On saturated classes of morphisms

Carles Casacuberta*and Armin Frei

Abstract

The term "saturated," referring to a class of morphisms in a category, is used in the literature for two nonequivalent concepts. We make precise the relationship between these two concepts and show that the class of equivalences associated with any monad is saturated in both senses.

0 Introduction

In [1], we drew attention to the fact that the concept of "saturation" of a class of morphisms appears in the literature with two different meanings. In [1], [2], and [3], the saturation of a class of morphisms S in a category denotes the double orthogonal $S^{\perp\perp}$ in the sense of Freyd–Kelly [8]. On the other hand, in the book by Gabriel–Zisman [9] and in subsequent articles such as [4], the saturation of a class of morphisms S in a category $\mathcal{C}$ consists of the morphisms rendered invertible by the canonical functor from $\mathcal{C}$ to the category of fractions $\mathcal{C}[S^{-1}]$.

In the present paper we show that, although the two concepts do not coincide in general, the saturation of a class of morphisms S in the first sense contains the saturation of S in the second sense. We also prove that the class of equivalences associated with any monad is saturated in both senses. In fact, whenever a functor F has a right adjoint, the class of morphisms rendered invertible by F is saturated in both senses.

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1 Terminology

Most of the following terminology is taken from [2], [3], [6], [8], and [9]. For any functor $F: \mathcal{C} \to \mathcal{D}$ between two given categories, we define

$$\mathcal{S}(F) = \{\text{morphisms } f \text{ in } \mathcal{C} \text{ such that } Ff \text{ is invertible}\}, \tag{1.1}$$

and say that morphisms in $\mathcal{S}(F)$ are *F-equivalences*.

A morphism $f: A \to B$ and an object X in a category $\mathcal{C}$ are called *orthogonal*, as in [8], if the function

$$\mathcal{C}(f, X): \mathcal{C}(B, X) \to \mathcal{C}(A, X)$$

is bijective. For a class of morphisms S (resp. a class of objects D), we denote by $S^{\perp}$ the class of objects orthogonal to every f in S (resp. by $D^{\perp}$ the class of morphisms orthogonal to all X in D). Objects in $S^{\perp}$ were called *left closed* for S in [4], [5], or in [9, I.4]. We call $S^{\perp\perp}$ the *internal saturation* of S, and say that S is *internally saturated* if $S^{\perp\perp} = S$. Observe that every class of the form $D^{\perp}$ is internally saturated, since $D^{\perp\perp\perp} = D^{\perp}$.

Given a class of morphisms S in a category $\mathcal{C}$, let $\mathcal{C}[S^{-1}]$ denote the category of fractions of $\mathcal{C}$ with respect to S (see [9]). There is a canonical functor $F_S: \mathcal{C} \to \mathcal{C}[S^{-1}]$ such that $F_S f$ is invertible for every f in S and, if a functor $F: \mathcal{C} \to \mathcal{D}$ renders all the morphisms in S invertible, then there is a unique functor $G: \mathcal{C}[S^{-1}] \to \mathcal{D}$ such that $GF_S = F$.

The *external saturation* $\hat{S}$ of a class S of morphisms is the class of all morphisms rendered invertible by the canonical functor $F_S: \mathcal{C} \to \mathcal{C}[S^{-1}]$. Thus, according to (1.1),

$$\hat{S} = \mathcal{S}(F_S). \tag{1.2}$$

The class S is said to be *externally saturated* if $S = \hat{S}$. We find that this language is justified by the fact that this kind of saturation is not intrinsic in the category $\mathcal{C}$, as internal saturation is.

The universal property of the category of fractions implies the following, which was already pointed out in [4, Proposition 1.1].

Proposition 1.1 *A class of morphisms S is externally saturated if and only if $S = \mathcal{S}(F)$ for some functor F.* $\square$

For any functor $K: \mathcal{A} \to \mathcal{C}$ between two given categories, the *shape category* $\mathbf{Sh}_K$ of K has the same objects as $\mathcal{C}$, and morphisms in $\mathbf{Sh}_K$ from X to Y are natural transformations

$$\mathcal{C}(Y, K-) \longrightarrow \mathcal{C}(X, K-).$$

There is a canonical functor $D_K: \mathcal{C} \to \mathbf{Sh}_K$ which is the identity on objects and is defined as $D_K f = \mathcal{C}(f, K-)$ on morphisms. Additional information about shape categories can be found e.g. in [6], [7]. This concept is relevant in our context, since $S^{\perp\perp}$ is precisely the inverse image under D_K of the invertible morphisms in $\mathbf{Sh}_K$ when K is the full embedding of $S^{\perp}$ into $\mathcal{C}$. In other words, using the notation (1.1),

$$S^{\perp\perp} = \mathcal{S}(D_K), \tag{1.3}$$

where $K: S^{\perp} \hookrightarrow \mathcal{C}$ is the full embedding. (By a standard abuse of terminology, we often denote by the same symbol a class of objects and the full subcategory with these objects.)

2 Comparing internal and external saturation

Let $\mathcal{C}$ be any category.

Theorem 2.1 *If a class S of morphisms in $\mathcal{C}$ is internally saturated, then it is externally saturated.*

Proof. If S is internally saturated in $\mathcal{C}$, then $S = S^{\perp\perp} = \mathcal{S}(D_K)$, as pointed out in (1.3). Hence, by Proposition 1.1, S is externally saturated. □

The converse of Theorem 2.1 does not hold, as the following two examples illustrate.

Example 2.2 Let $\mathcal{A}$ be the category of Abelian groups and $T: \mathcal{A} \to \mathcal{A}$ the functor taking each object of $\mathcal{A}$ to its torsion subgroup. Then $\mathcal{S}(T)$ is externally saturated but not internally saturated. Indeed, consider the zero morphism $z: \mathbf{Z} \to 0$, which satisfies $\{z\}^{\perp} = \{0\}$ and therefore $\{z\}^{\perp\perp}$ is the class of all morphisms in $\mathcal{A}$. Since z is in $\mathcal{S}(T)$, we have $\{z\}^{\perp\perp} \subseteq \mathcal{S}(T)^{\perp\perp}$, so $\mathcal{S}(T)^{\perp\perp}$ is the class of all morphisms in $\mathcal{A}$, which is strictly larger than $\mathcal{S}(T)$.

Example 2.3 Let $\mathcal{C}$ be the multiplicative monoid of the integers $\mathbf{Z}$, viewed as a category with a single object. Let p be any prime, and Let $\mathbf{Z}_{(p)}$ denote the multiplicative monoid of the rationals whose denominator is not divisible by p in their reduced form. Then $\mathbf{Z}_{(p)}$ is isomorphic to the category of fractions $\mathcal{C}[S^{-1}]$, where S is the set of integers not divisible by p. Thus, S is externally saturated. However, S is not internally saturated. In fact, any category with a single object has only two internally saturated classes of morphisms; namely, the class of all morphisms and the class of the invertible morphisms.

Although they do not coincide in general, the two saturations are related as follows. This result implies of course Theorem 2.1.

Theorem 2.4 *Let S be any class of morphisms in a category $\mathcal{C}$. Then*

(a) $S \subseteq \hat{S} \subseteq S^{\perp\perp}$;

(b) $(\hat{S})^{\perp} = S^{\perp}$.

Proof. Let K denote the full embedding $S^{\perp} \hookrightarrow \mathcal{C}$. The canonical functor $D_K\colon \mathcal{C} \to \mathbf{Sh}_K$ renders all the morphisms in S invertible, and hence it factors through the canonical functor $F_S\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$. This implies that $\mathcal{S}(F_S) \subseteq \mathcal{S}(D_K)$, so part (a) follows from (1.2) and (1.3). Then we also obtain

$$S^{\perp\perp\perp} \subseteq (\hat{S})^{\perp} \subseteq S^{\perp},$$

and, since $S^{\perp\perp\perp} = S^{\perp}$, we infer (b). $\square$

In some cases, the two concepts of saturation coincide. The following situation is especially relevant.

Theorem 2.5 *If a functor F has a right adjoint, then $\mathcal{S}(F)$ is both internally and externally saturated.*

Proof. If G is right adjoint to F, then [2, Lemma 1.2] and [2, Theorem 1.3] say that $\mathcal{S}(F) = \mathcal{S}(GF) = \mathcal{D}(G)^{\perp}$, where $\mathcal{D}(G)$ denotes the class of objects which are isomorphic to GX for some X. Hence, $\mathcal{S}(F)$ is internally saturated, and it is also externally saturated by Proposition 1.1. $\square$

Recall from [10, Ch. VI] that, if (T, η, μ) is any monad (also called a triple), then $T = GF$ for some pair of adjoint functors G, F, which are not uniquely determined in general. By [2, Theorem 1.3], we then have $\mathcal{S}(T) = \mathcal{S}(F)$. Therefore, we obtain the following.

Corollary 2.6 *If* (T, η, μ) *is any monad, then* $\mathcal{S}(T)$ *is both internally and externally saturated.* $\square$

This applies e.g. to the case when F is a *localization*; that is, $F: \mathcal{C} \to \mathcal{D}$ is left adjoint to the embedding K of some full subcategory $\mathcal{D}$ into $\mathcal{C}$. The functor $T = KF$ is then part of an idempotent monad. The class $\mathcal{S}(F)$ admits a calculus of left fractions and the canonical functor from $\mathcal{C}$ to $\mathcal{C}[\mathcal{S}(F)^{-1}]$ has a right adjoint. Moreover, the category of fractions $\mathcal{C}[\mathcal{S}(F)^{-1}]$, the shape category $\mathbf{Sh}_K$, and the Kleisli category of KF are isomorphic, and they are equivalent to $\mathcal{D}$; see [5, § 2], [6, Corollary 2.3], and [9, I.4].

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Universitat Autònoma de Barcelona, E–08193 Bellaterra, Spain
`casac@mat.uab.es`

Università della Svizzera Italiana, Via Ospedale 13, CH–6900 Lugano, Switzerland
`Armin.Frei@lu.unisi.ch`