Solid Abelian Groups

Idea: Want some kind of "completeness" condition on Cond Ab gps.

why? Peter explained that

$$\mathrm{Hom}(A,B),$$
$$\underline{\mathrm{RHom}}(A,B)$$

in CondAb are "good"

$\exists$ tensor product $-\otimes-$ on Cond(Ab) s.t.

$$\mathbb{Z}[S]\otimes\mathbb{Z}[T]=\mathbb{Z}[S\times T]$$

s.t. $\mathbb{Z}_p\otimes\mathbb{Z}_p$, $\mathbb{Z}_p\otimes\mathbb{Z}_\ell$

we only a wild

cond. ab sps.

In fact, $(\mathbb{Z}_p \otimes \mathbb{Z}_p)(*) =$

$$\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}_p$$

(algebraic tensor product)

We'd rather have

$$\mathbb{Z}_p \hat{\otimes} \mathbb{Z}_p = \mathbb{Z}_p$$

$$\mathbb{Z}_p \hat{\otimes} \mathbb{Z}_\ell = 0$$

$\mathrm{Solid}_{\mathbb{Z}} \subseteq \mathrm{CondAb}$

closed under lim & colim,

$\exists$ left adjoint $M \mapsto M^{\blacksquare}$

Completed $\hat{\otimes}$ will be

$$M \otimes^{\blacksquare} N := (M \otimes N)^{\blacksquare}.$$

In fact, $\mathrm{Solid}_{\mathbb{Z}}$ will be

an abelian category

[illegible] category generated by compact projectives.

$$\mathbb{Z}[\beta S]^{\circledast} \cong \prod_I \mathbb{Z}$$

Abstract framework:

Let $\mathcal{A}$ be an abelian category generated by compact projectives, and suppose given a functor

$$\mathcal{A}^{cp} \xrightarrow{L} \mathcal{A}$$

w/ nat. tr. $\mathrm{id} \to L$ s.t.:

(*) If $M \in \mathcal{A}$ is a cokernel of a map $\bigoplus L(M_i) \to \bigoplus L(N_i)$

then

$$\mathrm{RHom}(C, M) \xleftarrow{\sim} \mathrm{RHom}(L(C), M)$$

$$\forall\ C \in \mathcal{A}^{cp}.$$

Then: 1) The following full subcategories of $\mathcal{A}$ agree:

$X := \{ M \in \mathcal{A} \text{ a cokernel as above} \}$

$$Y := \left\{ M \in \mathcal{A} \text{ s.t. } \begin{tikzcd} C \arrow[r, "\forall f"] \arrow[d] & M \\ L(C) \arrow[ur, dashed, "\exists!"'] & \end{tikzcd} \right\}$$

& form abelian subcategories closed under all lims colims, extensions.

gen by $L($ cpct proj$)$, $L(M)$, $M \in \mathcal{A}^{cp}$

2) L extends uniquely to a colim-preserving functor

$$L : \mathcal{A} \to \mathcal{A} \ ;$$

ess. im of L =
the full subcat of $\mathcal{H}$,
and L is the left
adj to the inclusion.

3) ~~Obj~~ $L: D(A) \to D(A)$
colim-preserving extension
(derived functor)
has essential image
the full subcat of
$D(A)$ characterized
by either the
derived analogs of
c), or
that all its
$\pi_i(\ )$ lie in the
full subcat of 1)
$= D($ab cat in 1)$)$

Pf: Suppose Y. Need X.

$$\begin{tikzcd}
 & \bigoplus_{i\in I} C_i \arrow[rr, two heads] & & M \\
 & & \bigoplus_{i\in I} LC_i \arrow[ul, hook] \arrow[ur] & \leftarrow \text{satisfies } Y \\
\bigoplus_{j\in J} LD_j \arrow[r, two heads] & M' \arrow[ur, hook] & \leftarrow \text{satisfies } Y &
\end{tikzcd}$$

Refinement: If A has a $-\otimes-$ commuting w/ colims in each variable, and if have the condition $(*)$ not just for RHom but for $\underline{\mathrm{RHom}}$, then we get a unique tensor product on

$L(A) \subseteq \mathcal{A}$

making $\mathcal{A} \overset{\subset}{\dashv} L(A)$ symmetric monoidal, $L(X \otimes Y)$.

Rk: (*) $\Leftarrow$ same statement, but with kernel instead of cokernel

$\Rightarrow$ same statement, but with complex w/ terms $\bigoplus_{i \in I} L(M_i)$

So we need a functor from cpct proj cond ab gps to CondAb.

$\mathbb{Z}[M$ $\mathbb{Z}[S]$, S extr. disc.

Def: If S extr disc, profinite set, set

$$\mathbb{Z}[S]^{\blacksquare} = \varprojlim_{S_i} \mathbb{Z}[S_i]$$

$S \twoheadrightarrow S_i$ finite set

Note: $\mathbb{Z}[S] \to \mathbb{Z}[S]^{\blacksquare}$

So what we need is $(*)$,
need to control $\underline{R\mathrm{Hom}}(\mathbb{Z}[S]^{\blacksquare}, -)$

Thm:

Note: $$\mathbb{Z}[S]^{\blacksquare} = R\underline{\mathrm{Hom}}_{\mathrm{Cond(Ab)}}\big(\overset{\varinjlim_i C(S_i;\mathbb{Z})}{\overset{\|}{C(S;\mathbb{Z})}}, \mathbb{Z}\big)$$

Thm: (Specker) $C(S,\mathbb{Z}) \simeq \bigoplus_I \mathbb{Z}$:
a free ab gp
(f prof set S)

(for proj. self S)

$$\Rightarrow \quad \mathbb{Z}[S]^{\blacksquare} \simeq \prod_I \mathbb{Z}$$

Most basic case of (✱):

$$\Big(\bigoplus_J \prod_I \mathbb{R}\Big)(S) \simeq R\mathrm{Hom}\Big(\mathbb{Z}[S]^{\blacksquare}, \bigoplus_J \prod_I \mathbb{R}\Big)$$

$$\begin{matrix} \| & & \| \\ \bigoplus \prod_I \mathrm{cont}(S, \mathbb{R}) & & \boxed{R\mathrm{Hom}\Big(\boxed{\prod_I \mathbb{Z}}, \bigoplus_J \prod_I \mathbb{R}\Big)} \end{matrix}$$

Rete: $R\mathrm{Hom}\big(\prod_I \mathbb{Z}, \mathbb{R}\big) = \bigoplus_I \mathbb{R}$

$$\Rightarrow R\mathrm{Hom}\big(\prod_I \mathbb{Z}, \prod_J \mathbb{R}\big) = \prod_J \bigoplus_I \mathbb{R}$$

$$\big(\prod_I - \text{ is exact}\big)$$

$$\prod \mathbb{Z} \longrightarrow \boxed{\prod \mathbb{R}} \longrightarrow \boxed{\prod \mathbb{R}/\mathbb{Z}}$$

Recall: Breen-Deligne resolution ⇒ any compact ab gp A is resolved by copies of $\mathbb{Z}[A]$.

$\mathbb{Z}(A)$ can be resolved by $\mathbb{Z}[S]$, S extr. disc, by choosing hypercover.

$\Rightarrow$ A is resolved by cpct projectives.

"pseudo-coherent"

~~almost comp~~

$\Rightarrow$ $\underline{\mathrm{Ext}}^i(A, -)$ commute w/ filtered colimits. (infinite $\oplus$)

So only need to control

$$R\underline{\mathrm{Hom}}\Big(\prod \mathbb{R}, \bigoplus \prod \mathbb{Z}\Big).$$

But again, $R\underline{\mathrm{Hom}}(\mathbb{R}, \bigoplus \prod \mathbb{Z}) = 0$

($\mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z}$

$\Rightarrow \mathbb{R}$ pseudo-coherent)

Conclusion: $\mathrm{RHom}(\prod \mathbb{Z}, \bigoplus \prod \mathbb{Z})$

$$= \bigoplus \mathrm{RHom}(\prod \mathbb{Z}, \prod \mathbb{Z})$$

$$= \bigoplus \prod \bigoplus \mathbb{Z}$$

The general case is more complicated

How to calculate $- \overset{L}{\otimes} -$?

Resolve by $\prod_I \mathbb{Z}$'s, then use

$$\prod_I \mathbb{Z} \overset{L}{\otimes} \prod_J \mathbb{Z}$$

$$\parallel$$

$$\prod_{I \times J} \mathbb{Z}$$

(infinite distributivity)

Ex: $\mathbb{Z}_p \overset{\wedge L}{\otimes} \mathbb{Z}_p = \mathbb{Z}_p$

$$0 \to \mathbb{Z}[[T]] \xrightarrow{T-p} \mathbb{Z}[[T]] \to \mathbb{Z}_p \to 0$$

More generally, if A, B: profinite ab gps,

$A \overset{\wedge}{\otimes} B$ commutes w/ filtered colimits,

$$\mathbb{Z}_p \overset{\wedge}{\otimes} \mathbb{Z}_\ell = 0 \qquad \ell \neq p$$

$$\prod_I \mathbb{Z} \otimes \mathbb{Z}_p = \prod \mathbb{Z}_p$$

$$\prod_I \mathbb{Z} \otimes \mathbb{Q}_p = \left(\prod \mathbb{Z}_p\right)\left[\tfrac{1}{p}\right]$$

$$\cap$$

$$\prod \mathbb{Q}_p$$

Note If V, W are

Frechet spaces / $\mathbb{Q}_p$,

$\cap$

$\mathrm{Solid}_{\mathbb{Q}}$ closed under lim & colim.

$$V \overset{\blacksquare}{\otimes} W = W \hat{\otimes}_{\pi} V$$

$\Rightarrow$ In practice, $-\overset{\blacksquare}{\otimes}-$ does work like a completed tensor product.

Rk: $\mathbb{R}^{\blacksquare} = 0$

~~If solidizatn~~

Moral: Solidization is a non archimedean kind of completion.

$\approx$ $\boxed{\text{completion}}$ w.r.t. linear topology

Why is solid like complete?

$$\begin{tikzcd}
S \arrow[d, mapsto] \arrow[dd, bend right] \arrow[drr, "f"] & & \\
\mathbb{R}(S) \arrow[d] \arrow[rr] & & M \\
\mathbb{R}[S]^{\blacksquare} \arrow[urr, "\boxed{\exists!}"'] \arrow[d, equal] & & \\
\underline{\mathrm{Hom}}(C(S,\mathbb{R}),\mathbb{R}) \arrow[d, equal] & & \\
\text{“measures on } S\text{”} & &
\end{tikzcd}$$

Why is it <u>not</u> like completeness?

1) It's only complete "on the compacts"

(↔) quasi-completeness

2) No connection to Hausdorff!

$\bigoplus \mathbb{Z}$ $\prod \mathbb{Z}$

$$0 \to \bigoplus_I \mathbb{Z} \to \prod_I \mathbb{Z} \to Q \to 0$$

$\uparrow$ non-Hausdorff but solid

$$M^{\wedge}_{p} \overset{\blacksquare}{\otimes} N^{\wedge}_{p} = (M \overset{\blacksquare}{\otimes} N)^{\wedge}_{p}$$

Rk: By construction,

$$\mathbb{Z}[S]^{\blacksquare} = \varprojlim_{S_i} \mathbb{Z}[S_i]$$

for S extr. disc, profinite.

what about more general S?

Rk: M pseudo-coherent solid ab. gp.

$$\Rightarrow M \simeq \underline{R\mathrm{Hom}}(\underline{R\mathrm{Hom}}(M, \mathbb{Z}), \mathbb{Z})$$

complex of discrete ab gps

$M = \mathbb{Z}[S]^{\bullet,L}$, S cpct Haus space

$$\Rightarrow \mathbb{Z}[S]^{\bullet,L} = \underline{R\mathrm{Hom}}\left(R\mathrm{Hom}(\mathbb{Z}[S],\mathbb{Z}),\mathbb{Z}\right)$$

$$= \underline{R\mathrm{Hom}}\left(R\Gamma(S,\mathbb{Z}),\mathbb{Z}\right)$$

<u>Ex:</u> S profinite $= \varprojlim S_i$

$$\Rightarrow R\Gamma(S,\mathbb{Z}) = \varinjlim_{i\in I} \bigoplus_{S_i}\mathbb{Z}$$

$$\Rightarrow \mathbb{Z}[S]^{\bullet,L} = \varprojlim_{S\to S_i} \mathbb{Z}[S_i]$$

<u>Ex:</u> S finite CW cx $\Rightarrow$

$\mathbb{Z}[S]^{\bullet,L} =$ complex computing

$H_i(S;\mathbb{Q})$

$\mathbb{Z}[S]^{\blacksquare,L}$: Homology of S.

Solidification commutes w/ colims
$\Rightarrow$ Same statement true
for arbitrary CW cx

Synthetic approach to Solidity:

$A^{cp} \subseteq A$ cpct proj generators

$\Rightarrow$

$$\begin{tikzcd}
\bigoplus D_j \arrow[r] & \bigoplus C_i \arrow[r] \arrow[dr, bend left] & M \arrow[d] \\
\bigoplus D_j' \arrow[r] & \bigoplus C_i' \arrow[r] & N \\
 & & C \arrow[u] \arrow[ul, bend left] \arrow[ull, bend left]
\end{tikzcd}$$

completely formally determined by maps between cpct proj.

If A^{cp} is small,

$$\Rightarrow A = \mathrm{Fun}^{add}((A^{cp})^{op}, Ab)$$

$$\overset{\in}{M} \mapsto (C \mapsto \mathrm{Hom}(C, M))$$

$\mathrm{Solid}_{\mathbb{Z}}$ (κ-condensed)

Each map $\prod_I \mathbb{Z} \to \prod_J \mathbb{Z}$ is dual of $\bigoplus_J \mathbb{Z} \to \bigoplus_I \mathbb{Z}$

$$\mathrm{Solid}_{\mathbb{Z}}^{\kappa} = \mathrm{Fun}^{add}(\text{Free Ab gps}_{<\kappa}, Ab)$$

Could forget that it sits in CondAb & just work w/ this.

G solid ab gp

Whitehead's problem:

Is the following true:

$A \in \mathrm{Ab}$, $\mathrm{Ext}^1(A, \mathbb{Z}) = 0$

$\Downarrow ?$

$A \cong \bigoplus_I \mathbb{Z}$?

Answer (Shelah): It depends!
(on axioms beyond ZFC)

Thm: $A \in \mathrm{Ab}$, then

(in ZFC)

$$\underline{\mathrm{Ext}}^1_{\mathrm{CondAb}}(A, \mathbb{Z}) = 0$$

$$\Rightarrow A \cong \bigoplus_I \mathbb{Z}.$$

Note: $\underline{\mathrm{Ext}}^1_{\mathrm{CondAb}}(A, \mathbb{Z}) \in$ Solid

w/ underlying ab gp

$$= \mathrm{Ext}^i(A, \mathbb{Q}).$$

Pf: $0 \to \bigoplus_J \mathbb{Q} \to \bigoplus_I \mathbb{Q} \to A \to 0$

take $\underline{R\mathrm{Hom}}(A, \mathbb{Q})$ $\Rightarrow$

$$\underline{R\mathrm{Hom}}(A, \mathbb{Q}) \to \prod_I \mathbb{Q} \to \prod_J \mathbb{Q}$$

If $\underline{\mathrm{Ext}}^1(A, \mathbb{Q}) = 0$, then this is SES

$$\begin{tikzcd} 0 \arrow[r] & \underline{\mathrm{Hom}}(A, \mathbb{Q}) \arrow[r] & \prod_I \mathbb{Q} \arrow[r] & \prod_J \mathbb{Q} \arrow[r] \arrow[l, dashed, bend left] & 0 \end{tikzcd}$$

$\exists$ splitting

$\Rightarrow$ $\underline{R\mathrm{Hom}}(A, \mathbb{Q})$ is a summand of $\prod_I \mathbb{Q}$

Dualize back again

$\Rightarrow$ A summand of $\bigoplus_I \mathbb{Q}$.

There are some mysteries here.

1) $\boxed{\prod_I \mathbb{Z}}$: cpct proj generator.

what $\prod$ do

$$\mathrm{colim}\left(\prod_I \mathbb{Z} \to \prod_I \mathbb{Z}\right)$$

look like?

2) $\boxed{\prod_I \mathbb{Z}}$ are flat?

i.e. $M \in \mathrm{Solid}_{\mathbb{Z}}$,

$$\prod_I \mathbb{Z} \otimes^{L} M = \prod_I \mathbb{Z} \otimes M ?$$

$\mathrm{Solid}_{\mathbb{Z}_p} :=$ $\mathbb{Z}_p$-modules in $\mathrm{Solid}_{\mathbb{Z}}$

$$\cancel{\otimes^{\mathbb{Z}_p}} \quad \cap| \quad \mathrm{Solid}_{\mathbb{Z}}$$

$$\mathbb{Z}_p \otimes^{L} \mathbb{Z}_p = \mathbb{Z}_p$$

---

solid$_{\mathbb{Z}_p}$ is an ab cat w/ enough cpct projectives

$$\Big(\prod_I \mathbb{Z}\Big) \overset{L}{\otimes}{}^{\blacksquare} \mathbb{Z}_p \cong \boxed{\prod_I \mathbb{Z}_p}$$

Here 1) & 2) can be answered.

1) cpct objects $=$ cokernels$(\prod \mathbb{Z}_p \to \prod \mathbb{Z}_p)$ are exactly the pro p abelian gps

2) $\prod \mathbb{Z}_p$ is flat for $\otimes^{\blacksquare}$.

1) $\Rightarrow$ Solid$_{\mathbb{Z}_p}$ $=$ Ind(prop-ab gps) $=$ Ind(Pro(finite p-ab gps))

How to prove 2)?

$\prod \mathbb{Z}_p \otimes M$ lives in deg 0!

By 1, can assume M is pro-p.

But then $M \overset{\blacksquare L}{\otimes} \prod \mathbb{Z}_p$
$= \prod M$

More general version of 2):
If M is qsep &
p torsion free, then
M is flat.

e.g., M qsep $\mathbb{Q}_p$-module

Note! there is no
p-completion
condition on
a solid $\mathbb{Z}_p$-module.

$M \not\simeq R\lim M/p^n$.

However, can consider

$$\mathcal{D}_{\geq 0}(\mathrm{Solid}_{\mathbb{Z}})^{\overset{\text{derived}}{p\text{-cplete}}} \simeq \mathcal{D}_{\geq 0}(\mathrm{Solid}_{\mathbb{Z}_p})$$

this is closed under all limits, not colimits, but (surprisingly) closed under $-\overset{\blacksquare L}{\otimes}-$!

$$M_{\hat{p}} \overset{\blacksquare L}{\otimes} N_{\hat{p}} \simeq \lim M/p^n \overset{\blacksquare}{\underset{\mathbb{Z}/p^n}{\otimes}} N/p^n .$$

$\Rightarrow$ can reduce to understanding

$$M/p \overset{\blacksquare}{\underset{\mathbb{F}_p}{\otimes}} N/p$$

Eg.: $\boxed{\left(\bigoplus_I \mathbb{Z}\right)_{\hat{p}}} \overset{\blacksquare}{\otimes} \left(\bigoplus_J \mathbb{Z}\right)_{\hat{p}}$

$$= \left(\bigoplus_{I\times J} \mathbb{Z}\right)_{\hat{p}}$$

Pf of eg: $S \longrightarrow \left(\bigoplus \mathbb{Z}\right)_{\hat{}}$

$I \quad p$

$$\mapsto \left(S \longrightarrow \underset{\uparrow}{\bigoplus_I} \mathbb{Z}/p^n \right)_n$$

finite image.

p-adic valuations of image have to $\to \infty$ with the index set.

$$\Rightarrow \left(\bigoplus_I \mathbb{Z}\right)^\wedge_p = \bigcup_{\substack{f: I \to \mathbb{N} \cup \{\infty\} \\ \to \infty \text{ as } i \to \infty}} \left(\left(\bigoplus_I \mathbb{Z}\right)^\wedge_p\right)^{\geq f}$$

$$= \bigcup_{\substack{f: I \to \mathbb{N}_{\geq 0} \cup \{\infty\} \\ \text{decreasing}}} \prod_{i \in I} \mathbb{Z}_p \cdot p^{f(i)}$$

then to take the tensor product of this with itself $\Rightarrow$

$$\bigcup_{\substack{f: I \to \mathbb{N} \cup \{\infty\} \\ g: J \to \mathbb{N} \cup \{\infty}} \prod_{\substack{i \in I \\ j \in J}} \mathbb{Z}_p \cdot p^{f(i)+g(j)}$$

cofinal in

$$\bigcup_{h: I\times J \to \mathbb{N}\cup\{\infty\}} \left(\prod_{i,j} \mathbb{Z}_p \cdot p^{h(i,j)}\right) \quad ✓$$

Let's verify that the solid theory exists. Need (a):

If we have a complex

$$(M_\bullet) = \dots \to M_1 \to M_0$$

where each term is $\cong \bigoplus \prod \mathbb{Z}$, then we need

$$R\mathrm{Hom}(\mathbb{Z}[S], M^\bullet) \simeq R\mathrm{Hom}(\prod \mathbb{Z}, M_\bullet)$$

We did this already for $M_\bullet$ in degree 0

$\Rightarrow$ get it also if $M_\bullet$ is a bounded complex.

complex.

Stupid truncations:

$$M_{\geq n} \to M \to M_{<n}$$ ✓

enough to show that $R\mathrm{Hom}^i(\pi^*\mathbb{Z}, M_{\geq n})$ lives in degree at most e.g. $n-2$ $(n-1)$

(can shift $M_{\geq n}$ to $M_{\geq 0}$.

$\Rightarrow$ enough to see that

$$\mathrm{Ext}^i(\pi^*\mathbb{Z}, M) = 0 \text{ for } i \geq 2.$$

Now take the Postnikov truncations

$$M \to \tau_{\leq n} M \to \tau_{\leq n-1} M_{\leq n}$$

complex w/ same

terms as M

except in single degree n,

where it's $\boxed{\text{ker of } d_n}$.

<u>Upshot:</u> need

$$\mathrm{Ext}^i(\prod \mathbb{Z}, \ker d_n) = 0$$

$i \geq 2$.

$\bigwedge \bigwedge U$

$$\begin{tikzcd} \oplus \prod \mathbb{Z} \arrow[d, "d_n"] \\ \oplus \prod \mathbb{Z} \end{tikzcd} \qquad \left(\begin{tikzcd} \prod \mathbb{Z} \arrow[d] \\ \oplus \prod \mathbb{Z} \end{tikzcd} \right) = \cup\cup \left(\begin{tikzcd} \prod \mathbb{Z} \arrow[d] \\ \prod \mathbb{Z} \end{tikzcd} \right)$$

$\Rightarrow$ cpml d_n to

$$\begin{tikzcd} \oplus \prod \mathbb{Z} \arrow[r] \arrow[d] \arrow[dr, bend right] & \oplus \prod \mathbb{R} \arrow[d, "d_n^{\mathbb{R}}"] \\ {} & {} \end{tikzcd}$$

$\oplus \prod \mathbb{Z} \to \oplus \prod \mathbb{R}$

using "same matrix".

Then this extension $d_n^{\mathbb{R}}$ is unique b/c

$R\mathrm{Hom}(\oplus \prod \mathbb{R}/\mathbb{Z}, \oplus \prod \mathbb{R}) = 0$

$\Rightarrow$ $d_n^{\mathbb{R}}$ organize into a cplx

$$M_\bullet \to M_\bullet^{\mathbb{R}} \to M_\bullet^{\mathbb{R}/\mathbb{Z}}$$

It's enough to show?

$$\mathrm{Ext}^i(\prod \mathbb{Z}, \ker(d_n^{\mathbb{R}}))$$

$$\boxed{\ker(d_n^{\mathbb{R}/\mathbb{Z}})}$$

$$\|$$

$$\ker\left(\begin{array}{c} \oplus \prod \mathbb{R}/\mathbb{Z} \\ \downarrow \\ \oplus \prod \mathbb{R}/\mathbb{Z} \end{array}\right)$$