COFREENESS IN REAL BORDISM THEORY AND THE SEGAL CONJECTURE

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ABSTRACT. We prove that the genuine C_{2^n}-spectrum $N_{C_2}^{C_{2^n}} MU_{\mathbb{R}}$ is cofree, for all n. Our proof is a formal argument using chromatic hypercubes and the Slice Theorem of Hill, Hopkins, and Ravenel. We show that this gives a new proof of the Segal conjecture for C_2, independent of Lin's theorem.

1. INTRODUCTION

In this paper, we establish the following result:

Theorem 1.1. *For all $n > 0$, the C_{2^n}-spectrum $N_{C_2}^{C_{2^n}} MU_{\mathbb{R}}$ is cofree, i.e. the map*

$$N_{C_2}^{C_{2^n}} MU_{\mathbb{R}} \to F(EC_{2^n +}, N_{C_2}^{C_{2^n}} MU_{\mathbb{R}})$$

is an equivalence.

The equivariant spectra $N_{C_2}^{C_{2^n}} MU_{\mathbb{R}}$ play a central role in the solution to the Kervaire Invariant One problem by Hill, Hopkins, and Ravenel [7]. Their detecting spectrum Ω is the homotopy fixed point spectrum of a localization $\Omega_{\mathbb{O}} := D^{-1} N_{C_2}^{C_8} MU_{\mathbb{R}}$ of $N_{C_2}^{C_8} MU_{\mathbb{R}}$. An essential piece of their argument is the Homotopy Fixed Point Theorem ([7], 1.10), which states that this homotopy fixed point spectrum coincides with the *genuine* fixed point spectrum, i.e. that $\Omega_{\mathbb{O}}$ is cofree. Our result shows that this holds even *before* localization away from D.

In the case $n = 1$, Hu and Kriz show that $MU_{\mathbb{R}}$ is cofree via direct computation [8]. They compute the C_2-homotopy fixed point and Tate spectral sequences for $BP_{\mathbb{R}}$, and deduce that $(BP_{\mathbb{R}})^{tC_2} = H\mathbb{F}_2$, so that the result is an immediate consequence of the Tate square for $BP_{\mathbb{R}}$. We give a new, more conceptual proof of their result that generalizes readily to $n > 1$. The idea is that $BP_{\mathbb{R}}[\overline{v_i}^{-1}]$ is cofree for formal reasons, so one can take an approach via local cohomology and form cartesian cubes

$$\begin{tikzcd} \tilde{L}_2 BP_{\mathbb{R}} \arrow[r] \arrow[d] & BP_{\mathbb{R}}[\overline{v_1}^{-1}] \arrow[d] \\ BP_{\mathbb{R}}[\overline{v_2}^{-1}] \arrow[r] & BP_{\mathbb{R}}[(\overline{v_1}\overline{v_2})^{-1}] \end{tikzcd}$$

$$\begin{tikzcd}
& BP_{\mathbb{R}}[\overline{v_3}^{-1}] \arrow[rr] \arrow[dd] & & BP_{\mathbb{R}}[(\overline{v_2}\overline{v_3})^{-1}] \arrow[dd] \\
\tilde{L}_3 BP_{\mathbb{R}} \arrow[ur] \arrow[rr] \arrow[dd] & & BP_{\mathbb{R}}[\overline{v_2}^{-1}] \arrow[ur] \arrow[dd] & \\
& BP_{\mathbb{R}}[(\overline{v_1}\overline{v_3})^{-1}] \arrow[rr] & & BP_{\mathbb{R}}[(\overline{v_1}\overline{v_2}\overline{v_3})^{-1}] \\
BP_{\mathbb{R}}[\overline{v_1}^{-1}] \arrow[ur] \arrow[rr] & & BP_{\mathbb{R}}[(\overline{v_1}\overline{v_2})^{-1}] \arrow[ur] &
\end{tikzcd}$$

and so on, and $\tilde{L}_n BP_{\mathbb{R}}$ is cofree for all n. Applying the slice tower to each vertex $BP_{\mathbb{R}}[(\overline{v_{i_1}} \cdots \overline{v_{i_j}})^{-1}]$, one forms a cartesian cube in filtered C_2-spectra, and the limit term gives a modified slice filtration of $\tilde{L}_n BP_{\mathbb{R}}$. It is then a formal consequence of the Hill-Hopkins-Ravenel (HHR) slice theorem [7] that, taking the limit in n, one recovers the slice tower of $BP_{\mathbb{R}}$.

The $n = 1$ case may then be used as the base case for an induction argument which allows us to reduce to showing that

$$(N_{C_2}^{C_{2^n}} MU_{\mathbb{R}})^{C_{2^n}} \to (N_{C_2}^{C_{2^n}} MU_{\mathbb{R}})^{hC_{2^n}}$$

is an equivalence. This map may be analyzed by a separate induction argument that originates in various inductive proofs of versions of the Segal Conjecture. In [17], Ravenel showed that the Segal conjecture for C_{p^n} follows from the case $n = 1$; he provided both a computational approach via a modified Adams spectral sequence as well as an approach using explicit geometric constructions. Bokstedt, Bruner, Lunoe-Nielsen, and Rognes [2] generalized the geometric approach and proved the following:

Theorem 1.2. ([2], Theorem 2.5) *Let X be a C_{p^n}-spectrum. Suppose for each $Y \in \{X, \Phi^{C_p}(X), \ldots, \Phi^{C_{p^{n-1}}}(X)\}$ that $\pi_*(Y)$ is bounded below, $H_*(Y)$ is of finite type, and $Y^{C_p} \to Y^{hC_p}$ is a p-complete equivalence. Then $X^G \to X^{hG}$ is a p-complete equivalence.*

In [16], Nikolaus and Scholze strengthen this result by giving a description of the subcategory of genuine C_{p^n}-spectra whose geometric fixed points spectra are bounded below in terms of iterated pullbacks and gluing maps. In the case of $MU^{((C_{2^n}))}$, we identify these gluing maps with either the map in the $n = 1$ case, or maps of the form

$$MO^{\wedge k} \to (N_e^{C_2}(MO^{\wedge k}))^{tC_2}$$

Each of these in the latter case is an equivalence by the Segal conjecture:

Theorem 1.3. *For any bounded below spectrum X, the Tate diagonal*

$$X \to (N_e^{C_2}(X))^{tC_2}$$

is a 2-complete equivalence.

This theorem was shown for X with finitely generated homotopy groups by Lunøe-Nielsen and Rognes ([11], 5.13) and for all X bounded below by Nikolaus and Scholze ([16], III.1.7). These both rest on the the case $X = S^0$, due to Lin:

Theorem 1.4. [10] *Let γ denote the canonical line bundle over $\mathbb{RP}^\infty$, and for each integer $n > 0$, let $\mathbb{RP}^\infty_{-n}$ denote the Thom spectrum of $-n\gamma$. Then there is an equivalence of spectra*

$$\mathbb{RP}^\infty_{-\infty} = \mathrm{holim}_n \mathbb{RP}^\infty_{-n} \simeq (S^{-1})\hat{}_2$$

We refer the reader to the introduction of [5] for a discussion of the different forms of the Segal conjecture for C_2 and their relation to Lin's theorem. Lin's theorem follows from a difficult calculation of a continuous Ext group

$$\widehat{\mathrm{Ext}}_{\mathcal{A}}(H^*(\mathbb{RP}^\infty_{-\infty}; \mathbb{F}_2); \mathbb{F}_2)$$

where $\mathcal{A}$ is the Steenrod algebra. Nikolaus and Scholze showed, however, that 1.3 follows formally for all X bounded below from the case $X = H\mathbb{F}_2$. Hahn and Wilson [5] used this to show that 1.3 can be established by analysis of the descent spectral sequence for the map

$$N_e^{C_2} H\mathbb{F}_2 \to H\underline{\mathbb{F}_2}$$

which reduces to a continuous Ext group calculation over a much smaller polynomial coalgebra $\mathbb{F}_2[x]$. We use Lin's theorem to prove the following, from which 1.1 follows.

Theorem 1.5. *Let Y be a bounded below C_2-spectrum. If $Y^{\wedge 2^k}$ is a cofree C_2-spectrum for all $0 \leq k < n$, then $N_{C_2}^{C_{2^n}} Y$ is cofree.*

Our argument for 1.1 may be reversed: knowing that

$$(MU^{((C_{2^n}))})^{C_{2^n}} \to (MU^{((C_{2^n}))})^{hC_{2^n}}$$

is an equivalence may be used to show that the corresponding gluing maps are equivalences, and, using the reduction of Nikolaus and Scholze to the case of $X = H\mathbb{F}_2$, we may deduce the Segal conjecture for C_2. This gives a proof of the Segal conjecture for C_2 that involves no homological algebra - apart from the Tate orbit lemma of Nikolaus and Scholze - and proceeds from a chromatic approach. In particular, the main piece of our argument that is not formal is the use of the HHR slice theorem.

Remark 1.6. Essential to our proof is the identification $\Phi^{C_2}(N_{C_2}^{C_4} BP_{\mathbb{R}}) \simeq N_e^{C_2} H\mathbb{F}_2$. In [15], Meier, Shi, and Zeng use this identification to deduce differentials in the homotopy fixed point spectral sequence of $N_e^{C_2} H\mathbb{F}_2$ from differentials in the slice spectral sequence of $N_{C_2}^{C_4} BP_{\mathbb{R}}$. Our results should shed light on these spectral sequences.

In particular, the map from the Slice SS of $N_{C_2}^{C_4} BP_{\mathbb{R}}$ to its HFPSS is an isomorphism below a line of slope 3 (see [18]). The Slice SS vanishes above this line, but there are many classes above this line in the HFPSS. By Theorem 1.1, the map between them must give an isomorphism on their E_∞-pages, so there must be some pattern of differentials killing all the classes above this line in the HFPSS.

Summary. In Section 2, we show that the cofreeness of $MU^{((G))}$ follows formally from (and is equivalent to) the Hu-Kriz $n = 1$ case together with Lin's theorem. This is the most direct way to Theorem 1.1, using these known results. In Section 3, we withhold knowledge of these theorems and give a different proof - via chromatic hypercubes - that $BP^{((C_4))}$ is cofree. In turn, this result implies the $n = 1$ case and Lin's Theorem, which then gives the result for $n > 2$ by the same induction used in Section 2.

Notation and Conventions. We use Sp^G to denote the category of orthogonal G-spectra or the associated ∞-category given by taking the homotopy coherent nerve of bifibrant objects in the stable model structure of Mandell and May [14]. We use the notation $MU^{((G))}$ and $BP^{((G))}$ to denote $N_{C_2}^G MU_{\mathbb{R}}$ and $N_{C_2}^G BP_{\mathbb{R}}$ respectively, as in HHR.

Acknowledgments. The $n = 1$ case of our chromatic hypercubes result - namely that $BP_{\mathbb{R}} = \text{holim}_n \tilde{L}_n BP_{\mathbb{R}}$ - is due to Mike Hill. It was his idea to use this approach to establish the $n > 1$ cases. We thank him for introducing us to this problem and for his guidance throughout the project.

2. Cofreeness and Gluing Maps

2.1. **Cofreeness.** We begin by reviewing the notion of cofreeness for a genuine G-spectrum.

Proposition 2.1. *For $X \in Sp^G$, the following are equivalent*

(1) $X \to F(EG_+, X)$ *is an equivalence of G-spectra.*
(2) $X^H \to X^{hH}$ *is an equivalence of spectra for all $H \subset G$.*
(3) X *is G_+-local.*

Proof. For $1 \iff 3$, it suffices to show that $L_{G_+}(X) = F(EG_+, X)$. The map

$$X \to F(EG_+, X)$$

becomes an equivalence after smashing with G_+ by the Frobenius relation, and the target is G_+-local because if $Z \wedge G_+ \simeq *$, then

$$[Z, F(EG_+, X)]^G = [Z \wedge EG_+, X]^G = 0$$

as EG_+ is in the localizing subcategory generated by G_+. $1 \iff 2$ follows from the fact that the fixed point functors $(-)^H$ are jointly conservative, and

$$i_H^G(F(EG_+, X)) = F(EH_+, i_H^G X)$$

as can be seen from the more general statement

$$i_H^G(L_E(X)) = L_{i_H^G E}(i_H^G X)$$

(see [3], 3.2). $\square$

Definition 2.2. We say a G-spectrum X is *cofree* if any of the equivalent conditions in 2.1 hold.

Corollary 2.3. *The category of cofree G-spectra is closed under homotopy limits.*

Proof. This is true of any category of E-locals. $\square$

Remark 2.4. Cofree G-spectra are often called *Borel complete*, or just *Borel*. The source of this terminology is the fact that there is a forgetful functor

$$Sp^G \to \text{Fun}(BG, Sp)$$

from genuine G-spectra to so-called Borel G-spectra. For formal reasons, this functor admits a right adjoint, and it is not hard to show that this right adjoint is an equivalence onto the full subcategory of cofree G-spectra.

We will make use of the slice filtration on G-spectra, introduced for C_2-spectra by Dugger [4] and generalized to all finite groups G by HHR [7]. To fix notions, we use the *regular* slice filtration, as in Ullman [18], although for the G-spectra we consider, using the original slice filtration in HHR would not change anything. Let $X \geq n$ denote that a G-spectrum is slice $\geq n$, i.e. X is slice $(n-1)$-connected. We need the following useful lemma:

Lemma 2.5. *Suppose $\{X_i\}_{i\in\mathbb{N}}$ is a family of G-spectra such that, for all $n \in \mathbb{Z}$, all but finitely many X_i have the property that $X_i \geq n$. Then the canonical map*

$$\bigvee_i X_i \to \prod_i X_i$$

is an equivalence.

Proof. It suffices to show that, for all $n \in \mathbb{Z}$, the map of Mackey functors

$$\bigoplus_i \underline{\pi}_n(X_i) \cong \underline{\pi}_n\left(\bigvee_i X_i\right) \to \underline{\pi}_n\left(\prod_i X_i\right) \cong \prod_i \underline{\pi}_n(X_i)$$

is an isomorphism. This follows immediately from the observation that for all but finitely many i, $\underline{\pi}_n(X_i) = 0$. Indeed, by ([7], 4.40), if $Y \geq n$, then $\underline{\pi}_k(Y) = 0$ for $k < \lceil n/|G| \rceil$. $\square$

Proposition 2.6. *If $MU_{\mathbb{R}}$ is cofree, then $MU_{\mathbb{R}}^{\wedge n}$ is cofree for all $n \geq 1$, and similarly for $BP_{\mathbb{R}}^{\wedge n}$.*

Proof. We proceed by induction on n. Since $MU_{\mathbb{R}}^{\wedge(n-1)}$ is Real-oriented, we have

$$MU_{\mathbb{R}}^{\wedge n} = MU_{\mathbb{R}}^{\wedge(n-1)}[\overline{b_1}, \overline{b_2}, \ldots] = \bigvee_{m\in M} S^{\frac{|m|}{2}\rho} \wedge MU_{\mathbb{R}}^{\wedge(n-1)}$$

where M is a monomial basis of $\mathbb{Z}[b_1, b_2, \ldots]$. By the lemma, the canonical map

$$\bigvee_{m\in M} S^{\frac{|m|}{2}\rho} \wedge MU_{\mathbb{R}}^{\wedge(n-1)} \to \prod_{m\in M} S^{\frac{|m|}{2}\rho} \wedge MU_{\mathbb{R}}^{\wedge(n-1)}$$

is an equivalence, as $MU_{\mathbb{R}}^{\wedge(n-1)} \geq 0$ and $S^{k\rho} \geq 2k$, so that $S^{k\rho} \wedge MU_{\mathbb{R}}^{\wedge(n-1)} \geq 2k$ by ([7], 4.26). This completes the proof, as the category of cofree C_2-spectra is closed under limits and smashing with a dualizable C_2-spectrum, hence the target is cofree. $\square$

2.2. **Gluing maps and cofreeness.** We set up an inductive argument to prove Theorem 1.5. To fix notation, we use $\Phi^{C_{2^k}}$ to denote the functor $Sp^{C_{2^n}} \to Sp$ and $\widetilde{\Phi}^{C_{2^k}}$ to denote the functor $Sp^{C_{2^n}} \to Sp^{C_{2^{n-k}}}$, so that $i_e^{C_{2^{n-k}}} \circ \widetilde{\Phi}^{C_{2^k}} = \Phi^{C_{2^k}}$. Nikolaus and Scholze use a result of Hesselholt and Madsen ([6], 2.1) along with their Tate orbit lemma, to show:

Proposition 2.7. ([16], Corollary II.4.7) *If $X \in Sp^G$ has the property that $\Phi^{C_{2^k}} X \in Sp$ is bounded below for all $0 \leq k < n$, there is a homotopy limit diagram*

$$\begin{tikzcd}
X^{C_{2^n}} \arrow[rrrr] \arrow[dddd] & & & & \Phi^{C_{2^n}} X \arrow[d] \\
& & & (\widetilde{\Phi}^{C_{2^{n-1}}} X)^{hC_2} \arrow[r] \arrow[d] & (\widetilde{\Phi}^{C_{2^{n-1}}} X)^{tC_2} \\
& & (\widetilde{\Phi}^{C_4} X)^{hC_{2^{n-2}}} \arrow[r] \arrow[d] & \cdots & \\
& \left(\widetilde{\Phi}^{C_2} X\right)^{hC_{2^{n-1}}} \arrow[r] \arrow[d] & \left((\widetilde{\Phi}^{C_2} X)^{tC_2}\right)^{hC_{2^{n-2}}} & & \\
X^{hC_{2^n}} \arrow[r] & \left(X^{tC_2}\right)^{hC_{2^{n-1}}} & & &
\end{tikzcd}$$

Theorem 2.8. *Let Y be a bounded below C_2-spectrum. If $Y^{\wedge 2^k}$ is a cofree C_2-spectrum for all $0 \leq k < n$, then $N_{C_2}^{C_{2^n}} Y$ is cofree.*

Proof. Set $X := N_{C_2}^{C_{2^n}} Y$. We proceed by induction on n, with the base case $n = 1$ being tautological. For all $1 \leq k < n$,

$$i_{C_{2^{n-k}}}^{C_{2^n}} X = N_{C_2}^{C_{2^{n-k}}} (Y^{\wedge 2^k})$$

is cofree by induction, so it suffices to show the map $X^{C_{2^n}} \to X^{hC_{2^n}}$ is an equivalence. Since Y is bounded below, so is X, and this map is an equivalence if all of the short vertical maps in 2.7 are equivalences. Each such map is of the form

$$(f)^{hC_{2^{n-k}}} : \left(\widetilde{\Phi}^{C_{2^k}} X\right)^{hC_{2^{n-k}}} \to \left((\widetilde{\Phi}^{C_{2^{k-1}}} X)^{tC_2}\right)^{hC_{2^{n-k}}}$$

for $k > 0$, which is induced by the map in $Sp^{C_{2^{n-k}}}$

$$f : \widetilde{\Phi}^{C_{2^k}} X \to (\widetilde{\Phi}^{C_{2^{k-1}}} X)^{tC_2}$$

It therefore suffices to show that for all $k > 0$, f is an equivalence of Borel $C_{2^{n-k}}$-spectra, which by definition is simply an underlying equivalence. The underlying map is the natural map

$$\Phi^{C_2}\left(i_{C_2}^{C_{2^{n-k+1}}} \tilde{\Phi}^{C_{2^{k-1}}} X\right) \to \left(i_{C_2}^{C_{2^{n-k+1}}} \tilde{\Phi}^{C_{2^{k-1}}} X\right)^{tC_2}$$

so it suffices to show $i_{C_2}^{C_{2^{n-k+1}}} \tilde{\Phi}^{C_{2^{k-1}}} X$ is a cofree C_2-spectrum. When $k = 1$, we have

$$i_{C_2}^{C_{2^{n-k+1}}} \tilde{\Phi}^{C_{2^{k-1}}} X \simeq Y^{\wedge 2^{n-1}}$$

and for $k > 1$, one has

$$i_{C_2}^{C_{2^{n-k+1}}} \tilde{\Phi}^{C_{2^{k-1}}} X \simeq i_{C_2}^{C_{2^{n-k+1}}} (N_e^{C_{2^{n-k+1}}} (\Phi^{C_2} Y)) \simeq N_e^{C_2}\left(\Phi^{C_2}(Y^{\wedge 2^{n-k}})\right)$$

using the identification

$$\widetilde{\Phi}^{C_{2^k}} X \simeq N_e^{C_{2^{n-k}}} (\Phi^{C_2} Y)$$

(see [15], Theorem 2.2). $N_e^{C_2} \Phi^{C_2}(Y^{\wedge 2^{n-k}})$ is cofree by Lin's theorem: since $Y^{\wedge 2^{n-k}}$ is bounded below and cofree,

$$\Phi^{C_2}(Y^{\wedge 2^{n-k}}) \simeq (Y^{\wedge 2^{n-k}})^{tC_2}$$

is bounded below and 2-complete. □

Remark 2.9. This result has various converses. For example, if Y is a bounded below C_2-spectrum, then $N_{C_2}^{C_{2^k}} Y$ is cofree for all $1 \leq k \leq n$ if and only if $Y^{\wedge 2^k}$ is a cofree C_2-spectrum for all $0 \leq k < n$. The other direction follows because if $N_{C_2}^{C_{2^{k+1}}} Y$ is cofree, then $Y^{\wedge 2^k} = i_{C_2}^{C_{2^{k+1}}} N_{C_2}^{C_{2^{k+1}}} Y$ is also cofree.

If Y is also a ring spectrum, then the direct converse of 2.8 is true: $N_{C_2}^{C_{2^n}} Y$ is cofree if and only if $Y^{\wedge 2^k}$ is a cofree C_2-spectrum for all $0 \leq k < n$. This follows because $Y^{\wedge 2^k}$ is a retract of $Y^{\wedge 2^{n-1}} = i_{C_2}^{C_{2^n}} N_{C_2}^{C_{2^n}} Y$ in this case.

Corollary 2.10. *For all $n \geq 1$, $MU^{((C_{2^n}))}$ is cofree, and similarly for $BP^{((C_{2^n}))}$.*

Proof. $MU_{\mathbb{R}}$ is bounded below, so this follows immediately from 2.6, the Hu-Kriz $n = 1$ case, and the theorem. □

We have shown that the case $n = 1$, due to Hu and Kriz, along with Lin's theorem, implies that $MU^{((C_{2^n}))}$ is cofree for all $n \geq 1$. The argument can be reversed to point to another proof of Lin's theorem, namely:

Proposition 2.11. *For any $n > 1$, the cofreeness of $MU^{((C_{2^n}))}$ implies both Lin's theorem and the $n = 1$ case.*

Proof. If for any $n > 1$, $MU^{((C_{2^n}))}$ is cofree, then a smash power of $BP^{((C_4))}$ is cofree, and it follows that $BP^{((C_4))}$ is cofree, as a retract; similarly for $BP_{\mathbb{R}}$ and therefore for its smash powers by 2.6. In this case, the limit diagram in 2.7 is as follows:

$$\begin{tikzcd}
(BP^{((C_4))})^{C_4} \arrow[rr] \arrow[dd] & & \Phi^{C_4}(BP^{((C_4))}) \arrow[d] \\
& (\widetilde{\Phi}^{C_2} BP^{((C_4))})^{hC_2} \arrow[r] \arrow[d] & (\widetilde{\Phi}^{C_2} BP^{((C_4))})^{tC_2} \\
(BP^{((C_4))})^{hC_4} \arrow[r] & ((BP^{((C_4))})^{tC_2})^{hC_2} &
\end{tikzcd}$$

The lefthand vertical arrow is an equivalence by assumption, and the middle arrow is an equivalence since $BP_{\mathbb{R}} \wedge BP_{\mathbb{R}}$ is cofree. We find that the righthand vertical map is an equivalence, and this is the Tate diagonal $H\mathbb{F}_2 \to (N_e^{C_2} H\mathbb{F}_2)^{tC_2}$, which is an equivalence if and only if Lin's theorem holds, by ([16], III.1.7). □

3. Chromatic Hypercubes

3.1. **Generalities on Hypercubes.** We give some general results on hypercubes that look like (summands of) our chromatic hypercubes. In this section, we use the language of ∞-categories following [13]; in particular, we work in the model of quasicategories, and use stable ∞-categories following [12]. For a discussion of cubical diagrams in the context of ∞-categories, see ([12], Section 6) or [1].

We fix $\mathcal{C}$ a stable ∞-category that has all finite limits. Let $[n]$ denote the totally ordered set $\{1, \ldots, n\}$. For T a totally ordered set, let $\mathcal{P}(T)$ denote its power set, regarded as a poset under inclusion. Let $\mathcal{P}_0(T)$ denote the sub-poset $\mathcal{P}(T) \setminus \{\emptyset\}$.

Definition 3.1. An n-cube $\mathcal{X}$ in $\mathcal{C}$ is a functor $\mathcal{X} : \mathcal{P}([n]) \to \mathcal{C}$, and a partial n-cube is a functor $\mathcal{P}_0([n]) \to \mathcal{C}$. We say an n-cube $\mathcal{X}$ is *cartesian* if the map

$$\mathcal{X}(\emptyset) \to \mathrm{holim}_{T \in \mathcal{P}_0([n])} \mathcal{X}(T)$$

is an equivalence.

Construction 3.2. *Suppose for each $T \in \mathcal{P}([n])$, one has an object $C_T \in \mathcal{C}$. We construct inductively an n-cube $\mathcal{X}$ in $\mathcal{C}$ as follows:*

(1) *When $n = 1$, $\mathcal{X}$ is the canonical inclusion $C_\emptyset \to C_\emptyset \oplus C_{\{1\}}$.*
(2) *We may assume inductively that we have constructed $(n-1)$-cubes $\mathcal{Y}_0$ and $\mathcal{Y}_1$ with*

$$\mathcal{Y}_0(T) = \bigoplus_{S \leq T} C_S \quad \text{and} \quad \mathcal{Y}_1(T) = \bigoplus_{S \leq T} C_{S \cup \{n\}}$$

for $T \in \mathcal{P}_0([n-1])$, where the maps in $\mathcal{Y}_0$ and $\mathcal{Y}_1$ are the canonical inclusions. $\mathcal{X}$ is then given by the canonical inclusion of $(n-1)$-cubes $\mathcal{Y}_0 \to \mathcal{Y}_0 \oplus \mathcal{Y}_1$, via the identification $\mathrm{Fun}(\mathcal{P}([n]), \mathcal{C}) = \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\mathcal{P}([n-1]), \mathcal{C}))$.

Definition 3.3. Suppose for each $T \in \mathcal{P}([n])$, one has an object $C_T \in \mathcal{C}$ and $C_\emptyset = *$. Let $\mathcal{X}$ be the associated n-cube as in 3.2, and define a partial n-cube

$$\mathcal{Y} : \mathcal{P}_0([n]) \hookrightarrow \mathcal{P}([n]) \xrightarrow{\mathcal{X}} \mathcal{C}$$

We say a partial n-cube in $\mathcal{C}$ is *built from disjoint split inclusions* if it is equivalent to $\mathcal{Y}$ for some choice of objects $\{C_T\}_{T \in \mathcal{P}_0([n])}$. If $\mathcal{X}$ is a cartesian n-cube such that the corresponding partial n-cube is built from disjoint split inclusions, we say $\mathcal{X}$ is a cartesian n-cube built from disjoint split inclusions.

To make this definition clearer, note that any partial 2-cube built from disjoint split inclusions is equivalent to one of the form

$$\begin{tikzcd} & C_2 \arrow[d] \\ C_1 \arrow[r] & C_1 \oplus C_2 \oplus C_{12} \end{tikzcd}$$

and any partial 3-cube built from disjoint split inclusions is equivalent to one of the form

$$\begin{tikzcd} C_3 \arrow[rr] \arrow[dd] & & C_2 \oplus C_3 \oplus C_{23} \arrow[dd] \\ & C_2 \arrow[ur] \arrow[dd] & \\ C_1 \oplus C_3 \oplus C_{13} \arrow[rr] & & C_1 \oplus C_2 \oplus C_3 \oplus C_{12} \oplus C_{13} \oplus C_{23} \oplus C_{123} \\ C_1 \arrow[u] \arrow[r] & C_1 \oplus C_2 \oplus C_{12} \arrow[ur] & \end{tikzcd}$$

where the inclusions are the canonical ones. We want to identify the limit of a diagram of this form, and we use a result of Antolín-Camarena and Barthel on computing limits of cubical diagrams inductively:

Proposition 3.4. ([1], 2.4) *Let $\mathcal{X} : \mathcal{P}_0([n]) \to \mathcal{C}$ be a partial n-cube in $\mathcal{C}$. One has a pullback square*

$$\begin{tikzcd} \mathrm{holim}_{S\in\mathcal{P}_0([n])}\mathcal{X}(S) \arrow[r] \arrow[d] & \mathrm{holim}_{S\in\mathcal{P}_0([n-1])}\mathcal{X}(S) \arrow[d] \\ \mathcal{X}(\{n\}) \arrow[r] & \mathrm{holim}_{S\in\mathcal{P}_0([n-1])}\mathcal{X}(S\cup\{n\}) \end{tikzcd}$$

Proposition 3.5. *Let $\mathcal{X}$ be a partial n-cube in $\mathcal{C}$ built from disjoint split inclusions so that it is equivalent to $\mathcal{Y}$ for some choice of objects $\{C_T\}_{T\in\mathcal{P}_0([n])}$ as in 3.3. Then $\mathcal{X}$ satisfies*

(1) $\mathrm{holim}_{S\in\mathcal{P}_0([n])}\mathcal{X}(S) \simeq \Omega^{n-1}C_{\{1,\dots,n\}}$
(2) *The map*

$$\mathrm{holim}_{S\in\mathcal{P}_0([n])}\mathcal{X}(S) \to \mathrm{holim}_{S\in\mathcal{P}_0([n-1])}\mathcal{X}(S)$$

is nullhomotopic.

Proof. We proceed by induction on n. For $n = 1$, a cartesian 1-cube is an equivalence

$$\mathrm{holim}_{S\in\mathcal{P}_0([1])}\mathcal{X}(S) \xrightarrow{\simeq} \mathcal{X}(\{1\})$$

and the map in (2) is the map to the terminal object. It is straightforward to show that the partial $(n-1)$-cube

$$\mathcal{P}_0([n-1]) \to \mathcal{P}_0([n]) \xrightarrow{\mathcal{X}} \mathcal{C}$$

is built from disjoint split inclusions, and

$$\mathcal{P}_0([n-1]) \xrightarrow{-\cup\{n\}} \mathcal{P}_0([n]) \xrightarrow{\mathcal{X}} \mathcal{C}$$

is of the form $C_{\{n\}} \oplus \mathcal{Z}$ where $\mathcal{Z}$ is a partial $(n-1)$-cube built from disjoint split inclusions using the objects $\{C_T \oplus C_{T\cup\{n\}}\}_{T\in\mathcal{P}_0([n-1])}$, as in 3.3. By induction, 3.4 gives a pullback square

$$\begin{tikzcd} \mathrm{holim}_{S\in\mathcal{P}_0([n])}\mathcal{X}(S) \arrow[r] \arrow[d] & \Omega^{n-2}C_{\{1,\dots,n-1\}} \arrow[d] \\ C_{\{n\}} \arrow[r] & C_{\{n\}} \oplus \Omega^{n-2}C_{\{1,\dots,n-1\}} \oplus \Omega^{n-2}C_{\{1,\dots,n\}} \end{tikzcd}$$

which is a cartesian 2-cube built from disjoint split inclusions. It therefore suffices to prove the proposition in the case $n = 2$, which is the claim that for objects $C_1, C_2, C_{12} \in \mathcal{C}$, there is a pullback square of the form

$$\begin{tikzcd} \Omega C_{12} \arrow[r, "0"] \arrow[d, "0"'] & C_2 \arrow[d] \\ C_1 \arrow[r] & C_1 \oplus C_2 \oplus C_{12} \end{tikzcd}$$

One may form a morphism of partial 2-cubes

$$\begin{tikzcd} & * \arrow[d] \\ * \arrow[r] & C_{12} \end{tikzcd} \implies \begin{tikzcd} & C_2 \arrow[d] \\ C_1 \arrow[r] & C_1 \oplus C_2 \oplus C_{12} \end{tikzcd}$$

via 3.2 which, taking limits, constructs such a square. Taking fibers along the vertical maps, one has the identity map of $\Omega C_1 \oplus \Omega C_{12}$; the square is therefore cartesian by ([1], 2.2). □

3.2. **Chromatic Hypercubes and Slice Towers.** We introduce the chromatic n-cubes we need to prove Theorem 1.1 and show they split as a summand that is constant in n and a cartesian n-cube built from disjoint split inclusions.

Definition 3.6. Consider the following hypercubes:

(1) Let $\mathcal{H}_n$ be the cartesian n-cube so that for $\{i_1, \ldots, i_j\} \in \mathcal{P}_0([n])$

$$\mathcal{H}_n(\{i_1, \ldots, i_j\}) = BP^{((C_4))}[(N(\overline{t_{i_1}}) \cdots N(\overline{t_{i_j}}))^{-1}]$$

One may form this cube inductively in a manner similar to 3.2 by working in the category of $MU^{((C_4))}_{(2)}$-modules and applying the functors $(-)[N(\overline{t_i})^{-1}]$. See ([1], 3.1) for a similar construction.

(2) Let $\mathcal{S}_{n,d}$ be the cartesian n-cube defined on $\mathcal{P}_0([n])$ by

$$\mathcal{S}_{n,d} : \mathcal{P}_0([n]) \xrightarrow{\mathcal{H}_n} Sp^{C_4} \xrightarrow{P^{2d}_{2d}} Sp^{C_4}$$

where P^{2d}_{2d} is the $2d$-slice functor.

To understand the n-cubes $\mathcal{S}_{n,d}$, we need to determine the slices of

$$BP^{((C_4))}_{i_1,\ldots,i_j} := BP^{((C_4))}[(N(\overline{t_{i_1}}) \cdots N(\overline{t_{i_j}}))^{-1}]$$

and this follows as expected from the HHR slice theorem for $BP^{((C_4))}$. We follow the discussion in ([7], Section 6) and use their notation: in $\pi^u_*(BP^{((C_4))}) = \pi_*(BP \wedge BP)$, there are classes $\{t_i\}_{i\geq 1}$ with the property that

$$\pi^u_*(BP^{((C_4))}) = \mathbb{Z}_{(2)}[t_i, \gamma(t_i) : i \geq 1]$$

as a C_4-algebra, where γ is the generator of C_4 and $\gamma^2(t_i) = -t_i$. Inverting the classes above, we see that

$$\pi^u_*(BP^{((C_4))}_{i_1,\ldots,i_j}) = \mathbb{Z}_{(2)}[t_i, \gamma(t_i) : i \geq 1][(t_{i_1} \cdots t_{i_j} \gamma(t_{i_1}) \cdots \gamma(t_{i_j}))^{-1}]$$

and the restriction map

$$\pi^{C_2}_{*\rho_2}(BP^{((C_4))}_{i_1,\ldots,i_j}) \to \pi^u_{2*}(BP^{((C_4))}_{i_1,\ldots,i_j})$$

is a split surjection (in fact an isomorphism, but we don't need this). Lifting the classes t_i along this map, we have an associative algebra map

$$A := S^0[\overline{t_i} : i \geq 1][(\overline{t_{i_1}} \cdots \overline{t_{i_j}})^{-1}] \to i^{C_4}_{C_2} BP^{((C_4))}_{i_1,\ldots,i_j}$$

Using the method of twisted monoid rings (and the C_4-commutative ring structure on $MU^{((C_4))}_{(2)}$), this gives a map

$$S^0[C_4 \cdot \overline{t_i} : i \geq 1][C_4 \cdot (\overline{t_{i_1}} \cdots \overline{t_{i_j}})^{-1}] \to BP^{((C_4))}_{i_1,\ldots,i_j}$$

Proposition 3.7. *The above map*

$$S^0[C_4 \cdot \overline{t_i} \,:\, i \geq 1][C_4 \cdot (\overline{t_{i_1}} \cdots \overline{t_{i_j}})^{-1}] \to BP^{((C_4))}_{i_1,\ldots,i_j}$$

gives a refinement of homotopy. Let M_d be the monomial ideal in A consisting of the slice sphere summands of underlying dimension $\geq d$, and set

$$K_d = BP^{((C_4))}_{i_1,\ldots,i_j} \wedge_A M_d$$

Then the cofiber sequences

$$P_{2d+1}(BP^{((C_4))}_{i_1,\ldots,i_j}) \to BP^{((C_4))}_{i_1,\ldots,i_j} \to P^{2d}(BP^{((C_4))}_{i_1,\ldots,i_j})$$

are equivalent to

$$K_{2d+2} \to BP^{((C_4))}_{i_1,\ldots,i_j} \to BP^{((C_4))}_{i_1,\ldots,i_j}/K_{2d+2},$$

$P^{2d+1}BP^{((C_4))}_{i_1,\ldots,i_j} \simeq P^{2d}BP^{((C_4))}_{i_1,\ldots,i_j}$, *and*

$$K_{2d}/K_{2d+2} \simeq H\underline{\mathbb{Z}}_{(2)} \wedge M_{2d}/M_{2d+2}$$

Proof. The proof is identical to that of ([7], Section 6), where the last identification follows from the key computation: the reduction theorem. □

Remark 3.8. The previous proposition should be interpreted as follows: the slice tower for $BP^{((C_4))}_{i_1,\ldots,i_j}$ forgets to the ordinary Postnikov tower of $i_e^{C_4} BP^{((C_4))}_{i_1,\ldots,i_j}$, which has $P^{2d-1}_{2d-1} \simeq *$ and

$$P^{2d}_{2d} \simeq H\mathbb{Z}_{(2)} \wedge W_{2d}$$

where W_{2d} is a wedge of S^{2d}'s over the set of monomials of degree $2d$ in

$$\pi^u_*(BP^{((C_4))}_{i_1,\ldots,i_j}) = \mathbb{Z}_{(2)}[t_i, \gamma(t_i) \,:\, i \geq 1][(t_{i_1} \cdots t_{i_j} \gamma(t_{i_1}) \cdots \gamma(t_{i_j}))^{-1}]$$

The slice tower is an equivariant refinement of this wherein the odd slices vanish, $H\mathbb{Z}_{(2)}$ is replaced with $H\underline{\mathbb{Z}}_{(2)}$, the spheres in W_{2d} corresponding to a summand of the above C_4-module with stabilizer C_2 are grouped with their conjugates in a

$$C_{4+} \wedge_{C_2} S^{d\rho_2},$$

the spheres corresponding to a C_4-fixed summand are replaced with $S^{\frac{d}{2}\rho_4}$, and there are no free summands. For $BP^{((C_4))}_{i_1,\ldots,i_j}$, we let $\widehat{W}^{i_1,\ldots,i_j}_{2d}$ denote the quotient M_{2d}/M_{2d+2} as above, and $\widehat{W}_{2d}$ the corresponding quotient for $BP^{((C_4))}$.

Note that for any $i_1, \ldots, i_j$, $\widehat{W}^{i_1,\ldots,i_j}_{2d}$ has $\widehat{W}_{2d}$ as a split summand, corresponding to the split inclusion

$$\pi^u_{2d}(BP^{((C_4))}) \hookrightarrow \pi^u_{2d}(BP^{((C_4))}_{i_1,\ldots,i_j})$$

This splitting is natural in $\{i_1, \ldots, i_j\}$, so we see that there is a splitting

$$\mathcal{S}_{n,d} \simeq (H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{W}_{2d}) \oplus \mathcal{X}_{n,d}$$

where $\mathcal{X}_{n,d}$ is a cartesian n-cube satisfying

$$\mathcal{X}_{n,d}(\{i_1, \ldots, i_j\}) = H\underline{\mathbb{Z}}_{(2)} \wedge (\widehat{W}^{i_1,\ldots,i_j}_{2d}/\widehat{W}_{2d})$$

We have the following connection to the generalities in 3.1:

Proposition 3.9. *$\mathcal{X}_{n,d}$ is a cartesian n-cube built from disjoint split inclusions.*

Proof. $\mathcal{X}_{n,d}$ is cartesian by definition. The result - and the terminology - follows from the fact that for any $\{i_1, \ldots, i_j\}$, the maps

$$\pi^u_*(BP^{((C_4))}_{i_k}) \hookrightarrow \pi^u_*(BP^{((C_4))}_{i_1,\ldots,i_j})$$

are split inclusions, and after factoring out $\pi^u_*(BP^{((C_4))})$, the maps

$$\iota_k : \frac{\pi^u_*(BP^{((C_4))}_{i_k})}{\pi^u_*(BP^{((C_4))})} \hookrightarrow \frac{\pi^u_*(BP^{((C_4))}_{i_1,\ldots,i_j})}{\pi^u_*(BP^{((C_4))})}$$

are split inclusions with the property that $\mathrm{im}(\iota_k) \cap \mathrm{im}(\iota_{k'}) = \{0\}$ for $k \neq k'$. In particular, for $T \leq T'$ in $\mathcal{P}_0(n)$, the map

$$\mathcal{X}_{n,d}(T) \to \mathcal{X}_{n,d}(T')$$

is the split inclusion of the free $H\underline{\mathbb{Z}}_{(2)}$-module on wedges of slice spheres corresponding to the split inclusion

$$\frac{\pi^u_{2d}(BP^{((C_4))}_T)}{\pi^u_{2d}(BP^{((C_4))})} \hookrightarrow \frac{\pi^u_{2d}(BP^{((C_4))}_{T'})}{\pi^u_{2d}(BP^{((C_4))})}$$

so the claim follows from the fact that

$$\frac{\pi^u_*(BP^{((C_4))}_{i_1,\ldots,i_j})}{\pi^u_*(BP^{((C_4))})} = \left(\bigoplus_{\substack{T<\{i_1,\ldots,i_j\} \\ T\in\mathcal{P}_0(n)}} \frac{\pi^u_*(BP^{((C_4))}_T)}{\pi^u_*(BP^{((C_4))})} \right) \oplus \frac{(t_{i_1} \cdots t_{i_j}\gamma(t_{i_1}) \cdots \gamma(t_{i_j}))^{-1}\pi^u_*(BP^{((C_4))})}{\pi^u_*(BP^{((C_4))})}$$

where the latter summand denotes the subgroup of $\frac{\pi^u_*(BP^{((C_4))}_{i_1,\ldots,i_j})}{\pi^u_*(BP^{((C_4))})}$ generated by monomials containing $(t_{i_1} \cdots t_{i_j}\gamma(t_{i_1}) \cdots \gamma(t_{i_j}))^{-1}$. $\square$

The following is an immediate consequence of 3.5 and 3.9:

Corollary 3.10. *The map $\mathcal{S}_{n,d}(\emptyset) \to \mathcal{S}_{n-1,d}(\emptyset)$ can be identified with*

$$(H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{W}_{2d}) \oplus \mathcal{X}_{n,d}(\emptyset) \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} (H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{W}_{2d}) \oplus \mathcal{X}_{n-1,d}(\emptyset)$$

3.3. **Proof of Main Theorem.** The canonical map $BP^{((C_4))} \to BP^{((C_4))}_{i_1,\ldots,i_j}$, by universal property, determines compatible maps $BP^{((C_4))} \to \mathcal{H}_n(\emptyset)$ so that there is a map

$$BP^{((C_4))} \to \mathrm{holim}_n \mathcal{H}_n(\emptyset)$$

We will show this map is an equivalence, and this will complete the proof that $BP^{((C_4))}$ is cofree by the following:

Proposition 3.11. $\mathrm{holim}_n \mathcal{H}_n(\emptyset)$ *is cofree.*

Proof. The category of cofree C_4-spectra is closed under limits, hence it suffices to show that each $\mathcal{H}_n(\emptyset)$ is cofree. There is by definition an equivalence

$$\mathcal{H}_n(\emptyset) \xrightarrow{\simeq} \mathrm{holim}_{T\in\mathcal{P}_0([n])}\mathcal{H}_n(T) = \mathrm{holim}_{\{i_1,\ldots,i_j\}\in\mathcal{P}_0([n])} BP^{((C_4))}_{i_1,\ldots,i_j}$$

so it suffices to show each $BP^{((C_4))}_{i_1,\ldots,i_j}$ is cofree. This is as in ([7], Section 10): we have that $\Phi^{C_4}(BP^{((C_4))}_{i_1,\ldots,i_j}) \simeq \Phi^{C_2}(BP^{((C_4))}_{i_1,\ldots,i_j}) \simeq *$, as

$$\Phi^{C_4}(N^{C_4}_{C_2}(\overline{t_{i_1}})) = \Phi^{C_2}(\overline{t_{i_1}}) = 0$$

and similarly

$$\Phi^{C_2}(N_{C_2}^{C_4}(\overline{t_{i_1}})) = \Phi^{C_2}(i_{C_2}^{C_4} N_{C_2}^{C_4}(\overline{t_{i_1}})) = \Phi^{C_2}(\overline{t_{i_1}} \cdot \overline{\gamma(t_{i_1})}) = 0$$

□

To show that the map

$$BP^{((C_4))} \to \text{holim}_n \mathcal{H}_n(\emptyset)$$

is an equivalence, we use the slice filtration to work in the ∞-category $\text{Fun}(\mathbb{Z}^{op}, Sp^G)$ of filtered G-spectra (see [12], 1.2.2). We refer to [19] for a treatment of the slice filtration in an ∞-categorical context. Let

$$\mathcal{T} : Sp^G \to \text{Fun}(\mathbb{Z}^{op}, Sp^G)$$

be the functor which associates to a G-spectrum its slice tower, which may be obtained as in ([12], 1.2.1.17). Functoriality gives a map of filtered C_4-spectra

$$f : \mathcal{T}(BP^{((C_4))}) \to \text{holim}_n \Big(\text{holim}_{\{i_1,\ldots,i_j\} \in \mathcal{P}_0([n])} \mathcal{T}(BP^{((C_4))}_{i_1,\ldots,i_j}) \Big)$$

Theorem 3.12. *$BP^{((C_4))}$ is cofree, independent of Lin's theorem.*

Proof. Let $\text{holim} : \text{Fun}(\mathbb{Z}^{op}, Sp^{C_4}) \to Sp^{C_4}$ be the functor sending a tower to its homotopy limit. Limits are computed pointwise in functor categories, so we find that

$$\text{holim}\Big(\text{holim}_{\{i_1,\ldots,i_j\} \in \mathcal{P}_0([n])} \mathcal{T}(BP^{((C_4))}_{i_1,\ldots,i_j}) \Big) \simeq \mathcal{H}_n(\emptyset)$$

It therefore suffices to show that f is an equivalence. Note that the filtration induced on $\mathcal{H}_n(\emptyset)$ is not its slice filtration, hence we use the notation

$$\widetilde{P}^k : \text{Fun}(\mathbb{Z}^{op}, Sp^{C_4}) \xrightarrow{\text{ev}_k} Sp^{C_4}$$

and

$$\widetilde{P}^k_k = \text{fib}(\widetilde{P}^k \to \widetilde{P}^{k-1})$$

Note that

$$\begin{aligned} \widetilde{P}^k_k \Big(\text{holim}_{\{i_1,\ldots,i_j\} \in \mathcal{P}_0([n])} \mathcal{T}(BP^{((C_4))}_{i_1,\ldots,i_j}) \Big) &\simeq \text{holim}_{\{i_1,\ldots,i_j\} \in \mathcal{P}_0([n])} \widetilde{P}^k_k \Big(\mathcal{T}(BP^{((C_4))}_{i_1,\ldots,i_j}) \Big) \\ &\simeq \begin{cases} * & k = 2d-1 \\ \mathcal{S}_{n,d}(\emptyset) & k = 2d \end{cases} \end{aligned}$$

The map $\widetilde{P}^{2d}_{2d}(f)$ is then identified with the map

$$H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{W}_{2d} \to \text{holim}_n((H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{W}_{2d}) \oplus \mathcal{X}_{n,d}) \simeq \text{holim}_n(H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{W}_{2d}) \oplus \text{holim}_n \mathcal{X}_{n,d}(\emptyset)$$

By 3.10, the lefthand summand is constant in n, and the righthand summand is pro-zero, hence the map is an equivalence.

To establish that f is an equivalence, it therefore suffices to show that

$$\text{colim}_k \widetilde{P}^k \Big(\text{holim}_n \Big(\text{holim}_{\{i_1,\ldots,i_j\} \in \mathcal{P}_0([n])} \mathcal{T}(BP^{((C_4))}_{i_1,\ldots,i_j}) \Big) \Big) \simeq *$$

i.e. that the filtration on $\text{holim}_n \mathcal{H}_n(\emptyset)$ *strongly* converges. Note that by ([7], 4.40), if $X \in Sp^{C_4}$, then $\underline{\pi}_l(P^k X) = 0$ for $l \geq \lceil (k+1)/4 \rceil$. Taking limits, it follows that

$$\underline{\pi}_l \Big(\widetilde{P}^k \Big(\text{holim}_{\{i_1,\ldots,i_j\} \in \mathcal{P}_0([n])} \mathcal{T}(BP^{((C_4))}_{i_1,\ldots,i_j}) \Big) \Big) = 0$$

for $l \geq \lceil (k+1)/4 \rceil$, and so

$$\underline{\pi}_l\left(\widetilde{P}^k\left(\mathrm{holim}_n\left(\mathrm{holim}_{\{i_1,\dots,i_j\}\in\mathcal{P}_0([n])}\mathcal{T}(BP^{((C_4))}_{i_1,\dots,i_j})\right)\right)\right)=0$$

for $l \geq \lceil (k+1)/4 \rceil$ by the Milnor sequence. It follows that, for any l, taking the colimit as $k \to -\infty$ of $\underline{\pi}_l$ gives zero. □

Remark 3.13. This result recovers the Hu-Kriz result that $BP_{\mathbb{R}}$ is cofree: since $BP^{((C_4))}$ is cofree, $i^{C_4}_{C_2}BP^{((C_4))} = BP_{\mathbb{R}} \wedge BP_{\mathbb{R}}$ is cofree, hence so is the retract $BP_{\mathbb{R}}$. Alternatively, as discussed in the introduction, one may argue similarly to 3.12 to show that $BP_{\mathbb{R}}$ is cofree, and the result in this case is due to Mike Hill.

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