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RESEARCH ARTICLE

The connective K-theory of the Eilenberg–MacLane space $K(\mathbb{Z}_p, 2)$

Donald M. Davis[1] and W. Stephen Wilson[2]

[1]Department of Mathematics, Lehigh University, Bethlehem, PA, 18015, USA
[2]Department of Mathematics, Johns Hopkins University, Baltimore, MD, 01220, USA
Corresponding author: Donald M. Davis; Email: dmd1@lehigh.edu

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Abstract
We compute $ku^*\left(K(\mathbb{Z}_p, 2)\right)$ and $ku_*\left(K(\mathbb{Z}_p, 2)\right)$, the connective KU-cohomology and connective KU-homology groups of the mod-p Eilenberg–MacLane space $K(\mathbb{Z}_p, 2)$, using the Adams spectral sequence. We obtain a striking interaction between h_0-extensions and exotic extensions. The mod-p connective KU-cohomology groups, computed elsewhere, are needed in order to establish higher differentials and exotic extensions in the integral groups.

1. Introduction

Algebraic topologists try to turn homotopy theory questions into algebraic ones. We do this by assigning algebraic objects to topological spaces. There are many standard topological spaces that occur all the time and several algebraic theories that are in standard use. Eilenberg–MacLane spaces are important building blocks in homotopy theory, and any new information about them is potentially useful. This paper focuses on the second mod p Eilenberg–MacLane space, $K_2 = K(\mathbb{Z}_p, 2)$. We use $\mathbb{Z}_p$ to denote $\mathbb{Z}/p$, the integers mod p. The algebraic tool we use is complex K-theory. It has long been known that $KU^*(K_2)$ is trivial [2]. Although interesting, this gives limited information. But if we move to the connective version of complex K-theory, $ku^*(-)$, we suddenly obtain an overwhelming amount of new information about K_2.

Because $KU^*(K_2)$ is trivial, we know that the homotopy maps $[K_2, BU]$ and $[K_2, U]$ are trivial. Consider the connective Omega spectrum for BU, $\underline{bu}_k$ with $\underline{bu}_0 = Z \times BU$. We have $ku^n(X) \simeq [X, \underline{bu}_n]$ and $\underline{bu}_n$ is $(n-1)$-connected for $n > 0$.

Let $v \in ku^{-2}$ be the Bott periodicity element. It gives maps $\underline{bu}_{n+2} \longrightarrow \underline{bu}_n$. In this paper, we give a complete computation of $ku^*(K_2)$. Our result shows that there are many nontrivial elements in most $\left[K_2, \underline{bu}_n\right]$, but mapping any such element a finite number of times with v results in the trivial map.

To simplify our discussion, let $K_n = K(\mathbb{Z}_p, n)$ and $K(\mathbb{Z}_p)$ be the stable Eilenberg–MacLane spectrum.

There are a couple of interesting directions in which this research could go. First, $ku^*(K_1)$ is well known and has no v-torsion, so the suspension map $ku^*(K_2) \longrightarrow ku^*(K_1)$ is trivial ($ku^*(K_2)$ is all v-torsion). On the other hand, it is easy to compute the stable result $ku^*\left(K(\mathbb{Z}_p)\right)$. Every element here is killed by multiplication with a single v, so the suspension image must lie in the trivial part of $ku^*(K_2)$, a part to which we pay little attention. However, it is easy to see that only one element is in the image and it is in degree $2p+2$. Our computation of $ku^*(K_2)$ is just the first step in interpolating between $ku^*(K_1)$ and $ku^*\left(K(\mathbb{Z}_p)\right)$. The results and the suspension maps would be most interesting.

With such results, one could go after $ko^*(K_n)$ and $ko_*(K_n)$ using the exact sequences that come from the usual maps:

$$\cdots \longrightarrow \underline{bo}_{n+1} \longrightarrow \underline{bo}_n \longrightarrow \underline{bu}_n \longrightarrow \underline{bo}_{n+2} \longrightarrow \cdots.$$

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In [14] and [6], the authors use very partial results to give new information about non-immersions of spin manifolds. More complete results would allow us to go much further on this problem.

Our computation of $ku^*(K_2)$ is done with the Adams spectral sequence (ASS), but we have a second tool to use as well. We already know the mod p connective complex K-theory of K_2 from [8]. Many (perhaps most) ASS computations result only in an associated graded object because solving the extension problems for the multiplication by p can be very difficult. However, using the long exact sequence for $ku^*(-)$ and its mod p version, we are able to solve all of these extension problems giving an unusually complete answer.

In general, the more algebraic invariants we have for standard spaces in homotopy theory, the better off we are.

In [14] and [6], the authors initiated a partial computation of the connective KU-homology groups, $ku_*(K(\mathbb{Z}_2, 2))$, of the mod-2 Eilenberg–MacLane space $K(\mathbb{Z}_2, 2)$ in separate studies of Stiefel–Whitney classes of manifolds. We eventually turned to the associated cohomology groups, $ku^*(K(\mathbb{Z}_2, 2))$, and were able to give a complete determination, via the ASS. This generalized nicely to the odd primes, and then we found a duality result ([5]) relating these homology and cohomology groups which enabled us to determine the homology groups $ku_*\left(K\left(\mathbb{Z}_p, 2\right)\right)$.

Notation 1.1. We need to establish some notation. Whenever we have ku, we mean it to be localized at the prime p. Adjustments must be made for odd primes because we don't work directly with ku, but with an Adams' summand. It is well known that BU splits at an odd prime. This splitting lifts $\underline{bu}_k$. The original source for BU is [1, Corollary 8, p. 91]. A stable version is proven in [9, Proposition 2.7]. We'll skip Adams' notation. In the literature, the stable cohomology summand is often denoted by ℓ. In a context where $BP\langle n\rangle$ is around for all n, the summand is naturally called $BP\langle 1\rangle$. We want something that reflects the obvious connection to $ku^*(-)$, and so we adopt for our notation $kup^*(-)$ for the stable summand. This gives an Omega spectrum, $\{\underline{bup}_*\}$ with $kup^n(X) \simeq [X, \underline{bup}_n]$. With this notation, Adams' original theorem says

$$BU \simeq \underline{bup}_2 \times \underline{bup}_4 \times \cdots \times \underline{bup}_{2p-2}.$$

There is a corresponding stable splitting:

$$bu \simeq bup \times \Sigma^2 bup \times \Sigma^4 bup \times \cdots \times \Sigma^{2p-4} bup$$

Consequently, if we compute $kup^*(X)$, we also know $ku^*(X)$. Note that for $p = 2$, there is no spliting. At $p = 2$, ku localized is kup. Because we are working with a p-local space, K_2, it is not really necessary to localize ku as well. But for us to work with just the one summand, it is. Again, we repeat ku and kup are always localized at a prime p.

We begin with a description of the kup^*-module $kup^*(K_2)$. Note that $kup^* = \mathbb{Z}_{(p)}[v]$ with $|v| = -2(p-1)$. We find that depiction via ASS charts is the most insightful way to envision the groups. There is a very nice interplay between extensions (multiplication by p) seen in Ext (h_0-extensions) and exotic extensions. We depict the ASS with cohomological (co)degrees increasing from right to left. We write $|x| = d$ if $x \in kup^d(K_2)$ or the associated E_2-term.

In $kup^*(K_2)$, there is a trivial submodule whose Poincaré series when $p = 2$ is described at the end of Section 2. It plays no role and will be ignored from now on. As a kup^*-module, $kup^*(K_2)$ is generated by certain products of elements of E_2^0:

$$y_0,\ y_i = y_0^{p^i},\ \text{with } |y_i| = 2p^i, \tag{1.2}$$

$$z_j \text{ for } j \geq 0 \text{ with } |z_j| = 2\left(p^{j+1} + 1\right), \tag{1.3}$$

and

$$q \text{ with } |q| = 9 \text{ if } p = 2 \text{ and } |q| = 4p - 1 \text{ if } p \text{ is odd}. \tag{1.4}$$

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We give two descriptions of our answer. In Theorem 1.16, we give the E_∞-term of the ASS and then describe the exotic extensions from multiplication by p. Our preferred description is to incorporate them together. That is done in Theorems 1.8 and 1.15.

Let $TP_i[v] := \mathbb{Z}_p[v]/(v^i)$, the truncated polynomial algebra. The even-graded part $kup^{\mathrm{ev}}(K_2)$ is formed from shifted copies of kup^*-modules A_k and B_k, which can be defined inductively as follows.

Definition 1.5. *Let $k_0 = 1$ if p is odd, and $k_0 = 2$ if $p = 2$. Let $B_{k_0-1} = 0$. Let $A_0 = \langle z_0 \rangle$ for all p. Inductively*

$$B_k \textit{ is built from } z_{k-1}^{p-1}B_{k-1},\ TP_{p^k-k}[v]z_k,\ \textit{and } y_{k-1}^{p-1}B_{k-1},\ \text{if } k \geq k_0$$

and

$$A_k \textit{ is built from } z_{k-1}^{p-1}B_{k-1},\ TP_{p^k}[v]z_k,\ \textit{and } y_{k-1}^{p-1}A_{k-1},\ \text{if } k \geq 1$$

with extensions determined by:

$$pz_k = vz_{k-1}^p \text{ for } k \geq 2, \text{ and } py_{k-1}^{p-1}z_{k-1} = v^{p^{k-1}(p-1)}z_k. \tag{1.6}$$

When we write something like zB, we mean that all elements of B are multiplied by the element z. Saying "is built from" means that these are successive quotients in a filtration as a kup^*-module. The extension formulas are only asserted up to multiplication by a unit in $\mathbb{Z}_p$ and can both occur on an element. For example, in Figure 1, we have, in grading 116 when $p = 2$, $2y_3z_3z_4 = vy_3z_2^2z_4 + v^8z_4^2$.

Figure 1 should enable the reader to envision A_k and B_k for $p = 2$ and $k \leq 5$, and, by extrapolating, for all k. Elements connected by dashed lines are in A_5 but not in B_5. The long red[1] lines, sometimes slightly curved, are the exotic extensions. The portion in gradings ≤ 102, not including the top v-tower or the extensions to it, is y_4A_4 (or y_4B_4 if the dashed part is omitted). The portion in gradings ≥ 106, not including the v-tower on z_5 or the h_0-extensions from it, is z_4B_4. The reader is encouraged to understand how the case $k = 5$ of Definition 1.5 is embodied in Figure 1. We have depicted z_4B_4 and y_4B_4 in green.

The portion in the lower right corner of Figure 1 in grading ≤ 84 and height ≤ 7 is $y_3y_4A_3$, and $y_2y_3y_4A_2$ is in gradings ≤ 74. In Figure 2, we present a schematic of A_3 and B_3 at the odd primes. Again the dashed portion is in A_3, but not B_3, and the triangle in the lower right portion is $y_1^{p-1}y_2^{p-1}A_1$.

A generating set as a $\mathbb{Z}_p[v]$-module for B_k is

$$\left\{ z_j \prod_{i=j}^{k-1} \left\{ z_i^{p-1}, y_i^{p-1} \right\} : k_0 \leq j \leq k \right\}, \tag{1.7}$$

while A_k has additional generators:

$$\begin{cases} z_1y_1 \cdots y_{k-1} & p = 2 \\ z_0y_0^{p-1} \cdots y_{k-1}^{p-1} & \text{all } p. \end{cases}$$

The notation here means a product over all choices of one of the two elements in each factor. For example,

$$\prod_{i=1}^{2} \left\{ z_i^{p-1}, y_i^{p-1} \right\} = \left\{ z_1^{p-1}z_2^{p-1},\ z_1^{p-1}y_2^{p-1},\ y_1^{p-1}z_2^{p-1},\ y_1^{p-1}y_2^{p-1} \right\}.$$

An empty product is defined to equal 1.

The following theorem explains how the portion of $kup^*(K_2)$ in even gradings is a direct sum of shifted versions of A_k and B_k.

[1]Colors are present in online versions, but not in the print version.

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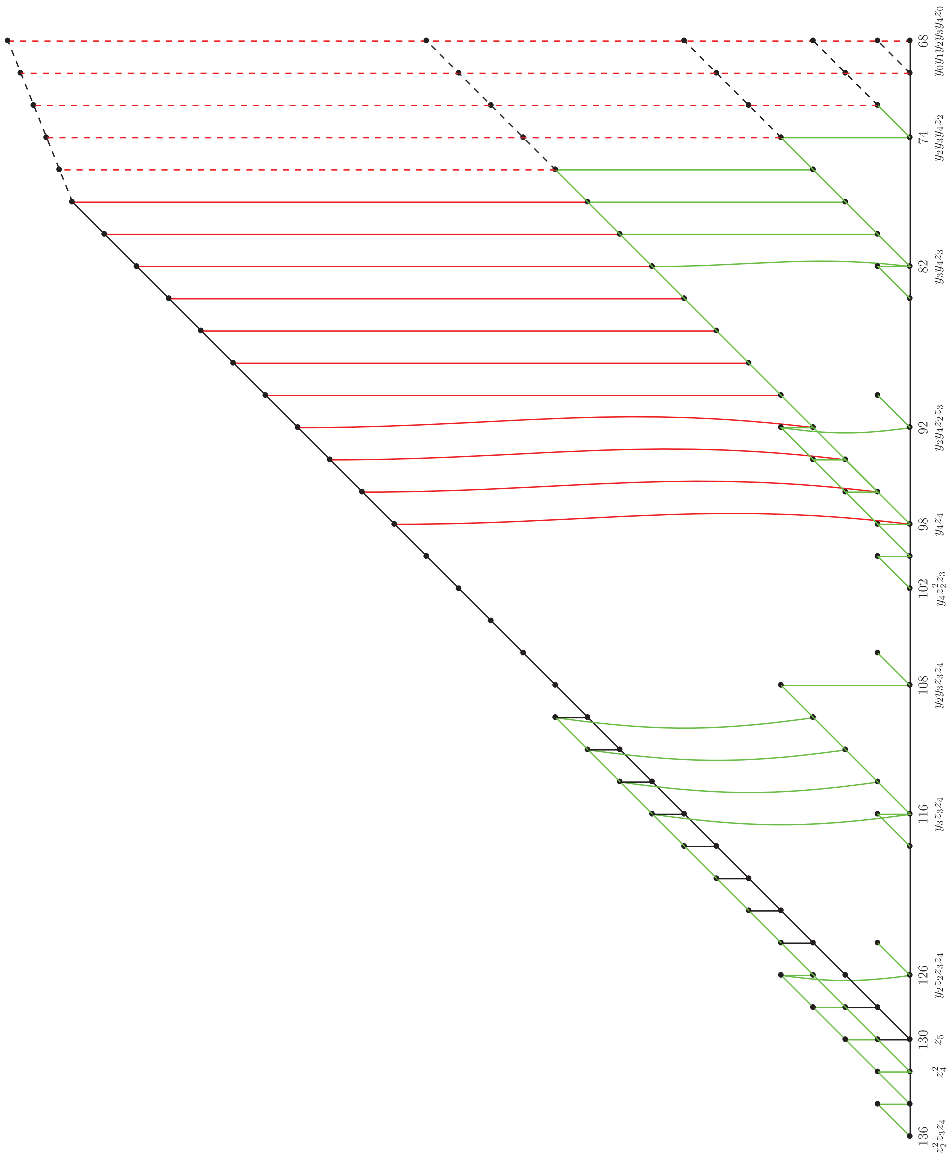

Figure 1. *B_5 and A_5 when $p = 2$.*

Theorem 1.8. *Let $M_p[S]$ denote the set of monomials in the elements of a set S raised to powers $< p$. Let*

$$\mathcal{M}_k = \left(M_p[z_k, y_k] - \left\{z_k^{p-1}, y_k^{p-1}\right\}\right) \cdot M_p\left[z_i, y_i : i > k\right], \tag{1.9}$$

where $M_p[z_k, y_k] - \left\{z_k^{p-1}, y_k^{p-1}\right\} = \left\{z_k^i y_k^j : 0 \le i, j \le p-1 \text{ and } \{i, j\} \ne \{0, p-1\}\right\}$, which is a set with $p^2 - 2$ elements. Let $\mathcal{M}_k^A$ be the set of monomials in $\mathcal{M}_k$ with no z-factors, and $\mathcal{M}_k^B = \mathcal{M}_k - \mathcal{M}_k^A$. Then,

$$kup^{\mathrm{ev}}(K_2) = \bigoplus_{k \ge 1} \left(\bigoplus_{M \in \mathcal{M}_k^A} M \cdot A_k \oplus \bigoplus_{M \in \mathcal{M}_k^B} M \cdot B_k \right)$$

plus a trivial kup^*-module.

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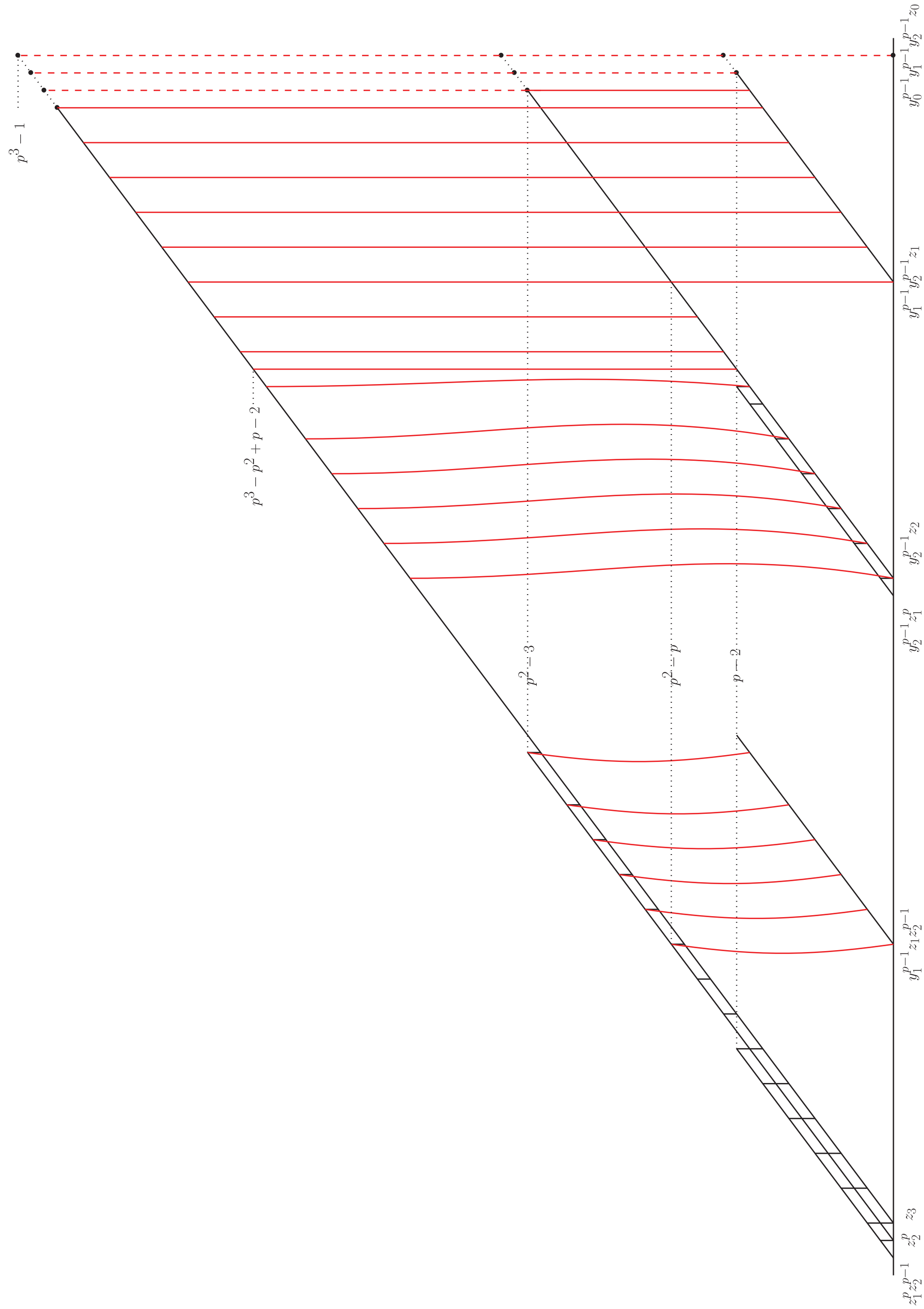

Figure 2. *Schematic of A_3 and B_3 for odd p.*

Note that the monomial 1 is in $\mathcal{M}_k^A$, so A_k appears by itself, but B_k does not. For example, if $p=2$, copies of B_k appear multiplied by each monomial of the form:

$$z_k^{\varepsilon_k}y_k^{\delta_k}z_{k+1}^{\varepsilon_{k+1}}y_{k+1}^{\delta_{k+1}}\cdots \text{ such that } \varepsilon_k=\delta_k \text{ and } \sum \varepsilon_i \geq 1.$$

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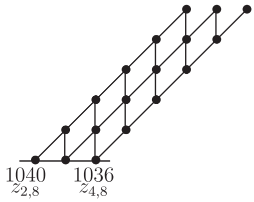

Figure 3. *$S_{5,8}$ if $p=2$.*

Now we describe the portion of $kup^*(K_2)$ in odd gradings. Let $P[S]$ denote the polynomial algebra on a set S, and $TP_i[S] = P[S]/(s^i : s \in S)$, the truncated polynomial algebra. Let $\Lambda_j = TP_p[z_i : i \ge j]$. Note that if $p = 2$, Λ_j is an exterior algebra. For $i \le j$, let

$$z_{i,j} = z_i\big(z_i \cdots z_{j-1}\big)^{p-1}. \tag{1.12}$$

If $j = i$, then $z_{i,j} = z_i$.

Definition 1.13. *For $\ell > k \ge 1$, let $S_{k,\ell} = TP_{k+1}[v]\langle z_{k_0,\ell}, \ldots, z_{\ell-k-1+k_0,\ell}\rangle$ with $pz_{i,\ell} = vz_{i-1,\ell}$ and $pz_{k_0,\ell} = 0$.*

For example, $S_{5,8}$ with $p = 2$ is depicted in Figure 3.

The following result describes the portion of $kup^*(K_2)$ in odd gradings. The exponent of p in an integer i is denoted simply by $\nu(i)$; the prime p is implicit. The element q here has grading 9 or $4p-1$, as mentioned earlier.

Theorem 1.15. *There is an isomorphism of kup^*-modules:*

$$kup^{\text{odd}}(K_2) \approx \bigoplus_{i\ge 1}\ \bigoplus_{\ell \ge \nu(i)+2} qy_1^{i-1} S_{\nu(i)+1,\ell} \otimes TP_{p-1}[z_\ell] \otimes \Lambda_{\ell+1}.$$

The nonvisual, formulaic form of our result is as follows.

Theorem 1.16. *The kup^*-module $kup^*(K_2)$ is isomorphic to a trivial kup^*-module plus a module whose associated graded is*

$$P[y_1]y_0^{p-1}z_0 \oplus \bigoplus_{t\ge 1} TP_{p^t}[v] \otimes P[y_t]z_t \tag{1.17}$$

$$\oplus \bigoplus_{t \ge k_0} TP_{p^t - t}[v] \otimes P[y_t]z_t\overline{\Lambda}_t \tag{1.18}$$

$$\oplus \bigoplus_{i\ge 1}\bigoplus_{\ell \ge 0} TP_{\nu(i)+2}[v]qy_1^{i-1}z_{k_0+\ell,\ell+\nu(i)+2}\Lambda_{\ell+\nu(i)+2}. \tag{1.19}$$

Multiplication by p in (1.17) and (1.18) is determined by (1.6) and in (1.19) as in Definition 1.13.

Our initial interest in this project was $kup_*(K_2)$ ([14,6]), but we first achieved success in computing $kup^*(K_2)$. In [5, Example 3.4], the following result was proved.

Theorem 1.20. *There is an isomorphism of kup_*-modules $kup_*(K_2) \approx \big(kup^{*+2p}K_2\big)^\vee$.*

Here, $M^\vee = \text{Hom}\,(M, \mathbb{Z}/p^\infty)$, the Pontryagin dual, localized at p. A homotopy chart for $kup_*(K_2)$ could be thought of as a shifted version of the homotopy chart of $kup^*(K_2)$ viewed upside-down and backward. For example, the element of $kup^{108}(K_2)^\vee$ dual to the element $v^4y_3z_3z_4$ in Figure 1 corresponds to the generator of a $\mathbb{Z}_4$ in $kup_{104}(K_2)$ on which v^4 acts nontrivially. This element can be seen in Figure 4.

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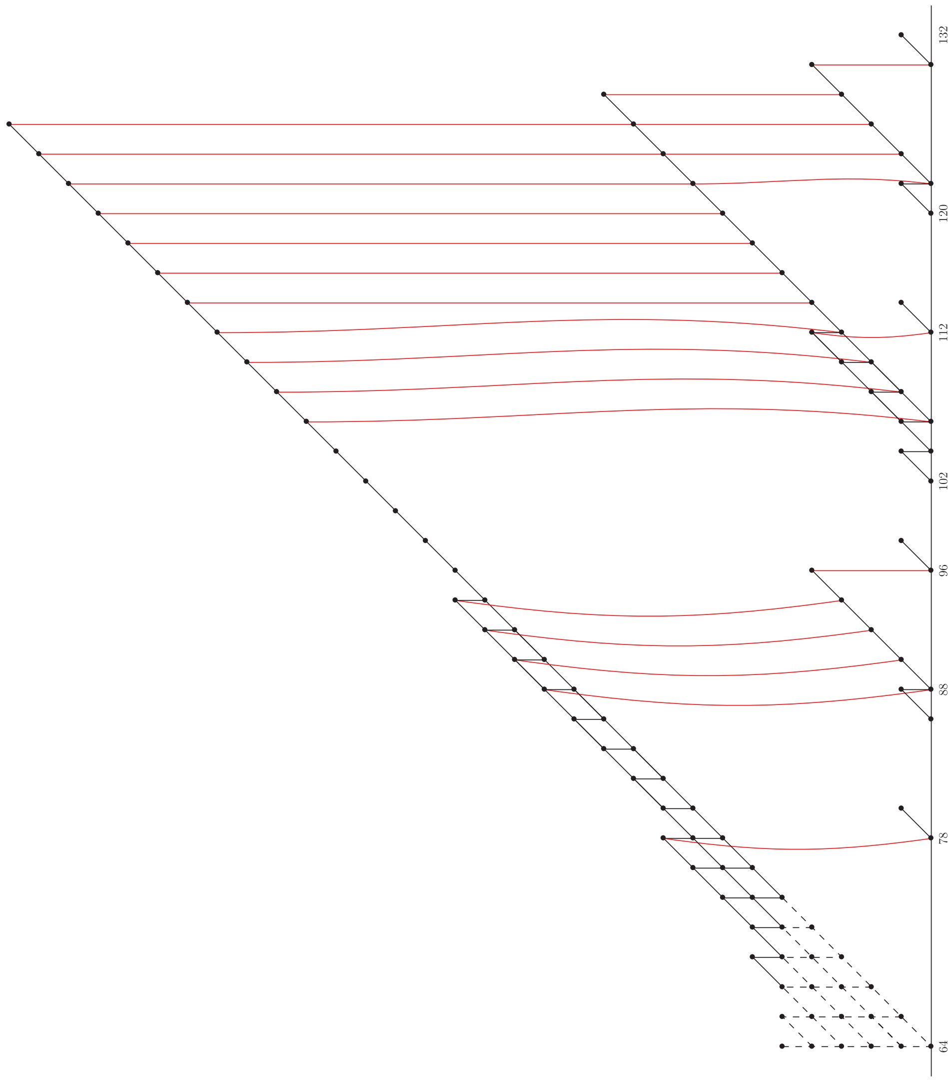

Figure 4. *Portion of $kup_*(K_2)$ corresponding to B_5 and A_5.*

A remarkable property, for which one explanation is given in Section 7, is that B_k is self-dual as a kup^*-module. One way of stating this is to let $\widetilde{B}_k$ denote B_k with its indices negated. Then there is an isomorphism of kup_*-modules:

$$\Sigma^{2\left(p^{k+1}+p^k+(k+1)p-k+1\right)}\widetilde{B}_k \approx B_k^\vee. \tag{1.21}$$

For example, with $p=2$, the second smallest generator Y of $\Sigma^{208}\widetilde{B}_5$ is in grading $208-134=74$ and has $2Y\neq 0$ and $v^4Y\neq 0$ (see Figure 1). The second generator Z of $B_5^\vee$ is dual to the class in position $(74,4)$ in Figure 1 and also satisfies $2Z\neq 0$ and $v^4Z\neq 0$. The isomorphism (1.21) can be proved by induction on k using Definition 1.5.

A complete description of the kup_*-module $kup_*(K_2)$ is immediate from Theorems 1.8, 1.15, and 1.20. However, one might like a complete description of its ASS. We can write formulas for the E_2-term and differentials but will not do so here. In Theorem 1.23, we give a complete description of the E_∞-term of the ASS of $kup_*(K_2)$ with exotic extensions included, in terms of the charts described in Section 1.

In [5], a comparison was made of a chart for A_3 and its kup_* analog. Here, we present in Figure 4 the kup_* analog of Figure 1. This presents the portion of the ASS of $kup_*(K_2)$ dual to A_5 with $p=2$ under the

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isomorphism of Theorem 1.20. The ASS chart dual to B_5 is obtained from this by removing the classes connected by dashed lines and lowering the remaining tower so that the bottom is in filtration 0. The resulting chart is isomorphic to the B_5 part of Figure 1.

We observe that in even gradings of the ASS for $kup_*(K_2)$, h_0-extensions exactly correspond to exotic extensions in the ASS of $kup^{*+2p}(K_2)$, and vice versa. As a typical example of the duality, the summands of $kup^{82}(K_2)$, $kup^{82}(K_2)^\vee$, and $kup_{78}(K_2)$ in Figures 1 and 4 are all isomorphic to $\mathbb{Z}_8 \oplus \mathbb{Z}_2$. But for the kup_*-module structure, it is $kup^{82}(K_2)^\vee$ and $kup_{78}(K_2)$ that correspond to, since in both, the element that is divisible by 4, in position $(82, 0)$ and $(78, 7)$, resp., is also divisible by v^7 for A_5 and by v^4 for B_5.

Theorem 1.23. *The E_∞-term of the ASS of $kup_*(K_2)$ with exotic extensions included contains exactly the following.*

- *There is a trivial kup_*-module, which when $p = 2$ has generators corresponding to those enumerated at the end of Section 2 with gradings decreased by 4, and similarly when p is odd.*
- *For every $S_{k,\ell}$ occurring in a summand of Theorem 1.15, there is a chart of the same form as Figure 3 with v-towers of height $k+1$ on generators in gradings $2p^{\ell+1} + 2(p-1)(i-k_0-1)$ for $1 \le i \le \ell - k$. One must add to this the grading of the other factors accompanying $S_{k,\ell}$ in Theorem 1.15.*
- *For each occurrence of B_k in Theorem 1.8, there is a summand*
$$\Sigma^{2(p^{k+1}+p^k+kp-k+1)}\widetilde{B}_k$$
with gradings increased by those of other factors accompanying B_k in 1.1. Here, $\widetilde{B}_k$ is as defined prior to (1.21).
- *For each summand $y_k^e A_k$ in Theorem 1.8, there is a variant of $\Sigma^{2(p^{k+1}+p^k+kp-k+1)}\widetilde{B}_k$ with gradings increased by $2ep^k$. In this variant, the initial v-towers are pushed up by k filtrations and surrounded with a triangle of classes of the sort appearing in the lower left corner of Figure 4. See Remark 1.22.*

Proof. Theorem 1.20 and our results for $kup^*(K_2)$ give the kup_*-module structure of $kup_*(K_2)$, but that is not the same as the ASS picture. Expanding on work done in [6] and [14] and using methods such as those in Section 2, we were able to write the E_2-term of the ASS for $kup_*(K_2)$ and had conjectured the differentials (but not the extensions) prior to embarking on our kup-cohomology project. We were unable to *prove* the differentials, probably because we had not taken sufficient advantage of the exact sequence with $k(1)_*(K_2)$. Now that we know the 2-orders and v-heights of generators (by grading, at least, if not by name), it is straightforward to see that the differentials must be as we expected. The isomorphism (1.21) plays an important role here; the left-hand side gives the ASS form of the right-hand side. □

Remark 1.24. *Regarding the unusual portion of the ASS chart for part of $kup_*(K_2)$ in the lower left of Figure 4, this is obtained from [6, Figure 4.2] with d_6-differentials on all odd-graded towers. For A_k, it will be a triangle going up to filtration k, with all but the first two dots on the top row being part of B_k.*

The structure of the rest of the paper is as follows. In Section 2, we compute the E_2-term of the ASS for $kup^*(K_2)$. In Section 3, we determine the differentials in this ASS. In order to do so, we need to compare with $k(1)^*(K_2)$, where $k(1)$ is a summand of the spectrum for mod-p connective KU-theory, using the exact sequence:

$$\to k(1)^{*-1}(K_2) \to kup^*(K_2) \xrightarrow{p} kup^*(K_2) \to k(1)^*(K_2) \to kup^{*+1}(K_2) \xrightarrow{p} . \qquad (1.25)$$

In Section 3, we restate results about $k(1)^*(K_2)$ from [8]. At the end of Section 3, we show how the descriptions of $kup^*(K_2)$ in Theorems 1.8 and 1.15 are obtained once we know the differentials and extensions. This exact sequence is also used in determining the exotic extensions of (1.6), which is done

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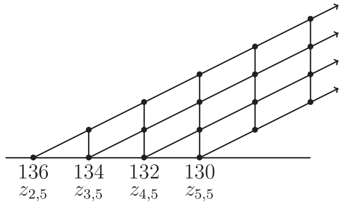

Figure 5. *A depiction of* $P[v] \otimes W_5$.

in Section 4. In Section 5, we propose complete formulas for the exact sequence (1.25), and then in Section 6, we show that our proposed formulas account for all elements of $k(1)^*(K_2)$ exactly once.

The main point of Section 6 is to prove that there are no additional exotic extensions in $kup^*(K_2)$. An exotic extension $p \cdot A = B$ implies that A is not in the image from $k(1)^{*-1}(K_2)$ and B does not map nontrivially to $k(1)^*(K_2)$, so once we have shown that all elements are accounted for, there can be no more extensions. Many of our formulas in Section 5 are forced by naturality. However, many others occur in regular families, but with surprising filtration jumps. We could probably prove that the homomorphisms *must* be as we claim, by showing that there are no other possibilities, but we prefer to forgo doing that. In the optional Section 7, we discuss in more detail how the charts are obtained and provide an explanation for the duality result (1.21).

2. The E_2-term of the ASS for $kup^*(K_2)$

We will need some notation. By H^*K_2, we understand $H^*\left(K(\mathbb{Z}_p, 2)\,; \mathbb{Z}_p\right)$. Let E denote an exterior algebra, P a polynomial algebra, and $TP_n[x] = P[x]/(x^n)$ the truncated polynomial algebra. In all cases these will be over $\mathbb{Z}_p$, the integers mod p. Let $\overline{E}$ denote the augmentation ideal of an exterior algebra, and $E_1 = E[Q_0, Q_1]$, where Q_i are the Milnor primitives. Because $Q_i^2 = 0$ we have homology groups, $H_*(-\,; Q_i)$, defined for E_1-modules. We let $\langle y_1, y_2, \ldots \rangle$ denote the $\mathbb{Z}_p$-span of classes y_i.

The ASS for $kup^*(K_2)$ has $E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(bup), H^*K_2)$, where $\mathcal{A}$ is the mod p Steenrod algebra and $H^*(bup) \approx \mathcal{A}/\mathcal{A}(Q_0, Q_1)$. Using a standard change of rings theorem [10], this is $\mathrm{Ext}_{E_1}^{s,t}(\mathbb{Z}_p, H^*K_2)$. This converges to $kup^{-(t-s)}(K_2)$. We depict this with $E_2^{s,t}$ in position $(t-s, s)$ as usual but label the axis with codegrees, the negative of the homotopical degree, so the left side of the chart will have positive gradings and refer to cohomological grading. In an attempt to avoid confusion, we rewrite this as $G_2^{-(t-s),s}$. With this notation, the differentials are $d_r : G_r^{a,b} \longrightarrow G_r^{a+1,b+r}$, multiplication by the element $v \in kup^{-2(p-1)}$ (also considered in $G_r^{-2(p-1),1}$) is $v : G_r^{a,b} \longrightarrow G_r^{a-2(p-1),b+1}$, and multiplication by the element representing $p \in kup^0$, $\left(h_0 \in G_r^{0,1}\right)$, is $h_0 : G_r^{a,b} \longrightarrow G_r^{a,b+1}$.

In the paragraph preceding Remark 2.17, we will define elements $z_j \in G_2^{2(p^{j+1}+1),0}$ for $j \geq 0$ and elements:

$$z_{i,j} \in G_2^{2(p^{j+1}+1+(p-1)(j-i)),0}$$

as in (1.12) satisfying the properties in Definition 1.13.

Definition 2.1. *For $j \geq k_0$, we define $W_j = \langle z_{j,j}, z_{j-1,j}, \ldots, z_{k_0,j} \rangle$.*

We also have $y_i \in G_2^{2p^i,0}$ for $i \geq 0$, and

$$q \in G_2^{9,0} \text{ if } p = 2, \text{ and in } G_2^{4p-1,0} \text{ if } p \text{ is odd.} \tag{2.2}$$

Cf. (1.3), (1.2), and (2.2). One last definition, let $\Lambda_{j+1} = TP_p[z_i : i \geq j+1]$.

A picture of $P[v] \otimes W_5$ as a $P[v, h_0]$-module with $p = 2$ appears in Figure 5.

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The remainder of this section is devoted to the proof of the following result.

Theorem 2.4. *The E_2 term of the Adams spectral sequence for the $kup^*(K_2)$ is isomorphic as a $P[h_0, v]$-module to*

$$P[v, y_1] \otimes E[q] \otimes \left(\bigoplus_{j \geq k_0} (W_j \otimes TP_{p-1}[z_j] \otimes \Lambda_{j+1}) \right)$$

$$\oplus \left(P[h_0, v, y_1] \otimes E\left[v^{k_0} q\right]\right) \oplus \left(P[y_1] \otimes \begin{cases} \langle y_0^{p-1} z_0 \rangle & p \text{ odd} \\ \langle y_0 z_0, z_1, h_0 y_0 z_0 = v z_1 \rangle & p = 2. \end{cases} \right)$$

plus a trivial $P[h_0, v]$-module.

Some of the algebra structure of this E_2 will be useful later. For example, the product structure among the z_j's will be clear, and also the formula

$$(v^2 q)^2 = v^4 z_2, \tag{2.5}$$

holds when $p = 2$ since, as we shall see, in $H^*(K_2)$, $x_9^2 - Q_0 x_{17} \in \mathrm{im}\,(Q_1)$.

We will give a detailed proof when $p = 2$ and then sketch the minor changes for odd p. There are two parts to proving this theorem. First, we must give a complete description of the E_1-module structure of $H^* K_2$. Second, we have to compute $\mathrm{Ext}_{E_1}^{*,*}(\mathbb{Z}_2, -)$ of this. We begin the first part.

Serre ([11]) showed that $H^* K_2$ is a polynomial algebra on classes u_{2^j+1} in degree $2^j + 1$ for $j \geq 0$ defined by $u_2 = \iota_2$ and $u_{2^{j+1}+1} = \mathrm{Sq}^{2^j} u_{2^j+1}$ for $j \geq 0$. We easily have

$$Q_0(u_2) = u_3, \;\; Q_0(u_3) = 0, \;\; Q_0(u_{2^j+1}) = u_{2^{j-1}+1}^2 \text{ for } j \geq 2,$$

and

$$Q_1(u_2) = u_5, \;\; Q_1(u_3) = u_3^2, \;\; Q_1(u_5) = 0, \;\; Q_1(u_{2^j+1}) = u_{2^{j-2}+1}^4 \text{ for } j \geq 3.$$

Let $x_5 = u_5 + u_2 u_3$ and write $H^* K_2$ as an associated graded object:

$$P\left[u_2^2\right] \otimes E[x_5] \otimes \left(E[u_2] \otimes P[u_3]\right) \otimes_{j \geq 2} \left(E[u_{2^{j+1}+1}] \otimes P\left[\left(u_{2^j+1}\right)^2\right]\right)$$

From this, we can read off

Lemma 2.6.

$$H_*\,(H^* K_2; Q_0) = P\left[u_2^2\right] \otimes E[x_5]$$

Letting $x_9 = u_9 + u_3^3$ and $x_{17} = u_{17} + u_2 u_5^3$, we rewrite again as:

$$P\left[u_2^2\right] \otimes TP_4[x_9] \otimes TP_4[x_{17}] \otimes_{j>4} E\left[(u_{2^j+1})^2\right]$$

$$\otimes \left(E[u_2] \otimes P[u_5]\right) \otimes \left(E[u_3] \otimes P\left[u_3^2\right]\right) \otimes_{j>4} \left(E[u_{2^j+1}] \otimes P\left[(u_{2^{j-2}+1})^4\right]\right).$$

Again we read off

Lemma 2.7.

$$H_*(H^* K_2; Q_1) = P\left[u_2^2\right] \otimes TP_4[x_9] \otimes TP_4[x_{17}] \otimes_{j>4} E\left[(u_{2^j+1})^2\right]$$

An associated graded version of this is

Lemma 2.8.

$$H_*\,(H^* K_2; Q_1) = P\left[u_2^2\right] \otimes E[x_9] \otimes E[x_{17}] \otimes_{j>2} E\left[(u_{2^j+1})^2\right]$$

The bulk of the work here is finding a nice splitting of $H^* K_2$ as an E_1-module.

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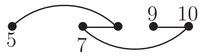

Figure 6. *An E_1-module N.*

Figure 7. *The E_1-module L_3.*

Let N be the E_1-submodule with single nonzero elements in gradings 5, 7, 8, 9, and 10 with generators $x_5 = u_5 + u_2u_3$, $x_7 = u_2u_5$, and $x_9 = u_9 + u_3^3$, satisfying $Q_0x_7 = Q_1x_5$ and $Q_0x_9 = Q_1x_7 = x_{10}$. It has a Q_0-homology class x_5 and a Q_1-homology class x_9. This class x_9 is called q in Theorem 2.4 and in all other sections. A picture of N is in Figure 6. In pictures such as this, straight lines indicate $Q_0 = \mathrm{Sq}^1$ and curved lines Q_1.

The E_1-submodule $P[u_2^2] \oplus P[u_2^2] \otimes N$ carries the Q_0-homology of H^*K_2, while the remaining Q_1-homology is, written in our usual way as an associated graded version,

$$P[u_2^2] \otimes E[x_9] \otimes \overline{E}[x_{17}, u^2_{2^j+1}, \ j > 2] . \tag{2.10}$$

We will exhibit a Q_0-free E_1-submodule R whose Q_1-homology is exactly the above $\overline{E}$. Moreover, $N \otimes R$ contains an E_1-split summand S which maps isomorphically to $\langle x_9 \rangle \otimes R$.

It is premature to state this because we haven't defined R and S yet, but for the record:

Proposition 2.11. *As an E_1 module, $\widetilde{H}^*K_2$ is isomorphic to $T \oplus F$ where F is free over E_1 and T is*

$$P[u_2^2] \otimes \left(\langle u_2^2 \rangle \oplus N \oplus R \oplus S\right)$$

A start on R and S. For this to make sense, we need to find R and S. The module R is a direct sum of shifted versions of modules L_k, $k \ge 0$, which have generators g_{2i}, $0 \le i \le k$, with $Q_1g_{2i} = Q_0g_{2i+2}$ for $0 \le i < k$, $Q_0g_0 \ne 0$, and $Q_1g_{2k} = 0$. For example, L_3 is depicted in Figure 7.

A splitting map, $\langle x_9 \rangle \otimes L_k \longrightarrow N \otimes L_k$, for the epimorphism $N \otimes L_k \to \langle x_9 \rangle \otimes L_k$ is defined by:

$$x_9g_{2i} \mapsto x_9 \otimes g_{2i} + x_7 \otimes g_{2i+2} + x_5 \otimes g_{2i+4} \text{ for } 0 \le i \le k-2,$$

$x_9g_{2k-2} \mapsto x_9 \otimes g_{2k-2} + x_7 \otimes g_{2k}$, and $x_9 \otimes g_{2k} \mapsto x_9 \otimes g_{2k}$.

The E_1-module M_j. Let

$$x_{2^j+1} = u_{2^j+1} + \begin{cases} u_2u_5^3 & j=4 \\ u_2u_3u_5^2u_9^2 & j=5 \\ u_3u_5^2u_9^2u_{17}^2 & j=6 \\ 0 & j>6 \end{cases} \quad \text{and } w_{2^j-1} = \begin{cases} u_2u_3u_5^2 & j=4 \\ u_3u_5^2u_9^2 & j=5 \\ 0 & j>5. \end{cases}$$

Then $Q_0x_{2^j+1} = u^2_{2^{j-1}+1} + Q_1w_{2^j-1}$, so $Q_0x_{2^j+1}$ and $u^2_{2^{j-1}+1}$ represent the same Q_1-homology class. Define E_1-modules M_j inductively by $M_3 = 0$, and for $j \ge 4$ there is a short exact sequence of E_1-modules:

$$0 \to u^2_{2^{j-2}+1}M_{j-1} \to M_j \to M'_j \to 0, \tag{2.13}$$

where $M'_j = \langle x_{2^j+1}, Q_0x_{2^j+1} \rangle$ and $Q_1x_{2^j+1} = u^2_{2^{j-2}+1}Q_0x_{2^{j-1}+1}$. The above definitions of the x_{2^j+1} are necessary to get this formula to work right.

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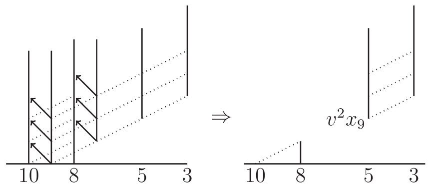

Figure 8. *The first computation of* $\mathrm{Ext}_{E_1}(\mathbb{Z}_2, N)$.

There is an isomorphism of E_1-modules $M_j \approx \Sigma^{2^j+1} L_{j-4}$ given by:

$$\Sigma^{2^j+1} g_{2i} \mapsto \begin{cases} x_{2^j+1} & i=0 \\ u^2_{2^{j-2}+1} x_{2^{j-1}+1} & i=1 \\ u^2_{2^{j-2}+1} u^2_{2^{j-3}+1} x_{2^{j-2}+1} & i=2 \\ u^2_{2^{j-2}+1} u^2_{2^{j-3}+1} \cdots u^2_{2^{j-i-1}+1} x_{2^{j-i}+1} & 2 < i \le j-4 \end{cases} \tag{2.14}$$

And we have

$$H_*(M_j; Q_1) = \begin{cases} \langle u_9^2, u_{17} \rangle & j=4 \\ \langle u_{17}^2, u_9^2 u_{17} \rangle & j=5 \\ \langle u_{33}^2, u_{17}^2 u_9^2 u_{17} \rangle & j=6 \\ \langle u^2_{2^{j-1}+1}, u^2_{2^{j-2}+1} \cdots u_9^2 x_{17} \rangle & j>6 \end{cases} \tag{2.15}$$

The E_1-module R. Let

$$R = \bigoplus_{j \ge 4} M_j \otimes E\left[u^2_{2^j+1}, u^2_{2^{j+1}+1}, \ldots\right]. \tag{2.16}$$

Then $H_*(R; Q_1) = \overline{E}\left[x_{17}, u_9^2, u_{17}^2, \ldots\right]$, since monomials in $\overline{E}$ without x_{17} appear from a first term (of the two in (2.15)) in $H_*(M_j \otimes E; Q_1)$, where j is minimal such that $u^2_{2^{j-1}+1}$ appears in the monomial, while those with x_{17}, and also containing a product $u_9^2 \cdots u^2_{2^{j-2}+1}$ of maximal length, occur as a second term in $H_*(M_j \otimes E; Q_1)$.

Proof of Proposition 2.11. We have the E_1-submodule T given in Proposition 2.11. Because this contains all of the Q_0 and Q_1 homology, what remains must be free over E_1 by [13]. □

Proof of Theorem 2.4. We compute $\mathrm{Ext}_{E_1}(\mathbb{Z}_2, T)$ with T as in Proposition 2.11. We will not be concerned with the free E_1-module F, but later we will give the Poincaré series for it. Each copy of E_1 in F gives a $\mathbb{Z}_2$ in $G^{*,0}$ that corresponds to Q_0Q_1.

That

$$\mathrm{Ext}^{*,*}_{E_1}\left(\mathbb{Z}_2, P\left[u_2^2\right]\right) = P[v, h_0, y_1]$$

with $y_1 \in G_2^{4,0}$ should be clear, given our labeling conventions. We normally work with the reduced cohomologies, so the y_1^0 generator above would be ignored. The y_1 notation is particularly useful when we consider all primes p. It is $y_0^{p^1}$ where $y_0 \in G_2^{2,0}$. So, $|y_1| = 2p$.

We compute $\mathrm{Ext}_{E_1}(\mathbb{Z}_2, N)$ in two ways using two different filtrations of N. From this, we see that the generator of the towers can be thought of either as v^2x_9 or $h_0^2x_5$.

Using Figure 6 as our guide, our first filtration is $\langle x_5, x_8\rangle$, $\langle x_7, x_{10}\rangle$, and $\langle x_9\rangle$. The Ext on $x_9 \in G^{9,0}$ is just $P[v, h_0]$. For the other two, we get h_0-towers on $x_{10} \in G^{10,0}$ and $x_8 \in G^{8,0}$. The extensions in N show these two h_0-towers are connected by multiplication by v. In addition, a d_1 is forced on us by the extensions. Figure 8 describes this completely.

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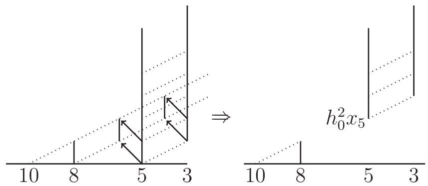

Figure 9. *The second computation of* $\mathrm{Ext}_{E_1}(\mathbb{Z}_2, N)$.

Again referring to Figure 6, our second filtration is $\langle x_9, x_{10}\rangle$, $\langle x_7, x_8\rangle$, and $\langle x_5\rangle$. Now our Ext groups are $P[v, h_0]$ on $x_5 \in G^{5,0}$ and $P[v]$ on $x_8 \in G^{8,0}$ and $x_{10} \in G^{10,0}$. Again, the d_1 is forced by the extensions in N. Figure 9 describes the result.

This concludes the computation of Ext for $P\left[u_2^2\right] \otimes (\langle u_2^2\rangle \oplus N)$ of Proposition 2.11. The result is the second line of Theorem 2.4.

We need to compute Ext for $P\left[u_2^2\right] \otimes (R \oplus S)$ and show it is the same as the top line in Theorem 2.4. Since $S \approx \langle x_9\rangle \otimes R$, all we need to do is $P\left[u_2^2\right] \otimes R$ and ignore the $E[x_9]$. Similarly, we can ignore the $P\left[u_2^2\right]$ and the $P[y_1]$ because for every power of u_2^2 we will have a copy of the answer indexed by powers of y_1. All we have left now is R, but R is just many copies of the various M_j and the indexing for the number of copies is given by the Λ_{j+1}.

All that remains is to show that $\mathrm{Ext}_{E_1}(\mathbb{Z}_2, M_j) \approx P[v] \otimes W_{j-2}$ with W_{j-2} as in Definition 2.1.[2] Recall that $M_j = \Sigma^{2^j+1} L_{j-4}$. We can filter L_{j-4} into pairs of elements $g_{2i}, Q_0 g_{2i}$, for $0 \le i \le j-4$. Then $\mathrm{Ext}_{E_1}(\mathbb{Z}_2, M_j)$ has a $P[v]$ on each element $\Sigma^{2^j+1} Q_0 g_{2i}$ which we denote by $z_{j-i-2,j-2} \in G^{2^j+2+2i,0}$. The element $z_{j-2,j-2}$ is often called z_{j-2}. There is no d_1, but undoing the filtration does solve the extension problem and gives us $h_0 z_{k,j-2} = v z_{k-1,j-2}$. This completes our computation and thus our proof. □

Remark 2.19. *To illustrate the last computation in the proof, consider the generators of the v-towers for* $\mathrm{Ext}_{E_1}(\mathbb{Z}_2, M_7)$. *They are* z_5, z_4^2, $z_3^2 z_4$, *and* $z_2^2 z_3 z_4$, *which is what we have called* $z_{5,5}$, $z_{4,5}$, $z_{3,5}$, *and* $z_{2,5}$, *as pictured in Figure 5. For future reference, we note that (with* $\sim$ *meaning homologous)*

$$z_j = Q_0 x_{2^{j+2}+1} \sim u^2_{2^{j+1}+1} = Q_0 u_{2^{j+2}+1} = Q_0 Q_{j+2} \iota_2 = Q_{j+2} Q_0 \iota_2. \tag{2.20}$$

We now describe briefly the changes required when p is odd. We have

$$H^*(K_2) = P[y_0] \otimes P[g_1, g_2, \ldots] \otimes E[u_0, u_1, \ldots],$$

with $|y_0| = 2$, $|g_j| = 2(p^j+1)$, $|u_i| = 2p^i+1$, $Q_0 y_0 = u_0$, $Q_0 u_i = g_i$, $Q_1 y_0 = u_1$, $Q_1 u_0 = g_1$, and $Q_1 u_i = g_{i-1}^p$, $i \ge 2$. Let $y_1 = y_0^p$. Then, similarly to the case $p = 2$,

$$H_*(H^* K_2, Q_0) = P[y_1] \otimes E\left[y_0^{p-1} u_0\right].$$

Let $N = \left\langle y_0^{p-1} u_0, q = y_0^{p-1} u_1, Q_0 q = Q_1\left(y_0^{p-1} u_0\right)\right\rangle$. Then, $P[y_1] \oplus P[y_1] \otimes N$ carries the Q_0-homology and part of the Q_1-homology. Similarly to (2.10), the rest of the Q_1-homology is

$$P[y_1] \otimes E[q] \otimes \overline{E[w_1] \otimes TP_p[g_2, g_3, \ldots]},$$

where $w_1 = u_2 + u_0 g_1^{p-1}$. There are E_1-submodules M_j for $j \ge 2$, defined inductively by $M_2 = \langle w_1, g_2 = Q_0 w_1\rangle$, $M_j' = \langle u_j, g_j = Q_0 u_j\rangle$ for $j \ge 3$, and for $j \ge 3$, there exists a short exact sequence of E_1-modules:

$$0 \to g_{j-1}^{p-1} M_{j-1} \to M_j \to M_j' \to 0,$$

with $Q_1 u_j = g_{j-1}^p$. There is an isomorphism of E_1-modules $M_j \approx \Sigma^{2p^j+1} L_{j-2}$, where L_j is similar to Figure 7, but with ith generator ($i \ge 0$) in grading $2(p-1)i$ rather than $2i$.

[2] The reason for this awkward shift is that the gradings for z_j which give the elegant statements in Definition 1.5 and elsewhere are not particularly convenient in developing the E_2 statement.

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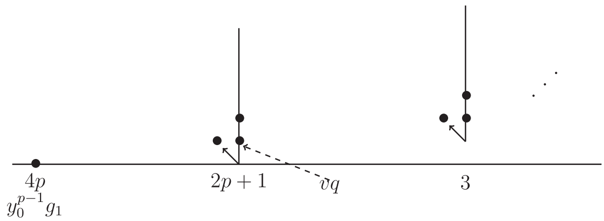

Figure 10. *Computation of* $\mathrm{Ext}_{E_1}(\mathbb{Z}_p, N)$.

Let

$$R = \bigoplus_{j\ge 2} M_j \otimes TP_{p-1}[g_j] \otimes TP_p[g_{j+1}, \ldots].$$

Then $H_*(R; Q_1) = \overline{E[w_1] \otimes TP_p[g_2, g_3, \ldots]}$, and so, similarly to Proposition 2.11, up to free E_1-modules:

$$H^*K_2 \approx P[y_1] \otimes (\langle y_1\rangle \oplus N \oplus R \oplus qR). \tag{2.21}$$

Similarly to Figure 9, $\mathrm{Ext}_{E_1}(\mathbb{Z}_p, N)$ can be read off from Figure 10. This gives the third summand and vq part of the second summand in Theorem 2.4, while the $\langle y_1\rangle$ part of (2.21) gives the non-vq part of the second summand. For the first summand in Theorem 2.4, we replace g_j by z_{j-1} and then note that $\mathrm{Ext}_{E_1}(\mathbb{Z}_p, M_j) \approx P[v] \otimes W_{j-1}$, similar to Figure 5. For example, M_3 has v-towers on g_3 and g_2^p, which are renamed $z_2 = z_{2,2}$ and $z_1^p = z_{1,2}$, the generators of the v-towers of W_2. This completes our sketch of proof of Theorem 2.4 when p is odd.

We explain here the reason for the k_0 in Definition 1.5. In Theorem 2.4, $y_0^{p-1}z_0$ and z_1 are in the part that is not multiplied by higher z's when $p = 2$, but when p is odd, they form the module M_2, whose Ext is $P[v] \otimes W_1$, which is multiplied by higher z's. Since B_k's are multiplied by higher z's, but A_k's are not, this explains why z_1 is in B_1 when p is odd, but not when $p = 2$. The reason for the split in Theorem 2.4 is the difference in the submodules N. Its second class is $y_0^{p-1}Q_1y_0$ in each. Applying Q_1 yields $y_0^{p-2}(Q_1y_0)^2$. This is 0 when p is odd, but not when $p = 2$. The reason that the portion of Ext corresponding to N is not multiplied by higher z's is that it gives part of the Q_0-homology, and this is not multiplied by higher z's.

We close this section with enumeration of the unimportant $\mathbb{Z}_2$-classes in $kup^*(K_2)$ when $p = 2$.

More on the E_1-free part when $p = 2$. If we compute the $\mathrm{Ext}_{E_1}(\mathbb{Z}_2, F)$ for the E_1 free part of H^*K_2, we just get a $\mathbb{Z}_2$ corresponding to the top element for each copy of E_1. If we find the Poincaré series (PS) for the free part, all we have to do to get the PS for these elements is to multiply by $\frac{x^4}{(1+x)(1+x^3)}$. The Poincaré series for free part is obtained by subtracting the PS for the non-free part of Proposition 2.11 from that of H^*K_2. This is

$$\prod_{k\ge 0}\frac{1}{(1-x^{2^k+1})} - \frac{1}{(1-x^4)}\left(1+x^5+x^7+x^8+x^9+x^{10}\right)$$
$$- \frac{1}{(1-x^2)(1-x^4)}\left(\bigoplus_{j\ge 4}\left(x^{2^j+1}(1+x^9)(1+x)\left(1-x^{2^j-6}\right)\prod_{k\ge j}\left(1+x^{2^{k+1}+2}\right)\right)\right)$$

The first term is the PS for H^*K_2. The second is the PS for $P\left[u_2^2\right] \otimes (\langle 1\rangle \oplus N)$. The last term is more complicated but does the S and R terms. The $(1-x^4)$ in the denominator is for the $P\left[u_2^2\right]$. The x^9 is the shift that takes R to S. The $(1+x)$ is because they are Q_0 free. The $x^{2^j+1}\left(1-x^{2^j-6}\right)/(1-x^2)$ is for the odd part of M_j and the remainder is for Λ.

This is easy to put into a computer and calculate. For example, the number of free generators in degree 79 is 245.

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3. Differentials in the ASS of $kup^*(K_2)$

The main theorem of this section determines the differentials in the ASS for $kup^*(K_2)$.

Theorem 3.1. *The differentials in the spectral sequence whose E_2-term was given in Theorem 2.1 are as follows. All v-towers are involved, either as source or target, in exactly one of these. Here, M refers to any monomial (possibly $=1$) in the specified algebra. Recall that $\Lambda_j = TP_p[z_i : i \geq j]$, which is an exterior algebra if $p = 2$. Also, recall $y_t = y_1^{p^{t-1}}$. We give reference numbers to the differentials when p is odd, but references to these also apply to the corresponding differential when $p = 2$, as the proofs are extremely similar.*

First with $p = 2$.

$$\begin{aligned} d_{\nu(i)+2}\left(y_1^i\right) &= h_0^{\nu(i)} v^2 q y_1^{i-1},\ i \geq 1; \\ d_{\nu(i)+2}\left(y_1^i z_j M\right) &= v^{\nu(i)+2} q y_1^{i-1} z_{j-\nu(i),j} M, \\ &\qquad j \geq \nu(i)+2,\ M \in \Lambda_j; \\ d_{2^t-t}\left(h_0^{t-2} v^2 q y_1^{2^{t-1}-1} M\right) &= v^{2^t} z_t M, \\ &\qquad t \geq 2,\ M \in P[y_t]; \\ d_{2^t-t}\left(q y_1^{2^{t-1}-1} z_{j-(t-2),j} M\right) &= v^{2^t-t} z_t z_j M, \\ &\qquad j \geq t \geq 2,\ M \in P[y_t] \otimes \Lambda_{j+1}. \end{aligned}$$

Now with p odd.

$$d_{\nu(i)+2}\left(y_1^i\right) = h_0^{\nu(i)+1} v q y_1^{i-1},\ i \geq 1; \tag{3.2}$$

$$\begin{aligned} d_{\nu(i)+2}\left(y_1^i z_j M\right) &= v^{\nu(i)+2} q y_1^{i-1} z_{j-\nu(i)-1,j} M, \\ &\qquad j \geq \nu(i)+2,\ M \in \Lambda_j; \end{aligned} \tag{3.3}$$

$$\begin{aligned} d_{p^t-t}\left(h_0^{t-1} v q y_1^{p^{t-1}-1} M\right) &= v^{p^t} z_t M, \\ &\qquad t \geq 1,\ M \in P[y_t]; \end{aligned} \tag{3.4}$$

$$\begin{aligned} d_{p^t-t}\left(q y_1^{p^{t-1}-1} z_{j-(t-1),j} M\right) &= v^{p^t-t} z_t z_j M, \\ &\qquad j \geq t \geq 1,\ M \in P[y_t] \otimes TP_{p-1}[z_j] \otimes \Lambda_{j+1}. \end{aligned} \tag{3.5}$$

The proof occupies the rest of this section, except that at the end of the section we explain briefly how this leads to our description of $kup^*(K_2)$ in Section 1, except for the exotic extensions.

By [12, Theorem A], $Q_j Q_0 \iota_2$ is in the image from $BP^*(K_2)$ and hence must be a permanent cycle in our ASS. Thus by (2.20), z_j is a permanent cycle, and so (3.3) follows from (3.2), and (3.5) follows from (3.4), using $pz_{i,\ell} = vz_{i-1,\ell}$, as noted in 1.13.

The differentials (3.2) follow from the result of [3] that $H^{2pi+1}\left(K_2;\mathbb{Z}\right) \approx \mathbb{Z}/p^{\nu(i)+2} \oplus \bigoplus \mathbb{Z}_p$. See also [4, Proposition 1.3.5] when $p = 2$. The ASS converging to $H^*\left(K_2;\mathbb{Z}\right)$ has $E_2 = \mathrm{Ext}_{A_0}\left(\mathbb{Z}_2, H^*K_2\right)$, where $A_0 = \langle 1, Q_0\rangle$. We depict this E_2 similarly to our ASS for $kup^*(K_2)$. It has an h_0-tower for each element of $H_*(H^*K_2, Q_0)$, which was described in Lemma 2.6. These come in pairs in grading $2pi$ and $2pi+1$ corresponding to y_1^i and $y_1^{i-1} y_0^{p-1} u_0$. In order to get the $\mathbb{Z}/p^{\nu(i)+2}$, there must be a $d_{\nu(i)+2}$-differential, as pictured on the right-hand side of Figure 11.

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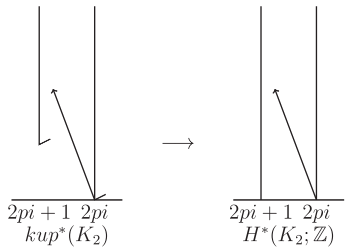

Figure 11. $kup^*(K_2) \to H^*\big(K_2; \mathbb{Z}\big)$.

Similarly to Figures 8 and 9, we have, for $p = 2$ and $i \geq 1$, an h_0-tower in the ASS for $kup^*(K_2)$ arising from $G^{4i+1,2}$, called either $h_0^2 y_1^{i-1} x_5$ or $v^2 y_1^{i-1} q$. There is also an h_0-tower arising from $y_1^i \in G^{4i,0}$. The classes y_1 and x_5 correspond to cohomology classes u_2^2 and $u_5 + u_2 u_3$. Under the morphism $kup^*(K_2) \to H^*\big(K_2; \mathbb{Z}\big)$, these towers map across, as suggested in Figure 11. We deduce the $d_{\nu(i)+2}$-differential claimed in (3.2), promulgated by the action of v. Note that $x_9 = q$.

The situation when p is odd is extremely similar, using Figure 10. The difference is that the h_0-tower in $2pi + 1$ in the kup^* ASS starts in filtration 1 rather than 2. Its generator can be called $vy_1^{i-1}q$.

In Figure 12, we depict many of the differentials asserted in Theorem 3.1 in grading ≤ 36 when $p = 2$. Regarding the third (final) summand in Theorem 2.4, which is $P[y_1] \otimes A_1$ when $p = 2$, we have included $y_1 A_1$, $y_1^3 A_1$, and $y_1^5 A_1$. Not included are the portions involving (3.2) and (3.3) when i is odd, as this portion self-annihilates. What is shown is (3.2) for $i = 2, 4$, and 6, (3.4) for $(t, k) = (1, 0), (1, 1), (1, 2)$, and $(2, 0)$, and (3.5) with $t = 1$, $k = 0$, and $j = 4$.

In order to establish the remaining differentials, we will need the following description of $k(1)^*(K_2)$, which is proved in [8]. We shift by 1 the subscripts of the classes z_j and w_j used there. The formulas for $r(j)$ and $r'(j)$ are as in [8]. We recapitulate some of their properties. Those stated here but not there are easily proved by induction.

Proposition 3.8. [8] *For $j \geq 0$, z_j is the reduction of the class in $kup^*(K_2)$ and satisfies $|z_j| = 2\big(p^{j+1} + 1\big)$. The classes w_j satisfy $|w_1| = 2p^2 + 1$, $|w_2| = 2p^3 - 2p^2 + 6p - 3$, and $w_{j+2} = y_j^{p-1} w_j z_{j+1}^{p-1}$. The integers $r(j)$ and $r'(j)$ satisfy the following properties:*

$$r(0) = 1,\ r(1) = p,\ r(j+2) = r(j) + p^{j+1}(p-1) + 1;$$

$$r'(0) = p - 1,\ r'(1) = p^2 - p, \tag{3.9}$$

$$r'(j+2) = r'(j) + p^{j+2}(p-1) - 1, \tag{3.10}$$

$$r(j) - r'(j-1) = j, \tag{3.11}$$

$$r(j) + r'(j) = p^{j+1}, \tag{3.12}$$

$$r(j+2) + r'(j) = p^{j+2} + 1, \tag{3.13}$$

$$(p-1)(r(j-1) + j - 1) < p^j, \tag{3.14}$$

$$p^{j+1} - p^j \leq r'(j) < p^{j+1} - p^{j-1}. \tag{3.15}$$

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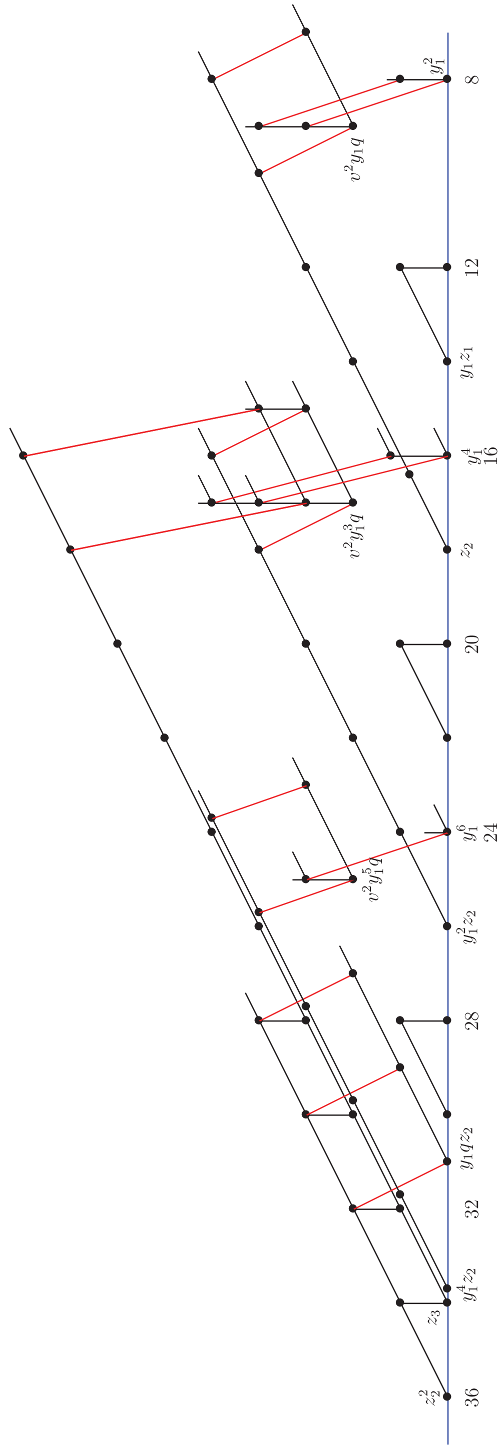

Figure 12. *Some differentials with* $p = 2$.

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Theorem 3.16. [8] *For any p, $k(1)^*(K_2)$ is a trivial $k(1)^*$-module plus*

$$\bigoplus_{j>0} TP_{r(j)}[v]\otimes P\big[y_{j+1}\big]\otimes TP_{p-1}\big[y_j\big]\otimes \overline{E}\big[w_j\big]\otimes E\big[w_{j+1}\big]\otimes \Lambda_{j+1}$$

$$\oplus\bigoplus_{j\ge 1} TP_{r'(j-1)}[v]\otimes P[y_j]\otimes E\big[w_j\big]\otimes \overline{TP}_p[z_j]\otimes \Lambda_{j+1}$$

$$\oplus\, P[y_1]\otimes\left(\overline{E}\big[y_0^{p-1}z_0\big]\oplus\begin{cases}\overline{E}[z_1] & p=2\\ 0 & p \text{ odd}\end{cases}\right)\oplus\bigoplus_{j\ge 1} P[y_1]\otimes E[q]\otimes \overline{E}\big[z_j^p\big]\otimes \Lambda_{j+1}.$$

The last line was not discussed in [8]; it is from free $E[Q_1]$ summands which are not part of free E_1 summands and plays a very important role.

Now we continue the proof of Theorem 3.1. We have already proved (3.2) and (3.3). As already noted, the z_j's are infinite cycles by [12], and so the differentials in (3.5) are implied as soon as the corresponding differential in (3.4) is proved.

As a warmup, we consider the cases $t=2$ and 3 of (3.4) when $p=2$. We make extensive use of the exact sequence (1.25). Referring to Figure 12 is useful.

In even gradings ≤ 14, $k(1)^*(K_2)=0$ in positive filtration, by Theorem 3.16. Thus, the map $kup^*(K_2)\to k(1)^*(K_2)$ implies that in the ASS for $kup^*(K_2)$, v^sz_2 must be hit by a differential or divisible by 2 for $s\ge 2$. In grading <8, there is nothing that can divide it, and the only odd-grading v-tower in that range is on v^2y_1q. Thus, $d_2\big(v^2y_1q\big)=v^4z_2$, the case $t=2$, $M=1$ of (3.4). Since $d_2\big(y_1^{2k}\big)=0$ by (3.2), the case $t=2$ of (3.4) follows for any M by the derivation property. An analogous argument does not work at the odd primes.

Similarly v^sz_3 must be hit or divisible for $s\ge 4$, and examination of options in Figure 12 shows that we must have $d_5\big(h_0v^2y_1^3q\big)=v^8z_3$, preceded by extensions. Since $d_5\big(y_1^8\big)=h_0^3v^2y_1^7q$, we deduce the case $t=3$, $M\in P\big[y_1^8\big]$ of (3.4) using the derivation property (2.5) and $h_0z_2=0$. We do not have *a priori* knowledge that $y_1^4z_3$ is a permanent cycle in the ASS of $kup^*(K_2)$. However, if it supported a nonzero differential, then the tower of v-height 4 on $y_1^4z_3$ in the ASS of $k(1)^*(K_2)$ would have to map to v^tC for $0\le t\le 3$ for some C in positive filtration in grading 51 in the ASS of $kup^*(K_2)$. Then, v^4C must be $d_r(B)$ with $r\ge 5$ and B in filtration 0 in grading 42. (B cannot have higher filtration since everything is v-towers, and v^3C cannot be hit.) But the only possible B is $y_1^6z_2$, and we already know that $v^4y_1^6z_2\in \operatorname{im}(d_4)$. (Ordinarily this would not preclude the possibility of B supporting a differential, but it does since everything is v-towers.) Thus, $y_1^4z_3$ is a permanent cycle, and consideration of its image in $k(1)^*(K_2)$ implies that $v^sy_1^4z_3$ is hit by a differential for some $s\ge 4$. The only element in odd grading <42 not yet accounted for is $h_0v^2y_1^7q$ in grading 33, and so this must be the source of the differential. This is the case $t=3$, $M=y_1^4$ of (3.4). The validity for all $M=y_1^{8i+4}$ (and $t=3$) now follows similarly to what we did for $M=y_1^{8i}$ at the beginning of this paragraph.

Now we switch our attention to the odd primes. The situation when $p=2$ is extremely similar. We want to prove the following version of (3.4):

$$d_{p^t-t}\left(h_0^{t-1}vqy_1^{(i+1)p^{t-1}-1}\right)=v^{p^t}y_1^{ip^{t-1}}z_t. \tag{3.17}$$

Now we work toward proving this. We illustrate with $p=5$, but it should be clear how it generalizes to an arbitrary prime. One new thing is the Divisibility Criterion as invoked in [8]. Each mod $(p-1)$ value of i can be considered separately. We will consider (3.17) with $p=5$ and $i=4\ell$; other congruences follow similarly. We index the differential (3.17) by (ℓ,t). We write T (for vertical tower) for the class $h_0^{t-1}vqy_1^{(4\ell+1)5^{t-1}-1}$, and M (for Monomial) is $y_1^{4\ell 5^{t-1}}z_t$. We will often afflict T and M with the parameters (ℓ,t). We write $|T|$ for $\frac{1}{2}(|T|+1)$. The $\frac{1}{2}$ avoids extraneous factors of 2 that always cancel out. The $+1$ is so that this indicates the grading (times $\frac{1}{2}$) of the class that it hits. $|M|$ denotes $\frac{1}{2}$ times the grading of M, and M' equals $\frac{1}{2}$ times the grading of v^hM, where h is the v-height of M in $k(1)^*(K_2)$. We wish to show that the differentials *must be* as claimed.

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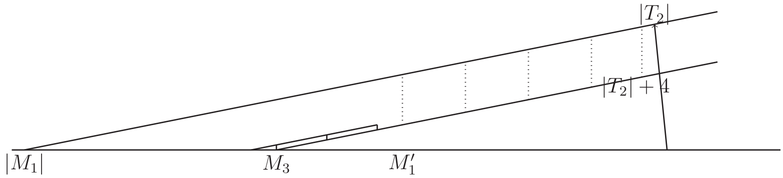

Figure 13. *The role of M_3.*

There are three types of constraints on the differentials involving these classes. Constraint C1 is that if $T \to M$ (by which we mean that a certain T class supports a differential hitting $v^i M$ for some i and a certain monomial M), then $|T| \le M'$. (This says that the v-tower on M cannot be hit while its image in $k(1)^*(K_2)$ is nonzero.)

Constraint C2 says that if $T(5\ell+1, t-1) \to M_1$ and $T(\ell, t) \to M_2$, then $|M_2| > |M_1|$. Since $|T(5\ell + 1, t-1)| = |T(\ell, t)|$, this says that as you move up an h_0 tower,[3] differentials must get longer (unless they are hitting into an h_0 tower, which is not the case here.)

Constraint C3 says that if $T_2 \to M_1$, then there exists M_3 such that $|M_1| \ge |M_3| \ge M_1'$ and either

$$M_3' \le |T_2|$$

or

$$T_3 \to M_3 \text{ has already been proved, and } |T_3| \le |T_2|.$$

The reason for C3 is that there must be extensions into the M_1-tower from grading M_1' to $|T_2| + 4$. The nonzero classes on the v-tower (on M_3) supporting the extensions must go to at least $|T_2| + 4$, and it has nonzero classes at least to $M_3' + 4$, and if $T_3 \to M_3$ was already proved, it has nonzero classes to $|T_3| + 4$. Note that we are saying that the v-tower on M_1 maps to 0 in $k(1)^*(K_2)$ once we get to grading M_1' (and hence in gradings $\le M_1'$ it is either hit by differentials or is divisible by p). There might be classes of higher filtration in $k(1)^*(K_2)$ to which it could map, but, if so, we can modify the generator of the M_1 tower by the class on the tower sitting above it. Also note that it is possible that extensions from the tower M_3 don't start from the generator, if there are h_0-extensions on the tower for awhile. See Figure 13. There is an exception to the C3 requirement for $T(\ell, 1) \to M(\ell, 1)$. Here, the extension into $v^4 y_1^{4\ell} z_1$ is obtained from the special class $y_1^{4\ell} y_0^4 z_0$.

With the above conventions, we have $|T| = 5^t(4\ell + 1) + 1$, $|M| = 5^t(4\ell + 5) + 1$, and $M' = |M| - 4r'(t-1)$, where $4r'(t-1)$ has the values 16, 80, 412, and 2076 for $t = 1$, 2, 3, and 4. Increasing from t to $t+2$ increases this by $4^2 \cdot 5^{t+1} - 4$. We consider the cases in order of increasing $|M|$ and, for equal values of $|M|$, increasing ℓ. We tabulate a representative sample in Table 1. We omit listing values of $\ell \equiv 3, 4 \bmod 5$ because they behave similarly to $\ell \equiv 2$.

Before presenting a general argument, we illustrate with an example, starting with $M_1 = M(1, 3)$. We will see that it builds a chart which is y_1^{100} times Figure 2. In Table 1, we have $|M_1| = 1126$. Its v-tower is truncated at height $p^3 = 125$ by a differential on $T(1, 3)$, with $|T(1, 3)| = 626$, using our grading conventions. Playing the role of M_3 is $M(6, 2)$ with $|M_3| = 726$. We have $M_1' = 714$. It is $v^3 M_3$ which supports the extension in "grading" 714. Note that for $0 \le i \le 2$, $h_0 v^i M_3 \ne 0$, and so $p \cdot v^i M_3$ is not a v-multiple of M_1. (In Figure 2, the class $y_2^{p-1} z_2$ corresponds to M_3.) From Table 1, we see that $M_3' = 646$, which means that in "grading" ≤ 646, the v-tower on M_3 is either hit by a differential or divisible by p. Table 1 says it is hit by a differential in 626. In "gradings" from 646 to 630, it is divisible by p. It has its own, distinct, M_3 class, namely $M(31, 1)$. In Figure 2, this latter class corresponds to $y_1^{p-1} y_2^{p-1} z_1$.

Now we start the proof. We begin with a lemma.

Lemma 3.19. *For $M = M(\ell', t')$ with $|M(5\ell + 1, t-1)| < |M| < |M(\ell, t)|$, we have $t' < t$, $|T(\ell, t)| < |T(\ell', t')|$, and $|M(5\ell + 1, t-1)| < M'$.*

[3] Note that $h_0 T(5\ell + 1, t-1) = T(\ell, t)$.

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Table 1. *Cases in order.*

ℓ	t	$\|T\|$	$\|M\|$	M'	ℓ	t	$\|T\|$	$\|M\|$	M'
0	1	6	26	10	36	1	726	746	730
1	1	26	46	30	37	1	746	766	750
2	1	46	66	50	7	2	726	826	746
0	2	26	126	46	40	1	806	826	810
5	1	106	126	110	41	1	826	846	830
6	1	126	146	130	42	1	846	866	850
7	1	146	166	150	8	2	826	926	846
1	2	126	226	146	45	1	906	926	910
10	1	206	226	210	46	1	926	946	930
11	1	226	246	230	47	1	946	966	950
12	1	246	266	250	9	2	926	1026	946
2	2	226	326	246	50	1	1006	1026	1010
15	1	306	326	310	51	1	1026	1046	1030
16	1	326	346	330	52	1	1046	1066	1050
17	1	346	366	350	1	3	626	1126	714
3	2	326	426	346	10	2	1026	1126	1046
20	1	406	426	410	55	1	1106	1126	1110
21	1	426	446	430	56	1	1126	1146	1130
22	1	446	466	450	57	1	1146	1166	1150
4	2	426	526	446	11	2	1126	1226	1146
25	1	506	526	510	60	1	1206	1226	1210
26	1	526	546	530	61	1	1226	1246	1230
27	1	546	566	550	62	1	1246	1266	1250
0	3	126	626	214			⋮		
5	2	526	626	546	154	1	3086	3106	3090
30	1	606	626	610	0	4	626	3126	1050
31	1	626	646	630	5	3	2626	3126	2714
32	1	646	666	650	30	2	3026	3126	3046
6	2	626	726	646	155	1	3106	3126	3110
35	1	706	726	710	156	1	3126	3146	3130

Proof. The given inequalities quickly force $t' < t$. The inequality $|T(\ell,t)| < |T(\ell',t')|$ follows immediately. Finally, the given inequalities prevent $M' \le |M(5\ell+1,t-1)|$. □

To prove the differentials, we use induction on our ordering of the M's. If the differentials are not as posed, consider the smallest $|M|$ such that $T(\ell,t) \to M$ with $M \ne M(\ell,t)$.

We cannot have $|M| > |M(\ell,t)|$, because $|M(\ell,t)|$ would contradict the minimality of $|M|$. We cannot have $|M| \le |M(5\ell+1,t-1)|$ by constraint C2.

If $|M(5\ell+1,t-1)| < |M| < |M(\ell,t)|$, by constraint C3 and the lemma, we must have M_3 with $|M(5\ell+1,t-1)| < M' \le |M_3| < |M|$. Because $|M_3| < |M|$, we know $T_3 \to M_3$ by induction. From the lemma, we get $|T(\ell,t)| < |T_3|$, but that contradicts constraint C3.

We must have $T(\ell,t) \to M(\ell,t)$, and $M(5\ell+1,t-1)$ is eligible for our M_3. This completes most of the proof of (3.17) and hence of Theorem 3.1.

Underlying the above analysis has been an assumption that the M-classes are always hit by T-classes. We show now that it could not have occurred that an M-class supported a differential. Assume that $M = y_1^{ip^{t-1}} z_t$ is the M-class of lowest grading which supports a differential. We now revert to letting $|x|$ denote the actual grading of a class x, not divided by 2.

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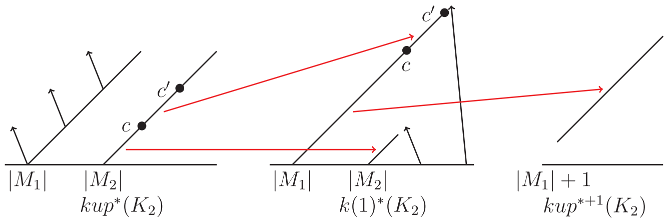

Figure 14. *An unwanted possibility.*

In $k(1)^*(K_2)$, M supports a v-tower of v-height $r'(t-1)$ by 3.2. We will show at the end of the proof that there is a number $\Delta \le t$ such that v^iM maps nontrivially to $kup^{*+1}(K_2)$ if and only if $i \le r'(t-1) - \Delta$. (Usually $\Delta = 1$.) The image of M in $kup^{|M|+1}(K_2)$ is a class C of positive filtration such that $v^{r'(t-1)-\Delta}C \neq 0$ and $v^{r'(t-1)-\Delta+1}C = 0 \in kup^*(K_2)$, so there must be a differential in the ASS of $kup^*(K_2)$ from a filtration-0 class hitting a class of filtration $\ge r'(t-1) - \Delta + 2$ in grading $|M| + 1 - 2(p-1)(r'(t-1) - \Delta + 1)$. (The reason that the differential must start from filtration 0 is that in even gradings, E_2 consists entirely of v-towers starting in filtration 0.) This differential cannot come from another such M because of our lowest-grading assumption. It cannot come from a product of one or more z's times one of these M's because z's are infinite cycles. We must rule out the possibility that this differential is one of type (3.3). They are distinguished by having the smallest z-subscript at least 2 greater than the p-exponent of the exponent of y_1.

The differential to C has subscript $\ge r'(t-1) - \Delta + 2$, and so the class in (3.3) would be $y_1^{\ell p^{r'(t-1)-\Delta}}Z$ for some positive integer ℓ, where Z is a product of z_j's with $j \ge r'(t-1) - \Delta + 2$, and each j appears at most $p-1$ times, except that the smallest j might appear p times. Equating this grading with $|M| - 2(p-1)(r'(t-1) - \Delta + 1)$ and canceling a common factor 2 from all terms yield

$$\ell p^{r'(t-1)-\Delta+1} + \sum_j \left(p^{j+1} + 1\right) = ip^t + p^{t+1} + 1 - (p-1)(r'(t-1) - \Delta + 1). \tag{3.20}$$

Using (3.12) and (3.14) and $\Delta \le t$, the right-hand side of (3.20) equals $p^t(i+1) + (p-1)(r(t-1) + \Delta - 1) + 1 \equiv (p-1)(r(t-1) + \Delta - 1) + 1 \mod p^t$, with $(p-1)(r(t-1) + \Delta - 1) + 1 \le p^t$ (strict if $t > 2$). Since $r'(t-1) - \Delta > t$, this implies that the $\sum\limits_j$ on the left-hand side of (3.20) must contain at least $(p-1)(r(t-1) + \Delta - 1) + 1$ summands. We obtain

$$\sum p^j \ge p \cdot p^{r'(t-1)-\Delta+2} + (p-1)\left(p^{r'(t-1)-\Delta+3} + \cdots + p^{r'(t-1)+r(t-1)}\right)$$
$$= p^{r'(t-1)+r(t-1)+1} = p^{p^t+1},$$

so $\sum p^{j+1} \ge p^{p^t+2}$, and hence $p^t(i+1) > p^{p^t+2}$. Thus $i \ge p^{p^t-t+2} > p^{p^t-2t}$.

Since $d_{p^t-t+1}\left(y_1^{p^{p^t-t-1}}\right)$ is defined,

$$d_r\left(y_1^{p^{p^t-t-1}}\right) = 0 \text{ for } r \le p^t - t, \tag{3.21}$$

and by the lowest-grading assumption, $d_{p^t-t}\left(h_0^{t-1}vqy_1^{\left(i-p^{p^t-2t}+1\right)p^{t-1}-1}\right) = v^{p^t}y_1^{(i-p^{p^t-2t})p^{t-1}}z_t$ and $y_1^{(i-p^{p^t-2t})p^{t-1}}z_t$ is a permanent cycle. Since

$$y_1^{ip^{t-1}}z_t = y_1^{\left(i-p^{p^t-2t}\right)p^{t-1}}z_t \cdot y_1^{p^{p^t-t-1}},$$

we deduce that $y_1^{ip^{t-1}}z_t$ survives to E_{p^t-t} and (3.17), using the derivation property of differentials.

Now we consider the need for Δ in the above argument. The worry is that maybe part of the v-tower on M in $k(1)^*(K_2)$ might be in the image from $kup^*(K_2)$, due to a filtration jump from a lower tower, as sketched in Figure 14, so that only a smaller part of the M-tower in $k(1)^*(K_2)$ maps to $kup^{*+1}(K_2)$.

The monomials $M_\varepsilon = y_{t_\varepsilon}^{i_\varepsilon} z_{t_\varepsilon}$ ($\varepsilon = 1, 2$) have $|M_\varepsilon| = 2(p^{t_\varepsilon}(i_\varepsilon + p) + 1)$ and are truncated in $k(1)^*(K_2)$ in grading $M'_\varepsilon = |M_\varepsilon| - 2(p-1)r'(t_\varepsilon - 1)$. In $kup^*(K_2)$, M_2 is truncated in grading $|T_2| = |v^{p^{t_2}} M_2| = 2(p^{t_2}(i_2 + 1) + 1)$. In Figure 14, elements c are in grading M'_2, and c' is in grading $M'_1 + 2(p-1)$. The necessary condition for nontrivial image in $k(1)^*(K_2)$ (and hence $\Delta > 1$) is

$$|T_2| + 2(p-1) \le M'_1 + 2(p-1) \le M'_2. \tag{3.23}$$

If this occurs, then we might have Δ as large as $\dfrac{M'_2 - M'_1}{2(p-1)} + 1$. We now show in Lemma 3.24 that if (3.23) holds, then $\left(M'_2 - M'_1\right)/(2(p-1)) < t$, establishing the claim made earlier about $\Delta \le t$.

We restrict to $p = 5$, $i = 4\ell$ for simplicity, and so that the reader can refer to Table 1 as an aid. The argument easily generalizes to any prime and any congruence. We divide everything by 2 as was done above and also subtract off the $+1$ which occurs in formulas for $|M|$ and $|T|$, so the numbers will be 1 smaller than those in the table.

Lemma 3.24. *If* $t_1 > t_2$ *and*

$$5^{t_2}(4\ell_2 + 1) + 4 \le 5^{t_1}(4\ell_1 + 5) - 4r'(t_1 - 1) + 4 \le 5^{t_2}(4\ell_2 + 5) - 4r'(t_2 - 1),$$

then

$$\frac{1}{4}\Big(5^{t_2}(4\ell_2 + 5) - 4r'(t_2 - 1) - (5^{t_1}(4\ell_1 + 5) - 4r'(t_1 - 1))\Big) < t_1 - 1.$$

Proof. If there is a counterexample to this, then there is one with $\ell_1 = 0$, since ℓ_2 could be decreased by $5^{t_1 - t_2}\ell_1$, so it suffices to use $\ell_1 = 0$. Let $Q(k) = (5^{2k} - 1)/24$ (called $q(k)$ in [8, Lemma 5.3]). Then, using [8, Lemma 5.5] for $t = 2k + \delta$ with $\delta = 1$ or 2,

$$5^{t+1} - 4r'(t-1) = 5^{2k+\delta} + 16 \cdot 5^\delta Q(k) + 4k + 4 \cdot 5^{\delta - 1}.$$

Since $16 \cdot 5^\delta Q(k) + 4k + 4 \cdot 5^{\delta-1} < 3 \cdot 5^{2k+\delta}$, the hypothesis of the lemma says that $5^{t_1+1} - 4r'(t_1 - 1)$ mod $4 \cdot 5^{t_2}$ lies in the mod-$(4 \cdot 5^{t_2})$ interval $[5^{t_2}, 5^{t_2+1} - 4r'(t_2 - 1) - 4]$.

Let $t_1 = 2k_1 + \delta_1$ and $t_2 = 2k_2 + \delta_2$. The condition is restated as:

$$5^{2k_1+\delta_1} + 16 \cdot 5^{\delta_1} Q(k_1) + 4k_1 + 4 \cdot 5^{\delta_1 - 1} \tag{3.25}$$

lies in the mod-$(4 \cdot 5^{t_2})$ interval:

$$[5^{t_2}, 5^{t_2} + 16 \cdot 5^{\delta_2} Q(k_2) + 4k_2 + 4 \cdot 5^{\delta_2 - 1} - 4]. \tag{3.26}$$

Let $\delta_2 = 1$. The reduction mod $4 \cdot 5^{t_2}$ of (3.25) is

$$5^{t_2} + 16 \cdot 5^{\delta_1} Q(k_2) + 4k_1 + 4 \cdot 5^{\delta_1 - 1}. \tag{3.27}$$

Let $\delta_1 = 2$. Then, $5^{t_2} + 16 \cdot 5^{\delta_1} Q(k_2) > 4 \cdot 5^{t_2}$ and equals $5^{2k_2+2} - (2000Q(k_2 - 1) + 100)$, so (3.27) will first be in the interval (3.26) when $4k_1 + 20 = 2000Q(k_2 - 1) + 100$, hence $k_1 = 500Q(k_2 - 1) + 20$, so $t_1 = 1000Q(k_2 - 1) + 42$. The left-hand side of the conclusion of the lemma is $\frac{1}{8}\left(M'_2 - M'_1\right)$ with M'_1 and M'_2 as in (3.23). For $k_1 = 500Q(k_2 - 1) + 20$, the value of M'_1 is at the left end of the interval (3.26), and so $\frac{1}{8}\left(M'_2 - M'_1\right)$ equals $\frac{1}{4}$ times the length plus 4 of (3.26), which is

$$20Q(k_2) + k_2 + 1 = 500Q(k_2 - 1) + k_2 + 21 = \frac{1}{2}t_1 + k_2.$$

Since $k_2 \ll t_1$, this is less than $t_1 - 1$, as claimed. If k_1 is increased from the value $500Q(k_2 - 1) + 20$, the value of t_1 increases, while $M'_2 - M'_1$ decreases, since M'_1 is moving through the interval, so the inequality asserted in the lemma is satisfied more strongly.

Now, with $\delta_2 = 1$ continuing, let $\delta_1 = 1$. Since $k_1 > k_2$, (3.27) lies outside the interval (3.26) until $80Q(k_2) + 4k_1 + 4 = 4 \cdot 5^{t_2}$, so

$$k_1 = 5^{2k_2+1} - 20Q(k_2) - 1 = 100Q(k_2) + 4$$

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and $t_1 = 200Q(k_2) + 9$. Again $\frac{1}{8}\left(M_2' - M_1'\right) = 20Q(k_2) + k_2 + 1 \approx \frac{1}{10}t_2 + k_2$, so the conclusion of the lemma is satisfied more strongly.

A similar analysis works when $\delta_2 = 2$. In this case, $\frac{1}{8}\left(M_2' - M_1'\right) \approx \frac{1}{2}t_1 + k_2$ if $\delta_1 = 1$, and $\frac{1}{8}\left(M_2' - M_1'\right) \approx \frac{1}{10}t_1 + k_2$ if $\delta_1 = 2$. □

We close this section by explaining how Theorems 2.4 and 3.1 lead to the descriptions of $kup^*(K_2)$ given in Theorems 1.8 and 1.15, modulo exotic extensions. We begin with the portion in even gradings and restrict our attention to odd p. All elements in the $P[h_0, v, y_1]$ part of Theorem 2.4 support differentials of type (3.2). Note that $y_0^{p^k-1} = y_0^{p-1}y_1^{p^{k-1}-1} = \prod_{j=0}^{k-1} y_j^{p-1}$. The first is easiest to write, the second occurs in Theorem 2.4, and the third in 1.5 and Figure 2. From 1.5, $y_0^{p^k-1}z_0$ is in A_k for $k \geq 1$, the bottom right element in Figure 2. Then,

$$P[y_1]y_0^{p-1}z_0 = \bigoplus \mathcal{M}_k^A \cdot y_0^{p^k-1}z_0 \subset \bigoplus \mathcal{M}_k^A A_k. \tag{3.28}$$

The first part occurs in Theorem 2.4 and the last part in Theorem 1.8.

Now we consider $P[y_1] \otimes \bigoplus_{j\geq 1} W_j \otimes TP_{p-1}[z_j] \otimes \Lambda_{j+1}$ in Theorem 2.4. The $\bigoplus$ part is all monomials $z_\ell M$ with $\ell \geq 1$ and $M \in \Lambda_\ell$. From Theorem 3.1, $y_1^i z_\ell M$ supports a differential (3.3) if $\ell \geq \nu(i) + 2$, while those with $\nu(i) \geq \ell - 1$ are hit by differentials (3.4) and (3.5), yielding v-towers with heights as given in 1.5. These are all monomials in $\bigoplus_{\ell\geq 1} P[y_\ell, y_{\ell+1}, \ldots]z_\ell\Lambda_\ell$. From 1.5 or (1.7), the generators of the v-towers in B_k are all

$$z_j \prod_{i=j}^{k-1} \left\{z_i^{p-1}, y_i^{p-1}\right\}, \quad 1 \leq j \leq k.$$

Let $(z_\ell M)_i$ be the $y_i^e z_i^{e'}$ factors of M. Then, $\mathcal{M}_k B_k$ consists of all monomials $z_\ell M$ such that $(z_\ell M)_i$ equals y_i^{p-1} or z_i^{p-1} for $\ell \leq i < k$, but not for $i = k$, and so every monomial $z_\ell M$ is in a unique $\mathcal{M}_k B_k$. From Theorem 3.1, $z_\ell M$ has v-height p^ℓ if and only if M contains no z-factors, which explains the split into $\mathcal{M}_k^A$ and $\mathcal{M}_k^B$ in Theorem 1.8.

Now we address the odd gradings. The $P[h_0, v, y_1]vq$ part of Theorem 2.4 is totally removed either as sources (3.4) or targets (3.2) of differentials. See grading 17 in Figure 12 for a nice illustration. The $qy_1^{i-1}S_{\nu(i)+1,\ell}$ part of Theorem 1.15 is formed from $TP_{\nu(i)+2}[v]qy_1^{i-1}W_\ell$ in 2.1 using (3.3). The generators of $S_{\nu(i)+1,\ell}$ are $z_{1,\ell}, \ldots, z_{\ell-\nu(i)-1,\ell}$, but to see the differential from (3.3), one should write $z_{t,\ell} = z_{t,t+\nu(i)+1}Z_{t+\nu(i)+1}^\ell$, where

$$Z_i^j = (z_i \cdots z_{j-1})^{p-1} \text{ for } j > i, \text{ with } Z_i^i = 1. \tag{3.29}$$

The remaining generators of $qy_1^{i-1}W_\ell$, namely $qy_1^{i-1}z_{j,\ell}$ with $\ell - \nu(i) \leq j \leq \ell$, support differentials (3.5). There can be no unexpected exotic extensions among these summands for the reason noted at the end of Section 1. The ker (p) elements in the S summands play a very important role in the exact sequence.

4. Exotic extensions

In this section, we prove the following expansion of (1.6).

Theorem 4.1. *If $i \geq 0$ and $k \geq k_0$,*

$$py_k^i y_{k-1}^{p-1} z_{k-1} = v^{p^{k-1}(p-1)} y_k^i z_k$$

with an additional term $vy_k^i y_{k-1}^{p-1} z_{k-2}^p$ if $k \geq k_0 + 2$.

The additional term is seen in Ext and will be ignored in the rest of this section. We have included the factor y_k^i, which is not automatic since y_k^i is not a permanent cycle. Since, for example, $y_{k+1} = y_k^p$, we need not consider y_i for $i > k$. It is automatic that this formula can be multiplied by z_j's, since they do survive the spectral sequence.

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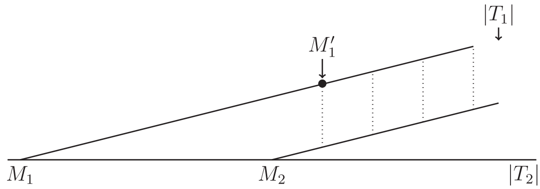

Figure 15. *Conditions for extension.*

The extension is deduced from the exact sequence:

$$kup^*(K_2) \xrightarrow{\cdot p} kup^*(K_2) \longrightarrow k(1)^*(K_2)$$

and the fact that $v^{r'(k-1)}y_k^i z_k = 0$ in $k(1)^*(K_2)$ with $r'(k-1) \geq p^k(p-1)$. Thus, $v^{r'(k-1)}y_k^i z_k$ must be divisible by p in $kup^*(K_2)$, and, as we will show, the v-tower on $y_k^i y_{k-1}^{p-1} z_{k-1}$ provides the only classes that can do the dividing. Once we know the division formula toward the end of the v-tower, we can deduce that it holds earlier in the tower, as well. For example, $r'(2) = p^3 - p^2 + p - 2$, which is the height in the top v-tower in Figure 2 where the extensions into it do not also involve an h_0-extension. We deduce the extensions from the earlier part of the v-tower on $y_2^{p-1}z_2$ by naturality.

We illustrate in Figure 15, using the notation of the preceding section. Thus, T_i is the class satisfying $d_r(T_i) = v^r M_i$, Here, the portion of the top tower to the right of M_1' must be divisible by p. The tower providing the extension must have $M_1' \leq |M_2| < |M_1|$ and $|T_2| \leq |T_1|$.

As we did for the differentials in the previous section, we will perform the argument for $p = 5$. It will be clear that it generalizes to an arbitrary odd prime, and with minor modification to $p = 2$. Also, we use $i = 4\ell$ in Theorem 1.15. If instead we used $i = 4\ell + d$ for $1 \leq d \leq 3$, it will just add the same amount to the quantities $|M|$, $|T|$, and M' involved in the argument. We can use Table 1 to envision the analysis, with the t there replaced by k. For a monomial $M(\ell, k) = y_k^{4\ell}z_k$, we have, after dividing by 2, $|M| = 5^k(4\ell + 5) + 1$, $|T| = 5^k(4\ell + 1) + 1$, and $5^k(4\ell + 1.16) + 1 < M' \leq 5^k(4\ell + 1.8) + 1$, using (3.15). With M_1 and M_2 as in Figure 15, we will show that $M_2(5\ell + 1, k - 1)$ is the unique monomial satisfying the inequalities stated just before Figure 15 for $M_1(\ell, k)$. Note that $M(5\ell + 1, k - 1) = y_k^{4\ell}y_{k-1}^4 z_{k-1}$. We omit the $+1$ in all the formulas.

The inequalities are satisfied by $M_2(5\ell + 1, k - 1)$ since

$$5^k(4\ell + 1.8) \leq 5^{k-1}(4(5\ell + 1) + 5) < 5^k(4\ell + 5) \text{ and } 5^{k-1}(4(5\ell + 1) + 1) \leq 5^k(5\ell + 1).$$

If $k_2 \geq k$, then the first inequality, after dividing by 5^k, becomes

$$4\ell + 1.8 \leq 5^{k_2 - k}(4\ell_2 + 5) < 4\ell + 5,$$

which cannot be satisfied since the middle term is $\equiv 1 \bmod 4$. If $k_2 < k - 1$, then

$$M_1' - |T_1| > 5^k \cdot .16 \geq 4 \cdot 5^{k_2} = |M_2| - |T_2|,$$

which is inconsistent with two of the inequalities. Let $k_2 = k - 1$. If $\ell_2 < 5\ell + 1$, then

$$|M_2| = 5^{k-1}(4\ell_2 + 5) \leq 5^{k-1}(4 \cdot 5\ell + 5) < 5^k(4\ell + 1.16) < M_1',$$

contradicting one of the inequalities. If $k_2 = k - 1$ and $\ell_2 > 5\ell + 1$, then

$$|T_2| \geq 5^{k-1}(4(5\ell + 2) + 1) > 5^k(4\ell + 1) = |T_1|,$$

contradicting one of the inequalities.

We deduce that $M_2 = y_k^{4\ell}y_{k-1}^4 z_{k-1}$, as claimed. We should perhaps have noted that the extensions could not have come from classes with more than one z_j-factor, because these are z_j times a class on which the extensions have already been determined.

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5. Proposed formulas for the exact sequence (1.25)

In this section, we propose what we conjecture must be the correct complete formulas for the exact sequence (1.25). Some homomorphisms are forced by naturality, but many others involve significant filtration jumps. However, they all occur in several families with nice properties. The 10-term exact sequence (5.2) shows how the $S_{k,\ell}$ portions and the exotic extensions yield compatibility of the differing v-tower heights in $kup^*(K_2)$ and $k(1)^*(K_2)$. In Section 6, we show that all elements of $k(1)^*(K_2)$ are accounted for exactly once in these homomorphisms, which implies that there can be no more exotic extensions. This does not require us to prove that our homomorphism formulas are actually correct, as discussed at the end of Section 1. We will focus on the case when p is odd. We could incorporate all primes together at the expense of involving the parameter k_0, but things are complicated enough without that. In an earlier version of this paper ([7]), a thorough analysis when $p=2$ was performed.

We propose that (1.25) can be split into exact sequences of length 4 and 10 (not including 0's at the end). There are subgroups of $k(1)^*(K_2)$ called G_k^1 and G_k^2 for $k\geq 1$ and $G_{k,\ell}^i$ for $3\leq i\leq 6$ and $1\leq k<\ell$ such that there are exact sequences:

$$0\to G_k^1\to A_k\xrightarrow{p}A_k\to G_k^2\to 0 \tag{5.1}$$

for $k\geq 1$, and, for $1\leq k<\ell$,

$$\begin{aligned}0\quad\to\quad &G_{k,\ell}^3\to y_kB_kZ_k^\ell\xrightarrow{p}y_kB_kZ_k^\ell\to G_{k,\ell}^4\to y_1^{p^{k-1}-1}qS_{k,\ell}\\ \xrightarrow{p}\quad &y_1^{p^{k-1}-1}qS_{k,\ell}\to G_{k,\ell}^5\to B_kz_\ell\xrightarrow{p}B_kz_\ell\to G_{k,\ell}^6\to 0,\end{aligned} \tag{5.2}$$

with Z_k^ℓ as defined in (3.29). The sequence (5.1) can be tensored with $TP_{p-1}[y_k]\otimes P[y_{k+1}]$, while (5.2) can be tensored with $TP_{p-1}[y_k]\otimes P[y_{k+1}]\otimes TP_{p-1}[z_\ell]\otimes\Lambda_{\ell+1}$. If p is odd, there are also exact sequences:

$$0\to G_{k,e}^7\to B_kz_k^e\xrightarrow{p}B_kz_k^e\to G_{k,e}^8\to 0 \tag{5.3}$$

for $k\geq 1$ and $1\leq e\leq p-2$. This can be tensored with $P[y_k]\otimes\Lambda_{k+1}$.

One can verify that the totality of A_k and B_k groups in these exact sequences agrees with that in Theorem 1.8. We will study these exact sequences by breaking them up into short exact sequences and isomorphisms involving kernels and cokernels of $\cdot p$.

Let $K_k^A=\ker(\cdot p|A_k)$, $K_k^B=\ker(\cdot p|B_k)$, $C_k^A=\operatorname{coker}(\cdot p|A_k)$, and $C_k^B=\operatorname{coker}(\cdot p|B_k)$. There are important elements $g_k\in K_k^A$ and K_k^B defined (up to unit coefficients) by $g_1=z_1$, $g_2=v^{p-2}z_2$, and, for $k\geq 1$,

$$g_{k+2}=v^{r'(k)-1}z_{k+2}+g_ky_k^{p-1}z_{k+1}^{p-1}. \tag{5.4}$$

To see that this is in $\ker(\cdot p)$, we use (1.6) to see that $p\cdot v^{r'(k)-1}z_{k+2}=v^{r'(k)}z_{k+1}^p$, and that the $v^{r'(k-2)-1}z_k$ term in g_k yields $v^{r'(k-2)-1}v^{p^k(p-1)}z_{k+1}z_{k+1}^{p-1}$ in $p\cdot g_ky_k^{p-1}z_{k+1}^{p-1}$. Using (3.10), these terms cancel. Other terms in $p\cdot g_ky_k^{p-1}z_{k+1}^{p-1}$ yield 0 since $g_k\in\ker(\cdot p)$.

The v-towers in K_k^A are generated by:

$$g_k\text{ and }g_jz_j^{p-1}\prod_{i=j+1}^{k-1}\left\{z_i^{p-1},y_i^{p-1}\right\},\ 1\leq j\leq k-1. \tag{5.5}$$

For example, using Figure 2 when $k=3$, these are $g_3=v^{p^2-p-1}z_3+y_1^{p-1}z_1z_2^{p-1}$, $g_2z_2^{p-1}=v^{p-2}z_2^p$, $g_1z_1^{p-1}z_2^{p-1}$, and $g_1z_1^{p-1}y_2^{p-1}$. The v-heights are $p^k-(r'(k-2)-1)$ for g_k, and $p^j-j-(r'(j-2)-1)$ for the others, since they are determined by v-heights of z_j in B_k. The map $G_k^1\to K_k^A$ sends w_k to g_k and

$$w_jP\mapsto g_jP\text{ for }P=z_j^{p-1}\prod_{i=j+1}^{k-1}\left\{z_i^{p-1},y_i^{p-1}\right\}, \tag{5.6}$$

with w_j as in 3.8 and 3.16. The v-height of w_j is $r(j)$ if it is not accompanied by z_j, and $r'(j-1)$ if it is. By (3.13) and ((3.12) and (3.11)), the v-heights agree, so (5.6) is an isomorphism on v-towers.

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For $L = K_k^A$ or K_k^B or C_k^A or C_k^B, we say that a $\mathbb{Z}_p$ in L is a class of v-height 1 in L which is not part of a larger v-tower in L. There is one $\mathbb{Z}_p$ in K_3^A, as can be seen in Figure 2. This is the element $v^{p-2}y_1^{p-1}z_1z_2^{p-1}$. Note that for $i < p-1$, $v^iy_1^{p-1}z_1z_2^{p-1} + v^{i+p^2-p-1}z_3$ is part of a v-tower in K_3^A, which continues with the elements v^iz_3 for $i > p^2-3$, but $v^iy_1^{p-1}z_1z_2^{p-1}$ itself is in K_3^A only for $i = p-2$. Using 1.5, we find that the $\mathbb{Z}_p$'s in K_k^A are

$$v^{p^t-t-1}(y_t\cdots y_{j-1})^{p-1}z_tz_j^{p-1}\prod_{i=j+1}^{k-1}\left\{z_i^{p-1},y_i^{p-1}\right\} \text{ for } 1\le t<j<k. \tag{5.7}$$

For example, the elements $v^{p-2}(y_1y_2)^{p-1}z_1$ and $v^{p^2-3}y_2^{p-1}z_2$ in Figure 2 yield elements in K_4^A after being multiplied by z_3^{p-1}. The basic formula for the homomorphism from part of $k(1)^*(K_2)$ to $\mathbb{Z}_p$'s in various K_k^A and K_k^B, possibly tensored with other classes as in Theorem 1.8, is

$$\left(q(y_1\cdots y_t)^{p-1}z_{j-t,j}\mapsto v^{p^t-t-1}y_t^{p-1}z_tz_j\right)\otimes P\big[y_j\big]\otimes TP_{p-1}[z_j]\otimes\Lambda_{j+1} \text{ for } j>t\ge 1. \tag{5.8}$$

The domain elements are in the second half of the third line of Theorem 3.16. The ones that are in G_k^1 in the isomorphism $G_k^1 \to K_k^A$ can be extracted using (5.7).

The isomorphism $G_{k,\ell}^3 \to y_kK_k^BZ_k^\ell$ in (5.2) is given using formulas analogous to (5.6) and (5.8). There are several minor differences. One is that the v-tower on $y_kg_kZ_k^\ell$ is truncated due to $v^{p^k-k}z_k = 0$ in B_k (as opposed to $v^{p^k}z_k = 0$ in A_k). This is compatible with the fact that the v-height of w_kz_k in $k(1)^*(K_2)$ is k less than that of w_k, using Theorem 3.16 and (3.11). The other is that K_k^B has additional $\mathbb{Z}_p$'s:

$$v^{p^t-t-1}(y_t\cdots y_{k-1})^{p-1}z_t \text{ for } 1\le t\le k-1, \tag{5.9}$$

as seen in Figure 2 when $k=3$, but these are always multiplied by higher z's, and so (5.8) applies.

The isomorphisms $C_k^A \to G_k^2$ and $C_k^Bz_\ell \to G_{k,\ell}^6$ are defined simply by sending an element to one with the same name. Moreover, $C_k^A = C_k^B$ except for $(y_0\cdots y_{k-1})^{p-1}z_0 \in C_k^A - C_k^B$. When $k=3$, we see that the $\mathbb{Z}_p$'s in C_k^B are $\{z_1^pz_2^{p-1},\ z_2^p,\ y_2^{p-1}z_1^p\}$ in Figure 2.[4] For future reference,

$$\mathbb{Z}_p'\text{s in } C_k^B \text{ are } \left\{z_t^p\prod_{i=t+1}^{k-1}\left\{z_i^{p-1},y_i^{p-1}\right\}\ :1\le t<k\right\}. \tag{5.10}$$

The corresponding elements in $k(1)^*(K_2)$ are from the third line of 3.2.

The v-towers in $C_k^A = C_k^B$ are generated by:

$$z_k \text{ and } y_t^{p-1}z_t\prod_{i=t+1}^{k-1}\left\{z_i^{p-1},y_i^{p-1}\right\},\ 1\le t<k. \tag{5.11}$$

We will show that the v-height of z_k in C_k^B is $r'(k-1)$, which equals its v-height in $k(1)^*(K_2)$. It follows from 1.5 that the v-height of $y_t^{p-1}z_t\prod_{i=t+1}^{k-1}\left\{z_i^{p-1},y_i^{p-1}\right\}$ equals $r'(t-1)$, establishing the isomorphisms out of C_k^A and $C_k^Bz_\ell$. In Figure 2, the v-height of z_3 equalling $p^3-p^2+p-2 = r'(2)$ is apparent.

The proof of the claim about v-heights is by induction. By (3.10), $r'(k-1)-r'(k-3) = p^{k-1}(p-1)-1$. Let $D = \left(|z_k| - |y_{k-1}^{p-1}z_{k-1}|\right)/(2(p-1)) = p^{k-1}(p-1)$. This is the filtration on the z_k-tower above the element $y_{k-1}^{p-1}z_{k-1}$. We show that $v^{i-1+D}z_k$ is divisible by p if and only if v^iz_{k-2} is divisible by p. Thus, the difference of the v-heights in cokernels equals the difference of the corresponding r' values. From Theorem 4.1, we have

$$pv^{i-1}y_{k-1}^{p-1}z_{k-1} = v^{i-1+D}z_k + v^iy_{k-1}^{p-1}z_{k-2}^p.$$

The claim follows, since $v^iy_{k-1}^{p-1}z_{k-2}^p$ is divisible by p if and only if v^iz_{k-2} is, by 1.5.

The analysis of (5.3) is extremely similar.

Now $S_{k,\ell}$ becomes involved. Let $S_{k,\ell}^K = \ker(\cdot p|S_{k,\ell})$ and $S_{k,\ell}^C = \operatorname{coker}(\cdot p|S_{k,\ell})$. Then, $S_{k,\ell}^K$ consists of $TP_{k+1}[v]\langle z_{1,\ell}\rangle$ plus $\mathbb{Z}_p$'s on $v^kz_{i,\ell}$ for $2\le i\le \ell-k$, while $S_{k,\ell}^C$ has $TP_{k+1}[v]\langle z_{\ell-k,\ell}\rangle$ plus $\mathbb{Z}_p$'s on $z_{i,\ell}$ for

[4] The class $y_2^{p-1}z_1^{p-1}$ should really be called $y_2^{p-1}z_1^{p-1} + v^{p^2(p-1)-1}z_3$ so that v times it is divisible by p, hence 0 in C_k^B, but we will ignore this fine-tuning.

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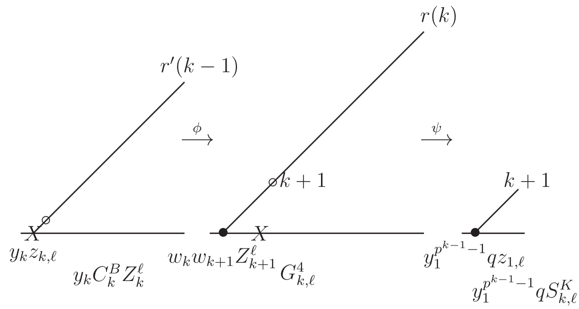

Figure 16. *Towers in exact sequence.*

$1 \le i < \ell - k$. Next, we consider the short exact sequence:

$$0 \to y_k C_k^B Z_k^\ell \xrightarrow{\phi} G_{k,\ell}^4 \xrightarrow{\psi} y_1^{p^{k-1}-1} q S_{k,\ell}^K \to 0. \tag{5.12}$$

The map ϕ sends everything except the v-tower on $y_k z_k Z_k^\ell$ to classes with the same name, and the heights of these v-towers agree, as seen above. The class $y_k z_k Z_k^\ell = y_k z_{k,\ell}$ maps to a $\mathbb{Z}_p$ with the same name in $k(1)^*(K_2)$. We have $\psi\left(w_k w_{k+1} Z_{k+1}^\ell\right) = q y_1^{p^{k-1}-1} z_{1,\ell}$. Then $v^{k+1} w_k w_{k+1} Z_{k+1}^\ell \in \ker(\psi)$, and we have

$$\phi\left(v y_k z_{k,\ell}\right) = v^{k+1} w_k w_{k+1} Z_{k+1}^\ell.$$

We illustrate this in the schematic Figure 16, in which X, $\circ$, and $\bullet$ map to elements with the same symbol. The expressions at the end of the v-towers are their v-heights. In particular, $v^{r'(k-1)} y_k z_{k,\ell} = 0$ in $y_k C_k^B Z_k^\ell$. The v-heights agree by (3.11), and the gradings match by an induction proof. The $\mathbb{Z}_p$'s in $y_1^{p^{k-1}-1} q S_{k,\ell}^K$ are hit by $\psi(y_k z_{i+k-1,\ell}) = y_1^{p^{k-1}-1} q v^k z_{i,\ell}$, $2 \le i \le \ell - k$, another interesting filtration jump.

Finally, we consider the short exact sequence:

$$0 \to y_1^{p^{k-1}-1} q S_{k,\ell}^C \xrightarrow{\phi'} G_{k,\ell}^5 \xrightarrow{\psi'} K_k^B z_\ell \to 0. \tag{5.14}$$

Similarly to (5.5), the generators of v-towers in K_k^B are g_k and, for $1 \le j < k$, elements of the form $g_j z_j^{p-1} \prod_{j+1}^{k-1} \left\{z_i^{p-1}, y_i^{p-1}\right\}$. The morphism ψ' is determined by $w_j \mapsto g_j$. The v-heights of the corresponding elements in $k(1)^*(K_2)$ and K_k^B both equal $r'(j-1)$ for $j < k$. However, the v-height of $w_k z_\ell$ is $r(k)$, which is k greater than $r'(k-1)$. We have $\phi'\left(v y_1^{p^{k-1}-1} q z_{\ell-k,\ell}\right) = v^{r'(k-1)} w_k z_\ell$. The class $y_1^{p^{k-1}-1} q z_{\ell-k,\ell}$ at the base of the v-tower maps to a $\mathbb{Z}_p$ with the same name. The picture is quite similar to Figure 16 with $k+1$ and $r'(k-1)$ interchanged.

The $\mathbb{Z}_p$ classes $y_1^{p^{k-1}-1} q z_{i,\ell}$ for $1 \le i < \ell - k$ are mapped by ϕ' to classes with the same name in $G_{k,\ell}^5 \subset k(1)^*(K_2)$. The $\mathbb{Z}_p$'s in $K_k^B z_\ell$ are of the same form as in (5.7) and are hit by analogs of (5.8).

6. All accounted for

In this section, we show that all elements of $k(1)^*(K_2)$ are involved in exactly one of the homomorphisms involving some G-group described in the preceding section. As discussed earlier, this implies that there can be no exotic extensions in $kup^*(K_2)$ other than those in (1.6), because an additional extension would decrease the number of elements in $\ker(\cdot p|kup^*(K_2))$ and $\operatorname{coker}(\cdot p|kup^*(K_2))$, and these must correspond to elements of $k(1)^*(K_2)$. It also provides an excellent check on our analysis.

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Let p be odd, G_k^i and $G_{k,\ell}^i$ as in (5.1) and (5.2), and

$$G^i = \begin{cases} \bigoplus\limits_{k\ge 1} G_k^i \otimes TP_{p-1}[y_k] \otimes P[y_{k+1}] & 1 \le i \le 2 \\ \bigoplus\limits_{1\le k<\ell} G_{k,\ell}^i \otimes TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes TP_{p-1}[z_\ell] \otimes \Lambda_{\ell+1} & 3 \le i \le 6 \\ \bigoplus\limits_{k\ge 1} \bigoplus\limits_{e=1}^{p-2} G_{k,e}^i \otimes P[y_k] \otimes \Lambda_{k+1} & 7 \le i \le 8. \end{cases}$$

Theorem 6.1. *$G^1 \oplus \cdots \oplus G^8$ equals $k(1)^*(K_2)$, as described in Theorem 3.16.*

As throughout the paper, $\mathbb{Z}_p$'s coming from E_1-free submodules of $H^*(K_2)$ are ignored here. The remainder of this section is devoted to the proof of Theorem 6.1. There are four parts of Theorem 3.16. We deal with them one at a time.

Case 1. $P[y_1]y_0^{p-1}z_0$. In (3.28), it is shown that these classes form a subset of $\bigoplus \mathcal{M}_k^A A_k$, and they map to classes with the same name in G^2.

Case 2. $\bigoplus_{j>0} TP_{r(j)}[v] \otimes P[y_{j+1}] \otimes TP_{p-1}\big[y_j\big] \otimes \overline{E}\big[w_j\big] \otimes E\big[w_{j+1}\big] \otimes \Lambda_{j+1}$. The generators of v-towers of height $r(j)$ occur in G^1, G^4, and G^5. From (5.6), only w_j is in G_j^1. So G^1 has $TP_{p-1}\big[y_j\big] \otimes P[y_{j+1}]w_j$. From Figure 16, $G_{j,\ell}^4$ has $w_j w_{j+1} Z_{j+1}^\ell$. Note that $\bigoplus_\ell Z_{j+1}^\ell TP_{p-1}[z_\ell] \otimes \Lambda_{\ell+1} = \Lambda_{j+1}$, since the ℓ-component gives the monomials whose smallest non-$(p-1)$-power is a power of z_ℓ, so G^4 contains $P[y_{j+1}] \otimes TP_{p-1}\big[y_j\big]w_j w_{j+1} \otimes \Lambda_{j+1}$. From the analysis following (5.14), $G_{j,\ell}^5$ has only $w_j z_\ell$ of v-height $r(j)$, so G^5 will have $P[y_{j+1}] \otimes TP_{p-1}\big[y_j\big]w_j \otimes \overline{\Lambda}_{j+1}$. Thus, $G^1 \oplus G^5$ contains the part without w_{j+1}, while G^4 contains the part with w_{j+1}.

Case 3. $\bigoplus_{j\ge 1} TP_{r'(j-1)}[v] \otimes P[y_j] \otimes E\big[w_j\big] \otimes \overline{TP}_p[z_j] \otimes \Lambda_{j+1}$. The generators of v-towers of height $r'(j-1)$ occur in each G^i as follows.

G^1: $w_j z_j^{p-1} \bigoplus\limits_{k\ge j+1} TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes \bigoplus\limits_{i=j+1}^{k-1} \{z_i^{p-1}, y_i^{p-1}\}$. This can be deduced from (5.6).

G^2: From (5.11),

$$z_j TP_{p-1}\big[y_j\big] \otimes P[y_{j+1}] \oplus y_j^{p-1} z_j \bigoplus_{k\ge j+1} TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes \prod_{i=j+1}^{k-1} \{z_i^{p-1}, y_i^{p-1}\}.$$

G^3: We use (5.5) and (5.6) and adapt some arguments used in Case 2 to obtain

$$w_j z_j^{p-1}\Big(\overline{TP}_p\big[y_j\big] \otimes P[y_{j+1}] \otimes \Lambda_{j+1} \oplus \bigoplus_{k\ge j+1} \overline{TP}_p[y_k]P[y_{k+1}]z_k^{p-1}\Lambda_{k+1} \prod_{i=j+1}^{k-1} \{z_i^{p-1}, y_i^{p-1}\}\Big).$$

G^4: We use (5.11) and (5.12) to obtain

$$y_j^{p-1} z_j \bigoplus_{k\ge j+1} \overline{TP}_p[y_k] \otimes P[y_{k+1}]z_k^{p-1}\Lambda_{k+1} \prod_{i=j+1}^{k-1} \{z_i^{p-1}, y_i^{p-1}\}.$$

G^5: We use (5.14) and $\bigoplus_{\ell>k} z_\ell TP_{p-1}[z_\ell] \otimes \Lambda_{\ell+1} \approx \overline{\Lambda}_{k+1}$ to obtain

$$w_j z_j^{p-1} \bigoplus_{k\ge j+1} TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes \overline{\Lambda}_{k+1} \otimes \prod_{i=j+1}^{k-1} \{z_i^{p-1}, y_i^{p-1}\}.$$

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G^6: We combine the analysis for G^2 and the observation used for G^5 to obtain

$$z_jTP_{p-1}\big[y_j\big]\otimes P[y_{j+1}]\otimes\overline{\Lambda}_{j+1}$$

$$\oplus\, y_j^{p-1}z_j\bigoplus_{k\geq j+1}TP_{p-1}[y_k]\otimes P[y_{k+1}]\otimes\overline{\Lambda}_{k+1}\otimes\prod_{i=j+1}^{k-1}\{z_i^{p-1},y_i^{p-1}\}$$

G^7: Similarly to G^3, we have

$$\bigoplus_{e=1}^{p-2}\left(w_jz_j^e\otimes P\big[y_j\big]\otimes\Lambda_{j+1}\oplus w_jz_j^{p-1}\bigoplus_{k\geq j+1}z_k^e\otimes P[y_k]\otimes\Lambda_{k+1}\otimes\prod_{i=j+1}^{k-1}\{z_i^{p-1},y_i^{p-1}\}\right).$$

G^8: Using (5.11), we get

$$\bigoplus_{e=1}^{p-2}\left(z_j^e\otimes P\big[y_j\big]\otimes\Lambda_{j+1}\oplus y_j^{p-1}z_j\bigoplus_{k\geq j+1}z_k^e\otimes P[y_k]\otimes\Lambda_{k+1}\otimes\prod_{i=j+1}^{k-1}\{z_i^{p-1},y_i^{p-1}\}\right).$$

We begin by analyzing the portion including the factor w_j. We will show that

$$G^1\oplus G^3\oplus G^5\oplus G^7=P\big[y_j\big]w_j\otimes\overline{TP}_p[z_j]\otimes\Lambda_{j+1}.$$

Here, and in the remainder of our analysis of Case 3, G^i refers just to the relevant portion of G^i, here the part with $TP_{r'(j-1)}[v]w_j$. The first part of G^7 gives all terms with z_j^e for $1\leq e\leq p-2$. The remaining part has factors $w_jz_j^{p-1}$, which we will omit writing. Combining G^1 and G^5 removes the bar in G^5. The first part of G^3 gives the part with positive exponent of y_j, which we now omit.

Let $E_\ell=P[y_\ell]\otimes\Lambda_\ell$, thought of as monomials in y_i and z_i for $i\geq\ell$ with exponents $\leq p-1$. The remaining parts of the G^i's under consideration combine to

$$\bigoplus_{k\geq j+1}\left(TP_{p-1}[y_k]\oplus y_kz_k^{p-1}TP_{p-1}[y_k]\oplus\bigoplus_{e=1}^{p-2}z_k^eTP_p[y_k]\right)\otimes E_{k+1}\otimes\prod_{i=j+1}^{k-1}\{z_i^{p-1},y_i^{p-1}\}. \tag{6.2}$$

We wish to show this equals E_{j+1}. The portion in parentheses is all monomials in $TP_p[y_k,z_k]$ except y_k^{p-1} and z_k^{p-1}. For a monomial M in E_{j+1}, let M_i denote its $y_i^sz_i^t$ factor. The k-summand in (6.2) is all monomials M in E_{j+1} for which k is the smallest i such that M_i is neither y_i^{p-1} nor z_i^{p-1}. Thus, the sum over all k yields all of E_{j+1}, as claimed.

A very similar argument shows that the $G^2\oplus G^4\oplus G^6\oplus G^8$ part for Case 3 equals the portion which includes just the 1 in $E[w_j]$, that is, $P\big[y_j\big]\otimes\overline{TP}_p[z_j]\otimes\Lambda_{j+1}$.

Case 4. $\bigoplus_{j\geq1}P[y_1]\otimes E[q]\otimes\overline{E}\big[z_j^p\big]\otimes\Lambda_{j+1}$. We first consider the part without the q, and fix j and omit writing the z_j^p. The desired answer is $P[y_1]\otimes\Lambda_{j+1}$. These come from the $\mathbb{Z}_p$'s in $G^2\oplus G^4\oplus G^6\oplus G^8$. Similarly to Case 3, G^2 and G^6 combine to give

$$\bigoplus_{k\geq j+1}TP_{p-1}[y_k]\otimes P[y_{k+1}]\otimes\Lambda_{k+1}\otimes\prod_{i=k+1}^{j-1}\{z_i^{p-1},y_i^{p-1}\}.$$

This, together with the portion of G^4 from $\operatorname{im}(\phi)$ in (5.12) obtained using (5.10), and the $\mathbb{Z}_p$'s in G^8 obtained using (5.10) give exactly (6.2), which we showed equals $P[y_{j+1}]\otimes\Lambda_{j+1}$.[5] The element X in Figure 16 with k replaced by j yields, from G^4,

$$y_jTP_{p-1}\big[y_j\big]\otimes P[y_{j+1}]\otimes\bigoplus_{\ell>j}Z_{j+1}^\ell TP_{p-1}[z_\ell]\otimes\Lambda_{\ell+1}$$

$$=y_jTP_{p-1}\big[y_j\big]\otimes P[y_{j+1}]\otimes\Lambda_{j+1},$$

[5] Here, the classes in (6.2) are $\mathbb{Z}_p$'s and are multiplied by z_j^p, whereas in Case 3 they were multiplied by $w_jz_j^{p-1}$ and were generators of v-towers of height $r'(j-1)$.

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which combines with the portion just obtained to yield $P[y_j] \otimes \Lambda_{j+1}$.

The last line of the $G^4_{k,\ell}$ discussion in Section 5 describes $\mathbb{Z}_p$'s in G^4 mapped by ψ in (5.12). Those with a z_j^p factor yield

$$\bigoplus_{k=1}^{j-1} y_k TP_{p-1}[y_k]P[y_{k+1}] \bigoplus_{\ell>j} Z^{\ell}_{j+1} TP_{p-1}[z_\ell]\Lambda_{\ell+1}$$
$$= \bigoplus_{k=1}^{j-1} (P[y_k] - P[y_{k+1}]) \otimes \Lambda_{j+1}$$
$$= (P[y_1] - P[y_j]) \otimes \Lambda_{j+1}.$$

Combining this with the result of the preceding paragraph yields the desired $P[y_1] \otimes \Lambda_{j+1}$.

We finish this section by showing that the $\mathbb{Z}_p$'s including a factor q are obtained exactly once. We omit writing the q. The classes which we must obtain are $P[y_1] \bigoplus_{j\ge 1} z_j^p \Lambda_{j+1}$. There are eight ways these appear in G^i-sets.

1. In G^1, using (5.7) and (5.8), for $1 \le i < j < k$,
$$y_1^{p^{j-1}-1} z_{i,j} z_j^{p-2} \prod_{s=j+1}^{k-1} \{z_s^{p-1}, y_s^{p-1}\} \otimes TP_{p-1}[y_k] \otimes P[y_{k+1}].$$
2. In G^3, using (5.9) and (5.8), for $1 \le i < k < \ell$,
$$y_1^{p^{k-1}-1} y_k z_{i,k} z_k^{p-2} Z^{\ell}_{k+1} \otimes TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes TP_{p-1}[z_\ell] \otimes \Lambda_{\ell+1}.$$
3. In G^3, using (5.7) and (5.8), for $1 \le i < j < k < \ell$,
$$y_1^{p^{j-1}-1} y_k z_{i,j} z_j^{p-2} \prod_{s=j+1}^{k-1} \{z_s^{p-1}, y_s^{p-1}\} Z^{\ell}_k \otimes TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes TP_{p-1}[z_\ell] \otimes \Lambda_{\ell+1}.$$
4. From im (ϕ') in (5.14), for $1 \le k < \ell$ and $1 \le i \le \ell - k$,
$$y_1^{p^{k-1}-1} z_{i,\ell} \otimes TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes TP_{p-1}[z_\ell] \otimes \Lambda_{\ell+1}.$$
5. From ψ' in (5.14), using (5.9) and (5.8), for $k < \ell$ and $\ell - k < i < \ell$,
$$y_1^{p^{k-1}-1} z_{i,\ell} \otimes TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes TP_{p-1}[z_\ell] \otimes \Lambda_{\ell+1}.$$
6. From ψ' in (5.14), using (5.7) and (5.8), for $i < j < k < \ell$,
$$y_1^{p^{j-1}-1} z_{i,j} z_j^{p-2} \prod_{s=j+1}^{k-1} \{z_s^{p-1}, y_s^{p-1}\} \cdot z_\ell \otimes TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes TP_{p-1}[z_\ell] \otimes \Lambda_{\ell+1}.$$
7. From (5.3), using (5.9) and (5.8), for $i < k$ and $1 \le e \le p-2$,
$$y_1^{p^{k-1}-1} z_{i,k} z_k^{e-1} P[y_k] \otimes \Lambda_{k+1}.$$
8. From (5.3), using (5.7) and (5.8), for $i < j < k$ and $1 \le e \le p-2$,
$$y_1^{p^{j-1}-1} z_{i,j} z_j^{p-2} \prod_{s=j+1}^{k-1} \{z_s^{p-1}, y_s^{p-1}\} \cdot z_k^e P[y_k] \otimes \Lambda_{k+1}.$$

First combine (1)+(6) to put a $\otimes\Lambda_{k+1}$ at the end of (1), and then, similarly to the simplification of (6.2), combine with (3)+(8) to get

$$\bigoplus_{i<j} y_1^{p^{j-1}-1} P[y_{j+1}] z_{i,j} z_j^{p-2} \Lambda_{j+1}. \tag{6.3}$$

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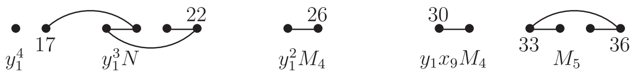

***Figure 17.** The E_1-module $\mathcal{M}_3$.*

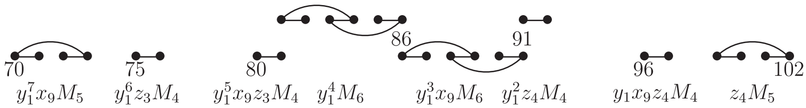

***Figure 18.** The E_1-module $\mathcal{M}_{3,4}$.*

We combine and relabel (4)+(5) to give

$$\bigoplus_{i<j} y_1^{p^{j-1}-1} TP_{p-1}\left[y_j\right] P[y_{j+1}] z_{i,j+1} \Lambda_{j+1} \tag{6.4}$$

together with

$$\bigoplus_{i\geq j\geq 1} y_1^{p^{j-1}-1} TP_{p-1}\left[y_j\right] P[y_{j+1}] z_i^p \Lambda_{i+1}. \tag{6.5}$$

Let $Y(s) = y_1^{p^s-1} TP_{p-1}[y_{s+1}]P[y_{s+2}] = \langle y_1^i : \nu(i+1) = s\rangle$. Then (6.5) is

$$\bigoplus_{i>s\geq 0} Y(s) z_i^p \Lambda_{i+1}. \tag{6.6}$$

We simplify and relabel (2) to

$$\bigoplus_{i<j} y_1^{p^{j-1}-1} y_j TP_{p-1}\left[y_j\right] P[y_{j+1}] z_{i,j} z_j^{p-2} \Lambda_{j+1}. \tag{6.7}$$

(6.3), (6.7), and (7) combine to give

$$\bigoplus_{i<j} y_1^{p^{j-1}-1} P\left[y_j\right] z_{i,j} TP_{p-1}[z_j] \Lambda_{j+1} = \bigoplus_{i\leq j-1\leq t} Y(t) z_{i,j} TP_{p-1}[z_j]\Lambda_{j+1}.$$

For any $t \geq i$, the coefficient of $Y(t)z_i^p$ in (6.4) plus this is

$$Z_{i+1}^{t+2}\Lambda_{t+2} \oplus \bigoplus_{j=i+1}^{t+1} Z_{i+1}^{j} TP_{p-1}[z_j]\Lambda_{j+1} = \Lambda_{i+1},$$

as the second part has all monomials not divisible by Z_{i+1}^{t+2}. Combining this with (6.6) yields the desired result,

$$\bigoplus_{s\geq 0} Y(s) \bigoplus_{i\geq 1} z_i^p \Lambda_{i+1}.$$

7. An explanation of self-duality of B_k

In this optional section, we discuss some observations about the ASS of $kup^*(K_2)$ and $kup_*(K_2)$ which, among other things, provide an explanation of the self-dual nature of the B_k summands which occur in both $kup^*(K_2)$ and $kup_*(K_2)$. We restrict to $p = 2$.

We first observe that, for $k \geq 1$, there is an E_1-submodule, $\mathcal{M}_k$, of $H^*(K_2)$ such that $\mathrm{Ext}_{E_1}(\mathbb{Z}_2, \mathcal{M}_k)$ (resp. $\mathrm{Ext}_{E_1}(\mathcal{M}_k, \mathbb{Z}_2)$) is closed under the differentials in the ASS converging to $kup^*(K_2)$ (resp. $kup_*(K_2)$), yielding the chart A_k (resp. the kup-homology analog of A_k discussed in Theorem 1.23). For example, with M_j as in (2.13) and N as in Figure 6, $\mathcal{M}_3$ is as depicted in Figure 17. The two ASSs for $\mathcal{M}_3$ will yield the charts for A_3 and its homology analog pictured in [5].

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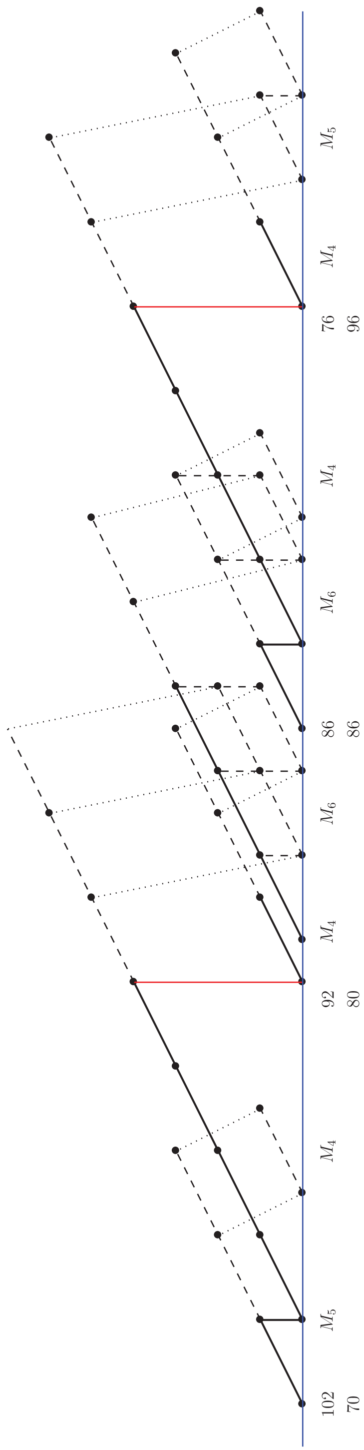

Figure 19. *Two ASSs for $\mathcal{M}_{2,3}$.*

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The situation for B_k is slightly more complicated. There is no E_1-submodule of $H^*(K_2)$ which, by itself, can give a chart $B_k z_\ell$. Some of the differentials that truncate v-towers in $B_k z_\ell$ come from classes that are part of a summand that includes $y_1^{2^{k-1}-1} qS_{k,\ell}$. We find that, for $2 \le k < \ell$, there is an E_1-submodule $\mathcal{M}_{k,\ell}$ of H^*K_2 such that $\mathrm{Ext}_{E_1}(\mathbb{Z}_2, \mathcal{M}_{k,\ell})$ is closed under the differentials in the ASS converging to $kup^*(K_2)$ and yields the chart:

$$B_k z_\ell \oplus y_1^{2^{k-1}-1} qS_{k,\ell} \oplus y_k B_k Z_k^\ell.$$

Note that these three subsets of $kup^*(K_2)$ appeared together in the 10-term exact sequence (5.2).

This $\mathcal{M}_{k,\ell}$ is symmetric, that is, there is an integer D such that $\Sigma^D \mathcal{M}_{k,\ell}^*$ and $\mathcal{M}_{k,\ell}$ are isomorphic E_1-modules, where $\mathcal{M}_{k,\ell}^*$ is obtained from $\mathcal{M}_{k,\ell}$ by negating gradings and dualizing Q_0 and Q_1. This implies that the v-towers in $\mathrm{Ext}_{E_1}(\mathbb{Z}_2, \mathcal{M}_{k,\ell})$ and $\mathrm{Ext}_{E_1}(\mathcal{M}_{k,\ell}, \mathbb{Z}_2)$ correspond nicely. Moreover, the differentials in the two ASSs correspond to obtaining isomorphic charts, although the gradings in one decrease from left to right, while in the other they increase.

We illustrate with an example, $\mathcal{M}_{3,4}$, and then discuss the implication for self-duality of B_k. In Figure 18, we depict $\mathcal{M}_{3,4}$.

In Figure 19, we depict the ASS chart for both $\mathrm{Ext}_{E_1}(\mathbb{Z}_2, \mathcal{M}_{3,4})$ and $\mathrm{Ext}_{E_1}(\mathcal{M}_{3,4}, \mathbb{Z}_2)$. They are isomorphic except that, from left to right, the gradings start with 102 for the first and 70 for the second. We label the portions of the chart corresponding to the eight summands of $\mathcal{M}_{3,4}$ just by the M-factor, since accompanying factors differ for the two versions. For example, the M_5 on the left-hand side is $z_4 M_5$ for the first spectral sequence and is $y_1^7 x_9 M_5$ for the second.

For the $kup^*(K_2)$ version, $B_3 z_4$ is on the left-hand side of Figure 19 and $y_3 B_3 z_3$ on the right-hand side, with $y_1^3 qS_{3,4}$ separating them. The duality isomorphism in Theorem 1.20 says that the Pontryagin dual of $B_3 z_4$ is isomorphic as a kup_*-module to Σ^4 of the right-hand side of the $kup_*(K_2)$ version of Figure 19, and we see that this is isomorphic to a shifted version of B_3 with indices negated. This is the self-duality statement that the Pontryagin dual of B_k is isomorphic as a kup_*-module to a shifted version of B_k with indices negated.

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