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Equivariant stable homotopy and Segal's Burnside ring conjecture

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Introduction

In 1960, M. F. Atiyah proved the following:

THEOREM [7]. *Let BG denote the classifying space of a finite group G, and let KU^* denote representable complex periodic K theory (so $KU^0(X) = [X, BU \times \mathbf{Z}]$). Then we have*

$$KU^0(BG) \cong \hat{R}[G]$$

and

$$KU^1(BG) = 0$$

where $\hat{R}[G]$ denotes the completion of the complex representation ring at its augmentation ideal.

Analogous results were proved later for KO, in the generality of compact Lie groups, by Atiyah and Segal [8], and for $K\mathbf{F}_q$, the algebraic K-theory spectrum associated to the finite field $\mathbf{F}_q$, by Rector [29] using Quillen's [26] computation of $\pi_*(K\mathbf{F}_q)$.

In each case, the answer involves an appropriate completed representation ring of G, and the cohomology theory in question is constructed from the permutative category of finite dimensional vector spaces over a field (see [31]). If one considers cohomology theories constructed from other permutative categories, one expects to find analogous computations in terms of a "completed representation ring" of G in the given category, appropriately defined. In particular, stable cohomotopy, π_S^*, is constructed from the category of finite sets [31], and for this category the analogue of the representation ring is a well-known object, the *Burnside ring* $A(G)$ [13]. $A(G)$ is a commutative ring with augmentation, so one may speak of $\hat{A}(G)$, the completed Burnside ring. Moreover, there is

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a natural map $\hat{A}(G) \to \pi_S^0(BG^+)$ ($\pi_S^*(X)$ is defined for infinite complexes by taking inverse limits over skeletons of X; it is thus equipped with a topology). G. B. Segal was led to make the following conjecture:

SEGAL'S CONJECTURE (*weak form*) (see [1], [2]). *The map* $A(G) \to \pi_S^0(BG^+)$ *is an isomorphism.*

As it stands, the conjecture is very difficult to approach. Since it only involves π_S^0, and not π_S^n, an induction on the order of the group seems impossible. One is led to make a conjecture describing the entire structure of $\pi_S^*(BG^+)$.

To motivate the generalization, we recall that in [8], Atiyah and Segal developed a much simpler proof of Atiyah's original result in [7] by comparing $KU^*(BG)$ with an appropriately defined equivariant K-theory group $KU_G^*(pt)$, which is canonically isomorphic to the representation ring $R[G]$. One would like to construct an equivariant stable cohomotopy group $\pi_G^*(S^0)$, which should be computable and map to $\pi_S^*(BG^+)$, and hope to prove that this map becomes an isomorphism after completion. This construction was accomplished by Segal in [30], where he states the result $\pi_G^{-*}(S^0) \cong \bigoplus_K \pi_*^S(BW(K)^+)$, where the sum is over all conjugacy classes of subgroups of G and $W(K) = N_G(K)/K$. In particular, $\pi_G^0(S^0) \cong A(G)$, and $\pi_G^*(X)$ is an $A(G)$-module. We are led to:

SEGAL'S CONJECTURE (*strong form*). *The map* $\hat{\pi}_G^*(S^0) \to \pi_S^*(BG^+)$ *is an isomorphism, where* $\hat{\pi}_G^*(S^0)$ *denotes* $\pi_G^*(S^0)$ *completed at the augmentation ideal in* $A(G)$.

It is this form of the conjecture that we prove. We now outline the history of the conjecture. Having this formulation, one might hope to perform an induction on the order of the group using the Atiyah-Hirzebruch spectral sequence associated to an exact sequence of groups. This was precisely the technique used originally by Atiyah, but his proof relies heavily on the computability of the groups $KU^*(pt)$, using Bott periodicity. Of course, no such computation is available for $\pi_S^*(S^0)$, and the spectral sequence is not well-behaved, having non-zero groups in all positive and negative dimensions at the E_2-level. Therefore it was felt that one should attempt to calculate in some special cases, to determine whether the conjecture was even plausible. The case $G = \{e\}$ being trivial, the next simplest case is $G = \mathbf{Z}/2\mathbf{Z}$, where one is interested in computing $\pi_S^*(\mathbf{R}P^{\infty +})$. For reasons unrelated to Segal's conjecture, J. F. Adams [3] and M. E. Mahowald had suggested that if one was interested in $\mathbf{R}P^\infty$, one should study the Ext-groups $\mathrm{Ext}_{\mathscr{A}(2)}^{**}(L; \mathbf{Z}/2\mathbf{Z})$, where L is the graded ring of Laurent series in a one-dimensional generator x, equipped with an $\mathscr{A}(2)$-action compatible with that on $\mathbf{Z}/2\mathbf{Z}[x] \cong H^*(\mathbf{R}P^\infty; \mathbf{Z}/2\mathbf{Z})$. Here $\mathscr{A}(2)$ denotes the mod 2 Steenrod algebra. Although L is not the cohomology of a spectrum, it is a direct limit associated with an inverse system of spectra. There is in fact a map $\mathrm{Ext}^{*,*}_{\mathscr{A}(2)}(\mathbf{Z}/2\mathbf{Z}; \mathbf{Z}/2\mathbf{Z}) \to \mathrm{Ext}^{*+1,*+1}_{\mathscr{A}(2)}(L, \mathbf{Z}/2\mathbf{Z})$, and W. H. Lin [18] observed that in order to prove Segal's conjecture for $G = \mathbf{Z}/2\mathbf{Z}$, it was sufficient to prove that this map is an isomorphism. That this is the case was originally proved by Lin by performing intricate calculations in the Λ-algebra. The proof was much simplified by Lin, Davis, Mahowald, and Adams [19].

At this point, it seemed that one should attempt to determine how far Lin's methods could be extended. J. H. C. Gunawardena [14] was able to prove the conjecture for Laurent series in that case. D. Ravenel [28] settled the case $G = \mathbf{Z}/p^r\mathbf{Z}$, by modifying the Adams filtration and producing a new spectral sequence whose E_2-term was computable using the results of Lin and Gunawardena. From Ravenel's work, however, it was becoming clear that it would be very difficult to produce a proof of the full conjecture using these methods, since the Adams spectral sequence was unworkable even for $G = \mathbf{Z}/p^r\mathbf{Z}$, and Ravenel's modification of it depended heavily on the simple structure of the Burnside ring of $\mathbf{Z}/p^r\mathbf{Z}$.

Several other results concerning abelian groups were also proved. E. Laitinen [15] proved that the map $\hat{A}(G) \to \pi_S^*(BG^+)$ is injective for G elementary abelian, i.e. $G \cong (\mathbf{Z}/p\mathbf{Z})^k$. Segal and Stretch [33], [34] extended this to the case of all abelian groups. Laitinen calculated using ordinary characteristic classes of permutation representations, Segal and Stretch using MU-classes. This author [12] proved the weak form of the conjecture for $G = (\mathbf{Z}/2\mathbf{Z})^k$, using Brown-Gitler spectra and the Adams spectral sequence. Although only the weak form was proved here, the methods appear to be of independent interest. In particular, H. Miller has used them in his proof of the Sullivan conjecture, an unstable version of the Segal conjecture. Adams, Gunawardena, and Miller [6] were able to prove the strong form of the conjecture for $G = (\mathbf{Z}/p\mathbf{Z})^k$, using some ingenious calculations in the Adams spectral sequence, generalizing Lin's methods.

In order to proceed much further from this point, one must dispense with pure calculation and consider seriously the map $\pi_G^*(S^0) \to \pi_S^*(BG^+)$. That is, one must actually use properties of π_G^*, rather than simply computing $\pi_S^*(BG^+)$ and observing that the map is an isomorphism, since $\pi_S^*(BG^+)$ becomes increasingly uncomputable as G increases in complexity. It is this idea we carry out in this paper.

The method of proof is by reduction to a known case, namely $G = (\mathbf{Z}/p\mathbf{Z})^k$. May and McClure [21] show that Segal's conjecture (strong form) holds for all finite groups if and only if it holds for all p-groups. We show in this paper that it holds for all p-groups, if and only if it holds for $G = (\mathbf{Z}/p\mathbf{Z})^k$. Since this has been treated by Adams, Miller, and Gunawardena [6], the conjecture is confirmed.

In Section I we provide needed preliminaries from equivariant homotopy theory, and outline the proof. In Section II, we reduce the conjecture to the study of $\pi_G^*(S^{\infty V})$, where V is a particular orthogonal representation of G, $S^{\infty V} = \varinjlim_k S^{kV}$, and S^{kV} denotes the one point compactification of kV. Section III proves one half of the main inductive step (Theorem B of § I), using the Adams spectral sequence in a *non-computational* way. Sections IV, V, and VI prove the other half, using a careful study of the *singular locus*

$$\sum(X) = \bigcup_{\substack{H \neq e \\ H \subseteq G}} X^H$$

of a G-space under the action of a non-elementary abelian p-group G. Here, the work of Quillen [25] on the posets of subgroups of finite groups plays a crucial role. We supply two appendices. Appendix A constructs Thom spectra of virtual representations of finite groups, a construction which is required in Section III. Appendix B shows precisely how the inductive step fails for $G = (\mathbf{Z}/p\mathbf{Z})^k$, and how one can prove the conjecture for $(\mathbf{Z}/p\mathbf{Z})^k$ in this context, using only one of the calculations of [6]. An argument along these lines has also been developed by May and Priddy [22].

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I. Preliminaries

In this section, we will present the necessary preliminaries from equivariant stable homotopy theory. References are [5], [13], [16], [30].

Let G be a finite group. By a G-complex, we mean a based G-CW complex in the sense of [10], with only finitely many cells in each dimension. For any G-space X, let X^+ denote X with a disjoint base point, fixed under the action of G, added. Every G-complex is thus obtained by attaching cells of the form $G/H^+ \wedge D^n$, where $H \subseteq G, G/H$ denotes the left G-set of left cosets of H, and D^n denotes the standard n-disc, with a fixed choice of basepoint in $\partial D^n = S^{n-1}$, equipped with trivial action. Throughout, S^k will denote the k-sphere equipped with trivial G-action. X is equipped with a preferred choice of basepoint fixed under the G-action. If X and Y are G-complexes, $[X,Y]^G$ will denote the based G-homotopy classes of based G-maps from X to Y.

PROPOSITION I.1 (*see* [10]). *Let* $X \hookrightarrow Y$ *be an inclusion of G-complexes, and let Z denote the quotient complex* Y/X. *Then we have a long exact sequence of pointed sets*

$$\cdots \to [S^1 \wedge X, W]^G \to [Z,W]^G \to [Y,W]^G \to [X,W]^G$$

for any G-complex W. That is, the Puppe sequence holds for equivariant mapping sets.

For any G-complex X and subgroup $H \subseteq G$, let

$$X^H = \{x \in X | hx = x \forall h \in H\}.$$

Bredon has related properties of equivariant mapping sets with properties of X^H. For any CW complex X, let $\langle X \rangle = \min\{i | \pi_i(X) \neq 0\}$. Let $\dim(X)$ denote the dimension of X.

THEOREM I.2 (Bredon, see [10]). *Let X, Y, and Z be G-complexes. Suppose* $f\colon Y \to Z$ *is a G-map so that for all* $H \subseteq G$, $\pi_i(f^H)$ *is an isomorphism for all* $i \leq \dim(X^H) + 1$, *where* $f^H = f|Y^H$. *Then* $[\ \ ,f]^G\colon [X,Y]^G \to [X,Z]^G$ *is a bijection.*

COROLLARY I.3. *Suppose X and Z are G-complexes, with* $\langle Z^H \rangle > \dim(X^H) + 1$ *for all* $H \subseteq G$. *Then* $[X,Z]^G = *$.

Proof. Apply Theorem I.2 with $Y = *$, and $f\colon Y \to Z$ the inclusion of the base point. □

If V is a finite dimensional real G-module, we let S^V denote its one point compactification. S^V becomes a G-complex, and we choose the point ∞ as base point. One sees that $S^{U \oplus V} = S^U \wedge S^V$. Let $\mathcal{U}_G$ denote a real G-module which is a countable direct sum of finite dimensional real G-modules so that every irreducible representation of G occurs infinitely often. We assume $\mathcal{U}_G$ equipped with a G-invariant positive definite real valued inner product $\langle\ ,\ \rangle$. For $U \subseteq \mathcal{U}_G$, we let $U^\perp = \{v \in \mathcal{U}_G | \langle v, u\rangle = 0\ \forall u \in U\}$. For finite G-complexes (i.e., with finitely many cells) and arbitrary G-complexes Y, we define

$$\{X,Y\}^G = \varinjlim_{U \subseteq \mathcal{U}_G} [S^U \wedge X, S^U \wedge Y]^G.$$

Here, the direct limit is taken over the ordered set of all finite dimensional G-subspaces of $\mathcal{U}_G$ under inclusion, and the maps in the directed system are given by

$$[S^{U_1} \wedge X, S^{U_1} \wedge Y]^G \xrightarrow{\mathrm{id}_{U_1^\perp \cap U_2} \wedge} \left[S^{U_1^\perp \cap U_2} \wedge S^{U_1} \wedge X, S^{U_1^\perp \cap U_2} \wedge S^{U_1} \wedge Y\right]^G$$
$$\cong [S^{U_2}X, S^{U_2} \wedge Y]^G,$$

for $U_1 \subseteq U_2$.

The second morphism is defined using the inner product, which identifies $S^{U_1^\perp \cap U_2} \wedge S^{U_1}$ with S^{U_2}. One checks that this definition is independent of $\mathscr{U}_G$ and $\langle\, ,\rangle$, using the fact that the direct limit is attained, by Hausschild's suspension theorem (see [5] or [13]).

We define $\pi_n^G(X) = \{S^n, X\}^G$, $\pi_G^n(X) = \{X, S^n\}^G$, where X is required to be finite in the definition of π_G^n. The definition is extended to negative values of n in the usual way, so that we obtain graded groups π_*^G and π_G^*. π_*^G and π_G^* are equivariant homology and cohomology theories, respectively; in particular, they yield long exact sequences when applied to G-cofibration sequences. (A G-cofibration sequence is a sequence $X \xrightarrow{i} Y \to Z$, where i is the inclusion of a G-subcomplex of Y, and $Z = Y/i(X)$. See [10] for a complete discussion.) Suppose X is a G-complex with trivial G-action; then there are natural "inflation" maps i_*: $\pi_*^S(X) \to \pi_*^G(X)$ and i^*: $\pi_S^*(X) \to \pi_S^*(X)$; here π_*^S and π_S^* denote reduced ordinary stable homotopy and cohomotopy. Now i_* and i^* are defined by recognizing that the direct limit systems defining $\pi_*^S(X)$ and $\pi_S^*(X)$ are subsystems of those defining $\pi_*^G(X)$ and $\pi_G^*(X)$, since the trivial representation occurs infinitely often in $\mathscr{U}_G$. We say that a G-complex X is *free* if no element of G except the identity fixes any point of X except $*$. For free G-complexes, π_G^* and π_*^G may be evaluated non-equivariantly.

PROPOSITION I.4 (*see* [5]). (a) *Let X be a finite free G-complex. Then $\pi_G^*(X)$ is naturally isomorphic to $\pi_S^*(X/G)$. Moreover, the isomorphism is given by the composite $\pi_S^*(X/G) \xrightarrow{i^*} \pi_G^*(X/G) \to \pi_G^*(X)$, where the second arrow is induced by the projection $X \to X/G$.*

(b) *Let X be an arbitrary G-complex. Then $\pi_*^G(X)$ is naturally isomorphic to $\pi_*^S(X/G)$. The isomorphism is given on finite complexes by the composite $\pi_*^S(X/G) \xrightarrow{i_*} \pi_*^G(X/G) \to \pi_*^G(X)$, where the second arrow is induced by the equivariant transfer associated to the projection $X \to X/G$ (see [5]). For arbitrary complexes, one passes to direct limits over skeletons.*

We now wish to discuss "change of groups" results.

Definition 1.5. *Let $H \subseteq G$, and suppose X is an H-complex. Then define $e_H^G(X) = G^+ \wedge X/\simeq$, where $\simeq$ is the equivalence relation generated by the* *equivalences* $gh \wedge x \simeq g \wedge hx$, $g \in G$, $h \in H$. *G acts on* $e_H^G(X)$ *by the multiplication on the first coordinate.*

PROPOSITION I.6. (a) $[e_H^G(X), Y]^G \cong [X, Y]^H$.
(b) *If X is a G-complex, then* $e_H^G(X) \cong G/H^+ \wedge X$.

Proof. (a) is a standard result; see [5]. For (b), a G homeomorphism is given by $g \wedge x \to [g] \wedge gx$, where $[g]$ denotes the coset gH. □

PROPOSITION I.7 (see [5]). *Let X and Y be G-complexes, X finite. Then there are natural isomorphisms* $\{X \wedge G/H^+, Y\}^G \cong \{X, Y\}^H$ *and* $\{X, Y \wedge G/H^+\}^G \cong \{X, Y\}^H$.

Suppose $H \subseteq G$. Let $N_G(H)$ denote the normalizer of H in G. If X is any G-complex, X^H is invariant under the action of $N_G(H)$ and $H \subseteq N_G(H)$ acts trivially on X^H. Thus, X^H becomes an $N_G(H)/H$-space. Define $W(H)$, the "Weyl group" of H, by $W(H) = N_H(H)/H$. In order to evaluate π_*^G on complexes which are not necessarily free, we have a theorem of tom Dieck.

THEOREM I.8 (*see* [13]). *For any G-complex X,*

$$\pi_*^G(X) \cong \bigoplus_H \pi_*^{W(H)}\left(EW(H)^+ \wedge X^H\right).$$

Here H ranges over a set of representatives of the conjugacy classes of subgroups of G, and EW(H) denotes a contractible space on which W(H) acts freely.

Proposition I.4(b) and Theorem I.8 together give

COROLLARY I.9. *For any G-complex X,*

$$\pi_*^G(X) \cong \bigoplus_H \pi_*^S\left(EW(H)^+ \underset{W(H)}{\wedge} X^H\right).$$

In particular, $\pi_*^G(S^0) \cong \oplus_H \pi_*^S(BW(H)^+)$, *where B denotes the classifying space functor.*

COROLLARY I.10. $\pi_*^G(X)$ *and* $\pi_G^*(Y)$ *are finitely generated for each value of* $*$, *if X is an arbitrary G-complex, and Y is a finite G-complex.*

Proof. Using the fact that π_*^G and π_G^* are homology and cohomology theories respectively, one reduces to the case where $X = Y = G/H^+$. Proposition I.7 reduces this to the case where $X = Y = S^0$, for which the result is standard, by Corollary I.9. The proof for $\pi_*^G(X)$ uses also the fact that $\pi_n^G(X) = \pi_n^G(X^{(k)})$, for some k-skeleton $X^{(k)}$ of X, which is a finite G-complex. □

Composition of maps gives $\pi_*^G(S^0) \cong \pi_G^{-*}(S^0)$ the structure of a graded-commutative ring. The graded groups $\pi_*^G(X)$ and $\pi_G^*(X)$ become modules over this ring. Tom Dieck [31] has described the ring structure of $\pi_0^G(S^0)$. We first recall the definition of the Burnside ring of G, $A(G)$. Let $M(G)$ denote the monoid whose elements are isomorphism classes of finite G-sets, and whose addition is given by disjoint union, so that the empty G-set becomes an identity element for $M(G)$. $A(G)$, as an additive group, is the group completion of this abelian monoid. It becomes a free abelian group on basis elements $[G/H]$, as H ranges over a set of representatives of conjugacy classes of subgroups of G. Direct product of G-sets gives $A(G)$ a ring structure. $A(G)$ is equipped with an augmentation $A(G) \to \mathbf{Z}$, which takes $[G/H]$ to the integer $|G|/|H|$. $I(G)$, the augmentation ideal of $A(G)$, is defined to be the kernel of ε. For further information on $A(G)$, see [13].

THEOREM I.11 (*tom Dieck*, [13]). $\pi_0^G(S^0) \cong A(G)$ *as rings.*

We now formulate Segal's Burnside ring conjecture equivariantly. Let EG denote a free contractible G-CW complex, in the sense of [10]. (EG is not a G-complex, in our terminology, since it does not have a basepoint. Freeness here means $EG^H = \varnothing$ for all $H \subseteq G$, $H \neq \{e\}$). We have a natural map $EG^+ \to S^0$, which takes EG to the non-base point. Let $EG^{(k)}$ denote the k-skeleton of EG.

Conjecture I.12 (*Segal*). *The map* $\pi_G^*(S^0) \to \varprojlim_k \pi_G^*(EG^{(k)+})$ *becomes an isomorphism after* $I(G)$*-adic completion.*

One may observe that $\varprojlim_k \pi_G^*(EG^{(k)+}) = \varprojlim_k \pi_S^*(BG^{(k)+})$, using Proposition I.4(a), and that the map $\pi_G^*(S^0) \to \varprojlim_k \pi_S^*(BG^{(k)+})$ is a map of rings, when $\pi_S^*(BG^{(k)+})$ is given its usual ring structure. This shows that a corollary of the above conjecture is

Conjecture I.12′ (*Segal; weak form*). $\varprojlim_k \pi_S^0(BG^{(k)+}) \cong \hat{A}(G)$, *where* $\hat{A}(G)$ *denotes* $A(G)$ *completed at* $I(G)$.

The following result is an important preliminary reduction in the proof of Conjecture I.12.

PROPOSITION I.13 (*Lewis, May, McClure* [17]). *Conjecture* I.12 *holds for all finite groups if and only if it holds for all finite p-groups.*

One may also show (May, McClure [21]) that for p-groups, $I(G)$-adic completion is essentially p-adic completion. Thus, let $\hat{\pi}_*^G$ and $\hat{\pi}_G^*$ denote p-adically completed equivariant stable homotopy and cohomotopy.

PROPOSITION I.14 (*May, McClure*, [21]). *Conjecture* I.12 *holds for a p-group G if and only if the natural map*

$$\hat{\pi}_G^*(S^0) \to \varprojlim_k \hat{\pi}_G^*(EG^+)$$

is an isomorphism.

This is the formulation of the conjecture which we shall prove. Before proceeding to outline the proof, we shall fix some conventions. Let X be any G-complex, finite or otherwise. Then we define $\{X, Y\}_*^{G,\wedge}$ to be $\varprojlim_k \{X^{(k)}, Y\}_*^G \otimes_{\mathbf{Z}} \hat{\mathbf{Z}}_p$, where $X^{(k)}$ denotes the k-skeleton of X. This definition is clearly independent of the choice of CW-structure on X; in fact, any increasing sequence $\{X^{(k)}\}$ of finite G-complexes which exhausts X will do. $\hat{\pi}_G^*(X)$ is now defined for all G-complexes X. Since finitely generated $\hat{\mathbf{Z}}_p$-modules are compact groups, and $\varprojlim$ is exact for inverse systems of compact groups, Corollary I.10 shows that $\{X;\ \}_*^{G,\wedge}$ and $\{\ ; Y\}_*^{G,\wedge}$ give long exact sequences when applied to cofibrations involving arbitrary G-complexes.

We define $\tilde{E}G$ to be the mapping cone of the G-map $EG^+ \to S^0$ described above. $\tilde{E}G$ may be viewed as the unreduced suspension of EG, with trivial G-action on the suspension coordinate.

LEMMA I.15. *Conjecture* I.12 *holds for all p-groups G if and only if* $\hat{\pi}_G^*(\tilde{E}G) = 0$.

Proof. By Proposition I.14, we need only show that $\hat{\pi}_G^*(S^0) \to \hat{\pi}_G^*(EG^+)$ is an isomorphism. Since we have a cofiber sequence $EG^+ \to S^0 \to \tilde{E}G$, this is equivalent to the assertion that $\hat{\pi}_G^*(\tilde{E}G) = 0$. □

For any G-complex X, let $\mathrm{Iso}(X) = \{H \subseteq G \mid X^H \neq *\}$.

PROPOSITION I.16. *Let X and Y be G-complexes. Suppose* $\hat{\pi}_H^*(Y) = 0$ *for all* $H \in \mathrm{Iso}(X)$. *Then* $\hat{\pi}_G^*(X \wedge Y) = 0$.

Proof. By the definition of Iso(X), X may be filtered by skeletons $X^{(k)}$ so that $X^{(k+1)}/X^{(k)} \cong \vee_i S^{k+1} \wedge G/H_i^+$, with all $H_i \in \mathrm{Iso}(X)$. It is easily checked that $\hat{\pi}_G^*(X \wedge Y) \cong \varprojlim_k \hat{\pi}_G^*(X^{(k)} \wedge Y)$. Using the long exact sequences

$$\to \hat{\pi}_G^*(X^{(k+1)} \wedge Y) \to \hat{\pi}_G^*(X^{(k)} \wedge Y) \to \hat{\pi}_G^*(X^{(k+1)}/X^{(k)} \wedge Y) \to$$

we see that to deduce the result it will suffice to prove that $\hat{\pi}_G^*(X^{(k+1)}/X^{(k)} \wedge Y)$

$= 0$ for all k. But,

$$\hat{\pi}_G^*(X^{(k+1)}/X^{(k)} \wedge Y) \cong \bigoplus_i \hat{\pi}_G^*(S^{k+1} \wedge G/H_i^+ \wedge Y)$$
$$\cong \bigoplus_i \hat{\pi}_{H_i}^*(S^{k+1} \wedge Y) \cong \bigoplus_i \hat{\pi}_{H_i}^{*-k-1}(Y),$$

where each H_i is in $\mathrm{Iso}(X)$. The second identification uses Proposition I.7. But now by hypothesis, $\hat{\pi}_{H_i}^*(Y) = 0$ for $H_i \in \mathrm{Iso}(X)$; so the proposition is proved. □

We now summarize the proof. We say that a real G-module V is fixed-point free if it contains no summands isomorphic with the trivial representation. Let kV denote a direct sum of k copies of V, and let $S^{\infty V} = \varprojlim_k S^{kV}$, the limit taken over the obvious inclusions $kV \to (k+1)V$. We first prove a preliminary reduction.

THEOREM A. (a) *If Conjecture* I.12 *holds for p-group G and all p-groups H, with* $|H| < |G|$, *then* $\hat{\pi}_G^*(S^{\infty V}) = 0$ *for all representations V of G,* $V \neq 0$.

(b) *Conjecture* I.12 *holds for a p-group G if it holds for all p-groups H, with* $|H| < |G|$, *and if* $\hat{\pi}_G^*(S^{\infty V}) = 0$ *for some fixed-point free representation V of G.*

We then study the groups $\hat{\pi}_G^*(S^{\infty V})$ by mapping $S^{\infty V}$ into the cofibre sequence $EG^+ \to S^0 \to \tilde{E}G$. We obtain the long exact sequence

$$\text{(A)} \qquad \cdots \{S^{\infty V}, EG^+\}_*^{G,\wedge} \to \hat{\pi}_G^*(S^{\infty V}) \to \{S^{\infty V}, \tilde{E}G\}_*^{G,\wedge} \to \cdots$$

of the cofibration. The main theorem can now be stated as follows.

THEOREM B. *Suppose the p-group G is not elementary abelian, and that Conjecture* I.12 *holds for all p-groups H, with* $|H| < |G|$. *Then there is a fixed point free representation V of G so that*

(a) $\{S^{\infty V}, EG^+\}_*^{G,\wedge} = 0$,

(b) $\{S^{\infty V}, \tilde{E}G\}_*^{G,\wedge} = 0$.

From this, we derive

THEOREM C. *Conjecture* I.12 *holds for all groups G.*

Proof. We prove that $\hat{\pi}_G^*(\tilde{E}G) = 0$ for all p-groups G; according to Propositions I.13 and I.14 together with Lemma I.15, this will give the result. We note first that Adams, Miller, and Gunawardena [6] have proved Conjecture I.12 in the case $G = (\mathbf{Z}/p\mathbf{Z})^k$. In fact, they prove their theorem in a non-equivariant context; that their formulation implies the equivariant formulation is proved by Lewis, May, and McClure in [17]. We show in Appendix B how to prove the theorem for $G = (\mathbf{Z}/p\mathbf{Z})^k$ using only one computational result from [6]. If

$G \cong (\mathbf{Z}/p\mathbf{Z})^k$, we suppose that Conjecture I.12 holds for all p-groups H, with $|H| < |G|$. Then Theorem B shows that there is a fixed point free representation V so that $\hat{\pi}_G^*(S^{\infty V}) = 0$, by use of the long exact sequence (A) above. Now, Theorem A implies the validity of the conjecture. □

Theorem A is proved in Section II; its proof is an immediate application of Proposition I.16. Theorem B, part (a) is proved in Section III; its proof requires the use of Thom spectra over BG of virtual representations of G. This construction is discussed in Appendix A. The proof also requires a non-computational use of the Adams spectral sequence applied to inverse systems of such Thom spectra. Theorem B, part (b) is proved in Sections IV, V, and VI. Its proof uses the notion of S-functor, introduced in Section IV, and a result of Quillen's (see [25]) concerning the classifying space of the partially ordered set of proper, non-trivial subgroups of a group G.

To make notation less cumbersome we assume throughout the remainder of the paper that G is a p-group, and that all groups $\{X, Y\}_*^G$ are p-adically completed. Thus, $\{X, Y\}_*^G$ and π_*^G will denote $\{X, Y\}_*^{G,\wedge}$ and $\hat{\pi}_*^G$ throughout.

II. Proof of Theorem A

We prove Theorem A of Section I. Recall its statement.

THEOREM A. (a) *If Conjecture* I.12 *holds for* G *and for all* p*-groups* H, *with* $|H| < |G|$, *then* $\pi_G^*(S^{\infty V}) = 0$ *for all non-trivial* G*-representations* V.

(b) *Conjecture* I.12 *holds for* G *if it holds for all* p*-groups* H, *with* $|H| < |G|$, *and if* $\pi_G^*(S^{\infty V}) = 0$ *for some fixed-point free representation of* G.

Proof. We prove part (a) first. We study π_G^* applied to the cofibre sequence

$$EG^+ \wedge S^{\infty V} \to S^{\infty V} \to \tilde{E}G \wedge S^{\infty V}.$$

It will suffice to prove that $\pi_G^*(EG^+ \wedge S^{\infty V}) = \pi_G^*(\tilde{E}G \wedge S^{\infty V}) = 0$. We first prove that $\pi_G^*(EG^+ \wedge S^{\infty V}) = 0$. In the notation of Proposition I.16, Iso(EG^+) consists of the trivial subgroup. By Proposition I.16, with $X = EG^+$ and $Y = S^{\infty V}$, it suffices to prove that $\pi_S^*(S^{\infty V}) = 0$. But this is clear, since $S^{\infty V}$ is non-equivariantly contractible. Next, we show that $\pi_G^*(\tilde{E}G \wedge S^{\infty V}) = 0$. By the hypothesis for part (a), $\pi_H^*(\tilde{E}G) = 0$ for all $H \subseteq G$, since restricted to any subgroup H of G, $\tilde{E}G$ has the homotopy type of $\tilde{E}H$; hence $\pi_H^*(\tilde{E}G) = \pi_H^*(\tilde{E}H) = 0$. Applying I.16 with $X = S^{\infty V}$, $\tilde{E}G = Y$, we obtain the result. To prove (b), we consider the cofibre sequence

$$\tilde{E}G \to \tilde{E}G \wedge S^{\infty V} \to \tilde{E}G \wedge (S^{\infty V}/S^0).$$

It will suffice to prove that

$$\pi_G^*(\tilde{E}G \wedge S^{\infty V}) = \pi_G^*\big(\tilde{E}G \wedge (S^{\infty V}/S^0)\big) = 0.$$

To show $\pi_G^*(\tilde{E}G \wedge S^{\infty V}) = 0$, we note that $\operatorname{Iso}(\tilde{E}G)$ consists of the trivial subgroup and G itself. We thus need to show that $\pi_S^*(S^{\infty V}) = 0$ and that $\pi_G^*(S^{\infty V}) = 0$. The first is true since $S^{\infty V}$ is non-equivariantly contractible, the second by hypothesis. To show $\pi_G^*(\tilde{E}G \wedge (S^{\infty V}/S^0)) = 0$, we note that $\operatorname{Iso}(S^{\infty V}/S^0)$ is contained in the collection of all proper subgroups of G. This uses the fact that $S^{\infty V}$ is fixed point free, which gives that $(S^{\infty V})^G = S^0$, so that $(S^{\infty V}/S^0)^G = *$. Thus, I.16 asserts that we need only show that $\pi_H^*(\tilde{E}G) = 0$ for all proper subgroups. Again, $\pi_H^*(\tilde{E}G) = \pi_H^*(\tilde{E}H) = 0$, by the hypothesis that Conjecture I.12 holds for all p-groups H, with $|H| < |G|$.

III. Proof of Theorem B, part (a)

In this section, we will prove that for any p-group G, with G not elementary abelian, i.e. $G \neq (\mathbf{Z}/p\mathbf{Z})^k$, there is a fixed-point free complex G-representation V so that $\{S^{\infty V}, EG^+\}^G = 0$. This is Theorem B, part (a) of Section I.

Given G, let ρ_G denote the reduced regular complex representation of G: that is, the kernel of the map $\mathbf{C}[G] \to \mathbf{C}$ of G-modules, which sends each $g \in G$ to 1. Now ρ_G is a fixed-point free representation of G. Let $n = \dim_{\mathbf{C}}\rho_G = |G| - 1$.

LEMMA III.1. *The n-th Chern class of ρ_G, $c_n(\rho_G)$, is nilpotent as an element of $H^{2n}(BG;\mathbf{Z}/p)$.*

Proof. We observe that if $H \subsetneq G$, $\rho_G|H$ contains a trivial summand. For, $\mathbf{C}[G]$, regarded as an H-module, is isomorphic to a direct sum of $|G|/|H|$ copies of $\mathbf{C}[H]$, and hence contains $|G|/|H|$ trivial summands. Now, the splitting $\mathbf{C}[G] \cong \rho_G \oplus \varepsilon$, where ε is a one dimensional trivial summand, shows that as an H-module, ρ_G contains $|G|/|H| - 1$ trivial summands. Since $H \neq G$, $\rho_G|H$ contains at least one trivial summand, so that $c_n(\rho_G|H) \in H^{2n}(BH;\mathbf{Z}/p) = 0$. But now $0 = c_n(\rho_G|H) = i^*c_n(\rho_G)$, where i: $BH \to BG$ is induced by the inclusion, so that $c_n(\rho_G)$ restricts trivially on every proper subgroup. Since G is not elementary abelian, $c_n(\rho_G)$ restricts trivially to every elementary abelian subgroup of G. By the Quillen-Venkov theorem [27], $c_n(\rho_G)$ is nilpotent. □

Proof of Theorem B, *part* (a). In Appendix A, it is shown that

$$\{S^{\infty\rho_G}, EG^+\}_*^G \cong \varprojlim_k \{S^{k\rho_G}, EG^+\}_*^G \cong \varprojlim_k \pi_*^s(BG^{-k\rho_G}).$$

According to Proposition A.8, $H^*(BG^{-k\rho_G};\mathbf{Z}/p\mathbf{Z})$ is a free $H^*(BG;\mathbf{Z}/p\mathbf{Z})$-module on a generator in dimension $-2kn$. It is a standard result (see [4]) that the Adams spectral sequence for computing the groups $\pi_*^s(BG^{-k\rho_G})$ is convergent; we have $E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}(p)}^{s,t-s}(H^*(BG^{-k\rho_G}, \mathbf{Z}/p\mathbf{Z}); \mathbf{Z}/p\mathbf{Z})$, where $\mathscr{A}(p)$ denotes the mod p Steenrod algebra. Corresponding to the inverse system of homotopy groups, there is an inverse system of spectral sequences; at the E_2-level, it is induced by the directed system of cohomology groups

$$\cdots H^*(BG^{-k\rho_G}; \mathbf{Z}/p\mathbf{Z}) \to H^*(BG^{-(k+1)\rho_G}; \mathbf{Z}/p\mathbf{Z}) \to \cdots$$

Since $c_n(\rho_G)$ is nilpotent, say $c_n(\rho_G)^d = 0$, Proposition A.9 shows that any d-fold composite in the directed system is zero. Applying $\mathrm{Ext}_{\mathscr{A}(p)}$, we find that on E_∞-terms, any d-fold composite in the inverse system is zero. This means that the maps $BG^{-(k+d)\rho_G} \to BG^{-k\rho_G}$ strictly increase Adams filtration. Therefore, if any element in $\pi_*^s(BG^{-k\rho_G})$ is in the image of $\pi_*^s(BG^{(k+l)\rho_G})$ for all $l \geq 0$, it must have infinite Adams filtration, hence be zero. Therefore, $\varprojlim_k \pi_*^s(BG^{-k\rho_G}) = 0$, which was to be shown. $\square$

IV. S-functors

In this section, we define the notion of S-functor, which will be crucial in our analysis of $\{S^{\infty V}, \tilde{E}G\}_*^G$. Let $\mathscr{S}_G$ denote the category of G-complexes.

Definition IV.1. *An S-functor is a pair (T, τ), where $T: \mathscr{S}_G \to \mathscr{S}_G$ is a functor, and τ is a natural transformation $\tau(X,Y): T(X \wedge Y) \to X \wedge T(Y)$, such that the following three conditions hold*:

(a) *The diagrams*

$$\begin{array}{ccc} T(X \wedge Y \wedge Z) & \xrightarrow{\tau(X, Y\wedge Z)} & X \wedge T(Y \wedge Z) \\ & {\scriptstyle \tau(X\wedge Y, Z)} \searrow & \downarrow {\scriptstyle \mathrm{id} \wedge \tau(Y,Z)} \\ & & X \wedge Y \wedge T(Z). \end{array}$$

commute.

(b) $\tau(S^0, X) = \mathrm{id}_{T(X)}$.

(c) *$\tau(X,Y)$ is a homeomorphism when X is a G-complex with trivial G-action.*

Examples. (a) $T(X) = X \wedge A$, where A is any G-complex. Here, $\tau(X,Y)$ is the homeomorphism $(X \wedge Y) \wedge A \to X \wedge (Y \wedge A)$.

(b) $T(X) = X^H$, where $H \subseteq G$ is any normal subgroup. $\tau(X,Y)$ is the natural inclusion $(X \wedge Y)^H \to X \wedge Y^H$.

(c) $T(X) = \Sigma(X) = \bigcup_{\substack{H \subseteq G \\ H \neq \{e\}}} X^H$, the *singular locus* functor. $\tau(X,Y)$ is the natural inclusion $\Sigma(X \wedge Y) \to X \wedge \Sigma(Y)$.

We will construct an S-functor which will be useful in the next section. Let e_H^G be defined as in Section I.

Definition IV.2. *Let* $\omega = (H, K)$ *be a pair of subgroups of* G, *with* $H \subseteq N_G(K)$. *Define a functor* T_ω: $\mathcal{S}_G \to \mathcal{S}_G$ *by* $T_\omega(X) = e_H^G(X^K)$. (*Note that the action of* H *on* X *preserves* X^K, *since* $H \subseteq N_G(K)$.)

LEMMA IV.3. T_ω *admits the structure of an S-functor. That is, there is a natural transformation* τ *satisfying the requirements of Definition* IV.1.

Proof. We must construct a G-map

$$\tau(X,Y)\colon e_H^G(X^K \wedge Y^K) \to X \wedge e_H^G(Y^K);$$

by Proposition I.6(a), this is equivalent to constructing an H-map $\hat{\tau}$: $X^K \wedge Y^K \to X \wedge e_H^G(Y^K)$. Let i_K: $X^K \to X$ be the inclusion, and let j_K: $Y^K \to e_H^G(Y^K)$ be the H-map given by $j_K(y) = 1 \wedge y$. Then $i_K \wedge j_K$ is the required $\hat{\tau}$. It is routine to verify that τ, defined in this way, is a natural transformation, and that it satisfies the requirements of Definition IV.1. □

In terms of coordinates, $\tau(X,Y)$ is defined by $g \wedge x \wedge y \to gx \wedge g \wedge y$, for $g \in G^+$, $x \in X^K$, $y \in Y^K$. When discussing S-functors we will often suppress mention of τ, when no confusion will arise. In particular, T_ω will now refer to the S-functor (T_ω, τ), where τ is as constructed in Lemma IV.3.

Definition IV.4. *Let* T *be an S-functor. We define groups* $\pi_G^*(X;T)$ *for any finite G-complex* X *as follows. For* $n \geq 0$, *set* $\pi_G^n(X;T) = \varinjlim_U [T(S^U \wedge X), S^U \wedge S^n]^G$, *where the direct limit is over the finite-dimensional G-subspaces* $U \subseteq \mathcal{U}_G$ ($\mathcal{U}_G$ *is defined in* § I), *and the maps in the directed system are the composites*

$$\begin{aligned}\left[T(S^U \wedge X), S^U \wedge S^n\right]^G &\xrightarrow{f \to \mathrm{id} \wedge f} \left[S^{V \cap U^\perp} \wedge T(S^U \wedge X), S^{V \cap U^\perp} \wedge S^U \wedge S^n\right]^G \\ &\xrightarrow{[\tau(S^{V\cap U^\perp}, S^U \wedge X),\,]^G} \left[T(S^{V \cap U^\perp} \wedge S^U \wedge X), S^{V \cap U^\perp} \wedge S^U \wedge S^n\right]^G \\ &\cong \left[T(S^V \wedge X), S^V \wedge S^n\right]^G\end{aligned}$$

for $V \supseteq U$. *For the* $\perp$ *notation, see Section* I; *the last identification is determined by the inner product* $\langle\, ,\rangle$ *of Section* I. *This definition extends to negative values of* n, *using condition* (c) *in Definition* IV.1, *by setting* $\pi_G^{-k}(X;T) = \pi_G^0(S^k \wedge X;T)$ *and gives graded groups* $\pi_G^*(X;T)$. *By completing at* p *and taking inverse limits as before, we obtain groups* $\hat{\pi}_G^*(X;T)$ *for arbitrary G-complexes* X. *From this point on, we will assume all groups* $\pi_G^*(\ ;T)$ *to be p-adically completed. Thus,* $\hat{\pi}_G^*(\ ;T) = \pi_G^*(\ ;T)$.

We now prove a proposition connecting $\pi_G^*(X; T)$ with $\{X; \tilde{E}G\}_{-*}^G$, where T is the singular locus functor described above. The point of this is that $\pi_G^*(X; T)$ is much easier to filter in a useful way than is $\{X; \tilde{E}G\}_{-*}^G$.

PROPOSITION IV.5. *There is a natural isomorphism* $\{X, \tilde{E}G\}_{-*}^G \to \pi_G^*(X; \Sigma)$ *for any G-complex X.*

Proof. We claim first that for any G-complexes X and Y, with X finite, the restriction map $[X; \tilde{E}G \wedge Y]^G \to [\Sigma(X), \tilde{E}G \wedge Y]^G$, $f \to f|\Sigma(X)$, is a bijection. For, X is obtained from $\Sigma(X)$ by attaching cells of the form $G^+ \wedge D^n$ along maps from $G^+ \wedge S^{n-1}$. The obstruction to extending a map over $G^+ \wedge D^n$ is an element in $[G^+ \wedge S^{n-1}, \tilde{E}G \wedge Y]^G = [S^{n-1}, \tilde{E}G \wedge Y]$, by Proposition I.6. Now, $\tilde{E}G$ is the mapping cone of the map $EG^+ \to S^0$, which is (non-equivariantly) a homotopy equivalence. Hence $\tilde{E}G$ is non-equivariantly contractible; therefore so is $\tilde{E}G \wedge Y$, so that $\pi_{n-1}(\tilde{E}G \wedge Y) = 0$, and all G-maps $\Sigma(X) \to \tilde{E}G \wedge Y$ extend to X. Homotopies extend similarly; so we have the above bijection. We now define a natural transformation $\{X, \tilde{E}G\}^G \to \pi_G^0(X; \Sigma)$ by constructing a morphism α of directed systems as follows. For each set $[S^U \wedge X, S^U \wedge \tilde{E}G]^G$, define $\hat{\alpha}(f) \in [\Sigma(S^U \wedge X), S^U \wedge \tilde{E}G]^G$ by $\hat{\alpha}(f) = f|\Sigma(S^U \wedge X)$. Now note that for every $H \subseteq G$, $H \neq \{e\}$, $\tilde{E}G^H = S^0$; so $\Sigma(S^U \wedge \tilde{E}G) \cong \Sigma(S^U)$. Thus,

$$[\Sigma(S^U \wedge X), S^U \wedge \tilde{E}G]^G \cong [\Sigma(S^U \wedge X), \Sigma(S^U \wedge \tilde{E}G)]^G$$
$$\cong [\Sigma(S^U \wedge X), \Sigma(S^U)]^G \cong [\Sigma(S^U \wedge X), S^U]^G.$$

Here, we use the fact that for any two G-complexes, $[\Sigma(X), Y]^G \cong [\Sigma(X), \Sigma(Y)]^G$, since any G-map takes $\Sigma(X)$ into $\Sigma(Y)$. We now define α to be $i \circ \hat{\alpha}$, where i is the composite of the above chain of bijections. One easily checks that α gives a map of directed systems, which is an isomorphism by the above remarks. □

We now wish to study the S-functors T_ω defined above. For a CW complex X, let $\langle X \rangle = \min\{i | \pi_i(X) \neq 0\}$, as in Section I.

LEMMA IV.6. *Let T be an S-functor, and suppose that, for all finite G-complexes X and integers N, there is a representation V of G so that for all* $H \subseteq G$, $\dim(T(S^W \wedge X)^H) < \dim((S^W)^H) - N$ *whenever W is a representation of G containing V as a direct summand. Then* $\pi_G^*(X, T) = 0$ *for all G-complexes X.*

Proof. We assume X is a finite complex; the general case follows by passing to inverse limits. Consider the set $[T(S^U \wedge X); S^U \wedge S^k]^G$. Note that $\pi_G^k(X; T) = \varinjlim_U [T(S^U \wedge X), S^U \wedge S^k]^G$. We show that there is a representation V so that $[T(S^W \wedge X), S^W \wedge S^k]^G = *$ whenever W contains V. Let $N > 1 - k$, and suppose V is chosen so that $\dim(T(S^W \wedge X)^H) < \dim((S^W)^H) - N$ for all $H \subseteq G$, and all W containing V as a summand, as in the hypotheses of the lemma. Then,

$$\langle (S^W \wedge S^k)^H \rangle - 1 = \langle (S^W)^H \rangle + k - 1 > \dim\big(T(S^W \wedge X)^H\big) + N + k - 1$$
$$> \dim\big(T(S^W \wedge X)^H\big),$$

for all $H \subseteq G$, and all W containing V as a summand. By Corollary I.3, $[T(S^W \wedge X), S^W \wedge S^k]^G = 0$ for all W containing V as a summand. Consequently, $\pi_G^k(X; T) = 0$ for all positive values of k. To obtain the result for negative values of k, note that $\pi_G^{-k}(X; T) \cong \pi_G^0(S^k \wedge X; T)$, by Definition IV.1, (c). □

Proposition IV.7. *Let $\omega = (H, K)$ be a pair, as in Definition IV.2. Suppose $K \not\subseteq H$. Then $\pi_G^*(X; T) = 0$ for all G-complexes X.*

Proof. We will show that T_ω satisfies the hypotheses of Lemma IV.6. We must first examine the fixed point sets $e_H^G(X^K)^L$, where $L \subseteq G$, and X is a finite G-complex. We find that

$$e_H^G(X^K)^L = \bigvee_{\substack{g \in G \\ g^{-1}Lg \subseteq H}} g \wedge (X^K)^{g^{-1}Lg}.$$

Since $(X^K)^{g^{-1}Lg} = X^{K \cdot g^{-1}Lg}$, we find that

$$\dim\big(e_H^G(X^K)^L\big) \leq \max_{\substack{g \in G \\ g^{-1}Lg \subseteq H}} \dim\big(X^{K \cdot g^{-1}Lg}\big).$$

Note that for all g such that $g^{-1}Lg \subseteq H$, the inclusion $g^{-1}Lg \subseteq K \cdot g^{-1}Lg$ is proper, since $g^{-1}Lg \subseteq H$, and $K \not\subseteq H$ by hypothesis. We will now prove the existence of a G-representation V so that $\dim((S^W \wedge X)^{K \cdot \tilde{H}}) < (S^W)^{\tilde{H}} - N$ for all subgroups $\tilde{H}$ of H, whenever W contains V as a summand. By the above upper bound for $\dim(e_H^G(X^K)^L)$, and Lemma IV.6, this will prove the proposition. Select a representation V so that $\dim(V^{H_1}) < \dim(V^{H_2}) - N - \dim X$ for all proper inclusions $H_2 \subseteq H_1$ of subgroups of G. This is possible since it is easily checked that $\dim(V^{H_1}) - \dim(V^{H_2})$ can be made arbitrarily large. Now,

$$\dim\big((S^W \wedge X)^{K \cdot \tilde{H}}\big) = \dim\big((S^W)^{K \cdot \tilde{H}}\big) + \dim(X^{K \cdot \tilde{H}})$$
$$\leq \dim\big((S^W)^{K \cdot \tilde{H}}\big) + \dim(X)$$
$$< \dim\big((S^W)^{\tilde{H}}\big) - N - \dim X + \dim X$$
$$= \dim\big((S^W)^{\tilde{H}}\big) - N$$

□

PROPOSITION IV.8. *Let $\omega = (H, K)$ as above, with $K \subseteq H$. Then $\pi_G^*(X; T_\omega) \cong \pi_{H/K}^*(X^K)$ for all G-complexes X.*

Proof. As usual, we assume X finite and the general case follows by passing to inverse limits. For $k \geq 0$,

$$\begin{aligned}
\pi_G^k(X\colon T_\omega) &= \varinjlim_W \left[T_\omega(S^W \wedge X); S^W \wedge S^k\right]^G \\
&= \varinjlim_W \left[e_H^G\big((S^W)^K \wedge X^K\big); S^W \wedge S^k\right]^G \\
&= \varinjlim_W \left[(S^W)^K \wedge X^K; S^W \wedge S^k\right]^H \\
&= \varinjlim_W \left[(S^W)^K \wedge X^K; (S^W)^K \wedge S^k\right]^H \\
&= \varinjlim_W \left[(S^W)^K \wedge X^K, (S^W)^K \wedge S^k\right]^{H/K} \\
&= \varinjlim_W \left[S^{(W^K)} \wedge X^K, S^{(W^K)} \wedge S^k\right]^{H/K} \\
&\cong \pi_{H/K}^k(X^K).
\end{aligned}$$

To obtain the final isomorphism, one must observe that the representations of H/K of the form W^K for some representation W over G are cofinal among all representations of G/H. Finally, one passes to negative values of k using the usual isomorphism

$$\pi_G^{-n}(X\colon T_\omega) = \pi_G^0(S^n \wedge X; T_\omega). \qquad \square$$

Finally, we will need to discuss filtrations of S-functors.

Definition IV.9. *A natural transformation between S-functors (T_1, τ_1) and (T_2, τ_2) is a natural transformation $N\colon T_1 \to T_2$ so that the diagrams*

$$\begin{array}{ccc}
T_1(X \wedge Y) & \xrightarrow{N(X \wedge Y)} & T_2(X \wedge Y) \\
\Big\downarrow {\scriptstyle \tau_1(X,Y)} & & \Big\downarrow {\scriptstyle \tau_2(X,Y)} \\
X \wedge T_1(Y) & \xrightarrow{\mathrm{id} \wedge N(Y)} & X \wedge T_2(Y)
\end{array}$$

commute.

Definition IV.10. *A natural transformation N between S-functors T_1 and T_2 is said to be a cofibration if for each G-complex X the map $N(X)\colon T_1(X) \to T_2(X)$*

is a cofibration. Given a cofibration N: $T_1 \to T_2$, we define the quotient S-functor, T_2/T_1, by $T_2/T_1(X) = T_2(X)/T_1(X)$. The associated τ is induced by τ_2 in the natural way.

LEMMA IV.11. *For any cofibration N: $T_1 \to T_2$, between S-functors, we obtain a long exact sequence*

$$\cdots \to \pi_G^*(X; T_2/T_1) \to \pi_G^*(X; T_2) \to \pi_G^*(X; T_1) \to \cdots .$$

Proof. One first checks that natural transformations of S-functors induce natural transformations on the groups $\pi_G^*(\ ; T)$. The long exact sequence is now an immediate consequence of Proposition I.1. We use condition (c) in Definition IV.1 to identify $T(S^1 \wedge X)$ with $S^1 \wedge T(X)$. □

PROPOSITION IV.12. *Let $\{T_i\}_{i=0}^n$ be a family of S-functors, $T_0(X) = *$, and suppose we are given cofibrations $T_i \to T_{i+1}$. Suppose further that $\pi_G^n(X; T_{i+1}/T_i)$ is a finitely generated $\hat{\mathbf{Z}}_p$-module for all finite G-complexes and all values of n and i. If, for a given G-complex X, $\pi_G^*(X; T_{i+1}/T_i) = 0$ for all i, then $\pi_G^*(X, T_n) = 0$.*

Proof. By taking the long exact sequences associated to the cofibations $T_i \to T_{i+1} \to T_{i+1}/T_i$, we obtain an exact couple of inverse systems

$$\begin{array}{ccc} \pi_G^*\big(X^{(k)}, T_i\big) & \longrightarrow & \pi_G^*\big(X^{(k)}; T_{i+1}\big) \\ & \nwarrow \quad \swarrow & \\ & \pi_G^*\big(X^{(k)}; T_{i+1}/T_i\big) & \end{array}$$

where $X^{(k)}$ denotes the k-skeleton in some CW decomposition of X. The hypothesis on T_{i+1}/T_i shows that all the inverse systems are inverse systems of finitely generated $\hat{\mathbf{Z}}_p$-modules. (That $\pi_G^n(X^{(k)}; T_i)$ is finitely generated is proved by induction from the fact that $\pi_G^n(X^{(k)}; T_{i+1}/T_i)$ is.) Under these circumstances, $\varprojlim$ is exact, so we obtain a convergent spectral sequence whose E_1-term is a direct sum of groups $\pi_G^*(X^k, T_{i+1}/T_i)$ and for which E is an associated graded version of $\pi_G^*(X, T_n)$. The spectral sequence is convergent since $E_{n+1} = E_\infty$ and $\pi_G^*(X, T_0) = 0$. Since $E_1 = 0$ by hypothesis, the proposition is proved. □

V. "Blowing up" the singular locus

In this section, we will construct a G-homotopy equivalent model for the singular locus

$$\Sigma(X) = \bigcup_{\substack{H \subseteq G \\ H \neq \{G\}}} X^H$$

of a G-complex X, which admits a manageable filtration. We will be relying heavily on work of Quillen's [24], [25]; in fact all the results of this section are contained in [25] implicitly.

Let C be any category. Then recall from [24] the definition of the *nerve* of C, NC_*, as the simplicial set for which $NC_0 = \text{ob}(C)$, and NC_n consists of n-tuples of morphisms $[f_1, f_2, \ldots, f_n]$, so that f_{i+1} and f_i are composable, whose face and degeneracy operators are obtained by composing arrows and inserting identity maps, respectively. Recall from [24] that a functor $f\colon C_1 \to C_2$ induces a map $Nf\colon NC_1 \to NC_2$, and that a natural transformation T between functors $f, g\colon C \to D$ induces a homotopy from Nf_1 to Nf_2. Any partially ordered set $\mathscr{P}$ may be viewed as a category with a unique morphism $p \to q$ whenever $p \leq q$, and $\text{Mor}(p, q) = \varnothing$ if $p \nleq q$. We write $N\mathscr{P}$ for the nerve of this category. Note that a G-action on a partially ordered set $\mathscr{P}$ induces an action on $N\mathscr{P}$, and that $(N\mathscr{P})^G = N(\mathscr{P}^G)$, where $\mathscr{P}^G$ denotes the partially ordered set of elements fixed by G. The verification of this fact uses the observation that if a simplex $[x_1 \leq x_2 \leq \cdots \leq x_k]$ of $N\mathscr{P}_k$ is invariant under the action of G, then each x_i is fixed by the G-action.

Let $H \subseteq G$. We say a subgroup $K \subseteq G$ is H-invariant if $hKh^{-1} = K$ for all $h \in H$. Define $\mathcal{O}(G, H)$ ($\mathcal{O}^*(G, H)$) to be the partially ordered set of nontrivial (non-trivial and proper) H-invariant subgroups of G.

We will also need some preliminaries concerning p-groups.

PROPOSITION V.1. *Let G be a p-group, and suppose G is not elementary abelian, i.e. $G \neq (\mathbf{Z}/p\mathbf{Z})^k$ for any k. Then there is a non-trivial central element x of order p so that x projects trivially in $\overline{G}$, the quotient of G by the subgroup generated by all commutators and p-th powers.*

Proof. Consider the subgroup C of the center of G consisting of all central elements of order p. This is non-trivial since the center of G is non-trivial, G being a p-group. Let $\overline{C}$ denote its image in $\overline{G}$. Selecting any section $S\colon \overline{C} \to C$ of the projection $C \to \overline{C}$, we find that $G = G' \times \overline{C}$, via the map $\overline{C} \xrightarrow{S} C \to G$, where all central elements of G' project trivially to $\overline{G}'$. G' is nontrivial since G is not elementary abelian; so select x to be any central element of order p in $G' \subset G$. This is possible since G is also a p-group. □

LEMMA V.2. *For $H \subseteq G$, G a p-group, let $\overline{H}$ denote the image of H in $\overline{G}$. Then H is a proper subgroup of G if and only if $\overline{H}$ is a proper subgroup of $\overline{G}$.*

Proof. Since G is a p-group, it is nilpotent. If $\overline{H} = \overline{G}$, we find that if $\Gamma_i(G)$ denotes the i-th term in the mod p lower central series for G (see [35]), $\Gamma_i(H)/\Gamma_{i+1}(H) \to \Gamma_i(G)/\Gamma_{i+1}(G)$ is surjective. Since $\Gamma_N(G) = \{e\}$ for N sufficiently large, we conclude that $H \to G$ is surjective. □

COROLLARY V.3. *Let $H \subseteq G$, and let T be a cyclic subgroup of order p in the center of G, which projects trivially to $\overline{G}$, as was constructed in Proposition V.1. Then H is a proper subgroup of G if and only if $H \cdot T$ is.*

Proof. H is proper $\Leftrightarrow$ $\overline{H}$ is proper $\Leftrightarrow$ $\overline{HT} = \overline{H}$ is proper $\Leftrightarrow$ HT is proper. □

We now study the simplicial sets $N\mathcal{O}(G, H)$ and $N\mathcal{O}^*(G, H)$.

PROPOSITION V.4. (a) *For any group G and subgroup $H \subseteq G$, $N\mathcal{O}(G, H)$ is contractible.*

(b) *If a p-group G is not elementary abelian, $N\mathcal{O}^*(G, H)$ is contractible.*

Proof. (a) $\mathcal{O}(G, H)$ has a maximal element, namely G; hence, by [24], $N\mathcal{O}(G, H)$ is contractible.

(b) Let T be a central cyclic subgroup of G of order p, projecting trivially to $\overline{G}$, as was constructed in Proposition V.1. Define functors Φ, Φ': $\mathcal{O}^*(G, H) \to \mathcal{O}^*(G, H)$ (viewed as a category) by $\Phi(K) = KT$ and $\Phi'(K) = T$, for any $K \in \mathcal{O}^*(G, H)$. Note that $\Phi(K) \in \mathcal{O}^*(G, H)$ by Corollary V.3, and by the fact that T is central, hence H-invariant. The inequalities $T \leq TK \geq K$ give natural transformations $\mathrm{Id}_{\mathcal{O}^*(G,H)} \to \Phi$ and $\Phi' \to \Phi$, which shows that $\mathrm{Id}_{N\mathcal{O}^*(G,H)} \simeq N\Phi'$. But $N\Phi'$ is a constant simplicial map. □

Let X and Y be sets, and suppose X is given a preferred choice of basepoint. Then by $X \rtimes Y$ we mean $X \times Y/* \times Y$. Given any based simplicial set X_* and any simplicial set Y_*, we may form the bisimplicial set $(X \rtimes Y)_{**}$, whose (m, n) simplices are the elements of the set $X_m \rtimes Y_n$, and whose face and degeneracy operators are defined in the evident way. These definitions carry over without change if X_* and Y_* are simplicial G-sets. (See [9] for a discussion of the properties of bisimplicial sets.) Let X be a based G-simplicial set. Note that $N\mathcal{O}(G, e)$ and $N\mathcal{O}^*(G, e)$ are G-simplicial sets, with the G-action induced from the conjugation actions of G on $\mathcal{O}(G, e)$ and $\mathcal{O}^*(G, e)$. We will define bisimplicial G-sets, $B(X) \subseteq X \rtimes N\mathcal{O}(G, e)$ and $B^*(X) \subseteq X \rtimes N\mathcal{O}^*(G, e)$, by setting

$$B(X)_{m,n} = \left\{x \rtimes [K_0 \subseteq K_1 \subseteq \cdots \subseteq K_m] | x \in X^{K_m}\right\},$$
$$B^*(X)_{m,n} = \left\{x \rtimes [K_0 \subseteq K_1 \subseteq \cdots \subseteq K_m] | x \in X^{K_m}, K_m \subsetneq G\right\}.$$

We also define $B^0(X) \subseteq X \rtimes \{*\}$ by $B^0(X)_{m,n} = \{x \rtimes * \,|x \in \Sigma(X)\}$. There are evident bisimplicial maps p: $B(X) \to B^0(X)$ and p^*: $B^*(X) \to B^0(X)$, obtained by projection on the first coordinate.

PROPOSITION V.5. (a) $|p|$: $|B(X)| \to |B^0(X)|$ *is a G-homotopy equivalence for all X.*

(b) *Suppose a p-group G is not elementary abelian. Then* $|p^*|: |B^*(X)| \to |B^0(X)|$ *is a G-homotopy equivalence.*

Proof. We prove (a). By Theorem I.2, it will suffice to show that the map $|p|^H: |B(X)|^H \to |B^0(X)|^H$ is a homotopy equivalence, for all $H \subseteq G$. Now, $|B(X)|^H = |B(X)^H|$ and $|B^0(X)|^H = |B^0(X)^H|$, so that we must examine the bisimplicial set $B(X)^H$. For a point of x, let G_x denote its stabilizer. Then

$$B(X)^H_{m,n} = \{x \rtimes [K_0 \subseteq \cdots \subseteq K_n] | H \subseteq G_x, K_n \subseteq G_x, \text{and} \\ [K_0 \subseteq \cdots \subseteq K_n] \in N\mathcal{O}(G,H)_n\}.$$

Now, by the results of [9], it will suffice to prove that the simplicial map $|p^H_{m,*}|: |B(X)^H_{m,*}| \to |B^0(X)^H_{m,*}|$ is a homotopy equivalence. $B^0(X)^H_{m,*}$ is a discrete simplicial set, so we need only check that the inverse image of any point is contractible. Now, the simplicial set $p^{H^{-1}}_{m,*}(x)$ is isomorphic to $N\mathcal{O}(G_x, H)$, which is contractible by Proposition V.4(a). To prove (b), one repeats the above analysis, and finds that $(p^{*H}_{m,*})^{-1} \cong N\mathcal{O}(G_x, H)$ if $G_x \subseteq G$, and $(p^{*H}_{m,*})^{-1}(x) \cong N\mathcal{O}^*(G, H)$ if $G_x = G$. In either case, the simplicial sets are contractible by Proposition V.4. □

PROPOSITION V.6. $|B^0(X)| \cong \Sigma(|X|)$.

Proof. This is immediate from the definition of geometric realization. □

For any G-complex X, define G-subcomplexes

$$\hat{\Sigma}(X) \subseteq X \rtimes N\mathcal{O}(G,e) \text{ and}$$
$$\hat{\Sigma}^*(X) \subseteq X \rtimes N\mathcal{O}^*(G,e) \text{ by}$$
$$\hat{\Sigma}(X) = \bigcup_{\substack{H \subseteq G \\ H \neq e}} X^H \rtimes N\mathcal{O}(H,e) \text{ and}$$
$$\hat{\Sigma}^*(X) = \bigcup_{\substack{H \subseteq G \\ H \neq e}} X^H \rtimes N\mathcal{O}(H,e) \cap N\mathcal{O}^*(G,e).$$

There are natural maps $\pi: \hat{\Sigma}(X) \to \Sigma(X)$ and $\pi^*: \hat{\Sigma}^*(X) \to \Sigma(X)$.

PROPOSITION V.7. (a) $\pi: \hat{\Sigma}(|X|) \to \Sigma(X)$ *is a G-homotopy equivalence for all G-simplicial sets.*

(b) *If the p-group G is not elementary abelian,* $\pi^*: \hat{\Sigma}^*(|X|) \to \Sigma(|X|)$ *is a G-homotopy equivalence for all G-simplicial sets.*

Proof. $\hat{\Sigma}(|X|)$ and $\hat{\Sigma}^*(|X|)$ are the realizations of the diagonal simplicial sets associated to $B(X)$ and $B^*(X)$, and π and π^* are the realizations of p and p^*, respectively. Since the realization of the diagonal of a bisimplicial set is homeomorphic to the realization of the bisimplicial set, (see [9]), the result follows from Propositions V.5 and V.6. □

COROLLARY V.8. *The equivalences* $\hat{\pi}$ *and* $\hat{\pi}^*$ *hold for arbitrary G-complexes.*

Proof. Any G-complex has the G-homotopy type of the realization of a G-simplicial set. One need only check that $\hat{\Sigma}$ and $\hat{\Sigma}^*$ preserve G-homotopy equivalences, which is immediate. □

VI. Proof of Theorem B(b)

In this section, we prove that if G is not elementary abelian, if ρ_G is the reduced regular representation, and if Conjecture I.12 holds for all p-groups of H, with $|H| < |G|$, then $\{S^{\infty V}, \tilde{E}G\}_*^G = 0$. This will complete the proof of Theorem B(b), and therefore the proof of C njecture I.12. We will first note that the functor $\hat{\Sigma}^*$ of the previous section admits an S-functor structure, and that $\pi^*\colon \hat{\Sigma}^* \to \Sigma$ is a natural transformation of S-functors. We will then filter $\hat{\Sigma}^*(X)$, and relate the groups $\pi_G^*(X; \hat{\Sigma}^*) \cong \pi_G^*(X; \Sigma)$ to groups $\pi_{H/K}^*(X^K)$, for subgroups $K \subseteq H \subseteq N_G(K)$ of G. We then use Proposition IV.5 to conclude the proof of Theorem B.

Let $\mathscr{P}_G$ denote $|N\mathscr{O}^*(G,e)|$, where $N\mathscr{O}^*(G,e)$ is defined in Section V. For a G-complex X and a G-space Y (Y not necessarily based), we define $X \rtimes Y = X \times Y/* \times Y$, as we did for sets in Section V. $\hat{\Sigma}^*(X)$ is defined as a subcomplex of $X \rtimes \mathscr{P}_G$. We note that $\hat{\Sigma}^*(X \wedge Y)$ is contained as a subspace of $X \wedge \hat{\Sigma}^*(Y)$, via the inclusion $(x \wedge y) \rtimes z \to x \wedge (y \rtimes z)$, $x \in X$, $y \in Y$, $z \in \mathscr{P}_G$, and denote the inclusion by $\hat{\sigma}(X, Y)$.

LEMMA VI.1. $(\hat{\Sigma}^*, \hat{\sigma})$ *is an S-functor.*

Proof. This is immediate from the definitions. □

LEMMA VI.2. *The map* $\pi^*\colon \hat{\Sigma}^*(X) \to \Sigma(X)$ *is natural in* X, *and is a natural transformation of S-functors.*

Proof. Again, this is immediate from the definitions. □

PROPOSITION VI.3. *If the p-group G is not elementary abelian, the natural transformation* $\hat{\Sigma}^* \to \Sigma$ *induces an isomorphism of functors* $\pi_G^*(\ ;\Sigma) \to \pi_G^*(\ ;\hat{\Sigma}^*)$.

Proof. By Corollary V.8, $\hat{\Sigma}^*(X) \to \Sigma(X)$ is a G-homotopy equivalence; so the directed systems defining the groups are isomorphic. □

We now wish to filter $\hat{\Sigma}^*(X)$ in a useful way. Let $\mathscr{P}_G^{(k)}$ denote the union of all the k-faces of $\mathscr{P}_G$. $\mathscr{P}_G^{(k)}$ is a G-subcomplex of $\mathscr{P}_G$, and we define $\hat{\Sigma}_k^*(X)$ to be $X \rtimes \mathscr{P}_G^{(k)} \cap \hat{\Sigma}^*(X)$.

Proposition VI.4. *The S-functor structure on $\hat{\Sigma}^*(X)$ restricts to an S-functor structure on $\hat{\Sigma}_k^*(X)$. Moreover, each of the inclusions $\hat{\Sigma}_k^*(X) \to \hat{\Sigma}_{k+1}^*(X)$ is a natural transformation of S-functors.*

Proof. Immediate. □

Note that if $s = \#(\mathcal{O}^*(G, e))$, then $\hat{\Sigma}_s^*(X) = \hat{\Sigma}^*(X)$, and conventionally we set $\hat{\Sigma}_{-1}^*(X) = \{*\}$. We define $\Gamma_k(X)$ to be the G-complex $\hat{\Sigma}_k^*(X)/\hat{\Sigma}_{k-1}^*(X)$. As in Section IV, Γ_k admits an S-functor structure induced from that on $\hat{\Sigma}_k^*$. We wish to decompose the S-functor Γ_k in terms of the S-functors T_ω defined in Section IV. For each k, let Q_k denote the set of k-simplices of $\mathscr{P}_G^{(k)}$. These correspond to increasing chains $[K_0 \subseteq K_1 \subseteq \cdots \subseteq K_k]$ of proper, non-trivial subgroups of G. For a k-face σ of $\mathscr{P}_G$ corresponding to $[K_0 \subseteq K_1 \subseteq \cdots \subseteq K_k]$, define $\mathscr{H}(\sigma) = K_k$. Also for σ a k-face of $\mathscr{P}_G$, let $\bar{\sigma} = \sigma/\partial\sigma$. We see that as a space, $\Gamma_k(X)$ is homeomorphic to the wedge $\bigvee_{\sigma \in Q_k} X^{\mathscr{H}(\sigma)} \wedge \bar{\sigma}$. We wish to study the G-action on this space. Let $\{\sigma_1, \ldots, \sigma_s\}$ be a set of orbit representatives for the G-set Q_k. For each i, let H_i denote the stabilizer of σ_i. If σ_i corresponds to $[K_0 \subseteq K_1 \subseteq \cdots \subseteq K_k]$, then H_i normalizes each K_j. In particular, H_i normalizes $\mathscr{H}(\sigma_i)$. The inclusion map $X^{\mathscr{H}(\sigma_i)} \wedge \bar{\sigma}_i \to \bigvee_{\sigma \in Q_k} X^{\mathscr{H}(\sigma)} \wedge \bar{\sigma}$ is thus an H_i-equivariant map. Applying $e_{H_i}^G$, we obtain from Proposition I.6 a G-map from $e_{H_i}^G(X^{\mathscr{H}(\sigma_i)} \wedge \bar{\sigma}_i) \to \bigvee_{\sigma \in Q_k} X^{\mathscr{H}(\sigma)} \wedge \bar{\sigma}$, which is a G-homeomorphism to the wedge of all the summands corresponding to simplices in the orbit of σ_i. Performing this construction for all i, we obtain a G-homeomorphism

$$\bigvee_i e_{H_i}^G\left(X^{\mathscr{H}(\sigma_i)} \wedge \bar{\sigma}_i\right) \to \Gamma_k(X).$$

Let ω_i denote the pair $(H_i, \mathscr{H}(\sigma_i))$.

Proposition VI.5. *The above construction is natural in X, and is a natural transformation of S-functors. Thus, it produces an isomorphism of S-functors $\bigvee_i T_{\omega_i}(X \wedge \bar{\sigma}_i) \to \Gamma_k(X)$.*

Proof. Everything here is clear, except possibly the fact that we have a natural transformation of S-functors. We verify this. Points in $T_{\omega_i}(X \wedge Y \wedge \bar{\sigma}_i)$ are of the form $g \wedge x \wedge y \wedge s$, where $g \in G^+$, $x \in X^{\mathscr{H}(\sigma_i)}$, $y \in Y^{\mathscr{H}(\sigma_i)}$, and $s \in \bar{\sigma}_i$. From the remarks following Lemma IV.3, the formula defining the S-functor structure on $T_{\omega_i}(X \wedge Y \wedge \bar{\sigma}_i)$ is given by $g \wedge x \wedge y \wedge s \to$

$gx \wedge g \wedge y \wedge s$. Also, the S-functor structure on $\Gamma_k(X) = \bigvee_{\sigma \in Q_k} X^{\mathscr{H}(\sigma)} \wedge \bar{\sigma}$ is given by $x \wedge y \wedge s \to x \wedge (y \wedge s)$, for $s \in \bar{\sigma}$, $x \in X^{\mathscr{H}(\sigma)}$, $y \in Y^{\mathscr{H}(\sigma)}$. Now, for $g \in G^+$, $x \in X^{\mathscr{H}(\sigma_i)}$, $y \in Y^{\mathscr{H}(\sigma_i)}$, $s \in \bar{\sigma}_i$, we find that the map $T_{\omega_i}(X \wedge Y \wedge \bar{\sigma}_i) \to \Gamma_k(X \wedge Y)$ is given by $g \wedge x \wedge y \wedge s \to gx \wedge gy \wedge gs$, since it is the extension to $e_{H_i}^G(X \wedge Y\bar{\sigma}_i)$ of the H_i-map which takes $1 \wedge x \wedge y \wedge s$ to $x \wedge y \wedge s$. Now, we compute that the composite

$$T_{\omega_i}(X \wedge Y \wedge \bar{\sigma}_i) \to \Gamma_k(X \wedge Y) \to X \wedge \Gamma_k(Y),$$

is $g \wedge x \wedge y \wedge s \to gx \wedge (gy \wedge gs)$. On the other hand, the composite $T_{\omega_i}(X \wedge Y \wedge \bar{\sigma}_i) \to X \wedge T_{\omega_i}(Y \wedge \bar{\sigma}_i) \to X \wedge \Gamma_k(Y)$ is also given by $g \wedge x \wedge y \wedge s \to gx \wedge (gy \wedge gs)$, which gives the result. □

COROLLARY VI.6. *If G is not elementary abelian, there is a sequence of S-functors T_k, with $T_{-1}(X) = \{*\}$, and $T_s = \hat{\Sigma}^*$, and cofibrations $T_k \to T_{k+1}$, with $T_{k+1}/T_k(X) \cong \bigvee_i T_{\omega_i}(X \wedge S^k)$, where each $\omega_i = (H_i, K_i)$, and K_i is a proper non-trivial subgroup of G.*

Proof. This is just a restatement of the above discussion, since $\mathscr{H}(\sigma)$ is proper and nontrivial for any simplex σ of $|(N\mathcal{O}^*(G,e)|$. Also, we noted above that H_i acts trivially on $\bar{\sigma}_i$, since H_i normalizes all the subgroups in a chain $K_0 \subseteq K_1 \subseteq \cdots \subseteq K_k$ corresponding to σ_i; so $\bar{\sigma}_i \cong S^k$ as an H_i-space. □

PROPOSITION VI.7. *Suppose that Conjecture* I.12 *holds for all p-groups G', with $|G'| < |G|$. Suppose V is any finite-dimensional G-representation, and suppose $V^K \neq 0$, where $\{e\} \neq K \subseteq G$. Let $\omega = (H, K)$, where $K \subseteq H \subseteq N_G(K)$. Then $\pi_G^*(S^{\infty V}; T_\omega) = 0$.*

Proof. We first use Proposition IV.8 to identify $\pi_G^*(S^{\infty V}; T_\omega)$ with $\pi_{H/K}^*(S^{\infty(V^K)})$. Since $K \neq \{e\}$, H/K is a p-group of order strictly smaller than $|G|$, and V^k is a representation of H/K. Since $V^k \neq 0$, the inductive hypothesis and Theorem A, part (a) of Section I allow us to conclude that $\pi_{H/K}^*(S^{\infty(V^K)}) = 0$. □

PROPOSITION VI.8. *Let ρ_G be the reduced regular representation, as defined in Section* III. *Then $\rho_G^H \neq 0$ if H is any proper subgroup of G.*

Proof. In the proof of Lemma III.1, it was shown that for any proper subgroup H, ρ_G contains a trivial summand. □

THEOREM VI.9. *Suppose Conjecture* I.12 *holds for all p-groups G', with $|G'| < |G|$, and that G is not elementary abelian. Then $\pi_G^*(S^{\infty\rho_G}; \Sigma) = 0$.*

Proof. By Proposition VI.3, it suffices to prove that $\pi_G^*(S^{\infty\rho_G}; \hat{\Sigma}^*) = 0$. To do this, we use Proposition IV.12 and Corollary VI.6 to prove $\pi_G^*(S^{\infty\rho_G}; T_\omega) = 0$, where $\omega = (H, K)$, and K is a non-trivial proper subgroup of G. (That the finite generation hypothesis in Proposition IV.12 is satisfied follows from Proposition IV.8 and Corollary I.10.) Proposition VI.8 now shows that for $V = \rho_G$, $\omega = (H, K)$, the hypotheses of Proposition VI.7 are satisfied, and hence we have the result. □

COROLLARY VI.10. *Suppose that Conjecture* I.12 *holds for all p-groups* G', *with* $|G'| < |G|$, *and that G is not elementary abelian. Then* $\{S^{\infty\rho_G}; \tilde{E}G\}_*^G = 0$.

Proof. This is immediate from Theorem VI.9 with Proposition IV.5. □

This completes the proof of Theorem B.

Appendix A. Thom spectra of virtual representation

In this appendix, we will construct spectra BG^{-V}, where V is any G-representation, and prove that $\{S^V, EG^+\}^G \cong \pi_*^s(BG^{-V})$. Here, the term *G-space* will be used to mean a G-CW complex in the sense of [10], without a preferred choice of base point. A G-space X is said to be free if $X^H = \varnothing$ for all $H \neq \{e\}$, $H \subseteq G$.

We first need a result of M. Atiyah [7]. Recall from [7] the definition of complex K-theory K^* and its reduced version $\tilde{K}^*$. Let G be a finite group.

PROPOSITION A.1. $|G|$ *annihilates* $\operatorname{im}(\tilde{K}^*(BG^{(l+1)}) \to \tilde{K}^*(BG^{(l)}))$, *where* $BG^{(l)}$ *denotes the l-skeleton of the classifying space BG.*

Proof. This is an easy consequence of Proposition 2.4 of [7], together with the fact that $|G|$ annihilates $\operatorname{im}(\tilde{H}^*(BG^{(l+1)}; \mathbf{Z}) \to \tilde{H}^*(BG^{(l)}; \mathbf{Z}))$. □

Let V be any complex representation of G. We obtain an associated vector bundle $EG \times_G V \to BG$, which we also call V. We denote by $V|BG^{(l)}$ the restriction of the bundle V to the l-skeleton of BG. If k is an integer, kV will denote a direct sum of k copies of V.

LEMMA A.2. $|G|^m V|BG^{(l)}$ *is a trivial bundle, for all* $m \geq l$.

Proof. By Proposition A.1, $|G|^m V|BG^{(l)}$ is stably trivial. Since the dimension of $|G|^m V$ is greater than l, it is actually trivial. □

Choose any trivialization α of $|G|^{l+1}V|BG^{(l+1)}$. By $|G|^l\alpha$ we mean the trivialization of $|G|^l \cdot |G|^{l+1}V|BG^{(l+1)} = |G|^{2l+1}V|BG^{(l+1)}$ which is a $|G|^l$-fold direct sum of α.

LEMMA A.3. $|G|^l\alpha$ *restricted to* $BG^{(l)}$ *is independent of the choice of* α, *as a homotopy class of trivializations of* $|G|^{2l+1}V|BG^{(l)}$.

Proof. If α and α' are distinct trivializations of $|G|^{l+1}V|BG^{(l+1)}$, $\alpha - \alpha'$ represents an element in $\tilde{K}^1(BG^{(l+1)})$. Now $|G|^l\alpha - |G|^l\alpha' = |G|^l(\alpha - \alpha')$

restricts to zero in $\tilde{K}^1(BG^{(l)})$, by Proposition A.1. Consequently, the class of $|G|^l\alpha$ as a stable trivialization is independent of the choice of α. But stable trivializations are equivalent to actual trivializations for dimensional reasons. □

Let $V\langle l\rangle$ denote $|G|^{2l+1}V|BG^{(l)}$, and let α_l denote the trivialization of $|G|^l\alpha|BG^{(l)}$ of $V\langle l\rangle$, for any trivialization α of $|G|^{l+1}V|BG^{(l+1)}$. Let $V^0\langle l\rangle$ denote $V\langle l\rangle$, equipped with a trivial G-action. Thus, α_l is a bundle isomorphism $V^0\langle l\rangle \to V\langle l\rangle$.

Lemma A.4. *The diagram of bundle isomorphisms*

$$\begin{array}{ccc} |G|^2V^0\langle l\rangle & \xrightarrow{\;i^0\;} & V^0\langle l+1\rangle|BG^{(l)} \\ \Big\downarrow{\scriptstyle |G|^2\alpha_l} & & \Big\downarrow{\scriptstyle \alpha_{l+1}|BG^{(l)}} \\ |G|^2V\langle l\rangle & \xrightarrow{\;i\;} & V\langle l+1\rangle|BG^{(l)} \end{array}$$

is homotopy commutative as a diagram of bundle isomorphisms; that is, $i\circ|G|^2\alpha_l$ *and* $(\alpha_{l+1}|BG^{(l)})i^0$ *are homotopic through bundle isomorphisms.* (*Here,* i *and* i^0 *identify the* j-th *copy of* $V\langle l\rangle$ *and* $V^0\langle l\rangle$ *with the* $(j\dim V+1)$-th *through* $(j+1)\dim V$-th *copies of* V *and* V^0 *in* $V\langle l+1\rangle$ *and* $V^0\langle l+1\rangle$.)

Proof. We show that both $i\circ|G|^2\alpha_l$ and $(\alpha_{l+1}|BG^{(l)})i^0$ are of the form $|G|^l\alpha|BG^{(l)}$, where α is a trivialization of $|G|^2|G|^{l+1}V|BG^{(l+1)}$. Then Lemma A.3 shows that the diagram is homotopy commutative as stated. Recall that α_l is constructed as $|G|^l\alpha|BG^{(l)}$, where α is some trivialization of $|G|^{l+1}V|BG^{(l+1)}$. If we take $\hat{\alpha} = |G|^2\alpha$, a trivialization of $|G|^2|G|^{l+1}V|BG^{(l+1)}$, we see that $|G|^2\alpha_l = (|G|^l\cdot\hat{\alpha})|BG^{(l)}$. On the other hand, α_{l+1} is constructed by setting $\alpha_{l+1} = (|G|^{l+1}\alpha')|BG^{(l+1)}$, where α' is a trivialization of $|G|^{l+2}V|BG^{(l+2)}$. Now, $\alpha_{l+1}|BG^{(l)} = (|G|^l\cdot\bar{\alpha})BG^{(l)}$, where $\bar{\alpha} = (|G|\alpha')|BG^{(l+1)}$. This proves the lemma. □

Corollary A.5. *The diagram*

$$\begin{array}{ccc} |G|^2V^0\langle l\rangle & \searrow & \\ \Big\downarrow & & \\ (|G|^2-1)V^0\langle l\rangle\oplus V^0\langle l\rangle & & V^0\langle l+1\rangle|BG^{(l)} \\ \Big\downarrow{\scriptstyle \mathrm{id}\oplus\alpha_l} & & \Big\downarrow{\scriptstyle \alpha_{l+1}|BG^{(l)}} \\ (|G|^2-1)V^0\langle l\rangle\oplus V\langle l\rangle & & \\ \Big\downarrow{\scriptstyle (|G|^2-1)\alpha_l\oplus\mathrm{id}} & & \\ |G|^2V\langle l\rangle & \longrightarrow & V\langle l+1\rangle|BG^{(l)} \end{array}$$

is homotopy commutative as a diagram of bundle isomorphisms.

We recall from the work of G. B. Segal [32] that for a free G-space, vector bundles over X/G and their trivializations correspond bijectively to G-equivariant vector bundles over X and their trivializations. Under this correspondence, an equivariant vector bundle E over X corresponds to the orbit space bundle $E/G \to X/G$. In particular, for $X = EG^{(l)}$, and a G-representation W, the equivariant bundle $W \to EG^{(l)} \times W \to EG^{(l)}$ corresponds to the ordinary vector bundle $W \to EG^{(l)} \times_G W \to BG^{(l)}$.

Given this, the bundle isomorphisms α_l: $V^0\langle l\rangle \to V\langle l\rangle$ constructed above yield isomorphisms of G-vector bundles, which we also call α_l,

$$\alpha_l\colon |G|^{2l+1}V^0 \times EG^{(l)} \to |G|^{2l+1}V \times EG^{(l)},$$

where V^0 denotes V with trivial G-action. This produces a G-homeomorphism of Thom complexes

$$T(\alpha_l)\colon T\big(|G|^{2l+1}V^0 \times EG^{(l)}\big) \to T\big(|G|^{2l+1}V \times EG^{(l)}\big),$$

or equivalently based G-homeomorphisms

$$\beta_l\colon S^{|G|^{2l+1}V^0} \wedge EG^{(l)^+} \to S^{|G|^{2l+1}V} \wedge EG^{(l)^+}.$$

Let $\bar{\beta}_l$: $S^{|G|^{2l+1}V^0} \underset{G}{\wedge} EG^{(l)^+} \to S^{|G|^{2l+1}V} \underset{G}{\wedge} EG^{(l)^+}$ be the induced map on orbit spaces. We now construct the spectrum BG^{-W}, where W is any finite dimensional orthogonal representation of G. To define a spectrum, it suffices to define a prespectrum (see [20]), i.e. a family of spaces $\{X_l\}_{l\geq 0}$, and maps $S^i \wedge X_l \to X_{l+1}$. To define a prespectrum, it actually suffices to define the spaces X_{l_i}, for l_i some strictly increasing sequence of integers, and maps $S^{l_{i+1}-l_i} \wedge X_i \to X_{l_i+1}$. For then, if $l_i \leq j \leq l_{i+1}$, we define $X_j = S^{j-l_i}X_{l_i}$. Let R denote the regular complex representation of G, and let W be embedded as a G-subspace in $\oplus_{i=0}^{\infty}\mathrm{Re}_i$. We define a positive definite, G invariant inner product on $\oplus_{i=0}^{\infty}\mathrm{Re}_i$ in the usual way. Let s be such that W is contained in $\oplus_{i=0}^{s}\mathrm{Re}_i$. We set $l_i = 2|G|^{2i+2}$, and define

$$\begin{cases} X_{l_i} = * \text{ if } |G|^{2i+1} < s, \\ X_{l_i} = S^{|G|^{2i+1}R-W} \underset{G}{\wedge} EG^{(i)+} \text{ otherwise.} \end{cases}$$

Here, $|G|^{2i+1}R - W = |G|^{2i+1}R \cap W^{\perp}$. We define the map

$$\sigma_i\colon S^{l_{i+1}-l_i} \wedge X_{l_i} \to X_{l_{i+1}}$$

to be the composite

$$\begin{aligned} S^{l_{i+1}-l_i} \wedge X_{l_i} &= S^{(|G|^2-1)|G|^{2i+1}R^0} \wedge \Big(S^{|G|^{2i+1}R-W} \wedge_G EG^{(i)^+}\Big) \\ &\xrightarrow{\phi} S^{|G|^{2i+3}R-W} \underset{G}{\wedge} EG^{(i)^+} \\ &\xrightarrow{\mathrm{id}\wedge j} S^{|G|^{2i+3}R-W} \underset{G}{\wedge} EG^{(i+1)^+} = X_{l_{i+1}}, \end{aligned}$$

where ϕ is the orbit space map associated to the equivariant map

$$S^{(|G|^2-1)|G|^{2i+1}R^0} \wedge S^{|G|^{2i+1}R-W} \wedge EG^{(i)^+} \to S^{|G|^{2i+3}R-W} \wedge EG^{(i)^+}$$

induced by the bundle isomorphism

$$\begin{aligned}(|G|^2-1)\alpha_i \oplus \text{id}\colon &(|G|^2-1)|G|^{2i+1}R^0 \oplus (|G|^{2i+1}R-W)\\ &\to (|G|^2-1)|G|^{2i+1}R \oplus (|G|^{2i+1}R-W) = |G|^{2i+3}R-W,\end{aligned}$$

and $j\colon EG^{(i)^+} \to EG^{(i+1)^+}$ is the inclusion.

We now state some well known facts related to the Thom isomorphism. We first recall that for any free based G-complex X, free off the basepoint, $\tilde{H}^*(X/G)$ is equipped with an $H^*(BG)$-module structure, which is respected by G-maps.

PROPOSITION A.6. *Let X be a free G-space, V a finite dimensional complex representation of G. Then $H^{*+2\dim V}(S^V \underset{G}{\wedge} X^+) \cong H^*(X^+/G)$; the isomorphism is an isomorphism of $H^*(BG)$-modules.*

Proof. This is just the Thom isomorphism associated to the bundle $V \underset{G}{\wedge} X \to X/G$; the Thom isomorphism exists since V is orientable, being complex. □

PROPOSITION A.7. *Let V and W be complex G representations, $v = 2\dim_{\mathbf{C}} V$, $w = 2\dim_{\mathbf{C}} W$, and let X be a free G-space. Then the diagram of $H^*(BG)$-modules*

$$\begin{array}{ccc}\tilde{H}^{*+v+w}\left(S^{V\oplus W} \underset{G}{\wedge} X^+\right) & \longrightarrow & \tilde{H}^{*+v+w}\left(S^{V} \underset{G}{\wedge} X^+\right)\\ \downarrow & & \downarrow\\ \tilde{H}^*(X^+/G) & \xrightarrow{\ \cdot C_d(W)\ } & \tilde{H}^{*+w}(X^+/G)\end{array}$$

commutes, where the upper horizontal arrow is induced by the inclusion $V \to V \oplus W$, the two vertical arrows are the Thom isomorphisms mentioned above, and $\cdot C_d(W)$ denotes multiplication by the d-th Chern class $C_d(W)$ of the representation W; $c_d(W) \in H^w(BG)$, $d = \dim_{\mathbf{C}} W$.

PROPOSITION A.8. *Let W be a complex G-representation. Then $H^*(BG^{-W})$ is equipped with an $H^*(BG)$-module structure. As a module over $H^*(BG)$, it is free on one generator in dimension $-2\dim_{\mathbf{C}} W$.*

Proof. This is an easy consequence of the preceding propositions and the definition of BG^{-W}. □

Let V and W be two complex representations; embed $V \oplus W$ in $\oplus_{i=0}^{\infty} \text{Re}_i$, so that V and W are perpendicular. Then we have the inclusions $\oplus_{i=0}^{s} R - (V \oplus W) \hookrightarrow \oplus_{i=0}^{s} R - V$, for sufficiently large s. This induces a map $BG^{-(V\oplus W)} \to BG^{-V}$.

Proposition A.9. *The homomorphism* $f\colon H^*(BG^{-V}) \to H^*(BG^{-(V\oplus W)})$ *induced by the map* $BG^{-(V\oplus W)} \to BG^{-V}$ *is a map of* $H^*(BG)$*-modules; after applying Thom isomorphisms,* f *is multiplication by the* $(\dim W)$*-th Chern class of* W.

Proof. This is an immediate consequence of Proposition A.7. □

It remains to produce the isomorphism

$$\Theta\colon \{S^V, EG^+\}^G{}_* \cong \pi^s{}_*(BG^{-V}).$$

We suppose, as before, that we have $V \subseteq \oplus_{i=0}^{s}\mathrm{Re}_i \subseteq \oplus_{i=0}^{\infty}\mathrm{Re}_i$.

First, we define a directed system of groups whose i-th group is

$$\Gamma^i_*\colon \left\{S^{R^0\langle i\rangle} \wedge S^V, S^{R\langle i\rangle} \wedge EG^{(i)^+}\right\}^G_*, \quad \text{for } i \geq s.$$

The maps $\gamma_i\colon \Gamma^i_* \to \Gamma^{(i+1)}_*$ are defined to be the composites

$$\left\{S^{R^0\langle i\rangle} \wedge S^V, S^{R\langle i\rangle} \wedge EG^{(i)^+}\right\}^G_*$$

$$\xrightarrow{f\to \mathrm{id}\wedge f}\{S^{R^0\langle i+1\rangle - R^0\langle i\rangle} \wedge S^{R^0\langle i\rangle} \wedge S^V, S^{R^0\langle i+1\rangle - R^0\langle i\rangle} \wedge S^{R\langle i\rangle} \wedge EG^{(i)^+}\}^G_*$$

$$\longrightarrow \{S^{R^0\langle i+1\rangle} \wedge S^V, S^{R^0\langle i+1\rangle - R^0\langle i\rangle} \wedge S^{R\langle i\rangle} \wedge EG^{(i)^+}\}^G_*$$

$$\xrightarrow{\text{(a)}} \{S^{R^0\langle i+1\rangle} \wedge S^V, S^{R\langle i+1\rangle - R\langle i\rangle} \wedge S^{R\langle i\rangle} \wedge EG^{(i)^+}\}^G_*$$

$$\longrightarrow \{S^{R^0\langle i+1\rangle} \wedge S^V, S^{R\langle i+1\rangle} \wedge EG^{(i)^+}\}^G_*$$

$$\longrightarrow \{S^{R^0\langle i+1\rangle} S^V, S^{R\langle i+1\rangle} EG^{(i+1)^+}\}^G_*.$$

All maps are self explanatory except (a), which is induced by the bundle isomorphism

$$(|G|^2 - 1)\alpha_i \oplus \mathrm{id}\colon R^0\langle i+1\rangle - R^0\langle i\rangle \oplus R\langle i\rangle \to R\langle i+1\rangle - R\langle i\rangle + R\langle i\rangle.$$

Define $\phi_i\colon \{S^V, EG^{(i)}\}^G_* \to \Gamma^i_*$ to be the composite

$$\{S^V, EG^{(i)^+}\}^G_* \xrightarrow{f\to \mathrm{id}\wedge f} \left\{S^{R^0\langle i\rangle} \wedge S^V, S^{R^0\langle i\rangle} \wedge EG^{(i)^+}\right\}^G_*$$

$$\xrightarrow{\text{(b)}} \left\{S^{R^0\langle i\rangle} \wedge S^V, S^{R\langle i\rangle} \wedge EG^{(i)^+}\right\}^G_*$$

where the arrow (b) is given by $f \to \beta_i \circ f$, where as above, β_i is $T(\alpha_i)$.

Lemma A.10. *The diagrams*

$$\begin{array}{ccc}
\{S^V, EG^{(i)^+}\}^G_* & \xrightarrow{\phi_i} & \Gamma^i_* \\
\downarrow & & \downarrow \gamma_i \\
\{S^V, EG^{(i+1)^+}\}^G_* & \xrightarrow{\phi_{i+1}} & \Gamma^{i+1}_*
\end{array}$$

commute.

Proof. This is an immediate consequence of Corollary A.5. □

Since each ϕ_i is an isomorphism, being the composite of a suspension map and a map induced by a G-homeomorphism, the ϕ_i's combine to give an isomorphism

$$\Phi\colon \{S^V, EG^+\}^G_* \xrightarrow{\sim} \varinjlim_i \{S^V, EG^{\langle i\rangle^+}\}^G_* \to \varinjlim_i \Gamma^i_*.$$

We now construct another directed system $\bar{\Gamma}_*$, defining

$$\bar{\Gamma}^i_* = \left\{S^{R^0\langle i\rangle}, S^{R\langle i\rangle - V} \wedge EG^{(i)^+}\right\}^G_*, \quad \text{whenever } i \geq s.$$

Maps $\bar{\gamma}_i\colon \bar{\Gamma}^i_* \to \bar{\Gamma}^{i+1}_*$ are defined in precisely the same way as γ_i, using the bundle isomorphism $(|G|^2 - 1)\alpha_i \oplus \mathrm{id}$, as above. We define $\bar{\phi}_i\colon \Gamma^i_* \to \bar{\Gamma}^i_*$ to be the inverse to the suspension isomorphism

$$\left\{S^{R^0\langle i\rangle}, S^{R\langle i\rangle - V} \wedge EG^{(i)^+}\right\}^G_* \xrightarrow{f \to f\wedge \mathrm{id}} \left\{S^{R^0\langle i\rangle} \wedge S^V, S^{R\langle i\rangle - V} \wedge EG^{(i)^+} \wedge S^V\right\}^G_*$$

$$\longrightarrow \left\{S^{R^0\langle i\rangle} \wedge S^V, S^{R\langle i\rangle} \wedge EG^{(i)^+}\right\}^G_*.$$

LEMMA A.11. *The diagrams*

$$\begin{array}{ccc} \Gamma^i_* & \xrightarrow{\phi_i} & \bar{\Gamma}^i_* \\ \downarrow{\scriptstyle \gamma_i} & & \downarrow{\scriptstyle \bar{\gamma}_i} \\ \Gamma^{i+1}_* & \xrightarrow{\phi_{i+1}} & \bar{\Gamma}^{i+1}_* \end{array}$$

commute.

Proof. This is immediate from the definitions. □

Let π^i_* denote the directed system

$$\pi^i_* = \pi^s_{*+2|G|^{2i+2}}\left(S^{R\langle i\rangle - V} \underset{G}{\wedge} EG^{(i)^+}\right)$$

with the maps $p_i\colon \pi^i_* \to \pi^{i+1}_*$ induced from the maps σ_i used in the definition of the spectrum BG^{-V}. Then Proposition I.4(b) gives isomorphisms $\bar{\Gamma}^i_* \to \pi^i_*$.

LEMMA A.12. *The diagrams*

$$\begin{array}{ccc} \bar{\Gamma}^i & \longrightarrow & \pi^i_* \\ \downarrow{\scriptstyle \bar{\gamma}_i} & & \downarrow{\scriptstyle p_i} \\ \bar{\Gamma}^{i+1}_* & \longrightarrow & \pi^{i+1}_* \end{array}$$

commute.

Proof. The maps σ_i are the induced maps on orbit spaces of the G-maps used to define $\bar{\gamma}_i$. The result then follows from the naturality of the isomorphism of Proposition I.4(b). □

COROLLARY A.13. $\varinjlim_i \bar{\Gamma}^i_* = \varinjlim_i \pi^i_*$.

PROPOSITION A.14. $\varinjlim_i \pi^i_* \cong \pi^s_*(BG^{-V})$.

Proof. $\pi^s_*(BG^{-V})$ is defined as

$$\varinjlim_i \pi_{*+2|G|^{2i+2}}\Big(S^{R\langle i\rangle - V} \wedge_G EG^{(i)^+}\Big);$$

$$\varinjlim_i \pi^i_* = \varinjlim_i \pi^s_{*+2|G|^{2i+2}}\Big(S^{R\langle i\rangle - V} \wedge_G EG^{(i)^+}\Big).$$

So we have a map $\pi^s_*(BG^{-V}) \to \varinjlim_i \pi^i_*$ obtained from the stabilization map

$$s_i\colon \pi_{*+2|G|^{2i+2}}\Big(S^{R\langle i\rangle - V} \wedge_G EG^{(i)^+}\Big) \to \pi^s_{*+2|G|^{2i+2}}\Big(S^{R\langle i\rangle - V} \wedge_G EG^{(i)^+}\Big).$$

Note that $S^{R\langle i\rangle - V} \wedge_G EG^{(i)^+}$ is $2|G|^{2i+2} - V$ connected, so that for large enough i, s_i is an isomorphism. □

COROLLARY A.15. $\{S^V, EG^+\}^G_* \cong \pi^s_*(BG^{-V})$.

PROPOSITION A.16. *The diagrams*

$$\begin{array}{ccc} \{S^{V\oplus W}, EG^+\}^G_* & \longrightarrow & \pi^s_*(BG^{-(V\oplus W)}) \\ \downarrow & & \downarrow \\ \{S^V, EG^+\}^G_* & \longrightarrow & \pi^s_*(BG^{-V}) \end{array}$$

commute.

Proof. This results from an easy diagram choice using the definitions. □

Appendix B. The elementary abelian case

In this appendix we will sketch a proof of Conjecture I.12 for the case $G = (\mathbf{Z}/p\mathbf{Z})^k$, using a homological calculation which appears in [6], [23]. We use the notation of Section V, and let V denote the reduced regular complex representation of G. Recall the exact sequence (A) of Section I:

$$\cdots \{S^{\infty V}, EG^+\}^G_* \to \pi^*_G(S^{\infty V}) \to \{S^{\infty V}, \tilde{E}G\}^G_* \to \cdots.$$

The groups $\{S^{\infty V}, EG^+\}^G_*$ and $\{S^{\infty V}, \tilde{E}G\}^G_*$ are no longer zero, as is the case when G is not elementary abelian. Thus, the object will be to determine the structure of the groups, and verify that the boundary map is an isomorphism. We first study the group $\{S^{\infty V}, \tilde{E}G\}_*^G$. We must first analyze the discrepancy between $\hat{\Sigma}^*(X)$ and $\Sigma(X)$. The following is a consequence of Quillen's analysis of $N\mathcal{O}^*(G, e)$ for $G = (\mathbf{Z}/p\mathbf{Z})^k$ (see [25]).

PROPOSITION B.1. *Let X be a G-complex. Then the cofibre of the map $\hat{\Sigma}^*(X) \to \Sigma(X)$ has the G-homotopy type of $X^G \wedge SB$, where B denotes the Tits building* (*see* [25]) *for G, and S denotes unreduced suspension.* (*Conventionally, $S(\varnothing) = S^0$. B is in fact $N\mathcal{O}^*(G, e)$.*)

We recall from [25] that B has the homotopy type of a wedge of $p^{\binom{k}{2}}$ $(k-2)$-spheres if $k \geq 2$, and is $\varnothing$ if $k = 1$.

COROLLARY B.2. *Suppose that Conjecture* I.12 *holds for all $(\mathbf{Z}/p\mathbf{Z})^l$, $l < k$. Then*

$$\{S^{\infty V}, \tilde{E}G\}_*^G \cong \pi_s^{-*}(SB) \cong p^{\binom{k}{2}} \cdot \pi_s^{-*}(S^{k-1}).$$

Proof. Let $\mathcal{B}$ denote the S-functor $\mathcal{B}(X) = X^G \wedge SB$. Proposition B.1 shows that there is an exact sequence

$$\cdots \to \pi_G^*(X; \mathcal{B}) \to \pi_G^*(X; \Sigma) \to \pi_G^*(X; \hat{\Sigma}^*) \xrightarrow{\partial} \cdots$$

and one readily checks that $\pi_G^*(X; \mathcal{B}) \cong \pi_S^*(X^G \wedge SB)$. An analysis identical to that in Section VI shows that $\pi_G^*(S^{\infty V}; \hat{\Sigma}^*) = 0$, by the inductive hypothesis. Thus, since $(S^{\infty V})^G = S^0$, we get that $\pi_G^*(S^{\infty V}; \Sigma) \cong \pi_s^*(SB)$. Now, Proposition IV.5 gives the corollary. □

We now turn to $\{S^{\infty V}, EG^+\}_*^G$.

PROPOSITION B.3. $\{S^{\infty V}, EG^+\}_*^G \cong \pi_s^{-*}(S^2B)$.

Proof. In Appendix A, it is shown that $\{S^{\infty V}, EG^+\}_*^G \cong \varprojlim_k \pi_*^s(BG^{-kV})$. Adams, Miller, and Gunawardena ([16]) prove that

$$\varprojlim_k \pi_*^s(BG^{-kV}) \cong p^{\binom{k}{2}} \pi_{*+k}(S^0) \cong p^{\binom{k}{2}} \pi_s^{-*}(S^k) \cong \pi_s^{-*}(S^2B). \qquad \square$$

Both $\{S^{\infty V}, \tilde{E}G\}_*^G$ and $\{S^{\infty V}, EG^+\}_*^G$ are $\pi_*^s(S^0)$-modules in an obvious way. $\{S^{\infty V}, \tilde{E}G\}_*^G$ is a free $\pi_*(S^0)$-module on $p^{\binom{k}{2}}(k-1)$-dimensional generators, and $\{S^{\infty V}, EG^+\}_*^G$ is free on $p^{\binom{k}{2}}$ k-dimensional generators. To verify that ∂ is an isomorphism, then, it will suffice to prove that

$$\partial\colon \{S^{\infty V}, \tilde{E}G\}_{-(k-1)}^G \to \{S^{\infty V}, EG^+\}_{-k}^G$$

is an isomorphism. Let $\bar{\partial}$ denote ∂ reduction mod p. Since we have completed at p, it will suffice to show that $\bar{\partial}$ is an isomorphism.

From this point on, we let $A = \{S^{\infty V}, \tilde{E}G\}^G_{-(k-1)}$ and $B = \{S^{\infty V}, EG^+\}^G_{-k}$. We will need to study generators for A. If we view A as $\pi_G^{k-1}(S^{\infty V}; \Sigma)$, we see that generators for A are given by the various projections from $\Sigma(S^{\infty V})$ to $S^{k-1} \wedge (S^{\infty V})^G = S^{k-1}$ given by Proposition B.1. To compute ∂ on a generator $\alpha\colon \Sigma(S^{\infty V}) \to S^{k-1}$, one first extends (uniquely up to G-homotopy by the proof of Proposition IV.5) the composite $\Sigma(S^{\infty V}) \to S^{k-1} \to S^{k-1} \wedge \tilde{E}G$ to $S^{\infty V} \xrightarrow{\hat{\alpha}} S^{k-1}\tilde{E}G$, and then projects to $S^{k-1} \wedge \tilde{E}G/S^{k-1} = S^k \wedge EG^+$.

LEMMA B.4. *For any G-complex X, $\{X, EG^+\}^G_* \cong \pi_G^{-*}(X; T)$, where T is the quotient S-functor $X \to X/\Sigma(X)$.*

Proof. This is an easy exercise, using the fact that $[X, Y]^G \cong [X/\Sigma(X), Y]^G$, when Y is free. □

We will now define a G-cohomology theory $\mathscr{H}^*$ on finite G-complexes. We note that if S^W is any complex representation of G, $C_*(S^W \wedge X; \mathbf{Z}/p\mathbf{Z})$ is $\mathbf{Z}_p[G]^*$-chain equivalent to $C_*(X; \mathbf{Z}/p\mathbf{Z})$ with a dimension shift of $2\dim W$. This is true since $C_*(S^W \wedge X; \mathbf{Z}/p\mathbf{Z}) = C_*(S^W, \infty, \mathbf{Z}/p\mathbf{Z}) \otimes_{\mathbf{Z}/p\mathbf{Z}} C_*(X; \mathbf{Z}/p\mathbf{Z})$, and the inclusion $\mathbf{Z}/p\mathbf{Z} \to \bar{C}_*(S^W, \infty; \mathbf{Z}/p\mathbf{Z})$ which takes 1 to a cycle representing the top class in $H_*(S^W; \mathbf{Z}/p\mathbf{Z})$ induces a chain equivalence $C_{*-2\dim W}(X; \mathbf{Z}/p\mathbf{Z}) \to C_*(S^W \wedge X; \mathbf{Z}/p\mathbf{Z})$. We denote the induced isomorphism $H^*(X/G; \mathbf{Z}/p\mathbf{Z}) \to H^{*+2\dim W}(S^W \wedge_G X; \mathbf{Z}/p\mathbf{Z})$ by t_w. We now define $\mathscr{H}^*(X; \mathbf{Z}/p\mathbf{Z}) = \varinjlim_W H^*(T(S^W \wedge X)/G; \mathbf{Z}/p\mathbf{Z})$. The maps in the directed system are the composites

$$
\begin{aligned}
&H^{*+2\dim W}\big(T(S^W \wedge X)/G; \mathbf{Z}/p\mathbf{Z}\big)\\
&\xrightarrow{t_U} H^{*+2\dim W+2\dim U}\Big(S^U \underset{G}{\wedge} T(S^W \wedge X); \mathbf{Z}/p\mathbf{Z}\Big)\\
&\to H^{*+2\dim(W+U)}\big(T(S^U \wedge S^W \wedge X)/G; \mathbf{Z}/p\mathbf{Z}\big)
\end{aligned}
$$

where the second map is induced by the S-functor map $T(S^U \wedge S^W \wedge X) \to S^U \wedge T(S^W \wedge X)$. $\mathscr{H}^*$ is extended to infinite complexes by passing to inverse limits over skeletons. Given any element f in $\{X, EG^+\}^G_*$, we may associate to it an element $h(f) \in \mathscr{H}^{-*}(X; \mathbf{Z}/p\mathbf{Z})$ where h is a homomorphism $\{X; EG^+\}^G_* \otimes_{\mathbf{Z}} \mathbf{Z}/p\mathbf{Z} \to \mathscr{H}^{-*}(X; \mathbf{Z}/p\mathbf{Z})$. Now, consider a basis $\alpha_1, \dots, \alpha_r$ for $\{S^{\infty V}, \tilde{E}G\}^G_{-(k-1)}$. As above, they may be represented by projections $\Sigma(S^{\infty V}) \xrightarrow{\alpha_i} S^{k-1}$, which are projections on wedge summands; hence the elements $\alpha_j^*(i_{k-1})$ form a basis for $H^{k-1}(\Sigma(S^{\infty V}; \mathbf{Z}/p\mathbf{Z}))$. Since $S^{\infty V}$ is contractible, the boundary map

$$\delta\colon H^{k-1}(\Sigma(S^{\infty V});\mathbf{Z}/p\mathbf{Z}) \to H^k(S^{\infty V}/\Sigma(S^{\infty V});\mathbf{Z}/p\mathbf{Z})$$

is an isomorphism. Thus, $\{\delta(\alpha_j^*(i_{k-1}))\}$ form a basis for $H^k(S^{\infty V}/\Sigma(S^{\infty V});\mathbf{Z}/p\mathbf{Z})$. But $\delta(\alpha_j^*(i_{k-1})) = \hat{\alpha}_j^*(\delta' i_{k-1})$, where $\delta'\colon H^{k-1}(S^{k-1};\mathbf{Z}/p\mathbf{Z}) \to H^k(S^k \wedge EG^+;\mathbf{Z}/p\mathbf{Z})$ is the boundary map associated to the cofibre sequence $S^{k-1} \to S^{k-1} \wedge \tilde{E}G \to S^k \wedge EG^+$, and $\hat{\alpha}_j$ is the element obtained as above by extending the composite $\Sigma(S^{\infty V}) \to S^{k-1} \to S^{k-1} \wedge \tilde{E}G$ to $S^{\infty V}$ and projecting to $S^k \wedge EG^+$. Thus, the elements $\hat{\alpha}_j^*(\delta' i_{k-1})$ form a basis for $H^k(S^{\infty V}/\Sigma(S^{\infty V});\mathbf{Z}/p\mathbf{Z})$, and from the earlier discussion, the $\hat{\alpha}_j$'s generate $\operatorname{im}(\partial\colon A \to B)$. If we can show that the set $\{h(\hat{\alpha}_j)\}_{j=1}^r$ is linearly independent in $\mathscr{H}^k(S^{\infty V};\mathbf{Z}/p\mathbf{Z})$, we will have proved that $\bar{\partial}$ is injective, hence an isomorphism, which will give the result. This fact is now established by the following lemma.

LEMMA B.5. *The homomorphism*

$$f_U\colon H^k(T(S^V)/G;\mathbf{Z}/p\mathbf{Z}) \to H^{k+2\dim U}\Big(S^U \underset{G}{\wedge} T(S^V);\mathbf{Z}/p\mathbf{Z}\Big) \to H^{k+2\dim U}(T(S^U \wedge S^V)/G;\mathbf{Z}/p\mathbf{Z})$$

is an injection for all k and U.

Proof. We dualize, and will show that each map

$$f_U^*\colon H_{k+2\dim U}(T(S^U \wedge S^V)/G;\mathbf{Z}/p\mathbf{Z}) \to H_k(T(S^V)/G;\mathbf{Z}/p\mathbf{Z})$$

is surjective. Recall that each of the graded groups $H_*(T(S^U \wedge S^V)/G;\mathbf{Z}/p\mathbf{Z})$ and $H_*(T(S^U)/G;\mathbf{Z}/p\mathbf{Z})$ are modules over the ring $H^*(G;\mathbf{Z}/p\mathbf{Z})$, with ring elements lowering degree. Consider first the case $G = \mathbf{Z}/p\mathbf{Z}$, and let U be a one-dimensional complex representation of G. Then $T(S^U)/G$ is the suspension of $(S^1)^+$; we note that the one-dimensional element in $H^*(G;\mathbf{Z}/p\mathbf{Z})$ acts non-trivially from $H_2(T(S^U)/G;\mathbf{Z}/p\mathbf{Z})$ to $H_1(T(S^U)/G;\mathbf{Z}/p\mathbf{Z})$. Now, let $\mathscr{B} = \{b_1,\dots,b_k\}$ be any basis for $G = (\mathbf{Z}/p\mathbf{Z})^k$; the dual basis $\mathscr{B}^*$ specifies a k-dimensional complex representation W of G. It is not hard to verify that $T(S^{W_{\mathscr{B}}})/G \simeq S^k(S^1)^+ \wedge \underset{k\text{-factors}}{\cdots} \wedge (S^1)^+$ where S^k denotes k-fold suspension, and that if $x_1,\dots,x_k \in H^1(G;\mathbf{Z}/p\mathbf{Z})$ are the cohomology classes corresponding to the dual basis $\mathscr{B}^*$, $x_1 \cdots x_k i_{\mathscr{B}} = \lambda j_{\mathscr{B}}$, where $i_{\mathscr{B}}$ is a generator for $H_{2k}(T(S^{W_{\mathscr{B}}})/G;\mathbf{Z}/p\mathbf{Z})$, $j_{\mathscr{B}}$ is a generator for $H_k(T(S^{W_{\mathscr{B}}})/G;\mathbf{Z}/p\mathbf{Z})$, and $\lambda \neq 0$. This follows, in fact, from the one-dimensional result given above. For any basis $\mathscr{B}$, $W_{\mathscr{B}}$ appears uniquely as a summand in the reduced regular representation V, and we obtain a map

$$g_{\mathscr{B}}\colon T(S^{W_{\mathscr{B}}})/G \to T(S^V)/G.$$

Using standard Mayer-Vietoris techniques, one finds that the map

$$\bigvee_{\mathscr{B}} T(S^{W_{\mathscr{B}}})/G \overset{Vg_{\mathscr{B}}}{\to} T(S^V)/G$$

induces a surjection on $H_k(\ ;\mathbf{Z}/p\mathbf{Z})$, where $\mathscr{B}$ ranges over all bases of G. Let $i \in H_{2\dim V}(T(S^V)/G;\mathbf{Z}/p\mathbf{Z})$. A standard fact about Chern classes shows that $g_{\mathscr{B}}(i_{\mathscr{B}}) = C(V - W_{\mathscr{B}}) \cdot i$, where $C(V - W_{\mathscr{B}})$ denotes the top Chern class of $V - W$. Let $x_{\mathscr{B}}$ denote the product $x_1 \cdots x_k$, where $x_1, \ldots, x_k$ are the cohomology classes corresponding to the dual basis $\mathscr{B}^*$. Then we have

$$g_{\mathscr{B}}(\lambda j_{\mathscr{B}}) = g_{\mathscr{B}}(x_{\mathscr{B}} i_{\mathscr{B}}) = x_{\mathscr{B}} g_{\mathscr{B}}(i_{\mathscr{B}}) = x_{\mathscr{B}} C(V - W_{\mathscr{B}}) i,$$

where $\lambda \neq 0$. Since as was remarked above the elements $g_{\mathscr{B}}(j_{\mathscr{B}})$ span $H_k(T(S^V)/G;\mathbf{Z}/p\mathbf{Z})$, we have shown that $H_k(T(S^V)/G;\mathbf{Z}/p\mathbf{Z})$ is contained entirely in the $H^*(G;\mathbf{Z}/p\mathbf{Z})$-module generated by i. Since the maps f_U^* are $H^*(G;\mathbf{Z}/p\mathbf{Z})$-modules and i is in the image of f_U^*, the result is proved.

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