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p-adic period map for the moduli space of deformations of a formal group

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Abstract

The moduli space of deformations of a formal group over a finite field is studied. We consider Lubin–Tate and Dieudonné approaches and find an explicit relation between them employing Hazewinkel's universal p-typical formal group, Honda's theory and rigid power series. The formula obtained allows to give an explicit description of the action of the automorphism group of the formal group on the moduli space. It essentially generalizes an analogous result of Gross and Hopkins [Contemp. Math. 158 (1994) 23–88].

Keywords: Deformations of formal groups; p-adic period maps

Introduction

There are two different approaches to the study of deformations of a formal group over a finite field considered up to $\star$-isomorphism, where a $\star$-isomorphism between two deformations means an isomorphism with identity reduction. First of them is due to Lubin and

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Tate [8] who constructed a universal deformation of a formal group over a finite field, i.e., a formal group which represents the functor of deformations over complete local algebras and thus provides a moduli space for it. Another approach is based on the Dieudonné theory (see [3]) that assigns to any formal group over a finite field a profinite module over the Dieudonné ring of this field which gives an equivalence between the category of formal groups and the category of Dieudonné modules. For any deformation of a formal group over a finite field which is defined over the ring of Witt vectors over this field, Fontaine introduced a submodule in the Dieudonné module of this formal group in such a way that $\star$-isomorphic deformations correspond to the same submodule (see [3]). Our aim is to compare these approaches and establish an explicit relation between them.

Let Φ be a formal group over the perfect field l of characteristic p. Hazewinkel [5] defined the universal p-typical formal group F_T and proved that any formal group over any $\mathbf{Z}_p$-algebra is isomorphic to a formal group which can be obtained from F_T by applying some $\mathbf{Z}_p$-homomorphism to its coefficients. Suppose that Φ is obtained from F_T in this way. Then we can use F_T to construct the canonical universal deformation Γ of Φ which is defined over the ring of formal power series in $h-1$ variables with coefficients from the ring of Witt vectors $W(l)$, where h is the height of Φ. It gives a parameterization of the Lubin–Tate moduli space and, in particular, allows us to fix the canonical deformation F_0 of Φ over $W(l)$ whose logarithm we denote by f_0.

Then we fix an algebraic extension k of l and study the deformations of Φ over $W(k)$. To this end, we consider the non-commutative ring E of formal power series in the variable Δ with coefficients in $W(k)$ and multiplication rule $\Delta a = \mathrm{Frob}(a)\Delta$ for $a \in W(k)$. The ring E acts on the left on the set of formal power series with coefficients in the fractions field of $W(k)$. Then we define the Dieudonné module D as the E-module Ef_0/P, where P is the E-submodule of Ef_0 consisting of the power series with coefficients in $pW(k)$. We denote $\widetilde{D} = E^*(f_0 + P) \subset D$ and introduce an equivalence relation on $\widetilde{D}$ as follows: $b \sim c$ iff $ab = c$ for some invertible $a \in W(k)$. We define the p-adic period map χ from the set of $\star$-isomorphism classes of deformations of Φ over $W(k)$ to the set of equivalence classes in $\widetilde{D}$ and prove that it is bijective.

Further, we observe that the former set can be supplied with a right action of the group of k-automorphisms of Φ, and the latter one has a natural right action of the multiplicative group of the ring E^{u_0}/Eu_0, where $E^{u_0} = E \cap u_0^{-1}Eu_0$ and $u_0 \in E$ is such that $u_0 \equiv p \bmod \Delta$, $u_0 f_0 \equiv 0 \bmod p$. But according to the results of Honda [6], the map from E^{u_0}/Eu_0 to the ring of k-endomorphisms of Φ which assigns to $w + Eu_0 \in E^{u_0}/Eu_0$ the reduction of the formal power series $f_0^{-1}(wf_0)$ is a ring isomorphism. Thus we obtain the right actions of the same group on the set of $\star$-isomorphism classes of deformations of Φ over $W(k)$ and on the set of equivalence classes in $\widetilde{D}$. We prove that the p-adic period map χ is equivariant with respect to these actions.

Then we pass to our main object, namely, to finding an explicit formula for the p-adic period map. For that purpose, we define an E-homomorphism α from Ef_0 to certain E-module in such a way that the image of P is equal to 0 and, moreover, its composition with the logarithm of the canonical universal deformation is a rigid analytic map on Lubin–Tate's moduli space which can be written down explicitly. Considering the first n coordinate functions of this composition, we obtain a linear system of equation on the coordinates of the p-adic period map χ that provides an explicit formula for it. We also remark that if Φ is the reduction of the Artin–Hasse formal group, then this formula can be essentially simplified.

As an application we suggest the following result. If we take for the definition of the p-adic period map χ our explicit formula, then the equivariance property of χ implies that it can be used for explicit description of the group action on Lubin–Tate's moduli space. To be more precise, we consider two moduli $\tau, \tau' \in pA \times \cdots \times pA$ and the deformations F, F' of Φ corresponding to τ, τ'. Then the k-automorphism of Φ which is equal to the reduction of $f_0^{-1}(wf_0)$, where $w \in E^{u_0 *}$, moves the $\star$-isomorphism class of F to that of F' iff for some $a \in A^*$, the equality $a\chi(\tau_1, \ldots, \tau_{h-1})C(w) = \chi(\tau'_1, \ldots, \tau'_{h-1})$ holds, where $C(w)$ is the matrix of the right multiplication in D by the element w with respect to the $W(k)$-basis $f_0, \Delta f_0, \ldots, \Delta^{h-1} f_0$. This theorem can be viewed as a generalization of the result of Gross and Hopkins [4] who proved such a formula in the case when Φ is the reduction of the Artin–Hasse formal group. On the other hand, it extends the previous authors' result [2] concerning the action of the automorphism group on the zero orbit of the Lubin–Tate polydisk.

The fact that the coordinate functions of the p-adic period mapping are rigid analytic implies the continuity of this mapping as well as the continuity of the action of the automorphism group on the moduli space of deformations. The canonical metric on the Lubin–Tate polydisk is defined in [9]. The explicit formula proved allows one to estimate the action of the natural filtration of the automorphism group with respect to this metric.

We will use the following notation. If B is a ring, we write B^* for its multiplicative group. We also denote by $B[\![x]\!]_0$ the B-module of the formal power series without constant term and by $B[\![x]\!]_p$ the B-module of the formal power series which have non-zero coefficients at x^n only if n is a power of p.

1. Lubin–Tate moduli space of formal group deformations and the action of the automorphism group on it

Let l be a perfect field of characteristic p, O the ring of Witt vectors over l and L the fraction field of O. We consider a one-parameter formal group Φ over l of finite height h. Let A be a complete Noetherian local O-algebra with maximal ideal $\mathcal{M} \supseteq pA$ and residue field $k = A/\mathcal{M} \supseteq l$. A formal group F over A such that its reduction modulo $\mathcal{M}$ is equal to Φ is called a deformation of Φ over A. If an isomorphism between deformations F and G over A has identity reduction modulo $\mathcal{M}$, we say that it is a $\star$-isomorphism. In this case F and G are called $\star$-isomorphic. The $\star$-isomorphism class of the deformation F will be denoted by $[F]$.

Lubin and Tate [8] constructed a moduli space for $\star$-isomorphism classes of deformations of Φ and defined an action of the automorphism group of Φ on it. Here we review their main results.

A formal group Φ over l of height h is said to be in normal form if

$$\Phi(x, y) \equiv x + y + aC_{p^h}(x, y) \mod \deg\bigl(p^h + 1\bigr)$$

for some non-zero $a \in l$, where $C_{p^i}(x, y) = ((x + y)^{p^i} - x^{p^i} - y^{p^i})/p$.

Lemma 1.1 [7, Lemma 6]. *Every formal group of finite height over l is isomorphic to a formal group in normal form.*

Suppose Φ to be in normal form. A formal group Γ over $O[\![t_1, \ldots, t_{h-1}]\!]$ is called a generic formal group for Φ, if the reduction of $\Gamma(0, \ldots, 0)$ modulo p is equal to Φ and

$$\Gamma(0, \ldots, 0, t_i, \ldots, t_{h-1})(x, y) \equiv x + y - t_i C_{p^i}(x, y) \mod \deg\bigl(p^i + 1\bigr)$$

for $1 \leqslant i \leqslant h-1$.

Theorem 1.2 [8, Theorem 3.1]. *Let Γ be a generic formal group for Φ and F be a deformation of Φ over A. Then there is a unique $(h-1)$-tuple $(\tau_1, \ldots, \tau_{h-1})$, $\tau_i \in \mathcal{M}$, such that F is $\star$-isomorphic to $\Gamma(\tau_1, \ldots, \tau_{h-1})$ and the $\star$-isomorphism is uniquely defined.*

Theorem 1.2 implies that Γ is a universal deformation of the formal group Φ, and the set of $\star$-isomorphism classes of deformations of Φ over A is in one-to-one correspondence with $\mathcal{M} \times \cdots \times \mathcal{M}$ ($h-1$ times). Thus we obtain a parameterization of the set of $\star$-isomorphism classes of deformations of Φ.

The group $\mathrm{Aut}_k \Phi$ acts on the right on the set of $\star$-isomorphism classes of deformations of Φ over A in the following way. If $\varphi \in \mathrm{Aut}_k \Phi$ and F is a deformation of Φ over A, then $[F]\varphi = [g^{-1} \circ F(g, g)]$, where $g \in A[\![x]\!]_0$, the reduction of g modulo $\mathcal{M}$ is equal to φ.

Proposition 1.3. *Let F be a deformation of Φ over A. Then* $\mathrm{Orb}[F]$ *is the set of the $\star$-isomorphism classes of deformations of Φ which are isomorphic to F over A.*

Proof. If $\varphi \in \mathrm{Aut}_k \Phi$ and $g \in A[\![x]\!]_0$, the reduction of g is equal to φ, then g provides an isomorphism between $g^{-1} \circ F(g, g) \in [F]\varphi$ and F. If g is an isomorphism between deformations G and F the reduction of g is an automorphism of Φ and it maps the class $[F]$ to the class $[G]$. $\square$

2. Hazewinkel's universal p-typical formal group and the canonical universal deformation

Hazewinkel used a universal p-typical formal group to get a parameterization of a large number of generic formal groups, which in particular allows to choose one of them canonically. We review here his construction.

Denote $\Lambda = \mathbf{Q}_p[t_1, t_2, \ldots]$, $\Omega = \mathbf{Z}_p[t_1, t_2, \ldots]$. Define a $\mathbf{Q}_p$-endomorphism σ of Λ by $\sigma(t_i) = t_i^p$. Let σ operate on the ring $\Lambda[\![x]\!]_0$ by the formula

$$\sigma\left(\sum_{n=1}^{\infty} c_n x^n\right) = \sum_{n=1}^{\infty} \sigma(c_n) x^{pn}.$$

There exists a unique formal power series $f_T \in \Lambda[\![x]\!]_p$ which satisfies the functional equation $pf_T - \sum_{i=1}^{\infty} t_i \sigma^i(f_T) = px$. We denote by d_n the coefficient of f_T at x^{p^n}. It is clear that $d_0 = 1$ and $d_n \in \mathbf{Q}_p[t_1, \dots, t_n] \subset \Lambda$.

Proposition 2.1 [5, Eq. (3.3.9)].

$$pd_n = \sum_{i=1}^{n} t_i^{p^{n-i}} d_{n-i}.$$

Lemma 2.2.

$$f_T \equiv x + (t_i/p)x^{p^i} \mod t_1, \dots, t_{i-1}, \deg\bigl(p^i + 1\bigr).$$

Proof. The functional equation for f_T implies $d_n \equiv t_n/p \bmod t_1, \dots, t_{n-1}$. The required formula follows immediately. □

Theorem 2.3.

(i) *f_T is the logarithm of a formal group F_T defined over Ω.*
(ii) $F_T(x, y) \equiv x + y - t_i C_{p^i}(x, y) \bmod t_1, \dots, t_{i-1}, \deg(p^i + 1)$.

Proof. The part (i) is an immediate consequence of Hazewinkel's functional equation lemma, see [5, Section 2.2(i) and Eq. (2.3.7)]. The part (ii) follows from Lemma 2.2. □

A formal group F over a $\mathbf{Z}_p$-algebra B is called p-typical if there is a homomorphism ξ from Ω to B such that $F = \xi_* F_T$. Evidently, F_T is a universal p-typical formal group.

Proposition 2.4 [5, Theorem 15.2.9]. *Every formal group over a $\mathbf{Z}_p$-algebra B is isomorphic to a p-typical one.*

Lemma 2.5. *Let ξ be a homomorphism from Ω to l. Then the height of the formal group $\xi_* F_T$ is equal to the minimal i satisfying $\xi(t_i) \neq 0$.*

Proof. It follows from Lemma 2.2. □

Corollary 2.6. *Every p-typical formal group over l is in normal form.*

Proof. It follows from Theorem 2.3(ii) and Lemma 2.5. □

Now suppose Φ to be a p-typical formal group. By Corollary 2.6, it is in normal form. Take a homomorphism ξ from Ω to l such that $\Phi = \xi_* F_T$. Let r_i be the multiplicative representative of $\xi(t_i)$ in O. Define a $\mathbf{Q}_p$-homomorphism η from Λ to $L[t_1, \dots, t_{h-1}]$ as follows: $\eta(t_i) = t_i$ for $i < h$; $\eta(t_i) = r_i \in O$ for $i \geqslant h$. Put $\Gamma = \eta_* F_T$. Then Γ is defined over $O[t_1, \dots, t_{h-1}]$, and its reduction modulo $p, t_1, \dots, t_{h-1}$ is equal to Φ. Moreover by

Theorem 2.3(ii), Γ is a generic formal group for Φ, and hence by Theorem 1.2, it is a universal deformation of Φ.

It is clear that the formal power series $\gamma = \eta_* f_T \in L[t_1, \dots, t_{h-1}][\![x]\!]_p$ is the logarithm of Γ. Denote $a_n = \eta(d_n) \in L[t_1, \dots, t_{h-1}]$. Then a_n is the coefficient of γ at x^{p^n}.

Proposition 2.7.

$$pa_n = \sum_{j=1}^{\min(n,h-1)} t_j^{p^{n-j}} a_{n-j} + \sum_{i=h}^{n} r_i^{p^{n-i}} a_{n-i}.$$

Proof. It follows immediately from Proposition 2.1. □

3. Honda's classification of formal groups and an explicit description of the automorphism group

Honda developed a theory of formal groups over the ring of Witt vectors over a perfect field of finite characteristic based on the properties of the logarithms of formal groups. That enables, in particular, to describe explicitly the automorphism group of a formal group over such field.

From now on, we suppose k to be an algebraic extension of l, and A to be the ring of Witt vectors over k. Let K be the fraction field of A and Δ denote the Frobenius automorphism of K. The reduction from A to k modulo p will be denoted by overline.

We denote by E the non-commutative ring of formal power series over A in the variable Δ with multiplication rule $\Delta a = a^{\Delta}\Delta$, $a \in A$. This ring has several common properties with the standard power series ring $A[\![x]\!]$, namely:

(1) a power series $s \in E$ is invertible iff the constant term of s is invertible in A;
(2) the non-commutative version of Weierstrass preparation lemma holds, i.e., for any $u \in E$ which is not divisible by p, there exists a unique invertible $s \in E$ such that su is a monic polynomial, and $\deg su$ is equal to the least power of Δ in the power series u which has an invertible coefficient;
(3) E admits uniquely defined left division transformation, it means that for any monic polynomial $u \in E$ and any $s \in E$, there exist unique $q, r \in E$ such that $s = qu + r$, r is a polynomial and $\deg r < \deg u$.

Let Δ operate on $K[\![x]\!]_0$ by the formula

$$\Delta \sum_{n=1}^{\infty} c_n x^n = \sum_{n=1}^{\infty} c_n^{\Delta} x^{pn}.$$

That determines a left E-module structure on $K[\![x]\!]_0$.

Let $u \in E$ be such that $u \equiv p \bmod \Delta$. A power series $f \in K[\![x]\!]_0$ is said to be of type u if $f(x) \equiv x \bmod x^2$ and $uf \equiv 0 \bmod p$.

Lemma 3.1 [6, Lemma 2.3]. *Let $u, s \in E$, $u \equiv p \bmod \Delta$, $f \in K[\![x]\!]_0$ be of type u and $g \in A[\![x]\!]_0$. Then $s(f \circ g) \equiv (sf) \circ g \bmod p$.*

Theorem 3.2 [6, Theorem 2]. *Let $u \in E$, $u \equiv p \bmod \Delta$ and $f \in K[\![x]\!]_0$ be of type u. Then f is the logarithm of a formal group over A.*

Lemma 3.3 [6, Proposition 2.6]. *Let $u_1, u_2 \in E$, $u_1 \equiv p \bmod \Delta$ and $f \in K[\![x]\!]_0$ be of type u_1. If $u_2 f \equiv 0 \bmod p$, then there exists $s \in E$ such that $u_2 = su_1$.*

Lemma 3.4 [6, Lemma 4.2]. *Let $u \in E$, $u \equiv p \bmod \Delta$, $f \in K[\![x]\!]_0$ be of type u, $\psi_1 \in K[\![x_1, \dots, x_n]\!]_0$ and $\psi_2 \in A[\![x_1, \dots, x_n]\!]_0$. Then $f \circ \psi_1 \equiv f \circ \psi_2 \bmod p$ iff $\psi_1 \equiv \psi_2 \bmod p$.*

Lemma 3.5 [6, Lemma 4.3]. *Let F be a formal group over A with the logarithm f and $g \in K[\![x]\!]_0$. Then $g \circ F(x, y) \equiv g(x) + g(y) \bmod p$ iff there exists $s \in E$ such that $g \equiv sf \bmod p$.*

Theorem 3.6 [6, Theorems 5 and 6]. *Let $w, w' \in E$, $u_i \in E$, $u_i \equiv p \bmod \Delta$, $f_i \in K[\![x]\!]_0$ be of type u_i and F_i be the formal group with the logarithm f_i for $i = 1, 2, 3$. Then*

(i) *$f_2^{-1}(wf_1)$ has coefficients in A iff there exists $z \in E$ such that $u_2 w = zu_1$;*
(ii) *if $f_2^{-1}(wf_1)$ has coefficients in A, then $\overline{f_2^{-1}(wf_1)} \in \mathrm{Hom}_k(\overline{F_1}, \overline{F_2})$;*
(iii) *if $f_2^{-1}(wf_1)$ and $f_3^{-1}(w'f_2)$ have coefficients in A, then $f_3^{-1}(w'wf_1)$ has coefficients in A and $\overline{f_3^{-1}(w'wf_1)} = \overline{f_3^{-1}(w'f_2)} \circ \overline{f_2^{-1}(wf_1)}$;*
(iv) *if $\varphi \in \mathrm{Hom}_k(\overline{F_1}, \overline{F_2})$, then there exists $w \in E$ such that $f_2^{-1}(wf_1)$ has coefficients in A and $\overline{f_2^{-1}(wf_1)} = \varphi$.*

Let $u \in E$, $u \equiv p \bmod \Delta$, $f \in K[\![x]\!]_0$ be of type u and F be the formal group with the logarithm f. Denote by E^u the set of power series $w \in E$ satisfying $uw = zu$ for some $z \in E$. Evidently E^u is a subring of E, and Eu is a two-sided ideal in E^u. Define a map μ from the ring E^u/Eu to $\mathrm{End}_k \overline{F}$ by the formula $\mu(w + Eu) = \overline{f^{-1}(wf)}$. Due to Lemma 3.4, the definition is correct.

Proposition 3.7. *μ is a ring isomorphism.*

Proof. By Theorem 3.6(iii), μ is a homomorphism. Theorem 3.6(iv) implies that μ is surjective. If $\overline{f^{-1}(wf)} = 0$ holds for some $w \in E^u$, then due to Lemma 3.4, we have $wf \equiv 0 \bmod p$. Now Lemma 3.3 implies that there exists $s \in E$ such that $w = su$, i.e., w belongs to the zero coset in E^u/Eu. Thus μ is injective. $\square$

4. Dieudonné module and p-adic period map χ

Denote $F_0 = \Gamma(0,\dots,0) \in O[\![x,y]\!]$ and $f_0 = \gamma(0,\dots,0) \in L[\![x]\!]_p$. Evidently f_0 is the logarithm of the formal group F_0, and the coefficient of f_0 at x^{p^n} is $a_n(0,\dots,0)$. Define a $\mathbf{Q}_p$-homomorphism θ from Λ to $L \subseteq K$ by the formula $\theta(t_i) = r_i$. Then $F_0 = \theta_* F_T$, $f_0 = \theta_* f_T$ and $a_n(0,\dots,0) = \theta(d_n) = d_n(r_1,\dots,r_n)$.

We define R to be the E-submodule of $K[\![x]\!]_p$ generated by f_0. Let P denote the E-submodule of R which consists of the formal power series with coefficients belonging to pA, and put $D = R/P$. The E-module D is nothing else than the Dieudonné module of the formal group Φ (see [3]).

Denote $u_0 = p - \sum_{i=1}^{\infty} r_i \Delta^i \in O[\![\Delta]\!] \subset E$.

Proposition 4.1. $u_0 f_0 = px$.

Proof. The coefficient of f_0 at x^{p^n} is $a_n(0,\dots,0)$. Taking into account that $r_i^{\Delta} = r_i^p$ and using the functional equation for f_T, we obtain

$$\begin{aligned} u_0 f_0 &= pf_0 - \sum_{i=1}^{\infty} r_i \Delta^i(f_0) = pf_0 - \sum_{i=1}^{\infty}\sum_{n=0}^{\infty} r_i a_n(0,\dots,0)^{\Delta^i} x^{p^{i+n}} \\ &= pf_0 - \sum_{i=1}^{\infty}\sum_{n=0}^{\infty} r_i d_n\bigl(r_1^{p^i},\dots,r_n^{p^i}\bigr) x^{p^{i+n}} = \theta_*\left(pf_T - \sum_{i=1}^{\infty} t_i \sigma^i(f_T)\right) = px. \qquad \square \end{aligned}$$

Let $\widetilde{D} = E^*(f_0 + P) \subset D$. We define on $\widetilde{D}$ the actions of the multiplicative groups E^* and $(E^{u_0}/Eu_0)^* = E^{u_0 *}/(1 + Eu_0)$.

Proposition 4.2. *The actions*

$$\begin{aligned} E^* \times \widetilde{D} \to \widetilde{D}: &\quad s(zf_0 + P) = szf_0 + P, \\ \widetilde{D} \times \bigl(E^{u_0 *}/(1 + Eu_0)\bigr) \to \widetilde{D}: &\quad (zf_0 + P)w(1 + Eu_0) = zwf_0 + P \end{aligned}$$

are well defined and commute. The left action is transitive, the right one is faithful, and the stabilizer of any element with respect to the right action is trivial.

Proof. We check that the right action is faithful. If $(zf_0 + P)w(1 + Eu_0) = zf_0 + P$ for any $z \in E^*$ then $(w - 1)f_0 \in P$ which implies $w \in 1 + Eu_0$ by Lemma 3.3. The left action is transitive therefore the stabilizer of any element with respect to the right action is trivial. $\square$

Finally, we denote by S the set of orbits in $\widetilde{D}$ with respect to the left action of the group $A^* \subset E^*$. Then S is equal to $\{A^* b(f_0 + P) \mid b \in E^*\}$. It inherits from $\widetilde{D}$ the right action of $E^{u_0 *}/(1 + Eu_0)$. The stabilizer of the element $A^*(f_0 + P) \in S$ with respect to this action is $A^* \cap E^{u_0 *} \subset E^{u_0 *}/(1 + Eu_0)$, and its orbit is $\{A^* b(f_0 + P) \mid b \in A^* E^{u_0 *}\}$.

We also observe that if $l = \mathbf{F}_p$ and $r_i = 0$ whenever i is not a multiple of the degree of k over $\mathbf{F}_p$, then u_0 belongs to the center of E, and hence $E^{u_0} = E$. Thus, in this case, the right action of $E^{u_0 *}/(1 + Eu_0) = E^*/(1 + Eu_0)$ on S is transitive. If, in addition, $k = \mathbf{F}_p$, then E is commutative, and the subgroup $A^* \cap E^{u_0} = A^*$ of $E^{u_0 *}/(1 + Eu_0) = E^*/(1 + Eu_0)$ is normal, i.e., S becomes a principal homogeneous space for the action of the group $E^*/(A^* + Eu_0)$.

Lemma 4.3. *Let F, G be formal groups over A with the logarithms f, g, respectively. Then the reductions of F and G are equal iff there exists $s \in E^*$ such that $g \equiv sf \bmod p$.*

Proof. According to Lemma 3.5, $g \equiv sf \bmod p$ for some $s \in E$ iff $g \circ F(x, y) \equiv g(x) + g(y) \bmod p$. But $g(x) + g(y) = g \circ G(x, y)$, and due to Lemma 3.4 $g \circ F \equiv g \circ G \bmod p$ iff $F \equiv G \bmod p$. □

We define the map χ from the set of $\star$-isomorphism classes of deformations of Φ to S. Let F be a deformation of Φ and f its logarithm. By Lemma 4.3, there exists $s \in E^*$ such that $f \equiv sf_0 \bmod p$. We put $\chi[F] = A^*(sf_0 + P)$. If f, g are logarithms of two $\star$-isomorphic deformations of Φ, then $g^{-1}(af) \equiv x \bmod p$ for some $a \in A^*$ and Lemma 3.4 implies that $g \equiv af \bmod p$. Hence, for any $s, s' \in E$ satisfying $f \equiv sf_0 \bmod p$ and $g \equiv s' f_0 \bmod p$, we have $A^*(sf_0 + P) = A^*(s' f_0 + P)$. Thus the definition of χ is correct.

Proposition 4.4. *χ is bijective.*

Proof. Any element of S is of the form $A^*(sf_0 + P)$ with $s \in 1 + \Delta E$. The power series sf_0 is of type $u_0 s^{-1}$, and then by Theorem 3.2, it is the logarithm of a formal group F defined over A. By Lemma 4.3, F is a deformation of Φ, and $\chi[F] = A^*(sf_0 + P)$. Thus χ is surjective. If the deformations F_1 and F_2 of Φ with the logarithms f_1 and f_2 are such that $\chi[F_1] = \chi[F_2]$, then $f_2 \equiv af_1 \bmod p$ for some $a \in A^*$. Due to Lemma 3.4, it implies that $f_2^{-1}(af_1) \equiv x \bmod p$, i.e., F_1 and F_2 are $\star$-isomorphic. Therefore χ is injective. □

Since the reduction of F_0 is equal to Φ, we can apply Proposition 3.7 to identify $\mathrm{End}_k \Phi$ and the ring E^{u_0}/Eu_0. Then $\mathrm{Aut}_k \Phi$ is identified with the multiplicative group $(E^{u_0}/Eu_0)^* = E^{u_0 *}/(1 + Eu_0)$. Thus we have right actions of the group $E^{u_0 *}/(1 + Eu_0)$ on both the set of $\star$-isomorphism classes of deformations of Φ and S.

Theorem 4.5. *χ is $E^{u_0 *}/(1 + Eu_0)$-equivariant.*

Proof. Let F be a deformation of Φ and let f be its logarithm. Then by Lemma 4.3, there exists $s \in E^*$ such that $f \equiv sf_0 \bmod p$. Then the image of $[F]$ with respect to χ is $A^*(sf_0 + P)$. Multiplying it by $w(1 + Eu_0) \in E^{u_0 *}/(1 + Eu_0)$ on the right, we obtain $A^*(swf_0 + P)$. On the other hand, the automorphism of Φ corresponding to $w(1 + Eu_0)$ is $\mu(w + Eu_0) = \overline{f_0^{-1}(wf_0)}$, and it sends $[F]$ to $[G]$, where G is the formal group with logarithm $g = a^{-1} f \circ f_0^{-1}(wf_0)$ and $a \in A^*$ is such that $w \equiv a \bmod \Delta$. By Lemma 3.1, we have $f \circ f_0^{-1}(wf_0) \equiv (sf_0) \circ f_0^{-1}(wf_0) \equiv swf_0 \bmod p$. Hence the images of $[G]$ with respect to χ is also $A^*(swf_0 + P)$ and χ is $E^{u_0 *}/(1 + Eu_0)$-equivariant. □

Now we are going to declare our main purpose. We can identify R with E so that f_0 corresponds to 1. It gives us the following identifications: D with E/Eu_0, $\widetilde{D}$ with $E^*/(1+Eu_0)$ and S with $\{A^*b(1+Eu_0) \mid b \in E^*\}$. By Lemma 2.5, the least power in the formal power series u_0 which has an invertible coefficient is h. Then according to Weierstrass preparation lemma, there exists a unique $s_0 \in E^*$ with coefficients in O such that s_0u_0 is a monic polynomial of degree h. Since E admits uniquely defined left division transformation, for any $s \in E$, there exist unique $q, r \in E$ such that $s = qs_0u_0 + r$, where r is a polynomial and $\deg r < h$. If s is invertible, then r is also invertible. Thus the cosets in D can be represented by the polynomials over A of degree less than h, and such a coset belongs to $\widetilde{D}$ iff the corresponding polynomial has an invertible constant term. Moreover, the polynomials over A of degree less than h with constant term equal to 1 can be chosen as representatives of the cosets in S. We call the polynomial of that sort belonging to the image of a $\star$-isomorphism class of deformations of Φ with respect to χ the Dieudonné polynomial of this class. The coefficients of the Dieudonné polynomial give a parameterization of the set of $\star$-isomorphism classes of deformations of Φ. Our purpose is to compare this parameterization with Lubin–Tate's one. To be more precise, we will prove an explicit formula expressing the coefficients of the Dieudonné polynomial through Lubin–Tate's parameters.

5. E-homomorphism α

Let $\pi_i \in O$, $0 \leqslant i \leqslant h$, be the coefficients of s_0u_0, i.e., $s_0u_0 = \sum_{i=0}^{h} \pi_i \Delta^i$ where $\pi_h = 1$ since s_0u_0 is monic and $\pi_i \in pO$ for $1 \leqslant i \leqslant h-1$ because s_0 is invertible. Define the sequence $\zeta_n \in O$ in the following way: $\zeta_0 = 1$, $\zeta_n = 0$ for $1 \leqslant n \leqslant h-1$ and

$$\zeta_n = -\sum_{i=0}^{h-1} \pi_i^{\Delta^{-h}} \zeta_{n+i-h}^{\Delta^{i-h}}$$

for $n \geqslant h$. Since $\pi_i \in pO$ for $0 \leqslant i \leqslant h-1$, we have $\nu(\zeta_n) \geqslant [n/h]$. In particular, $\lim \zeta_n = 0$.

Proposition 5.1. *Let $f = \sum_{i=0}^{\infty} c_i x^{p^i} \in R$. Then $pc_i - \sum_{j=1}^{i} r_j^{\Delta^{i-j}} c_{i-j} \equiv 0 \bmod p$ for any $i \geqslant 0$.*

Proof. If $f = f_0$, then $c_i = a_i(0, \ldots, 0)$, and Proposition 2.7 implies that the equality holds. Now, we are going to show that if the equality holds for $f = \sum_{i=0}^{\infty} c_i x^{p^i} \in R$, then it holds also for $\Delta f = \sum_{i=1}^{\infty} c_{i-1}^{\Delta} x^{p^i}$. Indeed

$$pc_{i-1}^{\Delta} - \sum_{j=1}^{i-1} r_j^{\Delta^{i-j}} c_{i-j-1}^{\Delta} = \left(pc_{i-1} - \sum_{j=1}^{i-1} r_j^{\Delta^{i-1-j}} c_{i-1-j} \right)^{\Delta} \equiv 0 \mod p.$$

Since f_0 generates R as E-module, we obtain the required statement. $\square$

For any $n \geqslant 0$, define an A-linear map $\alpha_n : R \to A$ in the following way:

$$\alpha_n\left(\sum_{i=0}^{\infty} c_i x^{p^i}\right) = p\zeta_n c_0 + \sum_{i=1}^{\infty} \zeta_{i+n}^{\Delta^i}\left(pc_i - \sum_{j=1}^{i} r_j^{\Delta^{i-j}} c_{i-j}\right).$$

Since $\lim_{i\to\infty} \zeta_{i+n}^{\Delta^i} = 0$, Proposition 5.1 implies that the definition is correct.

Proposition 5.2. *$\alpha_n(f_0) = p\zeta_n$ for any $n \geqslant 0$.*

Proof. The coefficient of f_0 at x^{p^i} is $a_i(0, \ldots, 0)$. Then the required formula follows immediately from Proposition 2.7. $\square$

Let $A\{y\}_p$ denote the A-submodule of $A[\![y]\!]_p$ consisting of the formal power series $\sum_{n=0}^{\infty} c_n y^{p^n}$ such that $\lim c_n = 0$. Let Δ operate on $A\{y\}_p$ by the formula

$$\Delta \sum_{n=0}^{\infty} c_n y^{p^n} = \sum_{n=0}^{\infty} c_{n+1}^{\Delta} y^{p^n}.$$

That determines a left E-module structure on $A\{y\}_p$.

Since $\lim_{n\to\infty} \zeta_{i+n}^{\Delta^i} = 0$, Proposition 5.1 implies that $\lim_{n\to\infty} \alpha_n(f) = 0$ for any $f \in R$. Therefore we can define the A-linear map $\alpha : R \to A\{y\}_p$ by the formula $\alpha(f) = \sum_{n=0}^{\infty} \alpha_n(f) y^{p^n}$.

Proposition 5.3. *α is a homomorphism of E-modules.*

Proof. If $f = \sum_{i=0}^{\infty} c_i x^{p^i} \in R$ then $\Delta f = \sum_{i=1}^{\infty} c_{i-1}^{\Delta} x^{p^i}$. We compute

$$\sum_{i=1}^{m} \zeta_{i+n}^{\Delta^i}\left(pc_{i-1}^{\Delta} - \sum_{j=1}^{i-1} r_j^{\Delta^{i-j}} c_{i-1-j}^{\Delta}\right)$$
$$= \left(p\zeta_{n+1}c_0 + \sum_{i=1}^{m-1} \zeta_{i+n+1}^{\Delta^i}\left(pc_i - \sum_{j=1}^{i} r_j^{\Delta^{i-j}} c_{i-j}\right)\right)^{\Delta}.$$

Now taking a limit over m, we obtain $\alpha_n(\Delta f) = \alpha_{n+1}(f)^{\Delta}$. Therefore $\alpha(\Delta f) = \sum_{n=0}^{\infty} \alpha_{n+1}(f)^{\Delta} y^{p^n} = \Delta\alpha(f)$, i.e., α is a homomorphism of E-modules. $\square$

Proposition 5.4. $\alpha(P) = 0$.

Proof. Let $f \in P$. Then $f = uf_0$ for some $u \in E$ and Lemma 3.3 implies that there exists $s \in E$ such that $u = su_0$. Further, applying Propositions 5.3, 5.2 and the definition of ζ_n, we obtain

$$\alpha(f) = su_0\alpha(f_0) = ss_0^{-1}s_0u_0\sum_{n=0}^{\infty} p\zeta_n y^{p^n} = pss_0^{-1}\sum_{n=0}^{\infty}\sum_{i=0}^{h}\pi_i\zeta_{n+i}^{\Delta^i}y^{p^n} = 0. \qquad \square$$

6. Rigid analytic functions ρ_n and explicit calculation of the p-adic period map

We define the ring of rigid analytic functions $K\{\{t_1, \dots, t_{h-1}\}\}$ as the subring of $K[[t_1, \dots, t_{h-1}]]$ consisting of the formal power series

$$f = \sum_{I=(i_1,\dots,i_{h-1})\in\mathbf{N}^{h-1}} f_I t^I$$

such that $\nu(f_I) + |I| \to \infty$ as $|I| \to \infty$, where $f_I = f_{i_1,\dots,i_{h-1}}$, $t^I = t_1^{i_1}t_2^{i_2}\cdots t_{h-1}^{i_{h-1}}$, $|I| = i_1 + \cdots + i_{h-1}$ and ν is the normalized valuation of K. Such power series converge if the variables t_i, $1 \leqslant i \leqslant h-1$, take value in pA. It means that the rigid analytic functions can be evaluated on the set $pA \times \cdots \times pA$ ($h-1$ times).

Define a norm on $K\{\{t_1, \dots, t_{h-1}\}\}$ by

$$\|f\| = p^{-\min_{I\in\mathbf{N}^{h-1}}\{\nu(f_I)+|I|\}}.$$

This norm provides $K\{\{t_1, \dots, t_{h-1}\}\}$ with the structure of Banach K-algebra (see [1, Section 6.1.5, Proposition 1]). If a sequence of rigid analytic functions converges to $f \in K\{\{t_1, \dots, t_{h-1}\}\}$, then the sequence of their values at a point in $pA \times \cdots \times pA$ ($h-1$ times) converges to the value of f at this point.

Proposition 6.1. *The sequence of polynomials*

$$p\zeta_n a_0 + \sum_{i=1}^{m}\zeta_{i+n}^{\Delta^i}\left(pa_i - \sum_{j=1}^{i} r_j^{\Delta^{i-j}} a_{i-j}\right) \in L[t_1, \dots, t_{h-1}]$$

converges in $L\{\{t_1, \dots, t_{h-1}\}\}$ to the formal power series

$$\rho_n = p\zeta_n + \sum_{i=1}^{\infty}\sum_{j=1}^{\min(i,h-1)}\zeta_{i+n}^{\Delta^i}t_j^{p^{i-j}}a_{i-j} = p\zeta_n + \sum_{i=0}^{\infty}\sum_{j=1}^{h-1}\zeta_{i+j+n}^{\Delta^{i+j}}t_j^{p^i}a_i.$$

Proof. By Proposition 2.7, we have

$$\zeta_{i+n}^{\Delta^i}\left(pa_i - \sum_{j=1}^{i} r_j^{\Delta^{i-j}} a_{i-j}\right) = \sum_{j=1}^{\min(i,h-1)}\zeta_{i+n}^{\Delta^i}t_j^{p^{i-j}}a_{i-j}.$$

Furthermore, the recursive formula for a_i implies that $p^i a_i \in O[t_1, \dots, t_{h-1}]$. Therefore we have

$$\left\| \sum_{j=1}^{\min(i,h-1)} \zeta_{n+i}^{\Delta^i} t_j^{p^{i-j}} a_{i-j} \right\| \leqslant p^{i-p^{i-h+1}},$$

i.e., the sequence of polynomials under consideration is a Cauchy sequence in $L\{\{t_1, \dots, t_{h-1}\}\}$. $\square$

Proposition 6.2. $\rho_n(\tau_1, \dots, \tau_{h-1}) = \alpha_n(\gamma(\tau_1, \dots, \tau_{h-1}))$ *for any* $\tau_i \in pA$ *and* $n \geqslant 0$.

Proof. The coefficient of $\gamma(\tau_1, \dots, \tau_{h-1})$ at x^{p^i} is $a_i(\tau_1, \dots, \tau_{h-1})$. Since the evaluation of a rigid analytic function at a point in $pA \times \dots \times pA$ ($h-1$ times) commutes with taking limit, Proposition 6.1 implies the required equality. $\square$

Lemma 6.3. *Let* $f \in K\{\{t_1, \dots, t_{h-1}\}\}$, $f(0) = 1$ *and* $\|f - 1\| < 1$, *i.e.,* $\nu(f_J) + |J| > 0$ *for any* $J \in \mathbf{N}^{h-1}$, $J \neq (0, \dots, 0)$. *Then* f *is invertible in* $K\{\{t_1, \dots, t_{h-1}\}\}$.

Proof. Let $f'(t_1, \dots, t_{h-1}) = f(pt_1, \dots, pt_{h-1})$. Then we have $f \in K\{\{t_1, \dots, t_{h-1}\}\}$ iff $\nu(f'_J) \to \infty$ as $|J| \to \infty$, i.e., $f' = \sum f'_J t^J$ belongs to the Tate algebra $T_{h-1}(K)$. By [1, Section 5.1.3, Proposition 1], an element $f' \in T_{h-1}(K)$ such that $f'(0) = 1$ is invertible iff $\nu(f'_J) > 0$ for any $J \in \mathbf{N}^{h-1}$, $J \neq (0, \dots, 0)$. Since f' is invertible iff f is invertible, we are done. $\square$

Proposition 6.4. ρ_0 *is invertible in* $L\{\{t_1, \dots, t_{h-1}\}\}$.

Proof. $\rho_0 \equiv p\zeta_0 = p \bmod t_1, \dots, t_{h-1}$. We will prove that ρ_0/p is invertible in $K\{\{t_1, \dots, t_{h-1}\}\}$. By Lemma 6.3, it is enough to show that $\|\rho_0/p - 1\| < 1$. Since $\zeta_n = 0$ for $1 \leqslant n \leqslant h-1$ and $\nu(\zeta_n) \geqslant [n/h]$, we obtain $\nu(\zeta_n) \geqslant 1$ for any $n \geqslant 1$. From the recursive formula for a_i, it follows that $\|a_i\| \leqslant p^i$. Therefore $\| \sum_{j=1}^{h-1} \zeta_{i+j}^{\Delta^{i+j}} t_j^{p^i} a_i \| \leqslant p^{-1-p^i+i} \leqslant p^{-2}$ for any $i \geqslant 0$, and thus $\|\rho_0 - p\| \leqslant p^{-2}$, i.e., $\|\rho_0/p - 1\| \leqslant p^{-1} < 1$. $\square$

Now let $1 + \sum_{i=1}^{h-1} \beta_i \Delta^i$ be the Dieudonné polynomial of the $\star$-isomorphism class of deformations of Φ which contains the formal group $\Gamma(\tau_1, \dots, \tau_{h-1})$. Then the formal group with the logarithm $(1 + \sum_{i=1}^{h-1} \beta_i \Delta^i) f_0$ is $\star$-isomorphic to $\Gamma(\tau_1, \dots, \tau_{h-1})$, i.e., there exists $\varepsilon \in A^*$ such that

$$\gamma(\tau_1, \dots, \tau_{h-1})^{-1}\left(\varepsilon\left(1 + \sum_{i=1}^{h-1} \beta_i \Delta^i\right) f_0\right) \equiv x \mod p.$$

Then by Lemma 3.4, it implies $\gamma(\tau_1, \dots, \tau_{h-1}) \equiv \varepsilon(1 + \sum_{i=1}^{h-1} \beta_i \Delta^i) f_0 \bmod p$. Propositions 6.2, 5.4, 5.3 and 5.2 give

$$\sum_{n=0}^{\infty}\rho_n(\tau_1,\dots,\tau_{h-1})y^{p^n}=\alpha\big(\gamma(\tau_1,\dots,\tau_{h-1})\big)=\alpha\left(\varepsilon\left(1+\sum_{i=1}^{h-1}\beta_i\Delta^i\right)f_0\right)$$
$$=\varepsilon\left(1+\sum_{i=1}^{h-1}\beta_i\Delta^i\right)\alpha(f_0)=p\varepsilon\left(1+\sum_{i=1}^{h-1}\beta_i\Delta^i\right)\sum_{n=0}^{\infty}\zeta_n y^{p^n}.$$

Writing down the coefficients at y^{p^n}, $0\leqslant n\leqslant h-1$, in the left and right parts of this equality, we obtain the following system of h linear equations on $p\varepsilon, p\varepsilon\beta_1,\dots,p\varepsilon\beta_{h-1}$:

$$\big(\rho_0(\tau_1,\dots,\tau_{h-1}),\rho_1(\tau_1,\dots,\tau_{h-1}),\dots,\rho_{h-1}(\tau_1,\dots,\tau_{h-1})\big)$$
$$=(p\varepsilon,p\varepsilon\beta_1,\dots,p\varepsilon\beta_{h-1})Z,$$

where $Z=\{\zeta_{i+j}^{\Delta^i}\}_{i,j=0}^{h-1}$ is an $h\times h$ matrix with entries from O.

Since $\zeta_0=1$, $\zeta_n=0$ for $1\leqslant n\leqslant h-1$ and $\zeta_h=-\pi_0^{\Delta^{-h}}\neq 0$, the matrix Z becomes after an obvious permutation of rows a triangle matrix with non-zero elements on its diagonal. Thus Z is invertible, and the linear system has a unique solution. Taking into account Proposition 6.4, we can summarize our results in the following theorem.

Theorem 6.5. *Let $\tau_j\in pA$ and $\beta_i\in A$, $1\leqslant i,j\leqslant h-1$, be such that*

$$\chi\big[\Gamma(\tau_1,\dots,\tau_{h-1})\big]=A^*\left(\left(1+\sum_{i=1}^{h-1}\beta_i\Delta^i\right)f_0+P\right)\in S.$$

Then β_i can be explicitly expressed through τ_i with the aid of the following rigid analytic functions with coefficients in L: $\beta_i=((\rho_0,\dots,\rho_{h-1})Z^{-1})_i/\rho_0$, where $Z=\{\zeta_{i+j}^{\Delta^i}\}_{i,j=0}^{h-1}$.

7. Applications

According to Theorem 6.5, β_i can be considered as rigid analytic functions on the Lubin–Tate polydisk $pA\times\cdots\times pA$ ($h-1$ times). One can use them for checking several properties of deformations of Φ with given moduli.

Proposition 7.1. *Let $\tau_j\in pA$ for $1\leqslant j\leqslant h-1$. Then $\Gamma(\tau_1,\dots,\tau_{h-1})$ is isomorphic to F_0 iff $1+\sum_{i=1}^{h-1}\beta_i\Delta^i\in A^*E^{u_0*}$, where $\beta_i=((\rho_0,\dots,\rho_{h-1})Z^{-1})_i/\rho_0$, $1\leqslant i\leqslant h-1$.*

Proof. The element $A^*(sf_0+P)\in S$ belongs to the orbit of $A^*(f_0+P)$ with respect to the right action of $E^{u_0*}/(1+Eu_0)$ on S iff $s\in A^*E^{u_0*}$. The required statement follows from Theorems 4.5, 6.5 and Proposition 1.3. □

Proposition 7.2. *Let $w\in E^{u_0*}$ and $\tau_j^{(k)}\in pA$, $\beta_i^{(k)}\in A$ for $k=1,2$, $1\leqslant i,j\leqslant h-1$, be such that $\chi[\Gamma(\tau_1^{(j)},\dots,\tau_{h-1}^{(j)})]=A^*((1+\sum_{i=1}^{h-1}\beta_i^{(j)}\Delta^i)f_0+P)\in S$. Then*

$$\big[\Gamma\big(\tau_1^{(1)},\dots,\tau_{h-1}^{(1)}\big)\big]\mu(w+Eu_0)=\big[\Gamma\big(\tau_1^{(2)},\dots,\tau_{h-1}^{(2)}\big)\big]$$
$$\textit{iff}\quad a\big(1,\beta_1^{(1)},\dots,\beta_{h-1}^{(1)}\big)C(w)=\big(1,\beta_1^{(2)},\dots,\beta_{h-1}^{(2)}\big)$$

for some $a\in A^*$, where $C(w)$ is the matrix of the right multiplication by the element w on E/Eu_0 in the basis $1,\Delta,\dots,\Delta^{h-1}$.

Proof. It is an immediate consequence of Theorem 4.5. □

Let us consider a metric d on the set of $\star$-isomorphism classes of deformations of Φ over A defined by the formula

$$d\big(\big[\Gamma\big(\tau_1^{(1)},\dots,\tau_{h-1}^{(1)}\big)\big],\big[\Gamma\big(\tau_1^{(2)},\dots,\tau_{h-1}^{(2)}\big)\big]\big)=\exp\Big(-\min_{1\leqslant j\leqslant h-1}\nu\big(\tau_j^{(1)}-\tau_j^{(2)}\big)\Big).$$

According to Yu [9, Section 2], the number $\min_{1\leqslant j\leqslant h-1}\nu(\tau_j^{(1)}-\tau_j^{(2)})$ has the following interpretation: it is the maximal i such that the identity homomorphism between the reductions of $\Gamma(\tau_1^{(1)},\dots,\tau_{h-1}^{(1)})$ and $\Gamma(\tau_1^{(2)},\dots,\tau_{h-1}^{(2)})$ can be deformed to an isomorphism over A/p^iA. Further, we define a filtration $\{U_n\}_{n\geqslant 1}$ on the group $E^{u_0*}/(1+Eu_0)$ as follows: let U_n be the image of the subgroup $(A^*+p^nE)\cap E^{u_0*}$ of E^{u_0*} with respect to the factorization by $1+Eu_0$. Remark that the intersection of the groups U_n is the stabilizer of the class $[F_0]$. Evidently, any element of U_n can be uniquely written in the form $w_0+p^n\sum_{i=1}^{h-1}w_i\Delta^i+Eu_0$, where $w_0\in A^*$, $w_i\in A$ for $1\leqslant i\leqslant h-1$.

Proposition 7.3. *Let $p\neq 2$, $w\in E^{u_0*}$, $\tau_j^{(k)}\in pA$ for $k=1,2$ and $1\leqslant j\leqslant h-1$. Let formal groups $F_k=\Gamma(\tau_1^{(k)},\dots,\tau_{h-1}^{(k)})$, $k=1,2$, be such that $[F_2]=[F_1]\mu(w+Eu_0)$. If $w+Eu_0\in U_m$ then $d([F_1],[F_2])\leqslant\exp(-m-1)$.*

Proof. We assume $w=w_0+p^m\sum_{i=1}^{h-1}w_i\Delta^i$ with $w_0\in A^*$, $w_i\in A$ for $1\leqslant i\leqslant h-1$. If $C(w)$ is the matrix of the right multiplication by the element w on E/Eu_0 with respect to the basis $1,\Delta,\dots,\Delta^{h-1}$ then $C(w)\equiv w_0I_h \bmod p^m$ where I_h is the identity $h\times h$ matrix. Therefore by Proposition 7.2 we have $\beta_i^{(1)}\equiv\beta_i^{(2)}\bmod p^m$ for $1\leqslant i\leqslant h-1$. It was shown in the proof of Theorem 6.5 that $\rho_0^{(k)}=p\varepsilon^{(k)}$ and $\rho_{h-i}^{(k)}=p\varepsilon^{(k)}\sum_{j=0}^{h-i-1}\beta_{i+j}^{(k)}\zeta_{h+j}^{\Delta^{i+j}}$ for $1\leqslant i\leqslant h-1$, $k=1,2$. Hence $\rho_0^{(1)}-\rho_0^{(2)}=p(\varepsilon^{(1)}-\varepsilon^{(2)})$ and $\rho_{h-i}^{(1)}\varepsilon^{(2)}-\rho_{h-i}^{(2)}\varepsilon^{(1)}=p\varepsilon^{(1)}\varepsilon^{(2)}\sum_{j=0}^{h-i-1}(\beta_{i+j}^{(1)}-\beta_{i+j}^{(2)})\zeta_{h+j}^{\Delta^{i+j}}\equiv 0\bmod p^{m+2}$ for $1\leqslant i\leqslant h-1$.

Suppose our claim is false, i.e., $\min_{1\leqslant j\leqslant h-1}\nu(\tau_j^{(1)}-\tau_j^{(2)})\leqslant m$. Let $1\leqslant n\leqslant h-1$ be such that $\nu(\tau_n^{(1)}-\tau_n^{(2)})\leqslant\nu(\tau_j^{(1)}-\tau_j^{(2)})$ for any $1\leqslant j\leqslant h-1$ and $\nu(\tau_n^{(1)}-\tau_n^{(2)})<\nu(\tau_j^{(1)}-\tau_j^{(2)})$ for $j>n$. Denote $\lambda=\nu(\tau_n^{(1)}-\tau_n^{(2)})$. As $\nu(\zeta_h)=1$, we have

$$\nu\big(\zeta_h^{\Delta^n}\big(\tau_n^{(1)}-\tau_n^{(2)}\big)\big)=\lambda+1.$$

Further, $\nu(\zeta_{h+j-n}^{\Delta^j}(\tau_j^{(1)}-\tau_j^{(2)}))\geqslant\lambda+2$ for $j>n$ and

$$\nu\left(\sum_{j=1}^{h-1}\zeta_{i+j+h-n}^{\Delta^{i+j}}\bigl(\tau_j^{(1)\,p^i}-\tau_j^{(2)\,p^i}\bigr)a_i\right)\geqslant 1+\lambda+\bigl(p^i-1\bigr)-i\geqslant\lambda+2$$

for $i\geqslant 1$ since $p^i a_i\in A$ and $p\geqslant 3$. It gives

$$\begin{aligned}\nu\bigl(\rho_{h-n}^{(1)}-\rho_{h-n}^{(2)}\bigr)&=\nu\Biggl(\zeta_h^{\Delta^n}\bigl(\tau_n^{(1)}-\tau_n^{(2)}\bigr)+\sum_{j=n+1}^{h-1}\zeta_{h+j-n}^{\Delta^j}\bigl(\tau_j^{(1)}-\tau_j^{(2)}\bigr)\\&\qquad+\sum_{i=1}^{\infty}\sum_{j=1}^{h-1}\zeta_{i+j+h-n}^{\Delta^{i+j}}\bigl(\tau_j^{(1)\,p^i}-\tau_j^{(2)\,p^i}\bigr)a_i\Biggr)\\&=\lambda+1.\end{aligned}$$

Similarly

$$\nu\bigl(\rho_0^{(1)}-\rho_0^{(2)}\bigr)=\nu\left(\sum_{i=1}^{\infty}\sum_{j=1}^{h-1}\zeta_{i+j}^{\Delta^{i+j}}\bigl(\tau_j^{(1)\,p^i}-\tau_j^{(2)\,p^i}\bigr)a_i\right)\geqslant\lambda+2.$$

Since

$$\nu\left(\sum_{j=1}^{h-1}\zeta_{i+j+h-n}^{\Delta^{i+j}}\tau_j^{(2)\,p^i}a_i\right)\geqslant 1+p^i-i\geqslant 2$$

for $i\geqslant 0$, we obtain

$$\nu\bigl(\rho_{h-n}^{(2)}\bigr)=\nu\left(p\zeta_{h-n}+\sum_{i=0}^{\infty}\sum_{j=1}^{h-1}\zeta_{i+j+h-n}^{\Delta^{i+j}}\tau_j^{(2)\,p^i}a_i\right)\geqslant 2.$$

Finally taking into account that $\nu(\varepsilon^{(2)})=0$ we deduce

$$\nu\bigl(\bigl(\rho_{h-n}^{(1)}-\rho_{h-n}^{(2)}\bigr)\varepsilon^{(2)}-p^{-1}\rho_{h-n}^{(2)}\bigl(\rho_0^{(1)}-\rho_0^{(2)}\bigr)\bigr)=\lambda+1.$$

Since $\lambda\leqslant m$, it contradicts the congruence $(\rho_{h-n}^{(1)}-\rho_{h-n}^{(2)})\varepsilon^{(2)}-p^{-1}\rho_{h-n}^{(2)}(\rho_0^{(1)}-\rho_0^{(2)})=\rho_{h-n}^{(1)}\varepsilon^{(2)}-\rho_{h-n}^{(2)}\varepsilon^{(1)}\equiv 0 \bmod p^{m+2}$. Thus we proved that

$$d\bigl([F_1],[F_2]\bigr)=\exp\Bigl(-\min_{1\leqslant j\leqslant h-1}\nu\bigl(\tau_j^{(1)}-\tau_j^{(2)}\bigr)\Bigr)\leqslant\exp(-m-1).\qquad\square$$

For a deformation F and an integer $n\geqslant 0$ denote by $B(F,n)$ the open ball in the set of $\star$-isomorphism classes of deformations of Φ with center $[F]$ of radius $\exp(-n)$, i.e.,

$$B(F,n)=\bigl\{[G]\mid d\bigl([G],[F]\bigr)<\exp(-n)\bigr\}.$$

Corollary 7.4. *Let* $p \neq 2$, $w \in E^{u_0*}$ *and deformations* F_1, F_2 *of* Φ *be such that* $[F_1] \in B(F,n)$, $w + Eu_0 \in U_m$ *and* $[F_2] = [F_1]\mu(w + Eu_0)$. *Then* $[F_2] \in B(F, \min\{n, m+1\})$.

Now we are going to consider the case when Φ is the reduction of the Artin–Hasse formal group over $\mathbf{Z}_p$ of height h. It means that $\Phi = \xi_* F_T$, where the $\mathbf{Q}_p$-homomorphism ξ from Ω to $\mathbf{F}_p$ is defined as follows $\xi(t_i) = \delta_{i,h}$ and δ is the Kronecker delta. Then $r_i = \delta_{i,h}$, $u_0 = p - \Delta^h$, $s_0 = -1$, $\pi_i = -p\delta_{i,h}$ for $0 \leqslant i \leqslant h-1$, and hence, $\zeta_{ih+n} = p^i \delta_{n,0}$ for $0 \leqslant n \leqslant h-1$. Further, we calculate

$$\begin{aligned} &p\zeta_n a_0 + \sum_{i=1}^{mh} \zeta_{i+n}^{\Delta^i}\left(pa_i - \sum_{j=1}^{i} r_j^{\Delta^{i-j}} a_{i-j}\right) \\ &\quad = p\delta_{n,0}a_0 + p(pa_{h-n} - \delta_{n,0}a_0) + \sum_{i=2}^{m} p^i(pa_{ih-n} - a_{ih-n-h}) \\ &\quad = p^2 a_{h-n} + \left(p^{m+1}a_{mh-n} - p^2 a_{h-n}\right) = p^{m+1}a_{mh-n}. \end{aligned}$$

Thus $\rho_n = \lim_{m\to\infty} p^{m+1}a_{mh-n}$ for $0 \leqslant n \leqslant h-1$. The matrix $Z = \{\zeta_{i+j}^{\Delta^i}\}_{i,j=0}^{h-1}$ has 1 at the left upper corner and p below the non-main diagonal, all other entries are 0. Therefore $\beta_i = p^{-1}\rho_{h-i}/\rho_0$ for $1 \leqslant i \leqslant h-1$. Finally, we obtain

$$\beta_i = p^{-1}\frac{\lim_{m\to\infty} p^{m+2}a_{(m+1)h-(h-i)}}{\lim_{m\to\infty} p^{m+1}a_{mh}} = \lim_{m\to\infty}\frac{a_{mh+i}}{a_{mh}}.$$

We notice that if the degree of k over $\mathbf{F}_p$ is equal to h, then $u_0 = p - \Delta^h$ belongs to the center of E, and hence, the group of k-automorphisms of Φ acts on the set of $\star$-isomorphism classes of deformations of Φ over A transitively. Moreover in this case, this group being identified with $E^{u_0*}/(1 + Eu_0) = E^*/(1 + E(p - \Delta^h))$ is isomorphic to the multiplicative group of the maximal order in the central division algebra over $\mathbf{Q}_p$ of rank h^2 and invariant $1/h$.

If Φ is the reduction of the Artin–Hasse formal group, the matrix $C(w)$ from Proposition 7.2 can be easily written in terms of the coefficients of w. Namely, if $w = \sum_{i=0}^{h-1} \omega_i \Delta^i$, then $C(w) = \{c_{ij}\}_{i,j=0}^{h-1}$, where $c_{i,j} = \omega_{j-i}^{\Delta^i}$ if $j \geqslant i$, and $c_{i,j} = p\omega_{h+j-i}^{\Delta^i}$ if $j < i$. Thus in this special case, Proposition 7.2 gives the result of Gross and Hopkins on the equivariance of a p-adic period map (see [4, Proposition 23.5]).

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