle w_1, \dots, w_{n-1}\rangle\rangle$ is a *P.D.* map. (See the appendix for a discussion of divided powers.) Now one might ask whether such an $\widetilde{F}$ exists over $W\langle\langle w_1, \dots, w_{n-1}\rangle\rangle$. This is the case; indeed, the existence of $\widetilde{F}$ is implied by Remark 2.11. However, we will provide an explicit construction of $\widetilde{F}$ (which is needed for our purposes) without using 2.11.

To determine the universal lift $\widetilde{F}$, we will compute what its logarithm must be and then show that this logarithm produces a formal group law over $W\langle\langle w_1, \dots, w_{n-1}\rangle\rangle$. The following result shows how to use the Cartier module of a formal group to compute a logarithm for it. It is implicit in [14], but never really stated, so we give a proof here.

PROPOSITION 3.6. *Let G be a one-dimensional formal group over a $\mathbb{Q}$-algebra R, and let $\gamma \in C_p(G)$ such that $\gamma : D \to G$ is an isomorphism of formal varieties. (In the language of [14], γ is called a basic curve.) Let π denote the composite*

$$C_pG \to \mathfrak{T}G \xrightarrow{\mathfrak{T}\gamma^{-1}} \mathfrak{T}D = R.$$

Then

$$\log_{G,\gamma}(x) = \sum_{k\geq 0} (\pi(F^k\gamma)/p^k)x^{p^k}$$

where $\log_{G,\gamma}$ is the logarithm of the formal group law defined by G and the coordinate system γ.

Proof. By naturality, we may assume that $G = D$ and $\gamma(x) = x$, so that $\log_{G,\gamma}$ is just the logarithm of G, regarded as a (p-typical) formal group law. Now $\log_{G,\gamma}$ is by definition the power series representation of the unique formal group isomorphism $G \to \mathfrak{T}G^+$ which is the identity on $\mathfrak{T}G$. Here $\mathfrak{T}G^+$ denotes the additive formal group whose tangent space is the R-module $\mathfrak{T}G = R$ (cf. [14; I, 5]). Thus

$$\log_{G,\gamma} : C_p(G) \to C_p(\mathfrak{T}G^+)$$

is a map of $\mathrm{Cart}_p\, R$-modules; the power series $\log_{G,\gamma}(x)$ is just the power series representation of the curve

$$\log_{G,\gamma}(\gamma) = \sum_{k\geq 0} a_k x^{p^k}, \ a_k \in R.$$

Then

$$\log_{G,\gamma}(F^i\gamma) = F^i\left(\sum_{k\geq 0} a_k x^{p^k}\right) = \sum_{k\geq i} p^i a_k x^{p^{k-i}}$$

by the computation of the action of the frobenius operators on the curves of the additive group ([14; III, 3.16]). Since $\log_{G,\gamma}$ is the identity on tangent spaces, it follows that

$$\pi(F^i\gamma) = p^i a_i.$$

This completes the proof.

The next result proves the existence of $\widetilde{F}$ and gives its logarithm.

THEOREM 3.7. *Let*

$$\ell(x) = \sum_{i\geq 0} x^{p^{ni}}/p^i,$$

and let $\widetilde{F}$ be the formal group law over $(W\otimes\mathbb{Q})[[w_1,\ldots,w_{n-1}]]$ with

$$\log_{\widetilde{F}}(x) = \ell(x) + \frac{1}{p}\cdot[w_1\ell(x^p) + w_2\ell(x^{p^2}) + \cdots + w_{n-1}\ell(x^{p^{n-1}})].$$

Then

i. *$\widetilde{F}$ is a lift of Γ_n to $W\langle\langle w_1,\ldots w_{n-1}\rangle\rangle$.*

ii. *$(\mu(G,\delta,\varepsilon)_*\widetilde{F},\ \mu(G,\delta,\varepsilon)(w))$ is $\star$-isomorphic to (G,δ,ε) for all $(G,\delta,\varepsilon)\in(lifts_n^\star)_*(W)$.*

Proof. We assume i for now and prove ii. Let $s: M\to C_pG$ be the (W,F)-linear section of 2.14, and let $\gamma\in M$ be the usual basic curve. Then $s(\gamma)$ is a basic curve; furthermore, since s is a section of π_* (see 2.14), the formal group law F obtained from G using the coordinate system $s(\gamma)$ is a lift of the formal group law Γ_n, and $(G,\delta,\mathfrak{T}s(\gamma)^{-1})$ is $\star$-isomorphic to $(F,1)$. By Proposition 3.6,

$$\log_F(x) = \sum_{k\geq 0}[\pi(F^k s(\gamma))/p^k]x^{p^k},$$

where now

$$\pi: C_pG \to \mathfrak{T}G \xrightarrow{\mathfrak{T}s(\gamma)^{-1}} \mathfrak{T}D = W.$$

But $\pi(F^k s(\gamma)) = t(F^k\gamma)$, where

$$t: M\xrightarrow{s} C_pG\to\mathfrak{T}G\xrightarrow{\mathfrak{T}s(\gamma)^{-1}}\mathfrak{T}D = W.$$

t is the map corresponding to $(G, \delta, \mathfrak{T}s(\gamma)^{-1})$ in Theorem 2.10. Now using the known structure of M (Proposition 2.18), we have

$$\begin{aligned}
\log_F(x) &= \sum_{k\geq 0} \left(t(F^k\gamma)/p^k\right) x^{p^k} \\
&= \sum_{r=0}^{n-1}\sum_{j\geq 0} \left(t(F^{jn+r}\gamma)/p^{jn+r}\right) (x^{p^r})^{p^{jn}} \\
&= \sum_{r=0}^{n-1}\sum_{j\geq 0} \left(p^{j(n-1)}/p^{jn+r}\right) \cdot t(V^{r(n-1)}\gamma)(x^{p^r})^{p^{jn}} \\
&= \sum_{j\geq 0} \left(t(\gamma)/p^j\right) x^{p^{jn}} + \sum_{r=1}^{n-1}\sum_{j\geq 0} \frac{1}{p^{r+j}} \cdot t\left(V^{(r-1)n+n-r}\gamma\right) (x^{p^r})^{p^{jn}} \\
&= t(\gamma)\ell(x) + \sum_{r=1}^{n-1}\sum_{j\geq 0} \left(p^{r-1}/p^{r+j}\right) \cdot t(V^{n-r}\gamma)(x^{p^r})^{p^{jn}} \\
&= t(\gamma)\ell(x) + \sum_{r=1}^{n-1} \frac{1}{p} \cdot t(V^{n-r}\gamma)\ell(x^{p^r}) \\
&= \ell(x) + \sum_{r=1}^{n-1} (\mu_0(w_r)/p)\ell(x^{p^r}),
\end{aligned}$$

where $\mu_0 = \mu_0(G, \delta)$. Thus $F = \mu_{0*}\widetilde{F}$. To complete the proof of ii, we need only show that

$$\mu(G, \delta, \varepsilon)(w)^{-1} = (\mathfrak{T}s(\gamma)^{-1} \circ \varepsilon^{-1})(1).$$

But this follows from the fact that $\mu(G, \delta, \mathfrak{T}s(\gamma)^{-1})(w) = 1$ together with Remark 3.2.

We now prove i. Write

$$\sum_{i\geq 0} \lambda_i x^{p^i} = \log_{\widetilde{F}}(x), \quad \lambda_i \in (W \otimes \mathbb{Q})[[w_1, \ldots, w_{n-1}]], \tag{3.8}$$

and let v_i also denote the image of the Hazewinkel generator v_i under the map $BP_* \to (W \otimes \mathbb{Q})[[w_1, \ldots, w_{n-1}]]$ classifying $\widetilde{F}$. These elements are related by

$$p\lambda_j = \sum_{0\leq i<j} \lambda_i v_{j-i}^{p^i} \tag{3.9}$$

for each $j > 0$([18; A2.2.1]). We must show that each v_i is in $W\langle\langle w_1, \ldots, w_{n-1}\rangle\rangle$ and that

$$v_i \equiv \begin{cases} 0 \bmod \mathfrak{m} & i \neq n \\ 1 \bmod \mathfrak{m} & i = n \end{cases}, \tag{3.10}$$

where $\mathfrak{m}$ is the maximal ideal of $W\langle\langle w_1, \ldots, w_{n-1}\rangle\rangle$.

Before proving 3.10, we single out some useful facts, which follow easily from 3.8 and our description of $\log_{\widetilde{F}}(x)$.

$$p\lambda_j = \lambda_{j-n} \qquad j \geq n \tag{3.11}$$

$$p^j\lambda_j \in \mathfrak{m} \qquad j \geq 1, n > 1. \tag{3.12}$$

If $n = 1$, it is easy to see from 3.9 and 3.11 that $v_1 = 1$ and $v_i = 0$ for all $i > 1$.

For $n > 1$, we proceed by induction on i. To start the induction, note that $\lambda_1 = w_1/p$, so that $v_1 = w_1$. Now assume that $j \leq n$ and that $v_k \in \mathfrak{m}$ for $k < j$. Then, since $\mathfrak{m}$ has divided powers, $v_k^p \in p\mathfrak{m}$. But $p\lambda_i = w_i \in \mathfrak{m}$ for $0 < i < n$; hence

$$p\lambda_j - v_j = \sum_{i=1}^{j-1} \lambda_i v_{j-i}^{p^i} \in \mathfrak{m}.$$

Therefore, if $j < n$, $v_j \in \mathfrak{m}$. If $j = n$, $p\lambda_j = 1$ by 3.11, so that $v_n \equiv 1 \bmod \mathfrak{m}$. Next assume that $j > n$ and that v_k satisfies 3.10 for $k < j$. If $0 < i < j$ and $j - i \neq n$, then $v_{j-i} \in \mathfrak{m}$, so

$$v_{j-i}^{p^i} \in (p^i)!\mathfrak{m} \subset p^i\mathfrak{m}.$$

Thus, by 3.12,

$$\lambda_i v_{j-i}^{p^i} \in \mathfrak{m}; \quad 0 < i < j \text{ and } j - i \neq n.$$

Equation 3.9 then becomes:

$$p\lambda_j - v_j - \lambda_{j-n} v_n^{p^{j-n}} \in \mathfrak{m}.$$

Moreover,

$$(v_n \equiv 1 \bmod \mathfrak{m}) \Rightarrow (v_n^p \equiv 1 \bmod p\mathfrak{m}) \Rightarrow (v_n^{p^{j-n}} \equiv 1 \bmod p^{j-n}\mathfrak{m}),$$

with the first implication holding because $\mathfrak{m}$ has divided powers. But $p^{j-n}\lambda_{j-n} \in \mathfrak{m}$; this implies that

$$p\lambda_j - v_j - \lambda_{j-n} \in \mathfrak{m}.$$

As $p\lambda_j = \lambda_{j-n}$, we conclude that $v_j \in \mathfrak{m}$, completing the induction and the proof.

4. Comparison of universal lifts. Continue to write $W \equiv W_{\mathbb{F}_{p^n}}$. Let $\widehat{F}$ be the universal Lubin-Tate lift of Γ_n to $W[[u_1, \ldots, u_{n-1}]]$ described in 1.3, and let $\widetilde{F}$ be the universal lift of Γ_n to $W\langle\langle w_1, \ldots, w_{n-1}\rangle\rangle$ described in 3.7. In this section we explicitly determine the unique continuous graded homomorphism

$$\iota : W[[u_1, \ldots, u_{n-1}]][u, u^{-1}] \to W\langle\langle w_1, \ldots, w_{n-1}\rangle\rangle[w, w^{-1}]$$

such that $(\iota_*\widehat{F}, \iota(u))$ is $\star$-isomorphic to $(\widetilde{F}, w)$. By 1.4 and 3.7ii, there is then the commutative diagram

$$\begin{tikzcd}
 & (\mathrm{lifts}_n^\star)_* W \arrow[dl, "="'] \arrow[dr, "\mu"] & \\
\mathrm{Hom}^c_{W-alg}(W[[u_1, \ldots, u_{n-1}]][u, u^{-1}], W) & & \mathrm{Hom}^{P.D.}_{W-alg}(W\langle\langle w_1, \ldots, w_{n-1}\rangle\rangle[w, w^{-1}], W) \arrow[ll, "\iota^*"]
\end{tikzcd}$$

This implies that, for each $g \in S_n$ and *P.D.* W-algebra homomorphism $h : W\langle\langle w_1, \ldots, w_{n-1}\rangle\rangle[w, w^{-1}] \to W$, the diagram

$$\begin{tikzcd}
W[[u_1, \ldots, u_{n-1}]][u, u^{-1}] \arrow[r, "\iota"] \arrow[dd, "g"'] & W\langle\langle w_1, \ldots, w_{n-1}\rangle\rangle[w, w^{-1}] \arrow[dr, "g"] & \\
 & & W\langle\langle w_1, \ldots, w_{n-1}\rangle\rangle[w, w^{-1}] \arrow[d, "h"] \\
W[[u_1, \ldots, u_{n-1}]][u, u^{-1}] \arrow[r, "\iota"] & W\langle\langle w_1, \ldots, w_{n-1}\rangle\rangle[w, w^{-1}] \arrow[r, "h"] & W
\end{tikzcd}$$

commutes. (Here g also denotes the algebra homomorphism induced by the action of g.) From this diagram, it follows from Corollary A.7 that

$$\begin{tikzcd}
W[[u_1, \ldots, u_{n-1}]][u, u^{-1}] \arrow[r, "\iota"] \arrow[d, "g"'] & W\langle\langle w_1, \ldots, w_{n-1}\rangle\rangle[w, w^{-1}] \arrow[d, "g"] \\
W[[u_1, \ldots, u_{n-1}]][u, u^{-1}] \arrow[r, "\iota"] & W\langle\langle w_1, \ldots, w_{n-1}\rangle\rangle[w, w^{-1}]
\end{tikzcd} \tag{4.1}$$

also commutes. Futhermore, we will show that ι induces an isomorphism

$$\bar{\iota} : W\langle\langle u_1, \ldots, u_{n-1}\rangle\rangle[u, u^{-1}] \to W\langle\langle w_1, \ldots, w_{n-1}\rangle\rangle[w, w^{-1}]$$

(cf. A.2); since we have described the action of S_n on

$$W\langle\langle w_1, \ldots, w_{n-1}\rangle\rangle[w, w^{-1}],$$

we therefore obtain a description of the action of S_n on $W[[u_1, \ldots, u_{n-1}]][u, u^{-1}]$.

In order to state our main result, we need a little preparation. Let

$$\log_{\widehat{F}}(x) = \sum_{k\geq 0} m_k x^{p^k}, \quad m_k \in W[[u_1, \ldots, u_{n-1}]] \otimes \mathbb{Q}. \tag{4.2}$$

By 1.3 and the relation between the Hazewinkel generators and the coefficients of the logarithm of the universal p-typical formal group law (cf. 3.9), we have the relations

$$\begin{cases} pm_k = \sum_{i=1}^{k} m_{k-i} u_i^{p^{k-i}}, & k < n \\ pm_k = \sum_{i=1}^{n-1} m_{k-i} u_i^{p^{k-i}} + m_{k-n}, & k \geq n. \end{cases} \tag{4.3}$$

We will denote by u_i and m_k the image of these elements under ι (or $\iota \otimes \mathbb{Q}$), and, as usual, $\mathfrak{m}$ will denote the maximal ideal of $W\langle\langle w_1, \ldots, w_{n-1}\rangle\rangle$.

THEOREM 4.4. *There is a unique continuous graded homomorphism*

$$\iota : W[[u_1, \ldots, u_{n-1}]][u, u^{-1}] \to W\langle\langle w_1, \ldots, w_{n-1}\rangle\rangle[w, w^{-1}]$$

such that $(\iota_*\widehat{F}, \iota(u))$ is $\star$-isomorphic to $(\widetilde{F}, w)$. This homomorphism satisfies the relations

$$\begin{aligned} w &\equiv (p^i m_{ni})u && \bmod p^i\mathfrak{m} \\ ww_j &\equiv (p^{i+1} m_{j+ni})u && \bmod p^{i+1}\mathfrak{m} \end{aligned}$$

for $1 \leq j \leq n-1$ and all $i \geq 0$. In addition, the map

$$\bar{\iota} : W\langle\langle u_1, \ldots, u_{n-1}\rangle\rangle[u, u^{-1}] \to W\langle\langle w_1, \ldots, w_{n-1}\rangle\rangle[w, w^{-1}]$$

induced by ι is an isomorphism.

Remark 4.5. Observe that the required map ι is obtained by first constructing a continuous homomorphism

$$\iota_0 : W[[u_1, \ldots, u_{n-1}]] \to W\langle\langle w_1, \ldots, w_{n-1}\rangle\rangle$$

such that $(\iota_0)_*\widehat{F}$ is $\star$-isomorphic to $\widetilde{F}$. The extension ι must then be given by defining $\iota(u) = h'(0)w$, where h is the unique $\star$-isomorphism from $(\iota_0)_*\widehat{F}$ to $\widetilde{F}$. By

[1; proof of Theorem 1.8] (see also Remark 1.2), such an ι_0 exists and is unique; however, we shall not use this result, since it is not much more work, and perhaps makes the proof of Theorem 4.4 clearer, to prove existence and uniqueness along the way.

Before proving this theorem, we separate off the following lemmas, the first of which is needed even to give sense to the assertions of the theorem.

LEMMA 4.6. *For each $i \geq 0$ and $1 \leq j \leq n$,*

$$p^{i+1} m_{ni+j} \in W\langle\langle u_1, \ldots, u_{n-1}\rangle\rangle,$$

and

$$p^{i+1} m_{ni+j} \equiv p^{i+2} m_{n(i+1)+j} \bmod p^{i+1} I,$$

where I is the maximal ideal of $W\langle\langle u_1, \ldots, u_{n-1}\rangle\rangle$.

Proof. Begin by noting that, from 4.3 and induction,

$$p^i m_i \in W[[u_1, \ldots, u_{n-1}]]$$

for all $i \geq 0$. (This is indeed a special case of a general fact, valid for any p-typical one dimensional formal group law over a torsion free ring.) Then, using 4.3 once more, we have

$$p^{i+2} m_{n(i+1)+j} = p^{i+1} m_{ni+j} + p^{i+1} \sum_{t=1}^{n-1} m_{k-t} u_t^{p^{k-t}}, \quad k = n(i+1)+j.$$

But

$$u_t^{p^{k-t}} \in (p^{k-t})! I \subset p^{k-t} I,$$

so, from above,

$$m_{k-t} u_t^{p^{k-t}} \in I,$$

and therefore,

$$p^{i+2} m_{n(i+1)+j} - p^{i+1} m_{ni+j} \in p^{i+1} I.$$

To finish the proof, we need only show that $pm_j \in W\langle\langle u_1, \ldots, u_{n-1}\rangle\rangle$. This follows easily from 4.3.

Now define

$$(4.7)\qquad \begin{cases} f_j(u_1,\ldots,u_{n-1}) = \lim_{i\to\infty} p^{i+1}m_{ni+j}, & 1\le j\le n-1 \\ f(u_1,\ldots,u_{n-1}) = \lim_{i\to\infty} p^{i+1}m_{n(i+1)}. \end{cases}$$

These are power series in $W\langle\langle u_1,\ldots,u_{n-1}\rangle\rangle$; note that

$$(4.8)\qquad \begin{cases} f_j(u_1,\ldots,u_{n-1}) \equiv u_j & \bmod I^2 \\ f(u_1,\ldots,u_{n-1}) \equiv 1 & \bmod I^2 \end{cases}$$

This immediately implies the following result.

LEMMA 4.9. *Let $T : W\langle\langle w_1,\ldots,w_{n-1}\rangle\rangle[w,w^{-1}] \to W\langle\langle u_1,\ldots,u_{n-1}\rangle\rangle[u,u^{-1}]$ be the continuous W-algebra homomorphism defined by*

$$T(w_j) = f_j/f, \quad 1\le j\le n-1$$
$$T(w) = f(u_1,\ldots,u_{n-1})\cdot u.$$

Then T is an isomorphism.

Proof of Theorem 4.4. We construct ι_0 and compute $h'(0)$ as in Remark 4.5. Since $\widetilde{F}$ and $(\iota_0)_*\widehat{F}$ are p-typical, $\widetilde{F}$ and $(\iota_0)_*\widehat{F}$ are $\star$-isomorphic if and only if there exists a homomorphism

$$(4.10)\qquad \sum_{i\ge 0}{}^{(\iota_0)_*\widehat{F}} a_i x^{p^i} : \widetilde{F} \to (\iota_{0*})\widehat{F}, \quad a_i \in W\langle\langle w_1,\ldots,w_{n-1}\rangle\rangle$$

with

$$(4.11)\qquad a_i \equiv \begin{cases} 1 \bmod \mathfrak{m} & i=0 \\ 0 \bmod \mathfrak{m} & i>0 \end{cases}.$$

Suppress ι_0 from the notation. 4.10 holds if and only if

$$\log_{\widehat{F}}\left(\sum_{i\ge 0}{}^{\widehat{F}} a_i x^{p^i}\right) = a_0 \log_{\widetilde{F}}(x).$$

We must find the values of the u_i in $W\langle\langle w_1,\ldots,w_{n-1}\rangle\rangle$ which allow the above equation to be solved for the a_i. By Theorem 3.7, this equation becomes

$$\sum_{i,j\ge 0} m_j a_i^{p^j} x^{p^{i+j}} = \sum_{i\ge 0}(a_0/p^i)x^{p^{ni}} + \sum_{j=1}^{n-1}\sum_{i\ge 0}(a_0 w_j/p^{i+1})x^{p^{ni+j}}.$$

Equivalently, the following equations must be satisfied for all $i \geq 0$ and $1 \leq j \leq n-1$:

$$
\begin{aligned}
(4.12) \qquad a_0/p^{i+1} &= m_{n(i+1)}a_0^{p^{n(i+1)}} + \sum_{k=0}^{n(i+1)-1} m_k a_{n(i+1)-k}^{p^k} \\
a_0 w_j/p^{i+1} &= m_{ni+j}a_0^{p^{ni+j}} + \sum_{k=0}^{ni+j-1} m_k a_{ni+j-k}^{p^k}
\end{aligned}
$$

Using 4.11 together with the sort of argument in the proofs of Theorem 3.7i and Lemma 4.6, one sees that 4.12 cannot be solved unless

$$
\begin{aligned}
(4.13) \qquad a_0 &\equiv p^{i+1}m_{n(i+1)} \bmod p^{i+1}\mathfrak{m} \\
a_0 w_j &\equiv p^{i+1}m_{ni+j} \bmod p^{i+1}\mathfrak{m}
\end{aligned}
$$

for all $i \geq 0$. Conversely, if 4.13 is satisfied, one can then solve inductively for the remaining a_k.

But, by Lemma 4.9, the above equations are satisfied if and only if

$$\iota_0 = T^{-1}|W[[u_1, \ldots, u_{n-1}]]$$

and

$$a_0 \cdot T^{-1}u = w.$$

This proves the existence and uniqueness of ι, and, since $a_0^{-1} = h'(0)$, we have that $\bar{\iota} = T^{-1}$ and is therefore an isomorphism. The relations between the canonical and the Lubin-Tate coordinates are just a restatement of 4.13.

5. Brown-Comenetz self-duality. In this section, we present our application of these formulas to Brown-Comenetz duality. We first recall the context of this application.

Let I_n be the Brown-Comenetz dual [3] of L_nS^0; it is characterized by a natural isomorphism

$$(5.1) \qquad [X, I_n]_0 \to \operatorname{Hom}\left(\pi_0(X \wedge L_nS^0), \mathbb{Q}/\mathbb{Z}_{(p)}\right)$$

for all spectra X. Hom here denotes the group of group homomorphisms between two abelian groups; in general, Hom_R will denote the R-module of module homomorphisms between two R-modules. Note that since $L_nX = X \wedge L_nS^0$ (see [20; 7.5.6]), it follows that $F(X, I_n)$, the function spectrum of maps from X to I_n, is the Brown-Comenetz dual of L_nX.

In our approach to the $E(n)$-approximate generating hypothesis (see [5]), it is important to have an explicit understanding of the spectrum I_n. Without going into details here, the first step in achieving this understanding is determining $I_n \wedge V_j$ for an appropriate direct system of finite spectra $\{\sum^{-n_j} V_j\}$. Each of these spectra should satisfy

$$BP_* V_j = BP_*/(p^{j_0}, v_1^{j_1}, \dots v_{n-1}^{j_{n-1}}) \tag{5.2}$$

as BP_*BP-comodules, where $v_k^{j_k}$ is invariant $\mathrm{mod}\,(p^{j_0}, v_1^{j_1}, \dots, v_{k-1}^{j_{k-1}})$ for $1 \le k \le n-1$. One also requires that

$$\varinjlim_j BP_*\Sigma^{-n_j} V_j = BP_*/(p^\infty, v_0^\infty, \dots, v_{n-1}^\infty), \quad n_j = \sum_{i=1}^{n-1} 2j_i(p^i - 1),$$

or, more precisely, that

$$L_n \varinjlim_j \Sigma^{-n_j} V_j = M_n S^0.$$

(See [17; 5,6] and [19] for the definition and properties of $M_n S^0$.)

Other conditions may additionally be imposed on the system $\{\Sigma^{-n_j} V_j\}$; for instance, one may require that each V_j be Spanier-Whitehead self-dual (up to suspension) and even that each V_j be a ring spectrum. The existence of such sequences is guaranteed by nilpotence technology ([11], [12]; also cf. [5; 2]). Indeed, we can and will assume that, for each V_j, there is a sequence of cofibrations

$$\Sigma^{|g_i|} X_i \xrightarrow{g_i} X_i \to X_{i+1}, \quad 0 \le i \le n-1,$$

with $X_0 = S^0$, $X_n = V_j$, and with $BP_* g_i$ given by multiplication by $v_i^{j_i}$. Furthermore, we will assume that $g_i \wedge X_i = X_i \wedge g_i$ for all i—which guarantees that each X_i is Spanier-Whitehead self-dual—and that a ring spectrum structure has been inductively constructed on each X_i using the method of [6].

Write V for a complex V_j of the sort above, and call such a spectrum n-admissible. We will say that $L_n V$ is Brown-Comenetz self-dual if there exists a map $f : \Sigma^{|f|} L_n V \to I_n$ such that the adjoint

$$\Sigma^{|f|} L_n V \to F(V, I_n)$$

of the map

$$\Sigma^{|f|} L_n V \wedge V \xrightarrow{L_n m} \Sigma^{|f|} L_n V \xrightarrow{f} I_n$$

is an equivalence. Here $m : V \wedge V \to V$ is the ring spectrum pairing.

A priori, our definition of Brown-Comenetz self-duality is stronger than it apparently should be. Namely, one would expect to call V Brown-Comenetz self-dual if there merely existed an equivalence $\Sigma^i L_n V \xrightarrow{\simeq} F(V, I_n)$. Although it seems likely that these two definitions are equivalent, we have chosen ours so that Theorem 5.4 holds and is easy to prove.

If $n = 1$ and p is odd, then $I_1 \cong L_1((S^2)^\wedge_p)$, the $E(1)_*$-localization of the p-completion of S^2 ([5; 1.5]). Hence Brown-Comenetz duality becomes essentially Spanier-Whitehead duality and so if V is any 1-admissible spectrum—in this case, a mod (p^i) Moore spectrum—then $L_1 V$ is Brown-Comenetz self-dual. The map f is just projection onto the top cell.

In this section, we will prove the following result.

THEOREM 5.3. *Let $p \geq 5$ and let V be 2-admissible. If $pBP_*V \neq 0$, then $L_2 V$ is not Brown-Comenetz self-dual.*

As remarked in the introduction, this theorem is proved by first reducing it to a problem in pure algebra.

If N is a (discrete) continuous Galois equivariant twisted $S_n - E^\wedge_{n*}$ module (see 0.2) of finite type, then so is

$$N^\sim \equiv \mathrm{Hom}_W(N, \mathbb{Q}/\mathbb{Z}_{(p)} \otimes W),$$

where $W = W\mathbb{F}_{p^n}$, Gal acts in the evident way on $\mathbb{Q}/\mathbb{Z}_{(p)} \otimes W$, and S_n acts trivially on $\mathbb{Q}/\mathbb{Z}_{(p)} \otimes W$. $N^\sim$ is called the Pontryagin dual of N. Note that $(N^\sim)^\sim = N$ as Galois equivariant twisted $S_n - E^\wedge_{n*}$ modules. If there exists a $t \in \mathbb{Z}$ such that

$$N^\sim \approx \Sigma^t N$$

as Galois equivariant twisted $S_n - E^\wedge_{n*}$ modules, we say that N is Pontryagin self-dual. We will prove the following result.

THEOREM 5.4. *Suppose $p > \max\{(n^2+1)/2, n+1\}$, and let V be n-admissible. Then $L_n V$ is Brown-Comenetz self-dual if and only if $E^\wedge_{n*} \otimes_{BP_*} BP_*V$ is Pontryagin self-dual.*

The next result will allow us to prove Theorem 5.3 by making a single computation.

LEMMA 5.5. *Suppose $(p^{i_0}, v_1^{i_1}, \ldots v_{n-1}^{i_{n-1}})$ and $(p^{j_0}, v_1^{j_1}, \ldots, v_{n-1}^{j_{n-1}})$ are ideals in BP_* with $v_k^{i_k}$ (resp. $v_k^{j_k}$) invariant* mod $(p^{i_0}, \ldots, v_{k-1}^{i_{k-1}})$ *(resp. $(p^{j_0}, \ldots, v_{k-1}^{j_{k-1}})$) for all $1 \leq k \leq n-1$. Suppose further that $i_k \geq j_k$ whenever $0 \leq k \leq n-1$. Now let*

$$M = E^\wedge_{n*} \otimes_{BP_*} BP_*/(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})$$
$$N = E^\wedge_{n*} \otimes_{BP_*} BP_*/(p^{j_0}, \ldots, v_{n-1}^{j_{n-1}})$$

Then, if M is Pontryagin self-dual, so is N.

If V is any 2-admissible complex with $pBP_*V \neq 0$, then $BP_*V = BP_*/(p^{j_0}, v_1^{j_1})$ with $j_0 \geq 2$ and $j_1 \geq p$. Thus, the next result, taken together with 5.4 and 5.5, implies Theorem 5.3.

LEMMA 5.6. *The module*

$$E_{2*}^{\wedge} \otimes_{BP_*} BP_*/(p^2, v_1^p) = E_{2*}^{\wedge}/(p^2, v_1^p) = W[u_1, u, u^{-1}]/(p^2, u_1^p)$$

is not Pontryagin self-dual.

We now prove 5.4, 5.5, and 5.6 in turn. We begin be recalling some properties of Poincaré pro-p-groups. The reader is referred to [21; I] for more details.

Let G be a Poincaré pro-p-group of dimension d, and let I be its dualizing module. I is a discrete G-module; it comes equipped with a homomorphism

$$i : H_c^d(G; I) \to \mathbb{Q}/\mathbb{Z}_{(p)},$$

such that, if A is any finite G-module, the map

$$\text{(5.7)} \qquad \operatorname{Hom}(A, I)^G \to \operatorname{Hom}(H_c^d(G; A), \mathbb{Q}/\mathbb{Z}_{(p)})$$

which sends $f \in \operatorname{Hom}(A, I)^G$ to the composite

$$H_c^d(G; A) \xrightarrow{f_*} H_c^d(G; I) \to \mathbb{Q}/\mathbb{Z}_{(p)}$$

is an isomorphism ([21; I 3.5]). In fact, as an abelian group $I \approx \mathbb{Q}/\mathbb{Z}_{(p)}$, and furthermore, i is an isomorphism ([21; I 4.5]). The isomorphism 5.7 can also be generalized.

PROPOSITION 5.8. *Let G, I be as above, and let $n \geq 1$. Write $W = W\mathbb{F}_{p^k}$, $k \geq 1$. Suppose that M is a finite discrete $W[G]$-module; set $M^{\sim} = \operatorname{Hom}_W(M, I \otimes W)$ and note that it is also a finite discrete $W[G]$-module. Then the "evaluation" map*

$$H_c^*(G; M) \otimes_W H_c^{d-*}(G; M^{\sim}) \to H_c^d(G; I \otimes W) \xrightarrow{\cong} \mathbb{Q}/\mathbb{Z}_{(p)} \otimes W$$

is a perfect pairing—that is, the adjoint

$$\text{(5.9)} \qquad H_c^{d-i}(G; M^{\sim}) \to \operatorname{Hom}_W(H_c^i(G; M), \mathbb{Q}/\mathbb{Z}_{(p)} \otimes W)$$

is an isomorphism for all $i \geq 0$.

Proof. Mimic the necessary parts of the proof of Proposition 30 in [21; I 4.5].

If $p-1 \nmid n$, it is known (see, for example, [18, 6.2.10]) that S_n^0, the p-Sylow subgroup of S_n consisting of those matrices of the form 2.20 with $a_0 \equiv 1 \bmod (p)$, is a Poincaré pro-p-group. We would like, however, to have analogous results for the full stabilizer group S_n. This is indeed the case.

PROPOSITION 5.10. *Assume $p-1 \nmid n$. The conclusions of Proposition 5.8 hold with $G = S_n$ and $I = \mathbb{Q}/\mathbb{Z}_{(p)}$ with trivial S_n action.*

This proposition will be proved in §6. Note that it implies in particular that our notation for the Pontryagin dual is not abusive.

We will also need to understand how the Galois action on S_n behaves in the above perfect pairing. Let I be the dualizing module for S_n. By the universal property of the dualizing module, there exists a unique $\mathbb{Z}_p[S_n \rtimes \mathrm{Gal}]$-module structure on I, extending the (trivial) S_n-action on I, such that

$$\begin{tikzcd} H_c^{n^2}(S_n;I) \arrow[r, "i"] \arrow[d, "\sigma_*"'] & \mathbb{Q}/\mathbb{Z}_{(p)} \\ H_c^{n^2}(S_n;I) \arrow[ur, "i"'] & \end{tikzcd}$$

commutes for each $\sigma \in \mathrm{Gal}$. Then

$$H_c^{n^2}(S_n; I \otimes W) \xrightarrow{i \otimes W} \mathbb{Q}/\mathbb{Z}_{(p)} \otimes W$$

is Galois equivariant, where $W = W\mathbb{F}_{p^n}$ with the usual Galois action. But $I \otimes W \approx \mathbb{Q}/\mathbb{Z}_{(p)} \otimes W$ as $\mathbb{Z}_p[\mathrm{Gal}]$-modules, where Gal acts trivially on $\mathbb{Q}/\mathbb{Z}_{(p)}$ (cf. [10; VI, 11.7]). In particular,

$$(I \otimes W)^{\mathrm{Gal}} \approx \mathbb{Q}/\mathbb{Z}_{(p)}.$$

Now it is a consequence (see e.g. [7]) of the fact $H^1(\mathrm{Gal}; GL_k(\mathbb{F}_{p^n})) = 0$ (see [23; X, Proposition 3]) that, if N is any finite Galois equivariant W module, then the inclusion $N^{\mathrm{Gal}} \hookrightarrow N$ induces a Galois equivariant W-module isomorphism

$$N^{\mathrm{Gal}} \otimes W \longrightarrow N. \tag{5.11}$$

Hence $I \otimes W \approx \mathbb{Q}/\mathbb{Z}_{(p)} \otimes W$ as Galois equivariant W-modules. Putting these facts together, it follows that the isomorphism

$$H^{n^2-i}(S_n; M^{\sim}) \longrightarrow \mathrm{Hom}_W(H_c^i(S_n; M), \mathbb{Q}/\mathbb{Z}_{(p)} \otimes W)$$

is Galois equivariant, whenever M is a (discrete) continuous Galois equivariant twisted $S_n - E^{\wedge}_{n*}$ module of finite type. Note that the Galois action on M is the one that is described above Theorem 5.4.

Proof of Theorem 5.4. For this proof, we use the following notations: If M is a discrete S_n-module, write $H^*(M)$ for $H^*_c(S_n; M)$. Also, if V is an n-admissible complex, write $E^{\wedge}_{n*}(V)$ for $E^{\wedge}_{n*} \otimes_{BP_*} BP_*(V)$.

Suppose first that L_nV is Brown-Comenetz self-dual, and let

$$f : \Sigma^m L_n V \to I_n$$

be the requisite map. Then regard

$$f \in \mathrm{Hom}\,(\pi_{-m} L_n V,\ \mathbb{Q}/\mathbb{Z}_{(p)})$$

and notice that

$$\pi_* L_n V \otimes \pi_{-m-*} L_n V \to \pi_{-m} L_n V \xrightarrow{f} \mathbb{Q}/\mathbb{Z}_{(p)}$$

is a perfect pairing. By 0.4, this can be rewritten as

$$H^{**}(E^{\wedge}_{n*}(V))^{\mathrm{Gal}} \otimes H^{**}(E^{\wedge}_{n*}(V))^{\mathrm{Gal}} \to H^{**}(E^{\wedge}_{n*}(V))^{\mathrm{Gal}} \xrightarrow{f} \mathbb{Q}/\mathbb{Z}_{(p)},$$

which yields, by 5.11 applied to $N = H^{**}(E^{\wedge}_{n*}(V))$, a Galois equivariant perfect pairing

$$(5.12) \qquad H^{**}(E^{\wedge}_{n*}(V)) \otimes_W H^{**}(E^{\wedge}_{n*}(V)) \to H^{**}(E^{\wedge}_{n*}(V)) \xrightarrow{f} \mathbb{Q}/\mathbb{Z}_{(p)} \otimes W$$

of W-modules.

Consider now the quotient map $E^{\wedge}_{n*}(V) \to \mathbb{F}_{p^n}[u, u^{-1}]$. Since $H^i M = 0$ whenever $i > n^2$ and M is any finite discrete S_n-module, it follows that

$$H^{n^2}(E^{\wedge}_{n*}(V)) \to H^{n^2}(\mathbb{F}_{p^n}[u, u^{-1}]) \to 0$$

is exact. But $H^{n^2}(\mathbb{F}_{p^n}[u, u^{-1}]) \neq 0$; therefore f must be concentrated on $H^{n^2, n^2 - m}(E^{\wedge}_{n*}(V))$. Thus, by 5.10 and the discussion following it, there exists a unique Galois equivariant S_n-map

$$g : (E^{\wedge}_n)_{n^2 - m}(V) \to \mathbb{Q}/\mathbb{Z}_{(p)} \otimes W$$

of W-modules such that

$$H^{n^2}(E_{n*}^{\wedge}(V)) \xrightarrow{g_*} H^{n^2}(\mathbb{Q}/\mathbb{Z}_{(p)} \otimes W) \xrightarrow{\cong} \mathbb{Q}/\mathbb{Z}_{(p)} \otimes W$$

is f.

Next consider the pairing

$$E_{n*}^{\wedge}(V) \otimes_W E_{n*}^{\wedge}(V) \to E_{n*}^{\wedge}(V) \xrightarrow{g} \mathbb{Q}/\mathbb{Z}_{(p)} \otimes W.$$

Its adjoint

$$h : E_{n*}^{\wedge}(V) \to \Sigma^{n^2-m} \operatorname{Hom}_W(E_{n*}^{\wedge}(V), \mathbb{Q}/\mathbb{Z}_{(p)} \otimes W)$$

is a Galois equivariant map of twisted $S_n - E_{n*}^{\wedge}$ modules, making the diagram

$$\begin{tikzcd} H^*(E_{n*}^{\wedge}(V)) \otimes_W H^*(E_{n*}^{\wedge}(V)) \arrow[r] \arrow[d, "id \otimes h_*"'] & Q/\mathbb{Z}_{(p)} \otimes W \\ H^*(E_{n*}^{\wedge}(V)) \otimes_W H^*((E_{n*}^{\wedge} V)^{\sim}) \arrow[ur] & \end{tikzcd}$$

commute. The top map is the pairing of 5.12 and the diagonal map is that of Proposition 5.10. Since both pairings are perfect, h_* is an isomorphism. We claim that this implies that h is an isomorphism.

Indeed, if h_* is an isomorphism, then $H^0(\ker h) = 0$. But

$$\ker h \neq 0 \Rightarrow p^{j_0-1} v_1^{j_1-1} \dots v_{n-1}^{j_{n-1}-1} \in \ker h,$$

where

$$BP_*V = BP_*/(p^{j_0}, \dots, v_{n-1}^{j_{n-1}}).$$

Since $p^{j_0-1} v_1^{j_1-1} \dots v_{n-1}^{j_{n-1}}$ is a primitive in BP_*V, it is S_n invariant in $E_{n*}^{\wedge}(V)$, proving that $\ker h = 0$. On the other hand, $(\operatorname{coker} h)^{\sim} = \ker(h^{\sim})$, and $(h^{\sim})_*$ is an isomorphism; therefore, $(\operatorname{coker} h)^{\sim}$ and thus coker h is trivial. This completes the proof that Brown-Comenetz self-duality implies Pontryagin self-duality.

To prove the converse, note that if one has a Galois equivariant isomorphism

$$h : E_{n*}^{\wedge}(V) \to \Sigma^{t} \operatorname{Hom}_W(E_{n*}^{\wedge}(V), \mathbb{Q}/\mathbb{Z}_{(p)} \otimes W)$$

of twisted $S_n - E_{n*}^{\wedge}$ modules, then $g = h(1)$ provides, by the same process as before, a map $f : \Sigma^{n^2-t} L_n V \to I_n$ making $L_n V$ Brown-Comenetz self-dual.

Proof of Lemma 5.5. Let

$$h : M \to \Sigma^t \operatorname{Hom}_W (M, \mathbb{Q}/\mathbb{Z}_{(p)} \otimes W)$$

be a Galois equivariant isomorphism of twisted $S_n - E_{n*}^{\wedge}$ modules. Define $\iota : \Sigma^s N \to M$ by

$$\iota(x) = p^{i_0 - j_0} v_1^{i_1 - j_1} \dots v_{n-1}^{i_{n-1} - j_{n-1}} x,$$

and let $\pi : M \to N$ be the usual quotient map. ι is a Galois equivariant twisted $S_n - E_{n*}^{\wedge}$ module map since the map

$$\iota' : \Sigma^s BP_*/(p^{j_0}, \dots, v_{n-1}^{j_{n-1}}) \to BP_*/(p^{i_0}, \dots, v_{n-1}^{i_{n-1}})$$

defined by

$$\iota'(x) = p^{i_0 - j_0} v_1^{i_1 - j_1} \dots v_{n-1}^{i_{n-1} - j_{n-1}} x$$

is a BP_*BP-comodule map. It is easy to check that there exists a unique Galois equivariant twisted $S_n - E_{n*}^{\wedge}$ module map h' such that the diagram

$$\begin{tikzcd} N \arrow[r, "h'"] & \Sigma^{t+s}(N^{\sim}) \\ M \arrow[u, "\pi"] \arrow[r, "h"] & \Sigma^t(M^{\sim}) \arrow[u, "\iota^{\sim}"'] \end{tikzcd}$$

commutes. Since h is an isomorphism and π, $\iota^{\sim}$ are surjective, h' is also an isomorphism.

Proof of Lemma 5.6. Since $E_{n*}^{\wedge}/(p^2, v_1^p)$ is generated as an $E_{2*}^{\wedge}$-module by the S_2 invariant element 1 it suffices to show that $(E_{2*}^{\wedge}/(p^2, v_1^p))^{\sim}$ has no S_2 invariant $E_{2*}^{\wedge}$-module generator.

Recall from A.3 that $W\langle\langle u_1 \rangle\rangle \subset (W \otimes \mathbb{Q})[[u_1]]$ and that if $\sum_{i \geq 0} a_i u_1^i \in W\langle\langle u_1 \rangle\rangle$, then $a_i \in W$ for $0 \leq i < p$. We therefore have a homomorphism

$$W\langle\langle u_1 \rangle\rangle[u, u^{-1}] \to E_{2*}^{\wedge}/(p^2, v_1^p)$$

defined by sending $u^k(\sum_{i \geq 0} a_i u_1^i)$ to $u^k(\sum_{i=0}^{p-1} a_i u_1^i)$. An easy computation using Theorem 4.4 shows that w is mapped to u and w_1 is mapped to u_1 under the (S_2 equivariant) composition

$$W\langle\langle w_1 \rangle\rangle[w, w^{-1}] \xrightarrow{\iota^{-1}} W\langle\langle u_1 \rangle\rangle[u, u^{-1}] \to E_{2*}^{\wedge}/(p^2, v_1^p).$$

Next consider the embedding of $W\mathbb{F}_{p^2}^{\times}$ in S_2 given by

$$a \mapsto g_a = \begin{bmatrix} a & 0 \\ 0 & a^{\sigma} \end{bmatrix} \qquad \text{(see 2.20)}.$$

Then, by 3.3,

$$\begin{aligned} g_a w &= aw \\ g_a w_1 &= a^{\sigma} a^{-1} w_1, \end{aligned}$$

and hence

$$\begin{aligned} g_a u &= au \\ g_a u_1 &= a^{\sigma} a^{-1} u_1 \end{aligned} \tag{5.13}$$

in $E_{2*}^{\wedge}/(p^2, v_1^p)$.

Now let h be an $E_{2*}^{\wedge}$-module generator of $(E_{2*}^{\wedge}/(p^2, v_1^p))^{\sim}$, say $h \in (E_{2*}^{\wedge}/(p^2, v_1^p))^{\sim}_{2k}$. Observe that $h(u_1^{p-1}u^k)$ then has order p^2. If h is S_2 invariant, then

$$h(g_a(u_1^{p-1}u^k)) = h(u_1^{p-1}u^k)$$

for all $a \in W^{\times}$. By 5.13, this means that

$$a^k a^{1-p}(a^{\sigma})^{p-1} \equiv 1 \bmod (p^2) \tag{5.14}$$

for all $a \in W^{\times}$.

Write

$$a = e_0 + e_1 p \bmod (p^2),$$

where each e_i is a member of the multiplicative system of representatives of $\mathbb{F}_{p^2}$ in W. Then

$$a^{\sigma} = e_0^p + e_1^p p \bmod (p^2),$$

and thus 5.14 is satisfied if and only if

$$\bar{e}_0^{p(p-1)+k+1-p} = 1$$

$$(p-1)\,\bar{e}_0^{(p-2)p+k+1-p}\,\bar{e}_1^{p} + (k+1-p)\,\bar{e}_0^{p(p-1)+k-p}\,\bar{e}_1 = 0$$

for all $\overline{e}_0 \in \mathbb{F}_{p^2}^{\times}$, $\overline{e}_1 \in \mathbb{F}_{p^2}$. But Galois automorphisms are linearly independent; therefore, the bottom equation implies that $(p-1)\overline{e}_0^{(p-2)p+k+1-p} = 0$, an impossibility. This shows that h cannot be S_2 invariant, completing the proof.

6. Proof of Proposition 5.10. We begin the proof by computing the action of S_n^0 on its dualizing module. We start with some generalities.

Let G be a Poincaré pro-p-group. Since $\mathbb{Z}_p = \mathrm{Hom}\,(\mathbb{Q}/\mathbb{Z}_{(p)}, \mathbb{Q}/\mathbb{Z}_{(p)})$ as rings, it follows that the action of G on its dualizing module is described by a continuous group homomorphism

$$\chi_G : G \longrightarrow \mathbb{Z}_p^{\times}.$$

In fact, since G is a pro-p-group, the image of χ_G lies in U, the group of units congruent to $1 \bmod (p)$. Note that if $p > 2$ (the only case of interest to us here), U is isomorphic to the additive group $\mathbb{Z}_p$ (cf. [18; A2.2.15]) and in particular is torsion free. If, in addition, G is an analytic group over $\mathbb{Q}_p$, χ_G has a convenient geometric description, which we now explain.

Let K be a local field, and let H be an analytic group over K with Lie algebra $\mathcal{L}(H)$ over K. If $h \in H$, let $\mathrm{ad}\,(h) : H \longrightarrow H$ be the automorphism defined by $\mathrm{ad}\,(h)(x) = hxh^{-1}$, and let $\mathrm{ad}\,(h)_* : \mathcal{L}(H) \longrightarrow \mathcal{L}(H)$ be the induced Lie algebra map. Define the group homomorphism

$$\mathrm{ad}_*^H : H \longrightarrow \mathrm{aut}_K\,(\mathcal{L}(H), \mathcal{L}(H))$$

by $\mathrm{ad}_*^H\,(h) = \mathrm{ad}\,(h)_*$, and consider the composition

$$H \xrightarrow{\mathrm{ad}_*^H} \mathrm{aut}_K\,(\mathcal{L}(H), \mathcal{L}(H)) \xrightarrow{\det} K^{\times},$$

where det is the determinant homomorphism.

THEOREM 6.1. ([13; V, 2.5.8], see also [22]). *Let G be an analytic group over $\mathbb{Q}_p$ and a Poincaré pro-p-group. Then $\chi_G = \det \circ \mathrm{ad}_*^G$.*

We will use this theorem to determine $\chi_{S_n^0}$.

PROPOSITION 6.2. *$\chi_{S_n^0}$ is the trivial homomorphism.*

Proof. S_n^0 is an open subgroup of the group of units of a central division algebra D over $\mathbb{Q}_p$ [18; A2.2.16] and hence $\mathcal{L}(S_n^0) = \mathcal{L}(D^{\times})$. Now there exists a finite extension K of $\mathbb{Q}_p$ such that

$$D_K \equiv D \otimes_{\mathbb{Q}_p} K \approx M_n(K),$$

the algebra of $n \times n$ matrices over K (see [25; IX]). Since $\mathcal{L}(D_K^{\times}) = \mathcal{L}(D^{\times}) \otimes_{\mathbb{Q}_p} K$, it follows from Theorem 6.1 that the diagram

$$
\begin{tikzcd}
S_n^0 \arrow[r, "\chi_{S_n^0}"] \arrow[d, hook] & U \arrow[r, hook] & \mathbb{Z}_p^\times \arrow[d, hook] \\
D^\times \arrow[r] & D_K^\times = GL_n(K) \arrow[r, "\det \circ\, \mathrm{ad}_*"] & K^\times
\end{tikzcd}
\tag{6.3}
$$

commutes.

But $\mathcal{L}(GL_n(K)) = M_n(K)$, and $\mathrm{ad}(a)_*$ is just conjugation by $a \in GL_n(K)$. Hence $\det \circ \mathrm{ad}_*$ is trivial and therefore so is $\chi_{S_n^0}$.

We next relate the cohomology of S_n^0 to that of S_n.

S_n^0 is a normal subgroup of S_n, and there is an extension

$$S_n^0 \longrightarrow S_n \longrightarrow \mathbb{F}_{p^n}^\times,$$

where the map $S_n \longrightarrow \mathbb{F}_{p^n}^\times$ is given by sending a matrix to the mod (p) reduction of a_0. This sequence splits; a homomorphism $\mathbb{F}_{p^n}^\times \longrightarrow S_n$ may be defined by

$$
a \mapsto \begin{bmatrix} e(a) & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & & \\ 0 & \cdots & \cdots & e(a)^{\sigma^{-(n-1)}} \end{bmatrix}
\tag{6.4}
$$

where $e(a) \in W\mathbb{F}_{p^n}$ is the multiplicative representative of $a \in \mathbb{F}_{p^n}$. With this splitting, we may write $S_n = S_n^0 \rtimes \mathbb{F}_{p^n}^\times$.

Suppose M is a discrete S_n-module over $\mathbb{Z}_{(p)}$. Then $\mathbb{F}_{p^n}^\times$ acts on M by restriction and also on S_n^0 by conjugation. This provides us with an action of $\mathbb{F}_{p^n}^\times$ on $H_c^*(S_n^0; M)$ and a canonical restriction map

$$H_c^*(S_n; M) \longrightarrow H_c^*(S_n^0; M)^{\mathbb{F}_{p^n}^\times}. \tag{6.5}$$

Since the $\mathbb{F}_{p^n}^\times$-fixed point functor is exact on the category of $\mathbb{Z}_{(p)}[\mathbb{F}_{p^n}^\times]$-modules, it follows from the Hochschild-Serre spectral sequence (see [24; II §4]) that this map is an isomorphism.

Now suppose $p-1 \nmid n$ and let I_0 be the dualizing module for S_n^0. Since $\mathbb{F}_{p^n}^\times$ acts on S_n^0 by automorphisms, there is a unique $\mathbb{Z}_{(p)}[S_n^0 \rtimes \mathbb{F}_{p^n}^\times]$-module structure on I_0, extending the (trivial) S_n^0-action, such that

$$\begin{tikzcd} H^{n^2}(S_n^0; I_0) \arrow[r, "i"] \arrow[d, "a_*"'] & \mathbb{Q}/\mathbb{Z}_{(p)} \\ H^{n^2}(S_n^0; I_0) \arrow[ur, "i"'] & \end{tikzcd}$$

commutes for all $a \in \mathbb{F}_{p^n}^\times$. This action is described by a group homomorphism

$$\tau : \mathbb{F}_{p^n}^\times \longrightarrow \mathbb{Z}_p^\times = \mathrm{Aut}\,(I_0, I_0).$$

Morava has shown [16; 2.2.2] that $\mathbb{F}_{p^n}^\times$ acts trivially on the elements of order p in I_0; therefore, $\tau(\mathbb{F}_{p^n}^\times) \subset U$. But U is torsion free, so τ must be trivial. We conclude from 6.2 and 6.5 that

$$\text{(6.6)} \qquad H_c^{n^2}(S_n; \mathbb{Q}/\mathbb{Z}_{(p)}) = H_c^{n^2}(S_n^0; \mathbb{Q}/\mathbb{Z}_{(p)}) \underset{i}{\overset{\approx}{\rightarrow}} \mathbb{Q}/\mathbb{Z}_{(p)},$$

with $\mathbb{Q}/\mathbb{Z}_{(p)}$ given the trivial S_n-action.

Proof of 5.10. Let $I = \mathbb{Q}/\mathbb{Z}_{(p)}$ with the trivial S_n-action. Using 6.6, we obtain a pairing

$$\text{(6.7)} \qquad H_c^*(S_n; M) \otimes_W H_c^{n^2-*}(S_n; M^\sim) \longrightarrow \mathbb{Q}/\mathbb{Z}_{(p)} \otimes W$$

for any finite discrete $W[S_n]$-module M, which we claim is in fact a perfect pairing.

Indeed, the isomorphism

$$H_c^{n^2-i}(S_n^0; M^\sim) \longrightarrow \mathrm{Hom}_W\,(H_c^i(S_n^0; M), \mathbb{Q}/\mathbb{Z}_{(p)} \otimes W)$$

is $\mathbb{F}_{p^n}^\times$-equivariant and the composition

$$\begin{aligned} H_c^{n^2-i}(S_n^0; M^\sim)^{\mathbb{F}_{p^n}^\times} &\xrightarrow{\approx} \mathrm{Hom}_W\,(H_c^i(S_n^0; M), \mathbb{Q}/\mathbb{Z}_{(p)} \otimes W)^{\mathbb{F}_{p^n}^\times} \\ &\longrightarrow \mathrm{Hom}_W\,(H_c^i(S_n^0; M)^{\mathbb{F}_{p^n}^\times}, \mathbb{Q}/\mathbb{Z}_{(p)} \otimes W) \end{aligned}$$

is the adjoint of 6.7. But the last map is an isomorphism by the following lemma, completing the proof.

LEMMA 6.8. *Let G be a finite group of order prime to p and let N be any $W[G]$-module, $W = W\mathbb{F}_{p^k}$. Then the restriction map*

$$\mathrm{Hom}_W(N, \mathbb{Q}/\mathbb{Z}_{(p)} \otimes W)^G \longrightarrow \mathrm{Hom}_W(N^G, \mathbb{Q}/\mathbb{Z}_{(p)} \otimes W)$$

is an isomorphism.

Proof. The idempotent $\frac{1}{|G|} \cdot \sum_{g \in G} g$ in $W[G]$ splits N as $N^G \oplus N_0$. The proof is completed upon observing that

$$\mathrm{Hom}_W(N_0, \mathbb{Q}/\mathbb{Z}_{(p)} \otimes W)^G = 0 = \mathrm{Hom}_W((N_0)^G, \mathbb{Q}/\mathbb{Z}_{(p)} \otimes W).$$

Appendix. In this appendix, we develop the theory of rings with divided power structure as far as we need it. For a more complete account, the reader is referred to [2, §3].

DEFINITION A.1. *Let A be a ring complete with respect to an ideal I. A divided power structure γ on (A, I) is a sequence of functions $\gamma_n : I \to I$, $n \geq 1$, such that, for all $x, y \in I$ and $a \in A$,*

i. $\gamma_1(x) = x$

ii. $\gamma_n(ax) = a^n \gamma_n(x)$

iii. $\gamma_n(x + y) = \gamma_n(x) + \gamma_n(y) + \sum_{i=1}^{n-1} \gamma_i(x)\gamma_{n-i}(y)$

iv. $\gamma_n(x)\gamma_m(x) = [(m+n)!/m!n!]\gamma_{n+m}(x)$

v. $\gamma_n(\gamma_m(x)) = [(mn!)/(m!)^n n!]\gamma_{nm}(x)$.

We also say that "(I, γ) is a P.D. ideal", or that "$(A, I; \gamma)$ is a P.D. ring", or that "γ is a P.D. structure on I". If $(B, J; \gamma')$ is another P.D. ring, then a P.D. morphism $f : (A, I; \gamma) \to (B, J; \gamma')$ is a ring homomorphism $f : (A, I) \to (B, J)$ such that

$$\gamma'_n(f(x)) = f(\gamma_n(x)) \text{ for all } x \in I.$$

(A ring homomorphism $f : (A, I) \to (B, J)$ is a ring homomorphism $f : A \to B$ with $f(I) \subset J$.)

Note that i and iv imply that

$$x^n = n!\gamma_n(x), \quad x \in I, n \geq 1.$$

Thus, if B is torsion free, then a P.D. structure on J, if it exists, is unique. Furthermore, a P.D. morphism $f : (A, I; \gamma) \to (B, J; \gamma')$ is just a ring homomorphism $f : (A, I) \to (B, J)$.

In the body of this paper, we discuss local rings with divided power structure. By this, we mean that the maximal ideal is given a P.D. structure. We also use the notion of the divided power envelope of an algebra over a P.D. ring.

DEFINITION A.2. *Let (A, I, γ) be a P.D. ring, let B be a ring complete with respect to the ideal J, and let $f : (A, I) \to (B, J)$ be a ring homomorphism; in other words, (B, J) is an (A, I)-algebra. The divided power envelope of (B, J) is a P.D. ring $(\overline{B}, \overline{J}; \overline{\gamma})$, together with a ring homomorphism $\iota : (B, J) \to (\overline{B}, \overline{J})$ such that:*

i. $\iota \circ f : (A, I; \gamma) \to (\overline{B}, \overline{J}; \overline{\gamma})$ *is a P.D. map.*

ii. *Whenever $(C, K; \delta)$ is a P.D. ring and $g : (B, J) \to (C, K)$ is a ring homomorphism with $g \circ f$ a P.D. map, there exists a unique P.D. map $\overline{g} : (\overline{B}, \overline{J}; \overline{\gamma}) \to (C, K; \delta)$ such that the diagram*

$$\begin{tikzcd} & (\overline{B}, \overline{J}; \overline{\gamma}) \arrow[dr, "\overline{g}"] & \\ (B, J) \arrow[ur, "\iota"] \arrow[rr, "g"'] & & (C, K; \delta) \\ & (A, I; \gamma) \arrow[ul, "f"] \arrow[ur, "g \circ f"'] & \end{tikzcd}$$

commutes.

Certainly, the divided power envelope is unique up to canonical isomorphism. As for existence, the following construction suffices for our purposes.

CONSTRUCTION A.3. *Let A be a torsion free complete local ring with divided powers. We now describe the divided power envelope $A\langle\langle x\rangle\rangle$ of the A-algebra $A[[x]]$. Consider the subring C of $(A \otimes \mathbb{Q})[x]$ generated by A and $S \equiv \{x^k/k! : k \geq 1\}$. Let $I \subset C$ be the ideal generated by S and the maximal ideal of A. Define*

$$A\langle\langle x\rangle\rangle = \varprojlim_j C/I^j.$$

$A\langle\langle x\rangle\rangle$ is canonically a subring of $(A \otimes \mathbb{Q})[[x]]$. One also sees that $A\langle\langle x\rangle\rangle$ is a complete local ring, whose residue field is the same as that of A, and that $A\langle\langle x\rangle\rangle$ has divided powers. Furthermore, the inclusion $A[[x]] \hookrightarrow A\langle\langle x\rangle\rangle$ expresses $A\langle\langle x\rangle\rangle$ as the divided power envelope of $A[[x]]$ regarded as an algebra over the divided power ring A.

By iterating Construction A.3, we obtain the divided power envelope of a power series algebra on a finite number of generators. For example, the divided power envelope $W\mathbb{F}_{p^n}\langle\langle w_1, \ldots, w_{n-1}\rangle\rangle$ of the $W\mathbb{F}_{p^n}$-algebra $W\mathbb{F}_{p_n}[[w_1, \cdots, w_{n-1}]]$ is given by

$$W\mathbb{F}_{p^n}\langle\langle w_1, \ldots, w_{n-1}\rangle\rangle = (W\mathbb{F}_{p^n}\langle\langle w_1, \ldots, w_{n-2}\rangle\rangle)\langle\langle w_{n-1}\rangle\rangle. \tag{A.4}$$

This ring is also a subring of $(W\mathbb{F}_{p^n} \otimes \mathbb{Q})[[w_1, \ldots, w_{n-1}]]$.

The (corollary of the) next result is needed in §4.

PROPOSITION A.5. *Let A be as in A.3, and assume in addition that A is a domain whose maximal ideal $\mathfrak{m}$ is infinite. For each $a \in \mathfrak{m}$, let $\bar{e}_a$ denote the extension of the A-algebra map*

$$e_a : A[[x]] \to A$$

given by $e_a(x) = a$ to a P.D. morphism

$$\bar{e}_a : A\langle\langle x\rangle\rangle \to A.$$

Suppose $P(x) \in A\langle\langle x\rangle\rangle \subset (A \otimes \mathbb{Q})[[x]]$ and that $P(a) \equiv \bar{e}_a(P) = 0$ for all $a \in \mathfrak{m}$. Then $P = 0$.

Proof. Write

$$P(x) = \sum_{i \geq 0} c_i(x^i/i!), \quad c_i \in A.$$

(The c_i's must also satisfy a certain convergence condition, but we need not spell it out here.) Fix $0 \neq b \in \mathfrak{m}$, and consider the power series

$$Q(x) \equiv P(bx) = \sum_{i \geq 0} c_i \gamma_i(b) x^i \in A[[x]].$$

$P(x) = 0$ if and only if $Q(x) = 0$. Since $Q(a) = 0$ for all $a \in \mathfrak{m}$, the next result implies our proposition.

LEMMA A.6. *Let A be as in A.5. and suppose that $Q(x) \in A[[x]]$ with $Q(a) = 0$ for all $a \in \mathfrak{m}$. Then $Q = 0$.*

Proof. Assume $Q \neq 0$. Then write

$$Q(x) = x^k Q_1(x)$$

with $Q_1(0) \neq 0$. Choose i such that $Q_1(0) \notin \mathfrak{m}^i$, and let $0 \neq c \in \mathfrak{m}^i$. (For example, we could take $c = a^i$, with $0 \neq a \in \mathfrak{m}$.) Then $Q_1(c) \neq 0$, and hence

$$Q(c) = c^k Q_1(c) \neq 0,$$

contradicting the assumption on Q. Therefore, $Q = 0$.

This proposition may also be iterated to give the following easy consequence.

COROLLARY A.7. *Let A be as in A.5, and let*

$$P(x_1, \dots, x_n) \in A\langle\langle x_1, \dots x_n \rangle\rangle \subset (A \otimes \mathbb{Q})[[x_1, \dots, x_n]].$$

if $P(a_1, \dots, a_n) = 0$ for all ordered n-tuples of elements of $\mathfrak{m}$, then $P = 0$.

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