Condensed Sets:

Idea: $\text{Cond Sets} \overset{\text{phenomena}}{\longleftrightarrow} \text{Top spaces}$,

but categorically,

$\text{Cond Sets} \longleftrightarrow \text{Sets}$.

$\Rightarrow$ easier to add-on algebraic structures

Basic building blocks: CHaus (compact Hausdorff)

We'll define a notion of covering on CHaus, giving a site, & then (modulo set-theory!)

$$\text{Cond Sets} = \text{Sh}(\text{CHaus})$$

Def: ~~a finite~~ a covering collection $\{X_i \to Y\}_{i \in I}$ is a finite collection of CHaus X_i mapping to Y w/ $\coprod X_i \twoheadrightarrow Y$

Strong enough so that you can work locally & that gets you something,

Not too strong: so that invariants you care about still are sheaves

↳ **Ex:** T: top space, $\mathcal{F}(X) = \mathrm{Cont}(X, T)$, $X \in \mathrm{CHaus}$ is a sheaf of sets.

<u>Ex:</u> In a suitable formulation cohomology of CHaus space also ~~the~~ satisfies descent for this topology.

<u>Strong enough?</u>

<u>Gleason:</u> The category CHaus "has enough projectives":

T "projective" $\iff$ ~~Con Act~~

$$\begin{tikzcd} & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd}$$

"enough" $\iff$ $\forall X \in \mathrm{CHaus}$, $\exists$ proj T & a $T \twoheadrightarrow X$

<u>Proof:</u> S : set $\Rightarrow$ βS is a projective object of CHaus. ✓

$$\beta S = \varprojlim_{\substack{S \twoheadrightarrow S_i \\ S_i \text{ finite}}} S_i$$

$$\begin{tikzcd} & X \arrow[d, two heads] \\ \beta S \arrow[r] & Y \end{tikzcd}$$

<u>Enough:</u> $\beta S \twoheadrightarrow X$ $\qquad$ $S \xrightarrow{\text{dense lt}} X$

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$\Rightarrow$ a sheaf on CHaus is uniquely determined by its restriction to βS's.

$$\begin{tikzcd}
\beta S' \arrow[r, two heads] & \beta S \times_X \beta S \arrow[d, shift right] \arrow[d, shift left] \\
& \beta S \arrow[d, two heads] \\
& X
\end{tikzcd}$$

$\leftarrow$ closed subset of $\beta S \times \beta S$.

! product of βS's is not a βS.

neither is a closed subspace

$$\Rightarrow \quad \mathcal{F}(X) \xrightarrow{\sim} \mathrm{eq}\left(\mathcal{F}(\beta S) \rightrightarrows \mathcal{F}(\beta S')\right).$$

Moreover, on ~~βS~~ βS's, the topology is simpler: (6)

sheaf condition $\iff$ $\mathcal{F}(\beta S \sqcup \beta S') \xrightarrow{\sim} \mathcal{F}(\beta S) \times \mathcal{F}(\beta S')$

$\mathcal{F}(\emptyset) = *$

Rk: Projective CHaus $\iff$ retracts of βS's.

"

"extremally disconnected CHaus spaces"

$\cap$ ~~$\neq$~~

totally disconnected $\iff$ "profinite"

0-dim

$$\text{CHaus} \supset \text{profinite sets} \supset \beta S\text{'s}$$

Any one of these can be taken as the defining site for Cond(Sets).

Want to define: $\text{Cond(Sets)} = \text{Sh(profinite sets)}$

But we're not allowed, because profinite Sets is not a small category.

(Hom Sets not sets.)

We need consider a truncated version of CHaus, where we bound the cardinality. (8)

Def: A cardinal κ is a <u>strong limit cardinal</u>

$$\Leftrightarrow \quad \lambda < \kappa \Rightarrow 2^\lambda < \kappa$$

It's easy to make examples:

$$\beth_{\alpha+1} = 2^{\beth_\alpha}$$

$$\beth_\alpha = \bigcup_{\alpha' < \alpha} \beth_{\alpha'} \quad \leftarrow \text{this is a strong limit cardinal.}$$

For all practical purposes, can take $\kappa = \beth_{\omega}$

Def: Category of κ-condensed sets =

$$\mathrm{Sh}\left(\kappa\text{-small}\ \begin{matrix}\beta S\text{'s} \\ \text{prof. sets} \\ \text{CHaus}\end{matrix}\right)$$

Barwick-Haine: κ strongly inaccessible

Condensed: take $\bigcup_{\kappa} \mathrm{Sh}(\kappa\text{-small } \beta S\text{'s})$

$\kappa < \kappa'$, there is a pullback map

$$\kappa\text{-small } \beta S\text{'s} \longrightarrow \kappa'\text{-small } \beta S\text{'s}$$

given by the fully faithful inclusion

$$\Rightarrow \quad \kappa\text{-Cond Sets} \longrightarrow \kappa'\text{-Cond sets}$$

also fully faithful.

$$X \longmapsto \Big(T \longmapsto \varinjlim_{\substack{T \to T' \\ \kappa\text{-small}}} X(T') \Big)^{sh}$$

What do we need to check?

Rk: There is an alternative def'n of Cond(Sets).

$$\mathrm{Cond}(\mathrm{Sets}) \subseteq \mathrm{PSh}(\beta S's)$$

is the full subcategory spanned by the sifted colimits of $\boxed{\beta S's}$.

sifted ↓ combination of filtered colims & reflexive equalizers

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Cond(Sets) satisfies Grand's axioms except existence of a small generating category.

So:

1) Finite lim's & arb. colims:
as in Sets (Topos)

2) Arb limits & sifted colims:
as in Sets. (Alg Thy)

(comes from compact proj generators:
T : extr. disconnected $\Rightarrow$
$\mathrm{Hom}(T, -) : \mathrm{Cond(Sets)} \to \mathrm{Sets}$

Pass to <u>Condensed Ab gps.</u>

$\quad\quad\quad\quad ||$

- Ab gp object in CondSets
- same, but with sheaves of ab gps.

$\Rightarrow$ Cond(Ab) is an abelian category
s.t. arbitrary lim's & colim's
interact exactly as in ab gps.

$\Rightarrow \bigoplus_{i \in I}$ exact, $\prod_{i \in I}$ exact

Cond(Ab) is an example of an ab. cat with enough compact projectives.

M cpct proj $\Leftrightarrow$ $\mathrm{Hom}(M, -) : \mathcal{A} \to Ab$ comm w/ all lims & colims

"enough": $\forall M \in \mathcal{A}$, $\exists$ cpct proj $P_i \in \mathcal{A}$,

$$\bigoplus_{i \in I} P_i \twoheadrightarrow M$$

$$\boxed{P = \mathbb{Z}[\beta S]}$$

<u>Rk</u>: $\overset{\text{Groth.}}{\text{Ab}}$ cat w/ enough cpct proj's
gen by a <u>single</u> cpct proj

$$= \mathrm{Mod}_R \ , \quad R \text{ rng.}$$

How to control limits in CondSets? Suppose we have a diagram. Put it in some common κ'-CondSet.

$$X^{\kappa' \text{ and } \kappa}(T) = \varinjlim_{\substack{T \to T_\alpha \\ \kappa'\text{-small}}} X(T_\alpha)$$

If we can guarantee that the colim is |diagram|-filtered, then this functor commutes with the limit over the diagram.

<u>Lemma:</u> Let $\mathrm{cof}(\kappa)$ be the cofinality of κ:

$\lambda < \mathrm{cof}(\kappa) \iff$ any λ-

indexed colim of κ-small sets is still κ-small.

<u>Claim 1</u>: $\mathrm{cof}(\kappa)$ can be made arbitrarily large

$\kappa = \beth_{\lambda^+}$ to make $\lambda < \mathrm{cof}(\kappa)$

<u>Claim 2</u>: Category of

$$T \to T_\alpha,$$

T_α κ-small is

$\mathrm{cof}(\kappa)$-filtered.

<u>Pf</u>: Enough to show a $\mathrm{cof}(\kappa)$-small limit of κ-small cardinal's is κ-small.

Enough to bound $\prod_{i \in I} X_i$

$$\prod_{i \in I} \kappa_i \leq \prod_{i \in I} 2^{\kappa_i} = 2^{\Sigma \kappa_i}$$

$$< 2^{\lambda}, \lambda < \kappa$$

if κ_i are κ-small. $< \kappa$

<u>Relation to top spaces.</u>

First, $\mathrm{Cont}(-, X)$ is a sheaf $\Rightarrow$

$$\boxed{\mathrm{CHaus}} \subseteq \mathrm{CondSets}.$$

Claim: CHaus is exactly the full subcategory of qcqs condensed sets.

Pf: Each object of CHaus is qc because of the finitary nature of the topology.

~~CHaus~~

- If X is qc, it admits a surjection from a CHaus space, and conversely.

$$\begin{tikzcd} Z \in \mathrm{CHaus} \arrow[r, phantom, "qc?"] & \\ qc? \arrow[r, two heads] \arrow[d, two heads] & Y \; qc \arrow[d, two heads] \\ Y' \; qc \arrow[r, two heads] & X \in \mathrm{CHaus} \end{tikzcd}$$

$Z(f)\ Y \times_X Y$ qcN

$$\begin{tikzcd} Y \times_X Y \arrow[d, shift left] \arrow[d, shift right] \\ Y \arrow[d, "f"] \\ X \end{tikzcd}$$

CHaus

qcqs

$\Rightarrow$ X is the quotient of Y by a closed eq'l rel'n $\Rightarrow$ CHaus.

X

Quasi-separated condensed sets:

"Hausdorff"

Claim: A condensed set is quasi-separated $\Longleftrightarrow$ it is a filtered colim of CHaus spaces along closed

injective maps ($\leftrightarrow$ inclusions)
if CHaus

Pf: $\Leftarrow$: qsep obj closed under filtered colims w/ inj transition maps

$\Rightarrow$: X qseparated.

Want: $\varinjlim_{\substack{K \hookrightarrow X \\ \rotatebox{90}{$\in$} \\ \text{CHaus}}} K \xrightarrow{\sim} X$

filtered

Key:
$$\begin{tikzcd} K \arrow[rr, "f"] \arrow[dr] & & X \\ & \operatorname{im}(f) \arrow[ur, hook] & \end{tikzcd}$$

$K \in$ CHaus $\Rightarrow$ $\operatorname{im}(f) \in$ CHaus

$$\operatorname{im}(f) = \operatorname{colim}\left(K \times_X K \rightrightarrows K \right)$$

eqs ⟹ CHaus,

⟹ thus the quotient of K by a closed eq rel'n ⟹ ∈ CHaus.

$$X \simeq \varinjlim_{\substack{K \to X \\ \text{CHaus}}} K \qquad \exists$$

$$\cong \varinjlim_{\substack{K \hookrightarrow X \\ \text{CHaus}}} K$$

Addendum: $\mathrm{Hom}\left(\varinjlim_{i \in I} K_i, \varinjlim_{j \in J} K_j\right)$

(inj) transition maps)

$$= \varprojlim_{i \in I} \varinjlim_{j \in J} \mathrm{Hom}(K_i, K_j)$$

$\Rightarrow$ QSep Cond Sets

$\subseteq \mathrm{Ind}(\mathrm{CHaus})$

spanned by those with injective transition maps.

= same as Waelbroeck's "compactological spaces"

<u>Cor:</u> CGWH top spaces $\qquad X$

$\qquad\qquad \cap|$ $\qquad\qquad\qquad \downarrow$

q-sep Cond Sets $\qquad$ TH Cond(T)

Pf: CGWH $\Leftrightarrow$ X is the colimit of its CHaus subspaces, (in Top!)

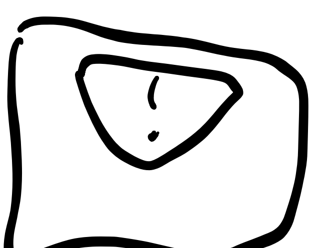
& $T \xrightarrow{} X$ (T CHaus) has closed image.

$\Uparrow$

- sequential
- locally compact

$$\mathrm{Cont}(-, X) = \bigcup_{\substack{K \subset X \\ \text{CHaus}}} \mathrm{Cont}(-, K)$$

$\Rightarrow$ assoc. cond set $= \bigcup$ all CHaus subspaces



It's not enough to take any CHaus subspace filtered unionly to X.

Ex: $[0,1]$ = filtered colim (in Top) of CHaus spaces $\cong$ $\left(\coprod_{\text{finite}} \mathbb{N} \cup \{\infty\}. \right)$

b/c top is sequential.

but this colim is not preserved by

$\mathrm{CGHW} \hookrightarrow \mathrm{Cond}(\mathrm{Set})$

OTOH the functor does preserve countable seq unions along closed subspaces & infinite disjoint unions

& pushouts of CHaus spaces where 1 map is an inclusion.

$\exists$ left adjoint to the inclusion

$$\mathrm{qSep} \subseteq \mathrm{CondSets},$$

"qSep-ification"

$$X \mapsto X^{\mathrm{qsep}}.$$

How to build it?

$$\coprod K_i \qquad K_i \in \mathrm{CHaus}$$
$$\downarrow$$
$$X$$

$$X = \coprod K_i / \sim,$$

$$\sim \subset \coprod_{i,j} K_i \times K_j$$

We can just enlarge $\sim$ by taking the smallest <u>closed</u> eq rel'n generated by it.

<u>Fact:</u> $X \mapsto X^{\mathrm{qtop}}$

preserves finite products!

$\Rightarrow$ induces analogous functors on CondAb, CondRings, ...

The above was about Hausdorff phenomena, but it's very important that we have lots of <u>non</u>-Hausdorff phenomena too.

Eg: CondAb.

$$\underset{\text{discrete top}}{\mathbb{Q}} \longrightarrow \underset{\text{usual top}}{\mathbb{R}}$$

in Top ab grps, this map is both a monomorphism & an epimorphism. but (obviously) not an iso.

When we pass to Cond(Ab) something different must happen: CondAb is an ab. cat.

$$\mathbb{Q} \hookrightarrow \mathbb{R}$$

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$\Downarrow$

$\mathbb{R}/\mathbb{Q}$

highly non-Hausdorff
cl-desed Ab gp.

<u>Even worse:</u>

$$\mathbb{R}^\delta \hookrightarrow \mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{R}^\delta$$

So non-Hausdorff that
$$(\mathbb{R}/\mathbb{R}^\delta)(*) = 0$$

$X(*)$: underlying set
of a cond set
X

But e.g.

$$\boxed{(\mathbb{R}/\mathbb{R}^\delta)(S^1) \neq 0}$$

= "closed 1-forms"

$$(\mathbb{R}/\mathbb{R}^\delta)(T) = \mathbb{R}(T)/\mathbb{R}^\delta(T)$$

$\uparrow$

extr. disc.

CHaus space

$\downarrow\downarrow\downarrow$

βT

$\downarrow\downarrow$

βS

$\downarrow$

S^1

$$= \mathrm{Cont}(T, \mathbb{R}) / \mathrm{Loc.Const}(T, \mathbb{R})$$

Rk: X top space

$\Rightarrow$ $\mathrm{Cont}(-, X)$

is a sheaf on CHaus.

When is it a cond set

(i.e. when is it the case

that $\mathrm{Cont}(-, X)$ is

a small colim of

representables?

Ans: $\Longleftrightarrow$ X is T_1.

(all pts are closed).

Proof in Peter's notes Condensed.pdf.