Math. Proc. Camb. Phil. Soc. (1999), **126**, 521

Printed in the United Kingdom

Operations in equivariant $\mathbb{Z}/p$-cohomology

By JEFFREY L. CARUSO

Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, 16 *Mill Lane, Cambridge CB*2 1*SB*
e-mail: `jlc@dpmms.cam.ac.uk`

(*Received* 16 *January* 1998; *revised* 19 *August* 1998)

Abstract

If G is a compact Lie group and $\mathbf{M}$ a Mackey functor, then Lewis, May and McClure [**4**] define an ordinary cohomology theory $H^*_G(-;\mathbf{M})$ on G-spaces, graded by representations. In this article, we compute the $\mathbb{Z}/p$-rank of the algebra of integer-degree stable operations $\mathscr{A}_{\mathbf{M}}$, in the case where $G = \mathbb{Z}/p$ and $\mathbf{M}$ is constant at $\mathbb{Z}/p$. We also examine the relationship between $\mathscr{A}_{\mathbf{M}}$ and the ordinary mod-p Steenrod algebra $\mathscr{A}_p$.

The main result implies that while $\mathscr{A}_{\mathbf{M}}$ is quite large, its image in $\mathscr{A}_p$ consists of only the identity and the Bockstein. This is in sharp contrast to the case with $\mathbf{M}$ constant at $\mathbb{Z}/p$ for $q \neq p$; there $\mathscr{A}_{\mathbf{M}} \cong \mathscr{A}_q$.

1. *The exact triangle for* $\mathscr{A}_{\mathbb{Z}/p}$

In their article [**4**] and in their book [**5**] (with Steinberger) Lewis, May and McClure promoted the point of view that equivariant cohomology theories and their representing spectra ought to be graded not on integers, but on representations of the ambient group. One may define an equivariant cohomology theory as a collection of G-homotopy-invariant functors $\{k_G^{m+V}(X)\}$ on pointed G-spaces, one for each integer m and each fixed-point-free (virtual) representation V, together with a coboundary homomorphism $\delta\colon k_G^{m+V} \to k_G^{m+1+V}$ satisfying long exact sequences; and suspension isomorphisms $\sigma_W : k_G^{m+V}(X) \to k_G^{m+V+W}(\Sigma^W X)$, where $\Sigma^W X = X \wedge S^W = X \wedge (W \cup \{\infty\})$. Thus for fixed V, the groups $k_G^{m+V}(X)$ form a $\mathbb{Z}$-graded equivariant cohomology theory in the sense of Bredon [**1**] and these various theories are knitted together via the suspension isomorphisms.

An *ordinary* equivariant cohomology theory is one satisfying Bredon's dimension axiom in integer degrees. That is, setting $V = 0$, we require $H_G^m(G/H) = 0$ for all $m \neq 0$ and all subgroups $H \leqslant G$.

To each G-map $f\colon G/H \to G/K$ is associated a morphism $f^*\colon H_G^0(G/K) \to H_G^0(G/H)$. Unlike Bredon cohomology, however, there is a natural transfer map $f_!\colon H_G^0(G/H) \to H_G^0(G/K)$ formed by embedding G/H in a large representation W and forming a Pontryagin–Thom construction, then using the suspension isomorphism σ_W. The various groups and maps $\{H_G^0(T), f^*, f_!\}$, with T ranging over orbits G/H, form an example of a *Mackey functor*.

For finite G, a Mackey functor can be defined as a pair of functors M^*, M_* from the category $\mathscr{F}_G$ of finite G-sets and G-maps to abelian groups, which

(*a*) agree on objects,
(*b*) carry disjoint unions to dirct sums and
(*c*) carry pullback squares

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\ Z \arrow[r, "i"'] & W \end{tikzcd}$$

to commutative squares

$$\begin{tikzcd} MX \arrow[r, "f_!"] & MY \\ MZ \arrow[u, "g^*"] \arrow[r, "i_!"'] & MW \arrow[u, "h^*"'] \end{tikzcd}$$

Any Mackey functor M gives rise to an ordinary cohomology theory $H^*_G(-;M)$ (uniquely on G-complexes and their equivalents) (see [**4**]).

Example 1. Let A be an abelian group and $\underline{A}$ the constant Mackey functor sending any orbit G/H to A, with f^* the identity and $f_!$ multiplication by the Euler characteristic of the fibre of f. Then $H^m_G(X;\underline{A}) = H^m(X/G;A)$ in integer degrees m. ($H^{m+V}_G(X;\underline{A})$ is more complicated for $V \neq 0$.)

Example 2. Let ΛA be the Mackey functor sending G/G to A and G/H to 0 for $H \neq G$, with f^* and $f_!$ both zero for all morphisms except the identity on G/G. Then $H^{m+V}_G(X;\Lambda A) = H^m(X^G;A)$ for all m and V.

Let $H\mathbb{Z}_p$ denote the representing spectrum of the ordinary theory associated to the constant Mackey functor $\underline{\mathbb{Z}/p}$. Thus $H\mathbb{Z}_p(m+V) = K(\mathbb{Z}_p, m+V)$ is an 'equivariant Eilenberg–Mac Lane space'. Note that when $V = 0$ this is the classical Eilenberg–Mac Lane space $K(\mathbb{Z}/p, m)$ with trivial G-action, so there is no real conflict of notation.

Let $\mathscr{A}_{\mathbb{Z}/p}$ denote the graded group $\{\tilde{H}^q_G(H\mathbb{Z}_p;\underline{\mathbb{Z}/p})\}_{q\in\mathbb{Z}}$. For an integer q, a G-spectrum map $\Sigma^q H\mathbb{Z}_p \to H\mathbb{Z}_p$ restricts on integer indices to a map of ordinary nonequivariant Eilenberg–Mac Lane spectra. Thus there is a homomorphism

$$\Omega\colon \mathscr{A}_{\mathbb{Z}/p} \to \mathscr{A},$$

where $\mathscr{A}$ is the (nonequivariant) mod-p Steenrod algebra.

Our main theorem is:

Theorem 3. *Let G be the cyclic group of prime order p.*

(*a*) *There is an exact triangle*

$$\begin{tikzcd} \mathscr{A}_{\mathbb{Z}/p} \arrow[rr, "\Omega"] & & \mathscr{A} \arrow[dl] \\ & \mathscr{A} \otimes \left(\bigoplus_{i=1}^{\infty} \tilde{H}^*(\Sigma^i B\mathbb{Z}/p;\mathbb{Z}/p) \right) \arrow[ul, "(+1)"] & \end{tikzcd}$$

where $B\mathbb{Z}/p$ is the usual classifying space of $\mathbb{Z}/p$.

(*b*) *The image of* Ω *is the graded* $\mathbb{Z}/p$*-vector subspace with basis consisting just of the identity operation in degree* 0 *and the Bockstein operation in degree* 1.

The structure of the proof is as follows. We give the fibre sequence leading to the proof of part (*a*) in Section 2, as well as statements of results necessary to understand the spaces involved and to generate the exact sequence. We give the proofs of these statements in Section 3, together with some fundamental lemmas on the fixed-point structure of $K(\mathbb{Z}_p, m+V)$.

The proofs in Section 3 are elementary; however, it is worth examining some of the concepts from the stable point of view. In Section 4 we discuss the relation of the proofs in Section 3 with the ideas of naive G-spectra and complete G-spectra.

In Section 5 we construct another fibre sequence involving $K(\mathbb{Z}_p, m+V)$ and in Section 6 we study the associated exact sequence. Part (*b*) of the main theorem then follows from a dimensional comparison of exact sequences.

This is a subject in which even the cohomology of a point cannot be taken for granted. The original computation for $\underline{R}$ coefficients (with R a ring) is due to Stong [**7**] and Waner [**8**]. This is reproduced in the Appendix, together with the cohomology of EG and $\tilde{E}G$.

The author is indebted to Peter May for his encouragement and for suggesting this line of inquiry and to Steve Costenoble, John Greenlees, Gaunce Lewis and Stefan Waner for enlightening conversations on the subject. The author also wishes to thank the referee for his valuable suggestions in rewriting part of the paper.

From now on the ambient group G will always be $\mathbb{Z}/p$ and all cohomology (equivariant and non-) will be with $\mathbb{Z}/p$ (or $\underline{\mathbb{Z}/p}$) coefficients unless otherwise indicated.

2. *G-homotopy of representing spaces*

To prove Theorem 3 we compute $\tilde{H}_G^{m+q+V}(K(\mathbb{Z}_p, m+V))$ in a stable range as m and V grow large and show that a similar exact triangle holds in this range for these groups. This will allow us to conclude that

$$(\mathscr{A}_{\mathbb{Z}/p})^q \cong \lim_{\substack{\longleftarrow \\ m+V}} [K(\mathbb{Z}_p, m+V), K(\mathbb{Z}_p, m+q+V)]_G$$

$$\cong \lim_{\substack{\longleftarrow \\ m+V}} \tilde{H}_G^{m+q+V}(K(\mathbb{Z}_p, m+V))$$

and that the triangle passes to the stable level.

The proof of this will follow from a series of lemmas, whose proof will be deferred to the end of this section.

LEMMA 4. *Let* $K\Lambda_m$ *be the representing* G*-space for the cohomology group* $H_G^m(-;\Lambda\mathbb{Z}/p)$ *of Example* 2. *If* V *is a fixed-point-free representation then*:

(*a*) $(K\Lambda_m)^e$ *is contractible*;

(*b*) $(K\Lambda_m)^G \simeq K(\mathbb{Z}_p, m)$;

(*c*) $K\Lambda_m/G \simeq \Sigma BG \wedge K(\mathbb{Z}_p, m)$;

(*d*) $\Sigma^V K\Lambda_m \simeq K\Lambda_m$;

(*e*) $K\Lambda_{m+V} \simeq K\Lambda_m \simeq \Omega^V K\Lambda_m$.

Since $K(\mathbb{Z}_p, m+V)$ is a space in a G-spectrum, there is a structure map

$$s_V : \Sigma^V K(\mathbb{Z}_p, m) \to K(\mathbb{Z}_p, m+V)$$

whose adjoint is a G-homeomorphism. Using the unit map $\eta : 1 \to \Omega^V\Sigma^V$ of the adjunction between Ω^V and Σ^V, it is easy to see that the map Ω above may be interpreted as the composite

$$\tilde{H}_G^{m+q+V}(K(\mathbb{Z}_p, m+V)) \xrightarrow{s_V^*} \tilde{H}_G^{m+q+V}(\Sigma^V K(\mathbb{Z}_p, m))$$
$$\underset{\cong}{\xrightarrow{\sigma_V}} \tilde{H}_G^{m+q}(K(\mathbb{Z}_p, m)) \underset{\cong}{\xrightarrow{u}} \tilde{H}^{m+q}(K(\mathbb{Z}_p, m))$$

where σ_V is the suspension isomorphism and u is the isomorphism of Example 1.

Proposition 5. *Let V be a fixed-point-free representation of dimenion n. The G-homotopy fibre Fs_V of s_V is G-$(2m+n)$-equivalent to the product $K\Lambda_m \times \cdots \times K\Lambda_{m+n-1}$.*

Part (a) of Theorem 3 will follow upon examination of the exact sequence of the associated fixed-point spaces of the G-$(2m+n)$-fibreing

$$K\Lambda_m \times \cdots \times K\Lambda_{m+n-1} \to \Sigma^V K(\mathbb{Z}_p, m) \to K(\mathbb{Z}_p, m+V).$$

For this, we need the following two lemmas.

Lemma 6. *Let $f : X \to Y$ be a G-q-equivalence, $q > 0$, and let V be a fixed-point-free representation with $\dim V = n$. Then the induced homomorphism $f^* : \tilde{H}_G^{i+V}(Y; \underline{\mathbb{Z}/p}) \to \tilde{H}_G^{i+V}(X; \underline{\mathbb{Z}/p})$ is an isomorphism for $i \leqslant (q-n)$.*

Lemma 7. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a G-fibreing. Any G-homotopy h from $p \circ i$ to a constant map induces a map $i(h)$ from the G-mapping cone Ci of i to B. If B is G-$(s-1)$-connected with $s > 2$ and F is G-$(t-1)$-connected, then $i(h)$ is a G-$(s+t-1)$-equivalence.*

Proof of Theorem $3(a)$. For sufficiently large m,

$$\begin{aligned}
\tilde{H}_G^{m+q+V}(Fs_V) &\cong \tilde{H}_G^{m+q+V}(K\Lambda_m \times \cdots \times K\Lambda_{m+n-1}) \\
&\cong \tilde{H}^{m+q}((K\Lambda_m \times \cdots \times K\Lambda_{m+n-1})/G) \\
&\cong \bigoplus_{i=0}^{n-1} \tilde{H}^{m+q}(K(\mathbb{Z}_p, m+i) \wedge \Sigma BG),
\end{aligned}$$

by Proposition 5, Lemmas 6, $4(b)$ and $4(c)$ and Example 1. Thus there is a long exact sequence

$$\cdots \to \tilde{H}_G^{m+q+V}(Ci) \xrightarrow{i'^*} \tilde{H}_G^{m+q+V}(\Sigma^V K(\mathbb{Z}_p, m))$$
$$\to \bigoplus_{i=0}^{n-1} \tilde{H}^{m+q}(K(\mathbb{Z}_p, m+i) \wedge \Sigma BG) \to \cdots, \quad (1)$$

where i' is the induced map from $\Sigma^V K(\mathbb{Z}_p, m)$ to the cofibre of $i : Fs_V \to \Sigma^V K(\mathbb{Z}_p, m)$.

One checks that Fs_V and $\Sigma^V K(\mathbb{Z}_p, m)$ are G-$(m-1)$-connected, so perforce $K(\mathbb{Z}_p, m+V)$ is as well. It follows by Lemma 7 that in the commutative diagram

$$\begin{tikzcd}
\tilde{H}_G^{m+q+V}(Ci) \arrow[r, "i'^*"] & \tilde{H}_G^{m+q+V}(\Sigma^V K(\mathbb{Z}_p, m)) \\
\tilde{H}_G^{m+q+V}(K(\mathbb{Z}_p, m+V)) \arrow[u, "i(h)^*"] \arrow[r, "\Omega"] & \tilde{H}^{m+q}(K(\mathbb{Z}_p, m)), \arrow[u, "\cong"', "\sigma_V"]
\end{tikzcd}$$

$i(h)$ is a G-$(2m-1)$-equivalence, hence $i(h)^*$ is an isomorphism for $q < m-n$. Thus in (1) we may replace i'^* by Ω and part (a) of the theorem follows by stabilizing with respect to m and V. $\square$

Remark 8. Actually more has been proven. Denote by $\mathscr{A}^*_{\mathbb{Z}/p}$ the whole $RO(G)$-graded algebra of cohomology operations on $H\mathbb{Z}_p$. Let pt denote the one-point space with trivial G-action and $\tilde{E}G$ the unreduced double cone on EG with G acting trivially on the cone points. Then there is an exact triangle

$$\begin{tikzcd}
\mathscr{A}^*_{\mathbb{Z}/p} \arrow[rr, "\Omega^*"] & & \mathscr{A} \otimes H_G^*(\mathrm{pt}) \arrow[dl, "\gamma^*"] \\
& \mathscr{A} \otimes \Big(\bigoplus\limits_{i=1}^{\infty} \Sigma^i \tilde{H}_G^*(\tilde{E}G) \Big) \arrow[ul] &
\end{tikzcd}$$

in which the objects are all $RO(G)$-graded and the unlabelled arrow has degree $+1$.

Here, if X is a $\mathbb{Z}$-graded object and Y is an $RO(G)$-graded object, we denote by $XFCPi\,Y$ the $RO(G)$-graded object whose $(m+V)_{\mathrm{th}}$ constituent is given by

$$(X \otimes Y)_{m+V} = \bigoplus_{i+j=m} (X_i \otimes Y_{j+V}).$$

In this context Ω^* is a ring homomorphism and γ^* is a morphism of $H_G^*(\mathrm{pt})$-modules.

Proof of Lemma 4. (a) First note that for any fixed-point-free representation V of dimension n,

$$H_G^{m+V}(G; \Lambda\mathbb{Z}/p) \cong H_G^{m+n}(G; \Lambda\mathbb{Z}/p)$$

(since $S^V \wedge G_+ \approx S^n \wedge G_+$) and the latter is identically zero by Example 2.

Thus

$$\begin{aligned}
\pi_i((K\Lambda_{m+V})^e) &\cong [S^i, (K\Lambda_{m+V})^e] \\
&\cong [S^i \wedge G_+, K\Lambda_{m+V}]_G \\
&\cong H_G^{m-i+V}(G; \Lambda\mathbb{Z}/p) \\
&\cong 0, \quad \text{for all } i.
\end{aligned}$$

(b) Similarly note that $\pi_i((K\Lambda_m)^G) \cong H_G^{m-i}(\mathrm{pt}; \Lambda\mathbb{Z}/p)$ is $\mathbb{Z}/p$ if $m = i$ and 0 otherwise.

(c) Let EG be a contractible space on which G acts freely. The unique map $EG \to \mathrm{pt}$ induces a cofibre sequence

$$EG_+ \to S^0 \to \tilde{E}G.$$

For $G = \mathbb{Z}/p$ one may construct an explicit model for $\tilde{E}G$ by letting ξ be a nontrivial fixed-point-free representation and letting

$$S^{n\xi} = S^{\xi} \wedge \overset{\leftarrow n \rightarrow}{\cdots} \wedge S^{\xi}. \tag{2}$$

Then one sees that

$$\tilde{E}G \simeq \varinjlim S^{n\xi}. \tag{3}$$

Since, obviously, $\tilde{E}G/G \simeq \Sigma BG$, part (c) follows if we show that $K\Lambda_m \simeq \tilde{E}G \wedge K(\mathbb{Z}_p, m)$.

For any G-space X, let Φ be the composite isomorphism

$$[X, K\Lambda_m]_G \cong \tilde{H}^m_G(X; \Lambda\mathbb{Z}/p) \cong \tilde{H}^m(X^G) \cong [X^G, K(\mathbb{Z}_p, m)], \tag{4}$$

so that $\Phi(f) = f^G$. Thus there exists a G-map $f\colon \tilde{E}G \wedge K(\mathbb{Z}_p, m) \to K\Lambda_m$ such that $f^G = \Phi(f)$ is the identity on $K(\mathbb{Z}_p, m) = (\tilde{E}G \wedge K(\mathbb{Z}_p, m))^G$.

Since $(\tilde{E}G \wedge K(\mathbb{Z}_p, m))^e$ is contractible, f is a homotopy equivalence on fixed-point sets and thus is an equivalence by the G-Whitehead theorem.

(d) This follows at once from (3) since the inclusion $\tilde{E}G \wedge S^0 \to \tilde{E}G \wedge S^{\xi}$ is a G-equivalence.

(e) Consider the G-cofibre sequence

$$S(V)_+ \xrightarrow{\subseteq} D(V)_+ \to S^V, \tag{5}$$

where $S(V)$ is the unit sphere in V, $D(V)$ is the unit disc and $S^V \approx D(V)/S(V)$ is the 1-point compactification of V with G acting trivially on the point at infinity. This dualizes to a G-fibre sequence

$$\Omega^V X \xrightarrow{\epsilon_0} X \to \operatorname{Map}(S(V)_+, X). \tag{6}$$

Taking $X = K\Lambda_{m+V}$ gives us

$$K\Lambda_m \to K\Lambda_{m+V} \to M = \operatorname{Map}(S(V)_+, K\Lambda_{m+V});$$

and, since $S(V)$ is a free G-space, one computes that

$$\pi_i(M^G) \cong H^{m-i+V}_G(S(V); \Lambda\mathbb{Z}/p) = 0$$

for all i. Of course it is also true that

$$M^e = \operatorname{Map}((S(V)_+)^e, \quad (K\Lambda_{m+V})^e)$$

is contractible, so M is G-contractible. Thus ϵ_0 is a G-equivalence. The rest of the statement follows identically with $X = K\Lambda_m$.

3. *Structure of* $K(\mathbb{Z}_p, m+V)$

Lemma 9. *Let V be as above with* $\dim V = n > 0$.

(a)

$$H^{m+V}_G(G; \underline{\mathbb{Z}/p}) = \begin{cases} \mathbb{Z}/p & \text{if } m = -n, \\ 0 & \text{if } m \neq -n. \end{cases}$$

(*b*)

$$H_G^{m+V}(\mathrm{pt}; \underline{\mathbb{Z}/p}) = \begin{cases} \mathbb{Z}/p & \textit{if } -n \leqslant m \leqslant 0, \\ 0 & \textit{otherwise}. \end{cases}$$

(*c*) *The map* $G \to \mathrm{pt}$ *induces an isomorphism in cohomology in dimensions* $m = -n < 0$.

Note that (*c*) does not hold for 'virtual' representations V with $n < 0$.

This lemma parallels an unpublished calculation of Stong [**7**] and is reproduced in the Appendix. See also Lewis [**3**], Waner [**8**] and Costenoble [**2**].

Corollary 10. (*a*) $K(\mathbb{Z}_p, m+V)^e \simeq K(\mathbb{Z}_p, m+n)$

(*b*) $K(\mathbb{Z}_p, m+V)^G \simeq K(\mathbb{Z}_p, m) \times \cdots \times K(\mathbb{Z}_p, m+n)$

(*c*) *There is a commutative diagram*

$$\begin{tikzcd} K(\mathbb{Z}_p, m+V)^G \arrow[r, "\simeq"] \arrow[d, "\subseteq"'] & K(\mathbb{Z}_p, m) \times \cdots \times K(\mathbb{Z}_p, m+n) \arrow[d, "\mathrm{pr}_n"] \\ K(\mathbb{Z}_p, m+V)^e \arrow[r, "\simeq"'] & K(\mathbb{Z}_p, m+n) \end{tikzcd}$$

where pr_n *is projection on the nth factor*.

Proof. (*a*) This follows at once from Lemma 9 by examining $\pi_i(K(\mathbb{Z}_p, m+V)^e) \cong H_G^{m-i+V}(G)$.

(*b*) and (*c*) Consider the G-fibre sequence (6) with $X = K(\mathbb{Z}_p, m+k\xi)$ and with V replaced by ξ, where ξ is an irreducible representation and k is a positive integer. Then

$$K(\mathbb{Z}_p, m+(k-1)\xi)^G \xrightarrow{(\epsilon_0)^G} K(\mathbb{Z}_p, m+k\xi)^G \xrightarrow{p^G} \mathrm{Map}\,(S(\xi)_+, K(\mathbb{Z}_p, m+k\xi))^G$$

is a non-equivariant fibre sequence.

If $p = 2$, then $S(\xi) \approx G/e, n = k$ and the last term is just $K(\mathbb{Z}_2, m+n\xi)^e = K(\mathbb{Z}_2, m+n)$. Inductively assume the conclusion for $(n-1)\xi$. By starting with m one higher and looping everything, we see that $K(\mathbb{Z}_2, m+n\xi)^G$ is the homotopy fibre of the connecting map

$$K(\mathbb{Z}_2, m+n) \to K(\mathbb{Z}_2, m+1) \times \cdots \times K(\mathbb{Z}_2, m+n).$$

This can only be 0 or $(0, \ldots, 0, 1)$; but it cannot be the latter, for then we would get the wrong answer (compared with Lemma 9(*b*)) for

$$\pi_i(K(\mathbb{Z}_2, m+V)^G) = H_G^{m-i+V}(\mathrm{pt}).$$

Thus the connecting map is zero and p^G is a product fibreing.

If $p > 2$, there are two complications: first $S(\xi)$ is not G/e and secondly V is not necessarily $k\xi$ for some integer k.

Let ξ be any nontrivial irreducible representation and suppose $2k = n$ is the dimension of V. In the appendix we shall see that there is a cohomology class $\alpha_{V-k\xi} \in H_G^{V-k\xi}(\mathrm{pt})$ such that multiplication by $\alpha_{V-k\xi}$ is an isomorphism of $H_G^*(\mathrm{pt})$ and of $H_G^*(G)$. This corresponds to a self-equivalence of the G-spectrum $H\mathbb{Z}_p$ which changes the dimension by $V-k\xi$ and this restricts to a G-equivalence of $K(\mathbb{Z}_p, m+V)$ to $K(\mathbb{Z}_p, m+k\xi)$. Thus it does suffice to consider the case $V = k\xi$.

Although $S(\xi)$ is not G/e, it is a free two-cell G-complex; that is, there is a G-cofibre sequence

$$\mathbb{Z}/p_+ \to S(\xi)_+ \to \Sigma(\mathbb{Z}/p_+).$$

Dualizing and passing to G-fixed sets we obtain the fibreing

$$(\Omega X)^e \to \mathrm{Map}\,(S(\xi)_+, X)^G \to X^e$$

and arguments similar to the ones used for the case $p=2$ show that for $X = K(\mathbb{Z}_p, m+k\xi)$, both this and (6) must be trivial fibreings.

Proof of Proposition 5. The homotopy-fibre construction is equivariant, so

$$(Fs_V)^G \to (\Sigma^V K(\mathbb{Z}_p, m))^G \to K(\mathbb{Z}_p, m+V)^G$$

is a fibre sequence. Since V is fixed-point-free, the middle term is just $K(\mathbb{Z}_p, m)$ and s_V^G corresponds to the map

$$\epsilon_0^G\colon K(\mathbb{Z}_p, m) = (\Omega^V K(\mathbb{Z}_p, m+V))^G \to K(\mathbb{Z}_p, m+V)^G$$

of (6) and, by the proof of Corollary 10, this is equivalent to the inclusion of the first factor in $K(\mathbb{Z}_p, m) \times \cdots \times K(\mathbb{Z}_p, m+n)$. Thus the homotopy fibre is

$$(Fs_V)^G \simeq K(\mathbb{Z}_p, m) \times \cdots \times K(\mathbb{Z}_p, m+n-1).$$

It follows from the isomorphism (4) that there is a G-map

$$f\colon Fs_V \to K\Lambda_m \times \cdots \times K\Lambda_{m+n-1}$$

such that f^G is homotopic to the identity. But note that the underlying set of the main G-fibre sequence is

$$(Fs_V)^e \to \Sigma^n K(\mathbb{Z}_p, m) \xrightarrow{s_V^e} K(\mathbb{Z}_p, m+n),$$

where s_V^e is the structure map for the non-equivariant Eilenberg–Mac Lane spectrum $K(\mathbb{Z}/p)$. Classically $(Fs_V)^e$ is a $(2m+n)$-connected space, so f^e is a $(2m+n)$-equivalence, and the result follows.

Proof of Lemma 6. This follows at once by applying the Bredon spectral sequence to the integer-graded theory $H_G^{*+V}(-;\underline{\mathbb{Z}/p})$. It says that there is a natural spectral sequence with

$$E_2^{s,t} \cong H_G^s(X; \mathbf{M}^{(t+V)}) \Rightarrow H_G^{s+t+V}(X; \underline{\mathbb{Z}/p}),$$

where $\mathbf{M}^{(t+V)}$ is the Mackey functor sending G/H to $H_G^{t+V}(G/H; \underline{\mathbb{Z}/p})$. Since f is a G-q-equivalence, $H_G^s(f)$ is an isomorphism for $s \leqslant q$, by Bredon (see [**1**]). Since $H_G^{t+V}(G/H) = 0$ for $t < -n$, the result follows by comparison of that part of the spectral sequences.

Proof of Lemma 7. This is true for general groups G. Let H be a subgroup of G. Then

$$F^H \to E^H \to B^H$$

is a fibre sequence. By hypothesis F^H is $(t-1)$-connected and B^H is $(s-1)$-connected, so by the classical Serre exact sequence $i(h^H): C(i^H) \to B^H$ is an $(s+t-1)$-equivalence. Since the cofibre construction is equivariant, this is the same as the map $i(h)^H: (Ci)^H \to B^H$ and the result follows.

4. *Complete G-spectra and naive G-spectra*

The results of the previous section are perhaps more naturally seen in the context of equivariant spectra. A G-spectrum as defined in [**5**] is a set of G-spaces, one for each subspace of a given G-inner-product space $\mathscr{U}$ of countable dimension. If $\mathscr{U}$ is $\mathbb{R}^\infty$ with trivial G-action, then we have what is known as a naive G-spectrum. If $\mathscr{U}$ contains countably many copies of each irreducible G-representation, then it is called a complete G-universe and the corresponding spectra are called complete G-spectra. Restriction to the trivial part of $\mathscr{U}$ gives us a functor from complete G-spectra to naive G-spectra $l: G\mathscr{S}\mathscr{U} \to G\mathscr{S}\mathscr{U}^G$. From now on fix $\mathscr{U}$ as a complete G-universe. Denote the functor l by $E \mapsto E[0]$ and define $E[V]$ to be $(\Sigma^V E)[0]$. (This is not to be confused with $E(V)$, the V-th space of the spectrum.)

Let E be a complete G-spectrum. Then the algebra of stable operations in equivariant E-cohomology is the set $\{E, E\}_G^*$ of G-homotopy classes of self-maps of E and, for any given integer q, $\{E, E\}_G^q \to \varprojlim_{m+V}[E(m+V), E(m+q+V)]_G$ is an isomorphism. But this is equivalent to $\varprojlim_V \varprojlim_m [E[V]_m, E[V]_{m+q}]_G \cong \varprojlim_V \{E[V], E[V]\}_G^q$. Thus Theorem 3($a$) follows by showing that there are compatible exact triangles

$$\begin{tikzcd}[column sep=small] \{E[V], E[V]\}_G^* \arrow[rr, "\Omega^V"] & & \{E[0], E[0]\}_G^* \arrow[dl] \\ & \{E[0], E[0]\}_G^* \otimes \left(\bigoplus_{i=1}^{\infty} \tilde{H}^*(\Sigma^i B\mathbb{Z}/p; \mathbb{Z}/p)\right) \arrow[ul, "(+1)"] & \end{tikzcd} \tag{7}$$

for all V, and taking inverse limits over V.

In this context the lemma corresponding to Lemma 6 is simply a matter of noting that as S^V is an n-dimensional CW-complex, its dual S^{-V} is perforce $(-n-1)$-connected and Lemma 7 is a complete triviality, since in the category of naive G-spectra a G-cofibreing is the same as a G-fibreing. Also, Proposition 5 could be stated as

Proposition 11. *Let V be a fixed-point-free representation of dimension n and let $s_V: H\mathbb{Z}_p[0] \wedge S^V \to H\mathbb{Z}_p[V]$ be the naive G-spectrum map induced by the structure map of the complete G-spectrum $H\mathbb{Z}_p$. Then the G-homotopy fibre s_V is G-equivalent to* $\bigvee_{i=0}^{n-1} \Sigma^i H\Lambda[0]$.

This is proved in the same way as Proposition 5 and (7) follows directly.

5. *A fibre sequence for $K(\mathbb{Z}_p, m+V)$*

Proposition 12. *There is a G-fibre sequence*

$$K(\mathbb{Z}_p, m+n) \xrightarrow{J_n} K(\mathbb{Z}_p, m+V) \xrightarrow{\pi_n} K\Lambda_m \times \cdots \times K\Lambda_{m+n-1}.$$

Proof. Again we apply (4). Let p be a map

$$K(\mathbb{Z}_p, m+V) \to K\Lambda_m \times \cdots \times K\Lambda_{m+n-1}$$

with p^G equivalent to projection on the first n factors. Then $(Fp)^G \simeq K(\mathbb{Z}_p, m+n)$. From Corollary 10($c$) and since $(Fp)^e \simeq K(\mathbb{Z}_p, m+V)^e \simeq K(\mathbb{Z}_p, m+n)$, we see that the inclusion $K(\mathbb{Z}_p, m+n) = (Fp)^G \hookrightarrow Fp$ is a G-equivalence. $\square$

The proof of Theorem 3(b) will proceed from a calculation of the cohomology of $K(\mathbb{Z}_p, m+V)$. This is done by studying the Serre exact sequence of the fibreing in Proposition 12. To be useful, this approach depends on being able to evaluate the connecting homomorphism

$$\delta^*\colon \tilde{H}_G^{*+V}(K(\mathbb{Z}_p, m+n)) \to \tilde{H}_G^{*+1+V}(K\Lambda_m \times \cdots \times K\Lambda_{m+n-1}).$$

This homomorphism is the composite of the inverse of the cohomology suspension

$$\Omega^1\colon \tilde{H}_G^{k+1+V}(K(\mathbb{Z}_p, m+n+1)) \to \tilde{H}_G^{k+V}(K(\mathbb{Z}_p, m+n))$$

and of the homomorphism induced by the homotopy fibre of

$$J_n\colon K(\mathbb{Z}_p, m+n+1) \to K(\mathbb{Z}_p, m+1+V),$$

which is a map

$$\tau_n\colon K\Lambda_m \times \cdots \times K\Lambda_{m+n-1} \to K(\mathbb{Z}_p, m+n+1).$$

We will characterize the homotopy class τ_n in terms of certain cohomology classes in $\tilde{H}_G^{m+n+1}(K\Lambda_m \times \cdots \times K\Lambda_{m+n-1})$. First we make note of the following standard fact, in which the graded tensor product follows the convention of Remark 8.

Lemma 13. *Let X be a G-CW-complex and Y a CW-complex with trivial G-action. If R is a field, then $\tilde{H}_G^*(X \wedge Y; \underline{R}) \cong \tilde{H}_G^*(X; \underline{R}) \otimes \tilde{H}^*(Y; R)$ as $\tilde{H}_G^*(X; \underline{R})$-modules.*

Proof. In integer degrees, this is true by Example 1 and the Künneth theorem. In non-integer degrees it follows by examining the Bredon spectral sequence of the integer-graded equivariant cohomology theory $\tilde{H}_G^*(- \wedge Y; \underline{R})$.

Corollary 14. (a) $\tilde{H}_G^*(K\Lambda_m) \cong \tilde{H}_G^*(\tilde{E}G) \otimes \tilde{H}^*(K(\mathbb{Z}_p, m))$ *as* $H_G^*(\mathrm{pt})$-*modules.*
(b) $\tilde{H}_G^*(K(\mathbb{Z}_p, m+n+1)) \cong \tilde{H}_G^*(\mathrm{pt}) \otimes \tilde{H}^*(K(\mathbb{Z}_p, m+n+1))$ *as* $H_G^*(\mathrm{pt})$-*modules.*

Note that the map J_n of Proposition 12 induces multiplication by the generator in $H_G^{V-n}(\mathrm{pt})$. For J_n is nonzero and $\tilde{H}_G^{m+V}(K(\mathbb{Z}_p, m+n))$ is

$$H_G^{V-n}(\mathrm{pt}) \otimes \tilde{H}^{m+n}(K(\mathbb{Z}_p, m+n)).$$

Thus J_n corresponds to a generator in $H_G^{V-n}(\mathrm{pt}) \cong \mathbb{Z}/p$, tensored with the identity class.

For what follows, let $\alpha \in H^1(B\mathbb{Z}/p)$ and $\beta \in H^2(B\mathbb{Z}/p)$ denote the usual generators (so that $\alpha^2 = \beta$ in the $p = 2$ case and $\alpha^2 = 0$ in the $p > 2$ case) and also their suspensions in $\tilde{H}^*(\Sigma B\mathbb{Z}/p)$. We will abbreviate $K(\mathbb{Z}_p, m+V)$ by K_{m+V} and let $\iota = \iota_m \in \tilde{H}^m(K_m)$ denote the identity class.

Lemma 15. *As usual, let V be an n-dimensional fixed-point-free representation. Let $i_q\colon K\Lambda_{m+q} \to K\Lambda_m \times \cdots \times K\Lambda_{m+n-1}$ be the inclusion as the qth factor. If $(n-q) = 2r+\epsilon$ (with $\epsilon = 0$ or 1), then the homotopy class $\{\tau_n \circ i_q\}$ corresponds to the cohomology class $\alpha^\epsilon\beta^r \times \iota$, where $\times$ denotes the cohomology cross product in $\tilde{H}^*(\Sigma BG \wedge K(\mathbb{Z}_p, m+q))$.*

Proof. Suppose $\tau_n \circ i_q \simeq 0$. Then $\mathrm{pr}_q \circ \pi_n$ is a product fibreing and $K\Lambda_{m+q}$ splits off

$K(\mathbb{Z}_p, m+V)$ as a factor. This would imply that $\Omega^V K\Lambda_{m+q} = K\Lambda_{m+q}$ splits off of $\Omega^V K_{m+V} = K_m$ as a factor, which is impossible. Thus $\{\tau_n \circ i_q\} \neq 0$.

Restrict to the case $p = 2$. Let $\eta\colon \Sigma^{(n-q)\xi} K_{m+q} \to K\Lambda_{m+q}$ be induced by the inclusion $S^{(n-q)\xi} \hookrightarrow S^{\infty\xi} \simeq \tilde{E}G$ (by (3)). Then $\tau_n i_q \eta \simeq 0$ if and only if we can fill in the diagram

$$\begin{tikzcd}
\Sigma^{(n-q)\xi} K_{m+q} \arrow[r, dashed] \arrow[d, "\eta"'] & K_{m+n\xi} \arrow[d, "\pi_n"] \\
K\Lambda_{m+q} \arrow[r] & K\Lambda_m \times \cdots \times K\Lambda_{m+n-1}
\end{tikzcd}$$

or, equivalently, the adjoint diagram

$$\begin{tikzcd}
K_{m+q} \arrow[r, dashed] \arrow[d, "\eta'"'] & K_{m+q\xi} \arrow[d, "\Omega^{(n-q)\xi}\pi_n"] \\
\Omega^{(n-q)\xi} K\Lambda_{m+q} \arrow[r] \arrow[d, "\omega'"', "\simeq"] & \Omega^{(n-q)\xi}(K\Lambda_m \times \cdots \times K\Lambda_{m+n-1}) \arrow[d, "\simeq"', "\omega'"] \\
K\Lambda_{m+q} \arrow[r, "i_q"'] & K\Lambda_m \times \cdots \times K\Lambda_{m+n-1}.
\end{tikzcd}$$

Here η' is nonzero so $\omega'\eta'$ must be G-homotopic to the inclusion $K(\mathbb{Z}/2, m+q) = (K\Lambda_{m+q})^G \hookrightarrow K\Lambda_{m+q}$. Thus the G-fixed-point diagram is

$$\begin{tikzcd}
K_{m+q} \arrow[r, dashed] \arrow[d, "\approx"'] & K_m \times \cdots \times K_{m+q} \arrow[d, "\subseteq"] \\
K_{m+q} \arrow[r] & K_m \times \cdots \times K_{m+n-1}.
\end{tikzcd}$$

It is clear that $(\eta')^G$ lifts for $q < n$, so η' does also; thus $\tau_n i_q \eta \simeq 0$.

We have shown that $\{\tau_n \circ i_q\}$ is a nonzero element in the kernel of the map

$$\eta^*\colon \tilde{H}_G^{m+n+1}(K\Lambda_{m+q}) \to \tilde{H}_G^{m+n+1}(\Sigma^{(n-q)\xi} K_{m+q}).$$

One sees that η^* sends $\alpha^j \times \sigma$ to 0 if and only if $j \geqslant n-q$. Thus the only nonzero element of degree $(m+n)$ in the kernel of this map is $\alpha^{n-q} \times \iota_{m+q}$, which corresponds to the statement in the lemma for $p = 2$.

If $p > 2$, it suffices as in Corollary 10 to consider the case $V = k\xi$, for some irreducible representation ξ.

If $\epsilon = 0$, then one must consider the map

$$\eta\colon \Sigma^{r\xi} K(\mathbb{Z}_p, m+q) \to K\Lambda_{m+q}$$

and the proof is formally the same as the case for $p = 2$.

If $\epsilon = 1$, then let T be the G-mapping cone of the map $\mathbb{Z}/p \to \mathrm{pt}$, considered as the one-skeleton of S^ξ. T consists of two fixed points connected by p line segments which are permuted cyclically by $\mathbb{Z}/p$. The map in question is

$$\eta\colon T \wedge \Sigma^{r\xi} K(\mathbb{Z}_p, m+q) \to K\Lambda_{m+q}$$

and, if $n = 2k$, the fixed-point diagram is

$$\begin{array}{ccc} K(\mathbb{Z}_p, m+q) & \dashrightarrow & \mathrm{Map}\,(T, K(\mathbb{Z}_p, m+(k-r)\xi))^G \\ {\scriptstyle \eta'}\downarrow{\scriptstyle \approx} & & \downarrow \\ K(\mathbb{Z}_p, m+q) & \longrightarrow & K(\mathbb{Z}_p, m) \times \cdots \times K(\mathbb{Z}_p, m+n-1) \end{array}$$

since $\mathrm{Map}\,(T, K\Lambda_{m+i}) \simeq K\Lambda_{m+i}$ by essentially the same argument as in Lemma 4(d). Similarly one sees that

$$\mathrm{Map}\,(T, K(\mathbb{Z}_p, m+(k-r)\,\xi))^G \simeq K(\mathbb{Z}_p, m) \times \cdots \times K(\mathbb{Z}_p, m+n-2r-1)$$

and the vertical arrow is the inclusion of the first $n-2r$ factors. Thus it is clear that η' lifts here also.

6. *The connecting map* τ_n^*

We are now in a position to evaluate τ_n^* and thus δ^*. Recall from Corollary 14(b) that in the stable range $\tilde{H}_G^*(K(\mathbb{Z}_p, m+n+1))$ is generated by classes $a\theta(\iota)$, where $a \in H_G^*(\mathrm{pt})$ and $\theta \in \mathscr{A}$. Considering such a class as a map $K(\mathbb{Z}_p, m+n+1) \to K(\mathbb{Z}_p, q+V)$ and thus as a cohomology operation $H_G^{m+n+1}(-) \to H_G^{q+V}(-)$ we have the following lemma.

Lemma 16.

$$i^*_{n-2r-\epsilon}\tau_n^*(a\theta(\iota)) = a \cdot \theta(\alpha^\epsilon \beta^r \times \iota),$$

where θ *operates on* $\tilde{H}_G^{m+n+1}(K\Lambda_{m+n-2r-\epsilon})$ *via the usual Steenrod algebra action on* $\tilde{H}^*(\Sigma BG \wedge K(\mathbb{Z}_p, m+n-2r-\epsilon))$.

Proof. Any map $\zeta\colon K\Lambda_r \to K(\mathbb{Z}_p, s)$ factors through the orbit space of $K\Lambda_r$. Thus $\zeta^*(\theta(\iota))$ is the composite

$$K\Lambda_r \to K\Lambda_r/G \xrightarrow{\zeta'} K(\mathbb{Z}_p, s) \xrightarrow{\theta} K(\mathbb{Z}_p, t)$$

which corresponds to the cohomology class $\theta(\{\zeta'\})$. The statement follows from Lemma 15.

Proposition 17. *Let* V *be an* n*-dimensional fixed-point-free representation. Then there is an exact triangle*

$$\begin{array}{ccc} \bigoplus\limits_{q=0}^{\infty} \varprojlim_m\, [K_{m+V}, K_{m+q+V}]_G & \longrightarrow & H_G^{*+V}(\mathrm{pt}) \otimes \Sigma^n \mathscr{A} \\ & {\scriptstyle (+1)}\nwarrow \quad \swarrow{\scriptstyle \delta^*} & \\ & \tilde{H}_G^{*+V}(\tilde{E}G) \otimes (\Sigma^{-1}\mathscr{A} \oplus \mathscr{A} \oplus \cdots \oplus \Sigma^{n-2}\mathscr{A}) & \end{array} \tag{8}$$

in which the $(n-2r-\epsilon)$*th factor* $\delta_{n-2r-\epsilon}$ *of* δ^* *sends an element* $a\theta$ *to* $a \cdot \theta(\alpha^\epsilon\beta^r \times \iota)$, *where the Steenrod algebra action on the tensor product satisfies the usual Cartan formula.*

Proof. Apply Lemmas 6 and 7 to the fibre sequence of Proposition 12 and take limits over m. The result is still exact and the right and bottom corners come from the isomorphisms in Corollary 14. The homomorphism δ^* is associated with the map

τ_n, so that the qth factor corresponds to $i_q^* \tau_n^*$ and the formula is immediate from Lemma 16.

COROLLARY 18. *Let V be a fixed-point-free two-dimensional representation. Then in the exact triangle* (8) *for V, the kernel of δ^* is generated by $k \cdot 1$ and $k \cdot \delta$, where k is the generator in $H_G^{V-2}(\mathrm{pt})$ and δ is the Bockstein operation.*

Proof. For $p = 2$, k is the class κ^2 in the appendix, while for $p > 2$, k is the class h_V. We shall give the proof for the case $p > 2$; the proof for $p = 2$ is similar.

Since V is a two-dimensional representation, $\tilde{H}_G^{*+V}(\mathrm{pt})$ is generated by τ_V, κ_V and h_V, in dimensions V, $V-1$ and $V-2$ respectively. The map δ^* is given by

$$\delta_1(a\theta) = a \cdot \theta(\alpha \times \iota)$$

and

$$\delta_0(a\theta) = a \cdot \theta(\beta \times \iota)$$

for $a \in H_G^{*+V}(\mathrm{pt})$.

By our knowledge of the operation of $\mathscr{A}$ on $H^*(\Sigma B\mathbb{Z}/p)$, $\theta(\alpha \times \iota)$ must be a sum of terms of the form

$$\alpha \times \theta + \beta \times \theta' + \beta^p \times \theta'' + \beta^{p^2} \times \theta''' + \cdots$$

and similarly $\theta(\beta \times \iota)$ has the form

$$\beta \times \theta + \beta^p \times \theta_1 + \beta^{p^2} \times \theta_2 + \cdots.$$

Here we have written elements in $\tilde{H}_G^*(\tilde{E}G) \otimes \mathscr{A}$ as cohomology cross products, since that is their origin, and the Cartan formula respects this interpretation.

From the appendix, $H_G^{*+V}(\mathrm{pt})$ acts on $\tilde{H}_G^*(\tilde{E}G) \cong \mathbb{Z}/p[t, t^{-1}] \otimes \tilde{H}^*(\Sigma BG)$ as follows:

$$\tau_V \cdot \alpha^\epsilon \beta^r = t \otimes \alpha^\epsilon \beta^r,$$

$$\kappa_V \cdot \alpha^\epsilon \beta^r = \begin{cases} t \otimes \alpha\beta^{r-1} & \text{if } \epsilon = 0, \\ 0 & \text{if } \epsilon = 1, \end{cases}$$

$$h_V \cdot \alpha^\epsilon \beta^r = \begin{cases} t \otimes \alpha^\epsilon \beta^{r-1} & \text{if } r > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\zeta = \tau_V \theta + \kappa_V \psi + h_V \phi$ be an element of $H_G^{*+V}(\mathrm{pt}) \otimes \Sigma^n \mathscr{A}$. Then, combining the formulae above,

$$\delta_0(\zeta) = (t \otimes \alpha) \times \psi + (\text{terms involving } \beta),$$

so for ζ to be in $\operatorname{Ker}(\delta^*)$, ψ must be zero. Thus

$$\delta_1(\zeta) = \delta_1(\tau_V \theta + h_V \phi) = (t \otimes \alpha) \times \theta + (\text{terms involving } \beta),$$

so θ must be zero. Thus $\zeta = h_V \phi$ and

$$\delta_0(h_V \phi) = (t \otimes \beta^{p-1}) \times \phi_1 + (t \otimes \beta^{p^2-1}) \times \phi_2 + \cdots$$

and

$$\delta_1(h_V \phi) = (t \otimes \beta^{p-1}) \times \phi'' + (t \otimes \beta^{p^2-1}) \times \phi''' + \cdots.$$

Setting these to zero forces $\phi^{(k)} = 0$ for $k \geqslant 2$ and $\phi_k = 0$ for $k \geqslant 1$. Therefore the cross product formulae for ϕ are

$$\phi(\alpha \times \iota) = \alpha \times \phi + \beta \times \phi'$$

and

$$\phi(\beta \times \iota) = \beta \times \phi.$$

A simple calculation in the Milnor basis [**6**] reveals that the only operations ϕ satisfying these identities are multiples of the identity and the Bockstein operation.

7. *Comparison of exact triangles*

Lemma 19. *Let V be a fixed-point-free two-dimensional representation. There is an exact triangle of $\mathbb{Z}/p$-vector spaces*

$$\begin{tikzcd} \bigoplus_{q=0}^{\infty} \varprojlim_m [K_{m+V}, K_{m+q+V}]_G \arrow[rr, "\Omega^V"] & & \mathscr{A} \arrow[dl] \\ & \tilde{H}^*(\Sigma BG) \otimes (\mathscr{A} \oplus \Sigma\mathscr{A}) \arrow[ul] & \end{tikzcd} \tag{9}$$

Proof. This is the exact sequence of (1) stabilized by inverse limits over m, with $\varprojlim_m \tilde{H}_G^{m+q+V}(K\Lambda_m \times K\Lambda_{m+1})$ replaced by the isomorphic $\tilde{H}^*(\Sigma BG) \otimes (\mathscr{A} \oplus \Sigma\mathscr{A})$.

Corollary 20. *In* (9), *the image of Ω^V is generated by* 1 *and* δ.

Proof. Compare Corollary 18 and Lemma 19 and count generators. Specifically, let

$$B = \tilde{H}^*(\Sigma BG) \otimes (\mathscr{A} \oplus \Sigma\mathscr{A}) = (\Sigma^2\mathscr{A} \oplus \Sigma^3\mathscr{A} \oplus \cdots) \oplus (\Sigma^3\mathscr{A} \oplus \Sigma^4\mathscr{A} \oplus \cdots)$$

$$C = \bigoplus_{q=0}^{\infty} \varprojlim_m [K(\mathbb{Z}_p, m+V), K(\mathbb{Z}_p, m+q+V)]_G.$$

Then $\Sigma^{-1}B \cong B \oplus (\Sigma\mathscr{A} \oplus \Sigma^2\mathscr{A}) = B \oplus D$.

Thus the two exact triangles may be rewritten:

$$\begin{tikzcd} C \arrow[rr, "j"] & & \mathscr{A} \oplus D \arrow[dl, "\delta^*"] \\ & B \oplus D \arrow[ul, "p"] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C \arrow[rr, "\Omega^V"] & & \mathscr{A} \arrow[dl, "\gamma"] \\ & B \arrow[ul, "\partial^*"] & \end{tikzcd}$$

A simple induction now shows that the rank of $\mathrm{Ker}\,(\gamma_q)$ equals the rank of $\mathrm{Ker}\,(\delta_q^*)$ for all q. Thus

$$\mathrm{rank}\,(\mathrm{Im}\,(\Omega_q^V)) = \mathrm{rank}\,(\mathrm{Im}\,(j_q)) = \begin{cases} 1, & \text{if } q = 0, 1; \\ 0, & \text{if } q \neq 0, 1. \end{cases}$$

Proof of 3(*b*). Let V be a fixed-point-free two-dimensional representation, as in Lemma 19. Of course, the triangle

$$\begin{tikzcd} \varprojlim_{m,W} [K_{m+W}, K_{m+q+W}]_G \arrow[rr, "\Omega"] \arrow[dr, "\Omega'"'] & & \varprojlim_m [K_m, K_{m+q}] \\ & \varprojlim_m [K_{m+V}, K_{m+q+V}]_G \arrow[ur, "\Omega^V"'] & \end{tikzcd}$$

commutes and so $\mathrm{Im}\,(\Omega) \subseteq \mathrm{Im}\,(\Omega^V) = \langle 1, \delta \rangle$. But in $\mathscr{A}_{\mathbb{Z}/p} = [H\mathbb{Z}_p, H\mathbb{Z}_p]_G^*$ we certainly have elements 1 corresponding to the identity map of the spectrum $H\mathbb{Z}_p$ and δ

corresponding to the connecting map in the G-fibreing $H\mathbb{Z}_p \to H\underline{\mathbb{Z}/p^2} \to H\mathbb{Z}_p$ and these clearly map to the nonequivariant identity and Bockstein operations. Thus $\mathrm{Im}(\Omega) \supseteq \langle 1, \delta \rangle$.

Appendix. Cohomology of points

Here we will compute the cohomology of G-orbits as graded rings and of the spaces EG and $\tilde{E}G$ as modules over the cohomology of a point. The ring structure of $H^*_G(\mathrm{pt}; \underline{\mathbb{Z}/p})$ is not only necessary to the computation of the cohomology of a point, but the module action is what allows us to determine the kernel of the map in Corollary 18.

Define $H^{(+)}_G(X)$ to be the part of $H^*_G(X; \underline{\mathbb{Z}/p})$ in dimensions $H^{m+V}_G(X)$, where V is an honest representation and m is any integer.

LEMMA 21. *$H^{(+)}_G(G)$ is a commutative polynomial algebra on generators $t_\xi \in H^{\xi - \dim \xi}_G(G)$, where ξ ranges over nontrivial irreducible representations.*

LEMMA 22. *If $p = 2$, $H^{(+)}_G(\mathrm{pt})$ is a commutative polynomial algebra on generators $\tau \in H^{\xi}_G(\mathrm{pt})$ and $\kappa \in H^{\xi-1}_G(\mathrm{pt})$, where ξ is the canonical irreducible one-dimensional representation.*

In each of these the commutativity relation holds without the introduction of any signs. In general, the commutativity relation in $H^*_G(X)$ is given by the action of a unit in the Burnside ring of G, as follows. Let $\chi_H : A(G) \to \mathbb{Z}$ denote the H-fixed point characteristic homomorphism, defined on generators by letting $\chi_H(G/K)$ be the Euler character of $(G/K)^H$. It is standard that $(\oplus_H \chi_H) : A(G) \to \bigoplus_{H \leqslant G} \mathbb{Z}$ is a ring monomorphism. Then if $a \in H^{\alpha}_G(X)$ and $b \in H^{\beta}_G(X)$, we have that $(a \cup b) = u(a, b) . (b \cup a)$, where $u(a, b)$ is the unit characterized by $\chi_H(u(a, b)) = (-1)^{\dim(\alpha^H)\dim(\beta^H)}$ for all $H \leqslant G$.

LEMMA 23. *If $p > 2$, $H^{(+)}_G(\mathrm{pt})$ is a signed-commutative algebra on generators $\tau_\xi \in H^{\xi}_G(\mathrm{pt})$ $\kappa_\xi \in H^{\xi-1}_G(\mathrm{pt})$, and $h_\xi \in H^{\xi-2}_G(\mathrm{pt})$, where ξ ranges over all nontrivial irreducible representations, subject only to the following relations:*

(*a*) $\kappa_\xi^2 = 0$;

(*b*) $\tau_\xi \kappa_\sigma = \tau_\sigma \kappa_\xi$;

(*c*) $h_\xi \kappa_\sigma = h_\sigma \kappa_\xi$;

(*d*) $\tau_\xi h_\sigma = \tau_\sigma h_\xi$.

In fact, in $H^*_G(\mathrm{pt})$, there is a class $\alpha_{\xi-\sigma} \in H^{\xi-\sigma}_G(\mathrm{pt})$ such that $\alpha_{\xi-\sigma}\alpha_{\sigma-\xi} = 1$ and $\tau_\xi = \alpha_{\xi-\sigma}\tau_\sigma$, $h_\xi = \alpha_{\xi-\sigma} h_\sigma$ and $\kappa_\xi = \alpha_{\xi-\sigma}\kappa_\sigma$.

The whole of $H^*_G(\mathrm{pt})$ cannot be described so neatly. Even for $p = 2$ there is an element $u_{-2} \in H^{2-2\xi}_{\mathbb{Z}/2}(\mathrm{pt})$ which is infinitely divisible by both τ and κ and for which $\tau u_{-2} = \kappa u_{-2} = u_{-2}^2 = 0$. Our results in the body of this article require calculating $H^{m+V}_G(\mathrm{pt})$ for honest representations V and so we avoid the messier part of the calculations.

COROLLARY 24. *Let V be an n-dimensional irreducible representation. Then*

$$H^{m+V}_G(\mathrm{pt}) = \begin{cases} \mathbb{Z}/p & \text{if } -n \leqslant m \leqslant 0; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose $p > 2$. (The case $p = 2$ is basically the same, but simpler.) Choose any nontrivial irreducible representation ξ. Then multiplication by $\alpha_{V-k\xi}$ (where $k = n/2$) induces an isomorphism of $H_G^{*+k\xi}(\mathrm{pt})$ onto $H_G^{*+V}(\mathrm{pt})$. Thus one might as well assume that $V = k\xi$. But $H_G^{*+k\xi}(\mathrm{pt})$ has generators $h_\xi^q \tau_\xi^{k-q}$ for each $0 \leqslant q \leqslant k$ and $h_\xi^q \kappa_\xi \tau_\xi^{k-q-1}$ for $0 \leqslant q \leqslant k-1$. There is one such generator in each dimension $-n \leqslant m \leqslant 0$.

Proof of Lemma 21. We make note of the following trick. If X is any G-space then there is a G-homeomorphism $G_+ \wedge X \approx G_+ \wedge X^e$. Applying this with $X = S^V$, we have for any representation degree α,

$$H_G^\alpha(G) \underset{\cong}{\xrightarrow{\text{susp.}}} \tilde{H}_G^{\alpha+V}(\Sigma^V(G_+)) \underset{\cong}{\rightarrow} \tilde{H}_G^{\alpha+V}(\Sigma^n(G_+)) \underset{\cong}{\xleftarrow{\text{susp.}}} H_G^{\alpha+V-n}(G) \tag{10}$$

is an isomorphism of $H_G^*(G)$-modules. Thus it is given by multiplication by an invertible element t_V in $H_G^{V-n}(G)$.

By the Bredon dimension axiom, $H_G^0(G) = \mathbb{Z}/p$ and $H_G^m(G) = 0$ when m is a nonzero integer. It follows by the above that

$$H_G^\alpha(G) = \begin{cases} \mathbb{Z}/p & \text{if } \dim(\alpha) = 0, \\ 0 & \text{if } \dim(\alpha) \neq 0. \end{cases}$$

If $p = 2$, we are done already, for t_ξ^n is a nonzero element of $H_G^{n(\xi-1)}(G)$ and by the preceding statement it must generate that group.

With regard to the claim about commutativity, the generators t_V are concentrated in degrees of even dimension and thus for nontrivial irreducible representations V and W, $\chi_H(u(t_V, t_W)) = 1$ for all H, whence $u(t_V, t_W)$ is 1 and so $H_G^*(G)$ is strictly commutative.

Proof of Lemma 22. Consider the cofibre sequence

$$S(\xi) \to \mathrm{pt} \to S^\xi$$

from (5). This gives rise to an exact triangle

$$\begin{tikzcd} \tilde{H}^*_{\mathbb{Z}/2}(S^\xi) \arrow[rr] & & H^*_{\mathbb{Z}/2}(\mathrm{pt}) \arrow[dl] \\ & H^*_{\mathbb{Z}/2}(\mathbb{Z}/2) \arrow[ul] & \end{tikzcd}$$

or equivalently

$$\begin{tikzcd} H^*_{\mathbb{Z}/2}(\mathrm{pt}) \arrow[rr, "\tau\cdot"] & & H^*_{\mathbb{Z}/2}(\mathrm{pt}) \arrow[dl, "\phi"] \\ & H^*_{\mathbb{Z}/2}(\mathbb{Z}/2) \arrow[ul, "\psi"] & \end{tikzcd}$$

where ϕ is a ring homomorphism, ψ is a homomorphism of $H^*_{\mathbb{Z}/2}(\mathrm{pt})$-modules (via ϕ) of degree $(+1-\xi)$, τ is an element of $H^\xi_{\mathbb{Z}/2}(\mathrm{pt})$ and $\tau\cdot$ is translation by τ. (We have not yet proved that $\tau \neq 0$, but we will.)

A picture will help. The generators of the cohomology of G look like this:

$$
\begin{array}{ccccccccccc}
 & & & & \uparrow m & & & & & \\
t^{-4} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & t^{-3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & t^{-2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & t^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \xrightarrow{k\xi} \\
\cdot & \cdot & \cdot & \cdot & \cdot & t & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t^{2} & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t^{3} & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t^{4} &
\end{array}
$$

Thus $\tau\cdot$ is an isomorphism except when the source is $H^{n(\xi-1)}_{\mathbb{Z}/2}(\mathrm{pt})$ or $H^{n(\xi-1)-1}_{\mathbb{Z}/2}(\mathrm{pt})$. By the Bredon dimension axiom, $H^{m}_{\mathbb{Z}/2}(\mathrm{pt}) = 0$ for nonzero integers m, and this forces whole areas of $H^{m*}_{\mathbb{Z}/2}(\mathrm{pt})$ to be zero:

$$
\begin{array}{ccccccccccc}
 & & & & \uparrow m & & & & & \\
? & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
? & ? & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
? & ? & ? & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
? & ? & ? & ? & \cdot & \cdot & \cdot & \cdot & \cdot & \\
? & ? & ? & ? & 1 & ? & ? & ? & ? & \xrightarrow{k\xi} \\
\cdot & \cdot & \cdot & \cdot & \cdot & ? & ? & ? & ? & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & ? & ? & ? & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & ? & ? & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & ? &
\end{array}
$$

where the ?s are the only possible places for nonzero groups. The exact triangle yields the exact sequence

$$0 \to H^{\xi-1}_{\mathbb{Z}/2}(\mathrm{pt}) \xrightarrow{\phi} H^{\xi-1}_{\mathbb{Z}/2}(\mathbb{Z}/2) \xrightarrow{\psi} H^{0}_{\mathbb{Z}/2}(\mathrm{pt}) \xrightarrow{\tau\cdot} H^{\xi}_{\mathbb{Z}/2}(\mathrm{pt}) \to 0.$$

A simple geometric argument allows us to identify ψ with $f_!\colon \mathbb{Z}/2 \to \mathbb{Z}/2$, the transfer associated to the unique map $f\colon G/e \to G/G$, via the isomorphism $t\cdot$ in $H^*_{\mathbb{Z}/2}(\mathbb{Z}/2)$. This is zero by the definition of $\underline{\mathbb{Z}/2}$ in Example 1 and thus ϕ and $\tau\cdot$ are isomorphisms in this sequence.

Therefore $H^{\xi}_{\mathbb{Z}/2}(\mathrm{pt}) \cong \mathbb{Z}/2$, generated by $\tau \neq 0$, and $H^{\xi-1}_{\mathbb{Z}/2}(\mathrm{pt}) \cong \mathbb{Z}/2$, generated by a class κ such that $\phi(\kappa) = t \in H^{\xi-1}_{\mathbb{Z}/2}(\mathbb{Z}/2)$.

Since ϕ is a ring homomorphism, the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & H^{n\xi-n}_{\mathbb{Z}/2}(\mathrm{pt}) \arrow[r, "\phi"] & H^{n\xi-n}_{\mathbb{Z}/2}(\mathbb{Z}/2) \arrow[r] & \cdots \\
& H^0_{\mathbb{Z}/2}(\mathrm{pt}) \arrow[u, "\kappa^n\cdot"] \arrow[r, "\phi", "\cong"'] & H^0_{\mathbb{Z}/2}(\mathbb{Z}/2) \arrow[u, "\cong"', "t^n\cdot"] \arrow[r] & \cdots
\end{tikzcd}$$

Hence $H^{n\xi-n}_{\mathbb{Z}/2}(\mathrm{pt}) \cong \mathbb{Z}/2$, generated by κ^n. Following the exact sequence at the top further on, one sees $\psi = 0$ and so $\tau\cdot$ is an isomorphism on $H^{(n-1)(\xi-1)}_{\mathbb{Z}/2}(\mathrm{pt})$.

The final picture is:

$$\begin{array}{llllll}
\uparrow m & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \\
1 & \tau & \tau^2 & \tau^3 & \tau^4 & \xrightarrow{k\xi} \\
\cdot & \kappa & \kappa\tau & \kappa\tau^2 & \kappa\tau^3 & \\
\cdot & \cdot & \kappa^2 & \kappa^2\tau & \kappa^2\tau^2 & \\
\cdot & \cdot & \cdot & \kappa^3 & \kappa^3\tau & \\
\cdot & \cdot & \cdot & \cdot & \kappa^4 &
\end{array}$$

The result follows. $\square$

The proof of Lemma 23 depends on understanding the cohomology of $S(\xi)$ for an irreducible ξ. In the present case, $(p > 2)$, $S(\xi)$ is *not* $\mathbb{Z}/p$, so we need the following lemma.

Lemma 25. *Let p be an odd prime and let ξ be any nontrivial irreducible representation of $\mathbb{Z}/p$. Then the part of $H^*_{\mathbb{Z}/p}(S(\xi))$ in dimensions $m + r\xi$ with $r \geqslant 0$ and $m \in \mathbb{Z}$ is a signed-commutative algebra on generators $k \in H^{\xi-1}_{\mathbb{Z}/p}(S(\xi))$ and $t \in H^{\xi-2}_{\mathbb{Z}/p}(S(\xi))$ subject only to the relation $k^2 = 0$.*

Proof. There is an equivariant inclusion $\mathbb{Z}/p \hookrightarrow S(\xi)$ as the pth roots of unity. The cofibre is a bouquet of p circles, permuted cyclically by $\mathbb{Z}/p$, and thus is equivalent to $\Sigma((\mathbb{Z}/p)_+)$. One sees directly that the connecting map δ is nullhomotopic on passage to orbit spaces and so, by Example 1, $\delta^* = 0$ in integer degrees. But since $S(\xi)$ is G-free, δ^* is a morphism of $H^*_{\mathbb{Z}/p}(\mathbb{Z}/p)$-modules and it follows that $\delta^* = 0$ in all degrees. So $H^*_{\mathbb{Z}/p}(S(\xi)) \cong H^*_{\mathbb{Z}/p}(\mathbb{Z}/p) \oplus \tilde{H}^*_{\mathbb{Z}/p}(\Sigma((\mathbb{Z}/p)_+))$, with the generator $t \in H^{\xi-2}_{\mathbb{Z}/p}(S(\xi))$ mapping to $t_\xi \in H^{\xi-2}_{\mathbb{Z}/p}(\mathbb{Z}/p)$, and so t^k generates $H^{k(\xi-2)}_{\mathbb{Z}/p}(S(\xi))$. The generator $k \in H^{\xi-1}_{\mathbb{Z}/p}(S(\xi))$ comes from $\sigma(t_\xi) \in \tilde{H}^{\xi-1}_{\mathbb{Z}/p}(\Sigma((\mathbb{Z}/p)_+))$, and so is translated isomorphically by $\cdot t^k$. Naturally $k^2 = 0$ since $H^{2\xi-2}_{\mathbb{Z}/p}(S(\xi)) = 0$.

Proof of Lemma 23. Let V and W be nontrivial representations of $\mathbb{Z}/p$. Each of these may be thought of as the complex plane $\mathbb{C}$, with $\mathbb{Z}/p$ acting via multiplication by a pth root of unity ϵ_V or ϵ_W. Thus there is a nontrivial equivariant map of the form $f: z \mapsto z^j$ sending $\mathbb{C}\{V\}$ to $\mathbb{C}\{W\}$. This induces a map $S^V \to S^W$ and so a map

$$H^*_{\mathbb{Z}/p}(\mathrm{pt}) \cong \tilde{H}^*_{\mathbb{Z}/p}(S^W) \to \tilde{H}^*_{\mathbb{Z}/p}(S^V) \cong H^*_{\mathbb{Z}/p}(\mathrm{pt})$$

changing dimension by $V - W$. Similarly there is a map back the other way $f': z \mapsto z^{j'}$ where $jj' \equiv 1 \,(\mathrm{mod}\, p)$. Since $(S^V)^G = (S^W)^G = S^0$, $(f \circ f')^G$ is the identity and $(f \circ f')^e$ is a map of degree jj'. It follows that these induce isomorphisms in all ordinary mod-p cohomology theories and by a comparison of Bredon spectral sequences $f \circ f'$ induces an isomorphism of $H^*_{\mathbb{Z}/p}(\mathrm{pt})$. Thus f^* is an $H^*_{\mathbb{Z}/p}(\mathrm{pt})$-module isomorphism and so comes from multiplication by an invertible class $\alpha_{V-W} \in H^{V-W}_{\mathbb{Z}/p}(\mathrm{pt})$. $\square$

The presence of these classes simplifies matters. It now suffices to prove the following lemma.

Lemma 26. *Let $p > 2$ and let ξ be a nontrivial irreducible representation. Then the part of $H^*_{\mathbb{Z}/p}(\mathrm{pt})$ in dimensions $m + k\xi$ with $k \geqslant 0$ and $m \in \mathbb{Z}$ is a signed-commutative algebra on generators $\tau \in H^{\xi}_{\mathbb{Z}/p}(\mathrm{pt})$, $\kappa \in H^{\xi-1}_{\mathbb{Z}/p}(\mathrm{pt})$ and $h \in H^{\xi-2}_{\mathbb{Z}/p}(\mathrm{pt})$, subject only to the relation $\kappa^2 = 0$.*

Proof. Again we consider the cofibreing $S(\xi) \to \mathrm{pt} \to S^\xi$ and obtain an exact triangle

$$\begin{tikzcd}
H^*_{\mathbb{Z}/p}(\mathrm{pt}) \arrow[rr, "\tau\cdot"] & & H^*_{\mathbb{Z}/p}(\mathrm{pt}) \arrow[dl, "\phi"] \\
& H^*_{\mathbb{Z}/p}(S(\xi)) \arrow[ul, "\psi"] &
\end{tikzcd}$$

where the composite

$$H^0_{\mathbb{Z}/p}(\mathbb{Z}/p) \overset{t\cdot}{\cong} H^{\xi-2}_{\mathbb{Z}/p}(\mathbb{Z}/p) \overset{\text{susp.}}{\cong} \tilde{H}^{\xi-1}_{\mathbb{Z}/p}(\Sigma(\mathbb{Z}/p_+)) \cong H^{\xi-1}_{\mathbb{Z}/p}(S(\xi)) \overset{\psi}{\to} H^0_{\mathbb{Z}/p}(\mathrm{pt})$$

is the morphism $f_!$, where $f\colon \mathbb{Z}/p \to \mathrm{pt}$.

Lemma 25 implies that the generator diagram for $H^*_{\mathbb{Z}/p}(S(\xi))$ is

$$
\begin{array}{cccccccccc}
 & & & & \uparrow m & & & & & \\
\cdot & \cdot & t^{-2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & kt^{-2} & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & t^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & kt^{-1} & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \xrightarrow{k\xi} \\
\cdot & \cdot & \cdot & \cdot & \cdot & k & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & t & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & kt & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t^2 & \cdot & \cdot &
\end{array}
$$

Thus $\tau\cdot : H^{m+k\xi}_{\mathbb{Z}/p}(\mathrm{pt}) \to H^{m+(k+1)\xi}_{\mathbb{Z}/p}(\mathrm{pt})$ is an isomorphism away from $-2k-2 \leqslant m \leqslant -2k-1$. Near there we chase the exact sequences just as in Lemma 22 to show $\tau \neq 0$, and to obtain generators $\kappa \in H^{\xi-1}_{\mathbb{Z}/p}(\mathrm{pt})$ and $h \in H^{\xi-2}_{\mathbb{Z}/p}(\mathrm{pt})$ such that $\phi(\kappa) = k$ and $\phi(h) = t$. The rest follows. $\square$

We will need to know something about the cohomology of $\tilde{E}G$ as a $H^*_G(\mathrm{pt})$-module. This is most easily computed by first studying the cohomology of EG.

LEMMA 27. (i) *Let* $p = 2$. *Then* $H^*_G(EG) \cong \mathbb{Z}/2[k, k^{-1}] \otimes H^*(B\mathbb{Z}/2)$, *where* $k \in H^{\xi-1}_G(EG)$. *The action of* $H^*_G(\mathrm{pt})$ *is given by*

$$\tau\cdot(k^i\alpha^j) = k^{i+1}\alpha^{j+1} \quad \textit{and}$$
$$\kappa\cdot(k^i\alpha^j) = k^{i+1}\alpha^j.$$

(ii) *Let* $p > 2$. *Then* $H^*_G(EG) \cong \bigotimes_{\xi\ \mathrm{irreducible}} \mathbb{Z}/p[k_\xi, k_\xi^{-1}] \otimes H^*(B\mathbb{Z}/p)$, *where* $k_\xi \in H^{\xi-2}_G(EG)$. *The action of* $H^*_G(\mathrm{pt})$ *is given by*

$$\tau_\xi\cdot(k_\xi^i\alpha^\epsilon\beta^j) = k_\xi^{i+1}\alpha^\epsilon\beta^{j+1}$$
$$\kappa_\xi\cdot(k_\xi^i\alpha^\epsilon\beta^j) = \begin{cases} k_\xi^{i+1}\alpha\beta^j, & \textit{if } \epsilon = 0, \\ 0, & \textit{if } \epsilon = 1; \quad \textit{and} \end{cases}$$
$$h_\xi\cdot(k_\xi^i\alpha^\epsilon\beta^j) = k_\xi^{i+1}\alpha^\epsilon\beta^j.$$

Proof. By Example 1 we already know $H^m_G(EG) = H^m(BG)$. By the fibre sequence in Proposition 12 and the note after Corollary 14, there is an exact sequence

$$\cdots \to H^{*+2}_G(EG) \xrightarrow{\kappa\cdot} H^{*+\xi}_G(EG) \to H^m_G(EG; \Lambda\mathbb{Z}/p) \oplus H^{m+1}_G(EG; \Lambda\mathbb{Z}/p) \to \cdots$$

where κ is the generator in $H^{\xi-2}_G(\mathrm{pt})$, and $\kappa\cdot$ is multiplication by κ. Since EG is G-free, $H^{m+i}_G(EG; \Lambda\mathbb{Z}/p) \cong 0$ for $i = 0, 1$, and so $\kappa\cdot$ is an isomorphism. This establishes the identity of the groups $H_G^*(EG)$ and the action of κ. The action of the other generators is established by studying the cofibreing

$$EG_+ \wedge S(\xi)_+ \to EG_+ \to \Sigma^\xi(EG_+)$$

and recalling that $G_+ \wedge EG_+$ is G-equivalent to G_+.

Lemma 28. *$\tilde{H}_G^*(\tilde{E}G) \cong (\bigotimes_{\xi \text{ irreducible}} \mathbb{Z}/p[t_\xi, t_\xi^{-1}]) \otimes \tilde{H}^*(\Sigma BG)$, where $t_\xi \in H_G^\xi(\mathrm{pt})$ and ξ ranges over nontrivial irreducible representations.*

If $p = 2$, the action of $H_G^*(\mathrm{pt})$ is given by

$$\tau \cdot (t^i \alpha^j) = t^{i+1}\alpha^j, \quad \textit{and}$$

$$\kappa \cdot (t^i\alpha^j) = \begin{cases} t^{i+1}\alpha^{j-1} & \textit{if } j > 1, \\ 0 & \textit{if } j = 1; \end{cases}$$

where we have written α^j for the suspended element in $H^{j+1}(\Sigma BG)$.

If $p > 2$ then the action is given by

$$\tau_\xi \cdot (t_\xi^i \alpha^\epsilon \beta^j) = t_\xi^{i+1}\alpha^\epsilon\beta^j;$$

$$\kappa_\xi \cdot (t_\xi^i \alpha^\epsilon \beta^j) = \begin{cases} t_\xi^{i+1}\alpha\beta^{j-1} & \textit{if } \epsilon = 0, \\ 0 & \textit{otherwise; and} \end{cases}$$

$$h_\xi \cdot (t_\xi^i \alpha^\epsilon \beta^j) = \begin{cases} t_\xi^{i+1}\alpha^\epsilon\beta^{j-1} & \textit{if } j > 1, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We already know that $\tilde{H}_G^m(\tilde{E}G) \cong \tilde{H}^m(\Sigma BG)$ from Example 1. By the proof of Lemma 4(d) the inclusion $\tilde{E}G \to \Sigma^\xi \tilde{E}G$ is a G-equivalence. Thus the induced homomorphism, which is multiplication by τ_ξ, is an isomorphism and this determines the groups $\tilde{H}_G^*(\tilde{E}G)$. The action of the other generators follows by studying the exact sequence of the cofibreing $EG_+ \to \mathrm{pt}_+ \to \tilde{E}G$.

REFERENCES

[**1**] G. E. Bredon. *Equivariant cohomology theories*. Springer Lecture Notes in Mathematics, vol. 34 (Springer, 1967).

[**2**] S. R. Costenoble. The structure of some equivariant Thom spectra. *Trans. Amer. Math. Soc.* **315** (1989), 231–254.

[**3**] L. G. Lewis, Jr. The equivariant cohomology of G-orbits. Unpublished notes.

[**4**] L. G. Lewis, J. P. May and J. E. McClure. Ordinary $RO(G)$-graded cohomology. *Bull. Amer. Math. Soc.* **4** (1981), 208–212.

[**5**] L. G. Lewis, J. P. May and M. Steinberger (with contributions by J. E. McClure). *Equivariant stable homotopy theory*. Springer Lecture Notes in Mathematics, vol. 1213 (Springer, 1986).

[**6**] J. Milnor. The Steenrod algebra and its dual. *Ann. of Math.* **67** (1958), 150–171.

[**7**] R. E. Stong. Ordinary cohomology of points. Unpublished manuscript (1980).

[**8**] S. Waner. Notes on the cohomology of a point. Manuscript (1980).