

On Lurie’s theorem and applications

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Introduction

One goal of spectral algebraic geometry is to unify common techniques from algebraic geometry and homotopy theory. An example of this is a deep theorem of Lurie which constructs the universal elliptic cohomology theory, originally constructed using chromatic homotopy theory, using the derived moduli stack of elliptic curves. This article discusses, proves, and applies *Lurie’s theorem* (Th.A)—a vast generalisation of the above statement for the moduli stack of elliptic curves to the moduli stack of *p-divisible groups*. Lurie’s theorem unifies many algebro-geometry constructions in stable homotopy theory, such as topological *K*-theory, topological modular and automorphic forms, Lubin–Tate theories, and various endomorphisms of these theories. The goal of this article is to address a proof of Lurie’s theorem (currently not available in the literature) and to advertise its utility with a handful of applications. To motivate the statement of Lurie’s theorem, let us recall Quillen’s theorem and the birth of elliptic cohomology.

Let $E^*(-)$ be a multiplicative cohomology theory, so the underlying spectrum E is a homotopy commutative ring spectrum. If $E^*(-)$ has a theory of Chern classes, meaning E has a *complex orientation* ([Ada74, §2]), then the formal spectrum $E^0\mathbf{CP}^\infty$ has the structure of a *formal group*. Quillen’s theorem is the remarkable fact that the cohomology theory MU of complex cobordism carries the universal complex orientation and the universal formal group lives over $\pi_*\mathrm{MU}$. This passage, from algebraic topology to arithmetic geometry has proven extremely useful in the homotopy theory, especially in providing global structure for the stable homotopy category; see [BB20].

The original constructions of *elliptic cohomology theories* were done in reverse. Given an elliptic curve \mathbf{X} over a ring R , its associated formal group $\widehat{\mathbf{X}}$ is an algebro-geometric construction ([Sil86, §IV]) resembling the Lie algebra of \mathbf{X} . If $\widehat{\mathbf{X}}$ is nicely behaved, one can apply the *Landweber exact functor theorem* (LEFT) to obtain an elliptic cohomology theory $E^*(-)$ with a complex orientation $\mathrm{MU} \rightarrow E$ such that its associated formal group can be identified with $\widehat{\mathbf{X}}$ through an isomorphism $E^0(*) \simeq R$; see [Lan88]. A ring spectrum E that can be constructed using the LEFT is said to be *Landweber exact*.

Such elliptic cohomology theories connect stable homotopy theory to arithmetic and algebraic geometry, provide examples of spectra of chromatic height 2, and also possess connections to physics, differential geometry, and the theory of genera. However, these theories also have their defects. For example, given an elliptic curve \mathbf{X} over a scheme S , one might hope to apply the LEFT to the restriction of \mathbf{X} to an open affine cover of S , and glue together the associated elliptic cohomology theories to obtain a cohomology theory “over the scheme S ”. This idea is not possible using the LEFT, as this theorem only produces a cohomology theory represented by an object in the stable homotopy category, and neither the category of cohomology theories nor the stable homotopy category have enough limits or colimits for gluing. This suggests that to adapt these algebro-geometric ideas to elliptic cohomology theories one needs to work with an enhanced version of the stable homotopy category. One suitable enhancement is the ∞ -category of spectra Sp and the associated ∞ -category of commutative algebras CAlg , whose

objects are known as \mathbf{E}_∞ -rings.

A major achievement in homotopy theory is a theorem of Goerss–Hopkins–Miller ([Goe10, Th.1.2]) which produces a functor of ∞ -categories \mathcal{O}^{top} from a category of elliptic curves to the ∞ -category of \mathbf{E}_∞ -rings. This functor does have the ability to glue together elliptic cohomology theories as \mathbf{E}_∞ -rings now, in other words, \mathcal{O}^{top} is a *sheaf*. For example, one can glue together all elliptic cohomology theories to obtain the universal theory TMF of *topological modular forms*; see [Beh20, §6].

In [SUR09], Lurie sketches an alternative construction of \mathcal{O}^{top} from the structure sheaf of a *derived moduli stack of oriented elliptic curves*, $\mathcal{M}_{\text{Ell}}^{\text{or}}$, and recently this construction was carried out in detail; see [EC2, §7]. This alternative construction uses different methods to those of Goerss–Hopkins–Miller, and also suggests a vast generalisation from elliptic curves to arbitrary *p-divisible groups*. The following is often referred to as *Lurie’s theorem*, which first appeared without proof in [BL10, Th.8.1.4]; see Th.1.6 for a more precise statement.

Theorem A. *Fix a prime p and an integer $n \geq 1$. There is a sheaf of \mathbf{E}_∞ -rings $\mathcal{O}_{\text{BT}_n}^{\text{top}}$ from a category of p -divisible groups of height n such that its value on a p -divisible group \mathbf{G} over a ring R is an \mathbf{E}_∞ -ring \mathcal{E} with the following properties:*

1. \mathcal{E} has a complex orientation and is Landweber exact.
2. There is an isomorphism of rings $\pi_0\mathcal{E} \simeq R$.
3. The homotopy groups $\pi_*\mathcal{E}$ vanishes for all odd integers and otherwise $\pi_{2k}\mathcal{E}$ is the k -fold tensor product of a line bundle on R .
4. There is an isomorphism between the formal group of the p -divisible group \mathbf{G} and the formal group of \mathcal{E} .

The sheaves $\mathcal{O}_{\text{BT}_n}^{\text{top}}$ are constructed using spectral algebraic geometry analogous to Lurie’s construction of \mathcal{O}^{top} . Interest in this theorem stems from its applications, all originally due to Lurie [EC2] or Behrens–Lawson [BL10], which we discuss in §5.

- The cohomology theory of *p-complete complex K-theory* KU_p can be recovered by applying $\mathcal{O}_{\text{BT}_1}^{\text{top}}$ to the multiplicative p -divisible group μ_{p^∞} over the p -adic integers \mathbf{Z}_p ; see §5.1. In fact, this reproduces KU_p as an \mathbf{E}_∞ -ring, and a variation also produces *p-complete real K-theory* KO_p .
- All of the *Lubin–Tate cohomology theories* associated to a perfect field κ and a formal group $\widehat{\mathbf{G}}$ of exact height n can be recovered from $\mathcal{O}_{\text{BT}_n}^{\text{top}}$; see §5.2. The functoriality of this construction with respect to automorphisms of formal groups recovers the action of the *extended Morava stabiliser group* on such \mathbf{E}_∞ -rings, as studied in [GH04].
- An elliptic curve E is an abelian variety of dimension 1 and its collection of p -power torsion produces a p -divisible group $E[p^\infty]$ of height two. Applying $\mathcal{O}_{\text{BT}_2}^{\text{top}}$ to the moduli

stack of elliptic curves produces the p -completion of the Goerss–Hopkins–Miller functor \mathcal{O}^{top} and also the p -completion of the universal elliptic cohomology theory TMF_p ; see §5.3.

- A more in-depth study of dimension g abelian varieties with PEL structure yields cohomology theories called *topological automorphic forms* due to Behrens–Lawson; see §5.4. The only known construction of these theories requires Th.A.
- Finally, the cohomology theories from Th.A come with functorality predicted by the LEFT; the sections of the sheaves $\mathcal{O}_{\text{BT}_n^p}^{\text{top}}$ have an action of the underlying p -divisible groups. Using this idea, we construct *stable Adams operations* (§5.5) and show these agree with the classical Adams operations for complex K -theory.

Outline

The proof of Lurie’s theorem found here is broadly based on Lurie’s construction of TMF in [EC2, §7]. In short, we want to define our sheaves using the instructive formula $\mathcal{O}_{\text{BT}_n^p}^{\text{top}} = \mathfrak{D}^* \Omega_* \mathcal{O}_{\text{BT}_n^p}^{\text{or}}$; our actual precise definition looks slightly different. In §2 we define \mathfrak{D} , in §3 we define Ω and $\mathfrak{D}_{\text{BT}_n^p}^{\text{or}}$, and in §4 we use this definition to prove Th.A. For some discussion on the technical background used in this proof, see the Conventions section below. The reader is also invited to the *Leitfaden* of the proof of Lurie’s theorem; see §1.2. In a little more detail, this paper is divided up into the following five sections (plus an appendix).

- (§1) We begin by introducing a precise statement of Lurie’s theorem and its supporting cast. This is followed by a *Leitfaden* for the proof, which gives a synopsis of the following three subsections.
- (§2) Here we focus on building some foundations for the phrase *formally étale* in spectral algebraic geometry; a manifestation of the deformation theory of Lurie ([SAG, §17-18]). These techniques are then used to lift classical to spectral p -divisible groups; see Th.2.34.
- (§3) Next, we explore the orientation theory for p -divisible groups à la [EC2, §4]. Using this we define a sheaf $\mathfrak{D}_{\text{BT}_n^p}^{\text{or}}$ which takes a p -divisible group over a p -complete \mathbf{E}_∞ -ring and produces its *orientation classifier*; see Df.3.13 and Pr.3.15.
- (§4) Finally, we define the sheaf $\mathcal{O}_{\text{BT}_n^p}^{\text{top}}$ by first applying the process of §2 followed by the sheaf $\mathfrak{D}_{\text{BT}_n^p}^{\text{or}}$ of §3. We are left to prove this sheaf satisfies the conditions of Th.A, and the arguments here follow those by Lurie in [EC2, §7.3].
- (§5) In this last section, we construct a variety of well-known \mathbf{E}_∞ -rings as well as some operations and actions thereof.
- (§A) In this appendix, we summarise some technical facts about formal spectral Deligne–Mumford stacks are used elsewhere in this article.

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Conventions

Now, and forever, fix a prime p .

Higher categories and higher algebra

We will make free and extensive use of the language of ∞ -categories, higher algebra, and spectral algebraic geometry, following [HTT09], [HA], [SAG], and especially the conventions listed in [EC2]. In particular:

- For an ∞ -category \mathcal{C} and two objects X and Y of \mathcal{C} , we will write $\mathrm{Map}_{\mathcal{C}}(X, Y)$ for the mapping space and $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ if \mathcal{C} happens to be the nerve of a 1-category.
- Commutative rings and abelian groups will be treated as discrete \mathbf{E}_{∞} -rings and spectra. Moreover, the smash product of spectra will be written as \otimes , even if the spectra involved are discrete (this does not mean the output will be discrete). The same goes for completions, and in this case the ∞ -categorical completions will be written as $(-)_I^{\wedge}$ following [SAG, §7].
- All module categories Mod_R refer to the stable ∞ -category of R -modules, where R is an \mathbf{E}_{∞} -ring. In particular, if R is a discrete commutative ring, then Mod_R will be the stable ∞ -category of R -module spectra, and **not** the abelian 1-category of R -modules. The same holds for ∞ -categories of quasi-coherent sheaves.
- Following [EC2] (and contrary to [SAG] and [EC1]), we will write $\mathrm{Spec} R$ for the non-connective spectral Deligne–Mumford stack associated to an \mathbf{E}_{∞} -ring R .

Moreover, all n -categories are $(n, 1)$ -categories, for $n = 1, 2, \infty$.

Sites and sheaves

Lurie’s theorem concerns sheaves between ∞ -categories. The ∞ -categories which we want to consider as sites are not necessarily (essentially) small, so we a priori do need to be careful

about potential size issues. However, we are interested in constructing particular functors and proving they are sheaves, so we only really need to step into a large universal to quantify our definition of a sheaf.

Definition 0.1. Given ∞ -category \mathcal{T} with a Grothendieck topology τ ([HTT09, Df.6.2.2.1]) and an ∞ -category \mathcal{C} then a functor $F: \mathcal{T}^{\text{op}} \rightarrow \mathcal{C}$ is a \mathcal{C} -valued τ -sheaf on \mathcal{T} if for all τ -sieves $\mathcal{T}_{/X}^0 \subseteq \mathcal{T}_{/X}$, the composite

$$\left((\mathcal{T}_{/X}^0)^{\triangleright} \right)^{\text{op}} \rightarrow \left((\mathcal{T}_{/X})^{\triangleright} \right)^{\text{op}} \rightarrow \mathcal{T}^{\text{op}} \xrightarrow{F} \mathcal{C}$$

is a limit diagram inside \mathcal{C} .

A *hypercouver* is a generalisation of a cover in a Grothendieck site. In general, our sheaves, including the sheaf occurring in the statement of Th.A will be hypersheaves. Following [SAG, §A], this variation on a sheaf comes with a more concrete description.

Definition 0.2 ([SAG, Df.A.5.7.1]). Let $\Delta_{s,+}$ denote the 1-category whose objects are linearly ordered sets of the form $[n] = \{0 < 1 < \dots < n\}$ for $n \geq -1$, and whose morphisms are strictly increasing functions. We will omit the $+$ when considering the full ∞ -subcategory with $n \geq 0$. If \mathcal{T} is an ∞ -category, we will refer to a functor $X_{\bullet}: \Delta_{s,+}^{\text{op}} \rightarrow \mathcal{T}$ as an *augmented semisimplicial object of \mathcal{T}* . When \mathcal{T} admits finite limits, then for each $n \geq 0$, we can associate to an augmented semisimplicial object X_{\bullet} the *n*th *matching object* and its associated *matching map*

$$X_n \rightarrow \lim_{[i] \hookrightarrow [n]} X_i = M_n(X_{\bullet})$$

where the limit above is taken over all injective maps $[i] \hookrightarrow [n]$ such that $i < n$. Given a collection of morphisms S inside \mathcal{T} , we call an augmented semisimplicial object X_{\bullet} is an *S-hypercover* (for $X_{-1} = X$) if all the natural matching maps belongs to S , for every $n \geq 0$. Given a Grothendieck topology τ on \mathcal{T} , then a presheaf of spectra F on \mathcal{T} is called a τ -*hypersheaf* if for all τ -hypercovers $X_{\bullet} \rightarrow X$, the natural map

$$F(X) \rightarrow \lim_{\Delta_s^{\text{op}}} F(X_{\bullet})$$

is an equivalence of spectra. Some useful general references for the prefix *hyper* in the homotopy theory of sheaves are [CM21], [DHI04], and [SAG, §A-D].

Our favourite examples will be when \mathcal{T} is the ∞ -category Aff^{cn} , Aff , \mathcal{C}_{A_0} , or \mathcal{C}_A , and S is either fpqc or étale covers. When we discuss these concepts with respect to \mathbf{E}_{∞} -rings, we will implicitly be talking about their opposite categories.

Given \mathcal{T} and τ from Df.0.2, then for each τ -covering family $\{C_i \rightarrow C\}$ in \mathcal{C} one can associate a Čech nerve C_{\bullet} which is a τ -hypercover of C . It is then clear that τ -hypersheaves are τ -sheaves. It is also obvious that if $S \subseteq S'$ then S' -hypersheaves are S -hypersheaves. We find the following diagram of implications useful, and they will often be used implicitly:

$$\begin{array}{ccc} \text{fpqc hypersheaf} & \implies & \text{fpqc sheaf} \\ \Downarrow & & \Downarrow \\ \text{étale hypersheaf} & \implies & \text{étale sheaf} \end{array}$$

Let us now state two useful lemmata regarding hypersheaves.

Lemma 0.3. *Let \mathcal{T} be an ∞ -category with a Grothendieck topology τ and let $F: \mathcal{T}^{\text{op}} \rightarrow \mathcal{C}at_{\infty/S}$ be a τ -sheaf such that the composite*

$$G: \mathcal{T}^{\text{op}} \xrightarrow{F} \mathcal{C}at_{\infty/S} \rightarrow \mathcal{C}at_{\infty}$$

is also a τ -sheaf, where the second functor is the canonical projection. Then the functor H defined by the composite

$$\mathcal{T}^{\text{op}} \xrightarrow{F} \mathcal{C}at_{\infty/S} \xrightarrow{\text{Un}} \mathcal{C}at_{\infty}$$

is a τ -sheaf. If F and G are τ -hypersheaves, then H is a τ -hypersheaf.

More informally, applying a Grothendieck construction to a sheaf is a sheaf.

Proof. Write $\coprod_{\alpha} C_{\alpha} \rightarrow C$ for a τ -cover of an object C in \mathcal{T} . We then note the following composite of natural equivalences is equivalent to the natural map $H(C) \rightarrow \lim H(C_{\alpha})$:

$$\begin{aligned} H(C) &= \text{Un}(F(C): G(C) \rightarrow S) \xrightarrow{\cong} \text{Un}(\lim F(C_{\alpha}): \lim G(C_{\alpha}) \rightarrow S) \\ &\xrightarrow{\cong} \lim \text{Un}(F(C_{\alpha}): G(C_{\alpha}) \rightarrow S) = \lim H(C_{\alpha}) \end{aligned}$$

The first equivalence comes from the fact that F and G are both τ -sheaves second equivalence from the fact that Un is a right adjoint. The proof for τ -hypersheaves is the same, with τ -covers replaced with τ -hypercovers. \square

Lemma 0.4 ([SAG, Cor.D.6.3.4 & Th.D.6.3.5]). *The identity functor $\text{CAlg} \rightarrow \text{CAlg}$ is a hypercomplete CAlg -valued sheaf (with respect to the fpqc topology). In particular, for any \mathbf{E}_{∞} -ring R and any fpqc hypercover R^{\bullet} of R , the natural map*

$$R \xrightarrow{\cong} \lim R^{\bullet}$$

is an equivalence.

Notice that if $R \rightarrow R^{\bullet}$ is an fpqc hypercover of an \mathbf{E}_{∞} -ring R , then there are natural equivalences

$$\tau_{\geq 0} R \xrightarrow{\cong} \tau_{\geq 0} \lim R^{\bullet} \xrightarrow{\cong} \lim \tau_{\geq 0} R^{\bullet} \tag{0.5}$$

from the above lemma and as $\tau_{\geq 0}$ commutes with limits as a right adjoint.

Topological rings and formal stacks

With experience, one knows that the study of deformation theory comes hand-in-hand with the study of rings with a topology and the associated algebraic geometry. We will follow the definition of an adic \mathbf{E}_{∞} -ring from [EC2, Df.0.0.11], except we will only consider the *connective* case.

Definition 0.6. An *adic ring* A is a discrete ring with a topology defined by an I -adic topology for some finitely generated ideal of definition $I \subseteq A$. Morphisms between adic rings are continuous ring homomorphisms, defining an ∞ -subcategory $\mathrm{CAlg}_{\mathrm{ad}}^{\heartsuit}$ of $\mathrm{CAlg}^{\heartsuit}$. An *adic \mathbf{E}_{∞} -ring* is a connective \mathbf{E}_{∞} -ring A such that $\pi_0 A$ is an adic ring. We define the *∞ -category of adic \mathbf{E}_{∞} -rings* as the following fibre product:

$$\mathrm{CAlg}_{\mathrm{ad}}^{\mathrm{cn}} = \mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{CAlg}^{\heartsuit}} \mathrm{CAlg}_{\mathrm{ad}}^{\heartsuit}$$

An adic \mathbf{E}_{∞} -ring A is said to be *complete* if it is complete with respect to an ideal of definition I ; see [SAG, Df.7.3.1.1 & Th.7.3.4.1]. An \mathbf{E}_{∞} -ring R is *local* if $\pi_0 R$ is a local ring, and we call an adic \mathbf{E}_{∞} -ring R *local* if the topology on $\pi_0 R$ is defined by the maximal ideal of $\pi_0 R$. We give $\mathrm{CAlg}_{\mathrm{ad}}^{\heartsuit}$ and $\mathrm{CAlg}_{\mathrm{ad}}^{\mathrm{cn}}$ the usual Grothendieck topologies (fpqc, étale, etc.) via the forgetful functors to $\mathrm{CAlg}^{\heartsuit}$ and $\mathrm{CAlg}^{\mathrm{cn}}$, respectively.

The geometric definition of a formal (spectral) Deligne–Mumford stack follows.

Definition 0.7. Let $\mathrm{Spf}: \mathrm{CAlg}_{\mathrm{ad}}^{\mathrm{cn}} \rightarrow \infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}^{\mathrm{sHen}}$ be the functor described in [SAG, Con.8.1.1.10 & Pr.8.1.2.1]. A spectrally ringed ∞ -topos \mathfrak{X} is said to be an *affine formal spectral Deligne–Mumford stack* if it lies in the essential image of Spf . A *formal spectral Deligne–Mumford stack* is a spectrally ringed ∞ -topos with a cover by affine formal spectral Deligne–Mumford stacks; see [SAG, Df.8.1.3.1]. Let fSpDM denote the full ∞ -subcategory of $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}^{\mathrm{loc}}$ spanned by formal spectral Deligne–Mumford stacks. Similarly, one can define a 2-category fDM of classical formal Deligne–Mumford stacks (Df.A.6) where we further assume all such objects are *locally Noetherian*.

Definition 0.8. Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathfrak{X}})$ be a formal spectral Deligne–Mumford stack. We call an object U inside \mathcal{X} *affine* if the locally spectrally ringed ∞ -topos $(\mathcal{X}|_U, \mathcal{O}_{\mathfrak{X}}|_U)$ is equivalent to $\mathrm{Spf} A$ for some adic \mathbf{E}_{∞} -ring A . We will also say that \mathfrak{X} is *locally Noetherian* if for every affine object U of \mathcal{X} , the \mathbf{E}_{∞} -ring $\mathcal{O}_{\mathfrak{X}}(U)$ is Noetherian in the sense of [HA, Df.7.2.4.30].

Note that $\mathrm{Spf} B$ is locally Noetherian if and only if B itself is a Noetherian \mathbf{E}_{∞} -ring; see [SAG, Pr.8.4.2.2].

Notation 0.9 (Fixed adic \mathbf{E}_{∞} -ring A). Let A denote some fixed complete local Noetherian adic \mathbf{E}_{∞} -ring with perfect residue field of characteristic p . Write A_0 for $\pi_0 A$, \mathfrak{m}_A for the maximal ideal of A_0 , and κ_A for the residue field.

The reader should keep in her mind the initial case of the p -complete sphere $A = \mathbf{S}_p$ with associated A_0 the p -adic integers \mathbf{Z}_p . Other choices include the spherical Witt vectors of a perfect field of characteristic p ; see [EC2, §5.1].

Functor of points

The classical moduli stack $\mathcal{M}_{\mathrm{BT}^p}^{\heartsuit}$ is neither a Deligne–Mumford nor an Artin stack. This necessitates our use of a functorial point of view, for both classical and spectral (formal) algebraic geometry.

Notation 0.10. Write $\text{Aff} = \text{CAlg}^{\text{op}}$ to which we will add super/subscripts such as $(-)^{\text{cn}}$, $(-)_{\text{ad}}$, and $(-)^{\heartsuit}$ as they apply to CAlg .

When working in $\mathcal{P}(\text{Aff}^{\heartsuit})$ or $\mathcal{P}(\text{Aff}^{\text{cn}})$, we will abuse notation and not distinguish between the objects representing functors and the functors themselves. This is justified by the following commutative diagram of fully faithful functors of ∞ -categories:

$$\begin{array}{ccccc}
 \text{Aff}_{\text{loc.N}}^{\heartsuit} & \xrightarrow{\hspace{10em}} & \text{Aff}^{\text{cn}} & & \\
 \downarrow (a) & \searrow & \downarrow (c) & \searrow & \\
 & \text{Aff}_{\text{ad,loc.N}}^{\heartsuit} & \xrightarrow{\hspace{10em}} & \text{Aff}_{\text{ad}}^{\text{cn}} & \\
 & \downarrow & \downarrow & \downarrow & \\
 \text{DM}_{\text{loc.N}} & \xrightarrow{\hspace{10em}} & \text{SpDM} & & \\
 & \searrow (b) & \searrow (d) & & \\
 & \text{fDM} & \xrightarrow{\hspace{10em}} & \text{fSpDM} & \longrightarrow \mathcal{P}(\text{Aff}^{\text{cn}})
 \end{array} \tag{0.11}$$

The loc.N subscript denotes those full ∞ -subcategories spanned by Noetherian or locally Noetherian objects; see Df.0.8. The definitions and fully faithfulness of the functors above are explained in Cor.A.10, except the functors (a)-(d), which can be justified as follows:

- (a) is fully faithful as this holds without the locally Noetherian hypotheses; see [SAG, Rmk.1.2.3.6] and restrict to the underlying 2-category.
- (b) is fully faithful by using part (d) below and Pr.A.9. Indeed, if $G \circ F$ and G are fully faithful, then so is F .
- (c) is fully faithful by making a connective version of [SAG, Rmk.1.4.7.1]; this is justified by [SAG, Cor.1.4.5.3].
- (d) is fully faithful as both SpDM and fSpDM being defined as full ∞ -subcategories of $\infty\text{Top}_{\text{CAlg}}^{\text{loc}}$ and the fact that spectral Deligne–Mumford stacks are examples of formal spectral Deligne–Mumford stacks by [SAG, p. 628].

Similarly, we will consider most of classical algebraic geometry as living in the 2-category $\text{Fun}(\text{CAlg}^{\heartsuit}, \mathcal{S}_{\leq 1})$ which we then embed inside the ∞ -category $\mathcal{P}(\text{Aff}^{\heartsuit})$ using the inclusion $\mathcal{S}_{\leq 1} \rightarrow \mathcal{S}$, which preserves limits.

Warning 0.12 (Quasi-coherent sheaves on formal spectral Deligne–Mumford stacks). When we consider quasi-coherent sheaves on a formal spectral Deligne–Mumford stack \mathfrak{X} , then what we write as $\text{QCoh}(\mathfrak{X})$ is what Lurie would write as $\text{QCoh}(h_{\mathfrak{X}})$, in other words, we consider the ∞ -categories of quasi-coherent sheaves of formal spectral Deligne–Mumford stacks through their functors of points. By [SAG, Cor.8.3.4.6], we see that these two notations are equivalent as long as one restricts to *almost connective* quasi-coherent sheaves on both sides. As all of our quasi-coherent sheaves of interest will be cotangent complexes, which are almost connective by definition ([SAG, Df.17.2.4.2]), this distinction does not matter to us.

Cotangent complexes

Given a natural transformation $X \rightarrow Y$ between functors in $\mathcal{P}(\text{Aff}^{\text{cn}})$ which admits a cotangent complex ([SAG, Df.17.2.4.2]), we will write this cotangent complex as $L_{X/Y}$ and consider it as an object of $\text{QCoh}(X)$; see [SAG, §6.2]. A few specific cases can be made more explicit.¹

1. If $\mathfrak{X} \rightarrow \mathfrak{Y}$ is a morphism of spectral Deligne–Mumford stacks and $X \rightarrow Y$ is the associated transformation of functors in $\mathcal{P}(\text{Aff}^{\text{cn}})$, then $L_{X/Y}$ is equivalent to $L_{\mathfrak{X}/\mathfrak{Y}}$ under the equivalence of categories $\text{QCoh}(X) \simeq \text{QCoh}(\mathfrak{X})$ by [SAG, Cor.17.2.5.4]. If $\mathfrak{X} = \text{Spec } B$ and $\mathfrak{Y} = \text{Spec } A$, then we have further identifications of $L_{X/Y}$ with $L_{B/A}$ under the equivalence of ∞ -categories $\text{QCoh}(\text{Spec } A) \simeq \text{Mod}_A$; see [SAG, Lm.17.1.2.5].
2. If \mathfrak{X} is a formal spectral Deligne–Mumford stack, and X is the associated functor in $\mathcal{P}(\text{Aff}^{\text{cn}})$, then L_X is equivalent to $L_{\mathfrak{X}}^\wedge$, the *completed cotangent complex* of [SAG, Df.17.1.2.8], under the equivalence of categories $\Theta_{\mathfrak{X}}: \text{QCoh}(\mathfrak{X})^{\text{acn}} \xrightarrow{\simeq} \text{QCoh}(X)^{\text{acn}}$ of [SAG, Cor.8.3.4.6], where the superscript *acn* indicates full ∞ -subcategories of almost connective objects. If $\mathfrak{X} = \text{Spf } A$ for an adic \mathbf{E}_∞ -ring A , then $L_{\text{Spf } A} \simeq (L_A)_I^\wedge$ (under the equivalence of ∞ -categories $\text{QCoh}(\text{Spf } A) \simeq \text{Mod}_A^{\text{Cpl}}$, where I is a finitely generated ideal of definition for the topology on $\pi_0 A$; see [SAG, Ex.17.1.2.9]).
3. If $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ is a morphism of formal Deligne–Mumford stacks and $F: X \rightarrow Y$ is the associated morphism of functors in $\mathcal{P}(\text{Aff}^{\text{cn}})$, then the cofibre $L_{\mathfrak{X}/\mathfrak{Y}}^\wedge$ of the natural map $f^* L_{\mathfrak{Y}} \rightarrow L_{\mathfrak{X}}$ is naturally equivalent to $L_{X/Y}$ under the equivalence of categories $\Theta_{\mathfrak{X}}: \text{QCoh}(\mathfrak{X})^{\text{acn}} \xrightarrow{\simeq} \text{QCoh}(X)^{\text{acn}}$; see [SAG, Df.17.1.2.8] for a definition of $L_{\mathfrak{X}/\mathfrak{Y}}^\wedge$. Indeed, the naturality of $\Theta_{\mathfrak{X}}$ in \mathfrak{X} ([SAG, Con.8.3.4.1]) yields an equivalence $\Theta_{\mathfrak{X}} \circ f^* \simeq F^* \circ \Theta_{\mathfrak{Y}}$ of functors. Our desired identification then follows from the existence of the (co)fibre sequences

$$f^* L_{\mathfrak{Y}} \rightarrow L_{\mathfrak{X}} \rightarrow L_{\mathfrak{X}/\mathfrak{Y}} \quad F^* L_{\mathfrak{Y}} \rightarrow L_X \rightarrow L_{X/Y},$$

the absolute case (2), and the fact that $\text{QCoh}(\mathfrak{X})^{\text{acn}}$ and $\text{QCoh}(X)$ are stable under (co)fibre sequences; see [SAG, Cor.8.2.4.13 & Pr.6.2.3.4], respectively.

Due to the equivalences above, we will drop the completion symbol from our notation for the cotangent complex between formal spectral Deligne–Mumford stacks. The following standard properties of the cotangent complex of functors will be used without explicit reference:

- For a map of connective \mathbf{E}_∞ -rings $A \rightarrow B$, we have a natural equivalence in $\text{Mod}_{\pi_0 B}$

$$\pi_0 L_{B/A} \simeq \Omega_{\pi_0 B/\pi_0 A}^1;$$

see [HA, Pr.7.4.3.9].

¹Thank you to an anonymous referee for vastly simplifying example 3 for us.

- For composable transformations of functors $X \rightarrow Y \rightarrow Z$ in $\mathcal{P}(\text{Aff}^{\text{cn}})$, where each functor (or each transformation) has a cotangent complex, we obtain a canonical (co) fibre sequence in $\text{QCoh}(X)$

$$L_{Y/Z}|_X \rightarrow L_{X/Z} \rightarrow L_{X/Y};$$

see [SAG, Pr.17.2.5.2].

- If we have transformations $X \rightarrow Y \leftarrow Y'$ of functors inside $\mathcal{P}(\text{Aff}^{\text{cn}})$, where $L_{X/Y}$ exists, then $L_{X \times_Y Y'/Y'}$ exists and is naturally equivalent to $\pi_1^* L_{X/Y}$; see [SAG, Rmk.17.2.4.6].

Warning 0.13 (Topological vs algebraic cotangent complexes). The cotangent complexes considered in this article are **not** the same as those developed by André and Quillen; see [Sta, 08P5]. In particular, for an ordinary commutative ring R considered as a discrete \mathbf{E}_∞ -ring, then L_R is what some call the *topological cotangent complex*. For more discussion, see [SAG, §25.3].

Deformation theory

We will be using ideas from classical deformation theory as well as Lurie's spectral deformation theory, so we take a moment here to clarify our definitions. What we discuss below is mostly taken from [EC2, §3].

Definition 0.14. Let \mathbf{G}_0 be a p -divisible group over a commutative ring R_0 and write $\text{CAlg}_{\text{ad}}^{\text{Cpl}}$ for the ∞ -subcategory of $\text{CAlg}_{\text{ad}}^{\text{cn}}$ spanned by complete connective adic \mathbf{E}_∞ -rings. Define a functor $\text{Def}_{\mathbf{G}_0} : \text{CAlg}_{\text{ad}}^{\text{Cpl}} \rightarrow \mathcal{S}$ by the formula

$$\text{Def}_{\mathbf{G}_0}(A) = \text{colim}_I \text{BT}^p(A) \times_{\text{BT}^p(\pi_0 A/I)} \text{Hom}_{\text{CRing}}(R_0, \pi_0 A/I)$$

where the colimits is indexed over all finitely generated ideals of definition I for $\pi_0 A$. A priori an ∞ -category, but [EC2, Lm.3.1.10] states this is an ∞ -groupoid. Let (R, \mathbf{G}) be a *deformation*² of \mathbf{G}_0 . We say \mathbf{G} is the *universal spectral deformation* of \mathbf{G}_0 with *spectral deformation ring* A if for every B in $\text{CAlg}_{\text{ad}}^{\text{Cpl}}$, the natural map

$$\text{Map}_{\text{CAlg}_{\text{ad}}^{\text{Cpl}}}(A, B) \xrightarrow{\cong} \text{Def}_{\mathbf{G}_0}(B)$$

is an equivalence. If R is discrete, we say \mathbf{G} is the *universal classical deformation* of \mathbf{G}_0 with *classical deformation ring* A if for every discrete B in $\text{CAlg}_{\text{ad}}^{\text{Cpl}}$, the natural map

$$\text{Map}_{\text{CAlg}_{\text{ad}}^{\text{Cpl}}}(A, B) \xrightarrow{\cong} \text{Def}_{\mathbf{G}_0}(B)$$

is an equivalence. If such universal spectral (or classical) deformations (R, \mathbf{G}) exist, they are evidently uniquely determined by the pair (R_0, \mathbf{G}_0) .

²Recall from [EC2, Df.3.1.4], a *deformation* of \mathbf{G}_0 is an adic \mathbf{E}_∞ -ring A , a finitely generated ideal of definition I of $\pi_0 A$, a ring homomorphism $R_0 \rightarrow \pi_0 A/I$, and an isomorphism of p -divisible groups $(\mathbf{G}_0)_{\pi_0 A/I} \simeq \mathbf{G}_{\pi_0 A/I}$. In other words, an object of $\text{Def}_{\mathbf{G}_0}(A)$.

The above definition agrees with that in [EC2, Df.3.1.11] in the cases that the A above is connective. Indeed, in this case, if B is a nonconnective complete adic \mathbf{E}_∞ -ring, the fact connective cover is a right adjoint and $\mathrm{BT}^p(B) = \mathrm{BT}^p(\tau_{\geq 0}B)$ by definition, we obtain the following:

$$\mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ad}}}(R, B) \simeq \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ad}}}(R, \tau_{\geq 0}B) \simeq \mathrm{Def}_{\mathbf{G}_0}(\tau_{\geq 0}B) \simeq \mathrm{Def}_{\mathbf{G}_0}(B)$$

The following will help us identify many classical deformation rings.

Remark 0.15. If a spectral deformation ring R exists for a pair (R_0, \mathbf{G}_0) , then a classical deformation ring also does, and it can be taken to be $\pi_0 R$. Indeed, if B is a discrete object of $\mathrm{CAlg}_{\mathrm{ad}}^{\mathrm{Cpl}}$ as in Df.0.14, then the fact the truncation functor is a left adjoint on connective objects yields the equivalences

$$\mathrm{Def}_{\mathbf{G}_0}(B) \simeq \mathrm{Map}_{\mathrm{CAlg}_{\mathrm{ad}}^{\mathrm{Cpl}}}(R, B) \simeq \mathrm{Map}_{\mathrm{CAlg}_{\mathrm{ad}}^{\mathrm{Cpl}}}(\pi_0 R, B)$$

showing that $\pi_0 R$ is the classical deformation ring of (R_0, \mathbf{G}_0) .

1 The statement of Lurie's theorem

The titular theorem promises the existence of a sheaf $\mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}$ on some site over the classical moduli stack of p -divisible groups satisfying certain properties. The idea behind the construction of $\mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}$ is to construct morphisms of stacks

$$\mathcal{M}_{\mathrm{BT}_n^p}^{\mathrm{or}} \xrightarrow{\Omega} \mathcal{M}_{\mathrm{BT}_n^p}^{\mathrm{un}} \xleftarrow{\mathfrak{D}} \mathcal{M}_{\mathrm{BT}_n^p}^{\heartsuit},$$

set $\mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}} = \mathfrak{D}^* \Omega_* \mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{or}}$, and check this possesses the desired properties. The maps of stacks above do not quite exist in our set-up, but the above formula for $\mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}$ is instructive. In this section, we state a precise version of Lurie's theorem and give a more detailed outline of the proof; in §2 we construct \mathfrak{D} (short for *deformation*) using spectral deformation theory disguised as the adjective *formally étale*; in §3 we construct Ω (short for *orientation*) using the orientation theory of Lurie; and in §4 we define $\mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}$ and check it satisfies the conditions of Lurie's theorem.

1.1 The precise statement

First, let us recall the definition of a p -divisible group over an \mathbf{E}_∞ -ring; see [EC2, Df.2.0.2] for this definition, and [EC1, §6] or [EC3, §2] for a wider discussion.

Definition 1.1. Let R be a connective \mathbf{E}_∞ -ring. A p -divisible (Barsotti–Tate) group over a connective \mathbf{E}_∞ -ring R is a functor $\mathbf{G}: \mathrm{CAlg}_R^{\mathrm{cn}} \rightarrow \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ with the following properties:

1. For every connective \mathbf{E}_∞ - R -algebra B , the \mathbf{Z} -module $\mathbf{G}(B)[1/p]$ vanishes.
2. For every finite abelian p -group M , the functor

$$\mathrm{CAlg}_R^{\mathrm{cn}} \rightarrow \mathcal{S} \quad B \mapsto \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(M, \mathbf{G}(B))$$

is corepresented by a finite flat \mathbf{E}_∞ - R -algebra.

3. The map $p: \mathbf{G} \rightarrow \mathbf{G}$ is locally surjective with respect to the finite flat topology.

A p -divisible group over a general \mathbf{E}_∞ -ring R , is a p -divisible group over its connective cover. The ∞ -category $\mathrm{BT}^p(R)$ of p -divisible groups over an \mathbf{E}_∞ -ring R is the full ∞ -subcategory of $\mathrm{Fun}(\mathrm{CAlg}_{\tau_{\geq 0}^{\mathrm{cn}} R}, \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}})$ spanned by p -divisible groups. Let $\mathcal{M}_{\mathrm{BT}^p}$ be the *moduli stack of p -divisible groups*, which is the functor inside $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$ defined on objects by sending R to the ∞ -groupoid core $\mathrm{BT}^p(R)^\simeq$; see [EC2, Df.3.2.1]. We say a p -divisible group \mathbf{G} has *height n* if the \mathbf{E}_∞ - R -algebra corepresenting the functor

$$\mathrm{CAlg}_R^{\mathrm{cn}} \rightarrow \mathcal{S} \quad B \mapsto \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\mathbf{Z}/p\mathbf{Z}, \mathbf{G}(B))$$

is finite of rank p^n ; see [EC1, §6.5]. Using this notion of height, we can further define a subfunctor $\mathcal{M}_{\mathrm{BT}^p_n}$ for all $n \geq 1$ consisting of all p -divisible groups of *height n* .

The reader is invited to check for herself that the definition above agrees with that of [Tat67, §2] when R is discrete.

Remark 1.2 (Height is an open condition). We claim $\mathcal{M}_{\mathrm{BT}^p_n} \rightarrow \mathcal{M}_{\mathrm{BT}^p}$ is an open embedding. Lurie’s definition of a commutative finite flat group scheme over $\mathrm{Spec} R$ ([EC1, Df.6.1.1]) states that $\mathbf{G}(\mathbf{Z}/p\mathbf{Z}) \simeq \mathrm{Spec} B$ is affine and $\pi_0 R \rightarrow \pi_0 B$ realises $\pi_0 B$ as a projective $\pi_0 A$ -module of finite rank equal to p^n . By [Sta, 00NX], this rank is locally (with respect to the Zariski topology on $|\mathrm{Spec} R| = |\mathrm{Spec} \pi_0 R|$) constant. In particular, if R is a local connective \mathbf{E}_∞ -ring then the commutative finite flat group scheme $\mathbf{G}(\mathbf{Z}/p\mathbf{Z})$ has a well-defined height and we obtain the formula:

$$\mathrm{Spec} R \times_{\mathcal{M}_{\mathrm{BT}^p}} \mathcal{M}_{\mathrm{BT}^p_n} \simeq \begin{cases} \mathrm{Spec} R & \mathrm{ht}(\mathbf{G}) = n \\ \emptyset & \mathrm{ht}(\mathbf{G}) \neq n \end{cases}$$

Definition 1.3. Let \mathfrak{X} be a formal spectral Deligne–Mumford stack. A p -divisible group over \mathfrak{X} is a natural transformation $\mathbf{G}: \mathfrak{X} \rightarrow \mathcal{M}_{\mathrm{BT}^p}$ in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$. We say \mathbf{G} has *height n* if this map factors through $\mathcal{M}_{\mathrm{BT}^p_n}$. By [EC2, Pr.3.2.2(4)], this is equivalent to a coherent family of p -divisible groups \mathbf{G}_{B_i} on $\mathrm{Spec}(B_i)_{J_i}^\wedge$, where the collection $\{\mathrm{Spf} B_i \rightarrow \mathfrak{X}\}_i$ form an affine étale cover of \mathfrak{X} and J_i is an ideal of definition for B_i .

Our main object of study is the spectral moduli stack $\mathcal{M}_{\mathrm{BT}^p_n}$, although we are also interested in its relationship to the underlying classical moduli stack.

Notation 1.4. For a functor $\mathcal{M}: \mathrm{CAlg}^{\mathrm{cn}} \rightarrow \mathcal{S}$, write \mathcal{M}^\heartsuit for its restriction along $\mathrm{CAlg}^\heartsuit \rightarrow \mathrm{CAlg}^{\mathrm{cn}}$. This commutes with finite products:

$$(X \times Y)^\heartsuit \xrightarrow{\simeq} X^\heartsuit \times Y^\heartsuit$$

Given an adic \mathbf{E}_∞ -ring B , write $\widehat{\mathcal{M}}_B$ for the product $\widehat{\mathcal{M}}_B = \mathcal{M} \times \mathrm{Spf} B$ in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$. The hat indicates a base-change over Spf , rather than Spec .

We can now define the sites upon which we will soon define our sheaves of \mathbf{E}_∞ -rings. Adjectives used below that have not yet been defined will be discussed after Th.1.6.

Definition 1.5. Recall the conventions of Nt.0.9, so fix a complete local Noetherian adic \mathbf{E}_∞ -ring A with perfect residue field of characteristic p . Let

$$\mathcal{C}_{A_0} \subseteq \mathcal{P}(\mathrm{Aff}^\heartsuit)_{/\widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A_0}^\heartsuit}$$

denote the full ∞ -subcategory spanned by those objects $\mathbf{G}_0: \mathfrak{X}_0 \rightarrow \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A_0}^\heartsuit$ where \mathfrak{X}_0 is a locally Noetherian qcqs³ formal Deligne–Mumford stack with perfect residue fields⁴ at all closed points, the cotangent complex⁵ $L_{\mathfrak{X}_0/\widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A_0}^\heartsuit}$ is *almost perfect*⁶ inside $\mathrm{QCoh}(\mathfrak{X}_0)$, and \mathbf{G}_0 is formally étale in $\mathcal{P}(\mathrm{Aff}^\heartsuit)$. Similarly, let

$$\mathcal{C}_A \subseteq \mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})_{/\widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A}}$$

denote the full ∞ -subcategory spanned by those objects $\mathbf{G}: \mathfrak{X} \rightarrow \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A}$ where \mathfrak{X} is a locally Noetherian qcqs formal spectral Deligne–Mumford stack with perfect residue fields at all closed points, and \mathbf{G} is formally étale in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$. We will endow \mathcal{C}_{A_0} and \mathcal{C}_A with both the fpqc and étale topologies through the forgetful map to $\mathcal{P}(\mathrm{Aff}^\heartsuit)$ and $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$, respectively.

A simplified criterion for an object $\mathfrak{X} \rightarrow \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A_0}^\heartsuit$ to lie in \mathcal{C}_{A_0} is discussed in Pr.1.8. The precise version of Lurie’s theorem (Th.A) can now be stated.

Theorem 1.6 (Lurie’s theorem). *Given an adic \mathbf{E}_∞ -ring A as in Nt.0.9, there is an étale hypersheaf of \mathbf{E}_∞ -rings $\mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}$ on \mathcal{C}_{A_0} such that for a formal affine $\mathbf{G}_0: \mathrm{Spf} B_0 \rightarrow \widehat{\mathcal{M}}_{\mathrm{BT}_n^p}^\heartsuit$ in \mathcal{C}_{A_0} the \mathbf{E}_∞ -ring $\mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}(\mathbf{G}_0) = \mathcal{E}$ has the following properties:*

1. \mathcal{E} is complex periodic⁷ and Landweber exact.⁸
2. There is a natural equivalence of rings $\pi_0 \mathcal{E} \simeq B_0$ and \mathcal{E} is complete with respect to an ideal of definition for B_0 . In particular, \mathcal{E} is \mathfrak{m}_A -complete, hence also p -complete.

³A locally Noetherian and quasi-compact scheme is called a Noetherian scheme. We choose to keep these two adjectives separate though, as they play different roles in this article.

⁴As our fixed A is assumed to be p -complete, all these residue fields are necessarily of characteristic p .

⁵This relative cotangent complex exists as one does for \mathfrak{X}_0 and $\widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A_0}^\heartsuit$ —a consequence of [SAG, Pr.17.2.5.1] and [EC2, Pr.3.2.2], respectively.

⁶Paraphrasing [SAG, §6.2.5], recall that a quasi-coherent sheaf \mathcal{F} on a functor $X: \mathrm{CAlg}^{\mathrm{cn}} \rightarrow \mathcal{S}$ is *almost perfect* if for all connective \mathbf{E}_∞ -rings R and all morphisms of presheaves $\eta: \mathrm{Spec} R \rightarrow X$, the R -module $\eta^* \mathcal{F}$ is almost perfect; see [HA, Df.7.2.4.10 & Pr.7.2.4.17] for the latter definition and a simple criterion for Noetherian \mathbf{E}_∞ -rings.

⁷Recall from [EC2, §4.1], that an \mathbf{E}_∞ -ring A is called *complex periodic* if A is *complex orientable* and *weakly 2-periodic*. An object E of $\mathrm{Sp}_{\mathbf{S}_/}$ is said to be *complex orientable* if the map given map $e: \mathbf{S} \rightarrow E$ admits a factorisation \bar{e} :

$$\mathbf{S} \simeq \Sigma^{-2} S^2 \simeq \Sigma^{-2} \mathbf{CP}^1 \rightarrow \Sigma^{-2} \mathbf{CP}^\infty \xrightarrow{\bar{e}} E;$$

see [Ada74, §II] or [EC2, §4.1.1]. An \mathbf{E}_∞ -ring A is *weakly 2-periodic* if $\Sigma^2 A$ is a locally free A -module of rank 1, or equivalently, that $\pi_2 A$ is a locally free $\pi_0 A$ -module of rank 1 and the natural map $\pi_2 A \otimes_{\pi_0 A} \pi_{-2} A \rightarrow \pi_0 A$ is an equivalence. Notice this is a **condition**, not data.

⁸A formal group $\widehat{\mathbf{G}}$ over a ring R is *Landweber exact* if the defining map from $\mathrm{Spec} R$ to the moduli stack of formal groups is flat. A complex periodic \mathbf{E}_∞ -ring is *Landweber exact* if its associated Quillen formal group is.

3. The groups $\pi_k \mathcal{E}$ vanish for all odd integers k . Otherwise, there are natural equivalences of B_0 -modules $\pi_{2k} \mathcal{E} \simeq \omega_{\mathbf{G}_0}^{\otimes k}$ where $\omega_{\mathbf{G}_0}^{\otimes k}$ is the dualising line⁹ of the identity component¹⁰ \mathbf{G}_0° of \mathbf{G}_0 .
4. There is a natural equivalence of formal groups $\mathbf{G}_0^\circ \simeq \widehat{\mathbf{G}}_{\mathcal{E}}^{\mathcal{Q}_0}$ over B_0 where the former is the identity component of \mathbf{G}_0 and the latter is the classical Quillen formal group¹¹ of \mathcal{E} .

We have included a few more details than in the original statement ([BL10, Th.8.1.4]) by incorporating some work of Behrens–Lawson involving Landweber exactness.

For transparency, let us explain the adjectives in the definition of \mathcal{C}_A and \mathcal{C}_{A_0} .

(Locally Noetherian) We assume our formal Deligne–Mumford stacks are locally Noetherian (Df.0.8) because completions of general rings in the classical world and derived world do not agree; see [SAG, Warn.8.1.0.4]. Moreover, even in the world of spectral algebraic geometry such objects are better behaved ([SAG, §8.4]), such as the existence of truncations; see Pr.A.1.

(Qcqs) This acronym stands for *quasi-compact and quasi-separated*; see Df.A.13. When a scheme X is qcqs, then it has a Zariski cover $\mathrm{Spec} A \rightarrow X$ (qc) and the fibre product $P = \mathrm{Spec} A \times_X \mathrm{Spec} A$ is also a Zariski cover $\mathrm{Spec} B \rightarrow P$ (qs). Eventually, we will define an étale (hyper) sheaf $\mathfrak{D}_{\mathrm{BT}_p}^{\mathrm{aff}}$ on the affine objects of \mathcal{C}_A , and to extend this to a formal Deligne–Mumford stack \mathfrak{X} inside \mathcal{C}_A , one will use the adjective qcqs; see Rmk.3.14. One could write this article again, with the word *separated* replacing the word *quasi-separated* and deleting all occurrences of the prefix *hyper*, although the extra generality of hypersheaves can be useful in practice.

(Formal geometry) One reason we work with *formal* spectral Deligne–Mumford stacks (§A and [SAG, §8]) is related to topological modular forms. In this case, one must appeal to the classical Serre–Tate theorem where one works with schemes where p is locally nilpotent, ie, over $\mathrm{Spf} \mathbf{Z}_p$; see Ex.2.7. Another, somewhat disjoint reason is for deformation theoretic purposes. As stated in [EC2, Rmk.3.2.7]:

“The central idea in the proof of Theorem 3.1.15 (of [EC2]) is (...) to guarantee the representability of $\mathcal{M}_{\mathrm{BT}^p}$ in a formal neighborhood of any sufficiently nice R -valued point.”

⁹Recall from [EC2, §4.2.5], the *dualising line* of a formal group $\widehat{\mathbf{G}}$ over a commutative ring R is the R -linear dual of its Lie algebra $\mathrm{Lie}(\widehat{\mathbf{G}})$. This *Lie algebra* of a formal group can be defined in multiple ways, but we will define it as the tangent space of $\widehat{\mathbf{G}}$ over R at the unit section $\mathcal{O}_{\widehat{\mathbf{G}}} \rightarrow R$; see [Zin84] for a discussion about Lie algebras associated to formal groups or here for an English translation.

¹⁰Recall from [EC2, Th.2.0.8], for each p -divisible group \mathbf{G} over a p -complete \mathbf{E}_∞ -ring R there is a unique formal group \mathbf{G}° over R such that on connective $\mathbf{E}_{\infty\text{-}\tau_{\geq 0}} R$ -algebras A which are truncated and p -nilpotent we can describe $\mathbf{G}^\circ(A)$ as the fibre of $\mathbf{G}(A) \rightarrow \mathbf{G}(A^{\mathrm{red}})$ induced by the quotient by the nilradical; see [Tat67, (2.2)] for a classical reference.

¹¹Recall from [EC2, Con.4.1.13], that a complex periodic \mathbf{E}_∞ -ring A comes with an associated *Quillen formal group* $\widehat{\mathbf{G}}_A^{\mathcal{Q}}$ over A . The *classical Quillen formal group* $\widehat{\mathbf{G}}_A^{\mathcal{Q}_0}$ is the image of $\widehat{\mathbf{G}}_A^{\mathcal{Q}}$ under the functor $\mathrm{FGroup}(A) \rightarrow \mathrm{FGroup}(\pi_0 A)$, or equivalently as the formal spectrum $\mathrm{Spf} A^0 \mathbf{CP}^\infty$. Notice the above definition is **independent** of the choice of complex orientation for A —that would yield a chosen coordinate for our formal group, ie, a *formal group law*; see [Goe08, §2].

As our moduli stack of interest is $\mathcal{M}_{\text{BT}^p}$, we embrace formal spectral algebraic geometry.

(Closed points have perfect residue fields) A crucial step in showing our definition of $\mathcal{O}_{\text{BT}_n^p}^{\text{top}}$ satisfies the conditions of Th.1.6 is to reduce ourselves to the closed points of the affine objects of \mathcal{C}_{A_0} , essentially reducing us to the Lubin–Tate theories of [EC2, §5]. It will then be important that these residue fields are perfect (they will already be of characteristic p as we are working over $\text{Spf } \mathbf{Z}_p$) to apply some of our formal arguments; see Pr.2.42.

(Formally étale over $\widehat{\mathcal{M}}_{\text{BT}_n^p}$) Again, one inspiration for Th.1.6 is the classical Serre–Tate theorem, which posits that $\widehat{\mathcal{M}}_{\text{Ell}, \mathbf{Z}_p}^{\heartsuit}$ is formally étale over $\widehat{\mathcal{M}}_{\text{BT}_2^p, \mathbf{Z}_p}^{\heartsuit}$. The phrase formally étale is used in this article to control and package our deformation theory; see §2.

(Cotangent complex conditions in \mathcal{C}_{A_0}) These conditions are finiteness hypotheses, however, they are necessary to apply a deep existence criterion of Lurie (Th.2.39).

Let us now discuss a simple criterion for checking if an object lies in \mathcal{C}_{A_0} .

Definition 1.7. A morphism $f: \mathfrak{X}_0 \rightarrow \text{Spf } A_0$ of classical formal Deligne–Mumford stacks is *locally of finite presentation* if for all étale morphisms $\text{Spf } B_0 \rightarrow \mathfrak{X}_0$, the induced morphisms of rings $A_0 \rightarrow B_0$ are of finite presentation. By the usual arguments, it suffices to check this on a fixed collection of étale morphisms $\text{Spf } B_0 \rightarrow \mathfrak{X}_0$ which cover \mathfrak{X}_0 . We say f is of *finite presentation* if f is locally of finite presentation and quasi-compact (Df.A.13).

Proposition 1.8. *Let A be as in Nt.0.9 and $\mathbf{G}_0: \mathfrak{X}_0 \rightarrow \widehat{\mathcal{M}}_{\text{BT}_n^p, A_0}^{\heartsuit}$ be a p -divisible group defined on a formal Deligne–Mumford stack \mathfrak{X}_0 of finite presentation over $\text{Spf } A_0$ such that the associated map into $\widehat{\mathcal{M}}_{\text{BT}_n^p, A_0}^{\heartsuit}$ is formally étale. Then \mathbf{G}_0 lies in \mathcal{C}_{A_0} .*

These simplified hypotheses are practical, but they do not apply to one of our favourite examples, Lubin–Tate theory, as power series rings $R[[x]]$ are simply **never** of finite presentation over R .

Proof. First we note that \mathfrak{X}_0 is locally Noetherian, qcqs, and has all residue fields corresponding to closed points perfect of characteristic p as the morphism $\mathfrak{X}_0 \rightarrow \text{Spf } A_0$ is of finite presentation.¹² It remains to show that the cotangent complex in question,

$$L = L_{\mathfrak{X}_0/\widehat{\mathcal{M}}_{\text{BT}_n^p, A}}$$

is almost perfect. To see this, we consider the composition in $\mathcal{P}(\text{Aff}^{\text{cn}})$

$$\mathfrak{X}_0 \xrightarrow{\mathbf{G}_0} \widehat{\mathcal{M}}_{\text{BT}_n^p, A} \xrightarrow{\pi_2} \text{Spf } A$$

¹²Indeed, for locally Noetherian one can use [Sta, 00FN], for qcqs one can use [GW10, §D], and the residue fields are perfect as finite field extensions of perfect fields are perfect by [Sta, 05DU].

which induces the following (co) fibre sequence in $\mathrm{QCoh}(\mathfrak{X}_0)$:

$$\mathbf{G}_0^* L_{\widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A}/\mathrm{Spf} A} \rightarrow L_{\mathfrak{X}_0/\mathrm{Spf} A} \rightarrow L$$

Abbreviating the above to $\mathbf{G}_0^* L_1 \rightarrow L_2 \rightarrow L$, we first focus on $\mathbf{G}_0^* L_1$. As a quasi-coherent sheaf on a formal spectral Deligne–Mumford stack \mathfrak{X}_0 , to see $\mathbf{G}_0^* L_1$ is almost perfect, it suffices to see that $\eta^* \mathbf{G}_0^* L_1$ is almost perfect inside $\mathrm{QCoh}(X)$ for every morphism $\eta: X \rightarrow \mathfrak{X}_0$ where X is a spectral Deligne–Mumford stack; see [SAG, Th.8.3.5.2]. Using the base-change equivalence

$$L_1 = L_{\widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A}/\mathrm{Spf} A} \simeq \pi_1^* L_{\mathcal{M}_{\mathrm{BT}_n^p}}$$

it suffices to show $L'_1 = \eta^* \mathbf{G}_0^* \pi_1^* L_{\mathcal{M}_{\mathrm{BT}_n^p}}$ is almost perfect. By [SAG, Cor.8.3.5.3], it suffices to check the affine case of $X = \mathrm{Spec} R$, where R is a connective \mathbf{E}_{∞} -ring. Note p is nilpotent in $\pi_0 R$ as $\mathrm{Spec} R$ maps into $\mathrm{Spf} A$, and $p \in \mathfrak{m}_A$ by assumption; see Nt.0.9. Our conclusion that L'_1 is almost perfect in Mod_R then follows from [EC2, Pr.3.2.5]. Therefore, $\mathbf{G}_0^* L_1$ is almost perfect.

Focusing on L_2 now, we consider the composition $\mathfrak{X}_0 \rightarrow \mathrm{Spf} A_0 \rightarrow \mathrm{Spf} A$ and the induced (co) fibre sequence of quasi-coherent sheaves over \mathfrak{X}_0 :

$$L_{\mathrm{Spf} A_0/\mathrm{Spf} A}|_{\mathfrak{X}_0} \rightarrow L_{\mathfrak{X}_0/\mathrm{Spf} A} = L_2 \rightarrow L_{\mathfrak{X}_0/\mathrm{Spf} A_0} \quad (1.9)$$

By Pr.A.12, we see $L_{\mathrm{Spf} A_0/\mathrm{Spf} A}$ is almost perfect in $\mathrm{QCoh}(\mathrm{Spf} A_0)$, and pullbacks preserve almost perfectness ([SAG, Cor.8.4.1.6]), hence the first term of (1.9) is almost perfect. To see the third term of (1.9) is almost perfect, we may work locally and replace \mathfrak{X}_0 with $\mathrm{Spf} B_0$ where B_0 is a complete discrete adic ring. In this case we use the assumption that $A_0 \rightarrow B_0$ is of finite presentation, which implies L_{B_0/A_0} is almost perfect in Mod_{B_0} ; see [HA, Th.7.4.3.18]. By [SAG, Pr.7.3.5.7], L_{B_0/A_0} is complete with respect to an ideal of definition J for B_0 , and it follows the B_0 -module

$$L_{B_0/A_0} \simeq (L_{B_0/A_0})_J^\wedge \simeq L_{\mathrm{Spf} B_0/\mathrm{Spf} A_0}$$

is almost perfect. Therefore L_2 is almost perfect, so L itself is almost perfect. \square

1.2 Leitfaden of the proof of Th.1.6

Our proof moves in three distinct, but connected, stages.

- (I) First, we move from classical algebraic geometry (in $\mathcal{P}(\mathrm{Aff}^\heartsuit)$) to spectral algebraic geometry (in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$) using deformation theory, presented here through the adjective *formally étale*. Given an object $\mathbf{G}_0: \mathfrak{X}_0 \rightarrow \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A_0}^\heartsuit$ inside \mathcal{C}_{A_0} , we consider the object X inside the following Cartesian diagram in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$:

$$\begin{array}{ccc} X & \longrightarrow & \tau_{\leq 0}^* \mathfrak{X}_0 \\ \downarrow \mathbf{G} & & \downarrow \mathbf{G}_0 \\ \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A} & \xrightarrow{f} & \tau_{\leq 0}^* \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A_0}^\heartsuit \end{array}$$

The functor $\tau_{\leq 0}^* \mathcal{P}(\text{Aff}^\heartsuit) \rightarrow \mathcal{P}(\text{Aff}^{\text{cn}})$ above is induced by precomposition with $\tau_{\leq 0}: \text{CAlg}^{\text{cn}} \rightarrow \text{CAlg}^\heartsuit$, and the maps $X(R) \rightarrow \tau_{\leq 0}^* X(R) = X(\pi_0 R)$ are induced by the truncation map $R \rightarrow \pi_0 R$. The assumption that \mathbf{G}_0 was formally étale in $\mathcal{P}(\text{Aff}^\heartsuit)$ implies that X is what Lurie calls the *de Rham space* of the map $\mathfrak{X}_0 \rightarrow \widehat{\mathcal{M}}_{\text{BT}_n^p, A}$ and that \mathbf{G} is formally étale; see Pr.2.35. Most of the adjectives defining \mathcal{C}_{A_0} then allow us to employ a powerful representability theorem of Lurie (Th.2.39), which identifies X as a formal spectral Deligne–Mumford stack, which we denote as \mathfrak{X} . Some analysis shows $\mathbf{G}: \mathfrak{X} \rightarrow \widehat{\mathcal{M}}_{\text{BT}_n^p, A}$ lies in \mathcal{C}_A and that the functor

$$\mathfrak{D}: \mathcal{C}_{A_0} \rightarrow \mathcal{C}_A, \quad (\mathfrak{X}_0, \mathbf{G}_0) \mapsto (\mathfrak{X}, \mathbf{G})$$

is an equivalence of ∞ -categories (Th.2.34).

- (II) Next, we apply the orientation theory of p -divisible groups devised by Lurie in [EC2]. This yields a moduli stack of *oriented p -divisible groups* $\mathcal{M}_{\text{BT}_n^p}^{\text{or}}$ and a map of presheaves on p -complete \mathbf{E}_{∞} -rings

$$\Omega: \mathcal{M}_{\text{BT}_n^p}^{\text{or}} \rightarrow \mathcal{M}_{\text{BT}_n^p}^{\text{un}};$$

see Df.3.6. The bulk of this section is formalising a global form of the constructions of [EC2, §4] and constructing the pushforward presheaf along Ω of the structure sheaf of $\mathcal{M}_{\text{BT}_n^p}^{\text{or}}$, which when restricted to \mathcal{C}_A becomes the functor $\mathfrak{D}_{\text{BT}_n^p}^{\text{or}}: \mathcal{C}_A^{\text{op}} \rightarrow \text{CAlg}$. It will follow rather formally that applying $\mathfrak{D}_{\text{BT}_n^p}^{\text{or}}$ to an affine object of \mathcal{C}_A yields the *orientation classifier* construction of Lurie; see [EC2, §4.3.3].

- (III) Finally, we set $\mathcal{O}_{\text{BT}_n^p}^{\text{top}}$ to be the composition of \mathfrak{D} followed by $\mathfrak{D}_{\text{BT}_n^p}^{\text{or}}$. In other words, we first send $(\mathfrak{X}_0, \mathbf{G}_0)$ to $(\mathfrak{X}, \mathbf{G})$ using \mathfrak{D} , and then take the orientation classifier of the identity component of \mathbf{G} ; see Df.4.1. To check this definition of $\mathcal{O}_{\text{BT}_n^p}^{\text{top}}$ satisfies the properties described in Th.1.6, we use descent ideas of Lurie.

The following three sections carry out these three steps given above.

2 Formally étale natural transformations

At the heart of spectral algebraic geometry is deformation theory—Lurie ([SAG, p.1385]) even goes as far as to state the heuristic principle:

$$\{\text{spectral algebraic geometry}\} = \{\text{classical algebraic geometry}\} + \{\text{deformation theory}\}$$

The adjective *formally étale* will help us navigate between the two worlds of classical and spectral algebraic geometry using Lurie’s spectral deformation theory. More concretely, given a (nice enough) formally étale morphism $\mathfrak{X}_0 \rightarrow \mathcal{M}$, where \mathfrak{X}_0 is a classical formal stack, there is a universal spectral deformation of \mathfrak{X}_0 , say \mathfrak{X} , such that \mathfrak{X}_0 can be viewed as the 0th truncation of \mathfrak{X} . This process allows us to lift objects in classical algebraic geometry to spectral algebraic geometry without changing the underlying classical object; see Th.2.34.

2.1 On presheaves of discrete rings

Let us first consider formally étale maps between presheaves of discrete rings.

Definition 2.1. A natural transformation $f: X \rightarrow Y$ of functors in $\mathcal{P}(\text{Aff}^\heartsuit)$ is said to be *formally étale* if, for all surjective maps of rings $\tilde{R} \rightarrow R$ whose kernel is square-zero, also called *square-zero extensions* of R , the following natural diagram of spaces is Cartesian:

$$\begin{array}{ccc} X(\tilde{R}) & \longrightarrow & X(R) \\ \downarrow & & \downarrow \\ Y(\tilde{R}) & \longrightarrow & Y(R) \end{array}$$

Moreover, we say that f is *formally unramified* if the fibres of the map

$$X(\tilde{R}) \rightarrow X(R) \times_{Y(R)} Y(\tilde{R})$$

are either empty or contractible.

Let us state some classical, useful, and also formal properties of formally étale morphisms; the reader may enjoy verifying them herself.

Proposition 2.2. *Formally étale morphisms in $\mathcal{P}(\text{Aff}^\heartsuit)$ are closed under composition. If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are composable morphisms in $\mathcal{P}(\text{Aff}^\heartsuit)$ such that g is formally unramified and gf is formally étale, then f is formally étale. Formally étale (resp. unramified) morphisms are closed under base-change.*

Let us now relate Df.2.1 to the definitions found in classical algebraic geometry.

Definition 2.3. A map $f: X \rightarrow Y$ between functors in $\mathcal{P}(\text{Aff}^\heartsuit)$ is *affine* if every ring R , and every R -point $\eta \in Y(R)$, the fibre product $\text{Spec } R \times_Y X$ is represented by an affine scheme.

Note that maps between (functors represented by) affines in $\mathcal{P}(\text{Aff}^\heartsuit)$ are always affine, as the Yoneda embedding $\text{Aff}^\heartsuit \rightarrow \mathcal{P}(\text{Aff}^\heartsuit)$ preserves limits.

Proposition 2.4. *Let $f: X \rightarrow Y$ be a natural transformation of functors in $\mathcal{P}(\text{Aff}^\heartsuit)$. Then f is formally étale if and only if for every ring R , every square-zero extension of rings $\tilde{R} \rightarrow R$, and every commutative diagram of the form*

$$\begin{array}{ccc} \text{Spec } R & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \text{Spec } \tilde{R} & \longrightarrow & Y \end{array} \tag{2.5}$$

the mapping space

$$\text{Map}_{\mathcal{P}(\text{Aff}^\heartsuit)_{R/Y}}(\text{Spec } \tilde{R}, X)$$

is contractible.¹³ Moreover, if f is affine, then f is formally étale if and only if for every ring A , and every A -point $\eta \in Y(A)$ such that the fibre product $\text{Spec } A \times_Y X$ is equivalent to any affine scheme $\text{Spec } B$, the natural projection map $A \rightarrow B$ is formally étale as a map of rings.¹⁴

Proof. Given a ring R , a square-zero extension $\tilde{R} \rightarrow R$, and a commutative diagram (2.5), consider the following diagram of spaces:

$$\begin{array}{ccccc}
\text{Map}_{R/Y}(\tilde{R}, X) & \longrightarrow & \text{Map}_{/Y}(\tilde{R}, X) & \longrightarrow & \text{Map}_{/Y}(R, X) \\
\downarrow & & \downarrow & & \downarrow \\
\text{Map}_{R/}(\tilde{R}, X) & \longrightarrow & \text{Map}(\tilde{R}, X) & \longrightarrow & \text{Map}(R, X) \\
\downarrow & & \downarrow & & \downarrow \\
\text{Map}_{R/}(\tilde{R}, Y) & \longrightarrow & \text{Map}(\tilde{R}, Y) & \longrightarrow & \text{Map}(R, Y)
\end{array}$$

By definition, the rows and columns are fibre sequences,¹⁵ we have abbreviated the categories above to express only the over/under categories, and we suppressed the functor Spec . By the Yoneda lemma, the bottom-right square is naturally equivalent to (2.5), hence f is formally étale if and only if this bottom right square is Cartesian. In turn, this is equivalent to the space in the top-left corner being contractible.

For the “moreover” statement, suppose that f is affine. If f is formally étale, then Pr.2.2 states that the map $\text{Spec } B \rightarrow \text{Spec } A$ is formally étale by base-change. Conversely, suppose we are given a diagram of the form (2.5), then by assumption the fibre product $\text{Spec } \tilde{R} \times_X Y \simeq \text{Spec } B$ is affine and $\text{Spec } B \rightarrow \text{Spec } \tilde{R}$ is formally étale, giving us the following diagram:

$$\begin{array}{ccc}
\text{Spec } R & & \\
\swarrow & \searrow & \\
& \text{Spec } B & \longrightarrow X \\
& \downarrow & \downarrow f \\
& \text{Spec } \tilde{R} & \longrightarrow Y
\end{array}$$

One then observes the following sequence of natural equivalences of spaces

$$\begin{aligned}
\text{Map}_{R/\tilde{R}}(\tilde{R}, B) &\simeq \text{Map}_{R/}(\tilde{R}, B) \times_{\text{Map}_{R/}(\tilde{R}, \tilde{R})} \{\text{id}_{\tilde{R}}\} \\
&\simeq \text{Map}_{R/}(\tilde{R}, X) \times_{\text{Map}_{R/}(\tilde{R}, Y)} \text{Map}_{R/}(\tilde{R}, \tilde{R}) \times_{\text{Map}_{R/}(\tilde{R}, \tilde{R})} \{\text{id}_{\tilde{R}}\} \simeq \text{Map}_{R/Y}(\tilde{R}, X)
\end{aligned}$$

¹³Ie, “there exists a unique lift $\text{Spec } \tilde{R} \rightarrow X$ for (2.5).”

¹⁴For the definition of a formally étale map of rings simply apply Df.2.1 to the transformation (co)representing this map of rings, or see [Sta, 02HF].

¹⁵The fibres in this diagram have been taken with respect to the maps from (2.5).

where we have used the same abbreviations from earlier in the proof. The first space above is contractible as $\mathrm{Spec} B \rightarrow \mathrm{Spec} \tilde{R}$ is formally étale, hence f is formally étale as the last space is contractible. \square

Let us list some instances of formally étale morphisms found in algebraic geometry.

Example 2.6 (Formally étale morphisms of schemes). In the setting of classical algebraic geometry, we usually take the existence of a unique map $\mathrm{Spec} \tilde{R} \rightarrow X$ (under $\mathrm{Spec} R$ and over Y) as the definition of a formally étale maps of rings (or schemes); see Pr.2.4. An object in $\mathcal{P}(\mathrm{Aff}^\heartsuit)$ represented by a scheme factors through $\mathrm{Fun}(\mathrm{CAlg}^\heartsuit, \mathrm{Set})$, as mapping spaces between classical schemes are discrete, and we see Pr.2.4 precisely matches [Sta, 02HG].

Example 2.7 (Classical Serre–Tate theorem). The classical Serre–Tate theorem (see [CS15, p.854] for the original source, or [EC1, Th.7.0.1] for statement of the spectral version) states that if $\tilde{R} \rightarrow R$ is a square-zero extension of commutative rings and p is nilpotent within them, then the diagram of 1-groupoids

$$\begin{array}{ccc} \mathrm{AVar}_g(\tilde{R})^\simeq & \longrightarrow & \mathrm{AVar}_g(R)^\simeq \\ \downarrow [p^\infty] & & \downarrow [p^\infty] \\ \mathrm{BT}_{2g}^p(\tilde{R})^\simeq & \longrightarrow & \mathrm{BT}_{2g}^p(R)^\simeq \end{array} \quad (2.8)$$

is Cartesian. This implies the morphism of classical moduli stacks $[p^\infty]: \mathcal{M}_{\mathrm{AVar}_g}^\heartsuit \rightarrow \mathcal{M}_{\mathrm{BT}_{2g}^p}^\heartsuit$ sending an abelian variety X to its associated p -divisible group $X[p^\infty]$ ([Tat67, §2]) is formally étale **after** base-change over $\mathrm{Spf} \mathbf{Z}_p$. This base-change is crucial, as there only exists a map $\mathrm{Spec} R \rightarrow \mathrm{Spf} \mathbf{Z}_p$ is when p is nilpotent inside R , as the continuous map of rings $\mathbf{Z}_p \rightarrow R$ must send $\{p^i\}_{i \geq 0}$ to a convergent sequence in R , where R is equipped with the discrete topology. If we fail to make this base-change, then (2.8) may not be Cartesian.¹⁶

Example 2.9 (Classical Lubin–Tate theorem). Another classical example of a formally étale map in $\mathcal{P}(\mathrm{Aff}^\heartsuit)$ comes from Lubin–Tate theory. The original source for this is [LT66] with respect to formal groups, but we will follow [EC2, §3] as our intended application is for p -divisible groups; see [EC2, Ex.3.0.5] for a statement of the dictionary between deformations of formal and p -divisible groups. Let \mathbf{G}_0 be a p -divisible groups of height $0 < n < \infty$ over a

¹⁶Indeed, consider the elliptic curve E over $\overline{\mathbf{F}}_3$ defined by the equation $y^2 = x^3 + x^2 + x + 1$. The 2^k -torsion subgroups of E are, by [KM85, Th.2.3.1], equivalent to the constant group schemes $(\mathbf{Z}/2^k\mathbf{Z})^2$ over $\overline{\mathbf{F}}_3$, hence the associated 2-divisible group $E[2^\infty]$ is equivalent to the constant 2-divisible group $(\mathbf{Q}_2/\mathbf{Z}_2)^2$ over $\overline{\mathbf{F}}_3$. Define two deformations E_1 and E_2 of E over the dual numbers $\overline{\mathbf{F}}_3[\epsilon]$ (augmented by the morphism $\epsilon \mapsto 0$), by the formulae

$$E_1 : y^2 = x^3 + x^2 + x + 1 + \epsilon, \quad E_2 : y^2 = x^3 + x^2 + x + 1 - \epsilon.$$

Once more, by [KM85, Th.2.3.1], we calculate $E_1[2^\infty]$ and $E_2[2^\infty]$ to both be the constant 2-divisible group $(\mathbf{Q}_2/\mathbf{Z}_2)^2$ over $\overline{\mathbf{F}}_3[\epsilon]$, and hence these 2-divisible groups also base-change to $E[2^\infty]$ over $\overline{\mathbf{F}}_3$. As a final observation, note that E_1 and E_2 are **not** equivalent as elliptic curves over $\overline{\mathbf{F}}_3[\epsilon]$, as one can calculate their j -invariants ([Sil86, §III.1]):

$$j(E_1) = \epsilon - 1 \neq \epsilon + 1 = j(E_2)$$

Hence $[2^\infty]: \mathcal{M}_{\mathrm{Ell}}^\heartsuit \rightarrow \mathcal{M}_{\mathrm{BT}_2^p}^\heartsuit$ is **not** formally étale over $\mathrm{Spec} \mathbf{Z}$.

perfect field κ of characteristic p . Then there exists a universal classical deformation \mathbf{G} of \mathbf{G}_0 over the classical deformation ring $R_{\mathbf{G}_0}^{\text{LT}}$; see [EC2, Df.3.1.4] or the proof of Pr.2.10.

This formally implies that the map into $\mathcal{M}_{\text{BT}_n^p}^\heartsuit$ defining \mathbf{G} is formally étale. In fact, we generalise the Lubin–Tate case above using [EC2, §3] to formally obtain:

Proposition 2.10. *Let R_0 be a discrete \mathbf{F}_p -algebra such that L_R is an almost perfect R -module¹⁷ and \mathbf{G}_0 is a nonstationary¹⁸ p -divisible group over R_0 of height n . Then the map $\text{Spf } R_{\mathbf{G}_0} \rightarrow \widehat{\mathcal{M}}_{\text{BT}_n^p, \mathbf{Z}_p}^\heartsuit$ induced by the universal classical deformation of \mathbf{G}_0 is formally étale. Conversely, if $\mathbf{G}: \text{Spf } R \rightarrow \widehat{\mathcal{M}}_{\text{BT}_n^p, A_0}^\heartsuit$ is formally étale for a complete Noetherian discrete ring R and A_0 from Nt.0.9, then for every maximal ideal $\mathfrak{m} \subseteq R$ such that the residue field $R/\mathfrak{m} = \kappa$ is perfect, then $\mathbf{G}_{B_{\mathfrak{m}}^\wedge}$ is the universal classical deformation of \mathbf{G}_κ .*

Proof. The existence of such an $R_{\mathbf{G}_0}$ follows by taking π_0 of the spectral deformation ring; the spectral deformation ring exists by [EC2, Th.3.4.1] and then we apply Rmk.0.15. Let $R \rightarrow R/J$ be the quotient map where R is discrete and J is a square-zero ideal. First, we wish to show the following commutative diagram of spaces is Cartesian:

$$\begin{array}{ccc} (\text{Spf } R_{\mathbf{G}_0})(R) & \longrightarrow & (\mathcal{M}_{\text{BT}_n^p}^\heartsuit)(R) \\ \downarrow l & & \downarrow r \\ (\text{Spf } R_{\mathbf{G}_0})(R/J) & \xrightarrow{b} & (\mathcal{M}_{\text{BT}_n^p}^\heartsuit)(R/J) \end{array} \quad (2.11)$$

Following [EC2, Df.3.1.4], for an adic \mathbf{E}_∞ -ring A the ∞ -category $\text{Def}_{\mathbf{G}_0}(A)$ is defined as

$$\text{Def}_{\mathbf{G}_0}(A) = \text{colim}_I \left(\text{BT}^p(A) \times_{\text{BT}^p(\pi_0 A/I)} \text{Hom}_{\text{CAlg}^\heartsuit}(\kappa, \pi_0 A/I) \right)$$

where the colimit is indexed over the filtered system of finitely generated ideals of definition $I \subseteq \pi_0 A$. By [EC2, Lm.3.1.10], if A is complete, then $\text{Def}_{\mathbf{G}_0}(A)$ is a space, which in particular holds if A is a discrete ring equipped with the discrete topology. As $R_{\mathbf{G}_0}$ is the universal deformation of \mathbf{G}_0 one obtains for any such A an equivalence of (discrete) spaces

$$\text{Hom}_{\text{CAlg}_{\text{ad}}^\heartsuit}(R_{\mathbf{G}_0}, A) \xrightarrow{\cong} \text{Def}_{\mathbf{G}_0}(A) = \text{colim}_{I \in \text{Nil}_0(A)} \left(\text{BT}^p(A) \times_{\text{BT}^p(A/I)} \text{Hom}_{\text{CAlg}^\heartsuit}(R_0, A/I) \right) \quad (2.12)$$

where the colimit is taken over all finitely generated nilpotent ideals I inside A ; see [EC2, Th.3.1.15]. By assumption, the cotangent complex L_{R_0} is almost perfect in Mod_{R_0} , and [EC2, Pr.3.4.3] then implies that the natural map

$$\text{Def}_{\mathbf{G}_0}(A) \xrightarrow{\cong} \text{colim}_{I \in \text{Nil}(A)} F_{A,I}$$

¹⁷See [EC2, Pr.3.3.7 & Th.3.5.1] for many equivalent conditions to L_R being almost perfect.

¹⁸Recall the definition of a *nonstationary* p -divisible group \mathbf{G}_0 from [EC2, Df.3.0.8], or the equivalent condition for \mathbf{G}_0 over a discrete Noetherian \mathbf{F}_p -algebra R_0 whose Frobenius is finite, that the cotangent complex $L_{\text{Spec } R/\mathcal{M}_{\text{BT}^p}}$ induced by the defining morphism of \mathbf{G}_0 is 1-connective; see [EC2, Rmk.3.4.4 & Th.3.5.1]. In particular, by [EC2, Ex.3.0.10], all p -divisible groups over \mathbf{F}_p -algebras R_0 whose Frobenius is surjective are nonstationary.

is an equivalence, where now the colimit is indexed over *all* nilpotent ideals $I \subseteq A$ and $F_{A,I}$ is the fibre product of (2.12). Given a fixed nilpotent ideal $J \subseteq A$, denote by $\text{Nil}_J(A)$ the poset of nilpotent ideals of A which contain J . We obtain a natural inclusion functor $\text{Nil}_J(A) \rightarrow \text{Nil}(A)$, which is cofinal, as any nilpotent ideal I lies within the nilpotent ideal $I+J$. Hence the natural map

$$\text{colim}_{I \in \text{Nil}_J(A)} F_{A,I} \xrightarrow{\simeq} \text{colim}_{I \in \text{Nil}(A)} F_{A,I}$$

is an equivalence. The map l of (2.11) is then equivalent to

$$\text{colim}_{I \in \text{Nil}_J(R)} F_{R,I} \xrightarrow{l} \text{colim}_{I \in \text{Nil}_J(R)} F_{R/J,I/J}$$

where we used the fact that ideals in R/J correspond to ideals in R containing J . If $(\text{Spf } R_{\mathbf{G}_0})(R/J)$ is empty, then so is $(\text{Spf } R_{\mathbf{G}_0})(R)$ and we are done. Otherwise, choose some x in $(\text{Spf } R_{\mathbf{G}_0})(R/J)$ and consider the fibre of l over x . As filtered colimits of spaces commute with finite limits we calculate this fibre as follows:

$$\text{fib}_x(l) \simeq \text{colim}_{I \in \text{Nil}_J(R)} \text{fib}_{x_I} \left(F_{R,I} \xrightarrow{g} F_{R/J,I/J} \right)$$

To simplify this further, we contemplate the following diagram in $\mathcal{C}\text{at}_\infty$:

$$\begin{array}{ccccc} \text{BT}^p(R) \times_{\text{BT}^p(R/I)} \text{Hom}(R_0, R/I) & \xrightarrow{g} & \text{BT}^p(R/J) \times_{\text{BT}^p(R/I)} \text{Hom}(R_0, R/I) & \longrightarrow & \text{Hom}(R_0, R/I) \\ \downarrow & & \downarrow & & \downarrow \\ \text{BT}^p(R) & \xrightarrow{f} & \text{BT}^p(R/J) & \longrightarrow & \text{BT}^p(R/I) \end{array}$$

The right square and outer rectangle above are Cartesian by definition, so the left square is also Cartesian. This means the natural map $\text{fib}(g) \rightarrow \text{fib}(f)$ is an equivalence in $\mathcal{C}\text{at}_\infty$, hence our fibre of l becomes

$$\text{fib}_x(l) \simeq \text{colim}_{I \in \text{Nil}_J(R)} \left(\text{fib}_{b(x_I)}(\text{BT}^p(R) \xrightarrow{f} \text{BT}^p(R/J)) \right) \simeq \text{fib}_{b(x)} \left(\text{BT}^p(R) \xrightarrow{f} \text{BT}^p(R/J) \right).$$

This shows the fibre of f lies in the essential image of $\mathcal{S} \rightarrow \mathcal{C}\text{at}_\infty$ as $\text{fib}_x(l)$ is. As r is f^\simeq we obtain a natural equivalence $\text{fib}(l) \simeq \text{fib}(r)$. As the fibres of l and r are naturally equivalent, we see that (2.11) is Cartesian, so the composition

$$\text{Spf } R_{\mathbf{G}_0} \rightarrow \widehat{\mathcal{M}}_{\text{BT}_n^p, \mathbf{Z}_p} \rightarrow \mathcal{M}_{\text{BT}_n^p} \rightarrow \mathcal{M}_{\text{BT}^p}$$

is formally étale. To see the first map in the composition above is formally étale, we use that the last map is open (Rmk.1.2) and hence formally étale, the second last map is the base-change of the formally unramified map $\text{Spf } \mathbf{Z}_p \rightarrow \text{Spec } \mathbf{Z}$, and the cancellation statement from Pr.2.2.

Let us omit a proof of the converse statement; the \mathbf{E}_∞ -version is Pr.2.42 and the proofs in both cases are analogous. \square

2.2 On presheaves of connective \mathbf{E}_∞ -rings

We are now in the position to make a spectral definition. See [HA, §7.4] for definition of the definition of (trivial) square-zero extension of \mathbf{E}_∞ -rings, and and [SAG, §17.2] for the definition of (infinitesimally) cohesively and nilcomplete functors in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$ and the definition of $L_{X/Y}$.

Definition 2.13. Let $f: X \rightarrow Y$ be a natural transformation of functors in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$. For an integer $0 \leq n \leq \infty$, we say f is n -formally étale if for all square-zero extensions of connective n -truncated \mathbf{E}_∞ -rings $\tilde{R} \rightarrow R$ the natural diagram of spaces

$$\begin{array}{ccc} X(\tilde{R}) & \longrightarrow & X(R) \\ \downarrow & & \downarrow \\ Y(\tilde{R}) & \longrightarrow & Y(R) \end{array}$$

is Cartesian. We abbreviate ∞ -formally étale to *formally étale*.

Remark 2.14. If f is n -formally étale, then f is also m -formally étale for all $0 \leq m \leq n \leq \infty$. In particular, for any $0 \leq n \leq \infty$, if f is n -formally étale then $X^\heartsuit \rightarrow Y^\heartsuit$ is formally étale inside $\mathcal{P}(\mathrm{Aff}^\heartsuit)$.

A converse statement also holds.

Remark 2.15. Write $\tau_{\leq 0}^*: \mathcal{P}(\mathrm{Aff}^\heartsuit) \rightarrow \mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$ for the functor induced by the truncation $\mathrm{CAlg}^{\mathrm{cn}} \rightarrow \mathrm{CAlg}^\heartsuit$. If $X \rightarrow Y$ is formally étale in $\mathcal{P}(\mathrm{Aff}^\heartsuit)$, then it follows that $\tau_{\leq 0}^*X \rightarrow \tau_{\leq 0}^*Y$ is (∞ -) formally étale inside $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$. Indeed, for each square-zero extension of connective \mathbf{E}_∞ -rings $\tilde{R} \rightarrow R$ we want to show the the diagram of spaces

$$\begin{array}{ccc} X(\pi_0\tilde{R}) & \longrightarrow & X(\pi_0R) \\ \downarrow & & \downarrow \\ Y(\pi_0\tilde{R}) & \longrightarrow & Y(\pi_0R) \end{array}$$

is Cartesian. If we can show the map $\rho: \pi_0\tilde{R} \rightarrow \pi_0R$ is a square-zero extension of classical rings, so we are done by our hypotheses. The (co)fibre sequence

$$M \rightarrow \tilde{R} \rightarrow R$$

of connective R -modules shows that ρ is surjective. Moreover, we see the kernel of ρ is not π_0M , but the image of the map $\pi_0M \rightarrow \pi_0\tilde{R}$. This does not worry us, as the multiplication map $M \otimes_{\tilde{R}} M \rightarrow M$ is nullhomotopic by [HA, Pr.7.4.1.14], hence the image of π_0M in $\pi_0\tilde{R}$ squares to zero, and we see ρ is a square-zero extension of rings.

Remark 2.16. If $\mathfrak{X} \rightarrow \mathfrak{Y}$ is a formally étale morphism of (locally Noetherian) classical formal Deligne–Mumford stacks inside $\mathcal{P}(\mathrm{Aff}^\heartsuit)$, then the corresponding morphism inside $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$ is 0-formally étale. This follows by the fully faithfulness of $\mathrm{fDM} \rightarrow \mathrm{fSpDM}$; see Pr.A.9.

Remark 2.17. Our definition of formally étale deviates from Lurie’s definition of étale morphisms ([HA, Df.7.5.0.4]) as there is no flatness assumption. However, even in $\mathcal{P}(\text{Aff}^{\heartsuit})$ a formally étale morphism of discrete rings need not be flat.¹⁹ This means there is no inherent descent theory for formally étale morphisms. For more in this direction, the reader is advised to make her way to Rmk.2.25.

The basic properties of Pr.2.2 also hold in $\mathcal{P}(\text{Aff}^{\text{cn}})$.

Proposition 2.18. *Let $0 \leq n \leq \infty$ and $X \xrightarrow{f} Y \xrightarrow{g} Z$ be composable morphisms in $\mathcal{P}(\text{Aff}^{\text{cn}})$ where g is n -formally étale. Then f is n -formally étale if and only if h is n -formally étale. Moreover, n -formally étale morphisms are closed under base-change.*

We would now like alternative ways to test if a map $X \rightarrow Y$ is formally étale in $\mathcal{P}(\text{Aff}^{\text{cn}})$. Although Lurie does not directly discuss the adjective formally étale in [SAG, §17], many of the techniques below follow his ideas.

Proposition 2.19. *Let $X \rightarrow Y$ be a natural transformation of functors in $\mathcal{P}(\text{Aff}^{\text{cn}})$ and $0 \leq n \leq \infty$.*

1. *The map $X \rightarrow Y$ is n -formally étale for finite n if and only if $X \rightarrow Y$ is 0-formally étale and for every connective n -truncated \mathbf{E}_{∞} -ring R the natural diagram of spaces*

$$\begin{array}{ccc} X(R) & \longrightarrow & X(\pi_0 R) \\ \downarrow & & \downarrow \\ Y(R) & \longrightarrow & Y(\pi_0 R) \end{array}$$

is Cartesian. If $X \rightarrow Y$ is nilcomplete, then the $n = \infty$ -case also holds.

2. *If $X \rightarrow Y$ is infinitesimally cohesive, then $X \rightarrow Y$ is formally étale if and only if for all trivial square-zero extensions of connective truncated \mathbf{E}_{∞} -rings $\tilde{R} \rightarrow R$ the natural diagram of spaces*

$$\begin{array}{ccc} X(\tilde{R}) & \longrightarrow & X(R) \\ \downarrow & & \downarrow \\ Y(\tilde{R}) & \longrightarrow & Y(R) \end{array}$$

is Cartesian.

¹⁹For example, the map of discrete rings $\mathbf{C}[t^q | q \in \mathbf{Q}, q > 0] \rightarrow \mathbf{C}$ sending $t \mapsto 0$ is formally étale but not flat. Indeed, one can always lift square-zero extensions of rings uniquely, as we have all square roots of t in the above ring, hence it is formally étale. To see this map is not flat, we can tensor it with the exact sequence

$$0 \rightarrow (t) \rightarrow \mathbf{C}[t^q] \rightarrow \mathbf{C}[t^q]/(t) \rightarrow 0,$$

which yields the clearly **not** exact sequence

$$0 \rightarrow \mathbf{C} \rightarrow \mathbf{C} \rightarrow \mathbf{C} \rightarrow 0.$$

3. If $X \rightarrow Y$ is infinitesimally cohesive and admits a connective cotangent complex $L_{X/Y}$, then $X \rightarrow Y$ is formally étale if and only if $L_{X/Y}$ vanishes.

If $X \rightarrow Y$ is infinitesimally cohesive, nilcomplete, and $L_{X/Y}$ exists and is connective, then $X \rightarrow Y$ is n -formally étale if certain Ext-groups $\mathrm{Ext}_R^m(\eta^* L_{X/Y}, M)$ vanish in a range, for certain discrete objects (R, η, M) of $\mathrm{Mod}_{\mathrm{cn}}^X$, à la the deformation theory of [Ill71]. There is also a sharpening of part 4 above which deals with an n -connective cotangent complex $L_{X/Y}$, which we note for the readers benefit is **not** equivalent to $X \rightarrow Y$ being n -formally étale. These ideas will not be used here though.

Thank you to an anonymous referee for correcting a previous version of (2) above.

Proof. Write f for the transformation $X \rightarrow Y$ in question.

1. Suppose f is n -formally étale for a finite $n \geq 0$, then f is 0-formally étale by Rmk.2.14. Given a connective n -truncated \mathbf{E}_∞ -ring R , then for any $0 \leq m \leq n$ we can consider the following diagram:

$$\begin{array}{ccccc}
 X(\tau_{\leq m+1}R) & \longrightarrow & X(\tau_{\leq m}R) & \longrightarrow & X(\pi_0R) \\
 \downarrow & & \downarrow & & \downarrow \\
 Y(\tau_{\leq m+1}R) & \longrightarrow & Y(\tau_{\leq m}R) & \longrightarrow & Y(\pi_0R)
 \end{array} \tag{2.20}$$

Above, the left square is always Cartesian by virtue of f being n -formally étale as $\tau_{\leq m+1}R \rightarrow \tau_{\leq m}R$ is a square-zero extension of \mathbf{E}_∞ -rings by [HA, Cor.7.1.4.28]. To show the outer rectangle Cartesian we use induction. The base case of $m = 0$ is tautological. For $m \geq 1$, the right square is Cartesian by our inductive hypotheses, hence the whole rectangle is Cartesian. Conversely, if the second condition of part 1 holds, we consider a square-zero extension of n -truncated connective \mathbf{E}_∞ -rings $\tilde{R} \rightarrow R$ and the following natural diagram of spaces:

$$\begin{array}{ccccc}
 X(\tilde{R}) & \longrightarrow & X(\pi_0\tilde{R}) & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & & X(R) & \longrightarrow & X(\pi_0R) \\
 & & \downarrow & & \downarrow \\
 Y(\tilde{R}) & \longrightarrow & Y(\pi_0\tilde{R}) & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & & Y(R) & \longrightarrow & Y(\pi_0R)
 \end{array}$$

The back and front faces are Cartesian by the second condition of part 1, and the rightmost face is Cartesian as the second condition of part 1 also assumes f is 0-formally étale. Hence by a base-change argument, we see the leftmost square is Cartesian, and we

are done. For the $n = \infty$ -case, suppose $X \rightarrow Y$ is nilcomplete, meaning that for every connective \mathbf{E}_∞ -ring R , the diagram of spaces

$$\begin{array}{ccc} X(R) & \longrightarrow & \lim X(\tau_{\leq n}R) \\ \downarrow & & \downarrow \\ Y(R) & \longrightarrow & \lim Y(\tau_{\leq n}R) \end{array}$$

is Cartesian. Combining this diagram with the finite case above yields the desired conclusion.

2. If f is formally étale, then logic implies the second condition holds. Conversely, let $e: \tilde{R} \rightarrow R$ be a square-zero extension of a connective \mathbf{E}_∞ -ring R by a connective R -module M and a derivation $d: L_R \rightarrow \Sigma M$. By definition ([HA, Df.7.4.1.6]) \tilde{R} is defined by the Cartesian diagram of connective \mathbf{E}_∞ -rings

$$\begin{array}{ccc} \tilde{R} & \xrightarrow{e} & R \\ \downarrow & & \downarrow \rho \\ R & \xrightarrow{0} & R \oplus \Sigma M \end{array}$$

where the bottom-horizontal map is induced by the zero map $L_R \rightarrow \Sigma M$ and the right-vertical map is induced by the derivation d . This Cartesian diagram of connective \mathbf{E}_∞ -rings then induces the following natural diagram of spaces:

$$\begin{array}{ccccccc} X(\tilde{R}) & \xrightarrow{X(e)} & X(R) & & & & \\ \downarrow & \searrow & \downarrow & \searrow^{X(\rho)} & & & \\ & & X(R) & \xrightarrow{\quad} & X(R \oplus \Sigma M) & \xrightarrow{\quad} & X(R) \\ & & \downarrow & & \downarrow & & \downarrow \\ Y(\tilde{R}) & \xrightarrow{\quad} & Y(R) & & & & \\ & \searrow & \downarrow & \searrow^{Y(\rho)} & & & \\ & & Y(R) & \xrightarrow{Y(0)} & Y(R \oplus \Sigma M) & \xrightarrow{\quad} & Y(R) \end{array} \quad (2.21)$$

The left cube is Cartesian from our assumption that f is infinitesimally cohesive. By assumption the rightmost square is Cartesian, and the only rectangle in the diagram is also Cartesian as the composition $R \rightarrow R \oplus \Sigma M \rightarrow R$ is equivalent to the identity, hence the left square in that same rectangle (the front face of the cube) is Cartesian. By a base-change argument²⁰ we see that the desired back square of the cube (containing $X(e)$ and $Y(e)$) is also Cartesian, and we are done.

²⁰This base-change argument is simple, but let us outline the argument. Write I for the poset of nonempty subsets of $\{1, 2, 3\}$, ordered by inclusion, and use this poset to index the cube in (2.21) by setting $F_\emptyset = X(\tilde{R})$, $F_1 = X(R)$ (in the top-right), $F_2 = X(R)$ (in the centre), $F_3 = Y(\tilde{R})$, etc. As the whole cube is Cartesian we have $F_\emptyset \simeq \lim_{I_0 \in I} F_{I_0}$ and as the front face is also Cartesian we have $F_2 \simeq \lim (F_{12} \rightarrow F_{123} \leftarrow F_{23})$. These two

3. Our proof outline here is essentially that of [SAG, Prs.17.3.9.3-4]. On the one hand, by [SAG, Pr.6.2.5.2(1)] and [SAG, Df.6.2.5.3], we see that for some fixed integer m , an object \mathcal{F} of $\mathrm{QCoh}(X)$ is m -connective if and only if for all connective \mathbf{E}_∞ -rings R and transformations $\eta: \mathrm{Spec} R \rightarrow X$, the object $\eta^*\mathcal{F}$ is m -connective inside $\mathrm{QCoh}(\mathrm{Spec} R) \simeq \mathrm{Mod}_R$. Furthermore, if \mathcal{F} is connective and $m \geq 0$, the object $\eta^*\mathcal{F}$ is m -connective if and only if the mapping space

$$\mathrm{Map}_{\mathrm{Mod}_R^{\mathrm{cn}}}(\eta^*\mathcal{F}, N) \simeq \mathrm{Map}_{\mathrm{Mod}_R^{\mathrm{cn}}}(\tau_{\leq m}\eta^*\mathcal{F}, N)$$

is contractible, for all connective $(m-1)$ -truncated R -modules N , by the Yoneda lemma. On the other hand, the object $L_{X/Y}$ in $\mathrm{QCoh}(X)$ exists if and only if the functor $F: \mathrm{Mod}_{\mathrm{cn}}^X \rightarrow \mathcal{S}$, given on objects by

$$F(R, \eta, M) = \mathrm{fib} \left(X(R \oplus M) \rightarrow X(R) \times_{Y(R)} Y(R \oplus M) \right) \quad (2.22)$$

is locally almost representable, meaning that we have a (locally almost; see [SAG, Df.17.2.3.1]) natural equivalence for all triples (R, η, M) in $\mathrm{Mod}_X^{\mathrm{cn}}$

$$F(R, \eta, M) \simeq \mathrm{Map}_{\mathrm{Mod}_R}(\eta^*L_{X/Y}, M)$$

where R is a connective \mathbf{E}_∞ -ring, $\eta: \mathrm{Spec} R \rightarrow X$ a map in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$, and M a connective R -module. If $L_{X/Y}$ vanishes, then we immediately see $F(R, \eta, M)$ is contractible for all triples (R, η, M) , which by part 3 implies $X \rightarrow Y$ is formally étale, courtesy of the definition (2.22) of F . Conversely, if $X \rightarrow Y$ is formally étale, then $F(R, \eta, M)$ is contractible for all triples (R, η, M) , hence the mapping space

$$\mathrm{Map}_{\mathrm{Mod}_R}(\eta^*L_{X/Y}, M) \simeq F(R, \eta, M)$$

is contractible for all triples (R, η, M) and $L_{X/Y}$ vanishes. \square

There are many examples of formally étale maps in spectral algebraic geometry.

Note that all formal spectral Deligne–Mumford stacks are cohesive, nilcomplete, and absolute cotangent complexes always exist, which follows by copying the proof of [SAG, Cor.17.3.8.5] (the same statement for SpDM), as all of the references made there also apply to fSpDM .

Example 2.23 (Étale morphisms of connective \mathbf{E}_∞ -rings). Let $A \rightarrow B$ be an étale morphism of connective \mathbf{E}_∞ -rings, then by [HA, Cor.7.5.4.5] we know $L_{B/A}$ vanishes, hence $A \rightarrow B$ is also a formally étale morphism of \mathbf{E}_∞ -rings by Pr.2.19.

facts, together with [MV15, Ex.5.3.8] give us the following natural chain of equivalences of spaces

$$F_\emptyset \simeq \lim_{I_0 \in I} F_{I_0} \simeq \lim (F_2 \rightarrow G_{123} \leftarrow G_{13}) \simeq G_{13},$$

where $G_{123} = \lim (F_{12} \rightarrow F_{123} \leftarrow F_{23}) \simeq F_2$ and $G_{13} = \lim (F_1 \rightarrow F_{13} \leftarrow F_3)$. This shows the back face of the cube (indexed by \emptyset , $\{1\}$, $\{3\}$, and $\{1, 3\}$), is Cartesian.

Example 2.24 (Relatively perfect discrete \mathbf{F}_p -algebras). Another classic example, which will not show up explicitly in this note but is at the heart of much of the work done in [EC2], is that a flat relatively perfect map of discrete commutative \mathbf{F}_p -algebras has a vanishing cotangent complex ([EC2, Lm.5.2.8]), and hence is formally étale.

Remark 2.25. In Rmk.2.17, we noted that formally étale morphisms of connective \mathbf{E}_∞ -rings were not necessarily flat. However, [EC2, Pr.3.5.5] states that morphisms of (not necessarily connective) *Noetherian* \mathbf{E}_∞ -rings with vanishing cotangent complex are flat. Combining this with Pr.2.19, we see formally étale morphisms of connective Noetherian \mathbf{E}_∞ -rings are flat. It also follows (as in classical algebraic geometry, see [Sta, 02HM]) that formally étale morphisms of almost finite presentation between connective Noetherian \mathbf{E}_∞ -rings are étale.

The functor $\mathcal{M}_{\mathrm{BT}^p}$ is cohesive, nilcomplete, and admits a cotangent complex by [EC2, Pr.3.2.2]. It follows that $\mathcal{M}_{\mathrm{BT}_n^p}$ (as well as all base-changes $\widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A}$) also satisfy these properties as $\mathcal{M}_{\mathrm{BT}_n^p} \rightarrow \mathcal{M}_{\mathrm{BT}^p}$ is open (Rmk.1.2).

Example 2.26 (Spectral Serre–Tate theorem). It follows from the spectral Serre–Tate theorem ([EC1, Th.7.0.1]) and Pr.2.19 that the map $[p^\infty]: \widehat{\mathcal{M}}_{\mathrm{AVar}_g, \mathbf{S}_p} \rightarrow \widehat{\mathcal{M}}_{\mathrm{BT}_{2g}^p, \mathbf{S}_p}$ is formally étale.

Example 2.27 (Spectral Lubin–Tate theory). For a nonstationary (18) p -divisible group \mathbf{G}_0 over a discrete ring R_0 where p is nilpotent and whose absolute cotangent complex L_{R_0} is almost perfect, Lurie uses his de Rham space formalism to construct a map $\mathbf{G}: \mathrm{Spf} R \rightarrow \mathcal{M}_{\mathrm{BT}_n^p}$ ([EC2, Th.3.4.1]) which is formally étale by [SAG, Cor.18.2.1.11(2)] and Pr.2.19. The p -divisible group \mathbf{G} is the universal spectral deformation of \mathbf{G}_0 and R its spectral deformation ring; see Df.0.14.

Example 2.28 (Formal spectral completions). Let \mathbf{X} be a spectral Deligne–Mumford stack and $K \subseteq |\mathbf{X}|$ be a cocompact closed subset, then the natural map from the formal completion of \mathbf{X} along K ([SAG, Df.8.1.6.1]) $\mathbf{X}_K^\wedge \rightarrow \mathbf{X}$ is formally étale by [SAG, Ex.17.1.2.10] and Pr.2.19.

Example 2.29 (Spectral de Rham space). Given a morphism $X \rightarrow Y$ of functors in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$, one can associate a *de Rham space* $(X/Y)_{\mathrm{dR}}$ inside $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$, whose value on a connective \mathbf{E}_∞ -ring is

$$(X/Y)_{\mathrm{dR}}(R) = \mathrm{colim}_I \left(Y(R) \times_{Y(\pi_0 R/I)} X(\pi_0 R/I) \right)$$

where the colimit is taken over all nilpotent ideals $I \subseteq \pi_0 R$, which we note is a discrete filtered system; see [SAG, §18.2.1]. By [SAG, Cor.18.2.1.11(2)], the natural map $(X/Y)_{\mathrm{dR}} \rightarrow Y$ is nilcomplete, infinitesimally cohesive, and admits a vanishing cotangent complex, so by Pr.2.19, it is formally étale.

This last example will help us study the moduli stack $\mathcal{M}_{\mathrm{BT}_n^p}$.

2.3 Applied to the moduli stack of p -divisible groups

Let us now apply the theory of formally étale natural transformations to the functor $\widehat{\mathcal{M}}_{\mathrm{BT}_n^p}$ and the categories \mathcal{C}_{A_0} and \mathcal{C}_A of Df.1.5.

Notation 2.30. Write $\iota: \mathrm{CAlg}^\heartsuit \rightarrow \mathrm{CAlg}^{\mathrm{cn}}$ for the inclusion (a right adjoint, inducing a left adjoint $(-)^{\heartsuit}$ on presheaf categories), and $\tau_{\leq 0}$ for the truncation functor (a left adjoint, inducing a right adjoint $\tau_{\leq 0}^*$ on presheaf categories) $\mathrm{CAlg}^{\mathrm{cn}} \rightarrow \mathrm{CAlg}^\heartsuit$. Also write $\tau_{\leq 0}$ for the

composition $(-)^{\heartsuit} \circ \tau_{\leq 0}$ (this should seldom cause confusion). For each functor \mathcal{M} in $\mathcal{P}(\text{Aff}^{\text{cn}})$ there is a natural unit $\mathcal{M} \rightarrow \tau_{\leq 0}^* \mathcal{M}$ induced by the truncation $R \rightarrow \pi_0 R$ of a connective \mathbf{E}_{∞} -ring R . One can also check the functor $\tau_{\leq 0}^* \mathcal{M}$ is the right Kan extension of \mathcal{M}^{\heartsuit} along ι .

Warning 2.31. In §A we introduce the truncation of a locally Noetherian formal spectral Deligne–Mumford stack $\tau_{\leq 0} \mathfrak{X}$ à la Lurie [SAG, §1.4.6] and we note that this is **not** equivalent to $\tau_{\leq 0}^* \mathfrak{X}$.

For mostly formal reasons, we obtain a functor $\mathcal{C}_A \rightarrow \mathcal{C}_{A_0}$.

Proposition 2.32. *The functor $(-)^{\heartsuit}: \mathcal{C}_A \rightarrow \mathcal{P}(\text{Aff}^{\heartsuit})_{/\widehat{\mathcal{M}}_{\text{BT}_n^p, A_0}^{\heartsuit}}$ factors through \mathcal{C}_{A_0} .*

Our proof of the above proposition relies on §A.

Proof. By definition, a object \mathfrak{X} of \mathcal{C}_A is qcqs and so has an affine étale hypercover $\mathfrak{U}_{\bullet} \rightarrow \mathfrak{X}$; see Pr.A.17. The formal spectral Deligne–Mumford stack $\tau_{\leq 0} \mathfrak{X} = \mathfrak{X}_0$ then lies in the essential image of $\text{fDM} \rightarrow \text{fSpDM}$ and hence can be considered as a classical spectral Deligne–Mumford stack. Moreover, $\mathfrak{X}^{\heartsuit}$ and $\mathfrak{X}_0^{\heartsuit}$ are naturally equivalent by Cor.A.5. As each affine formal spectral Deligne–Mumford stack \mathfrak{U}_n is Noetherian, $\mathfrak{X}^{\heartsuit} = \mathfrak{X}_0^{\heartsuit}$ has an affine étale hypercover by $\mathfrak{U}_{\bullet}^{\heartsuit} \rightarrow \mathfrak{X}^{\heartsuit} = \mathfrak{X}_0^{\heartsuit}$ inside fDM . By Pr.A.17, we see \mathfrak{X}_0 is qcqs. From Cor.A.5 we also see that \mathfrak{X}_0 is represented by $\tau_{\leq 0} \mathfrak{X}$ inside $\mathcal{P}(\text{Aff}^{\heartsuit})$, so \mathfrak{X}_0 and \mathfrak{X} have the same closed points. As $\mathfrak{U}_0^{\heartsuit}$ is a Noetherian affine classical formal Deligne–Mumford stack, we also see \mathfrak{X}_0 is locally Noetherian. It also follows from Rmk.2.14 that $\mathfrak{X}_0 \rightarrow \widehat{\mathcal{M}}_{\text{BT}_n^p, A_0}^{\heartsuit}$ is formally étale inside $\mathcal{P}(\text{Aff}^{\heartsuit})$. To see the cotangent complex L of the map $\mathfrak{X}_0 \rightarrow \widehat{\mathcal{M}}_{\text{BT}_n^p, A}$ is almost perfect inside $\text{QCoh}(\mathfrak{X}_0)$, consider the following composition of maps in $\mathcal{P}(\text{Aff}^{\text{cn}})$

$$\tau_{\leq 0} \mathfrak{X} \simeq \mathfrak{X}_0 \rightarrow \mathfrak{X} \rightarrow \widehat{\mathcal{M}}_{\text{BT}_n^p, A}$$

from which we obtain a (co)fibre sequence in $\text{QCoh}(\mathfrak{X}_0)$ of the form:

$$L_{\mathfrak{X}/\widehat{\mathcal{M}}_{\text{BT}_n^p, A}} \Big|_{\mathfrak{X}_0} \rightarrow L \rightarrow L_{\mathfrak{X}_0/\mathfrak{X}}$$

By part 3 of Pr.2.19, the first term in the above (co)fibre sequence vanishes, and our desired conclusion follows as $L_{\mathfrak{X}_0/\mathfrak{X}}$ is almost perfect by Pr.A.12. \square

To see $(-)^{\heartsuit}$ is an equivalence, we will construct an explicit inverse.

Definition 2.33. Define a functor $\mathfrak{D}: \mathcal{C}_{A_0} \rightarrow \mathcal{P}(\text{Aff}^{\text{cn}})_{/\widehat{\mathcal{M}}_{\text{BT}_n^p, A}}$ by sending an object $\mathbf{G}_0: \mathfrak{X}_0 \rightarrow \widehat{\mathcal{M}}_{\text{BT}_n^p, A_0}^{\heartsuit}$ of \mathcal{C}_{A_0} to the de Rham space of [SAG, §18.2.1] (and Ex.2.29):

$$\mathfrak{D}(\mathbf{G}_0) = \left(\mathfrak{X}_0 / \widehat{\mathcal{M}}_{\text{BT}_n^p, A} \right)_{\text{dR}}$$

The notation \mathfrak{D} is supposed to conjure the word “deformation”.

Theorem 2.34. *The functor \mathfrak{D} factors through \mathcal{C}_A , preserves affine objects and étale hypercovers, and is an inverse to $(-)^{\heartsuit}$.*

This equivalence of ∞ -categories fits into the general paradigm of spectral algebraic geometry—a well-behaved site over a classical moduli stack should be equivalent to the same site over the associated spectral moduli stack; see the example of the moduli stack of elliptic curves in [EC1, Rmk.2.4.2] and [EC2, §7], or the affine case in [HA, Th.7.5.0.6].

To prove Th.2.34, we will use the interaction of the de Rham space technology of Lurie ([SAG, §18.2.1]) with formally étale morphisms, and a representability theorem also due to Lurie.

Proposition 2.35. *Recall Nt.0.9. Let \mathfrak{X} be a formal spectral Deligne–Mumford stack and $\mathfrak{X} \rightarrow \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A}$ be a 0-formally étale map whose associated cotangent complex is almost perfect. Then the following natural diagram of functors in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$*

$$\begin{array}{ccc} (\mathfrak{X}/\widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A})_{\mathrm{dR}} & \xrightarrow{u} & \tau_{\leq 0}^* \mathfrak{X} \\ \downarrow \mathbf{G}_{\mathrm{dR}} & & \downarrow \\ \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A} & \xrightarrow{d} & \tau_{\leq 0}^* \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A} \end{array} \quad (2.36)$$

is Cartesian, the natural map $\mathfrak{X} \rightarrow (\mathfrak{X}/\widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A})_{\mathrm{dR}}$ induces an equivalence when evaluated on discrete \mathbf{E}_∞ -rings, and \mathbf{G}_{dR} is formally étale.

The above proposition and its proof generalise to a wider class of functors in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$ of which we could not find a neater formulation than our leading example—we leave the reader to exploit the general example as she wishes.

Proof of Pr.2.35. Recall the value of the de Rham space $(X/Y)_{\mathrm{dR}}$ on a connective \mathbf{E}_∞ -ring R from (2.37)

$$(X/Y)_{\mathrm{dR}}(R) = \mathrm{colim}_I \left(Y(R) \times_{Y(\pi_0 R/I)} X(\pi_0 R/I) \right) \quad (2.37)$$

where colimit is taken over all nilpotent ideals of $\pi_0 R$. Define a functor $(X/Y)_{\mathrm{dR}}^0 : \mathrm{CAlg}^{\mathrm{cn}} \rightarrow \mathcal{S}$ by the same formula as (2.37) above but index the colimit over finitely generated nilpotent ideals of $\pi_0 R$. One readily obtains a map of functors

$$(X/Y)_{\mathrm{dR}}^0 \rightarrow (X/Y)_{\mathrm{dR}}$$

and we claim this map is an equivalence for $X = \mathfrak{X}$ and $Y = \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A}$ in our hypotheses. Indeed, one can copy the proof of [EC2, Pr.3.4.3] *mutatis mutandis*, exchanging only R_0 for \mathfrak{X} ; the crucial step comes at the end and uses the almost perfect assumption on our cotangent complex. Writing $F_{R, I}$ for the fibre product within the colimit of (2.37) where $X = \mathfrak{X}$ and $Y = \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A}$, we place $F_{R, I}$ into the following commutative diagram of spaces:

$$\begin{array}{ccccc} F_{R, I} & \longrightarrow & \mathfrak{X}(\pi_0 R) & \longrightarrow & \mathfrak{X}(\pi_0 R/I) \\ \downarrow & & \downarrow f_{\pi_0 R} & & \downarrow f_{\pi_0 R/I} \\ \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A}(R) & \longrightarrow & \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A}(\pi_0 R) & \longrightarrow & \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A}(\pi_0 R/I) \end{array}$$

The outer rectangle is Cartesian by definition and we claim that the right square is also Cartesian. Indeed, this follows as I is finitely generated and hence is nilpotent of finite degree n for some integer $n \geq 2$, and our 0-formally étale hypotheses can be sequentially applied to the composition of square-zero extensions:

$$R \rightarrow R/I^n \rightarrow R/I^{n-1} \rightarrow \cdots \rightarrow R/I^2 \rightarrow R/I$$

This implies that the left square above is Cartesian, so the R -points of the de Rham space in question naturally take the form

$$\operatorname{colim}_I \left(\widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A}(R) \times_{\widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A}(\pi_0 R)} \mathfrak{X}(\pi_0 R) \right) \simeq \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A}(R) \times_{\widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A}(\pi_0 R)} \mathfrak{X}(\pi_0 R)$$

as the diagram indexing our colimit is filtered. This implies that (2.36) is Cartesian. For the second statement, we use the facts that (2.36) is Cartesian and d induces equivalences when evaluated on discrete rings to see that u induces an equivalence when evaluated on discrete rings. Noting that the maps

$$(\mathfrak{X}/\widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A})_{\mathrm{dR}} \rightarrow \tau_{\leq 0}^* \mathfrak{X} \leftarrow \mathfrak{X}$$

induce equivalences on discrete rings, we see the natural map $\mathfrak{X} \rightarrow (\mathfrak{X}/\widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A})_{\mathrm{dR}}$ induces an equivalence on discrete rings. Finally, to see $(\mathfrak{X}/\widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A})_{\mathrm{dR}} \rightarrow \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A}$ is formally étale we refer to Ex.2.29, or alternatively to the Cartesian diagram (2.36), Rmk.2.15, and Pr.2.18. \square

The proof of the above theorem exposes us to something quite useful.

Remark 2.38 (\mathfrak{D} produces universal spectral deformations). Recall that associated to a classical p -divisible group $\mathbf{G}_0: \operatorname{Spf} B_0 \rightarrow \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A_0}^\heartsuit$, such as those in \mathcal{C}_{A_0} , then we can ask if there exists a *universal spectral deformation* of \mathbf{G}_0 and its associated spectral deformation ring; see Df.0.14. It follows from the proof of Pr.2.35 above, that if \mathbf{G}_0 lies in \mathcal{C}_{A_0} , then the formal spectrum Spf of the spectral deformation ring of \mathbf{G}_0 is equivalent to the de Rham space $(\mathbf{G}_0: \operatorname{Spf} B_0 \rightarrow \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A})_{\mathrm{dR}}$. By Th.2.34, we see that this de Rham space is represented by a formal spectral Deligne–Mumford stack $\operatorname{Spf} B$. This means that $\mathfrak{D}(\mathbf{G}_0)$ is represented by the universal spectral deformation of \mathbf{G}_0 . This is even true in a nonaffine sense, but we will not need to venture further in that direction.

The following representability theorem of Lurie is crucial.

Theorem 2.39 ([SAG, Th.18.2.3.1]). *Let $f: \mathfrak{X} \rightarrow \mathcal{M}$ be a map of functors in $\mathcal{P}(\operatorname{Aff}^{\mathrm{cn}})$ such that \mathfrak{X} is a formal spectral Deligne–Mumford stack, \mathcal{M} is nilcomplete, infinitesimally cohesive, admits a cotangent complex, and is an étale sheaf, and $L_{\mathfrak{X}/\mathcal{M}}$ is 1-connective and almost perfect. Then $(\mathfrak{X}/\mathcal{M})_{\mathrm{dR}}$ is represented by a formal thickening²¹ of \mathfrak{X} .*

²¹Recall from [SAG, Df.18.2.2.1], a morphism $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ of formal spectral Deligne–Mumford stacks is called a *formal thickening* if the induced map on reductions $\mathfrak{X}^{\mathrm{red}} \rightarrow \mathfrak{Y}^{\mathrm{red}}$ is an equivalence ([SAG, §8.1.4]) and the map f is representable by closed immersions which are locally almost of finite presentation.

Importantly, we can apply this theorem to \mathcal{C}_{A_0} .

Remark 2.40. By definition the cotangent complex $L = L_{\mathfrak{X}_0/\widehat{\mathcal{M}}_{\text{BT}_n^p, A}}$ corresponding to an object inside \mathcal{C}_{A_0} is almost connective, meaning $\Sigma^n L$ is connective for some positive integer n ; see [SAG, Var.8.2.5.7 & Rmk.8.2.5.9]. However, we claim that L is actually 1-connective. Indeed, by [SAG, Cor.8.2.5.5] we may check this étale locally on \mathfrak{X}_0 , so let us replace \mathfrak{X}_0 with $\text{Spf } B_0$ for some complete Noetherian discrete adic ring B_0 . In particular, L is now an almost perfect J -complete B_0 -module, where J is an ideal of definition for B_0 . As L is almost perfect, the fibrewise connectivity criterion of [SAG, Cor.2.7.4.3] states that it suffices to check $L_{\mathfrak{m}}^\wedge$ is 1-connective for every maximal ideal $\mathfrak{m} \subseteq B_0$ which contains J . Moreover, considering the maps

$$\text{Spf}(B_0)_{\mathfrak{m}}^\wedge \rightarrow \text{Spf } B_0 \rightarrow \text{Spec } B_0$$

the composition is formally étale (as discussed for $\mathcal{P}(\text{Aff}^{\text{cn}})$ by Ex.2.28 and hence in $\mathcal{P}(\text{Aff}^\heartsuit)$ by Rmk.2.14), and the latter map is unramified, so by Pr.2.2 we see the first map is formally étale. We may then assume B_0 is a complete local Noetherian ring. The morphism $\mathbf{G}: \text{Spf } B_0 \rightarrow \widehat{\mathcal{M}}_{\text{BT}_n^p, A_0}^\heartsuit$ is formally étale, so by the converse statement in Pr.2.10, we see that B_0 is the classical deformation ring of \mathbf{G}_κ , where κ is the residue field of B_0 , which is necessarily perfect of characteristic p by assumption. For such a pair $(\mathbf{G}_\kappa, \kappa)$, there exists a spectral deformation ring B by [EC2, Th.3.1.15], as κ is perfect and \mathbf{G}_κ is nonstationary by [EC2, Ex.3.0.10], which implies $\pi_0 B \simeq B_0$ by Rmk.0.15. This means the map $\text{Spf } B_0 \rightarrow \widehat{\mathcal{M}}_{\text{BT}_n^p, A}$ in $\mathcal{P}(\text{Aff}^{\text{cn}})$ factors as

$$\text{Spf } B_0 \rightarrow \text{Spf } B \rightarrow \widehat{\mathcal{M}}_{\text{BT}_n^p, A} \quad (2.41)$$

where the first map is induced by the truncation. Associated to the above composition is the following (co)fibre sequence of complete B_0 -modules:

$$L_{\text{Spf } B/\widehat{\mathcal{M}}_{\text{BT}_n^p, A}} \Big|_{\text{Spf } B_0} \rightarrow L_{\text{Spf } B_0/\widehat{\mathcal{M}}_{\text{BT}_n^p, A}} \rightarrow L_{\text{Spf } B_0/\text{Spf } B}$$

The first object vanishes as $\text{Spf } B$ is the de Rham space for the composite (2.41) and such objects always vanish; see Ex.2.29. We then see the middle cotangent complex above is 1-connected and almost perfect as this holds for $L_{\text{Spf } B_0/\text{Spf } B}$ by Pr.A.12.

Proof of Th.2.34. First, let us check \mathfrak{D} factors through \mathcal{C}_A . Using Th.2.39 and Rmk.2.40, we see $\mathfrak{D}(\mathbf{G}_0)$ is represented by a formal thickening \mathfrak{X} of \mathfrak{X}_0 ; see [SAG, §18.2.2] or (21). To see \mathfrak{X} satisfies the conditions of Df.1.5, we note the following:

- \mathfrak{X} is locally Noetherian, as it is a formal thickening of the locally Noetherian \mathfrak{X}_0 ; see [SAG, Cor.18.2.4.4].
- \mathfrak{X} is qcqs as a formal thickening of a qcqs formal spectral Deligne–Mumford stack is qcqs; see Pr.A.19.
- \mathfrak{X} has perfect residue fields at closed points as this is true for \mathfrak{X}_0 and $\mathfrak{X}_0 = \tau_{\leq 0} \mathfrak{X}$ has the same residue fields as \mathfrak{X} .

- \mathbf{G} is formally étale, as $L_{\mathfrak{X}/\widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A}}$ vanishes, either by Ex.2.29 or Pr.2.35.

Notice that by [SAG, Cor.18.2.3.3], if $\mathfrak{X}_0 \simeq \mathrm{Spf} B_0$ is affine, then the image of any $\mathbf{G}_0: \mathrm{Spf} B_0 \rightarrow \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A_0}^\heartsuit$ in \mathcal{C}_{A_0} under \mathfrak{D} is also affine. To see \mathfrak{D} is inverse to $(-)^{\heartsuit}$, notice the composite $(-)^{\heartsuit}\mathfrak{D}$ is equivalent to the identity as $\mathbf{G}_0 \rightarrow \mathfrak{D}(\mathbf{G}_0)$ induces an equivalence on discrete rings by Pr.2.35. For the other composition, part 1 of Pr.2.19 states that the following diagram of spaces is Cartesian for every connective \mathbf{E}_∞ -ring R :

$$\begin{array}{ccc} \mathfrak{X}(R) & \longrightarrow & \mathfrak{X}(\pi_0 R) \\ \downarrow \mathbf{G} & & \downarrow \\ \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A}(R) & \longrightarrow & \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A}(\pi_0 R) \end{array}$$

By Pr.2.19, we then see the natural map $\mathfrak{D}((\mathbf{G})^{\heartsuit}) \rightarrow \mathbf{G}$ is an equivalence in \mathcal{C}_A . Finally, to see \mathfrak{D} preserves étale hypercovers, we first note this may be checked étale locally, so take an étale hypercover $\mathrm{Spf} C^\bullet \rightarrow \mathrm{Spf} B_0$ in \mathcal{C}_{A_0} and write $\mathrm{Spf} C^\bullet \rightarrow \mathrm{Spf} B$ for its image under \mathfrak{D} . From the above, we know that $\mathrm{Spf} C^\bullet \rightarrow \mathrm{Spf} B$ is an étale hypercover on zeroth truncations, so it suffices to see each map $B \rightarrow C^n$ is étale as morphism of \mathbf{E}_∞ -rings. Two applications of Pr.2.35 show the commutative diagram in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$

$$\begin{array}{ccccc} \mathfrak{Y}_\bullet & \longrightarrow & \mathfrak{X} & \longrightarrow & \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A} \\ \downarrow & & \downarrow & & \downarrow \\ \tau_{\leq 0}^* \mathfrak{Y}_\bullet^0 & \longrightarrow & \tau_{\leq 0}^* \mathfrak{X}_0 & \longrightarrow & \tau_{\leq 0}^* \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A} \end{array}$$

consists of Cartesian squares, hence $\mathrm{Spf} C^n \rightarrow \mathrm{Spf} B$ is formally étale by Rmk.2.15 and base-change Pr.2.18. It follows from Rmk.2.25 that $B \rightarrow C^n$ is étale as a map of \mathbf{E}_∞ -rings, hence also as a map of adic \mathbf{E}_∞ -rings. \square

Finally, let us solidify some of the connections between formally étale morphisms and universal deformations. The following is analogous to Pr.2.10 and our proof follows that of [EC2, Pr.7.4.2] (we will even copy some of Lurie's notation).

Proposition 2.42. *Recall Nt.0.9. Let $\mathbf{G}: \mathrm{Spf} B \rightarrow \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A}$ be formally étale map where B is a complete adic Noetherian \mathbf{E}_∞ -ring with ideal of definition J . Fix a maximal ideal $\mathfrak{m} \subseteq \pi_0 B$ containing J such that $\pi_0 B/\mathfrak{m}$ is perfect of characteristic p . Then the p -divisible group $\mathbf{G}_{B_\mathfrak{m}^\wedge}$ is the universal spectral deformation of \mathbf{G}_κ (in the sense of [EC2, Df.3.1.11]), where κ is the residue field of $B_\mathfrak{m}$.*

Proof. As κ is perfect of characteristic p , combining [EC2, Ex.3.0.10] with [EC2, Th.3.1.15] one obtains the spectral deformation ring $R_{\mathbf{G}_\kappa}^{\mathrm{un}} = B^{\mathrm{un}}$ with a universal p -divisible group \mathbf{G}^{un} . By definition $\mathbf{G}_{B_\mathfrak{m}^\wedge}$ is a deformation over \mathbf{G}_κ ([EC2, Df.3.0.3]), so from the universality of $(B^{\mathrm{un}}, \mathbf{G}^{\mathrm{un}})$ we obtain a canonical continuous morphism of adic \mathbf{E}_∞ -rings $B^{\mathrm{un}} \xrightarrow{\alpha} B_\mathfrak{m}^\wedge = \widehat{B}$ inducing the identity on the common residue field κ . By [EC2, Th.3.1.15], we see B^{un} belongs

to the full ∞ -subcategory \mathcal{C} of $(\mathrm{CAlg}_{\mathrm{ad}}^{\mathrm{cn}})_{/\kappa}$ spanned by complete local Noetherian adic \mathbf{E}_{∞} -rings whose augmentation to κ exhibits κ as its residue field. To see α is an equivalence in this ∞ -category, consider an arbitrary object C of \mathcal{C} and the induced map

$$\mathrm{Map}_{\mathrm{CAlg}_{/\kappa}}^{\mathrm{cont}}(\widehat{B}, C) \xrightarrow{\alpha^*} \mathrm{Map}_{\mathrm{CAlg}_{/\kappa}}^{\mathrm{cont}}(B^{\mathrm{un}}, C).$$

By writing C as the limit of its truncations we are reduced to the case where C is truncated, and by writing $\pi_0 C$ as a limit of Artinian subrings of $\pi_0 C$ we are further reduced to the case when $\pi_0 C$ is Artinian.²² In this situation, we have a finite sequence of maps

$$C = C_m \rightarrow C_{m-1} \rightarrow \cdots \rightarrow C_1 \rightarrow C_0 = k$$

where each map is a square-zero extension by an almost perfect connective module. Hence, it would suffice to show that for every $C \rightarrow \kappa$ in \mathcal{C} , and every square-zero extension $\tilde{C} \rightarrow C$ of C by an almost perfect connective C -module, with \tilde{C} also in \mathcal{C} , the natural diagram of spaces

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{CAlg}_{/\kappa}}^{\mathrm{cont}}(\widehat{B}, \tilde{C}) & \longrightarrow & \mathrm{Map}_{\mathrm{CAlg}_{/\kappa}}^{\mathrm{cont}}(B^{\mathrm{un}}, \tilde{C}) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{CAlg}_{/\kappa}}^{\mathrm{cont}}(\widehat{B}, C) & \longrightarrow & \mathrm{Map}_{\mathrm{CAlg}_{/\kappa}}^{\mathrm{cont}}(B^{\mathrm{un}}, C) \end{array} \quad (2.43)$$

is Cartesian, the $C = \kappa$ case being tautological. As \widehat{B} is the \mathfrak{m} -completion of $B_{\mathfrak{m}}$, then for any D in \mathcal{C} (which in particular is complete with respect to the kernel of its augmentation $D \rightarrow \kappa$) the map

$$\mathrm{Map}_{\mathrm{CAlg}_{/\kappa}}^{\mathrm{cont}}(\widehat{B}, D) \xrightarrow{\simeq} \mathrm{Map}_{\mathrm{CAlg}_{/\kappa}}^{\mathrm{cont}}(B_{\mathfrak{m}}, D)$$

induced by $B_{\mathfrak{m}} \rightarrow \widehat{B}$, is an equivalence. Moreover, for any D inside \mathcal{C} we have the following natural identifications:

$$\begin{aligned} \mathrm{Map}_{\mathrm{CAlg}_{/\kappa}}^{\mathrm{cont}}(B^{\mathrm{un}}, D) &\simeq \mathrm{fib}_{B^{\mathrm{un}} \rightarrow \kappa}(\mathrm{Map}_{\mathrm{CAlg}}^{\mathrm{cont}}(B^{\mathrm{un}}, D) \rightarrow \mathrm{Map}_{\mathrm{CAlg}}^{\mathrm{cont}}(B^{\mathrm{un}}, \kappa)) \\ &\simeq \mathrm{fib}_{B^{\mathrm{un}} \rightarrow \kappa}((\mathrm{Spf} B^{\mathrm{un}})(D) \rightarrow (\mathrm{Spf} B^{\mathrm{un}})(\kappa)) \simeq \mathrm{fib}_{\mathbf{G}^{\mathrm{un}}}(\mathrm{Def}_{\mathbf{G}_{\kappa}}(D) \rightarrow \mathrm{Def}_{\mathbf{G}_{\kappa}}(\kappa)) \\ &\simeq \mathrm{Def}_{\mathbf{G}_{\kappa}}(D, (D \rightarrow \kappa)) \simeq \mathrm{BT}_n^p(D) \times_{\mathrm{BT}_n^p(\kappa)} \{\mathbf{G}_{\kappa}\} \end{aligned}$$

The first equivalence is a categorical fact about over/under categories, the second is the identification of the R -valued points of $\mathrm{Spf} B^{\mathrm{un}}$ ([SAG, Lm.8.1.2.2]), the third is from universal property of spectral deformation rings ([EC2, Th.3.1.15]), and the fourth and fifth can be taken as two alternative definitions of $\mathrm{Def}_{\mathbf{G}_{\kappa}}(D, (D \rightarrow \kappa))$ ([EC2, Df.3.0.3 & Rmk.3.1.6]). These natural equivalences show (2.43) is equivalent to the upper-left square in the following

²²Our conventions demand that local adic \mathbf{E}_{∞} -rings have their topology determined by the maximal ideal.

natural diagram of spaces:

$$\begin{array}{ccccc}
\mathrm{Map}_{\mathrm{CAlg}/\kappa}^{\mathrm{cont}}(B_{\mathfrak{m}}, \tilde{C}) & \longrightarrow & \mathrm{BT}_n^p(\tilde{C}) \times_{\mathrm{BT}_n^p(\kappa)} \{\mathbf{G}_\kappa\} & \longrightarrow & \mathrm{BT}_n^p(\tilde{C}) \simeq \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Map}_{\mathrm{CAlg}/\kappa}^{\mathrm{cont}}(B_{\mathfrak{m}}, C) & \longrightarrow & \mathrm{BT}_n^p(C) \times_{\mathrm{BT}_n^p(\kappa)} \{\mathbf{G}_\kappa\} & \longrightarrow & \mathrm{BT}_n^p(C) \simeq \\
& & \downarrow & & \downarrow \\
& & \{\mathbf{G}_\kappa\} & \longrightarrow & \mathrm{BT}_n^p(\kappa) \simeq
\end{array} \tag{2.44}$$

The bottom-right square and right rectangle are both Cartesian by definition, so the upper-right square is Cartesian. It now suffices to see the upper rectangle is Cartesian, so we consider the following natural diagram of spaces:

$$\begin{array}{ccccc}
(\mathrm{Spf} B_{\mathfrak{m}})(\tilde{C}) & \longrightarrow & (\mathrm{Spf} B_{\mathfrak{m}})(C) & & \\
\downarrow & \searrow & \downarrow & \searrow & \\
& & \mathrm{BT}_n^p(\tilde{C}) \simeq & \longrightarrow & \mathrm{BT}_n^p(C) \simeq \\
& & \downarrow & & \downarrow \\
(\mathrm{Spf} B_{\mathfrak{m}})(\kappa) & \longrightarrow & (\mathrm{Spf} B_{\mathfrak{m}})(\kappa) & & \\
& \searrow & \downarrow & \searrow & \\
& & * & \longrightarrow & *
\end{array}$$

The top square is Cartesian as $\mathrm{Spf} B \rightarrow \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A}$ (and hence $\mathrm{Spf} B_{\mathfrak{m}} \rightarrow \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A}$) is formally étale, and the bottom square is trivially Cartesian. Taking the fibres of the vertical morphisms (at the given map $B_{\mathfrak{m}} \rightarrow \kappa$) we obtain the upper rectangle of (2.44), whence this upper rectangle is also Cartesian and we are done. \square

3 Orientations of p -divisible groups

The study of orientations of p -divisible (and formal) groups over \mathbf{E}_∞ -rings is the focus of [EC2]. Using Lurie’s work, we construct a “derived stack” classifying oriented p -divisible groups, $\mathcal{M}_{\mathrm{BT}_n^p}^{\mathrm{or}}$, defined on (not necessarily connective) p -complete \mathbf{E}_∞ -rings. The technical complications of this section stem from our movement between presheaves on connective and general \mathbf{E}_∞ -rings.

Let us suggest that the reader keeps a copy of [EC2] in her vicinity when reading this section.

3.1 The sheaf of oriented p -divisible groups

Recall the concept of an orientations of a *formal group*²³ over an \mathbf{E}_∞ -ring, as detailed in [EC2, §1.6 & 4.3].

Definition 3.1. Let R be an \mathbf{E}_∞ -ring and $\widehat{\mathbf{G}}$ be a formal group over R . A *preorientation* of $\widehat{\mathbf{G}}$ is an element e of $\Omega^2(\Omega^\infty \widehat{\mathbf{G}})(\tau_{\geq 0}R)$. Alternatively, assuming now that R is complex periodic (7), then an orientation of $\widehat{\mathbf{G}}$ is a morphism of formal groups $\widehat{\mathbf{G}}_R^{\mathcal{Q}} \rightarrow \widehat{\mathbf{G}}$ over R , where $\widehat{\mathbf{G}}_R^{\mathcal{Q}}$ is the Quillen formal group of R ; see (11). Such a preorientation $e: \widehat{\mathbf{G}}_R^{\mathcal{Q}} \rightarrow \widehat{\mathbf{G}}$ is an *orientation* if it is an equivalence of formal groups over R ; see [EC2, Pr.4.3.23]. Denote by $\text{OrDat}(\widehat{\mathbf{G}})$ the component of $\Omega^2(\Omega^\infty \widehat{\mathbf{G}})(\tau_{\geq 0}R)$ consisting of orientations—by definition this is empty if R is not complex periodic. An *orientation of a p -divisible group \mathbf{G}* over a p -complete \mathbf{E}_∞ -ring is an orientation of \mathbf{G}° , its identity component (10).

Recall that each time we associate to a functor $F: \mathcal{C} \rightarrow \mathcal{C}at_\infty$ (resp. $\mathcal{C} \rightarrow \mathcal{S}$) a coCartesian (resp. left) fibration $\int_{\mathcal{C}} F \rightarrow \mathcal{C}$, or visa versa, we are using the straightening–unstraightening adjunction of [HTT09, Th.3.2.0.1]—the ∞ -categorical Grothendieck construction.

- Definition 3.2.**
1. Let $\mathcal{M}_{\text{BT}^p}^{\text{nc}}: \text{CAlg} \rightarrow \mathcal{S}$ be the composite of $\tau_{\geq 0}: \text{CAlg} \rightarrow \text{CAlg}^{\text{cn}}$ and $\mathcal{M}_{\text{BT}^p}$; see [EC2, Var.2.0.6]. Define a functor $\mathcal{M}_{\text{BT}_n^p}^{\text{nc}}: \text{CAlg} \rightarrow \mathcal{S}$ analogously.
 2. Denote by CAlg^p the full ∞ -subcategory of CAlg consisting of p -complete \mathbf{E}_∞ -rings, and write Aff^p for $(\text{CAlg}^p)^{\text{op}}$. Let $\mathcal{M}_{\text{BT}^p}^{\text{un}}: \text{CAlg}^p \rightarrow \mathcal{S}$ be the composition of $\mathcal{M}_{\text{BT}^p}^{\text{nc}}$ with the inclusion $\text{CAlg}^p \rightarrow \text{CAlg}$. Define a functor $\mathcal{M}_{\text{BT}_n^p}^{\text{un}}: \text{CAlg}^p \rightarrow \mathcal{S}$ analogously.
 3. Let R be a p -complete \mathbf{E}_∞ -ring. Write $\text{OrBT}^p(R) \rightarrow \text{BT}^p(R)^\simeq$ for the left fibration associated to the following functor ([EC2, Rmk.4.3.4]):

$$\text{BT}^p(R)^\simeq \rightarrow \mathcal{S} \quad \mathbf{G} \mapsto \text{OrDat}(\mathbf{G}^\circ) = \text{OrDat}(\mathbf{G})$$

Define $\text{OrBT}_n^p(R)$ analogously.

We restrict to p -complete \mathbf{E}_∞ -rings above as we will often use [EC2, Th.2.0.8] to associate to a p -divisible group \mathbf{G} its identity component \mathbf{G}° . This is not strictly necessary, as demonstrated by [EC3, §2], however, we only care about the p -complete case to prove Th.1.6.

Our goal here is to define a moduli functor $\mathcal{M}_{\text{BT}^p}^{\text{or}}: \text{CAlg}^p \rightarrow \mathcal{S}$ sending R to $\text{OrBT}^p(R)^\simeq$, a sort of iterated Grothendieck construction. To do this honestly in the language of ∞ -categories, we will construct the associated left fibration; the reader is invited to skip the following technical construction for now, and only return if she is unconvinced by our heuristics.

²³Recall from [EC2, Df.1.6.1 & Var.1.6.2], a that a *formal group* over an \mathbf{E}_∞ -ring R is a functor $\widehat{\mathbf{G}}: \text{CAlg}_{\tau_{\geq 0}R}^{\text{cn}} \rightarrow \text{Mod}_{\mathbb{Z}}^{\text{cn}}$ whose post-composition with $\text{Mod}_{\mathbb{Z}}^{\text{cn}} \rightarrow \mathcal{S}$ is a formal hyperplane over $\tau_{\geq 0}R$ in the sense of [EC2, Df.1.5.10]. The latter can be identified as the essential image in $\mathcal{P}(\text{Aff}_{\tau_{\geq 0}R}^{\text{cn}})$ of certain cospectra of smooth coalgebras; see [EC2, §1.5].

Construction 3.3. Let $\mathrm{CAlg}_{\mathrm{co}}^p$ be the full ∞ -subcategory of CAlg^p spanned by those p -complete complex periodic \mathbf{E}_∞ -rings. Using [EC2, Rmk.1.6.4], define the functor $\mathcal{M}_{\mathrm{FGGroup}}(-)$

$$\mathrm{CAlg}_{\mathrm{co}}^p \xrightarrow{\mathrm{FGGroup}(-)} \mathcal{S} \quad R \mapsto \mathrm{FGGroup}(R)^\simeq$$

sending a p -complete \mathbf{E}_∞ -ring to the ∞ -groupoid core of its associated ∞ -category of formal groups ([EC2, Df.1.6.1]), and write $F: \mathcal{M}_{\mathrm{FGGroup}} \rightarrow \mathrm{CAlg}_{\mathrm{co}}^p$ for the associated left fibration. The functor F has a section \mathcal{Q} which sends a p -complete complex periodic \mathbf{E}_∞ -ring R to its Quillen formal group $\widehat{\mathbf{G}}_R^{\mathcal{Q}}$ ([EC2, Con.4.1.13]); which is functorial as taking the R -homology and then cospectrum are functorial. Let $\mathcal{M}_{\mathrm{OrFGGroup}}$ be the comma ∞ -category $(\mathcal{Q}F \downarrow \mathrm{id}_{\mathcal{M}_{\mathrm{FGGroup}}})$, in other words, there is a Cartesian diagram inside $\mathcal{C}at_\infty$

$$\begin{array}{ccc} \mathcal{M}_{\mathrm{OrFGGroup}} & \longrightarrow & (\mathcal{M}_{\mathrm{FGGroup}})^{\Delta^1} \\ \downarrow & & \downarrow (s,t) \\ \mathcal{M}_{\mathrm{FGGroup}} & \xrightarrow{(\mathcal{Q}F \times \mathrm{id}) \circ \Delta} & \mathcal{M}_{\mathrm{FGGroup}} \times \mathcal{M}_{\mathrm{FGGroup}} \end{array} \quad (3.4)$$

where Δ^1 is the 1-simplex, Δ is the diagonal map, and (s, t) sends an arrow in $\mathcal{M}_{\mathrm{FGGroup}}$ to its source and target. More informally, an object of $\mathcal{M}_{\mathrm{OrFGGroup}}$ is a complex periodic p -complete \mathbf{E}_∞ -ring R , a formal group $\widehat{\mathbf{G}}$ over R , and a equivalence $\widehat{\mathbf{G}}_R^{\mathcal{Q}} \simeq \widehat{\mathbf{G}}$ of formal groups over R . By [EC2, Pr.4.3.23], such a equivalence of formal groups over R is precisely the data of an orientation of $\widehat{\mathbf{G}}$, hence the name $\mathrm{OrFGGroup}$. The functor

$$\mathcal{M}_{\mathrm{OrFGGroup}} \rightarrow \mathcal{M}_{\mathrm{FGGroup}}, \quad (R, \widehat{\mathbf{G}}, e) \mapsto (R, \widehat{\mathbf{G}}) \quad (3.5)$$

is a left fibration with associated functor

$$\mathcal{M}_{\mathrm{FGGroup}} \rightarrow \mathcal{S} \quad (R, \widehat{\mathbf{G}}) \mapsto \mathrm{OrDat}(\widehat{\mathbf{G}}).$$

Indeed, this assignment is a functor by [EC2, Rmk.4.3.10] and the above identification comes by contemplating the fibre product of categories

$$\{(R, \widehat{\mathbf{G}})\}_{\mathcal{M}_{\mathrm{FGGroup}}} \times_{\mathcal{M}_{\mathrm{FGGroup}}} \mathcal{M}_{\mathrm{OrFGGroup}} \simeq \mathrm{Map}_{\mathcal{M}_{\mathrm{FGGroup}}(R)}(\widehat{\mathbf{G}}_R^{\mathcal{Q}}, \widehat{\mathbf{G}}) \simeq \mathrm{OrDat}(\widehat{\mathbf{G}}),$$

where the second equivalence again comes from [EC2, Pr.4.3.23]. Now, write $G: \mathcal{M}_{\mathrm{BT}^p}^{\mathrm{co}} \rightarrow \mathrm{CAlg}_{\mathrm{co}}^p$ for the left fibration associated to the following composition:

$$\mathcal{M}_{\mathrm{BT}^p}^{\mathrm{co}}(-): \mathrm{CAlg}_{\mathrm{co}}^p \xrightarrow{\mathrm{inc.}} \mathrm{CAlg}^p \xrightarrow{\mathcal{M}_{\mathrm{BT}^p}^{\mathrm{un}}} \mathcal{S} \quad R \mapsto \mathrm{BT}^p(R)^\simeq$$

The natural assignment sending a p -divisible group \mathbf{G} over a p -complete \mathbf{E}_∞ -ring R to its identity component induces a functor $(-)^{\circ}: \mathcal{M}_{\mathrm{BT}^p}^{\mathrm{co}} \rightarrow \mathcal{M}_{\mathrm{FGGroup}}$ between categories over $\mathrm{CAlg}_{\mathrm{co}}^p$. Define an ∞ -category $\mathcal{M}_{\mathrm{OrBT}^p}$ by the following Cartesian diagram of ∞ -categories:

$$\begin{array}{ccc} \mathcal{M}_{\mathrm{OrBT}^p} & \longrightarrow & \mathcal{M}_{\mathrm{OrFGGroup}} \\ \downarrow & & \downarrow \\ \mathcal{M}_{\mathrm{BT}^p}^{\mathrm{co}} & \xrightarrow{(-)^{\circ}} & \mathcal{M}_{\mathrm{FGGroup}} \end{array}$$

As (3.5) is a left fibration, then $\mathcal{M}_{\text{OrBT}^p} \rightarrow \mathcal{M}_{\text{BT}^p}^{\text{co}}$ is also a left fibration by base-change. Similarly, we define the ∞ -category $\mathcal{M}_{\text{BT}_n^p}^{\text{co}}$, which comes with a natural map $\mathcal{M}_{\text{BT}_n^p}^{\text{co}} \rightarrow \mathcal{M}_{\text{BT}^p}^{\text{co}}$ associated to the inclusion $\text{BT}_n^p(R)^\simeq \rightarrow \text{BT}^p(R)^\simeq$. Finally, define a left fibration $\mathcal{M}_{\text{OrBT}_n^p} \rightarrow \mathcal{M}_{\text{BT}^p}^{\text{co}}$ by the Cartesian diagram in $\mathcal{C}at_\infty$:

$$\begin{array}{ccc} \mathcal{M}_{\text{OrBT}_n^p} & \longrightarrow & \mathcal{M}_{\text{OrBT}^p} \\ \downarrow & & \downarrow \\ \mathcal{M}_{\text{BT}_n^p}^{\text{co}} & \longrightarrow & \mathcal{M}_{\text{BT}^p}^{\text{co}} \end{array}$$

In total, we have a left fibration

$$\mathcal{M}_{\text{OrBT}^p} \rightarrow \mathcal{M}_{\text{BT}^p} \rightarrow \text{CAlg}_{\text{co}}^p$$

and unravelling the construction above, one can calculate the functor associated to this composition:

$$\text{CAlg}_{\text{co}}^p \rightarrow \mathcal{S} \quad R \mapsto \text{OrBT}^p(R)$$

Similarly, we have $\mathcal{M}_{\text{OrBT}_n^p} \rightarrow \text{CAlg}_{\text{co}}^p$ and its associated functor.

Definition 3.6. Given a morphism $A \rightarrow B$ in CAlg^p , then if A is complex periodic, we see B is also complex periodic; see [EC2, Rmk.4.1.3]. Define a functor $\mathcal{M}_{\text{BT}^p}^{\text{or}}: \text{CAlg}^p \rightarrow \mathcal{S}$ first on $\text{CAlg}_{\text{co}}^p$ as the functor associated to the composition of left fibrations

$$\mathcal{M}_{\text{OrBT}^p} \rightarrow \mathcal{M}_{\text{BT}^p}^{\text{co}} \rightarrow \text{CAlg}_{\text{co}}^p$$

defined in Con.3.3, and then as the empty space on objects in CAlg^p who are not complex periodic. More informally, $\mathcal{M}_{\text{BT}^p}^{\text{or}}$ is the assignment:

$$R \mapsto \begin{cases} \text{OrBT}^p(R)^\simeq & \text{if } R \text{ is complex periodic} \\ \emptyset & \text{if } R \text{ is not complex periodic} \end{cases}$$

Define $\mathcal{M}_{\text{BT}_n^p}^{\text{or}}$ by the Cartesian square in $\mathcal{P}(\text{Aff}^p)$

$$\begin{array}{ccc} \mathcal{M}_{\text{BT}_n^p}^{\text{or}} & \longrightarrow & \mathcal{M}_{\text{BT}^p}^{\text{or}} \\ \downarrow \Omega & & \downarrow \Omega \\ \mathcal{M}_{\text{BT}_n^p}^{\text{un}} & \longrightarrow & \mathcal{M}_{\text{BT}^p}^{\text{un}} \end{array}$$

where right Ω is the functor naturally induced by Con.3.3.

The notation Ω is reminiscent of the word ‘‘orientation’’. sAt present, we have constructed a presheaf $\mathcal{M}_{\text{BT}_n^p}^{\text{or}}$, and a routine check shows this functor is a sheaf.

Proposition 3.7. *Let R be an \mathbf{E}_∞ -ring and n a positive integer. Then the functors*

$$\mathcal{M}_{\text{BT}^p}^{\text{un}}, \mathcal{M}_{\text{BT}_n^p}^{\text{un}}, \mathcal{M}_{\text{BT}^p}^{\text{or}}, \mathcal{M}_{\text{BT}_n^p}^{\text{or}}: \text{CAlg}^p \rightarrow \mathcal{S}$$

are all fpqc (hence also étale) hypersheaves.

As a first step, let us state a slight generalisation of [EC2, Pr.3.2.2(5)]; the proof is exactly as Lurie outlines in *ibid* but with fpqc hypercovers replacing fpqc covers.

Lemma 3.8. *The functors $\mathcal{M}_{\mathrm{BT}^p}, \mathcal{M}_{\mathrm{BT}^p_n} : \mathrm{CAlg}^{\mathrm{cn}} \rightarrow \mathcal{S}$ are fpqc hypersheaves.*

Proof of Theorem 3.7. To see $\mathcal{M}_{\mathrm{BT}^p}^{\mathrm{un}}$ in $\mathcal{P}(\mathrm{Aff}^p)$ is an fpqc hypersheaf, it suffices to see $\mathcal{M}_{\mathrm{BT}^p}^{\mathrm{nc}}$ in $\mathcal{P}(\mathrm{Aff})$ is an fpqc hypersheaf as the inclusion $\mathrm{CAlg}^p \rightarrow \mathrm{CAlg}$ sends fpqc hypercovers to fpqc hypercovers. To see $\mathcal{M}_{\mathrm{BT}^p}^{\mathrm{nc}}$ is an fpqc hypersheaf, take an \mathbf{E}_∞ -ring R and an fpqc hypercover $R \rightarrow R^\bullet$. From Lm.0.4, (0.5), and Lm.3.8 we see that the natural maps

$$\mathcal{M}_{\mathrm{BT}^p}^{\mathrm{nc}}(R) = \mathcal{M}_{\mathrm{BT}^p}(\tau_{\geq 0}R) \xrightarrow{\simeq} \mathcal{M}_{\mathrm{BT}^p}(\lim(\tau_{\geq 0}R)^\bullet) \xrightarrow{\simeq} \lim \mathcal{M}_{\mathrm{BT}^p}((\tau_{\geq 0}R)^\bullet) = \lim \mathcal{M}_{\mathrm{BT}^p}^{\mathrm{nc}}(R^\bullet)$$

are all equivalences. Hence $\mathcal{M}_{\mathrm{BT}^p}^{\mathrm{nc}}$, and also $\mathcal{M}_{\mathrm{BT}^p}^{\mathrm{un}}$, are fpqc hypersheaves. It follows that $\mathcal{M}_{\mathrm{BT}^p_n}^{\mathrm{un}}$ is also an fpqc hypersheaf as it is an open subfunctor of $\mathcal{M}_{\mathrm{BT}^p}^{\mathrm{un}}$; see Remark 1.2.

By Lm.0.3, it suffices to see that the functor $\mathcal{M}_{\mathrm{BT}^p}^{\mathrm{un}} : \mathrm{CAlg}^p \rightarrow \mathcal{S}$ is an fpqc hypersheaf, and that the functor F defined on objects by

$$F : \mathrm{CAlg}^p \rightarrow (\mathcal{C}\mathrm{at}_\infty)_{/\mathcal{S}}, \quad R \mapsto \left(\begin{array}{ccc} \mathrm{BT}^p(R)^\simeq & \rightarrow & \mathcal{S} \\ \mathbf{G} & \mapsto & \mathrm{OrDat}(\mathbf{G}^\circ) \end{array} \right) \quad (3.9)$$

is an fpqc hypersheaf; to define this functor honestly, one can use the standard techniques as done in Con.3.3. We have just seen this for $\mathcal{M}_{\mathrm{BT}^p}^{\mathrm{un}}$, so it suffices to see that (3.9) is an fpqc hypersheaf. Again, write $R \rightarrow R^\bullet$ for an fpqc hypercover of R in CAlg^p . As $\mathrm{BT}^p(-) : \mathrm{CAlg}^{\mathrm{cn}} \rightarrow \mathcal{S}$ is an fpqc hypersheaf (Lm.3.8), we obtain the following natural equivalence from the definition of $\mathrm{OrDat}(\lim R^\bullet)$:

$$\left(\begin{array}{ccc} \mathrm{BT}^p(\lim R^\bullet)^\simeq & \rightarrow & \mathcal{S} \\ \mathbf{G} & \mapsto & \mathrm{OrDat}(\mathbf{G}^\circ) \end{array} \right) \xrightarrow{\simeq} \left(\begin{array}{ccc} \lim \mathrm{BT}^p(R^\bullet)^\simeq & \rightarrow & \mathcal{S} \\ \mathbf{G}_\bullet & \mapsto & \mathrm{OrDat}((\lim \mathbf{G}_\bullet)^\circ) \end{array} \right) \quad (3.10)$$

Above, we have written \mathbf{G}_\bullet for the base-change of \mathbf{G} over R^\bullet . Using the characterising property of the identity component (as seen in [EC2, Th.2.0.8]), we take some $A \in \mathcal{E}$ (using the notation of [EC2, Th.2.0.8] and (10)) and obtain the following sequence of natural equivalences where all fibres are taken over the identity element:

$$\begin{aligned} (\lim \mathbf{G}_\bullet)^\circ(A) &= \mathrm{fib}(\lim(\mathbf{G}_\bullet)(A) \rightarrow \lim(\mathbf{G}_\bullet)(A^{\mathrm{red}})) \simeq \lim \mathrm{fib}(\mathbf{G}_\bullet(A) \rightarrow \mathbf{G}_\bullet(A^{\mathrm{red}})) \\ &= \lim(\mathbf{G}_\bullet^\circ(A)) \xleftarrow{\simeq} (\lim \mathbf{G}_\bullet^\circ)(A) \end{aligned}$$

The first equivalence comes from the fact that fibres commutes with small limits and the second equivalence from the fact that limits in functor ∞ -categories are computed levelwise. From this we see that (3.10) is naturally equivalent to

$$\left(\begin{array}{ccc} \lim \mathrm{BT}^p(R^\bullet)^\simeq & \rightarrow & \mathcal{S} \\ \mathbf{G}_\bullet & \mapsto & \mathrm{OrDat}(\lim(\mathbf{G}^\circ)_\bullet) \end{array} \right) \quad (3.11)$$

where $(\mathbf{G}^\circ)_\bullet$ is the base-change of \mathbf{G}° over R^\bullet . For a fixed pointed formal hyperplane X over an \mathbf{E}_∞ -ring R , the functor

$$\mathrm{CAlg}_R \rightarrow \mathcal{S}, \quad A \mapsto \mathrm{OrDat}(X_A)$$

is representable by [EC2, Pr.4.3.13], hence it commutes with small limits. In particular, this implies that the expression (3.11) is naturally equivalent to

$$\left(\begin{array}{ccc} \lim \mathrm{BT}^p(R^\bullet) \simeq & \rightarrow & \mathcal{S} \\ & \mathbf{G}_\bullet \mapsto & \lim \mathrm{OrDat}(\mathbf{G}_\bullet^\circ) \end{array} \right) = \lim F(R^\bullet)$$

Combining everything, we obtain our desired natural equivalence $F(R) \xrightarrow{\simeq} \lim F(R^\bullet)$. The corresponding statement for $\mathcal{M}_{\mathrm{BT}_n^p}^{\mathrm{or}}$ follows as it is a fibre product of fpqc hypersheaves. \square

3.2 Orientation classifiers

It is our goal now to try and understand *universal* orientations and their relation to $\mathcal{M}_{\mathrm{BT}_n^p}^{\mathrm{or}}$. We would like to formally construct a functor $\mathfrak{D}_{\mathrm{BT}_n^p}^{\mathrm{or}}$ which we think of as $\Omega_* \mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{or}}$. We perform this construction by first restricting ourselves to affines.

Definition 3.12. Recall Nt.0.9. Write $\mathcal{C}_{A_0}^{\mathrm{aff}}$ (resp. $\mathcal{C}_A^{\mathrm{aff}}$) for the full ∞ -subcategory of \mathcal{C}_{A_0} (reps. \mathcal{C}_A) spanned by affine objects.

We will now define a CAlg -valued presheaf $\mathfrak{D}_{\mathrm{BT}_n^p}^{\mathrm{aff}}$ on $\mathcal{C}_A^{\mathrm{aff}}$ as a composite of certain functors, which we describe now.

Definition 3.13. Write $\mathrm{Aff}_{/\mathcal{M}_{\mathrm{BT}_n^p}^{\mathrm{un}}}^p$ for the ∞ -subcategory of $\mathcal{P}(\mathrm{Aff}^p)_{/\mathcal{M}_{\mathrm{BT}_n^p}^{\mathrm{un}}}$ spanned by affines.

1. Define a functor $a: \mathcal{C}_A^{\mathrm{aff}} \rightarrow \mathrm{Aff}_{/\mathcal{M}_{\mathrm{BT}_n^p}^{\mathrm{un}}}^p$ by sending an object²⁴ $\mathbf{G}: \mathrm{Spf} B_J^\wedge \rightarrow \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A}$ first to the composite with the canonical maps $\widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A} \rightarrow \mathcal{M}_{\mathrm{BT}_n^p}$, then from $\mathrm{Spf} B_J^\wedge \rightarrow \mathcal{M}_{\mathrm{BT}_n^p}$ to its *algebraisation*²⁵ $\mathbf{G}^{\mathrm{alg}}: \mathrm{Spec} B_J^\wedge \rightarrow \mathcal{M}_{\mathrm{BT}_n^p}$ which naturally lives in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}, p})$ as \mathfrak{m}_A , and hence J , contains p , and then we apply $\tau_{\geq 0}^*: \mathcal{P}(\mathrm{Aff}^{\mathrm{cn}, p}) \rightarrow \mathcal{P}(\mathrm{Aff}^p)$.
2. Define a functor $\Gamma(\Omega_*(-)): (\mathrm{Aff}_{/\mathcal{M}_{\mathrm{BT}_n^p}^{\mathrm{un}}}^p)^{\mathrm{op}} \rightarrow \mathrm{CAlg}$ by pullback along $\Omega: \mathcal{M}_{\mathrm{BT}_n^p}^{\mathrm{or}} \rightarrow \mathcal{M}_{\mathrm{BT}_n^p}^{\mathrm{un}}$ followed by the global sections functor $(\mathrm{Aff}_{/\mathcal{M}_{\mathrm{BT}_n^p}^{\mathrm{or}}}^p)^{\mathrm{op}} \rightarrow \mathrm{CAlg}^p$, which is just a forgetful functor.

Let $\mathfrak{D}_{\mathrm{BT}_n^p}^{\mathrm{aff}}: (\mathcal{C}_A)^{\mathrm{op}} \rightarrow \mathrm{CAlg}$ be the composition of a followed by $\Gamma(\Omega_*(-))$, which is an étale hypersheaf as a sends étale hypercovers to étale hypercovers by construction, and $\mathcal{M}_{\mathrm{BT}_n^p}^{\mathrm{un}}$ and $\mathcal{M}_{\mathrm{BT}_n^p}^{\mathrm{or}}$ are étale hypersheaves by Pr.3.7. We also define $\mathfrak{D}_{\mathrm{BT}_n^p}^{\mathrm{or}}: \mathcal{C}_A^{\mathrm{op}} \rightarrow \mathrm{CAlg}$ by right Kan extension along the inclusion $(\mathcal{C}_A^{\mathrm{aff}})^{\mathrm{op}} \rightarrow \mathcal{C}_A^{\mathrm{op}}$. As a right Kan extensions preserve limits, we see $\mathfrak{D}_{\mathrm{BT}_n^p}^{\mathrm{or}}$ is an étale hypersheaf.

²⁴Recall that the formal spectrum $\mathrm{Spf} B$ is equivalent to $\mathrm{Spf} B_J^\wedge$ where J is a finitely generated ideal of definition for B ; see [SAG, Rmk.8.1.2.4].

²⁵Recall from [EC2, Th.3.2.2(4)], the map $\mathcal{M}_{\mathrm{BT}^p}(\mathrm{Spec} B) \rightarrow \mathcal{M}_{\mathrm{BT}^p}(\mathrm{Spf} B)$ is an equivalence of spaces if B is complete with respect to its ideal of definition. We call any $\mathbf{G}^{\mathrm{alg}}: \mathrm{Spec} B \rightarrow \mathcal{M}_{\mathrm{BT}^p}$ the *algebraisation* of the corresponding $\mathbf{G}: \mathrm{Spf} B \rightarrow \mathcal{M}_{\mathrm{BT}^p}$. This also implies the natural map $\mathcal{M}_{\mathrm{BT}_n^p}(\mathrm{Spec} B) \rightarrow \mathcal{M}_{\mathrm{BT}_n^p}(\mathrm{Spf} B)$ is an equivalence for B which are complete with respect to their ideal of definition, and we likewise use the phrase *algebraisation*.

Remark 3.14. The right Kan extension defining $\mathfrak{D}_{\mathrm{BT}_n^p}^{\mathrm{or}}$ on \mathcal{C}_A can be made more explicit. Indeed, by assumption, each object \mathfrak{X} in \mathcal{C}_A is qcqs, so by Pr.A.17, we have an étale hyper cover $\mathfrak{Y}_\bullet \rightarrow \mathfrak{X}$ such that each $\mathfrak{Y}_n = \mathrm{Spf} B_n$ is affine. The fact that $\mathfrak{D}_{\mathrm{BT}_n^p}^{\mathrm{or}}$ is an étale hypersheaf (as this is true étale locally on affines) then gives us a formula for $\mathfrak{D}_{\mathrm{BT}_n^p}^{\mathrm{or}}(\mathfrak{X})$:

$$\mathfrak{D}_{\mathrm{BT}_n^p}^{\mathrm{or}}(\mathfrak{X}) \simeq \lim \left(\mathfrak{D}_{\mathrm{BT}_n^p}^{\mathrm{aff}}(\mathrm{Spf} B^0) \rightrightarrows \mathfrak{D}_{\mathrm{BT}_n^p}^{\mathrm{aff}}(\mathrm{Spf} B^1) \rightrightarrows \cdots \right)$$

By Pr.3.15 below, the terms in the above limit take a known form.

The above is a formal construction of an étale hypersheaf $\mathfrak{D}_{\mathrm{BT}_n^p}^{\mathrm{or}}: \mathcal{C}_A^{\mathrm{op}} \rightarrow \mathrm{CAlg}$, however, we would like to be able to calculate with this functor as well.

Proposition 3.15. *Given a p -complete \mathbf{E}_∞ -ring R and an associated p -divisible group \mathbf{G} of height n , then there is a natural equivalence of p -complete \mathbf{E}_∞ -rings*

$$\Gamma(\Omega_*(\mathbf{G})) \simeq \widehat{\mathfrak{D}}_{\mathbf{G}^\circ}$$

where the latter is the p -completion of $\mathfrak{D}_{\mathbf{G}^\circ}$, the orientation classifier²⁶ for \mathbf{G}° .

Proof. From the definition of $\Gamma(\Omega_*(-))$, it suffices to show that the following natural square of presheaves of p -complete \mathbf{E}_∞ -rings is Cartesian:²⁷

$$\begin{array}{ccc} \mathrm{Spec} \widehat{\mathfrak{D}}_{\mathbf{G}^\circ} & \longrightarrow & \mathcal{M}_{\mathrm{BT}_n^p}^{\mathrm{or}} \\ \downarrow & & \downarrow \Omega \\ \mathrm{Spec} R & \xrightarrow{\mathbf{G}} & \mathcal{M}_{\mathrm{BT}_n^p}^{\mathrm{un}} \end{array} \quad (3.16)$$

Fix a p -complete \mathbf{E}_∞ -ring A and evaluating the above diagram at A . If there are no maps of p -complete \mathbf{E}_∞ -rings $R \rightarrow A$, then the two left-most spaces are empty and we are done, so let us then fix a map $\psi: R \rightarrow A$. We then note the following chain of natural equivalences between the fibres of the vertical morphisms from left to right:

$$\mathrm{Spec} \widehat{\mathfrak{D}}_{\mathbf{G}^\circ}(A) \simeq \mathrm{Map}_{\mathrm{CAlg}}(\mathfrak{D}_{\mathbf{G}^\circ}, A) \simeq \mathrm{OrDat}(\mathbf{G}_A^\circ) \simeq \{\psi\} \times_{\mathcal{M}_{\mathrm{BT}_n^p}^{\mathrm{un}}(A)} \mathcal{M}_{\mathrm{BT}_n^p}^{\mathrm{or}}(A)$$

The first equivalence follows as p -completion is a left adjoint, the second from [EC2, Pr.4.3.13], and the third from the construction of Ω ; see Con.3.3. As these equivalences are natural in A , this shows (3.16) is Cartesian. \square

Now that we can calculate $\mathfrak{D}_{\mathrm{BT}_n^p}^{\mathrm{or}}$ when restricted to affines, we are close to definition $\mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}$ and proving Th.1.6.

²⁶Recall from [EC2, §4.3.3], for a formal group $\widehat{\mathbf{G}}$ over an \mathbf{E}_∞ -ring R , the *orientation classifier* of $\widehat{\mathbf{G}}$ is the corepresenting R -algebra for the functor $\mathrm{CAlg}_R \rightarrow \mathcal{S} \quad A \mapsto \mathrm{OrDat}(\widehat{\mathbf{G}}_A)$.

²⁷We are abusing notation here and writing Spec for the Yoneda embedding $\mathrm{Aff}^p \rightarrow \mathcal{P}(\mathrm{Aff}^p)$.

4 The sheaf $\mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}$ and a proof of Th.1.6

The definition of $\mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}$ mirrors Lurie's definition of $\mathcal{O}^{\mathrm{top}}$ ([EC2, §7.3]) and the proof that this definition satisfies Th.1.6 also follows Lurie's proof.

Definition 4.1. Fix an adic \mathbf{E}_∞ -ring A as in Nt.0.9. Let $\mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}$ be the étale hypersheaf on \mathcal{C}_{A_0} defined by the composition

$$\mathcal{C}_{A_0}^{\mathrm{op}} \xrightarrow{\mathfrak{D}^{\mathrm{op}}} \mathcal{C}_A^{\mathrm{op}} \xrightarrow{\mathfrak{D}_{\mathrm{BT}_n^p}^{\mathrm{or}}} \mathrm{CAlg} \quad (4.2)$$

or in other words, first one calculates the universal spectral deformation of $\mathbf{G}_0: \mathfrak{X}_0 \rightarrow \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A_0}^\heartsuit$ giving $\mathfrak{D}(\mathbf{G}_0) = \mathbf{G}$ (Rmk.2.38), then the identity component \mathbf{G}° of \mathbf{G} , and $\mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}(\mathbf{G}_0)$ is then the p -completion orientation classifier of \mathbf{G}° ; we will see in the proof of Th.1.6 below that this p -completion is unnecessary. It follows from Th.2.34 and Df.3.13 that $\mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}$ is an étale hypersheaf.

With our sheaf in hand, we can prove Lurie's theorem; the following follows the outline of the proof of [EC2, Th.7.0.1].

Proof of Th.1.6. We have an étale hypersheaf $\mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}$ on \mathcal{C}_{A_0} from Df.4.1. It remains to show that when restricted to objects $\mathbf{G}_0: \mathrm{Spf} B_0 \rightarrow \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A_0}^\heartsuit$ in $\mathcal{C}_{A_0}^{\mathrm{aff}}$, where we may assume B_0 is complete with respect to its ideal of definition J , the \mathbf{E}_∞ -ring $\mathcal{E} = \mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}(\mathbf{G}_0)$ has the expected properties 1-4 of Th.1.6. As mentioned in Df.4.1, $\mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}$ can equivalently be described as applying the functor \mathfrak{D} followed by $\mathfrak{D}_{\mathrm{BT}_n^p}^{\mathrm{or}}$. Under \mathfrak{D} , the object \mathbf{G}_0 is sent to the affine object $\mathbf{G}: \mathrm{Spf} B \rightarrow \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A}^{\mathrm{aff}}$ of $\mathcal{C}_A^{\mathrm{aff}}$ such that $\pi_0 B \simeq B_0$ and \mathbf{G}_{un} is equivalent to \mathbf{G}_0 over $\mathrm{Spf} B_0$; see Df.2.33. By Pr.3.15 and (4.2), we see \mathcal{E} is the p -completion of the orientation classifier of the identity component \mathbf{G}° of \mathbf{G} , denoted by $\mathfrak{D}_{\mathbf{G}^\circ}$. First we will argue that the \mathbf{E}_∞ -ring $\mathfrak{D}_{\mathbf{G}^\circ}$ satisfies the desired properties 1-3, and then for \mathcal{E} .

Firstly, note that as $\mathfrak{D}_{\mathbf{G}^\circ}$ is an orientation classifier, [EC2, Pr.4.3.23] states that $\mathfrak{D}_{\mathbf{G}^\circ}$ is complex periodic (we will discuss Landweber exactness at the very end). It follows that \mathcal{E} is complex periodic as it receives an \mathbf{E}_∞ -ring homomorphism $\mathfrak{D}_{\mathbf{G}^\circ} \rightarrow \mathcal{E}$ from a complex oriented one; see [EC2, Rmk.4.1.10].

To see conditions 2 and 3 (except for the identification of $\pi_{2k}\mathcal{E}$), it suffices to show the formal group \mathbf{G}° is *balanced*²⁸ over B . Indeed, as we have proven condition 1 of Th.1.6 which states that $\Sigma^2\mathfrak{D}_{\mathbf{G}^\circ}$ is a locally free of rank 1 so each $\pi_{2k}\mathfrak{D}_{\mathbf{G}^\circ}$ is a line bundle over $\pi_0\mathfrak{D}_{\mathbf{G}^\circ}$. If \mathbf{G}° is balanced over B , then each $\pi_k\mathfrak{D}_{\mathbf{G}^\circ}$ is complete with respect to the ideal of definition J of $\pi_0\mathfrak{D}_{\mathbf{G}^\circ} \simeq B_0$, so $\mathfrak{D}_{\mathbf{G}^\circ}$ itself is J -complete, hence also \mathfrak{m}_A -complete and p -complete. This would also imply that $\mathcal{E} \simeq \mathfrak{D}_{\mathbf{G}^\circ}$. To show \mathbf{G}° is balanced over B , we use [EC2, Rmk.6.4.2]

²⁸Recall from [EC2, §6.4.1], that a formal group $\widehat{\mathbf{G}}$ over a connective \mathbf{E}_∞ -ring R is *balanced* if the unit map $R \rightarrow \mathfrak{D}_{\widehat{\mathbf{G}}}$ induces an equivalence on π_0 and the homotopy groups of $\mathfrak{D}_{\widehat{\mathbf{G}}}$ are concentrated in even degree.

(twice) to reduce ourselves to showing that $\mathbf{G}_{B_{\mathfrak{m}}^{\wedge}}^{\circ}$ is balanced over $B_{\mathfrak{m}}^{\wedge}$ for every maximal ideal $\mathfrak{m} \subseteq \pi_0 B \simeq B_0$; these ideals contain J as B_0 is J -complete. By Pr.2.42, we see $\mathbf{G}_{B_{\mathfrak{m}}^{\wedge}}$ is the universal spectral deformation of \mathbf{G}_{κ} , where κ is the residue field of $B_{\mathfrak{m}}^{\wedge}$, and a powerful statement of Lurie [EC2, Th.6.4.6] then implies the identity component $\mathbf{G}_{B_{\mathfrak{m}}^{\wedge}}^{\circ}$ of $\mathbf{G}_{B_{\mathfrak{m}}^{\wedge}}$ is balanced. Hence $\mathfrak{D}_{\mathbf{G}^{\circ}} \simeq \mathcal{E}$ satisfies conditions 2 and 3 (except for the identification of $\pi_{2k}\mathcal{E}$).

For condition 4, [EC2, Pr.4.3.23] states that the canonical orientation of the p -divisible group \mathbf{G} over \mathcal{E} supplies us with an equivalence $\widehat{\mathbf{G}}_{\mathcal{E}}^{\mathcal{Q}} \xrightarrow{\simeq} \mathbf{G}^{\circ}$ between the Quillen formal group of \mathcal{E} and the identity component of \mathbf{G} . In particular, this implies the classical Quillen formal group $\widehat{\mathbf{G}}_{\mathcal{E}}^{\mathcal{Q}^0}$ is isomorphic to the formal group \mathbf{G}_0° after an extension of scalars along the unit map $B_0 \simeq \pi_0 B \rightarrow \pi_0 \mathcal{E}$. As \mathbf{G}° is a balanced formal group over B , this unit map is an isomorphism, giving us property 4.

To round off condition 3 and calculate $\pi_{2k}\mathcal{E}$, we note this follows from the facts that \mathcal{E} is weakly 2-periodic, the p -divisible group \mathbf{G} over \mathcal{E} comes equipped with a canonical orientation and hence a chosen equivalence of locally free \mathcal{E} -modules of rank 1 $\beta: \omega_{\mathbf{G}} \rightarrow \Sigma^{-2}\mathcal{E}$, and the equivalence of B_0 -modules $\pi_0\omega_{\mathbf{G}} \simeq \omega_{\mathbf{G}_0}$:

$$\pi_{2k}\mathcal{E} \simeq (\pi_2\mathcal{E})^{\otimes k} \simeq (\pi_0\omega_{\mathbf{G}})^{\otimes k} \simeq \omega_{\mathbf{G}_0}^{\otimes k}$$

Finally, to finish condition 1 and the Landweber exactness of \mathcal{E} , we appeal directly to Behrens–Lawson’s arguments in [BL10, Lm.8.1.6 & Cor.8.1.7], as they are checking the same conditions on a sheaf with the same properties as ours above. \square

Remark 4.3. Let us close this section by stating that there have, of course, been other iterations of Lurie’s theorem; see [BL10, Th.8.1.4] and [Beh20, §6.7]. The statements made there are certainly not aesthetically identical²⁹ to our Th.1.6, however, we believe that the section to follow, detailing applications of Lurie’s theorem, justifies that all available statements of Lurie’s theorem apply to the same set of examples. In particular, as we can construct Lubin–Tate theories, TMF, and TAF, all using Th.1.6, we do not find any reason to compare all available statements in too much detail—neither would we know how to.

5 Applications of Lurie’s theorem

To advertise Lurie’s theorem to a wider audience and lay (known) groundwork for future applications, let us now discuss how the titular theorem of this article can be used. A vast majority of the applications below can be found in either [BL10], [EC2], or [Beh20, §6.7], in some form.

²⁹Phrases such as “(locally) fibrant in the Jardine model structure” can be translated to “étale hypersheaf”, and compatibility with checking fibres are universal deformation spaces and the adjective “formally étale” is explained in [Beh20, Rmk.6.7.5]; see Pr.2.10 for a similar iteration of that idea.

5.1 Complex topological K -theory

As our first application of Th.1.6, we would like to prove that one of the simplest p -divisible groups gives us an example of an \mathbf{E}_∞ -ring near and dear to stable homotopy theory: complex topological K -theory. To define the \mathbf{E}_∞ -ring \mathbf{KU} , we will follow the construction of [EC2, §6.5], which we will repeat here for the reader's convenience. Then we will discuss some specific classes in \mathbf{KU} -cohomology.

Construction 5.1. Denote by $\mathbf{Vect}_{\tilde{\mathbf{C}}}$ the 1-category of finite-dimensional complex vector spaces and complex linear isomorphisms. Considering this as a topologically enriched category with a symmetric monoidal structure given by the direct sum of vector bundles, the (topological) coherent nerve $\mathbf{N}(\mathbf{Vect}_{\tilde{\mathbf{C}}})$ is a Kan complex with an \mathbf{E}_∞ -structure. The inclusion

$$\coprod_{n \geq 0} \mathbf{BU}(n) \rightarrow \mathbf{N}(\mathbf{Vect}_{\tilde{\mathbf{C}}})$$

classified on each summand $\mathbf{BU}(n)$ by the universal n -dimensional complex vector bundle, is an equivalence of spaces and the \mathbf{E}_∞ -structure restricts to one on the domain. The group completion of this \mathbf{E}_∞ -space is the zeroth space of a connective spectrum \mathbf{ku} , *connective complex topological K -theory*, and the natural group completion map can be identified with the map

$$\xi: \coprod_{n \geq 0} \mathbf{BU}(n) \simeq \mathbf{N}(\mathbf{Vect}_{\tilde{\mathbf{C}}}) \rightarrow \Omega^\infty \mathbf{ku} \simeq \mathbf{Z} \times \mathbf{BU}$$

sending each $\mathbf{BU}(n)$ component to $\{n\} \times \mathbf{BU}$ via the canonical inclusion, which represents the universal complex vector bundle ξ_n over $\mathbf{BU}(n)$. There is also a multiplicative \mathbf{E}_∞ -structure on $\mathbf{N}(\mathbf{Vect}_{\tilde{\mathbf{C}}})$ given by the tensor product of vector bundles, which also gives the connective spectrum \mathbf{ku} the structure of a connective \mathbf{E}_∞ -ring; see [GGN15, Ex.5.3(ii)]. The map ξ is also a morphism of \mathbf{E}_∞ -spaces with respect to this multiplicative \mathbf{E}_∞ -structure. By identifying $\mathbf{CP}^\infty \simeq \mathbf{BU}(1)$ as a summand of $\mathbf{N}(\mathbf{Vect}_{\tilde{\mathbf{C}}})$, then \mathbf{CP}^∞ inherits the multiplicative \mathbf{E}_∞ -structure, as the tensor product of line bundles are line bundles. As ξ restricted to \mathbf{CP}^∞ lands in the identity component of $\Omega^\infty \mathbf{ku}$, that is $\{1\} \times \mathbf{BU}$, we obtain a map of \mathbf{E}_∞ -spaces $\mathbf{CP}^\infty \rightarrow \mathbf{GL}_1(\mathbf{ku})$. Under the adjunction³⁰

$$\Sigma_+^\infty: \mathbf{CMon} \rightleftarrows \mathbf{CAlg}: \mathbf{GL}_1$$

we obtain a morphism of \mathbf{E}_∞ -rings $\rho: \Sigma_+^\infty \mathbf{CP}^\infty \rightarrow \mathbf{ku}$. Furthermore, the based inclusion $\iota: S^2 \simeq \mathbf{CP}^2 \rightarrow \mathbf{CP}^\infty$ post-composed with the unit $\eta: \mathbf{CP}^\infty \rightarrow \Omega^\infty \Sigma^\infty \mathbf{CP}^\infty$ followed by Ω^∞ of the inclusion into the first summand $j: \Sigma^\infty \mathbf{CP}^\infty \rightarrow \Sigma^\infty \mathbf{CP}^\infty \oplus \mathbf{S} \simeq \Sigma_+^\infty \mathbf{CP}^\infty$ gives us an element β inside $\pi_2 \Sigma_+^\infty \mathbf{CP}^\infty$. The image of β under the map ρ is also called $\beta \in \pi_2 \mathbf{ku}$,

³⁰Recall the $(\Sigma_+^\infty, \mathbf{GL}_1)$ -adjunction (see [ABG⁺14, §2] for a modern reference) is the composite of two adjunctions:

$$\mathbf{CMon}^{\text{grp}} \begin{array}{c} \xrightarrow{\text{inc.}} \\ \xleftarrow{\mathbf{GL}_1} \end{array} \mathbf{CMon} \begin{array}{c} \xrightarrow{\Sigma_+^\infty} \\ \xleftarrow{\Omega^\infty} \end{array} \mathbf{CAlg}$$

The superscript $(-)^{\text{grp}}$ denotes those \mathbf{E}_∞ -spaces whose π_0 is a group. The functor $\mathbf{GL}_1: \mathbf{CMon} \rightarrow \mathbf{CMon}^{\text{grp}}$ sends an \mathbf{E}_∞ -space X to the subspace $\mathbf{GL}_1 X$ spanned by those path components of X with inverses in $\pi_0 X$.

which one can identify with the element $[\gamma_1] - 1$ inside $\widetilde{\text{ku}}^0(\mathbf{CP}^1)$, where γ_1 is the tautological line bundle over \mathbf{CP}^1 ; a consequence of Pr.5.2. We define the \mathbf{E}_∞ -ring of *periodic complex topological K-theory* as the localisation $\text{KU} = \text{ku}[\beta^{-1}]$; see [EC2, Pr.4.3.17] for a discussion about localising line bundles over \mathbf{E}_∞ -rings, and [HA, §7.2.3] for the \mathbf{E}_1 -ring case.

Proposition 5.2. *The composition*

$$\mathbf{CP}^\infty \xrightarrow{\eta} \Omega^\infty \Sigma^\infty \mathbf{CP}^\infty \xrightarrow{\Omega^\infty j} \Omega^\infty \Sigma_+^\infty \mathbf{CP}^\infty \xrightarrow{\Omega^\infty \rho} \Omega^\infty \text{ku}$$

represents the class $[\xi_1] - 1$ in $\widetilde{\text{ku}}^0(\mathbf{CP}^\infty)$, where ξ_1 is the universal line bundle over \mathbf{CP}^∞ .

Let us recall that for a spectrum E and a based space X , one defines the unreduced and reduced E cohomology groups of X as the abelian groups:

$$E^0(X) = \pi_0 \text{Map}_{\text{Sp}}(\Sigma_+^\infty X, E) \simeq \pi_0 \text{Map}_{\mathcal{S}}(X, \Omega^\infty E)$$

$$\widetilde{E}^0(X) = \pi_0 \text{Map}_{\text{Sp}}(\Sigma^\infty X, E) \simeq \pi_0 \text{Map}_{\mathcal{S}_*}(X, \Omega^\infty E)$$

Let us also state a lemma we will use regarding the $(\Sigma_+^\infty, \mathbf{GL}_1)$ -adjunction; we only state it to keep track of base points.

Lemma 5.3. *If R is an \mathbf{E}_∞ -ring, then the composite*

$$\mathbf{GL}_1(R) \rightarrow \mathbf{GL}_1(R)_+ \xrightarrow{\eta_+} \Omega^\infty \Sigma_+^\infty \mathbf{GL}_1(R) \xrightarrow{\Omega^\infty \epsilon} \Omega^\infty R$$

is homotopic to the inclusion $\mathbf{GL}_1(R) \rightarrow \Omega^\infty R$, where $\epsilon: \Sigma_+^\infty \mathbf{GL}_1(R) \rightarrow R$ is the counit of the $(\Sigma_+^\infty, \mathbf{GL}_1)$ -adjunction.

Note that the unit and the counit appearing in the lemma above do not come from the same adjunction.

Proof. The $(\Sigma_+^\infty, \mathbf{GL}_1)$ -adjunction is a composite of the adjunctions

$$\text{CMon}^{\text{grp}} \xrightleftharpoons[\mathbf{GL}_1]{\text{inc.}} \text{CMon} \xrightleftharpoons[\Omega^\infty]{\Sigma_+^\infty} \text{CAlg}$$

so the counit $\epsilon: \Sigma_+^\infty \mathbf{GL}_1(R) \rightarrow R$ factors as the composite $\Sigma_+^\infty \mathbf{GL}_1(R) \rightarrow \Sigma_+^\infty \Omega^\infty R \rightarrow R$ where the first map is induced by the defining inclusion $\mathbf{GL}_1(R) \rightarrow \Omega^\infty R$ and the second is the counit of the $(\Sigma_+^\infty, \Omega^\infty)$ -adjunction. This implies the diagram of spaces

$$\begin{array}{ccccccc} \mathbf{GL}_1(R) & \longrightarrow & \mathbf{GL}_1(R)_+ & \xrightarrow{\eta_+} & \Omega^\infty \Sigma_+^\infty \mathbf{GL}_1(R) & & \\ \downarrow & & \downarrow & & \downarrow & \searrow^{\Omega^\infty \epsilon} & \\ \Omega^\infty R & \longrightarrow & (\Omega^\infty R)_+ & \longrightarrow & \Omega^\infty \Sigma_+^\infty \Omega^\infty R & \longrightarrow & \Omega^\infty R \end{array}$$

commutes, where the vertical maps are all induced by the defining inclusion. Similarly, the first two maps in the bottom composition compose to the unit of the $(\Sigma_+^\infty, \Omega^\infty)$ -adjunction on $\Omega^\infty R$, and by the triangle identity for this adjunction, the bottom horizontal composite is the identity. This is what we wanted to prove. \square

Proof of Pr.5.2. Consider the natural commutative diagram of spaces

$$\begin{array}{ccccccc}
\mathbf{CP}^\infty & \longrightarrow & \mathbf{CP}_+^\infty & \xrightarrow{\eta_+} & \Omega^\infty \Sigma_+^\infty \mathbf{CP}^\infty & \xrightarrow{\Omega^\infty \rho} & \Omega^\infty \mathbf{ku} \\
\xi|_{\mathbf{BU}(1)} \downarrow & & \xi|_{\mathbf{BU}(1)_+} \downarrow & & \Omega^\infty \Sigma_+^\infty \xi|_{\mathbf{BU}(1)} \downarrow & & \\
\mathbf{GL}_1(\mathbf{ku}) & \longrightarrow & \mathbf{GL}_1(\mathbf{ku})_+ & \xrightarrow{\eta_+} & \Omega^\infty \Sigma_+^\infty \mathbf{GL}_1(\mathbf{ku}) & \xrightarrow{\Omega^\infty \epsilon} & \Omega^\infty \mathbf{ku}
\end{array}$$

where ϵ is the counit of the $(\Sigma_+^\infty, \mathbf{GL}_1)$ -adjunction. By Lm.5.3, the bottom horizontal composite is the inclusion $\mathbf{GL}_1(\mathbf{ku}) \rightarrow \Omega^\infty \mathbf{ku}$. This implies the composition $\mathbf{CP}^\infty \rightarrow \Omega^\infty \mathbf{ku}$ above corresponds to the morphism $\xi|_{\mathbf{BU}(1)}: \mathbf{CP}^\infty \rightarrow \Omega^\infty \mathbf{ku}$ which lands in $\{1\} \times \mathbf{BU}$ defining the universal line bundle ξ_1 over \mathbf{CP}^∞ . As this morphism represents $[\xi_1]$ in $\mathbf{ku}^0(\mathbf{CP}^\infty)$, it follows by the $(\Sigma_+^\infty, \Omega^\infty)$ -adjunction that ρ also represents the element $[\xi_1]$. Our desired composite is then represented by the image of ρ under the map

$$j^*: \mathbf{ku}^0(\mathbf{CP}^\infty) \rightarrow \widetilde{\mathbf{ku}}^0(\mathbf{CP}^\infty).$$

To identify j^* we write down the split (co)fibre sequence of spectra

$$\Sigma^\infty \mathbf{CP}^\infty \xrightarrow{j} \Sigma_+^\infty \mathbf{CP}^\infty \simeq \Sigma^\infty \mathbf{CP}^\infty \oplus \mathbf{S} \xrightarrow{q} \mathbf{S}$$

where q is induced by the unique map of pointed spaces $\mathbf{CP}_+^\infty \rightarrow S^0$ which is surjective on π_0 . We can calculate $q^*: \mathbf{ku}^0(*) \rightarrow \mathbf{ku}^0 \mathbf{CP}^\infty$ it induces a map of rings on \mathbf{ku}^0 -cohomology, and $\mathbf{ku}^0(*) \simeq \mathbf{Z}$, so q^* is the unique map. More explicitly, q^* sends an integer n to the n -dimensional virtual vector bundle on $\mathbf{ku}^0 \mathbf{CP}^\infty$. One can also calculate that the splitting i of q induces a map $i^*: \mathbf{ku}^0 \mathbf{CP}^\infty \rightarrow \mathbf{Z}$ sending a virtual vector bundle to its dimension. Indeed, this can be seen geometrically, as a class $x: \mathbf{CP}^\infty \rightarrow \mathbf{Z} \times \mathbf{BU}$ is sent to the composition $* \rightarrow \mathbf{CP}^\infty \rightarrow \mathbf{Z} \times \mathbf{BU}$ which only remembers which \mathbf{Z} -component the original x landed in, ie, its virtual dimension. We can then identify the map p^* induced by the splitting p of j with the inclusion of the kernel of i^* , ie, the inclusion of those virtual vector bundles over \mathbf{CP}^∞ with dimension 0. We rather formally see that j^* can then be identified by the formula:

$$j^*(x) = x - q^* i^*(x) = x - \dim(x)$$

Back to the question at hand, we wish to calculate $j^*(\rho)$. Using the above yields our desired conclusion:

$$j^*(\rho) = j^*([\xi_1]) = [\xi_1] - 1 \quad \square$$

The consequence of the above is that we obtain the usual complex orientation on \mathbf{KU} .

Remark 5.4. The map $j: \Sigma^\infty \mathbf{CP}^\infty \rightarrow \Sigma_+^\infty \mathbf{CP}^\infty$ defines a class

$$j \in (\Sigma_+^\infty \widetilde{\mathbf{CP}^\infty})^0(\mathbf{CP}^\infty).$$

Let us also write $j \in \widetilde{\mathcal{E}}^0(\mathbf{CP}^\infty)$ for the image of the above element under the localisation map:

$$\Sigma_+^\infty \mathbf{CP}^\infty \rightarrow \Sigma_+^\infty \mathbf{CP}^\infty[\beta^{-1}] = \mathcal{E}$$

A complex orientation $x_{\mathcal{E}}$ can be defined as $x_{\mathcal{E}} = \frac{j}{\beta} \in \tilde{\mathcal{E}}^2(\mathbf{CP}^{\infty})$ as we have $\iota^*(x_{\mathcal{E}}) = \beta \cdot \beta^{-1} = 1$. It follows that the image of $x_{\mathcal{E}}$ inside KU under the map

$$\rho[\beta^{-1}]: \mathcal{E} = \Sigma_+^{\infty} \mathbf{CP}^{\infty}[\beta^{-1}] \rightarrow \mathrm{ku}[\beta^{-1}] = \mathrm{KU}$$

is a complex orientation x_{KU} of KU, as complex orientations are sent to complex orientations by morphisms of \mathbf{E}_{∞} -rings; see [EC2, Rmk.4.1.3]. This complex orientation on KU is also the orientation we were all expecting, as by Pr.5.2 we obtain the equalities

$$x_{\mathrm{KU}} = \rho_*(x_{\mathcal{E}}) = \frac{[\xi_1] - 1}{\beta} \in \widetilde{\mathrm{KU}}^0(\mathbf{CP}^{\infty})$$

where ξ_1 is the universal line bundle over \mathbf{CP}^{∞} .

With this geometric definition and discussion of KU, let us define an algebraic object to compare it to.

Definition 5.5. Let $\mu_{p^{\infty}}^{\heartsuit}$ denote the *multiplicative p -divisible group* over $\mathrm{Spec} \mathbf{Z}$, whose R -valued points (for a discrete ring R) are defined as:

$$\mu_{p^n}^{\heartsuit}(R) = \{x \in R \mid x^{p^n} = 1\}$$

This lifts to a p -divisible group $\mu_{p^{\infty}}$ over $\mathrm{Spec} \mathbf{S}$ by [EC2, Pr.2.2.11].

Proposition 5.6. *The object $\mu_{p^{\infty}}^{\heartsuit}$ over $\mathrm{Spf} \mathbf{Z}_p$ is an object of $\mathcal{C}_{\mathbf{Z}_p}$ (for $n = 1$), and there is a natural equivalence of \mathbf{E}_{∞} -rings*

$$\mathcal{O}_{\mathrm{BT}_1^{\mathrm{top}}}(\mu_{p^{\infty}}^{\heartsuit}) \simeq \mathrm{KU}_p.$$

One can view this proposition as a special case of the Lubin–Tate example Pr.5.12, but a more direct comparison to the geometric discussion above is also possible using Lurie’s machinery from [EC2, §6.5]. Our argument below is a combination of [EC2, §3-4 & 6.5].

Proof. The fact that $\mu_{p^{\infty}}^{\heartsuit}$ lies in $\mathcal{C}_{\mathbf{Z}_p}$ follows immediately from Pr.1.8 and Pr.2.10. Alternatively, one can view this as a special case of Pr.5.12.

We now follow the argument of Lurie from [EC2]. First, notice the natural equivalence $\mathfrak{D}(\mu_{p^{\infty}}^{\heartsuit}/\mathrm{Spf} \mathbf{Z}_p) \simeq (\mu_{p^{\infty}}/\mathrm{Spf} \mathbf{S}_p)$. Indeed, [EC2, Cor.3.1.19] states that the universal spectral deformation of $\mu_{p^{\infty}}^{\heartsuit}$ over \mathbf{Z}_p is $\mu_{p^{\infty}}$ over \mathbf{S}_p which is identified with $\mathfrak{D}(\mu_{p^{\infty}}^{\heartsuit}/\mathrm{Spf} \mathbf{Z}_p)$ via Rmk.2.38.

By [EC2, Pr.2.2.12], we see that the identity component of $\mu_{p^{\infty}}$ over $\mathrm{Spf} \mathbf{S}_p$ is precisely the multiplicative formal group $\widehat{\mathbf{G}}_m$ over $\mathrm{Spf} \mathbf{S}_p$. It remains to compute our desired \mathbf{E}_{∞} -ring then takes the form of the orientation classifier \mathcal{E} of $\widehat{\mathbf{G}}_m$ over \mathbf{S}_p . By the p -completion of [EC2, Pr.4.3.25], the preorientation classifier of $\widehat{\mathbf{G}}_m$ over \mathbf{S}_p is $\Sigma_+^{\infty} \mathbf{CP}_p^{\infty}$. Taking a p -completion in Con.5.1, we obtain a map of \mathbf{E}_{∞} -rings $\rho_p: \Sigma_+^{\infty} \mathbf{CP}_p^{\infty} \rightarrow \mathrm{ku}_p$. Similarly, by [EC2, Cor.4.3.27], the localisation $\Sigma_+^{\infty} \mathbf{CP}_p^{\infty}[\beta^{-1}]$, where $\beta \in \pi_2 \Sigma_+^{\infty} \mathbf{CP}_p^{\infty}$ is the Bott element of Con.5.1 (we copied the definition from [EC2, §6.5]), is the orientation classifier \mathcal{E} we are after. Again, from Con.5.1, this naturally admits a map of \mathbf{E}_{∞} -rings $\rho_p[\beta^{-1}]: \mathcal{E} \rightarrow \mathrm{KU}_p$. We claim this map $\rho_p[\beta^{-1}]$ is an

equivalence.

Now we follow [EC2, §6.5]. As $\mu_{p^\infty}/\mathbf{S}_p$ is the universal spectral deformation of both $\mu_{p^\infty}^\heartsuit/\mathrm{Spf}\mathbf{Z}_p$ and $\mu_{p^\infty}^\heartsuit/\mathrm{Spec}\mathbf{F}_p$ (Pr.2.42), it follows from [EC2, Th.6.4.6] that $\widehat{\mathbf{G}}_m$ is balanced (28) over \mathbf{S}_p . This and the complex periodicity of \mathcal{E} yield an isomorphism of graded rings

$$\mathbf{Z}_p[\beta^\pm] \xrightarrow{\cong} \pi_*\mathcal{E}$$

defined by the invertible element $\beta \in \pi_2\mathcal{E}$. The p -completion of the classical Bott periodicity theorem then states the composite $\mathbf{Z}_p[\beta^\pm] \rightarrow \pi_*\mathrm{KU}_p$ through $\rho_p[\beta^{-1}]$ is an equivalence, hence $\rho_p[\beta^{-1}]$ is an equivalence. \square

There is a standard trick to obtain the integral \mathbf{E}_∞ -ring KU from $\mathcal{O}_{\mathrm{BT}_1}^{\mathrm{top}}(\mu_{p^\infty}^\heartsuit)$ by purely algebraic methods.

Remark 5.7. Consider the symmetric monoidal Schwede–Shipley equivalence of ∞ -categories

$$\mathrm{Mod}_R \simeq \mathcal{D}(R) \tag{5.8}$$

where R is a discrete commutative ring; see [SS03] or [HA, Th.7.1.2.13]. Replacing R with \mathbf{Q} , we note that the \mathbf{E}_∞ - \mathbf{Q} -algebra $\mathrm{ku}_\mathbf{Q}$, the rationalisation of ku , has homotopy groups $\pi_*\mathrm{ku}_\mathbf{Q} \simeq \mathbf{Q}[\beta]$, for $|\beta| = 2$. Define a map of \mathbf{Q} -cdgas $\Lambda_\mathbf{Q}[x_2] \rightarrow \mathrm{ku}_\mathbf{Q}$ from the free \mathbf{Q} -cdga on one element in degree 2 to $\mathrm{ku}_\mathbf{Q}$, defined by the element β . This is easily seen to be a equivalence of \mathbf{Q} -cdgas, and moreover, one obtains an equivalence upon localisations at x_2

$$\Lambda_\mathbf{Q}[x_2^{\pm 1}] \xrightarrow{\cong} \mathrm{ku}_\mathbf{Q}[\beta^{-1}] \simeq \mathrm{KU}_\mathbf{Q}$$

where $\mathrm{KU}_\mathbf{Q}$ is the rationalisation of KU . Carrying out the same construction in CAlg , we obtain a morphism $\Lambda[x_2^{\pm 1}] \rightarrow \mathrm{KU}_p$ of \mathbf{E}_∞ -rings from the free \mathbf{E}_∞ -ring on a single invertible generator in degree two to KU_p defined by $\beta \in \pi_2\mathrm{KU}_p$. Taking the product of these morphisms over all primes p and rationalising gives a morphism in $\mathrm{CAlg}_\mathbf{Q}$

$$\theta: \Lambda_\mathbf{Q}[x_2^{\pm 1}] \rightarrow \left(\prod_p \mathrm{KU}_p \right)_\mathbf{Q}$$

where we note that $(\Lambda[x_2^{\pm 1}])_\mathbf{Q}$ is naturally equivalent to $\Lambda_\mathbf{Q}[x_2^{\pm 1}]$. One then obtains KU from the following *Hasse* Cartesian square of \mathbf{E}_∞ -rings:

$$\begin{array}{ccc} \mathrm{KU} & \longrightarrow & \prod_p \mathrm{KU}_p \\ \downarrow & & \downarrow \\ \Lambda_\mathbf{Q}[x_2^{\pm 1}] & \xrightarrow{\theta} & (\prod_p \mathrm{KU}_p)_\mathbf{Q} \end{array}$$

where the two products are taken over all prime numbers p ; see [Bau14].

The \mathbf{E}_∞ -ring KO can also be obtained through these means. The following is a carbon copy of Con.5.1, replacing \mathbf{C} with \mathbf{R} .

Construction 5.9. Denote by $\text{Vect}_{\mathbf{R}}^{\simeq}$ the topological category of finite-dimensional real vector spaces and real linear isomorphisms. This category has two symmetric monoidal structures given by the direct sum and tensor product of vector bundles, the (topological) coherent nerve $N(\text{Vect}_{\mathbf{R}}^{\simeq})$ is a commutative monoid object in the ∞ -category of \mathbf{E}_{∞} -spaces. Moreover, the functor

$$c: \text{Vect}_{\mathbf{R}}^{\simeq} \rightarrow \text{Vect}_{\mathbf{C}}^{\simeq} \quad V \mapsto V \otimes_{\mathbf{R}} \mathbf{C}$$

is symmetric monoidal with respect to both monoidal structures, hence we obtain a morphism of commutative monoid objects in \mathbf{E}_{∞} -spaces:

$$c: N(\text{Vect}_{\mathbf{R}}^{\simeq}) \rightarrow N(\text{Vect}_{\mathbf{C}}^{\simeq})$$

The group completion (with respect to the direct sum \mathbf{E}_{∞} -structure) of $N(\text{Vect}_{\mathbf{R}}^{\simeq})$ is the zeroth space of the connective \mathbf{E}_{∞} -ring ko , *connective real topological K-theory*, and c induces a morphism $\text{ko} \rightarrow \text{ku}$ of \mathbf{E}_{∞} -rings. There is an element $\beta_{\mathbf{R}}$ inside $\pi_8 \text{ko}$, represented by an element which maps to the element β^4 inside $\pi_8 \text{ku}$; see [Ada74, §III] for example. We define the \mathbf{E}_{∞} -ring of *periodic real K-theory* as the localisation $\text{KO} = \text{ko}[\beta_{\mathbf{R}}^{-1}]$, and we notice this induces a morphism $c: \text{KO} \rightarrow \text{KU}$. By [HS14], the map c can be identified with the \mathbf{E}_{∞} -inclusion of the C_2 -fixed points of KU through the C_2 -action given by complex conjugation of vector bundles.

Definition 5.10. Let $\mathfrak{B}C_2$ be the quotient stack $\text{Spf } \mathbf{Z}_p / C_2$ with respect to the trivial action on $\text{Spf } \mathbf{Z}_p$. This formal spectral Deligne–Mumford stack has a cover $\text{Spf } \mathbf{Z}_p \rightarrow \mathfrak{B}C_2$ given by the canonical quotient map. By [LN14, A.3-4], this is the base-change over $\text{Spf } \mathbf{Z}_p$ of the *moduli stack of forms of the multiplicative group scheme* \mathbf{G}_m . The reason for the quotient by C_2 is to remove the automorphism on \mathbf{G}_m given by inversion. Moreover, the multiplicative p -divisible group $\mu_{p^\infty}^\heartsuit$ lives over $\mathfrak{B}C_2$, so we obtain a map $\mathfrak{B}C_2 \rightarrow \widehat{\mathcal{M}}_{\text{BT}_1^p, \mathbf{Z}_p}^\heartsuit$.

Proposition 5.11. *The map $\mathfrak{B}C_2 \rightarrow \widehat{\mathcal{M}}_{\text{BT}_1^p, \mathbf{Z}_p}^\heartsuit$ lives in $\mathcal{C}_{\mathbf{Z}_p}$. Moreover, the map $\mathcal{O}_{\text{BT}_1^p}^{\text{top}}(\text{Spf } \mathbf{Z}_p \rightarrow \mathfrak{B}C_2)$ is homotopic (as maps of spectra) to the p -completion of the map $c: \text{KO} \rightarrow \text{KU}$.*

The proof below uses some results about stable Adams operations which we discuss in §5.5.

Proof. As $\text{Spf } \mathbf{Z}_p \rightarrow \mathfrak{B}C_2$ is a finite étale cover and the composite $\text{Spf } \mathbf{Z}_p \rightarrow \widehat{\mathcal{M}}_{\text{BT}_1^p, \mathbf{Z}_p}^\heartsuit$ lies in \mathcal{C}_{A_0} , then so does $\mathfrak{B}C_2$. It suffices now to show that $\mathcal{O}_{\text{BT}_1^p}^{\text{top}}(\mathfrak{B}C_2) = \mathcal{E}$ is the inclusion of the C_2 -fixed points of KU_p with respect to the complex conjugation action on KU_p . We can rewrite \mathcal{E} using the fact that $\mathcal{O}_{\text{BT}_1^p}^{\text{top}}$ is an étale sheaf:

$$\mathcal{E} \simeq \lim \left(\mathcal{O}_{\text{BT}_1^p}^{\text{top}}(\mu_{p^\infty}^\heartsuit / \text{Spf } \mathbf{Z}_p) \Rightarrow \mathcal{O}_{\text{BT}_1^p}^{\text{top}}(\mu_{p^\infty}^\heartsuit / \text{Spf } \mathbf{Z}_p \times_{\mathfrak{B}C_2} \text{Spf } \mathbf{Z}_p) \Rightarrow \cdots \right)$$

As $\text{Spf } \mathbf{Z}_p \rightarrow \mathfrak{B}C_2$ is a C_2 -torsor by construction and using Pr.5.6, we can rewrite the above limit as

$$\lim \left(\text{KU}_p \Rightarrow \prod_{C_2} \text{KU}_p \Rightarrow \prod_{C_2 \times C_2} \text{KU}_p \cdots \right)$$

which is simply the homotopy fixed points $\mathrm{KU}_p^{hC_2}$. We are only left to check that this C_2 -action on KU_p is homotopic as maps of spectra to that given by complex conjugation. By the construction of $\mathfrak{B}C_2$, we see that $\mathrm{Spf} \mathbf{Z}_p \rightarrow \mathfrak{B}C_2$ is the quotient by the inversion action on the multiplicative group scheme \mathbf{G}_m , hence $\mathcal{E} \rightarrow \mathrm{KU}_p$ is the inclusion of the the C_2 -homotopy fixed point of KU_p with action given by $[-1]^*$. As we will discuss in Pr.5.30, this is homotopic to the action of spectra ψ^{-1} , and following arguments of *ibid* we see this is determined as a map of spectra by what it does on line bundles on KU_p -cohomology of finite spaces.³¹ We now refer to [MS74, p.168], which states that complex conjugate of a complex line bundle L is isomorphic to the dual of a complex line bundle L , the latter also being given by $\psi^{-1}(L)$. This finishes the proof. \square

5.2 Lubin–Tate and Barsotti–Tate theories

The above example of $\mathcal{O}_{\mathrm{BT}_1^p}^{\mathrm{top}}(\mu_{p^\infty}^\heartsuit) \simeq \mathrm{KU}_p$ can be extended to arbitrary heights. The following is a combination of [EC2, §5-6]. Recall the Lubin–Tate deformation theory of Ex.2.9.

Proposition 5.12. *Let $\widehat{\mathbf{G}}_0$ be a formal group of exact height n over a perfect field κ and \mathbf{G}_0 for a p -divisible group over κ whose identity component is equivalent to $\widehat{\mathbf{G}}_0$; see [EC2, Pr.4.4.22]. Write \mathbf{G} for the classical universal deformation of \mathbf{G}_0 , which is a p -divisible group over the discrete ring $R_{\widehat{\mathbf{G}}_0}^{\mathrm{LT}}$. The object $\mathbf{G}: \mathrm{Spf} R_{\widehat{\mathbf{G}}_0}^{\mathrm{LT}} \rightarrow \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, \mathbf{Z}_p}^\heartsuit$ lies in $\mathcal{C}_{\mathbf{Z}_p}$. Moreover, there is an equivalence of \mathbf{E}_∞ -rings $\mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}(\mathbf{G}) \simeq E_n$ where $E_n = E(\widehat{\mathbf{G}}_0)$ is the Lubin–Tate \mathbf{E}_∞ -ring of $\widehat{\mathbf{G}}_0$ (also known as Morava E -theory); see [EC2, §5].*

This will follow from a more general family of p -divisible groups in $\mathcal{C}_{\mathbf{Z}_p}$.

Proposition 5.13. *Let R_0 be a discrete Noetherian \mathbf{F}_p -algebra such that the Frobenius endomorphism on R_0 is finite and \mathbf{G}_0 be a nonstationary (18) p -divisible group of height n over R_0 . Write R for the universal spectral deformation adic \mathbf{E}_∞ -ring of \mathbf{G}_0 from [EC2, Th.3.4.1] and assume the residue fields of $\pi_0 R$ are perfect of characteristic p . Then the morphism $\mathbf{G}: \mathrm{Spf} \pi_0 R \rightarrow \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, \mathbf{Z}_p}^\heartsuit$ defined by the base-change of the universal spectral deformation of \mathbf{G}_0 along $R \rightarrow \pi_0 R$ lies in \mathcal{C}_{A_0} . Moreover, there is a natural equivalence of \mathbf{E}_∞ -rings $\mathfrak{D}(\mathbf{G}) \simeq R$.*

The \mathbf{E}_∞ -rings produced by applying $\mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}$ to the p -divisible groups \mathbf{G} occurring in Pr.5.13 seem interesting enough to name.

Definition 5.14. Let R_0 , \mathbf{G}_0 , and \mathbf{G} be as in Pr.5.13. We call $\mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}(\mathbf{G})$ the *Barsotti–Tate \mathbf{E}_∞ -ring* associated to (R_0, \mathbf{G}_0) .

Proof of Pr.5.13. Let us first see \mathbf{G} lies in \mathcal{C}_{A_0} by checking the conditions of Df.1.5. It is shown in Pr.2.10 that the morphism \mathbf{G} is formally étale. As R_0 is Noetherian, then [EC2, Th.3.4.1(6)] tells us that R and hence also $\pi_0 R$ are Noetherian as well. Consider the maps in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$

$$\mathrm{Spf} \pi_0 R \rightarrow \mathrm{Spf} R \rightarrow \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A}$$

³¹Recall from [HA, Nt.1.4.2.5], that the ∞ -category of *finite spaces* is the full ∞ -subcategory of \mathcal{S} generated by the terminal object under finite colimits.

and the associated (co)fibre sequence of complete $\pi_0 R$ -modules:

$$L_{\mathrm{Spf} R/\widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A}} \Big|_{\mathrm{Spf} \pi_0 R} \rightarrow L_{\mathrm{Spf} \pi_0 R/\widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A}} \rightarrow L_{\mathrm{Spf} \pi_0 R/\mathrm{Spf} R}$$

By construction ([EC2, Pr.3.4.3]), R corepresents the de Rham space of the map $\mathrm{Spec} R_0 \rightarrow \mathcal{M}_{\mathrm{BT}^p}$, or equivalently, the de Rham space of $\mathrm{Spec} R_0 \rightarrow \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, \mathbf{Z}_p}$, as R_0 is an \mathbf{F}_p -algebra and \mathbf{G}_0 is of height n . Identifying R as representing this de Rham space and using Ex.2.29, we see that $L_{\mathrm{Spf} R/\widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A}}$ vanishes. Hence $L_{\mathrm{Spf} \pi_0 R/\widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A}}$ is almost perfect as $L_{\mathrm{Spf} \pi_0 R/\mathrm{Spf} R}$ is almost perfect; see Pr.A.12. Rmk.2.38 identifies R with $\mathfrak{D}(\mathbf{G})$. \square

Proof of Pr.5.12. The fact that \mathbf{G} lies in $\mathcal{C}_{\mathbf{Z}_p}$ follows from Pr.5.13. The fact that $\mathcal{O}_{\mathrm{BT}^p}^{\mathrm{top}}(\mathbf{G})$ is equivalent to E_n follows as the universal spectral deformation of \mathbf{G}_0 is given by $\mathfrak{D}(\mathbf{G})$ (Pr.5.13) and the orientation classifier of $\mathfrak{D}(\mathbf{G})$ is E_n ([EC2, Cor.6.0.6]). \square

From the functoriality of $\mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}$ we obtain an action of the automorphism group of the pair $(\kappa, \widehat{\mathbf{G}}_0)$ on the \mathbf{E}_∞ -ring E_n . In other words, E_n obtains an action of the *extended Morava stabiliser group*; see [EC2, §5] and [GH04, §7]. It is not clear from these techniques that these account for all \mathbf{E}_∞ -endomorphisms of E_n ; this requires a dash of chromatic homotopy theory as done in [EC2, §5].

5.3 Topological modular forms

Another exciting application of Th.1.6 is to construct the \mathbf{E}_∞ -ring TMF of *periodic topological modular forms*. Of course, this also uses the ideas of Lurie from [EC2] and [SUR09], but reinterpreting TMF_p as a section of $\mathcal{O}_{\mathrm{BT}_2^p}^{\mathrm{top}}$ yields additional endomorphisms to those previously known. In particular, by §5.5, TMF_p will obtain stable Adams operations, and we also outline how TMF_p obtains stable Hecke operators.

Proposition 5.15. *The map $[p^\infty]: \widehat{\mathcal{M}}_{\mathrm{Ell}, \mathbf{Z}_p}^\heartsuit} \rightarrow \widehat{\mathcal{M}}_{\mathrm{BT}_2^p, \mathbf{Z}_p}^\heartsuit}$ lies inside $\mathcal{C}_{\mathbf{Z}_p}$.*

Proof. Using Pr.1.8, we only need to show that the map $[p^\infty]$ above is formally étale inside $\mathcal{P}(\mathrm{Aff}^\heartsuit)$ and that $\widehat{\mathcal{M}}_{\mathrm{Ell}, \mathbf{Z}_p}^\heartsuit}$ is finitely presented over $\mathrm{Spf} \mathbf{Z}_p$. The former follows from Ex.2.7; a consequence of the classical Serre–Tate theorem. The latter follows from [Ols16, Th.13.1.2], which states that $\mathcal{M}_{\mathrm{Ell}}^\heartsuit}$ is locally of finite presentation over $\mathrm{Spec} \mathbf{Z}$, the fact that the adjective “locally of finite presentation” is stable under base-change, and the fact that $\widehat{\mathcal{M}}_{\mathrm{Ell}, \mathbf{Z}_p}^\heartsuit}$ is qcqs. Indeed, to see $\widehat{\mathcal{M}}_{\mathrm{Ell}, \mathbf{Z}_p}^\heartsuit}$ is qc, we recall that it is also well known that after inverting an integer n that there is a surjective étale map $\mathcal{M}_1(n) \rightarrow \mathcal{M}_{\mathrm{Ell}}^\heartsuit} \times \mathrm{Spec} \mathbf{Z}[\frac{1}{n}]$ from the *moduli stack of elliptic curves with exact level n structure*, and for $n \geq 4$ the Deligne–Mumford stacks $\mathcal{M}_1(n)$ are affine, hence $\widehat{\mathcal{M}}_{\mathrm{Ell}, \mathbf{Z}_p}^\heartsuit}$ has a finite affine étale cover for all primes p ; see [KM85, Th.3.7.1]. Moreover, the iterated fibre products of this cover are affine too, as $\mathcal{M}_{\mathrm{Ell}}^\heartsuit}$ is separated over $\mathrm{Spec} \mathbf{Z}$; see [Ols16, Th.13.1.2] again. Hence we obtain a finite affine étale hypercover of $\widehat{\mathcal{M}}_{\mathrm{Ell}, \mathbf{Z}_p}^\heartsuit}$. By Pr.A.17, we see $\widehat{\mathcal{M}}_{\mathrm{Ell}, \mathbf{Z}_p}^\heartsuit}$ is qcqs. \square

As promised in the introduction, we should relate $\mathcal{O}_{\mathrm{BT}_2^p}^{\mathrm{top}}$ to a more classical object:

Definition 5.16. Let $\mathcal{O}^{\mathrm{top}}$ denote the Goerss–Hopkins–Miller sheaf of \mathbf{E}_∞ -rings on the étale site $\mathrm{DM}_{/\mathcal{M}_{\mathrm{EII}}^\heartsuit}^{\mathrm{ét}}$ of $\mathcal{M}_{\mathrm{EII}}^\heartsuit$; see [EC2, Th.7.0.1] or [Goe10, Th.1.2] for a version over the compactification of $\mathcal{M}_{\mathrm{EII}}^\heartsuit$.

We also have functors $\mathrm{DM}_{/\mathcal{M}_{\mathrm{EII}}^\heartsuit}^{\mathrm{ét}} \rightarrow \mathrm{fDM}_{/\widehat{\mathcal{M}}_{\mathrm{EII}, \mathbf{Z}_p}^\heartsuit}^{\mathrm{ét}}$ defined by base-change along the canonical map $\mathrm{Spf} \mathbf{Z}_p \rightarrow \mathrm{Spec} \mathbf{Z}$ and $\mathrm{fDM}_{/\widehat{\mathcal{M}}_{\mathrm{EII}, \mathbf{Z}_p}^\heartsuit}^{\mathrm{ét}} \rightarrow \mathcal{C}_{\mathbf{Z}_p}$ by post-composition along the map $[p^\infty]$ of Pr.5.15.

Theorem 5.17. *The following diagram of ∞ -categories commutes up to homotopy:*

$$\begin{array}{ccc} \left(\mathrm{DM}_{/\mathcal{M}_{\mathrm{EII}}^\heartsuit}^{\mathrm{ét}} \right)^{\mathrm{op}} & \xrightarrow{\mathcal{O}^{\mathrm{top}}} & \mathrm{CAlg} \\ \downarrow & & \downarrow (-)_p^\wedge \\ \left(\mathrm{fDM}_{/\widehat{\mathcal{M}}_{\mathrm{EII}, \mathbf{Z}_p}^\heartsuit}^{\mathrm{ét}} \right)^{\mathrm{op}} & \longrightarrow & (\mathcal{C}_{\mathbf{Z}_p})^{\mathrm{op}} \xrightarrow{\mathcal{O}_{\mathrm{BT}_2^p}^{\mathrm{top}}} \mathrm{CAlg} \end{array}$$

In particular, there is an equivalence of \mathbf{E}_∞ -rings:

$$\mathcal{O}_{\mathrm{BT}_2^p}^{\mathrm{top}} \left([p^\infty]: \widehat{\mathcal{M}}_{\mathrm{EII}, \mathbf{Z}_p}^\heartsuit \rightarrow \widehat{\mathcal{M}}_{\mathrm{BT}_2^p, \mathbf{Z}_p}^\heartsuit \right) \simeq \mathrm{TMF}_p$$

The following proof is essentially that of [EC2, Th.7.0.1] which proves an integral statement.

Proof. As done in the proof of [EC2, Th.7.0.1], we will conclude the proof by checking that for each affine object $E_0: \mathrm{Spec} B_0 \rightarrow \mathcal{M}_{\mathrm{EII}}^\heartsuit$ of $\mathrm{DM}_{/\mathcal{M}_{\mathrm{EII}}^\heartsuit}^{\mathrm{ét}}$, the \mathbf{E}_∞ -ring $\mathcal{E} = \mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}(E[p^\infty])$ satisfies the following conditions:

1. \mathcal{E} is weakly 2-periodic (7).
2. The homotopy groups $\pi_k \mathcal{E}$ vanish in odd degrees, so in particular \mathcal{E} is complex orientable.
3. There is a natural (in affines in $\mathrm{DM}_{/\mathcal{M}_{\mathrm{EII}}^\heartsuit}^{\mathrm{ét}}$) isomorphism of rings $(B_0)_p^\wedge \simeq \pi_0 \mathcal{E}$.
4. There is a natural (in affines in $\mathrm{DM}_{/\mathcal{M}_{\mathrm{EII}}^\heartsuit}^{\mathrm{ét}}$) isomorphism of formal groups $\widehat{\mathbf{E}}_{(B_0)_p^\wedge} \simeq \widehat{\mathbf{G}}_{\mathcal{E}}^{\mathcal{Q}_0}$ over $\mathrm{Spf}(B_0)_p^\wedge$.

Once one applies [EC2, Pr.7.4.1] to identify $\widehat{E} \simeq E[p^\infty]^\circ$, these conditions above are precisely the properties of $\mathcal{O}_{\mathrm{BT}_2^p}^{\mathrm{top}}$ by Th.1.6, hence they hold. The étale sheaf of \mathbf{E}_∞ -rings $\mathcal{O}^{\mathrm{top}}$ (followed by p -completion) is determined up to homotopy by the four conditions above, the desired diagram of ∞ -categories commutes up to homotopy; for a statement about the connectedness of the moduli space of such sheaves, see [EC2, Rmk.7.0.2] or [Goe10, Th.1.2] and the two paragraphs that follow. A proof of the connectedness of this moduli space, including the p -complete case used above, can be found in [Dav21b]. \square

As done in [Beh20, §6], we can use the collection of all p -complete \mathbf{E}_∞ -rings and a little rational information to construct integral TMF, similar to Rmk.5.7.

Remark 5.18. We have an étale hypersheaf of \mathbf{E}_∞ -rings on the small étale site over $\mathcal{M}_{\mathrm{EII}}^\heartsuit$ defined by the following composition:

$$\left(\mathrm{DM}_{/\mathcal{M}_{\mathrm{EII}}^\heartsuit}^{\mathrm{ét}}\right)^{\mathrm{op}} \xrightarrow{\pi_p^*} \prod_p \left(\mathrm{fDM}_{/\widehat{\mathcal{M}}_{\mathrm{EII}, \mathbf{Z}_p}^\heartsuit}^{\mathrm{ét}}\right)^{\mathrm{op}} \xrightarrow{\prod [p^\infty]} \prod_p (\mathcal{C}_{\mathbf{Z}_p}) \xrightarrow{\prod \mathcal{O}_{\mathrm{BT}_2^p}^{\mathrm{top}}} \prod_p \mathrm{CAlg} \rightarrow \mathrm{CAlg}$$

This sheaf comes with a canonical map into its rationalisation which we will use shortly. To construct $\mathcal{O}_{\mathbf{Q}}^{\mathrm{top}}$ algebraically, recall the symmetric monoidal equivalence of ∞ -categories (5.8). Define an étale hypersheaf of \mathbf{E}_∞ -rings $\mathcal{O}_{\mathbf{Q}}^{\mathrm{top}}$ first on affines by sending $E_0: \mathrm{Spec} B_0 \rightarrow \mathcal{M}_{\mathrm{EII}}^\heartsuit$ in $\mathrm{DM}_{/\mathcal{M}_{\mathrm{EII}}^\heartsuit}^{\mathrm{ét}}$ to the formal³² rational cdga:

$$\mathcal{O}_{\mathbf{Q}}^{\mathrm{top}}(E_0)_n = \begin{cases} \omega_{\widehat{E}_0}^{\otimes k} \otimes \mathbf{Q} & n = 2k \\ 0 & \text{else} \end{cases}$$

Extend this to the small étale site over $\mathcal{M}_{\mathrm{EII}}^\heartsuit$ by right Kan extension (ie, taking affine étale covers and using the sheaf condition). Let us now construct a morphism

$$\mathcal{O}_{\mathbf{Q}}^{\mathrm{top}} \rightarrow \prod_p \pi_*^p \left(\mathcal{O}_{\mathrm{BT}_2^p}^{\mathrm{top}} \circ [p^\infty] \right)_{\mathbf{Q}}$$

first on affines $E_0: \mathrm{Spec} B_0 \rightarrow \mathcal{M}_{\mathrm{EII}}^\heartsuit$ as the morphism

$$\omega_{\widehat{E}, \mathbf{Q}}^{\otimes *} \rightarrow \left(\prod_p \omega_{\widehat{E}, \mathbf{Z}_p}^{\otimes *} \right)_{\mathbf{Q}}$$

given by the rationalisation, of the product over all primes of the map from $\omega_{\widehat{E}}^{\otimes *}$ to its p -completion. Extend this to a morphism of sheaves on the whole small étale site by Kan extension. One can then recover $\mathcal{O}_{\mathbf{Q}}^{\mathrm{top}}$ itself as the pullback in the following Cartesian square of sheaves on the étale site of $\mathcal{M}_{\mathrm{EII}}^\heartsuit$:

$$\begin{array}{ccc} \mathcal{O}^{\mathrm{top}} & \longrightarrow & \prod_p \pi_*^p \left(\mathcal{O}_{\mathrm{BT}_2^p}^{\mathrm{top}} \circ [p^\infty] \right) \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathbf{Q}}^{\mathrm{top}} & \longrightarrow & \left(\prod_p \pi_*^p \left(\mathcal{O}_{\mathrm{BT}_2^p}^{\mathrm{top}} \circ [p^\infty] \right) \right)_{\mathbf{Q}} \end{array}$$

³²The rationalisation of $\mathcal{O}^{\mathrm{top}}$ is formal by construction if we use that of [Beh14], however, as discussed in [Mei21, Pr.4.8 & Cor.4.9] the formality of the rationalisation of such a sheaf on such a Deligne–Mumford stack is inevitable.

Taking global sections, one obtains the Cartesian square of \mathbf{E}_∞ -rings

$$\begin{array}{ccc} \mathrm{TMF} & \longrightarrow & \prod_p \mathrm{TMF}_p \\ \downarrow & & \downarrow \\ \mathrm{TMF}_{\mathbf{Q}} & \longrightarrow & \left(\prod_p \mathrm{TMF}_p \right)_{\mathbf{Q}} \end{array}$$

where everything in sight has been constructed using $\mathcal{O}_{\mathrm{BT}_2}^{\mathrm{top}}$ or a dash of rational information.

Let us also mention a few variations on TMF that one can obtain from $\mathcal{O}^{\mathrm{top}}$.

Definition 5.19. There exist moduli functors $\mathrm{CAlg}^{\heartsuit} \rightarrow \mathcal{S}$ denoted as \mathcal{M}_Γ for each congruence subgroup $\Gamma \leq \mathbf{SL}_2(\mathbf{Z})$. Of particular interest are $\Gamma = \Gamma(n)$, $\Gamma_1(n)$, and $\Gamma_0(n)$, which yield moduli stacks $\mathcal{M}(n)$, $\mathcal{M}_1(n)$, and $\mathcal{M}_0(n)$ for each $n \geq 1$. These are defined in [KM85, §3], and they sit in a commutative diagram in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$

$$\begin{array}{ccccc} \mathcal{M}(n) & \longrightarrow & \mathcal{M}_1(n) & \longrightarrow & \mathcal{M}_0(n) \\ & \searrow & \downarrow & \swarrow & \\ & & \mathcal{M}_{\mathrm{EII}} & & \end{array}$$

where all the transformations above are some kind of forgetful functor. Moreover, by [KM85, Th.3.7.1], we see that when working over $\mathrm{Spec} \mathbf{Z}[\frac{1}{n}]$, all of the morphisms above are finite étale. Using these maps one then defines the \mathbf{E}_∞ -rings $\mathrm{TMF}(\Gamma) = \mathcal{O}^{\mathrm{top}}(\mathcal{M}_\Gamma)$ called *periodic topological modular forms with level structure*. These \mathbf{E}_∞ -rings are naturally \mathbf{E}_∞ - $\mathrm{TMF}[\frac{1}{n}]$ -algebras for $\Gamma = \Gamma(n)$, $\Gamma_1(n)$, or $\Gamma_0(n)$.

Once again, the functoriality of these constructions ensures us that the natural $\mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z})$ -action on $\mathcal{M}(n)$ over $\mathcal{M}_{\mathrm{EII}, \mathbf{Z}[\frac{1}{n}]}$ yields a natural equivalence of \mathbf{E}_∞ -rings

$$\mathrm{TMF}[\frac{1}{n}] \xrightarrow{\simeq} \mathrm{TMF}(n)^{h\mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z})};$$

this is explained and explored in more detail in [MM15, §7]. We have another use for these moduli stacks. Let us further explain $\mathcal{M}_0(n)$; from now on, we assume a little familiarity with elliptic curves, from [Sil86] or [KM85, §2], for example.

Construction 5.20. The moduli stack of elliptic curves has R -valued points $\mathcal{M}_{\mathrm{EII}}(R)$ given by the space of elliptic curves over R and equivalences between them. The moduli stack $\mathcal{M}_0(n)$ has R -valued points $\mathcal{M}_0(n)(R)$ given by the space of elliptic curves E over R with a choice of cyclic subgroup $H \leq E$ of order n . As we will only study $\mathcal{M}_0(n)$ over $\mathbf{Z}[\frac{1}{n}]$ a cyclic subgroup $H \leq E$ of order n is a subgroup $H \leq E$ of order n , hence it is necessarily finite étale over R , and we demand that for any (or every) geometric point $\mathrm{Spec} \kappa \rightarrow \mathrm{Spec} R$, H_κ is a cyclic group of order n . There is a canonical map $\pi: \mathcal{M}_0(n) \rightarrow \mathcal{M}_{\mathrm{EII}, \mathbf{Z}[\frac{1}{n}]}$ given by forgetting the level structure. There is also a *quotient map* $q: \mathcal{M}_0(n) \rightarrow \mathcal{M}_{\mathrm{EII}, \mathbf{Z}[\frac{1}{n}]}$ defined on R -valued points by

sending a pair (E, H) to the elliptic curve E/H . It is explained on [Beh06, p.12] why q is étale, and this fact essentially comes down to decomposing q into a composition of étale maps of stacks. By construction there is an isomorphism $q^*\mathcal{E} \simeq \mathcal{E}_0(n)/\mathcal{H}$ of elliptic curves over $\mathcal{M}_0(n)$, where \mathcal{E} is the universal elliptic curve over $\mathcal{M}_{\text{Ell}, \mathbf{Z}[\frac{1}{n}]}$ and $(\mathcal{H} \leq \mathcal{E}_0(n))$ is the universal pair over $\mathcal{M}_0(n)$.

Using this quotient map q (which is induced by a quotient map of elliptic curves, rather than literally being a quotient map of stacks), we can construct spectral lifts of classical Hecke operators. Recall from [Mei22, Pr.2.13] that the morphism of \mathbf{E}_∞ -rings $\pi^*: \text{TMF}[\frac{1}{n}] \rightarrow \text{TMF}_0(n)$ witness the target as a compact object of $\text{Mod}_{\text{TMF}[\frac{1}{n}]}$. In particular, this implies the existence of a transfer map defined by the compsite

$$\text{tr}_p: \text{TMF}_0(n) \rightarrow F(\text{TMF}_0(n), \text{TMF}_0(n)) \xleftarrow{\simeq} F(\text{TMF}_0(n), \text{TMF}) \otimes_{\text{TMF}} \text{TMF}_0(n) \rightarrow \text{TMF}$$

the middle map being an equivalence coming from the compactness above, and where F indicates $\text{TMF}[\frac{1}{n}]$ -linear internal hom, and we have implicitly inverted n .

Definition 5.21 (Stable Hecke operators on TMF_p). Write $Q: \mathcal{E}_0(n) \rightarrow \mathcal{E}_0(n)/\mathcal{H}$ for the quotient homomorphism. For a prime³³ ℓ different to the ambient prime p , we define the ℓ th stable Hecke operator \mathcal{T}_ℓ on TMF_p as the composition

$$\begin{aligned} \mathcal{O}_{\text{BT}_2^p}^{\text{top}}(\mathcal{E}[p^\infty], \widehat{\mathcal{M}}_{\text{Ell}, \mathbf{Z}_p}) &\xrightarrow{q^*} \mathcal{O}_{\text{BT}_2^p}^{\text{top}}(\mathcal{E}_0(n)/\mathcal{H}[p^\infty], \widehat{\mathcal{M}}_0(\ell)_{\mathbf{Z}_p}) \\ &\simeq \downarrow Q^* \\ \mathcal{O}_{\text{BT}_2^p}^{\text{top}}(\mathcal{E}_0(n)[p^\infty], \widehat{\mathcal{M}}_0(\ell)_{\mathbf{Z}_p}) &\xrightarrow{\frac{1}{\ell} \text{tr}_\pi} \mathcal{O}_{\text{BT}_2^p}^{\text{top}}(\mathcal{E}[p^\infty], \widehat{\mathcal{M}}_{\text{Ell}, \mathbf{Z}_p}) \end{aligned}$$

where we have implicitly used Th.5.17. Put more plainly, this stable Hecke operator is given by the composition

$$\mathcal{T}_\ell: \text{TMF}_p \xrightarrow{q^*} \text{TMF}_0^q(n)_p \xrightarrow{Q^*} \text{TMF}_0(n)_p \xrightarrow{\frac{1}{\ell} \text{tr}_\pi} \text{TMF}_p$$

where the supscript $(-)^q$ reminds us that particular section has been defined using q rather than the projection π .

A key part of the above construction is the fact that Q induces an isomorphism on associated p -divisible groups, hence Q^* exists; this is not clear if we only consider \mathcal{O}^{top} as a sheaf over \mathcal{M}_{Ell} . The above construction can also be refined to one on $\text{TMF}[\frac{1}{\ell}]$ using Rmk.5.18.

³³We restrict ourselves to the stable Hecke operators \mathcal{T}_ℓ for a prime ℓ as the general definition requires a construction with more elaborate moduli stacks; we want to consider moduli stacks with level structure without the restriction to *cyclic* subgroups. However, inspired by [Bak98], we can also inductively define $\mathcal{T}_{p^{r+1}}$ for $r \geq 1$ by the formula:

$$\mathcal{T}_{p^{r+1}} = \mathcal{T}_{p^r} \mathcal{T}_p - \frac{1}{p} \psi^p \mathcal{T}_{p^{r-1}}$$

The formula for general positive integers follows by setting $\mathcal{T}_{mn} = \mathcal{T}_m \mathcal{T}_n$ for coprime m and n .

One can show that the above Hecke operators agree with those defined classically *on rings of modular forms* upon taking rational homotopy groups, but this will take us too far afield.

These Hecke operators have already been outlined in [BL10, §11.2] and when working with 6 inverted, such operations were originally defined and studied by Baker; see [Bak98] and [Bak90]. The above stable Hecke operators can be refined in three significant ways: first, one can hope to intrinsically define T_n with a moduli stack, rather than using T_p ; second, one can hope to define stable Hecke operators on the connective tmf and dualisable Tmf versions of TMF ; third, the functorality might be improved from a construction for each T_n individually, to a single construction of all Hecke operators T_n at once, encoding higher coherences of transfers and the like. These refinements have been undertaken by the author and will appear in [Dav22].

Another prominent use the maps $Q^* \circ q^*: \mathrm{TMF}_p \rightarrow \mathrm{TMF}_0(\ell)_p$ appear in the work of Behrens constructing his $Q(\ell)$ spectra, which form resolutions of the $K(2)$ -local sphere; see [Beh06].

5.4 Topological automorphic forms

The first examples of new cohomology theories constructed with Th.1.6 come from Behrens–Lawson [BL10]. The main idea is that the Serre–Tate theorem, which was vital in our construction of TMF_p from $\mathcal{O}_{\mathrm{BT}_2}^{\mathrm{top}}$, actually applies to the moduli stack of dimension g abelian varieties for any $g \geq 1$; the $g = 1$ case recovers the moduli stack of elliptic curves. A new problem now arises: we need our p -divisible groups to be of dimension 1, and then and only then can they have an orientation. To obtain a 1-dimensional p -divisible group from an abelian variety A of dimension $g \geq 2$, one needs more structure on A to split its associated p -divisible group into one of dimension 1 and another of dimension $g - 1$ (which we forget about). This comes in the form of **p**olarisations, **e**ndomorphisms, and **l**evel structure, leading us *PEL–Shimura varieties*; for a full introduction to the subject and the intended application to stable homotopy theory, see [BL10]. What appears below is simply a restatement of [BL10] and [Beh20].

Notation 5.22. Fixed an integer $n \geq 1$.

- Let F be a quadratic imaginary extension of \mathbf{Q} , such that p splits as $u\bar{u}$.
- Let \mathcal{O}_F be the ring of integers of F .
- Let V be an F -vector space of dimension n equipped with a \mathbf{Q} -valued nondegenerate Hermitian alternating form of signature $(1, n - 1)$.
- Let L be an \mathcal{O}_F -lattice in V such that the alternating form above takes integer values on L and makes $L_{(p)}$ self-dual.

Definition 5.23. Write $\mathcal{X}_{V,L}$ for the formal Deligne–Mumford stack over $\mathrm{Spf} \mathbf{Z}_p$ (of [BL10, Th.6.6.2] with $K^p = K_0^p$) where a point in $\mathcal{X}_{V,L}(S)$ for a locally Noetherian formal scheme S over $\mathrm{Spf} \mathbf{Z}_p$, is a triple (A, i, λ) where:

- A is an abelian scheme over S of dimension n .
- $\lambda: A \rightarrow A^\vee$ is a polarisation (principle at p), with Rosati involution \dagger on $\text{End}(A)_{(p)}$.
- $i: \mathcal{O}_{F,(p)} \rightarrow \text{End}(A)_{(p)}$ is an inclusion of rings satisfying $i(\bar{z}) = i(z)^\dagger$.

These triples have to satisfy two conditions assuring they are locally modelled by V and L ; see [Beh20, §6.7].

In the situation above, the splitting $p = u\bar{u}$ induces a splitting of p -divisible groups

$$A[p^\infty] \simeq A[u^\infty] \oplus A[\bar{u}^\infty]$$

and our assumption on (A, i, λ) ensure that $A[u^\infty]$ is a 1-dimensional p -divisible group. This yields a morphism of stacks $[u^\infty]: \mathcal{X}_{V,L} \rightarrow \widehat{\mathcal{M}}_{\text{BT}_n^p, \mathbf{Z}_p}^\heartsuit$ which sends (A, λ, i) to $A[u^\infty]$.

Proposition 5.24. *Given V and L as in Nt.5.22, then the morphism $[u^\infty]: \mathcal{X}_{V,L} \rightarrow \widehat{\mathcal{M}}_{\text{BT}_n^p, \mathbf{Z}_p}^\heartsuit$ is an object of $\mathcal{C}_{\mathbf{Z}_p}$.*

Proof. Pr.1.8 reduces us to show that $[u^\infty]$ is formally étale inside $\mathcal{P}(\text{Aff}^{\text{cn}})$ and that $\mathcal{X}_{V,L}$ is of finite presentation over $\text{Spf } \mathbf{Z}_p$. The first statement, that $[u^\infty]$ is formally étale, follows straight from the definitions of a formally étale morphism and [BL10, Th.7.3.1], which itself is a consequence the classical Serre–Tate theorem and an analysis of $\mathcal{X}_{V,L}$. We now use [BL10, Cor.7.3.3] to see $\mathcal{X}_{V,L}$ is of locally finite presentation over $\text{Spf } \mathbf{Z}_p$, so it suffices to show now that $\mathcal{X}_{V,L}$ is qcqs. To do this, we first use [BL10, Th.6.6.2], which states that $\mathcal{X}_{V,L}$ has an étale cover by a quasi-projective scheme. As a quasi-projective formal scheme X is separated and qc, we see X itself has a Zariski cover by an affine formal scheme $\text{Spf } B$, meaning $\mathcal{X}_{V,L}$ has an étale cover by $\text{Spf } B$. By Pr.A.17, this implies $\mathcal{X}_{V,L}$ is qcqs. \square

We can now define the spectra of topological automorphic forms as done in [BL10, §8.3].

Proposition 5.25. *Let V and L be as in Nt.5.22. Define the \mathbf{E}_∞ -ring of topological automorphic forms*

$$\text{TAF}_{V,L} = \mathcal{O}_{\text{BT}_n^p}^{\text{top}} \left(\mathcal{X}_{V,L} \xrightarrow{[u^\infty]} \widehat{\mathcal{M}}_{\text{BT}_n^p, \mathbf{Z}_p}^\heartsuit \right)$$

As with topological *modular* forms (Df.5.19), we can also define variants of $\text{TAF}_{V,L}$ which incorporate level structures. Such extra structure can then be used to define restriction maps, transfers, and Hecke operators on $\text{TAF}_{V,L}$; see [BL10, §11].

5.5 Stable Adams operations

The next example exploits the intrinsic functoriality of the sheaf $\mathcal{O}_{\text{BT}_n^p}^{\text{top}}$.

Definition 5.26. Let $k = (k_1, k_2, \dots)$ be a p -adic integer and \mathbf{G} be a p -divisible group over an arbitrary scheme (or stack) S . Write $[k]: \mathbf{G} \rightarrow \mathbf{G}$ for the endomorphism of \mathbf{G} given on p^n -torsion by the k_n -fold multiplication $[k_n]: \mathbf{G}_n \rightarrow \mathbf{G}_n$. These assemble to an endomorphism of \mathbf{G} as the sequence (k_1, k_2, \dots) represents a p -adic integer and the closed immersions $\mathbf{G}_n \rightarrow$

\mathbf{G}_{n+1} witness the equality $\mathbf{G}_n = \mathbf{G}_{n+1}[p^n]$. If k is a unit inside \mathbf{Z}_p then each $[k_n]$ is an isomorphism of finite flat groups schemes on S , hence $[k]$ is an automorphism of \mathbf{G} . If \mathbf{G} defines a morphism $S \rightarrow \widehat{\mathcal{M}}_{\mathrm{BT}_n^p, A_0}^\heartsuit$ inside \mathcal{C}_{A_0} and $k \in \mathbf{Z}_p^\times$, then write

$$[k]^*: \mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}(\mathbf{G}) \rightarrow \mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}(\mathbf{G})$$

for the induced endomorphism of \mathbf{E}_∞ -rings. These are the (p -adic) *stable Adams operations* $\mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}(\mathbf{G})$; we will justify this name shortly.

Many properties expected of Adams operations are formal.

Proposition 5.27. *Let l, k be two units in \mathbf{Z}_p , \mathbf{G} be an object of \mathcal{C}_{A_0} , and write $\mathcal{E} = \mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}(\mathbf{G})$. Then ψ^1 is homotopic to the identity map on the \mathbf{E}_∞ -ring \mathcal{E} , and the maps of \mathbf{E}_∞ -rings $\psi^l \psi^k$ and ψ^{lk} on \mathcal{E} are homotopic.*

The homotopy H between $\psi^l \psi^k$ and ψ^{lk} above are coherent in the following sense: if j is another p -adic unit, then the homotopy between $\psi^j \psi^l \psi^k$ and ψ^{jlk} factors through H . This follows straight from the fact that $\mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}: \mathcal{C}_{A_0}^{\mathrm{op}} \rightarrow \mathrm{CAlg}$ is first and foremost a functor of ∞ -categories, and the calculations $[l][k] = [lk]$ hold up to equality in \mathcal{C}_{A_0} .

Proof. As these facts hold for $[k]$ in \mathcal{C}_{A_0} and $\mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}$ is a functor, we obtain the result. \square

Using the information we already have at hand, we can calculate $[k]^*$ on the homotopy groups of the \mathbf{E}_∞ -rings $\mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}(\mathbf{G})$ over affine objects of \mathcal{C}_{A_0} .

Proposition 5.28. *Let k be a unit in \mathbf{Z}_p and \mathbf{G} be a p -divisible group defining an affine object in \mathcal{C}_{A_0} . Then for each integer n , we have the following equality of morphisms of \mathbf{Z}_p -modules:*

$$[k]^* = k^n: \pi_{2n} \mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}(\mathbf{G}) \rightarrow \pi_{2n} \mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}(\mathbf{G})$$

Proof. Using Th.1.6, we see that $\pi_{2n} \mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}(\mathbf{G})$ is naturally isomorphic to the line bundle $\omega_{\mathbf{G}}^{\otimes n}$ over $\pi_0 \mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}(\mathbf{G}) = B$. It then suffices to calculate the $n = 1$ case. As $\omega_{\mathbf{G}}$ is the dualising line for the identity component \mathbf{G}° of \mathbf{G} , we see the B -module $\omega_{\mathbf{G}}$ is naturally equivalent to the dual of the Lie algebra $\mathrm{Lie}(\mathbf{G}^\circ)$ (9), so it now suffices to calculate $[k]^*$ on this Lie algebra. This is quite elementary, but let us recall some details. The question can be answered by localising at each minimal ideal \mathfrak{m} of B containing its ideal of definition J , and over $B_{\mathfrak{m}}$ the 1-dimensional formal group \mathbf{G}° has coordinate t and an associated formal group law G —the choice of coordinates forms a line bundle over $B_{\mathfrak{m}}$ and line bundles over local rings are trivial; see [Goe08, §2]. Assume B is local then. If k is an integer, can write $[k]$ on $B[[t]]$, the global sections of \mathbf{G}° using the coordinate t , as the composite

$$[k]: B[[t]] \xrightarrow{c_k} B[[t_1, \dots, t_k]] \xrightarrow{\mu} B[[t]] \quad (5.29)$$

where the first map is the comultiplication on $B[[t]]$ induced by G and the second is the completed multiplication map. As $c_k(t) \equiv t_1 + \dots + t_k$ modulo higher degree terms, then

$[k](t) \equiv kt$ modulo higher degree terms. Finally, the Lie algebra $\text{Lie}(\mathbf{G}^\circ)$ can be written as a Zariski tangent space:

$$\text{Lie}(\mathbf{G}^\circ) \simeq \text{Hom}_{\text{Mod}_B}(tB[[t]]/(tB[[t]])^2, B)$$

It is now clear that $[k]^*: \text{Lie}(\mathbf{G}^\circ) \rightarrow \text{Lie}(\mathbf{G}^\circ)$ is simply multiplication by k if k is an integer. For a general p -adic unit k , we approximate k by integers using its p -adic expansion, and our conclusion then follows in this more general case by taking the limit. \square

The following justifies why we call the operations $[k]^*$ stable Adams operations.

Proposition 5.30. *For integers k not divisible by p , the map of \mathbf{E}_∞ -rings $[k]^*: \text{KU}_p \rightarrow \text{KU}_p$ is homotopic to classical stable Adams operation ψ^k ; see [Ati67, §3.2].*

Using a slight variant of Rmk.5.7, one can construct maps of \mathbf{E}_∞ -rings $[n]^*: \text{KU}[\frac{1}{n}] \rightarrow \text{KU}[\frac{1}{n}]$ for every integer n . It is well-known ([Ada74, §II.13]) that to construct a stable Adams operation ψ^n as a map of spectra, one **must** invert n . The same goes for stable Adams operations $\psi^n: \text{TMF}[\frac{1}{n}] \rightarrow \text{TMF}[\frac{1}{n}]$; these operations, as well as stable Adams operations on the connective tmf and dualisable Tmf variants, which have been further explored in [Dav21a].

Proof. By restricting ourselves to the case of an integer k not divisible by p , we have assured that $[k]: \mu_{p^\infty}^\heartsuit \rightarrow \mu_{p^\infty}^\heartsuit$ is an automorphism of p -divisible groups.

Let us write $\mathcal{E} = \mathcal{O}_{\text{BT}_1}^{\text{top}}(\mu_{p^\infty}^\heartsuit)$. We claim that $[k]^*$ can be calculated on the universal line bundle over \mathbf{CP}^∞ using just the algebraic geometry of $\widehat{\mathbf{G}}_m$. By (the proof of) Pr.5.6, the map $\rho_p[\beta^{-1}]: \mathcal{E} \rightarrow \text{KU}_p$ is an equivalence of \mathbf{E}_∞ -rings, and Rmk.5.4 states this equivalence sends the canonical complex orientation $x_\mathcal{E}$ of \mathcal{E} to the usual complex orientation x_{KU} of KU_p . We obtain orientations (now in the sense of Df.3.1) $e_\mathcal{E}$ and e_{KU} of the formal multiplicative group $\widehat{\mathbf{G}}_m$ over \mathcal{E} and KU_p , respectively, ([EC2, Ex.4.3.22]) such that $\rho(e_\mathcal{E}) = e_{\text{KU}}$. As these orientations of $\widehat{\mathbf{G}}_m$ determine morphisms from the associated Quillen formal group to $\widehat{\mathbf{G}}_m$ ([EC2, Pr.4.3.23]) and $\rho(e_\mathcal{E}) = e_{\text{KU}}$, we obtain the commutative diagram of equivalences of formal groups over \mathbf{Z}_p courtesy of [EC2, Pr.4.3.23]:

$$\begin{array}{ccc} \widehat{\mathbf{G}}_{\text{KU}_p}^{\mathcal{Q}_0} & \xrightarrow{\rho^*} & \widehat{\mathbf{G}}_{\mathcal{E}}^{\mathcal{Q}_0} \\ & \searrow & \swarrow \\ & \widehat{\mathbf{G}}_{m, \mathbf{Z}_p} & \end{array}$$

Focusing on KU_p now, let us rewrite the above diagram of equivalences of formal groups over $\text{Spf } \mathbf{Z}_p$:

$$\text{Spf } \text{KU}_p^0(\mathbf{CP}^\infty) \xrightarrow{\cong} \text{Spf } \mathbf{Z}_p[[t]] = \widehat{\mathbf{G}}_m \tag{5.31}$$

We know exactly how $[k]$ acts by taking k -fold multiplication, which on the multiplicative formal group is an operation represented by the map of rings:

$$[k]^*: \mathbf{Z}_p[[t]] \xrightarrow{c^k} \mathbf{Z}_p[[t_1, \dots, t_k]] \xrightarrow{\mu} \mathbf{Z}_p[[t]], \quad t \mapsto (t+1)^k - 1$$

Recall from (5.29) that the first map is the k -fold iteration of the comultiplication³⁴, and the second map is the completed multiplication map. As the map (5.31) induces a map of adic rings sending t to βx_{KU} , we then obtain the same formulae for $[k]^*$ in $\mathrm{KU}_p^0(\mathbf{CP}^\infty)$:

$$[k]^*(\beta x_{\mathrm{KU}}) = (\beta x_{\mathrm{KU}} + 1)^k - 1$$

As $\beta x_{\mathrm{KU}} \in \mathrm{KU}_p^0(\mathbf{CP}^\infty)$ is represented by $[\xi_1] - 1$ one obtains:

$$[k]^*([\xi_1]) = [k]^*(\beta x_{\mathrm{KU}} + 1) = [k]^*(\beta x_{\mathrm{KU}}) + 1 = (\beta x_{\mathrm{KU}} + 1)^k - 1 + 1 = [\xi_1^{\otimes k}]$$

It follows that for any finite space (31) X and any complex line bundle \mathcal{L} over X with corresponding map $g: X \rightarrow \mathbf{CP}^\infty$, the inherent naturality of $[k]^*$ gives us the formula:

$$[k]^*(\mathcal{L}) = [k]^*(g^*\xi_1) = g^*([k]^*(\xi_1)) = g^*[\xi_1]^k = [\mathcal{L}^{\otimes k}]$$

It follows from [Ati67, Pr.3.2.1(3)] that the operations $[k]^*$ on $\mathrm{KU}_p^0(X)$ are the Adams operations ψ^k as maps of cohomology theories.

To lift this statement from one about cohomology theories to one about the spectra that represent them, we now show there are no phantom maps of spectra $\mathrm{KU}_p \rightarrow \mathrm{KU}_p$, as this is the only obstacle to the fully faithfulness of the functor

$$\mathrm{hSp} \rightarrow \mathrm{CohomTh} \quad E \mapsto E^*(-)$$

where $\mathrm{CohomTh}$ denotes the 1-category cohomology theories on finite spaces; see [HS99, §2 & Cor.2.15] and [CHT10, Lec.17]. As KU_p represents an even periodic Landweber exact cohomology theory, it follows there exists no phantom endomorphisms of KU_p ; see [CHT10, Cor.7, Lec.17]. \square

Remark 5.32. There is a C_2 -action on the sections of $\mathcal{O}_{\mathrm{BT}_n}^{\mathrm{top}}$ coming from the inversion action on p -divisible groups, ie, coming from ψ^{-1} . Any C_2 -action on an \mathbf{E}_∞ -ring \mathcal{E} can be used to upgrade \mathcal{E} to a genuinely commutative C_2 -ring spectrum (the kind with norms); see [HM17, Th.2.4]. When $p = 2$, this has interesting results, for example, the C_2 -structure on sections of $\mathcal{O}_{\mathrm{BT}_n}^{\mathrm{top}}$ can be used to obtain a C_2 -equivariant refinement of part 1 of Th.1.6: the complex orientability and Landweber exactness of affine sections of $\mathcal{O}_{\mathrm{BT}_n}^{\mathrm{top}}$ can be upgraded to *Real orientability* and *Real Landweber exactness* à la [HM17, §3]. This essentially follows from the *regular homotopy fixed point spectral sequences* of [Mei22], the descent theory developed by Lurie in [EC2, §6], and the analogous result of Hahn–Shi [HS20] for Lubin–Tate spectra.

Remark 5.33. Let p be an odd prime. Using the Teichmüller character, a map of groups $\mathbf{F}_p^\times \rightarrow \mathbf{Z}_p^\times$, which sends d to the limit of the Cauchy sequence $\{d^{p^n}\}_{n \geq 0}$, one obtains an action

³⁴The comultiplication on the ring $\mathbf{Z}_p[[t]]$ representing the multiplicative formal group is given by

$$\mathbf{Z}_p[[t]] \rightarrow \mathbf{Z}_p[[x, y]], \quad t + 1 \mapsto xy + x + y + 1 = (x + 1)(y + 1).$$

of $\mathbf{F}_p^\times \simeq C_{p-1}$ on any sections of $\mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}$. In particular, for any \mathbf{G} in \mathcal{C}_{A_0} (it need not be just an affine object), then the \mathbf{E}_∞ -ring $\mathcal{E} = \mathcal{O}_{\mathrm{BT}_n^p}^{\mathrm{top}}(\mathbf{G})$ has an \mathbf{E}_∞ - \mathbf{F}_p^\times -action, and the homotopy fixed points $\mathcal{E}^{h\mathbf{F}_p^\times}$ split off a summand of \mathcal{E} using the idempotent map:

$$\frac{1}{p-1} \sum_{d \in \mathbf{F}_p^\times \subseteq \mathbf{Z}_p^\times} \psi^d: \mathcal{E} \rightarrow \mathcal{E}$$

In particular, if $\mathcal{E} = \mathrm{KU}_p$ as in §5.1, this summand is the *periodic Adams summand*.

A Appendix on formal spectral Deligne–Mumford stacks

Throughout this article we have used basic properties of formal spectral Deligne–Mumford stacks that are not explicitly contained in [SAG] (at least not obviously to the author), so we have arranged this appendix to prove these statements. Every single statement below is an extension of a proof in [SAG] and the author claims no originality for the ideas below.

Truncations

We would like to show that for locally Noetherian formal spectral Deligne–Mumford stacks, there is a well-defined truncation functor. The following is a generalisation of [SAG, Pr.1.4.6.3] to formal spectral Deligne–Mumford stacks; we will even use the same proof and notation.

Proposition A.1. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathfrak{X}})$ be a locally Noetherian formal spectral Deligne–Mumford stack. For each $n \geq 0$, the object $\tau_{\leq n} \mathfrak{X} = (\mathcal{X}, \tau_{\leq n} \mathcal{O}_{\mathfrak{X}})$ is a locally Noetherian formal spectral Deligne–Mumford stack. Moreover, for every $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ inside $\infty\mathrm{Top}_{\mathrm{CAlg}}^{\mathrm{sHen}}$, if $\mathcal{O}_{\mathcal{Y}}$ is connective and n -truncated, then the canonical map $\tau_{\leq n} \mathfrak{X} \rightarrow \mathfrak{X}$ induces an equivalence*

$$\mathrm{Map}_{\infty\mathrm{Top}_{\mathrm{CAlg}}^{\mathrm{sHen}}}((\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}), \tau_{\leq n} \mathfrak{X}) \rightarrow \mathrm{Map}_{\infty\mathrm{Top}_{\mathrm{CAlg}}^{\mathrm{sHen}}}((\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}), \mathfrak{X}).$$

Proof. The first half of the proof of [SAG, Pr.1.4.6.3] applies *mutatis mutandis*. That is, by copying that proof we see that for every strictly Henselian spectrally ringed ∞ -topos $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ which is connective and n -truncated, the canonical map

$$\mathrm{Map}_{\infty\mathrm{Top}_{\mathrm{CAlg}}^{\mathrm{sHen}}}((\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}), \tau_{\leq n} \mathfrak{X}) \rightarrow \mathrm{Map}_{\infty\mathrm{Top}_{\mathrm{CAlg}}^{\mathrm{sHen}}}((\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}), \mathfrak{X}) \quad (\mathrm{A}.2)$$

is an equivalence of spaces. Hence, we are left to show that $\tau_{\leq n} \mathfrak{X} = (\mathcal{X}, \tau_{\leq n} \mathcal{O}_{\mathfrak{X}})$ is a locally Noetherian formal spectral Deligne–Mumford stack. By [SAG, Prs.8.1.3.3 & 8.4.2.7], being a formal spectral Deligne–Mumford stack and being locally Noetherian are local conditions, hence we may assume $\mathfrak{X} = \mathrm{Spf} A$ for a complete Noetherian adic \mathbf{E}_∞ -ring A . Set $B = \tau_{\leq n} A$, equipped with the same topology as A induced by $I \subseteq \pi_0 A$ using the isomorphism $\pi_0 A \simeq \pi_0 B$. Here we need to show $\mathrm{Spf} B$ is connective, n -truncated, and construct an equivalence with $\tau_{\leq n} \mathfrak{X}$.

By [SAG, Pr.8.1.1.13], we see $\mathrm{Spf} B = (\mathcal{X}_{\mathrm{Spf} B}, \mathcal{O}_{\mathrm{Spf} B})$ is connective. For n -truncatedness, one can argue as follows: for affine objects U of $\mathcal{X}_{\mathrm{Spf} B}$ we have $\mathcal{O}_{\mathrm{Spf} B}(U) \simeq C_I^\wedge$ for some étale

B -algebra C . As C is an étale \mathbf{E}_∞ - B -algebra, then it is almost of finite presentation, and as B is Noetherian (as a truncation of the Noetherian \mathbf{E}_∞ -ring A), then the spectral Hilbert basis theorem ([HA, Pr.7.2.4.31]) implies that C is also Noetherian. It then follows from [SAG, Cor.7.3.6.9] that the natural map of \mathbf{E}_∞ - A -algebras $C \rightarrow C_I^\wedge$ is flat. As the composition

$$B \rightarrow C \rightarrow C_I^\wedge \simeq \mathcal{O}_{\mathrm{Spf} B}(U)$$

is flat, we see $\mathcal{O}_{\mathrm{Spf} B}(U)$ is n -truncated as B is so. The ∞ -topos $\mathcal{X}_{\mathrm{Spf} B}$ is generated by affine objects under small colimits ([SAG, Pr.8.1.3.7]) and the structure sheaf $\mathcal{O}_{\mathrm{Spf} B}: \mathcal{X}_{\mathrm{Spf} B}^{\mathrm{op}} \rightarrow \mathrm{CAlg}$ preserves limits, so it follows that $\mathcal{O}_{\mathrm{Spf} B}(X)$ is n -truncated for all $X \in \mathcal{X}_{\mathrm{Spf} B}$, hence $\mathrm{Spf} B$ is n -truncated; see [SAG, Rmk.1.3.2.6]. By (A.2), the natural map $\mathrm{Spf} B \rightarrow \mathrm{Spf} A = \mathfrak{X}$ factors as:

$$\mathrm{Spf} B \xrightarrow{\phi} \tau_{\leq n} \mathfrak{X} = (\mathcal{X}, \tau_{\leq n} \mathcal{O}_{\mathfrak{X}}) \rightarrow (\mathcal{X}, \mathcal{O}_{\mathfrak{X}}) = \mathfrak{X}$$

Using [SAG, Rmk.8.1.1.9], we see the map of underlying ∞ -topoi induced by $\phi: A \rightarrow \tau_{\leq n} A = B$ is an equivalence,

$$\mathrm{Shv}_{\pi_0 B/I}^{\mathrm{ét}} \simeq \mathrm{Shv}_B^{\mathrm{ad}} \xrightarrow{\phi_*} \mathrm{Shv}_A^{\mathrm{ad}} \simeq \mathrm{Shv}_{\pi_0 A/I}^{\mathrm{ét}}$$

where we used the notation of [SAG, Nt.8.1.1.8]. Under this map, the structure sheaf of $\mathrm{Spf} B$ is sent to the functor

$$\phi_* \mathcal{O}_{\mathrm{Spf} B}: \mathrm{CAlg}_A^{\mathrm{ét}} \rightarrow \mathrm{CAlg}^{\mathrm{cn}} \quad D \mapsto (D \otimes_A B)_I^\wedge \simeq (\tau_{\leq n} D)_I^\wedge. \quad (\text{A.3})$$

The equivalence above comes from the facts that $A \rightarrow D$ is étale and a degenerate Tor-spectral sequence calculation; see [HA, Pr.7.2.1.19]. To see ϕ is an equivalence, it therefore suffices to see that (A.3) is equivalent to $\tau_{\leq n} \mathcal{O}_{\mathrm{Spf} A}$. This is slight variation on an argument made above. As D is étale over the Noetherian \mathbf{E}_∞ -ring A , then the spectral Hilbert basis theorem implies that D is also Noetherian. It follows straight from the definition that the \mathbf{E}_∞ -ring $\tau_{\leq n} D$ is Noetherian, so the natural completion map of \mathbf{E}_∞ - A -algebras

$$\tau_{\leq n} D \rightarrow (\tau_{\leq n} D)_I^\wedge$$

is flat. This implies that $(\tau_{\leq n} D)_I^\wedge$ is n -truncated. As $\tau_{\leq n}(D_I^\wedge)$ is I -complete by [SAG, Cor.7.3.4.3], there is a natural equivalence of \mathbf{E}_∞ - A -algebras:

$$(\tau_{\leq n} D)_I^\wedge \simeq \tau_{\leq n}(D_I^\wedge)$$

Hence ϕ is an equivalence of spectrally ringed ∞ -topoi. □

The following is a formal generalisation of [SAG, Cor.1.4.6.4]:

Corollary A.4. *For each integer $n \geq 0$, write $\mathrm{fSpDM}_{\mathrm{loc.N}}^{\leq n}$ for the full ∞ -subcategory of $\mathrm{fSpDM}_{\mathrm{loc.N}}$ spanned by those n -truncated locally Noetherian formal spectral Deligne–Mumford stacks. The inclusion $\mathrm{fSpDM}_{\mathrm{loc.N}}^{\leq n} \hookrightarrow \mathrm{fSpDM}_{\mathrm{loc.N}}$ has a right adjoint, given on objects by*

$$\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathfrak{X}}) \mapsto \tau_{\leq n} \mathfrak{X} = (\mathcal{X}, \tau_{\leq n} \mathcal{O}_{\mathfrak{X}}).$$

Proof. This follows straight from the universal property of Pr.A.1 and the observation and truncations of locally Noetherian formal spectral Deligne–Mumford stacks remain locally Noetherian. \square

Corollary A.5. *Let \mathfrak{X} be a locally Noetherian formal spectral Deligne–Mumford stack. Then for any integer $n \geq 0$ the truncation $\tau_{\leq n}\mathfrak{X}$ and \mathfrak{X} represent the same functor on n -truncated \mathbf{E}_∞ -rings.*

Proof. Follows straight from Pr.A.1, as $\mathrm{Spec} R$ is a connective n -truncated spectrally ringed ∞ -topos when R is a connective n -truncated \mathbf{E}_∞ -ring; see [SAG, Ex.1.4.6.2]. \square

The fully faithful embedding $\mathrm{fDM} \rightarrow \mathrm{fSpDM}$

To formalise the relationship between the classical and spectral worlds of formal algebraic geometry, we need a functor $\mathrm{fDM} \rightarrow \mathrm{fSpDM}$. Let us begin by defining these categories.

Definition A.6. Let A be a discrete adic Noetherian ring with finitely generated ideal of definition $I \subseteq A$, cutting out a closed subset $V \subseteq |\mathrm{Spec} A|$.

1. Define the topos $\mathrm{Shv}_{\mathrm{Set}}^{\mathrm{ad}}(\mathrm{CAlg}_A^{\mathrm{ét}})$ is the full ∞ -subcategory of $\mathrm{Shv}_{\mathrm{Set}}^{\mathrm{ét}}(\mathrm{CAlg}_A^{\mathrm{ét}})$ spanned by those étale sheaves \mathcal{F} such that if the space $V \times_{|\mathrm{Spec} A|} |\mathrm{Spec} B|$ is empty, then $\mathcal{F}(B)$ is a point.
2. One has a sheaf of discrete rings $\mathcal{O}_{\mathrm{Spec} A}$ on $\mathrm{Shv}_{\mathrm{Set}}^{\mathrm{ét}}(\mathrm{CAlg}_A^{\mathrm{ét}})$ as in [SAG, Df.1.2.3.1], which we complete at I to obtain a sheaf $\widehat{\mathcal{O}}$. This sheaf factors through $\mathrm{Shv}_{\mathrm{Set}}^{\mathrm{ad}}(\mathrm{CAlg}_A^{\mathrm{ét}})$ as $\widehat{\mathcal{O}}(B) \simeq B_{\widehat{I}}$ vanishes if whenever the image of I generates the unit ideal of B .

Define the ringed topos $\mathrm{Spf} A = (\mathrm{Shv}_{\mathrm{Set}}^{\mathrm{ad}}(\mathrm{CAlg}_A^{\mathrm{ét}}), \widehat{\mathcal{O}})$, the *formal spectrum of A* , leaving the dependency on the specific topology on A implicit. A *locally Noetherian formal Deligne–Mumford stack* is a ringed topos $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathfrak{X}})$ such that \mathcal{X} has a cover U_α such that each ringed topos \mathfrak{X}/U_α is equivalent (in the 2-category of ringed topoi of [SAG, Df.1.2.1.1]) to $\mathrm{Spf} A_\alpha$ for some discrete adic Noetherian ring A_α . Write fDM for the full 2-category of $1\mathrm{Top}_{\mathrm{CAlg}^\heartsuit}^{\mathrm{loc}}$ spanned by locally Noetherian formal Deligne–Mumford stacks.

The ∞ -category of formal spectral Deligne–Mumford stacks fSpDM can be defined similarly; see Df.0.7 or [SAG, Df.8.1.3.1].

As in [SAG, §8], when dealing with classical formal Deligne–Mumford stacks, we restrict ourselves to the locally Noetherian case *by definition*, as opposed to the spectral case, when we only add this assumption when we need it. As mentioned in [SAG, Warn.8.1.0.4], this is due to the incompatibility between completions in the classical and derived worlds.

Remark A.7. If an adic discrete ring A has a nilpotent ideal of definition, then $\mathrm{Spf} B$ is naturally equivalent to $\mathrm{Spec} B$ by definition. In this way, we can see (Noetherian) affine Deligne–Mumford stacks as affine formal Deligne–Mumford stacks. It then also immediately follows from the definitions that $\mathrm{DM}_{\mathrm{loc}, \mathbf{N}}$ is a full 2-subcategory of fDM .

The following is [SAG, Rmk.1.4.1.5].

Construction A.8. There is a fully faithful embedding of ∞ -categories from classical ringed topoi to spectrally ringed ∞ -topoi

$$1\mathcal{T}\mathrm{op}_{\mathrm{CAlg}^\heartsuit} \hookrightarrow \infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}; \quad (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \mapsto (\mathcal{S}\mathrm{h}\mathcal{V}(\mathcal{X}), \mathcal{O}).$$

In other words, it associates to a classical Grothendieck topos \mathcal{X} the associated ∞ -topos $\mathcal{S}\mathrm{h}\mathcal{V}(\mathcal{X})$ (this is done using [HTT09, Pr.6.4.5.7]) and by [SAG, Rmk.1.3.5.6] we obtain a connective 0-truncated structure sheaf on $\mathcal{S}\mathrm{h}\mathcal{V}(\mathcal{X})$, denoted as \mathcal{O} . In fact, the essential image of the above embedding is spanned by the spectrally ringed ∞ -topoi $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ where \mathcal{X} is 1-localic and $\mathcal{O}_{\mathcal{X}}$ is connective and 0-truncated.

By definition [HTT09, Df.6.4.5.8], we see the ∞ -topoi $\mathcal{S}\mathrm{h}\mathcal{V}(\mathcal{X})$ produced by Con.A.8 are *1-localic*. By [SAG, Rmk.1.4.8.3], the fully faithful embedding of Con.A.8 restricts to the full-faithful embedding $\mathrm{DM} \rightarrow \mathrm{SpDM}$. Let us show that the same holds for *formal* Deligne–Mumford stacks.

Proposition A.9. *The functor of Con.A.8, when restricted to fDM factors through fSpDM . Moreover, the essential image of this fully faithful functor $\mathrm{fDM} \rightarrow \mathrm{fSpDM}$ consists of those locally Noetherian formal spectral Deligne–Mumford stacks $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathfrak{X}})$ for which the ∞ -topos \mathcal{X} is 1-localic ([HTT09, Df.6.4.5.8]) and the structure sheaf $\mathcal{O}_{\mathfrak{X}}$ is 0-truncated.*

Proof. The fully faithful functor of Con.A.8 descends to a fully faithful functor between (not full) ∞ -subcategories of local topoi:

$$1\mathcal{T}\mathrm{op}_{\mathrm{CAlg}^\heartsuit}^{\mathrm{loc}} \hookrightarrow \infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}^{\mathrm{loc}}$$

Indeed, we say $\mathcal{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ in $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ is local if $\pi_0\mathcal{O}_{\mathcal{X}}$ is local on \mathcal{X}^\heartsuit ([SAG, Df.1.4.2.1]), and given $\mathcal{X}_0 = (\mathcal{X}_0, \mathcal{O}_0)$ in $1\mathcal{T}\mathrm{op}_{\mathrm{CAlg}^\heartsuit}$, then the ringed topos $(\mathcal{S}\mathrm{h}\mathcal{V}(\mathcal{X})^\heartsuit, \pi_0\mathcal{O})$ is naturally equivalent to \mathcal{X}_0 by [HTT09, Pr.6.4.5.7]. Local morphisms between local spectrally ringed ∞ -topoi are morphisms of spectrally ringed ∞ -topoi whose underlying morphism of ringed topoi is local.

Let $\mathfrak{X}_0 = (\mathcal{X}_0, \mathcal{O}_0)$ be a classical formal Deligne–Mumford stack, and write $\mathfrak{X} = (\mathcal{X}, \mathcal{O})$ for the image of \mathfrak{X}_0 under Con.A.8, so $\mathcal{X} = \mathcal{S}\mathrm{h}\mathcal{V}(\mathcal{X}_0)$. By [SAG, Pr.8.1.3.3], the property of being a formal spectral Deligne–Mumford stack is a local one, so it suffices to show that there exists a cover U_α of \mathcal{X} such that each $\mathfrak{X}_{/U_\alpha}$ is in fSpDM . Consider a formal affine cover of \mathfrak{X}_0 in $1\mathcal{T}\mathrm{op}_{\mathrm{CAlg}^\heartsuit}$, so a collection of U_α inside \mathcal{X}_0 such that $\coprod U_\alpha \rightarrow \mathbf{1}_{\mathcal{X}_0}$ is an effective epimorphism and $(\mathfrak{X}_0)_{/U_\alpha}$ is equivalent in $1\mathcal{T}\mathrm{op}_{\mathrm{CAlg}^\heartsuit}$ to $\mathrm{Spf} A_\alpha$. Considering U_α as a discrete object V of \mathcal{X} (as in [HTT09, Pr.6.4.5.7]), then [SAG, Lm.1.4.7.7(2)] states that $\mathcal{X}_{/V}$ is 1-localic, as \mathcal{X} is 1-localic and V is 0-truncated in \mathcal{X} . One then notes the following natural equivalences:

$$\mathcal{X}_{/V} \xrightarrow{\cong} \mathcal{S}\mathrm{h}\mathcal{V}((\mathcal{X}_{/V})^\heartsuit) \simeq \mathcal{S}\mathrm{h}\mathcal{V}((\mathcal{X}_0)_{/U_\alpha}) \simeq \mathcal{S}\mathrm{h}\mathcal{V}(\mathcal{S}\mathrm{h}\mathcal{V}_{\mathrm{Set}}^{\mathrm{ad}}(\mathrm{CAlg}_{A_\alpha}^{\mathrm{ét}})) \xleftarrow{\cong} \mathcal{S}\mathrm{h}\mathcal{V}^{\mathrm{ad}}(\mathrm{CAlg}_{A_\alpha}^{\mathrm{ét}})$$

The first equivalence holds as $\mathcal{X}_{/V}$ is 1-localic, the second by identifying \mathcal{X}_0 as the underlying discrete objects of \mathcal{X} (and then [HTT09, Rmk.7.2.2.17]), the third from the choice of U_α as an affine object of \mathcal{X}_0 , and the fourth from the fact that affine formal spectral Deligne–Mumford

stacks are 1-localic; see [SAG, Rmk.8.1.1.9]. Furthermore, as \mathcal{O} was defined as the sheaf of connective 0-truncated \mathbf{E}_∞ -rings on \mathcal{X} associated to the commutative ring object \mathcal{O}_0 on \mathcal{X}_0 , we claim that by [SAG, Rmk.1.3.5.6] the spectrally ringed ∞ -topos $\mathfrak{X}/_{U_\alpha}$ is equivalent to $\mathrm{Spf} A_\alpha$. To see this, one notes that $\mathcal{O}(\mathrm{Spf} B) = B_1^\wedge$ for some étale morphism $\mathrm{Spf} B \rightarrow \mathrm{Spf} A_\alpha$ in $\mathcal{X}_0 \subseteq \mathcal{X}$, and one also has a natural equivalence $\mathcal{O}_{\mathrm{Spf} A_\alpha}(\mathrm{Spf} B) \simeq B_1^\wedge$ by [SAG, Con.8.1.1.10]. The “moreover” statement follows by [SAG, Rmk.1.4.1.5]. \square

Combining the functor of points approach with the above, we obtain the following:

Corollary A.10. *The following diagram of ∞ -categories and fully faithful functors commutes:*

$$\begin{array}{ccccc}
\mathrm{Aff}_{\mathrm{loc.N}}^\heartsuit & \xrightarrow{a} & \mathrm{Aff}_{\mathrm{ad,loc.N}}^\heartsuit & \xrightarrow{b} & \mathrm{fDM} \\
\downarrow c & & \downarrow d & & \downarrow e \\
\mathrm{Aff}^{\mathrm{cn}} & \xrightarrow{f} & \mathrm{Aff}_{\mathrm{ad}}^{\mathrm{cn}} & \xrightarrow{g} & \mathrm{fSpDM} \xrightarrow{h} \mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})
\end{array}$$

Warning A.11. One might want to place $\mathcal{P}(\mathrm{Aff}^\heartsuit)$ in the top-right corner of the diagram above, however, we do not see a functor $\mathcal{P}(\mathrm{Aff}^\heartsuit) \rightarrow \mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$ such that the diagram above commutes. Indeed, the right Kan extension mentioned in Nt.2.30 doesn’t commute with the other constructions above by inspection and a left Kan extension would not necessarily preserve sheaves. The existence of the functors c , d , and e above, are all due to nontrivial theorems of Lurie, and the lack of a similar functor $\mathcal{P}(\mathrm{Aff}^\heartsuit) \rightarrow \mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$ indicates one reason why we restrict our attention to (formal) Deligne–Mumford stacks.

Proof of Cor.A.10. The functors a , b , f , and g are all the inclusions of full ∞ -subcategories, c and d are the inclusions of ∞ -subcategories as shown by Lurie ([HA, Pr.7.1.3.18]), e is Con.A.8, and h is the functor of points functor. The diagram commutes as c and d are restrictions of e . To see why each functor is fully faithful, we have:

- By definition, we see that a , b , f , and g are fully faithful.
- By [HA, Pr.7.1.3.18], we see c and hence d are fully faithful.
- Pr.A.9 shows e is fully faithful.
- The fact that h is fully faithful is the content of [SAG, Th.8.1.5.1]. \square

Finiteness and compactness in fSpDM

Next, let us discuss finiteness and compactness conditions in fSpDM.

Proposition A.12. *Let \mathfrak{X} be a locally Noetherian formal spectral Deligne–Mumford stack. Then for any $n \geq 0$ the natural map $\tau_{\leq n} \mathfrak{X} \rightarrow \mathfrak{X}$ admits an $(n + 1)$ -connective and almost perfect cotangent complex.*

Proof. These are local conditions, so we may take $\mathfrak{X} = \mathrm{Spf} A$ for a complete Noetherian adic \mathbf{E}_∞ -ring A with finitely generated ideal of definition $I \subseteq \pi_0 A$. By the Hilbert basis theorem for connective \mathbf{E}_∞ -rings ([HA, Pr.7.2.4.31]) we see $\tau_{\leq n} A$ is almost finitely presented as an \mathbf{E}_∞ - A -algebra and the cofibre of map $A \rightarrow \tau_{\leq n} A$ is $(n+1)$ -connective. By [HA, Cor.7.4.3.2] and [HA, Th.7.4.3.18], we then see $L = L_{\tau_{\leq n} A/A}$ is $(n+1)$ -connective and almost perfect inside $\mathrm{Mod}_{\tau_{\leq n} A}$. It follows from [SAG, Pr.7.3.5.7] that L is in fact I -complete, hence we have a natural equivalence $L_1 \simeq L_{\mathrm{Spf} \tau_{\leq n} A / \mathrm{Spf} A}$ by [SAG, Df.17.1.2.8], and we are done. \square

Definition A.13. A formal spectral Deligne–Mumford stack $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathfrak{X}})$ is *quasi-compact* (qc) if the underlying ∞ -topos \mathcal{X} is quasi-compact, ie, every cover of \mathcal{X} has a finite subcover; see [SAG, Df.A.2.0.12]. A morphism of formal spectral Deligne–Mumford stacks

$$f: \mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathfrak{X}}) \rightarrow \mathfrak{Y} = (\mathcal{Y}, \mathcal{O}_{\mathfrak{Y}})$$

is qc if for any qc object U of \mathcal{Y} , the pullback $f^*(U)$ is qc in \mathcal{X} , meaning \mathcal{X}_{f^*U} is qc. A morphism of formal spectral Deligne–Mumford stacks is called *quasi-separated* (qs) if the diagonal map $\Delta: \mathfrak{Y} \rightarrow \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Y}$ is qc. We say \mathfrak{X} is qs if $\mathfrak{X} \rightarrow \mathrm{Spec} \mathbf{S}$ is qs.

It is a purely formal exercise that qc (and qs) maps are stable under base-change; a fact we will use without further reference.

Proposition A.14. *Let A be an adic \mathbf{E}_∞ -ring. Then $\mathrm{Spf} A$ is qc.*

Proof. By [SAG, Rmk.8.1.1.9], we see the underlying ∞ -topos of $\mathrm{Spf} A$ is equivalent to $\mathrm{Shv}_{\pi_0 A/I}^{\acute{e}t}$ where I is a finitely generated ideal of definition for the topology on $\pi_0 A$. As this is the same underlying ∞ -topos of $\mathrm{Spec}(\pi_0 A/I)$, it follows from [SAG, Pr.2.3.1.2] that $\mathrm{Spf} A$ is qc. \square

The following is a formal generalisation of a special case of [SAG, Pr.2.3.2.1].

Proposition A.15. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathfrak{X}})$ be a formal spectral Deligne–Mumford stack. Then the following are equivalent.*

1. \mathfrak{X} is qs.
2. For all qc objects U, V of \mathcal{X} , the product $U \times V$ in \mathcal{X} is qc.
3. For all affine objects U, V of \mathcal{X} , the product $U \times V$ is qc.

Proof. It is clear that 1 implies 2 as $U \times V = \Delta^*(U, V)$ inside $\mathcal{X} \times \mathcal{X}$, and 2 also implies 1 as the quasi-compact objects of $\mathcal{X} \times \mathcal{X}$ are all of the form (U, V) for U and V quasi-compact in \mathcal{X} . Pr.A.14 shows that 2 implies 3. Conversely, for two arbitrary qc objects U and V of \mathcal{X} , using the fact they are qc, there exists two effective epimorphisms $U' \rightarrow U$ and $V' \rightarrow V$ where U' and V' are affine. It then follows that $U \times V$ is qc as there is an effective epimorphism $U' \times V' \rightarrow U \times V$ from a qc object of \mathcal{X} . \square

Corollary A.16. *Let A be an adic \mathbf{E}_∞ -ring. Then $\mathrm{Spf} A$ is qcqs.*

Proof. By Pr.A.14, we see $\mathrm{Spf} A$ is qc, and by Pr.A.15 it suffices to see that for all affine objects $U = \mathrm{Spf} B$ and $V = \mathrm{Spf} C$ inside $\mathcal{X}_{\mathrm{Spf} A}$, that the product $U \times V$ in $\mathcal{X}_{\mathrm{Spf} A}$ is qc. This product can be recognised as the fibre product ([SAG, Lm.8.1.7.3])

$$\mathrm{Spf} B \times_{\mathrm{Spf} A} \mathrm{Spf} C \simeq \mathrm{Spf} \left(B \otimes_A C \right)_I^\wedge$$

where I is an ideal of definition for the topology on $\pi_0 A$, which is qc by Pr.A.15. \square

The following statement is why we care about the adjectives of Df.A.13.

Proposition A.17. *Let \mathfrak{X} be a formal spectral Deligne–Mumford stack. Then \mathfrak{X} is qcqs if and only if there exists an étale hypercover \mathfrak{U}_\bullet of \mathfrak{X} such that each \mathfrak{U}_n is an affine formal spectral Deligne–Mumford stack for every $n \geq 0$. In particular, the same holds for classical Deligne–Mumford stacks.*

Proof. First, let us assume \mathfrak{X} is qcqs and write $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathfrak{X}})$ and set $\mathfrak{U}_{-1} = \mathfrak{X}$. As a formal spectral Deligne–Mumford stack, there exists a collection of affine objects U_α in \mathcal{X} such that $\coprod_\alpha U_\alpha$ cover \mathcal{X} , and as \mathfrak{X} is qc, this collection can be taken to be finite. As $\mathcal{X}_{/U_\alpha} \simeq \mathrm{Spf} A_\alpha$ for some adic \mathbf{E}_∞ -ring A_α , we see the fact that $\coprod U_\alpha$ covers \mathcal{X} is equivalent to the statement that

$$\mathrm{Spf} A_0 = \mathrm{Spf} \left(\prod A_\alpha \right) \simeq \coprod \mathrm{Spf} A_\alpha \rightarrow \mathfrak{X}$$

is an étale surjection, where we have used the finiteness of the above (co)product. Set $\mathfrak{U}_0 = \mathrm{Spf} A_0$ and $\mathfrak{U}_0 \rightarrow M_0(\mathfrak{U}_\bullet^{\leq -1}) \simeq \mathfrak{U}_{-1} = \mathfrak{X}$ to be the étale surjection above. The rest of the proof can be summarised informally by inductively calculating $M_n(\mathfrak{U}_\bullet^{\leq n-1})$ which must be affine as they are defined by taking finites limits over mostly affine formal spectral Deligne–Mumford stacks, and using that affines are qcqs (Cor.A.16) we find \mathfrak{U}_{n+1} by taking an affine étale cover of $M_n(\mathfrak{U}_\bullet^{\leq n-1})$. To formalise this outline, we will need to play around with these matching objects more carefully, but the rest of this half of the proof is essentially index chasing.

Inductively, let us assume the following three hypotheses:

1. Suppose we have the n th stage of an étale hypercover $\mathfrak{U}_\bullet^{\leq n}$ such that $\mathfrak{U}_m \simeq \mathrm{Spf} A_m$ is affine for each $0 \leq m \leq n$.
2. Suppose that for every $0 \leq m \leq n$, $M_m(\mathfrak{U}_\bullet^{\leq m-1})$ is affine. The base case that $\mathfrak{U}_0 \simeq \mathrm{Spf} A_0$ is affine holds by construction.

For every $1 \leq k \leq m \leq n$, write $\mathfrak{U}_{\bullet+k}^{\leq m-k}$ for the functor defined by precomposition with the shift functor defined on objects by

$$\Delta_{s,+}^{\leq m-k} \rightarrow \Delta_{s,+}^{\leq m}, \quad [i] \mapsto [i+k],$$

and on morphisms by sending $\phi: [i] \rightarrow [j]$ to $\phi': [i+k] \rightarrow [j+k]$ which sends $a+k \mapsto \phi(a)+k$ for $a \geq 0$ and $-1 \mapsto -1$. The third hypothesis is then:

3. Suppose that for every $1 \leq k \leq m \leq n$, $M_{m-k+1}(\mathfrak{U}_{\bullet+k}^{\leq m-k})$ is affine. This condition is vacuous in the base-case.

We claim that there is a natural equivalence

$$M = M_{n+1}(\mathfrak{U}_{\bullet}^{\leq n}) = \lim_{[i] \hookrightarrow [n+1]} \mathfrak{U}_i^{\leq n} \simeq \mathfrak{U}_n \times_{M_n(\mathfrak{U}_{\bullet}^{\leq n-1})} M_n(\mathfrak{U}_{\bullet+1}^{\leq n-1}) \quad (\text{A.18})$$

which occurs in fSpDM, as by using [SAG, Pr.8.1.7.1] the ∞ -category fSpDM has finite limits. To see this (A.18) is an equivalence, recall that our diagram 1-category above is the poset of proper subsets of $[n+1]$. Using notation from [MV15, §5.1], we see the opposite of this poset is precisely the 1-category $\mathcal{P}_0(\underline{n+2})$ of nonempty subsets S of $\{1, \dots, n+2\}$. This yields the equivalence:

$$M \simeq \lim_{S \in \mathcal{P}_0(\underline{n+2})} \mathfrak{U}_{n-|S|+1}^{\leq n}$$

Using the cubical limit manipulations of [MV15, Lm.5.3.6], we obtain the natural equivalence of (A.18):

$$M \simeq \lim_{S \in \mathcal{P}_0(\underline{n+2})} \mathfrak{U}_{n-|S|+1}^{\leq n} \simeq \mathfrak{U}_n \times_{M_n(\mathfrak{U}_{\bullet}^{\leq n-1})} M_n(\mathfrak{U}_{\bullet+1}^{\leq n-1})$$

Note that the map $\mathfrak{U}_n \rightarrow M_n(\mathfrak{U}_{\bullet}^{\leq n-1})$ is an étale cover by our first inductive hypothesis and the natural map

$$M_n(\mathfrak{U}_{\bullet+1}^{\leq n-1}) \rightarrow M_n(\mathfrak{U}_{\bullet}^{\leq n-1})$$

is an étale cover by base-change. Indeed, this latter condition follows as $\mathfrak{U}_{m+1} \rightarrow \mathfrak{U}_m$ is an étale cover for each m , and $M_n(-)$ is a finite limit diagram of such covers. We also note that M is qcqs, which follows from (A.18), our inductive assumptions 1-3, and Cor.A.16. This guarantees the existence of an étale cover $\mathfrak{U}_{n+1} \rightarrow M_{n+1}(\mathfrak{U}_{\bullet}^{\leq n})$ with \mathfrak{U}_{n+1} an affine formal spectral Deligne–Mumford stack, from which we obtain our first inductive conclusion for $(n+1)$. We also saw $M = M_{n+1}(\mathfrak{U}_{\bullet}^{\leq n})$ is qcqs, so we also have our second inductive conclusion for $(n+1)$. For the last one, we consider $M(k) = M_{n-k+2}(\mathfrak{U}_{\bullet+k}^{\leq n+1-k})$, the only case left to consider; the others fall under part 3 of the the previous inductive step. We claim that $M(k)$ is affine. To see this, use an index shift of (A.18) to obtain:

$$M(k) \simeq \mathfrak{U}_{n-k+2} \times_{M_{n-k+1}(\mathfrak{U}_{\bullet+k}^{\leq n-k})} M_{n-k+1}(\mathfrak{U}_{\bullet+k+1}^{\leq n-k})$$

The left and bottom objects in the fibre product above are affine by our inductive hypotheses 1 and 2, respectively, so it suffices to show the right object in the above fibre product is affine. This can be done by applying (A.18) again, noting the left and bottom objects are affine by inductive hypotheses 1 and 2 again and again considering the right factor. Applying this process $(n-k)$ -many times, we are left with $M_0(\mathfrak{U}_{\bullet+n+2}^{\leq -1}) \simeq \mathfrak{U}_{n+1}$, which is affine by our construction above.

Conversely, assume that \mathfrak{X} has an étale hypercover $\mathfrak{U}_{\bullet} \rightarrow \mathfrak{X}$ where each \mathfrak{U}_n is affine, which we write as $U_{\bullet} \rightarrow \mathbf{1}$ when considered as objects in \mathcal{X} . Given an arbitrary cover $\{V_{\alpha}\}_{\alpha \in I}$ of \mathfrak{X} ,

so an effective epimorphism $\coprod V_\alpha \rightarrow \mathbf{1}$, then we can consider the Cartesian square inside \mathcal{X} of the form:

$$\begin{array}{ccc} W & \longrightarrow & U_0 \\ \downarrow & & \downarrow \\ \coprod_I V_\alpha & \longrightarrow & \mathbf{1} \end{array}$$

All of the maps above are effective epimorphisms either by assumption or as the class of such maps is stable under pullback; see [HTT09, Pr.6.2.3.15]. Products commute with colimits in an ∞ -topos as colimits in ∞ -topoi are universal,³⁵ hence we have a natural equivalence in $W \simeq \coprod_I W_\alpha$ in \mathcal{X} , where $W_\alpha = V_\alpha \times U_0$. As U_0 is quasi-compact (as an affine object of \mathcal{X} ; see Pr.A.14), we can choose a finite subset of I , say I_0 , such that $\coprod_{I_0} W_\alpha \rightarrow U_0$ is an effective epimorphism. We then consider the commutative diagram inside the ∞ -topos \mathcal{X} :

$$\begin{array}{ccc} \coprod_{I_0} W_\alpha & \longrightarrow & U_0 \\ \downarrow & & \downarrow \\ \coprod_{I_0} V_\alpha & \longrightarrow & \mathbf{1} \end{array}$$

The top and right maps are effective epimorphisms by assumption, and the bottom map is an effective epimorphism by [HTT09, Cor.6.2.3.12(2)], hence \mathcal{X} is qc. To see \mathcal{X} is qs, we look at the Cartesian diagram of formal spectral Deligne–Mumford stacks:

$$\begin{array}{ccc} \mathfrak{U}_0 & \xrightarrow{\Delta_{\mathfrak{U}_0}} & \mathfrak{U}_0 \times \mathfrak{U}_0 \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{\Delta_{\mathcal{X}}} & \mathcal{X} \times \mathcal{X} \end{array}$$

As $\mathfrak{U}_\bullet \rightarrow \mathfrak{X}$ is an étale hypercover, the map $\mathfrak{U}_0 \times \mathfrak{U}_0 \rightarrow \mathcal{X} \times \mathcal{X}$ is an effective epimorphism. As \mathfrak{U}_0 is the ∞ -topos of an affine formal Deligne–Mumford stack, then by Cor.A.16 we see \mathfrak{U}_0 is qs and the map $\Delta_{\mathfrak{U}_0}$ is qc. It follows from [SAG, Cor.A.2.1.5] that $\Delta_{\mathcal{X}}$ is qc; in *ibid*, a qc morphism is called *relatively 0-coherent*. Hence, \mathcal{X} , and therefore \mathfrak{X} , is qs. \square

Let us now show the *formal thickenings* of [SAG, §18.2.2] preserve the adjective qcqs.

Proposition A.19. *Let \mathfrak{X}_0 be a qcqs formal spectral Deligne–Mumford stack and $\mathfrak{X}_0 \rightarrow \mathfrak{X}$ a formal thickening. Then \mathfrak{X} is qcqs.*

Proof. The adjective qcqs depends only on the underlying ∞ -topoi, so it suffices to show if that $\mathfrak{X}_0 \rightarrow \mathfrak{X}$ is an equivalence of ∞ -topoi. To see this, consider the *reduction* of a formal spectral Deligne–Mumford stack of [SAG, Pr.8.1.4.4]. From this one obtains the following

³⁵We say that colimits in a presentable ∞ -category \mathcal{C} are *universal* if pullbacks commute with all small colimits; see [HTT09, Df.6.1.1.2]. This holds in an ∞ -topos due to the ∞ -categorical version of Giraud’s axioms; see [HTT09, Th.6.1.0.6].

commutative diagram of formal spectral Deligne–Mumford stacks:

$$\begin{array}{ccc} \mathfrak{X}_0^{\text{red}} & \longrightarrow & \mathfrak{X}_0 \\ \downarrow & & \downarrow \\ \mathfrak{X}^{\text{red}} & \longrightarrow & \mathfrak{X} \end{array}$$

We know the natural map from the reduction of a formal spectral Deligne–Mumford stack \mathfrak{X} back into \mathfrak{X} is an equivalence of underlying ∞ -topoi (by [SAG, Pr.8.1.4.4]), and the underlying ∞ -topoi of the reduction of a formal thickening is also an equivalence (by [SAG, Pr.18.2.2.6]). Hence the horizontal and the left vertical maps are equivalences of underlying ∞ -topoi, hence the right vertical map is as well. \square

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