Bousfield Localization Functors and Hopkins' Chromatic Splitting Conjecture

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Introduction

This paper arose from attempting to understand Bousfield localization functors in stable homotopy theory. All spectra will be p-local for a prime p throughhout this paper. Recall that if E is a spectrum, a spectrum X is E-acyclic if $E \wedge X$ is null. A spectrum is E-local if every map from an E-acyclic spectrum to it is null. A map $X \to Y$ is an E-equivalence if it induces an isomorphism on E_*, or equivalently, if the fibre is E-acyclic. In [**Bou79**], Bousfield shows that there is a functor called E-localization, which takes a spectrum X to an E-local spectrum $L_E X$, and a natural transformation $X \to L_E X$ which is an E-isomorphism. Studying $L_E X$ is studying that part of homotopy theory which E sees.

These localization functors have been very important in homotopy theory. Ravenel [**Rav84**] showed, among other things, that finite spectra are local with respect to the wedge of all the Morava K-theories $\bigvee_{n<\infty} K(n)$. This gave a conceptual proof of the fact that there are no non-trivial maps from the Eilenberg-MacLane spectrum $H\mathbf{F}_p$ to a finite spectrum X.

Hopkins and Ravenel later extended this to the chromatic convergence theorem [**Rav92**]. If we denote, as usual, the localization with respect to the first $n + 1$ Morava K-theories $K(0) \vee \cdots \vee K(n)$ by L_n, the chromatic convergence theorem says that for finite X, the tower $\ldots \pi_i L_n X \to \pi_i L_{n-1} X \ldots \to \pi_i L_0 X$ is pro-isomorphic to the constant tower $\{\pi_i X\}$. In particular, X is the inverse limit of the $L_n X$.

The major result of this paper is that finite torsion spectra are local with respect to any infinite wedge of Morava K-theories $\bigvee_{i<\infty} K(n_i)$. This has several interesting corollaries. For example, it implies that there are no maps from the Johnson-Wilson spectra $BP\langle n \rangle$ to a finite spectrum. It also implies that if E is a ring spectrum which detects all finite spectra, so that $E_*(X) \neq 0$ if X finite, then $L_E X$ is either X or X_p, the p-completion of X, for finite X. This in turn implies that the only smashing localization which detects all finite complexes is the identity functor.

In order to prove that finite torsion spectra are $\bigvee_i K(n_i)$ local, I show that BP_p is a wedge summand of $\prod_i L_{K(n_i)} BP_p$. This is saying that one does not have to reassemble the chromatic pieces of BP_p into an inverse limit to recover the homotopy theory of BP_p. This result is a BP analogue of the chromatic splitting conjecture of Hopkins. I will describe this conjecture in Section 4, but for now suffice it to say that the conjecture is that $L_{n-1} X_p$ is a wedge summand in $L_{n-1} L_{K(n)} X_p$. The chromatic splitting conjecture is actually stronger than that, for it also explains how this splitting occurs, but most of the corollaries I draw from the chromatic splitting conjecture only need the splitting itself. One corollary of the chromatic splitting conjecture would be that, for finite X, X_p is a wedge summand of $\prod_i L_{K(n_i)} X_p$, explaining how my result is a BP analogue of the chromatic splitting conjecture. I do not know if $L_{n-1} BP_p$ is a wedge summand of $L_{n-1} L_{K(n)} BP_p$.

In the first two sections of this paper, I describe some other results about Bousfield localization functors, this time with respect to spectra E which kill a finite spectrum. The pedigree of these results is somewhat confusing. Almost all of the results in Sections 1 and 2 have been known to Hopkins for some time. Others may have known them as well, but they have not appeared in print before. I feel that they warrant a larger audience. In addition, I discovered many of these results independently, and there are a couple of new results as well. For example, I show that $L_{K(n)}$ is a minimal localization functor. That is, if the natural transformation $X \to L_E X$ factors through $L_{K(n)} X$, then $L_E X$ is either the zero functor or is $L_{K(n)} X$ itself. I also provide some new examples of smashing localizations.

The last section of the paper discusses the consequences of the chromatic splitting conjecture on the homotopy groups of $L_n S^0$. We show that, given the chromatic splitting conjecture, the divisible summands in $\pi_* L_n S^0$ for $n \geq 1$ can be determined. There are 3^{n-1} of them, with 2^{n-1} of them occuring in dimension $-2n$, and the others spread out from dimension $-2n-1$ to dimension $-n^2-1$. This therefore explains part of the Shimomura-Yabe calculation of $\pi_* L_2 S^0$ for $p > 3$ in [**Sh-Y**].

This paper is written in the homotopy category of p-local spectra. In particular, '=' is equality in the homotopy category, namely homotopy equivalence. Similarly, I will often write that a map or spectrum is 0, by which I mean that it is null-homotopic or contractible.

1. Spectra with finite acyclics

Before describing the results of this section, I need to recall the definition of the Bousfield class of a spectrum [**Bou79**].

DEFINITION 1.1. *Two spectra E and F are Bousfield equivalent if, given any spectrum X,*

$$E \wedge X = 0 \text{ if and only if } F \wedge X = 0.$$

Denote the equivalence class of E by $\langle E \rangle$. Define $\langle E \rangle \leq \langle F \rangle$ if and only if $F \wedge X = 0$ implies $E \wedge X = 0$. Define

$$\langle E \rangle \wedge \langle F \rangle = \langle E \wedge F \rangle$$

and

$$\langle E \rangle \vee \langle F \rangle = \langle E \vee F \rangle.$$

There is a minimal Bousfield class $\langle * \rangle$, which we will often denote by 0, and a maximal Bousfield class $\langle S^0 \rangle$. I remind the reader that it is perfectly possible to have $\langle E \rangle \leq \langle F \rangle$ while nonetheless $\langle E \wedge F \rangle = 0$.

In this section we investigate Bousfield classes of spectra E which have finite acyclics, i.e. there is some finite X with $E \wedge X = 0$. Highlights of this section include the minimality of the Bousfield class of $K(n)$ (Corollary 1.7) and the new examples of smashing localizations (Proposition 1.5). We also show that every BP-module spectrum with finite acyclics has the Bousfield class of a wedge of Morava K-theories, and that a v_n-periodic Landweber exact spectrum has the same Bousfield class as $E(n)$.

First, we need to recall some corollaries of the nilpotence theorem [**DHS, HS**]. Recall that a finite spectrum X has type n if $K(i)_*(X) = 0$ for $i < n$ but $K(n)_*(X) \neq 0$. Every finite spectrum is of some finite type, and the periodicity theorem of J. Smith, written up in [**Rav92**], says that there is a spectrum of type n for all n. Let $\mathcal{C}_n$ denote the class of all finite spectra of type at least n. Then [**HS**] any nonempty collection of finite spectra that is closed under cofibrations and retracts is some $\mathcal{C}_n$.

LEMMA 1.2 (HOPKINS-SMITH). *All finite spectra of type n have the same Bousfield class, which we denote $F(n)$.*

PROOF. This is an easy application of the nilpotence theorem. Given an X of type n, let $\mathcal{C}$ consist of all finite spectra Y such that $\langle Y \rangle \leq \langle X \rangle$. It is easy to see that $\mathcal{C}$ is closed under retracts, cofibrations, and suspensions, so must be a

$\mathcal{C}_k$ for some k. Since $X \in \mathcal{C}$, $\mathcal{C} \subseteq \mathcal{C}_n$. In particular, if Y is type n, $\langle Y \rangle \leq \langle X \rangle$. Interchanging X and Y completes the proof. $\square$

A spectrum X in $\mathcal{C}_n$ has a v_n self-map, that is, a map inducing an isomorphism on $K(n)_*(X)$ [**HS**]. Any two such become equal upon iterating them enough times, so that there is a well-defined telescope $Tel_n(X)$. Tel_n is actually an exact functor on the category of finite spectra with v_n self maps. This follows from the fact that a map between two such finite spectra will commute with large enough iterates of the v_n self maps. By following a similar line of proof as in the above lemma, we get

LEMMA 1.3. *The telescopes of finite spectra of type n all have the same Bousfield class, which we denote $Tel(n)$.*

This lemma was also known to Hopkins and Smith, and it appears in [**MS**] as well.

Recall the lemma of [**Rav84**]: if f is a self-map of X and $Tel(X)$ is its telescope and Y its cofibre, then $\langle X \rangle = \langle Tel(X) \rangle \vee \langle Y \rangle$. Applying this repeatedly using v_n self maps, we get

(1) $\langle S^0 \rangle = \langle Tel(0) \rangle \vee \cdots \vee \langle Tel(n) \rangle \vee \langle F(n+1) \rangle$.

This decomposition is the key to most of our results in this section. Note that it is orthogonal, in the sense that $Tel(m) \wedge Tel(n) = Tel(m) \wedge F(n) = 0$ if $m < n$. This is proven in [**MS**].

Given any spectrum E, let

$$\mathrm{FA}(E) = \{ X \mid X \text{ is finite and } E \wedge X = 0 \}.$$

In this section, we will discuss spectra which have finite acyclics, so that we assume $\mathrm{FA}(E) \neq \{*\}$. It is easy to see that $\mathrm{FA}(E)$ is closed under cofibrations and retracts, so it must be $\mathcal{C}_{n+1}$ for some n. We then have the following observation.

LEMMA 1.4. *If $FA(E) = \mathcal{C}_{n+1}$, then*

$$\langle E \rangle = \langle E \wedge Tel(0) \rangle \vee \cdots \vee \langle E \wedge Tel(n) \rangle.$$

In particular, $\langle Tel(0) \vee \cdots \vee Tel(n) \rangle$ is the largest Bousfield class with finite acyclics $\mathcal{C}_{n+1}$, and therefore localization with respect to it, denoted L_n^f, is smashing.

PROOF. Just smash equation (1) with E. To see L_n^f is smashing, note that for any spectrum E, $\mathrm{FA}(L_E S^0) = \mathrm{FA}(E)$. Thus,

$$\langle L_n^f S^0 \rangle \leq \langle Tel(0) \vee \cdots \vee Tel(n) \rangle.$$

This implies by Prop. 1.27 of [**Rav84**] that L_n^f is smashing. $\square$

L_n^f has been investigated by many authors [**Bou92, MS, Mil, Rav92a**]. All of them noticed that it is smashing, though I think this is the most transparent proof. The telescope conjecture is usually stated as saying that if X is type n then $L_{K(n)}X = Tel(X)$. This is equivalent to $\langle Tel(n)\rangle = \langle K(n)\rangle$, and also to $L_n^f = L_n$. (For details, see [**MS**] or one of the other cited papers above.) This conjecture is now known to be false for $n = 2$ [**Rav92b, MRS**], and is presumed to be false for larger n as well.

As an amusing example of what the failure of the telescope conjecture means, we include the following proposition.

PROPOSITION 1.5. *Localization with respect to*

$$Tel(0) \vee \cdots \vee Tel(m) \vee K(m+1) \vee \cdots \vee K(n)$$

is smashing.

PROOF. Call this localization functor $L_{m,n}$. We have that

$$\langle Tel(0) \vee \cdots \vee Tel(m)\rangle \leq \langle L_{m,n}S^0\rangle \leq \langle Tel(0) \vee \cdots \vee Tel(n)\rangle.$$

We need to show that $\langle L_{m,n}S^0 \wedge Tel(i)\rangle = \langle K(i)\rangle$ if $m < i \leq n$. Note that

$$\langle L_{m,n}S^0 \wedge Tel(i)\rangle = \langle L_{m,n}S^0 \wedge F(i) \wedge Tel(i)\rangle = \langle L_{m,n}F(i) \wedge Tel(i)\rangle.$$

But $F(i)$ is $Tel(0) \vee \cdots \vee Tel(m)$ acyclic, so $L_{m,n}F(i) = L_nF(i)$. Since L_n is smashing [**Rav92**], $\langle L_nF(i)\rangle = \langle K(i) \vee \cdots \vee K(n)\rangle$, and the result follows. □

It is an old problem of Bousfield's to classify all smashing localization functors. We address another part of this problem in Section 3.

To measure the extent to which the telescope conjecture fails, note that there is a natural map $L_n^f X \to L_n X$. Let $A_n X$ be the fibre of this map. Note that if X is type n, this is also the fiber of the map $Tel(X) \to L_{K(n)}(X)$, for then $L_n^f(X) = Tel(X)$ ([**MS**]), and $L_n X = L_{K(n)}(X)$.

PROPOSITION 1.6. *If X is finite and type n, the Bousfield class of $A_n X$ does not depend on X. We denote it $A(n)$. $\langle Tel(n)\rangle = \langle K(n)\rangle \vee \langle A(n)\rangle$, and $A(n) \wedge K(m) = 0$ for all m.*

PROOF. First note that because L_n^f and L_n are both smashing (see [**Rav92**] for L_n), so is A_n. That is, $A_n X = A_n S^0 \wedge X$ for all X. In particular, if X and Y are Bousfield equivalent, so are $A_n X$ and $A_n Y$. This shows that $\langle A(n)\rangle$ is well-defined. The map $L_n^f X \to L_n X$ is an isomorphism on $K(m)$ homology for all m, (and also on BP homology as we will see below), so $A(n) \wedge K(m) = 0$ for all m. If X is type n, $\langle Tel(X)\rangle = \langle Tel(n)\rangle$ and $\langle L_{K(n)}X\rangle = \langle K(n)\rangle$, and it follows that $\langle Tel(n)\rangle = \langle K(n)\rangle \vee \langle A(n)\rangle$. □

$A(n)$ behaves very much like $M_n X$, the nth monochromatic layer, which is the fiber of $L_n X \to L_{n-1} X$. In particular, we have that $A(n) \wedge A(n)$ is homotopy equivalent to $A(n)$, and $L_{A(n)}A(n)$ is homotopy equivalent to $L_{A(n)}S^0$.

COROLLARY 1.7. $\langle K(n) \rangle$ *is a minimal Bousfield class. That is, if* $\langle E \rangle <$ $\langle K(n) \rangle$, *then* E *is null.*

PROOF. Suppose $\langle E \rangle \leq \langle K(n) \rangle$. Then

$$\langle E \wedge Tel(m) \rangle \leq \langle K(n) \wedge Tel(m) \rangle = 0$$

if $n \neq m$. Similarly, $\langle E \wedge F(n+1) \rangle \leq \langle K(n) \wedge F(n+1) \rangle = 0$. Thus, from equation (1), we have that $\langle E \rangle = \langle E \wedge T(n) \rangle$. But, also

$$\langle E \wedge A(n) \rangle \leq \langle K(n) \wedge A(n) \rangle = 0,$$

so by the preceeding proposition, we have $\langle E \rangle = \langle E \wedge K(n) \rangle$. Since $K(n)$ is a field spectrum, $E \wedge K(n)$ is a wedge of suspensions of $K(n)$, so there are only two possibilities for $\langle E \wedge K(n) \rangle$, 0 or $\langle K(n) \rangle$. □

Note that the corresponding result is not true for the other field spectrum, $H\mathbf{F}_p$. Indeed, in the proof of Theorem 2.2 of [**Rav84**], Ravenel shows that $\langle Y \rangle < \langle H\mathbf{F}_p \rangle$, where Y denotes the Brown-Comanetz dual of $BP \wedge M(p)$.

A similar argument to the above shows that if $\langle E \rangle$ is less than or equal to some finite wedge of Morava K-theories, then E must be Bousfield equivalent to a finite wedge of Morava K-theories. This says in particular that the chromatic tower is unrefineable. There is no localization functor L_E that fits between L_n and L_{n-1}.

In the light of this result and the failure of the telescope conjecture, one might ask if $A(n)$ is also a minimal Bousfield class. This would say that the telescope conjecture is not so badly wrong. I think this is likely to be true, but since I have no data, I will not be so bold as to conjecture it.

The following theorem will show that the telescope conjecture is true after smashing with BP. This has been known to Hopkins, Ravenel, and probably others, though it has not appeared before. First we need a lemma.

LEMMA 1.8. *Suppose* R *is a ring spectrum and the unit map* $S^0 \xrightarrow{\eta} R$ *factors through some spectrum* E. *Then* $\langle E \rangle \geq \langle R \rangle$.

PROOF. Since R is a ring spectrum, the composite

$$R = S^0 \wedge R \xrightarrow{\eta \wedge 1} R \wedge R \to R$$

is the identity. Since η factors through E, the identity map on R factors through $E \wedge R$. So if $E \wedge Z$ is null, so is $R \wedge Z$. □

Recall that $P(n)$ is a BP-module spectrum whose homotopy is

$$\pi_* P(n) = BP_*/(p, v_1, \dots, v_{n-1}).$$

The first part of the following theorem is Ravenel's theorem 2.1(g) in [**Rav84**]. We reprove it so as to make the second part clearer.

THEOREM 1.9.

$$\langle BP \wedge F(n) \rangle = \langle P(n) \rangle$$

and

$$\langle BP \wedge Tel(n) \rangle = \langle K(n) \rangle.$$

PROOF. If there were a spectrum $V(n-1)$ with

$$BP_* V(n-1) = BP_*/(p, v_1, \dots, v_{n-1}),$$

it would be type n and we would have $BP \wedge V(n-1) = P(n)$, so the result would be obvious. In general, there are not such spectra, but there are appropriate substitutes $M(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})$ constructed by Devinatz in [**Dev**]. These exist for sufficiently large $(i_0, i_1, \dots, i_{n-1})$, they are finite of type n, and they have the evident BP-homology. Furthermore, $BP \wedge M(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})$ can be constructed from $P(n)$ using cofibre sequences, in the same way that the mod p^n Moore space can be constructed from the mod p Moore space. Therefore

$$\langle BP \wedge M(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}}) \rangle \le \langle P(n) \rangle.$$

Note that there is a natural map of BP-module spectra

$$BP \wedge M(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}}) \to P(n).$$

The unit map $S^0 \to P(n)$ of the ring spectrum $P(n)$ factors through this map, so by the proceeding lemma,

$$\langle BP \wedge M(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}}) \rangle \ge \langle P(n) \rangle.$$

It can actually be shown using a variant of the Landweber exact functor theorem and Lemma 2.13 of [**Rav84**] that $P(n)$ is a module spectrum over $BP \wedge M(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})$, but we do not need this.

To see that $\langle BP \wedge Tel(n) \rangle = \langle K(n) \rangle$, we proceed similarly. A v_n self map on $M(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})$ induces multiplication by a power of v_n on BP-homology, so

$$BP \wedge Tel(M(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})) = v_n^{-1}(BP \wedge M(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})).$$

The maps that build $BP \wedge M(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})$ from $P(n)$ by cofibre sequences can all be chosen to be BP module maps. Thus they will also build $v_n^{-1}(BP \wedge M(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}}))$ from $v_n^{-1} P(n)$. Thus

$$\langle BP \wedge Tel(n) \rangle \le \langle v_n^{-1} P(n) \rangle = \langle K(n) \rangle.$$

The latter equality comes from Theorem 2.1 of [**Rav84**].

The unit map of $v_n^{-1} P(n)$ factors through $v_n^{-1}(BP \wedge M(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}}))$, so we also have $\langle BP \wedge Tel(n) \rangle \ge \langle K(n) \rangle$. $\square$

COROLLARY 1.10. $BP \wedge A(n) = 0$, *so that the natural map* $L_n^f X \to L_n X$ *is a BP equivalence.*

Proof.

$$\langle BP \wedge A(n)\rangle = \langle BP \wedge Tel(n) \wedge A(n)\rangle = \langle K(n) \wedge A(n)\rangle = 0.$$

□

Corollary 1.11. *Every BP-module spectrum with finite acyclics is Bousfield equivalent to a finite wedge of Morava K-theories.*

Proof. Suppose E is a BP-module spectrum with $FA(E) = \mathcal{C}_{n+1}$. Since E is a BP module spectrum, E is a retract of $BP \wedge E$, so

$$\langle E\rangle = \langle BP \wedge E\rangle = \langle BP \wedge E \wedge Tel(0)\rangle \vee \cdots \vee \langle BP \wedge E \wedge Tel(n)\rangle.$$

But $\langle BP \wedge E \wedge Tel(n)\rangle = \langle K(n) \wedge E\rangle$. Since $K(n)$ is a field spectrum, we have that $\langle K(n) \wedge E\rangle$ is either 0 or $\langle K(n)\rangle$. □

A particularly good kind of BP-module spectrum is a Landweber exact spectrum [**Land**]. Recall that E is Landweber exact if the natural map

$$BP_*(X) \otimes_{BP_*} E_* \to E_*(X)$$

is an isomorphism. The most common examples are $E(n)$ and elliptic cohomology. Call E v_n-periodic if $v_n \in BP_*$ maps to a unit in $E_*/(p, v_1, \ldots, v_{n-1})$.

Corollary 1.12. *If E is a v_n-periodic Landweber exact spectrum, then*

$$\langle E\rangle = \langle E(n)\rangle = \langle K(0) \vee \cdots \vee K(n)\rangle.$$

Proof. Recall that if E is v_n-periodic and Landweber exact then v_j is not a zero-divisor mod $(p, v_1, \ldots, v_{j-1})$ for $j < n$, and v_n is a unit mod $(p, v_1, \ldots, v_{n-1})$ [**Land**]. It suffices to show that $E \wedge K(j) \neq 0$ for $j \leq n$, and that $E \wedge F(n+1) = 0$. Since E is a BP-module spectrum, $\langle E \wedge K(j)\rangle = \langle E \wedge Tel(j)\rangle$, it suffices to show that $E \wedge v_j^{-1} M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{j-1}}) \neq 0$. But the homotopy of $E \wedge v_j^{-1} M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{j-1}})$ is $v_j^{-1} E_*/(p^{i_0}, v_1^{i_1}, \ldots, v_{j-1}^{i_{j-1}})$, which is not 0 by Landweber exactness.

Similarly, the homotopy of $E \wedge M(p^{i_0}, v_1^{i_1}, \ldots, v_n^{i_n})$ is $E_*/(p^{i_0}, v_1^{i_1}, \ldots, v_n^{i_n})$. We know that v_n is a unit mod $(p, v_1, \ldots, v_{n-1})$, and it follows that v_n is also a unit mod $(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$, so the homotopy is 0. □

2. Localizations with respect to finite spectra

In this section we consider what localization with respect to a finite spectrum looks like. We also determine the $K(n)$-localization of BP. All of the results in this section are known to Hopkins and possibly others. Special cases of some of these results have appeared in [**MS**].

We have already used the $M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$ in the previous section. We need them again here, and we need to know that they exist for sufficiently large $(i_0, \ldots, i_n)$. Furthermore, there are natural maps

$$M(p^{j_0}, v_1^{j_1}, \ldots, v_{n-1}^{j_{n-1}}) \to M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$$

for j_k sufficiently large compared to i_k, which induce the evident map on BP-homology. Notice that these maps fix the bottom cell.

The following result says that localization with respect to $F(n)$ is completion at $p, v_1, \ldots, v_{n-1}$.

THEOREM 2.1. *For arbitrary X, the map $X \to \varprojlim(X \wedge M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}}))$ induced by inclusion of the bottom cell is $F(n)$-localization.*

PROOF. First we verify that the right-hand side is $F(n)$-local. Suppose Z is $F(n)$-acyclic. Then

$$[Z, X \wedge M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})] = [Z \wedge DM(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}}), X],$$

where DY denotes the Spanier-Whitehead dual of Y. Since

$$K(i)_*(DY) = K(i)^*(Y) = \mathrm{Hom}_{K(i)_*}(K(i)_*(Y), K(i)_*),$$

$DM(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$ also has type n. Thus

$$Z \wedge DM(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}}) = 0,$$

so $X \wedge M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$ is $F(n)$-local. Then the inverse limit of $X \wedge M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$ is also $F(n)$-local.

Now we must check that the map is an $F(n)$-isomorphism. By Spanier-Whitehead duality, it suffices to show that

$$[F(n), X] \to [F(n), \varprojlim(X \wedge M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}}))]$$

is an isomorphism. We have an exact sequence

$$\varprojlim{}^1[F(n), X \wedge M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})] \hookrightarrow [F(n), \varprojlim(X \wedge M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}}))]$$

$$\to \varprojlim[F(n), X \wedge M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})] \to 0.$$

There is a dimension shift on the $\varprojlim^1$ term, but we will show it is 0 so that will not matter.

So we need to investigate $[F(n), X \wedge M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})]$, or equivalently, $[F(n) \wedge DM(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}}), X]$. Note that $DM(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$ is just a desuspension of $M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$, so that the top cell is in degree 0. (This is easy to see from the construction of the $M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$.) Also note that if X is type n, $X \wedge M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$ is a wedge of copies of X, for large enough indices $(i_0, \ldots, i_{n-1})$. Indeed, at each stage of the construction of $M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$, one takes the cofiber of a v_j self map on

$M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{j-1}})$. Since X is type n, that v_j self map must be nilpotent on $X \wedge M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{j-1}})$, so that if we take large enough indices, it will be null.

Thus $[F(n), X \wedge M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})]$ is a direct sum of copies of $[F(n), X]$ in dimensions corresponding to the cells of $DM(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$. The maps in the inverse system are all multiplication by a v_j to some power, except on the top cell, which is fixed. So they are nilpotent, and for large enough indices will be 0. Hence $\varprojlim[F(n), X \wedge M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})] = [F(n), X]$ as required. Furthermore, the system is Mittag-Leffler, so the $\varprojlim^1$ term vanishes as well. $\quad\square$

COROLLARY 2.2.
$$L_{F(n) \wedge E} X = L_{F(n)} L_E X.$$

PROOF. The map $X \to L_{F(n)} L_E X$ is an $F(n) \wedge E$-isomorphism, so it suffices to show that $L_{F(n)} L_E X$ is $F(n) \wedge E$-local. Since
$$L_{F(n)} L_E X = \varprojlim(L_E X \wedge M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})),$$
it will suffice to show that $L_E X \wedge M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$ is $F(n) \wedge E$-local. Suppose Z is $F(n) \wedge E$-acyclic, and consider
$$[Z, L_E X \wedge M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})] = [Z \wedge DM(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}}), L_E X].$$
$DM(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$ is type n, so $Z \wedge DM(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$ is E-acyclic, since Z is $F(n) \wedge E$-acyclic. Thus this group is 0 as required. $\quad\square$

Note that if X is finite,
$$L_{F(n)} L_E X = \varprojlim(L_E X \wedge M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}}))$$
$$= \varprojlim(X \wedge L_E M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})).$$
In particular, recalling from [**MS**] that
$$L_{\mathrm{Tel}(n)} M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}}) = Tel(M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})),$$
and taking $E = Tel(0) \vee \cdots \vee Tel(n)$, we recover their result that
$$L_{\mathrm{Tel}(n)} S^0 = \varprojlim(Tel(M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}}))).$$

We can use a similar argument to calculate $L_{K(n)} BP$.

LEMMA 2.3.
$$L_{K(n)} BP = L_{F(n)}(v_n^{-1} BP) = \varprojlim(v_n^{-1} BP \wedge M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})).$$

PROOF. Note that $v_n^{-1} BP$ is Landweber exact and v_n periodic, so has Bousfield class $\langle K(0) \vee \cdots \vee K(n) \rangle$. As a ring spectrum, it is self-local, so
$$L_n(v_n^{-1} BP) = v_n^{-1} BP.$$
Thus
$$L_{K(n)}(v_n^{-1} BP) = L_{F(n)} L_n(v_n^{-1} BP) = L_{F(n)}(v_n^{-1} BP).$$

So it suffices to show that $BP \to v_n^{-1}BP$ is a $K(n)$-isomorphism, or equivalently that $BP \overset{\times v_n}{\to} BP$ is a $K(n)$-isomorphism. (We have left out the evident suspension). Since $K(n)$ is a field spectrum and so has a Kunneth isomorphism, it will suffice to show that

$$BP \wedge M(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}}) \overset{\times v_n}{\to} BP \wedge M(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})$$

is a $K(n)$-isomorphism.

Note that $\times v_n$ induces multiplication by $\eta_R v_n$ on BP_*BP or $K(n)_*BP$. Here η_R is the right unit, discussed in [**Rav86**], where it is shown that

$$\eta_R v_n \equiv v_n \mod (p, v_1, \dots, v_{n-1}).$$

Thus, $\times v_n$ is an isomorphism on $K(n)_*P(n)$. But $BP \wedge M(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})$ can be built from $P(n)$ using cofiber sequences where the maps are BP-module maps. Thus $\times v_n$ is also an isomorphism on

$$K(n)_*(BP \wedge M(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})).$$

$\square$

The homotopy of $L_{K(n)}BP$ is then easily calculated to be $(v_n^{-1}BP_*)_{I_n}$, the completion of $v_n^{-1}BP_*$ at the ideal $I_n = (p, v_1, \dots, v_{n-1})$. Note that v_n is not a unit in $L_n BP$, but becomes one upon completion at I_n. In particular, one sees that $L_{K(n)}BP$ is Landweber exact, so we have

COROLLARY 2.4.

$$\langle L_{K(n)}S^0 \rangle = \langle K(0) \vee \cdots \vee K(n) \rangle$$

PROOF. $\langle L_{K(n)}S^0 \rangle \le \langle L_n S^0 \rangle$, sice $L_{K(n)}S^0$ is an $L_n S^0$ module spectrum. Since L_n is smashing, $\langle L_n S^0 \rangle = \langle K(0) \vee \cdots \vee K(n) \rangle$. On the other hand, $\mathbb{L}_{K(n)}BP$ is an $L_{K(n)}S^0$-module spectrum, and since $L_{K(n)}BP$ is Landweber exact,

$$\langle L_{K(n)}BP \rangle = \langle K(0) \vee \cdots \vee K(n) \rangle.$$

$\square$

3. Ring spectra without finite acyclics

In this section we prove our BP-version of the chromatic splitting conjecture and use it to deduce that finite torsion spectra are local with respect to any infinite wedge of Morava K-theories. A corollary of this is that localization with respect to a ring spectrum that has no finite acyclics must be the identity functor or p-completion on finite complexes.

Recall that all spectra are p-local, and X_p denotes the p-completion of X. Throughout this section (n_i) will be an infinite increasing sequence of nonnegative integers.

THEOREM 3.1. *The natural map*

$$BP_p \to \prod_{i=1}^{\infty} L_{K(n_i)} BP_p$$

is the inclusion of a wedge summand.

To prove this theorem, we use Brown-Comanetz duality. Recall that the Brown-Comanetz dual of a spectrum X is the spectrum IX which represents the functor $Y \to \text{Hom}(\pi_0(X \wedge Y), \mathbf{Q}/\mathbf{Z})$. In particular, if X has finitely generated homotopy groups, then $I^2 X = X_p$. Recall as well that a map $Y \to X$ is called f-phantom if, for all finite Z and maps $Z \to Y$, the composite $Z \to Y \to X$ is null. Recall the following lemma, on page 66 of [**Mar**].

LEMMA 3.2. *For any spectrum X, any f-phantom map into IX is null.*

Let F be the fibre of $BP_p \to \prod_{i=1}^{\infty} L_{K(n_i)} BP_p$. Since $BP_p = I(I(BP))$, we will have proved the theorem if we can show that the map $F \to BP_p$ is f-phantom.

First we remove the p-completion.

LEMMA 3.3. *Let F' be the fibre of*

$$BP \to \prod_{i=1}^{\infty} L_{K(n_i)} BP.$$

If $F' \to BP$ is f-phantom, then $F \to BP_p$ is null.

PROOF OF LEMMA. Let C be the fiber of $BP \to BP_p$. Then C is a rational space, so $L_{K(n)} C = *$, and $L_{K(n)} BP = L_{K(n)}(BP_p)$ for $n > 0$. Consider the following diagram.

$$
\begin{array}{ccccc}
C_1 & \longrightarrow & C & \longrightarrow & C_2 \\
\downarrow & & \downarrow & & \downarrow \\
F' & \longrightarrow & BP & \longrightarrow & \prod L_{K(n_i)} BP \\
\downarrow & & \downarrow & & \downarrow \\
F & \longrightarrow & BP_p & \longrightarrow & \prod L_{K(n_i)} BP_p
\end{array}
$$

We consider two cases. If $L_{K(0)}$ appears in the product, then

$$C_2 = \text{fiber}(L_{K(0)} BP \to L_{K(0)}(BP_p)) = C,$$

so that $F = F'$. Then if $F' \to BP$ is f-phantom, so is $F \to BP_p$, and so it is null.

On the other hand, if $L_{K(0)}$ does not appear in the product, then $C_2 = 0$, and we have a cofiber sequence

$$F' \to F \to \Sigma C.$$

If $F' \to BP$ is f-phantom, then $F' \to BP \to BP_p$ is null, so $F \to BP_p$ factors through ΣC. But C is $M(p)$-acyclic and BP_p is $M(p)$-local, so the map must be null. $\square$

So to complete the proof of the theorem, it will suffice to prove:

LEMMA 3.4. *If X is finite, the map $BP^*(X) \to (L_{K(n)}BP)^*(X)$ is injective for large n.*

PROOF. By using Spanier-Whitehead duality, it suffices to prove the lemma in homology rather than cohomology. Recall from the preceeding sections that $L_{K(n)}BP_* = (v_n^{-1}BP_*)_{I_n}$, where $I_n = (p, v_1, \ldots v_{n-1})$ as usual. Note that $L_{K(n)}BP$ clearly satisfies the hypotheses of the Landweber exact functor theorem, so that

$$(L_{K(n)}BP)_*(X) = BP_*(X) \otimes_{BP_*} L_{K(n)}BP_*.$$

The Landweber filtration theorem [**Land**] says that $BP_*(X)$ has a finite filtration by BP_*BP subcomodules M_i for $i = 1, \ldots, m$, such that the quotient M_{i+1}/M_i is isomorphic to BP_*/I_{m_i} for some m_i. Choose n larger than all the m_i. Then BP_*/I_{m_i} injects into $BP_*/I_{m_i} \otimes_{BP_*} (v_n^{-1}BP_*)_{I_n}$. The proof of the Landweber exact functor theorem actually shows that $(v_n^{-1}BP_*)_{I_n}$ is flat in the category of BP_*BP comodules which are finitely generated over BP_*. Now an easy induction on the M_i using the 5-lemma completes the proof. $\square$

COROLLARY 3.5. *BP_p is local with respect to $E = \bigvee K(n_i)$ for any infinite sequence (n_i) of integers. BP is E-local if and only if the sequence contains 0.*

PROOF. $L_{K(n_i)}BP_p$ is certainly E-local, and any product of local spectra is local. Thus BP_p is a retract of a local space, so is local. We have the cofiber sequence $C \to BP \to BP_p$, where C is rational. Thus, BP is E-local if and only if C is E-local if and only if $H\mathbf{Q}$ is E-local. This is true if and only if 0 is in the sequence. $\square$

It is natural to ask if the analogue of chromatic convergence holds. Define $X_j = L_{K(n_0) \vee \ldots \vee K(n_j)}BP_p$. One would then ask if BP_p is the inverse limit X of the X_j. I don't know the answer to this question. Note though that the map from $BP_p \to \prod L_{K(n_i)}BP_p$ factors through X, so that BP_p is a retract of X.

THEOREM 3.6. *Suppose R is a ring spectrum with no finite acyclics. If $H\mathbf{Q} \wedge R \neq *$, then $L_R X = X$ for all finite X. If $H\mathbf{Q} \wedge R = *$, then $L_R X = X_p$ for all finite X.*

First we show

LEMMA 3.7. *Suppose E is any spectrum such that $L_E X = X$ for some finite X. Then if If $H\mathbf{Q} \wedge E \neq *$, then $L_E X = X$ for all finite X. If $H\mathbf{Q} \wedge E = *$, then $L_E X = X_p$ for all finite X.*

PROOF. Consider the class $\mathcal{C}$ of all finite X that are local with respect to E. It is easy to see that $\mathcal{C}$ is closed under retracts, suspensions, and cofibrations. It is nonempty by hypothesis, so it must be a $\mathcal{C}_n$ for some n. Suppose $n > 1$, and let X be a space of type $n - 1$. Then X has a v_{n-1}-self map f, which must be of positive degree d. In the cofiber sequence $\Sigma^d X \to X \to Y$, Y has type n so is E-local. Thus, if Z is E-acyclic, any map $Z \xrightarrow{g} X$ factors through $\Sigma^d X$. Repeating this process, we find that g factors through the inverse limit of the $\Sigma^{kd} Y$, which is null. Thus X is E-local, which is a contradiction.

Thus $\mathcal{C} \supseteq \mathcal{C}_1$. In particular, the Moore space $M(p)$ is E-local. Consider the cofiber sequence $S_p^0 \xrightarrow{\times p} S_p^0 \to M(p)$. Again, if Z is E-acyclic, any map $Z \to S_p^0$ factors through the inverse limit of the times p map on the p-complete sphere, which is null. So S_p^0 is E-local.

Now consider the cofibre sequence

$$F \to S^0 \to S_p^0.$$

F is a rational space, so it is either E-acyclic or E-local according to whether $E \wedge H\mathbf{Q}$ is trivial or not. Localizing the cofibre sequence at E completes the proof of the lemma. $\square$

PROOF OF THEOREM. Thus to prove the theorem, we only need to show that some finite X is R-local. A corollary of the nilpotence theorem [**Hop**] tells us that any ring spectrum must be detected by one of the $K(n)$, for $0 \leq n \leq \infty$. If R is detected by $K(\infty) = H\mathbf{F}_p$ then the Bousfield class of R is at least as big as that of $H\mathbf{F}_p$. Since the Moore space $M(p)$ is $H\mathbf{F}_p$-local, it is also R-local, and we are done.

So suppose that $R \wedge H\mathbf{F}_p$ is null. I claim that $R \wedge K(n)$ must then be nonzero for infinitely many $n < \infty$. Indeed, for all n, there is a ring spectrum Y_n of type n. (see [**Dev**] for specific examples). Then $R \wedge Y_n$ is also a ring spectrum, which is nonzero since R has no finite acyclics. It is not detected by any $K(i)$ with $i < n$ or $i = \infty$, so it must be detected by some $K(i)$ for $i \geq n$.

This means by [**Rav84**, Thm 2.1] that the Bousfield class of R is as least as big as that of some infinite wedge of Morava K-theories. Thus it will suffice to show that $M(p)$ is local with respect to such a wedge, for then it will be R-local as well. To do this we follow the argument of [**Rav84**, Thm 4.4]. We know already that BP_p is local. It follows that any locally finite wedge of suspensions of $BP \wedge M(p)$ is local. We then use the Adams tower based on BP homology to write $M(p)$ as an inverse limit of spaces K_s of the form $BP \wedge \overline{BP}^{\wedge s} \wedge M(p)$. Here $\overline{BP}$ is the fiber of the unit map $S^0 \to BP$. Since $BP \wedge \overline{BP}$ is a locally finite wedge of suspensions of BP, each K_s is local. Then $M(p)$, as the inverse limit of local spectra, is also local. $\square$

Corollary 3.5 and Theorem 3.6 can be used to show that some mapping groups are 0. For example, they imply that $[BP\langle n \rangle, BP_p] = 0$ and $[BP\langle n \rangle, X_p] = 0$, where X is finite. Indeed, $BP\langle n \rangle$ has no $K(i)$ homology for $i > n$.

Recall the problem of Bousfield, mentioned in Section 1, which asks for a classification of smashing localization functors.

COROLLARY 3.8. *If the localization functor L_E is smashing and E has no finite acyclics, then L_E is the identity functor.*

PROOF. If L_E is smashing, then $\langle E \rangle = \langle L_E S^0 \rangle$, which is a ring spectrum. Since E has no finite acyclics, neither does $L_E S^0$. So the proceeding theorem tells us that $L_E S^0$, which is $L_{L_E S^0} S^0$, is either S^0 or S^0_p. But S^0_p has the same Bousfield class as the sphere itself. Indeed, suppose $S^0_p \wedge X$ is zero. Then, using the cofibre sequence

$$F \to S^0 \to S^0_p$$

we find that X is a rational space. But $S^0_p \wedge H\mathbf{Q}$ is not zero, so $S^0_p \wedge X$ can't be either. Thus $\langle E \rangle = \langle L_E S^0 \rangle = \langle S^0 \rangle$, as required. $\square$

This also proves the following conjecture in the case that E is a ring spectrum with no finite acyclics. Hopkins and possibly others have made this conjecture independently.

CONJECTURE 3.9. *If E is arbitrary, then $L_E S^0$ is smashing.*

This brings us to the question of localization with respect to an arbitrary spectrum with no finite acyclics. I make the following conjecture.

CONJECTURE 3.10. *If E has no finite acyclics, then $L_E S^0$ is either the sphere itself or S^0_p.*

Our method above relied on showing that BP_p is E-local. This will certainly not be true in general. There are E with no finite acyclics such that $BP \wedge E$ is zero. An example of such a spectrum is IS^0, the Brown-Comanetz dual of the sphere. It is a consequence of sections 2 and 3 of [**Rav84**] that $BP \wedge IS^0 = 0$. However, torsion finite spectra are local with respect to IS^0. In fact $I^2 X$ is always IX-local, since

$$[Z, I^2 X] = [Z, F(IX, IS^0)] = [Z \wedge IX, IS^0].$$

So $S^0_p = I^2 S^0$ is local with respect to IS^0.

4. The chromatic splitting conjecture

In this section, we describe Hopkins' chromatic splitting conjecture and deduce some corollaries of it. The conjecture is concerned with the fibre of the map $L_n S^0 \to L_{K(n)} S^0$. The following lemma is a generalization of a lemma of Hopkins.

LEMMA 4.1. *Suppose E, F are spectra such that $F \wedge L_E S^0$ is null. Then for arbitrary X, the fibre of the natural map $L_{E \vee F} X \to L_F X$ is the function spectrum $F(L_E S^0, L_{E \vee F} X)$.*

PROOF. Let Y denote the fibre. Then Y is $E \vee F$ local and F acyclic. We claim that Y is therefore E local. Consider the map $Y \to L_E Y$. This is an E isomorphism, and $F_* Y = 0$. Now $L_E Y$ is an $L_E S^0$ module spectrum, so since $F \wedge L_E S^0$ is null, so is $F \wedge L_E Y$. Thus the map $Y \to L_E Y$ is an $E \vee F$ isomorphism. Since both sides are $E \vee F$ local, it is therefore an equivalence, so Y is E local.

To show that Y is $F(L_E S^0, L_{E \vee F} X)$, it will suffice to show that Y has the same universal property, i.e. that

$$[Z, Y] = [Z \wedge L_E S^0, L_{E \vee F} X].$$

Since Y is E local, and the natural map $Z \to Z \wedge L_E S^0$ is an E isomorphism, we have $[Z, Y] = [Z \wedge L_E S^0, Y]$. Since $L_E S^0$ is F acyclic, so is $Z \wedge L_E S^0$. Applying $[Z, \ ]$ to the cofibre sequence

$$Y \to L_{E \vee F} X \to L_F X$$

we see that $[Z \wedge L_E S^0, Y] = [Z \wedge L_E S^0, L_{E \vee F} X]$, as required. $\square$

The main example we are interested in here is the cofibre sequence

$$F(L_{n-1} S^0, L_n X) \to L_n X \to L_{K(n)} X.$$

To describe the chromatic splitting conjecture, I must briefly describe some work of Hopkins-Ravenel and Hopkins-Miller based on Morava's philosophy. Unfortunately, little of this work has appeared. The idea is this: the Morava stabilizer group S_n is essentially the group of automorphisms of the formal group law over $K(n)_*$. This is not quite true: it is actually the automorphisms of the same formal group law, but considered over the ring $\mathbf{F}_{p^n}[u, u^{-1}]$. Here u has degree -2 and is a $-(p^n - 1)$-fold root of v_n. It is technically advantageous to use u instead of v_n. The work of Lubin and Tate gives an action of S_n on a complete ring whose residue field is $\mathbf{F}_{p^n}[u, u^{-1}]$. We take this ring to be the flat $E(n)_*$-module

$$E_{n_*} = W(\mathbf{F}_{p^n})[[u_1, \ldots, u_{n-1}]][u, u^{-1}].$$

Here the u_i have degree 0, u has degree -2, and $W(\mathbf{F}_{p^n})$ is the Witt vectors of the field with p^n elements. The map

$$E(n)_* = \mathbf{Z}_{(p)}[v_1, \ldots, v_n, v_n^{-1}] \to E_{n_*}$$

takes v_i to $u_i u^{1-p^i}$ and v_n to u^{1-p^n}. The residue field of the complete local ring E_{n_*} is then $\mathbf{F}_{p^n}[u, u^{-1}]$. Now given an element of S_n, it lifts to an isomorphism from the formal group F over E_{n_*} to a possibly different formal group F'. The work of Lubin-Tate [**LT**] shows that there is a well-defined automorphism of the ring E_{n_*} taking F' to a formal group law which is $*$-isomorphic to F, i.e. isomorphic by an isomorphism which reduces to the identity on the residue field $\mathbf{F}_{p^n}[u, u^{-1}]$. This gives a (continuous) action of S_n on E_{n_*}.

Now, E_{n_*} is actually the homotopy of a spectrum E_n. In fact, E_{n_*} is a flat $E(n)_*$-module, so one can simply tensor with it. In [**HM**] it is shown that S_n actually acts on the spectrum E_n, in fact by E_∞ maps. They show that the homotopy fixed point spectrum of this action is $L_{K(n)}S^0$. (Actually, one has to cope with the Galois group $\mathbf{Z}/n$ of the extension $W(\mathbf{F}_{p^n})$ over $\mathbf{Z}_p$ as well.) There is then a homotopy fixed point set spectral sequence

$$E_2 = H^{*,*}(S_n; E_{n_*})^{\mathbf{Z}/n} \implies \pi_*(L_{K(n)}S^0).$$

The group cohomology here must be taken to be continuous cohomology. This spectral sequence was known before the work of [**HM**]: I believe it is due to Hopkins-Ravenel, and a brief description of it appears in [**HMS**]. It collapses and there are no extensions when the prime p is large with respect to n.

Now, consider the inclusion

$$W(\mathbf{F}_{p^n}) \to E_{n,0}.$$

This is a map of S_n-modules, where S_n acts trivially on $W(\mathbf{F}_{p^n})$, so induces

$$H^*(S_n; W(\mathbf{F}_{p^n})) \to H^*(S_n; E_{n,0}).$$

(Here and below I always mean the $\mathbf{Z}/n$ invariants of cohomology groups.) Computations suggest that $H^*(S_n; W(\mathbf{F}_{p^n}))$ is, or at least contains, an exterior algebra on n generators $x_1, \ldots, x_n$. Of these, x_1 is most familiar: it is usually called ζ_n. It arises from the determinant map $S_n \to \mathbf{Z}_p$, where we are thinking of $\mathbf{Z}_p$ as a subgroup of its own group of units. This is a crossed homomorphism with respect to the trivial action of S_n on $Z_p \subseteq W(\mathbf{F}_{p^n})$, so gives rise to a class in $H^1(S_n; W(\mathbf{F}_{p^n}))$.

I will describe Hopkins' chromatic splitting conjecture first in the case $n = 2$ where it is simpler and also known to be true, at least for $p > 3$. This is all due to Hopkins, and uses essentially the calculations of Shimomura-Yabe in [**Sh-Y**]. In that case, $H^*(S_2; W(\mathbf{F}_{p^n}))$ is an exterior algebra on classes traditionally denoted ζ and ρ, in bidegrees $(1,0)$ and $(3,0)$. Both of these classes survive to homotopy classes

$$\zeta : S_p^{-1} \to L_{K(2)}S^0 \ \ \rho : S_p^{-3} \to L_{K(2)}S^0.$$

Multiplication also gives us a class

$$\zeta\rho : S_p^{-4} \to L_{K(2)}S^0.$$

Compose these maps with the map

$$L_{K(2)}S^0 \to F(L_1 S^0, L_2 S_p^0).$$

This is L_1-local, so we get maps $\zeta, \rho,$ and $\zeta\rho$ from $L_1 S_p^0$ to $F(L_1 S^0, L_2 S_p^0)$ of degrees $-2, -4, -5$. But, strangely enough, ρ and $\zeta\rho$ actually factor further through $L_0 S_p^0$. Thus we get a map

$$\Sigma^{-2} L_1 S_p^0 \vee \Sigma^{-4} L_0 S_p^0 \vee \Sigma^{-5} L_0 S_p^0 \to F(L_1 S^0, L_2 S_p^0).$$

This map is in fact a homotopy equivalence. This is the chromatic splitting conjecture for $n = 2$. The general case is more complicated and is stated below.

CONJECTURE 4.2 (HOPKINS' CHROMATIC SPLITTING CONJECTURE). *Fix an integer $n \geq 1$.*

(i) $H^*(S_n; W(\mathbf{F}_{p^n}))$ *contains the exterior algebra* $E(x_1, \dots, x_n)$.

(ii) *Each class of nonzero degree $x_{i_1} \cdots x_{i_j}$ in the exterior algebra survives to a homotopy class*

$$x_{i_1} \cdots x_{i_j} : S_p^{-2(\Sigma i_k)+j} \to L_{K(n)} S^0.$$

(iii) *The composite*

$$S_p^{-2(\Sigma i_k)+j} \overset{x_{i_1}\cdots x_{i_j}}{\to} L_{K(n)} S^0 \to \Sigma F(L_{n-1} S^0, L_n S_p^0)$$

factors through

$$L_{n-\max i_k} S_p^{-2(\Sigma i_k)+j}.$$

(iv) *The maps above split $F(L_{n-1}S^0, L_n S_p^0)$ into $2^n - 1$ summands.*

(v) *The cofibre sequence*

$$F(L_{n-1}S^0, L_n S_p^0) \to L_{n-1} S_p^0 \to L_{n-1} L_{K(n)} S^0$$

splits, so that

$$L_{n-1} L_{K(n)} S^0 \simeq L_{n-1} S_p^0 \vee \Sigma F(L_{n-1} S^0, L_n S_p^0).$$

As mentioned above, this conjecture is known to be true for $n = 1$ and for $n = 2, p > 3$. The only other thing known about this conjecture is that $x_1 = \zeta_n$ always survives to give a homotopy class [**HM**]. One would expect that part 1 should be possible to do, and that part 2 may be approachable using the techniques of [**HM**]. To this author at least, part 3 is a complete mystery. It seems to be suggesting that there is some interesting relationship between the different S_n. Note that part 5 of the chromatic splitting conjecture has not really been tested yet. The summand $\Sigma^{-2} L_{n-1} S_p^0$ of $F(L_{n-1} S^0, L_n S_p^0)$ corresponding to ζ_n always maps trivially to $L_{n-1} S_p^0$ by construction. For $n = 2$, the maps from the other 2 summands to $L_1 S_p^0$ are trivial for dimensional reasons. For $n = 3$ there are possible maps from the other summands to $L_2 S_p^0$. It is also part 5 from which the striking corollaries below can be derived.

THEOREM 4.3. *If the chromatic splitting conjecture is true, and if $f : X \to Y$ is a map between two finite spectra such that $L_{K(n)} f : L_{K(n)} X \to L_{K(n)} Y$ is null for infinitely many n, then f is null.*

PROOF. It suffices to show that $f : X \to Y_p$ is null. Note that $L_{K(n)} Y_p = L_{K(n)} Y$ if $n > 0$. We have the diagram

$$\begin{array}{ccc}
X \longrightarrow & Y_p & \longrightarrow & L_{K(n)}Y_p \\
 & \downarrow & & \downarrow \\
 & L_{n-1}Y_p & \longrightarrow & L_{n-1}L_{K(n)}Y_p
\end{array}$$

By the preceeding result, $L_{n-1}Y_p$ is a summand of $L_{n-1}L_{K(n)}Y_p$. Thus if $X \to L_{K(n)}Y_p$ is null, so is $X \to L_{n-1}Y_p$. The chromatic convergence theorem says that the tower $L_{n-1}Y$ is pro-isomorphic to the constant tower. It is easy to see that $L_{n-1}Y_p$ is also pro-isomorphic to the constant tower. Thus, since $X \to L_{n-1}Y_p$ is null for a cofinal sequence of n's, $X \to Y_p$ is null. $\square$

We can use the results in the previous section to prove that such a map must at least be null upon smashing with BP.

PROPOSITION 4.4. *If $f : X \to Y$ is a map between two finite spectra such that $L_{K(n)}f$ is null for infinitely many n, then the composite*

$$X \to Y \to BP \wedge Y$$

is null. In particular, if E is a BP-module spectrum, $E_*(f) : E_*(X) \to E_*(Y)$ is zero.

PROOF. First note that infinite products commute with smashing with finite spectra, by Spanier-Whitehead duality. Thus, $BP_p \wedge Y$ is a retract of $\prod L_{K(n_i)}BP_p \wedge Y$, for any infinite sequence (n_i). Since the map $X \to Y \to BP_p \wedge Y$ becomes null on localizaing with respect to $K(n)$ for infinitely many n, it is null. It follows from general facts about p-completions of spectra of finite type that the map

$$X \to Y \to BP \wedge Y$$

is null (see Chapter 9 of [**Mar**]). Smashing with E, we find that the composite

$$E \wedge X \to E \wedge Y \to E \wedge BP \wedge Y$$

is null. But if E is a BP-module spectrum, then E is a wedge summand of $E \wedge BP$, so in fact $E \wedge X \to E \wedge Y$ is null. $\square$

For several years, Hopkins has been saying that one does not need to reassemble the monochromatic parts of X to recover the homotopy theory of finite spectra. The following corollary indicates a precise sense in which this is true.

COROLLARY 4.5. *If the chromatic splitting conjecture is true, then the natural map $X_p \to \prod L_{K(n_i)}X_p$ is the inclusion of a summand. In particular, if Y is arbitrary, and $Y \to X$ is a map such that the composite $Y \to X \to L_{K(n)}X$ is null for infinitely many n, then $Y \to X \to X_p$ is null.*

PROOF. By the preceeding theorem, the map $X_p \to \prod_i L_{K(n_i)} X_p$ is injective on maps from finite complexes. Thus , if F denotes the fibre, the map $F \to X_p$ is f-phantom. Since there are no f-phantom maps to X_p, it is null. $\square$

5. Appendix: The p-completion

In this appendix, we investigate the consequences that the chromatic splitting conjecture would have on the structure of $\pi_* L_n S^0$. In particular, we show how to determine the divisible summands, and show that except for those summands and the free one in dimension 0, $\pi_* L_n S^0$ is a direct sum of cyclic groups which have bounded torsion in each dimension.

The idea is that the adjoint properties of function spectra together with the fact that L_n is smashing allow us to deduce the structure of $F(L_i S^0, L_n S_p^0)$ from just knowing it for $i = n - 1$. Indeed, we have

$$F(L_{i-1}S^0, L_n S_p^0) = F(L_{i-1}(L_i S^0), L_n S_p^0) = F(L_{i-1}S^0 \wedge L_i S^0, L_n S_p^0)$$

$$= F(L_{i-1}S^0, F(L_i S^0, L_n S_p^0)).$$

We illustrate the technique for $n = 2$, where we have

$$F(L_1 S^0, L_2 S_p^0) = \Sigma^{-2} L_1 S_p^0 \vee \Sigma^{-4} L_0 S_p^0 \vee \Sigma^{-5} L_0 S_p^0$$

Thus,

$$F(L_0 S^0, L_2 S_p^0) = \Sigma^{-4} L_0 S_p^0 \vee \Sigma^{-4} L_0 S_p^0 \vee \Sigma^{-5} L_0 S_p^0.$$

As we will see below, this splitting reflects the three $\mathbf{Q}/\mathbf{Z}_p$ summands in $\pi_* L_2 S^0$ as calculated by Shimomura-Yabe in [**Sh-Y**].

PROPOSITION 5.1. *Suppose the chromatic splitting conjecture is true. Then $F(L_0 S^0, L_n S_p^0)$ splits into a wedge of 3^{n-1} copies of $H\mathbf{Q}_p$, in dimensions ranging from $-2n$ to $-n^2 - 1$, with 2^{n-1} copies in dimension $-2n$.*

It is of course possible to explicitly calculate the locations of these summands for any specific n. For example, for $n = 3$, there are 4 summands in dimension -6, 3 in dimension -7, and 1 each in dimensions -9 and -10.

PROOF. We proceed by induction on n. Suppose that we know the result is true for all $k \le n - 1$. We have

$$F(L_0 S^0, L_n S_p^0) = F(L_0 S^0, F(L_{n-1}S^0, L_n S_p^0))$$

as above. We know by the chromatic splitting conjecture that $F(L_{n-1}S^0, L_n S_p^0)$ splits into $2^n - 1$ summands, with $L_k S_p^0$ occuring 2^{n-k-1} times. The highest dimensional occurence of $L_k S^0$ is in dimension $-2(n - k)$ (corresponding to $x_{n-k} \in H^{2k-1}(S_n; W(\mathbf{F}_{p^n}))$) and the lowest dimensional occurence of $L_k S^0$ is in $-(n - k)^2 - 1$ (corresponding to $x_{n-k} x_{n-k-1} \cdots x_1$).

Thus the total number of summands in $F(L_0 S^0, L_n S_p^0)$ is

$$3^{n-2} + 2 \times 3^{n-3} + \ldots + 2^{n-2} + 2^{n-1} = 3^{n-1}.$$

The only summands that can arise in dimension $-2n$ arise from the highest dimensional occurence of $L_k S^0$. Thus there are

$$2^{n-2} + 2^{n-3} + \ldots + 1 = 2^{n-1}$$

of them. The lowest dimensional summand arises from choosing $k = 0$ in the above paragraph, and occurs in dimension $-n^2 - 1$. $\square$

This proposition also explains why we need to complete the sphere in the chromatic splitting conjecture. If we did not, there would also be maps $\Sigma^{-1} H\mathbf{Q} \to L_n S^0$ coming from the fiber of the natural map $L_n S^0 \to L_n S_p^0$.

One certainly expects that $F(H\mathbf{Q}, L_n S_p^0)$ should correspond to the divisible summands in $\pi_* L_n S^0$. The rest of this section is devoted to proving that. Lemma 4.1 tells us that this function spectrum is the fibre of the map

$$L_n S_p^0 \to L_{K(1) \vee \cdots \vee K(n)} S_p^0.$$

Since

$$L_{K(1) \vee \cdots \vee K(n)} S_p^0 = L_{M(p)} L_n S_p^0,$$

we have a cofibre sequence

$$F(L_0 S^0, L_n S_p^0) \to L_n S_p^0 \to (L_n S_p^0)_p.$$

Now there are two things we need to do. First, we need to know something about how the homotopy groups of X_p are related to the homotopy groups of X. One might like them to be the p-completions of the homotopy groups of X. This is false in general, but the following proposition says that they are close to being p-complete. That something like this proposition might be true was first suggested to me by Hal Sadofsky.

For an abelian group G, let $p^\infty G = \bigcap p^n G$.

PROPOSITION 5.2. *For arbitrary X, Y, $[Y, X_p]$ is a module over $\mathbf{Z}_p$, has no divisible summands, and $[Y, X_p]/p^\infty[Y, X_p]$ is the p-completion of $[Y, X_p]$.*

PROOF. We can assume $X = X_p$ and $Y = Y_p$. Then Y is a module spectrum over S_p^0, so maps out of it are a module over $\pi_0 S_p^0 = \mathbf{Z}_p$. We will first show that $[Y, X]$ has no divisible summands. Consider the system of cofibre sequences whose nth and $n-1$st terms are displayed below.

$$
\begin{array}{ccccc}
X & \xrightarrow{\times p^n} & X & \longrightarrow & X \wedge M(p^n) \\
\times p \downarrow & & = \downarrow & & \downarrow \\
X & \xrightarrow{\times p^{n-1}} & X & \longrightarrow & X \wedge M(p^{n-1})
\end{array}
$$

If $f \in [Y, X]$ generates a divisible summand, there are maps $f_n \in [Y, X]$ for all n, such that $p f_n = f_{n-1}$, where $f_0 = f$. These will define a map into the inverse limit $Z = \varprojlim(\times p : X \to X)$ of the left column in the above diagram. Now inverse limits do not behave very well in general, but the inverse limit of

cofibre sequences is still a cofibre sequence, as we will prove below. Thus we get a cofibre sequence

$$Z \to X \to \varprojlim(X \wedge M(p^n)) = L_{M(p)}X.$$

Since X is already $M(p)$-local, Z must be null. Since f factors through Z, f is null too.

Now we will show that the map

$$[Y, X] \to [Y, X]_p = \varprojlim[Y, X]/p^n[Y, X]$$

is surjective. The proposition will then follow, since the kernel of the map $A \to A_p$ for abelian groups A is always $p^\infty A$. Suppose $(f_n) \in [Y, X]_p$, so $f_n \in [Y, X]/p^n[Y, X]$. Let $A = [Y, X]$ and $B = [Y, \Sigma X]$, and denote the elements of B killed by $\times p^n$ by $B(p^n)$. Then, using the cofibre sequence

$$X \xrightarrow{\times p^n} X \xrightarrow{i} X \wedge M(p^n),$$

we get a diagram of short exact sequences

$$
\begin{array}{ccccc}
A/p^n A & \xrightarrow{\ i\ } & [Y, X \wedge M(p^n)] & \longrightarrow & B(p^n) \\
\downarrow & & \downarrow & & {\scriptstyle \times p}\downarrow \\
A/p^{n-1} A & \xrightarrow{\ i\ } & [Y, X \wedge M(p^{n-1})] & \longrightarrow & B(p^{n-1})
\end{array}
$$

Since (f_n) is a compatible sequence, so is $(i(f_n))$, se we get an element of $\varprojlim[Y, X \wedge M(p^n)]$. The map

$$[Y, X] = [Y, \varprojlim(X \wedge M(p^n))] \to \varprojlim[Y, X \wedge M(p^n)]$$

is not an isomorphism in general, but it is always surjective. So we get a map $f \in [Y, X]$, and it is easy to see that f maps to $(f_n) \in [Y, X]_p$. $\quad\square$

This completes the proof of the proposition modulo the following lemma, which I learned from Hal Sadofsky.

LEMMA 5.3. *The inverse limit of cofibre sequences is a cofibre sequence.*

PROOF. It is easy to see that products of cofibre sequences are cofibre sequences. Thus, given cofibre sequences

$$
\begin{array}{ccccc}
A_n & \longrightarrow & B_n & \longrightarrow & C_n \\
\downarrow & & \downarrow & & \downarrow \\
A_{n-1} & \longrightarrow & B_{n-1} & \longrightarrow & C_{n-1}
\end{array}
$$

we get a diagram of cofibre sequences

$$
\begin{array}{ccccc}
\prod A_n & \longrightarrow & \prod B_n & \longrightarrow & \prod C_n \\
\downarrow & & \downarrow & & \downarrow \\
\prod A_n & \longrightarrow & \prod B_n & \longrightarrow & \prod C_n
\end{array}
$$

where the vertical arrows are the maps whose fibres are the inverse limits. Now it is not always true that the fibres in such a situation form a cofibre sequence, but it is true in this case since the map $\prod C_n \to \prod C_n$ is induced by the map $\prod B_n \to \prod B_n$. □

COROLLARY 5.4. *The kernel of the map*

$$[Y, X] \to [Y, X_p]$$

is precisely the divisible summands in $[Y, X]$.

PROOF. Any divisible summand in $[Y, X]$ must map to 0 in $[Y, X_p]$, by the proposition. To see the converse, note that we showed that the fibre of $X \to X_p$ is the rational spectrum $Z = \varprojlim(\times p : X \to X)$. So $[Y, Z]$ is a divisible group, and thus its image in $[Y, X]$ is also divisible. □

Now the second thing we need to do is to get some kind of control over the homotopy of $L_n S^0$.

LEMMA 5.5. $\pi_i L_n S^0$ *is a countable abelian group.*

PROOF. We will show this using the Adams-Novikov spectral sequence

$$E_2^{s,t} = \text{Ext}_{BP_*BP}^{s,s+i}(BP_*, BP_*(L_n S^0)) \Longrightarrow \pi_i L_n S^0$$

This spectral sequence converges in a very strong sense, in that $E_\infty^{s,s+i}$ is 0 for large enough s (and fixed i) [**Rav92**]. Thus, if $E_\infty^{s,s+i}$ is countable for all s, t, so is $\pi_i L_n S^0$. However, $E_\infty^{s,t}$ is a subquotient of $E_2^{s,s+i}$, so it will suffice to show that E_2 is countable in each bidegree.

One way to calculate the Ext groups of a BP_*BP comodule M is to use the cobar complex, made up out of

$$\Omega^s(M) = M \otimes_{BP_*} BP_*BP \otimes_{BP_*} \cdots \otimes_{BP_*} BP_*BP$$

where there are s factors of BP_*BP. Note that BP_*BP is countable in each degree. I claim that if M, N are countable in each degree their tensor product will be too. Indeed, the degree t part of their tensor product is a quotient of $\bigoplus M_k \otimes N_{t-k}$. The tensor product of two countable abelian groups is countable, as is the countable direct limit (or sum) of countable abelian groups. Thus, if M is countable in each degree, so is $\Omega^s M$, and thus also $\text{Ext}_{BP_*BP}^s(BP_*, M)$.

Thus it will suffice to show that $BP_*(L_n S^0)$ is countable in each degree. Since L_n is smashing, $BP_*(L_n S^0) = \pi_*(L_n BP)$. This is calculated by Ravenel in [**Rav84**]. His result is that

$$\pi_*(L_n BP) = BP_* \oplus \Sigma^{-n} N^{n+1}$$

for $n \geq 1$, where N_{n+1} is defined inductively by $N^0 = BP_*$ and the short exact sequence

$$0 \to N^k \to v_k^{-1} N^k \to N^{k+1} \to 0.$$

If N^k is countable in each degree, so is $v_k^{-1}N^k$, as it is a direct limit of countable groups. So by induction, N^{n+1} is countable in each degree, so is $BP_*(L_nS^0)$ and we are done. $\square$

Note that there is a sense in which countable torsion groups A are completely classified (Ulm's Theorem [**Kap**]). This classification is complicated, however, because $p^\infty A$ may not be 0. One certainly hopes that this complication does not arise in L_nS^0. We will see below that it does not if the chromatic splitting conjecture is true.

For the purposes of the theorem below, let n_k denote the number of summands in $F(L_0S^0, L_nS^0_p)$ in dimension k.

THEOREM 5.6. *Suppose the chromatic splitting conjecture is true. Then for all k, $\pi_k L_nS^0 = D_k \oplus T_k$, where $D_0 = \mathbf{Z}$, D_k is a direct sum of n_k copies of $\mathbf{Q}/\mathbf{Z}_{(p)}$, and T_k is a bounded torsion group which is a countable direct sum of cyclic groups.*

PROOF. It is clear that $D_0 = \mathbf{Z}$, since the composite $\pi_0 S^0 \to \pi_0 L_nS^0 \to \pi_0 L_1S^0$ is the identity. The rest of $\pi_i L_nS^0$ is all torsion. It suffices to prove the theorem for $X = L_nS^0_p$, which differs from L_nS^0 only in that $D_0 = \mathbf{Z}_p$ instead of $\mathbf{Z}$.

So we have the cofibre sequence

$$Y \to X \xrightarrow{f} X_p,$$

where $Y = F(L_0S^0, L_nS^0_p)$ is a finite wedge of suspensions of $H\mathbf{Q}_p$. We investigate the image of f on homotopy. We have a short exact sequence

$$0 \to A \to G \to H \to 0$$

where $A = \pi_{k+1}X/(\text{divisible summands}) = T_{k+1}$ is a countable torsion group, $G = \pi_{k+1}X_p$ is a $\mathbf{Z}_p$ module with no divisible summands whose p-completion is $G/p^\infty G$, and $H = \text{im } f$ is torsion-free. This means that $A = \text{Tor}(G)$, and that this is necessarily a short exact sequence of $\mathbf{Z}_p$ modules. There is an induced short exact sequence of $\mathbf{Z}_p$ modules

$$0 \to A/p^\infty A \to G/p^\infty G \to H'.$$

The induced map of $\mathbf{Z}_p$ modules $H \to H'$ is surjective, so since H is a $\mathbf{Z}_p$ submodule of a finite direct sum of $\mathbf{Q}_p$'s, H' can have at most only countable torsion.

Now, $B = A/p^\infty A$ is a countable torsion group, and $p^\infty B = 0$. Thus, by Theorem 11 of [**Kap**], it must be a direct sum of cyclics

$$B = \bigoplus_i \mathbf{Z}/p^{n_i}.$$

Further, it is sitting inside the p-complete group $G/p^\infty G$. Therefore, its p-completion B_p is also inside $G/p^\infty G$. If B is unbounded torsion, one can see

from the direct sum decomposition of B that $B_p/B \subseteq H$ is uncountable, and in fact even the torsion of B_p/B is uncountable. This is impossible, given the possibilities for H'. Thus B must be bounded torsion, say $p^N B = 0$. But in that case, we have $p^N A \subseteq p^\infty A$, and so we can deduce that the times p map from $p^N A$ to itself is surjective. This means $p^N A$ is divisible, and since A has no divisible summands, it must be 0. Therefore $A = T_{k+1}$ has bounded torsion, and is therefore a direct sum of cyclics, proving half of the proposition. But $A = \mathrm{Tor}(G)$, and a torsion subgroup which is bounded torsion always splits off. Thus $G = A \oplus H$. Thus $H = \mathrm{im}\, f$ can have no divisible summands. H is therefore a free $\mathbf{Z}_p$ module.

If the rank of H is less than n_k, then there is a $\mathbf{Q}_p$ summand in $\pi_k L_n S^0$. This summand survives to $\pi_k L_0 S^0$, which is impossible. It follows that the image of $\pi_k(Y)$ in $\pi_k L_n S^0$ is a direct sum of n_k copies of $\mathbf{Q}/\mathbf{Z}_{(p)}$, as required. $\quad\square$

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