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A SURVEY OF EQUIVARIANT STABLE HOMOTOPY THEORY

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INTRODUCTION

EQUIVARIANT stable homotopy theory was invented by G. B. Segal in the early 1970s [45]. He was motivated by his work with Atiyah [9] on equivariant K-theory, generalizing an earlier theorem of Atiyah's on the K-theory of classifying spaces of finite groups to compact Lie groups, and by his work on configuration space and discrete models for iterated loop spaces. His work also suggested to him the "Segal conjecture" (see 4.3), which asserts that the zero-dimensional stable cohomotopy group of the classifying space of a finite group is isomorphic to the completed Burnside ring of the group. The statement is a non-equivariant one, but the methods involved in the eventual proof of the conjecture require heavy use of the equivariant theory. In addition, J. P. May and his collaborators have pointed out that an equivariant version of spectra is the natural device for making spectrum level versions of space level constructions, such as the quadratic or p-adic construction used in defining Steenrod operations.

A first attempt to construct an equivariant stable homotopy theory would define the set of stable G-maps (G is finite) from X to Y to be the direct limit $\varinjlim_n [\Sigma^n X, \Sigma^n Y]^G$, where the G-actions on $\Sigma^n X$ and $\Sigma^n Y$ are obtained by directly suspending the actions on X and Y, and $[-,-]^G$ denotes G-homotopy classes of G-maps. Unfortunately, the theory one obtains this way does not allow for many of the familiar constructions one associates with stable homotopy theory, such as S-duality and transfer. To obtain these, one must allow the group G to act on the "suspension coordinate". Precisely, for any representation V of the group G, one can form the one point compactification S^V of V, which becomes a based G-space. S^V is a $dim(V)$-sphere.

The stable G-maps from X to Y are now defined to be the direct limit $\varinjlim_V [S^V \wedge X, S^V \wedge Y]^G$, where the actions on $S^V \wedge X$ and $S^V \wedge Y$ are diagonal actions, and the direct limit is taken over a certain ordering on the representations, so that $V \leq W \Rightarrow V$ is a summand of W. This construction permits S-duality and transfer.

Other topologists quickly began to study the properties of this theory, notably tom Dieck [21]. It gradually became clear that this theory was the natural way to stabilize G-homotopy theory; in this setting, it is possible to construct transfers and S-duality, and to prove an equivariant version of Thom's transversality theorem. More naive forms of stabilization do not permit these constructions.

Some comments about Frank Adams' contributions to this subject are in order. He became interested in Segal's conjecture at an early stage, and in [1] he pointed out the central role played by the spectrum $\mathbf{R}P^{\pm\infty}_{-\infty}$ in the study of the conjecture for $G = \mathbf{Z}/2\mathbf{Z}$.

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$\mathbf{R}P^{+\infty}_{-\infty}$ was then analyzed by W. H. Lin [37] to prove the conjecture in that case. After Lin's proof, Adams' interest in the conjecture became strong, and he, together with J. Gunawardena and H. Miller, proved the conjecture for elementary abelian groups. Since the proof of the conjecture for all p-groups involved equivariant homotopy theory, he also became interested in it and in characteristic fashion clarified a number of fuzzy points in the theory in his very valuable paper [2]. More recently, together with collaborators, he proved what must surely be the definitive forms of both the Segal conjecture and Atiyah's theorem [4], [5]. Of course, in many ways he invented non-equivariant stable homotopy theory, and his work in that area will certainly point out the right directions to follow in the equivariant theory in the future.

It is the aim of this paper to give the reader a feeling for some of the issues that come up as one introduces G-actions into stable homotopy theory. I have not attempted to give a complete survey of the area but rather have chosen some of the areas with which I am reasonably familiar and used them as illustrations. The paper is organized as follows. §1 gives a quick summary of the unstable equivariant theory. §2 discusses the equivariant version of the usual $\Omega^\infty S^\infty$ construction. It can be taken as a summary of the work of Segal [45], Adams [2], tom Dieck [21] and H. Hausschild [30]. §3 discusses the definitions of spectra in the equivariant setting, and some of their properties. Here, the book [35] by Lewis, May, and Steinberger is an excellent reference. §4 gives examples of various equivariant spectra, and gives some idea of what is known about them and how they are used. Finally, §5 contains a short list of open problems in the area, which seem to be of interest to the author.

1. A BRIEF DISCUSSION OF UNSTABLE EQUIVARIANT HOMOTOPY THEORY

We discuss certain fundamental notions of equivariant homotopy theory, which show how non-equivariant homotopy theory generalizes. We also introduce some ideas peculiar to equivariant theory. Throughout this paper, G will be a finite group.

Definition 1.1. Let G be a group. Then a left G-space is a space X together with a representation ρ of G into the self-homeomorphism group of X. We write gx for $\rho(g)(x)$. If $K \subseteq G$ is a subgroup, we define the K-fixed point set X^K to be $\{x \in X \mid kx = x \,\forall k \in K\}$. We also let X_G denote the orbit space of X, i.e. the identification space of X associated to the equivalence relation $x \sim y \Leftrightarrow \exists g \in G$ such that $gx = y$. If X and Y are G-spaces, a map $f: X \to Y$ is equivariant if $f(gx) = gf(x) \,\forall g, x$. A homotopy between two G-maps $f_1, f_2: X \to Y$ is a G-map $H: X \times I \to Y$, where I denotes the unit interval with trivial G-action and G acts on $X \times I$ by $g(x, t) = (gx, t)$, so that $H \mid X \times 0 = f_1$ and $H(x, 1) = f_2(x)$. A based G-space is a G-space X with preferred choice of basepoint in X^G; a based G-map is defined in the obvious way. A G-map $f: X \to Y$ is a G-homotopy equivalence if there is a G-map $g: Y \to X$ so that fg and gf are G-homotopic to the respective identity maps. There is of course the corresponding based notion. If X and Y are G-spaces, $X \times Y$ denotes the usual product space with G-action given by $g(x, y) = (gx, gy)$. Similarly, for based G-spaces, we have smash products, mapping cylinders, and mapping cones. We say a G-map $f: X \to Y$ is a weak equivalence if each map $f^K: X^K \to Y^K$ (f^K denotes the restriction of f to X^K for a subgroup $K \subseteq G$) is a weak equivalence.

Let G be a group, and let $K \subseteq G$ be a subgroup. We let G/K denote the collection of cosets gK, $g \in G$, it is acted on on the left by G, and is viewed as a discrete G-space. Let D^n denote the standard n-disc, and let ∂D^n denote its boundary $(n-1)$-sphere. Suppose that D^n

is equipped with the trivial G action, and that we have an equivariant G-map $f: G/K \times \partial D^n \to X$. Then we call the space $X \amalg G/K \times D^n/\cong$, where $\cong$ is the equivalence relation given by $(gK, x) \cong f(gK, x)$, $(gK, x) \in G/K \times \partial D^n$, the space obtained from X by adjoining a G/K-cell along f.

Definition 1.2. A $G - CW$ complex is a space obtained by iterated adjunction of G/K-cells for various different choices of K and dimensions of the cells. More precisely, a $G - CW$ complex is a G-space X equipped with a filtration $X^{(i)}$, so that $X^{(i)}$ is obtained from $X^{(i-1)}$ by adjoining i-cells of the form $G/K_\alpha \times D^i$, $\alpha \in A$, where A is an indexing set, and so that the topology on X is the direct limit topology. Fixed point sets of subgroups $K \subseteq G$ and orbit spaces of $G - CW$ complexes are CW complexes in the usual sense.

Definition 1.3. A G-simplicial set is a simplicial set with left G-action. This is equivalent to the notion of a simplicial object in the category of G-sets. All the usual notions of simplicial topology apply (realization, singular complex, simplicial maps, and simplicial homotopies). Also, all the notions defined in 1.1 have their simplicial versions. Note that the realization of a G-simplicial set is a $G - CW$ complex. We say a G-map of simplicial G-sets is a weak G-equivalence if its realization is a G-equivalence.

Certainly for finite groups, it is easy to check that the usual comparison theorems between the simplicial homotopy category and the homotopy theory of spaces hold. Particular examples are that all G-spaces have the weak G-homotopy type of a $G - CW$ complex, and every $G - CW$ complex has the G-homotopy type of the realization of a G-simplicial set. Further, Dwyer and Kan [23] have shown that the category of G-simplicial sets admits the structure of a closed model category in the sense of Quillen [43], in which the weak equivalences are the maps whose induced maps on fixed point sets are weak equivalences.

Let $K \subseteq G$ be a subgroup, and let X be a K-space. Then we can form the G-space $G \times_K X = G \times X/\simeq_K$, where $\simeq_K$ is the equivalence relation generated by $(gk, x) \simeq (g, kx)$, $k \in K$. This is a left G-space, with G acting by left multiplication on the G-factor. As a space, $G \times_K X$ is a disjoint union of copies of X, one for each left coset gK. If X is equipped with a G-action, restricting to the given K-action, then $G \times_K X \cong G/K \times X$ as G-spaces. A G-equivariant map from $G \times_K X$ into a G-space Y may be identified (by restriction to $e \times X$) with a K-equivariant map from X to Y, and this identification is bijective. There is a corresponding based analogue. For any G-space Z, let Z_+ denote Z with a disjoint fixed basepoint $+$ added. For $K \subseteq G$, and X a based K-space, we can now form $G_+ \wedge_K X$, defined by obvious analogy with the case of $G \times_K X$, and there is a similar identification of based G-maps with domain $G_+ \wedge_K X$ with based K-maps with domain X. Now one can see how $G - CW$ complexes are built up. Non-equivariantly, attaching data for an n-cell to a based complex X is an element of $\pi_{n-1}(X)$. Equivariantly, attaching data for a based cell $G/K_+ \wedge D^n \cong G_+ \wedge_K D^n$ is a based G-homotopy class of G-maps $G_+ \wedge_K S^{n-1} \to X$, or by the above adjunction, a homotopy class of K-equivariant based maps from S^{n-1} (with trivial K-action) into X. Such a homotopy class is clearly identified with an element of $\pi_{n-1}(X^K)$. This analysis of the obstructions to extending maps allows one to prove a useful result.

PROPOSITION 1.4. *Let $\mathscr{F}$ be a collection of subgroups of a finite group G, closed under conjugation and passage to subgroups. Suppose further that we are given a based G-space Y, with Y^K contractible for all $K \in \mathscr{F}$. Let X be a $G - CW$ complex, and let $X(\mathscr{F})$ denote $\bigcup_{K \notin \mathscr{F}} X^K$. Then the space $F_0^G(X, Y)$ of equivariant based maps from X to Y is weakly homotopy equivalent, via restriction, to the space $F_0^G(X(\mathscr{F}), Y) \cong F_0^G(X(\mathscr{F}), Y(\mathscr{F}))$.*

Proof. X is obtained from $X(\mathscr{F})$ by iteratively attaching cells of the form $G/K_+ \wedge D^n$, with $K \in \mathscr{F}$. From the above analysis, the obstruction to extending a given map over this cell is an element in $\pi_{n-1}(Y^k)$, and Y^K is contractible, so the restriction map is surjective on π_0. Injectivity is proved by showing that homotopies also extend, using a relative form of the result. To prove that the map induces isomorphisms on π_n, one uses the result for π_0 applied to the $G - CW$ complex $S^n \wedge X$. □

An important theme in equivariant homotopy theory is the reduction of equivariant questions to non-equivariant ones. If X and Y are G-spaces, and $F: X \to Y$ is a G-map, then if f is a G-homotopy equivalence it is certainly a weak G-equivalence in the sense of 1.1. When X and Y are $G - CW$ complexes, we have the following converse to this observation.

THEOREM 1.5. (see [13]). *Let G be finite, and suppose that $f: X \to Y$ is a weak G-equivalence. Then it is a G-homotopy equivalence.*

Recall that fibrations and cofibrations are defined in terms of homotopy lifting and homotopy extension properties, which make good sense in the equivariant setting, so the notions of G-fibration and G-cofibration are defined. If $F: X \to Y$ is a G-fibration or G-cofibration, then the induced maps $f^K: X^K \to Y^K$ are fibrations and cofibrations, respectively. If f is a G-fibration and $* \in Y^G$, then the usual homotopy fiber $F(f, *)$ makes sense, and is acted on by G.

We now recall some constructions which do not have counterparts in the non-equivariant theory. If G is a group, it is well known that there is a contractible space EG on which G acts freely. One can, for instance, take the infinite join of copies of G (Milnor construction), or the realization of the simplicial construction WG (see [39]). More generally, let $\mathscr{F}$ be any family of subgroups of a group G, closed under downward inclusion and conjugation. Then there is a G-space $E_{\mathscr{F}}G$ so that $(E_{\mathscr{F}}G)^K$ is contractible for $K \in \mathscr{F}$, and empty for $K \notin \mathscr{F}$. EG corresponds to the case $\mathscr{F} = \{\{e\}\}$. Moreover, $E_{\mathscr{F}}G$ can be taken to be a $G - CW$ complex, and any two $G - CW$ complexes satisfying the conditions above are naturally G-homotopy equivalent. See [22] for a discussion. The orbit space of $E_{\mathscr{F}}G$ is called $B_{\mathscr{F}}G$, the classifying space for G relative to the family $\mathscr{F}$.

Definition 1.6. We define $X_{hG} = EG \times_G X$, the "homotopy orbit space" of X. For any two G-spaces X and Y, let $F(X, Y)$ denote the space of functions from X to Y, with the compact-open topology. G acts on $F(X, Y)$ via the rule $(gf)(x) = gf(g^{-1}x)$. We also define $X^{hG} = F(EG, X)^G$, the "homotopy fixed point set" of X.

Note that X_{hG} and X^{hG} are orbit and fixed point sets of actions of G on spaces homotopy equivalent to X, namely $EG \times X$ and $F(EG, X)$, but with some "singularities" ironed out. In fact, the map $EG \to$ point induces equivariant maps $EG \times X \to X$ and $X = F(\text{point}, X) \to F(EG, X)$, hence maps $X_{hG} \to X_G$ and $X^G \to X^{hG}$. The analysis of these maps is often of some interest. For instance, $H_*(X_{hG})$ and $\pi_*(X^{hG})$ are computable from data about the ambient space X and information about the G-action on H_*X and π_*X, respectively, in the sense that there are spectral sequences with $E^2_{p,q} \cong H_p(G, H_q(X))$ and $E_2^{p,q} = H^{-p}(G; \pi_q(X))$ converging to $H_*(X_{hG})$ and $\pi_*(X^{hG})$, respectively. Also, X_{hG} and X^{hG} are often recognizable as well known non-equivariant spaces.

PROPOSITION 1.7. *Let X be a space with trivial G-action. Then $X^{hG} \cong F(BG, X)$.*

We say a G map $f: X \to Y$ is a quasi-equivalence if it is a weak equivalence as a map of spaces (ignoring the G-action). This is a much weaker notion of equivalence than G-weak equivalence, since it says nothing explicitly about fixed point sets. For instance, let $G = \mathbf{Z}$, let $X = \mathbf{R}$ with translation action, let Y denote a single point with trivial action, and let $f: X \to Y$ be the unique map from X to Y. Then $X^G = \varnothing$ and $Y^G = Y$, so f is clearly not a G-equivalence. However, the following proposition is easy to verify.

PROPOSITION 1.8. *If f is a quasi-equivalence, then the induced maps $f_{hG}: X_{hG} \to Y_{hG}$ and $f^{hG}: X^{hG} \to Y^{hG}$ are weak homotopy equivalences. More generally, if W is any free G-complex, the maps $W \times_G X \to W \times_G Y$ and $F(W, X)^G \to F(W, Y)^G$ are weak equivalences. The based version, where W is free off the basepoint, with based function complex, holds.*

If one is now able to prove a result showing that, e.g., a map $X^G \to X^{hG}$ is an equivalence in an appropriate sense, then one will be able to study X^G from data concerning the ambient space and the action of G on its homotopy groups.

2. STABLE EQUIVARIANT THEORY

(*A*) *Definitions of stable equivariant homotopy theory*

In attempting to understand what the right notion of equivariant stable homotopy theory should be, it is useful to recall three different points of view toward ordinary stable homotopy theory.

(a) We recall first that if X is a based CW complex, $Q(X)$ is defined to be $\varinjlim \Omega^n \Sigma^n X$, under the direct system given by suspension. Recall that $\pi_*^s X = \varinjlim \pi_{n+k}(\Sigma^n X)$, again under suspension maps, and it is clear that $\pi_*^s(X) \cong \pi_*(QX)$. More generally, if Y is a based complex, we define $\{Y, X\}$, the abelian group of homotopy classes of stable maps from Y to X to be $\pi_0(F_0(Y, QX))$, where $F_0(--, --)$ denotes based function space, equipped with compact open topology. The space $Q(S^0)$ plays a central role; its homotopy groups are the stable homotopy groups of spheres.

(b) It was discovered by Barratt, Priddy, and Quillen that the space $Q(S^0)$ could be constructed as follows. Let Σ_n denote the symmetric group on n letters, and let $B\Sigma_n$ denote the functorial simplicial construction of the classifying space for Σ_n (see [39]). Then we have homomorphisms $\Sigma_n \times \Sigma_m \to \Sigma_{n+m}$ arising from block sum of permutations, and we obtain simplicial maps $B\Sigma_n \times B\Sigma_m \to B\Sigma_{n+m}$ which turn $\coprod_{n \geq 0} B\Sigma_n$ into a simplicial monoid M. Then $Q(S^0)$ has the homotopy type of the realization of the simplicial group obtained from M by formally adjoining inverses levelwise. Equivalently, using Segal's machine [46] which associates an infinite loop space (indeed, a spectrum) to a symmetric monoidal category, the sphere spectrum is the spectrum associated to the symmetric monoidal category of finite sets.

(c) The Pontrjagin–Thom construction gives an isomorphism of the bordism group of framed n-manifolds with the group $\pi_n^s(S^0)$.

Now, let G be a finite group. We will work through the versions of equivariant stable homotopy theory which correspond to constructions (a), (b), and (c) above.

(a^G) Let V be the regular real representation of G. Let S^V denote its one point compactification, a based (at ∞) finite $G-CW$ complex. If X is a based G-complex, we can form the suspension $\Sigma^V X = S^V \wedge X$, also a based G-complex. Also, let $\Omega^V X$ denote the space of based maps from S^V into X, with action given by $(gf)(x) = gf(g^{-1}x)$. As in the non-equivariant setting, we have equivariant suspension maps $\Omega^{nV}\Sigma^{nV}X \to \Omega^{(n+1)V}\Sigma^{(n+1)V}X$, where nV denotes a direct sum of n copies of V. The direct limit $\varinjlim \Omega^{nV}\Sigma^{nV}X$ is now called $Q^G(X)$. The group of G-homotopy classes of stable

G-maps from Y to X, $\{Y, X\}^G$, is now defined to be $\pi_0(F_0(Y, Q^G X)^G)$. The graded group-valued functor $\pi_*^G X \cong \pi_*((Q^G X)^G)$ is a homology theory on the category of G-complexes, in the sense that it assigns to a pair $X \subseteq Y$ a long exact sequence

$$\cdots \to \pi_n^G(X) \to \pi_n^G(Y) \to \pi_n^G(Y, X) \to \cdots.$$

In fact, Q^G carries cofibration sequences into weak G-fiber sequences.

(b^G) Let X be any finite G-set. We let Σ_X denote the group of equivariant automorphisms of X. If X and Y are finite G-sets, we have a block sum homomorphism $\Sigma_X \times \Sigma_Y \to \Sigma_{X \amalg Y}$, where $X \amalg Y$ denotes the disjoint union of X and Y. By choosing a representative of each isomorphism class of finite G-sets appropriately, we obtain a simplicial monoid $\amalg_X B\Sigma_X$, whose group completion realizes to a space weakly equivalent to $(Q^G(S^0))^G$. In fact if we let $\tilde{\Sigma}_X$ denote the full automorphism group of X, including the non-equivariant automorphisms, then $\Sigma_X = (\tilde{\Sigma}_X)^G$, where G acts by conjugation on $\tilde{\Sigma}_X$. This gives that $\amalg_X B\tilde{\Sigma}_X = (\amalg_X B\Sigma_X)^G$, and the corresponding statement for the associated group completions. Moreover, one easily sees that the group completion of $\amalg_X B\tilde{\Sigma}_X$ has the same homotopy type as the group completion of $\amalg_n B\Sigma_n$, so we even have a discrete model for the G-space $Q^G S^0$, with its G-action.

H. Hausschild [30] has gone further to construct configuration space models for $Q^G X$, where X is a space, along the lines of May's models [40] for QX in the non-equivariant setting. We discuss what goes into these constructions. Let us recall how the non-equivariant version works. Let $\mathbf{R}^\infty$ denote the direct limit $\varinjlim \mathbf{R}^n$, with direct limit topology, and let X be a based space. Let $C_k \subseteq (\mathbf{R}^\infty)^k$ be the subspace $\{(v_1, \ldots, v_k) | v_i \neq v_j$ when $i \neq j\}$. Σ_k acts freely on C_k, and C_k is contractible. Further, for each i, $1 \leq i \leq k$, we have degeneracy maps $\sigma_i : C_k \to C_{k-1}$, defined by $\sigma_i(v_1, \ldots, v_k) = (v_1, \ldots, \hat{v}_i, \ldots, v_k)$. Let X^k be equipped with the permutation action, and let $C_k \times_{\Sigma_k} X^k$ denote the orbit space under the diagonal action. Define CX to be the identification space

$$\coprod_k C_k \underset{\Sigma_k}{\times} X^k / \cong,$$

where $\cong$ is the equivalence relation generated by the equivalences $(v_1, \ldots, v_k) \times (x_1, \ldots, x_{i-1}, *, x_{i+1}, \ldots, x_k) \cong (v_1, \ldots, \hat{v}_i, \ldots, v_k) \times (x_1, \ldots, \hat{x}_i, \ldots, x_k)$. This relation is respected by the permutation identifications. In order to obtain the correct configuration space model in the equivariant setting, we must only choose the right equivariant model for $\mathbf{R}^\infty$. We let $\mathbf{R}^\infty$ be $\varinjlim V^k$, where V denotes the regular representation of G. $\mathbf{R}^\infty$ thus contains each irreducible representation infinitely often. The analogue of C_k is now equipped with G-action, commuting with the Σ_k action, and the resulting CX construction is also equipped with a G-action. Hausschild shows that when X has connected fixed point sets X^K for all $K \subseteq G$, this gives a space with the weak G-homotopy type of $Q^G X$, and in general $Q^G X$ is obtained by a group completion procedure.

In May's model, the spaces C_n have a homotopy invariant meaning, namely $C_n \cong E\Sigma_n$ as Σ_n-spaces. We will give C_n in our setting a $G \times \Sigma_n$ homotopy invariant meaning. Let $\mathscr{F}$ be the family of subgroups of $G \times \Sigma_n$ consisting of all subgroups whose intersection with $\{e\} \times \Sigma_n \subseteq G \times \Sigma_n$ consists only of $\{e\}$. Recall the definition of $E_{\mathscr{F}}(G \times \Sigma_n)$ from §1. We wish to show that C_n is an $E_{\mathscr{F}}(G \times \Sigma_n)$-space. Thus, consider the fixed point sets C_n^K, $K \subseteq G \times \Sigma_n$. If $K \cap \Sigma_n \neq \{e\}$, then $C_n^K \simeq \varnothing$, since the action of Σ_n on C_n is free. Let $K \subseteq G \times \Sigma_n$ be a subgroup so that $K \cap \Sigma_n = \{e\}$, and let $\bar{K} \subseteq G$ be the image of K and G under the projection $G \times \Sigma_n \to G$. Of course, the projection $K \to \bar{K}$ is an isomorphism, since $K \cap \Sigma_n = \{e\}$. Consequently, for each $\bar{k} \in \bar{K}$, there is a unique permutation $f(\bar{k}) \in \sigma_n$ so that $(\bar{k}, f(\bar{k})) \in K$, and f is easily seen to be a homomorphism $\bar{K} \to \Sigma_n$. f determines and is determined by K. Let $(\bar{K}, f)$ be data for K. $C_n^K = \{(v_1, \ldots, v_n) | v_i = \bar{k}v_{f(\bar{k})(i)}$ for all $\bar{k} \in \bar{K}$, $i \in \{1, \ldots, n\}\}$. Note that $f: \bar{K} \to \Sigma_n$ determines an action of $\bar{K}$ on $\{1, \ldots, n\}$, and let $\{i_1, \ldots, i_s\}$ denote a non-redundant set of orbit representatives for this action. Then the coordinates $v_{i_1}, \ldots v_{i_s}$ clearly determine all the other coordinates. Let $\bar{K}_t$ denote the stabilizer of i_t in $\bar{K}$. Then we see that if $(v_1, \ldots, v_n) \in C_n^K$, $v_{i_t} \in (\mathbf{R}^\infty)^{K_t}$, and that the converse also holds. Thus, C_n^K may be viewed as the open subspace of $\prod_{t=1}^s (\mathbf{R}^\infty)^{\bar{K}_t}$, obtained by deleting the intersections C_n^K with the spaces C_n^L, $L \supseteq K$, which are linear subspaces of infinite codimension. Therefore, C_n^K is contractible, and $C_n \sim E_{\mathscr{F}}(G \times \Sigma_n)$.

(c^G) Let M^n be a smooth, compact, closed G-manifold. We say that M is G-framed if there is a bundle isomorphism $\tau(M) \oplus \mathbf{R}^k \to M \times \mathbf{R}^{n+k}$ of vector bundles respecting the G-action, where $\mathbf{R}^k$ and $\mathbf{R}^{n+k}$ are equipped with trivial G-action, and $\tau(M)$ denotes the tangent bundle of M. Framed bordism groups are defined in the evident way, and the k-dimensional framed G-bordism group of a point is the k-dimensional equivariant stable homotopy group of S^0. Note that a zero dimensional framed G-manifold is just a finite G-set with a choice of sign for each orbit. Let $K \subseteq G$ be a subgroup, and X a G-set. Let $\alpha_K^\pm(X)$ be the number of orbits of X of type G/K with sign ± 1. Then two zero dimensional framed G-manifolds X and Y are G-bordant if and only if $\alpha_K^+(X) - \alpha_K^-(X) = \alpha_K^+(Y) - \alpha_K^-(Y)$ for all $K \subseteq G$. The verification that these definitions are equivalent was outlined in Segal [45].

(a^G)$\Leftrightarrow$(c^G) uses equivariant transversality, which requires more hypotheses than the non-equivariant version. It is well-known that there are unstable obstructions to equivariant transversality. We will return to this point in §4. (a^G)$\Leftrightarrow$(b^G) can be obtained by explicitly describing the groups arising from (b^G) and (c^G), and using a Dyer–Lashof style map from the group-theoretic construction in (b^G) to the more homotopy theoretic construction in (a^G), together with the equivalence (a^G)$\Leftrightarrow$(c^G).

(*B*) *Description of the equivariant stable stems*

The easiest way to predict what the equivariant stable stems are is to use model (b^G). Let G be a finite group, and $K \subseteq G$ a subgroup. The group of G-equivariant automorphisms of the left G-set G/K is obtained as follows. Let $\alpha: G/K \to G/K$ be a G-automorphism; since G/K is a transitive G-set, α is determined by $\alpha(K)$. Since K is a fixed point under the K-action, so is $\alpha(K)$. gK is fixed under K if and only if $kgK = gK \, \forall k \in K$, or $g^{-1}kg \in K \, \forall k \in K$. Thus, $g \in N_G(K)$, the normalizer of K in G. If $g_1 K = g_2 K$, then $g_1 = g_2 k$ for some $k \in K$, so g_1 and g_2 correspond to the same automorphism of G/K if they differ by an element of K. Consequently, one verifies that $\mathrm{Aut}^G(G/K) \cong N_G(K)/K$, which we call the "Weyl group" of K in G, and denote by $W_G(K)$. Moreover, it is easy to see that $\mathrm{Aut}^G(\coprod_1^n G/K) \cong \Sigma_n \int W_G(K)$, where $\Sigma_n \int H$ denotes $\Sigma_n \tilde{\times} H^n$, with Σ_n acting on H by permuting coordinates. For any G-set X, let $X[K]$ be the union of the orbits of the form G/K, where $K \subseteq G$. Then $X = \coprod_K X[K]$, where the disjoint union runs over a set of orbit representatives of subgroups of G under conjugation. (Note that $X[K] = X[gKg^{-1}]$.) Finally, one sees that $\mathrm{Aut}^G(X) \cong \coprod_K \mathrm{Aut}^G(X[K])$, so we have a complete description of the automorphism groups of finite G-sets. Also, let H be any group. Then it is standard that $\amalg_n B\Sigma_n \int H$ is a simplicial monoid, with multiplication being given by the evident "block sum" homomorphisms $\Sigma_n \int H \times \Sigma_m \int H \to \Sigma_{n+m} \int H$. Further, the group completion of this simplicial monoid has the homotopy type of $Q(BH_+)$, where "plus" denotes disjoint basepoint as usual. From the description (b^G) of stable equivariant homotopy, it is now easy to use the above description of $\coprod_X B\Sigma_X$ to see that the G-equivariant stable homotopy groups of spheres are given by $\bigoplus_{K \subseteq G} \pi_*^s(BW(K)_+)$, where the direct sum is over the conjugacy classes of subgroups of G.

One can also see this description via model (c^G). The key observation here is that in a framed G-manifold M, if two points p and q are in the same component, then they have the same stabilizer. Let us look at the case $G = \mathbf{Z}/2\mathbf{Z}$ to give an idea of why this is so. Let M be a smooth G-manifold, and let $p \in M^G$. M^G is a submanifold, and the pullback of the tangent bundle of M to M^G breaks up, via choice of a Riemannian metric, into the sum of the tangent bundle of M^G and the normal bundle to M^G in M. $\tau_p(M)$ is thus isomorphic to $\tau_p(M^G) \oplus \nu_p(M^G)$, and we claim $\nu_p(M^G)$ contains no non-trivial fixed vectors. If it did, applying the exponential map to such a vector would yield an arc of G-fixed points normal to M^G, clearly an impossibility. Thus, if dim $M = n$, dim $M^G = g$, and ε and σ denote the one-dimensional trivial and sign representations, respectively, $\tau_p(M) \cong q\varepsilon \oplus (n-g)\sigma$ as G-modules. But, the definition of a framed G-manifold requires that $\tau_p(M)$ is a trivial G-module, so $n - g = 0$ and M^G is codimension zero in M. This readily shows that M breaks up as a disjoint union of M^G and a free framed G-manifold. More generally, it is not hard to see that an arbitrary framed G-manifold decomposes as a disjoint union $\amalg_K M[K]$, where the disjoint union ranges over all subgroups of G. Further, it can be decomposed as a disjoint union of G-manifolds $M = \amalg_{[K]} G \times_{N_G(K)} M[K]$, where the disjoint union is over all *conjugacy classes* of subgroups of G. Note that each $M[K]$ is a *free* $W_G(K)$ manifold, and is framed. It is now direct that bordism of G-framed, free G-manifolds is canonically isomorphic to the ordinary framed bordism groups of the classifying space BG. Consequently, we find that the G-equivariant framed bordism groups decompose as $\oplus_K \Omega_*^{fr}(BW(K)) \cong \oplus_K \pi_*^s(BW(K)_+)$, where the sum again ranges over conjugacy classes of subgroups of G. This description is clearly consistent with the one obtained from (b^G) above. The result may also be obtained via definition (a^G), but at a slightly higher technical price. We will return to this point later.

We now turn our attention to $\pi_0^G(S^0)$. A representative for an element of $\pi_0^G(S^0)$ is a G-map $S^V \to S^V$, for some multiple of the regular representation V. A first observation is that the fixed point sets $(S^V)^K$, $K \subseteq G$, are themselves spheres, namely the one-point compactifications of the vector subspaces V^K. Consequently, for each $K \subseteq G$, we get a function (easily seen to be a homomorphism) $\varphi^K : \pi_0^G(S^0) \to \mathbf{Z}$, given by $\varphi^K([f]) = \deg f^K$. The function depends only on the conjugacy class of K, and it turns out that $\oplus_K \varphi^K : \pi_0^G(S^0) \to \oplus_K \mathbf{Z}$ is injective. This homomorphism can also be seen in terms of the framed bordism description. Here, we define $\varphi^K : \Omega_0^{G,fr}(\text{point}) \to \Omega_0^{fr}(\text{pt}) \cong \mathbf{Z}$ by $\varphi^K(M) = M^K$, for a framed 0-dimensional G-manifold M. A second observation is that $\pi_0^G(S^0)$ is actually a ring. We describe the ring structure. Let $M(G)$ be the set of isomorphism classes of finite G-sets; it is a free commutative monoid under disjoint union, with basis given by the isomorphism classes of the G-sets G/K, as K ranges over the conjugacy classes of subgroups of G. We formally invert all elements in $M(G)$ to obtain a group $A(G)$. Note that if X and Y are G-sets, we may form the product G-set $X \times Y$, with diagonal G-action. This induces a bilinear map $A(G) \times A(G) \to A(G)$, and turns $A(G)$ into a ring, called the Burnside ring. We claim that $\pi_0^G(S^0)$ is isomorphic to $A(G)$ as a ring. To see this, one notes that we have an inclusion $M(G) \to \Omega_0^{G,fr}(\text{pt})$, since every finite G-set can be viewed as a zero dimensional framed G-manifold, by choosing the sign $+1$ for the framing. This map extends over $A(G)$ and gives a homomorphism $A(G) \xrightarrow{\theta} \Omega_0^{G,fr}(\text{pt})$ of abelian groups. Conversely, a zero dimensional framed G-manifold is just a discrete G-set together with a choice of stable framing of the (trivial) tangent bundle, i.e. a chosen selection for each orbit of an element in $\{\pm 1\}$. An inverse to θ is now given by sending the zero-dimensional framed G-manifold M to $\Sigma n_K[G/K]$, where n_K is the sum of the numbers associated to the orbits of type G/K. Note that $\Omega_0^{G,fr}(\text{pt})$ is also a ring, from products of manifolds, and it is clear that θ is a ring homomorphism. One then verifies, as in the non-equivariant case, that $\Omega_0^{G,fr}(\text{pt})$

is isomorphic to $\pi_0^G(S^0)$, as rings. It is now not hard to see that $\pi_n^G(X)$, for any n and G-space X, is a module over $A(G)$.

(C) *Analogues of some familiar theorems in stable homotopy theory*

The first important result in stable homotopy theory is the Freudenthal suspension theorem, which asserts that the suspension map $\pi_n X \to \pi_{n+1}\Sigma X$ is an isomorphism for X $[\frac{n}{2}+1]$-connected, where $[k]$ is the greatest integer less than or equal to k. This shows that the direct limit defining a particular stable homotopy group of X is actually attained. There is an analogue for equivariant mapping spaces, due to [29], but the hypotheses are a little more involved. Let Y be a G-space, and let $H \subseteq G$ be a subgroup. We define $C_H(Y) = \max\{n \mid \pi_r(Y^H) = 0 \text{ for } r \le n\}$. The equivariant Freudenthal theorem now reads as follows.

THEOREM 2.1. *Let X and Y be G-spaces, and suppose X is a $G-CW$ complex. Suppose*

(a) $\dim(X^H) \le 2C_H(Y)$ *for all* $H \subseteq G$

(b) *For any proper inclusion of subgroups* $K \subset H$, $\dim(X^H) \le C_K(Y) - 1$.

Then the suspension map $[X, Y]^G \xrightarrow{f \to \mathrm{id}_{S^V} \wedge f} [S^V \wedge X, S^V \wedge Y]^G$ *is bijective, for any representation V of G.*

Remark. In [29], an even more refined theorem of which this is a consequence was stated. We prefer to state just this slightly simpler result.

Note that (a) is the hypothesis for the non-equivariant Freudenthal theorem for maps from X^H to Y^H. Condition (b) does not have a non-equivariant analogue; it is related to the "gap hypothesis" which often enters in equivariant surgery and bordism. We will discuss this a bit more in §4.

Two constructions which are very useful in ordinary stable homotopy theory are Spanier–Whitehead duality and transfer. Both admit equivariant versions which we summarize.

One construction of S-duality for finite complexes proceeds by including a given complex X in a high dimensional sphere S^N, and letting $S^N - X$ be an appropriate suspension of the dual to X. The dual is an object in a category of spectra which admits formal desuspensions of the form Σ^{-n}. One then shows that $\{W, DX \wedge Z\} \cong \{X \wedge W, Z\}$, where $\{A, B\}$ denotes the group of homotopy classes of stable maps from A to B. In the equivariant theory, one can of course not embed an arbitrary finite G-complex in a high-dimensional sphere with trivial G-action. However, it is easy to see that one can embed a finite G-complex X in $S^{V\oplus 1}$ for an appropriately chosen representation V. If one does this, and takes the complement $Z = S^{V\oplus 1} - X$, one obtains an S-dual in the following sense. If U and W are G-complexes, there is a natural bijection from $\{S^V \wedge W, Z \wedge U\}^G$ to $\{X \wedge W, U\}^G$. Ideally, the dual should take its values in a stable category, which should admit desuspensions Σ^{-V} by arbitrary finite dimensional representations, and the actual dual would then be $\Sigma^{-V}Z$. In §3, we will see such a category. Note that since the fixed point sets of S^V are themselves spheres, the fixed point sets Z^H, $H \subseteq G$, are themselves suspensions of non-equivariant S-duals to X^H. The number of suspensions involved for Z^H depends on $\dim(V^H)$. See [2] and [50] for a discussion of this duality.

As for the transfer, recall how it is constructed for finite covering spaces. Let $\tilde{X} \xrightarrow{p} X$ be a covering map, where X is a finite complex. It can be shown that there is an inclusion

$\tilde{X} \subsetneq \xi$, over X, where ξ is a vector bundle over X. In fact, ξ may be taken to be a trivial bundle by adding an inverse to ξ, so we have an inclusion $\tilde{X} \to X \times \mathbf{R}^N$ over X for N sufficiently large. It is now possible to choose a "bundle of tubular neighborhoods" E of $\tilde{X}$ in $X \times \mathbf{R}^N$, over X, so that its closure is a bundle whose fiber over every point $x \in X$ is a disjoint union of closed discs in $\mathbf{R}^N$, one for every point in $p^{-1}(x)$. Note that as a space E is homeomorphic to $\tilde{X} \times \mathbf{R}^N$. Let E^∞ denote the fiberwise one point compactification of E, and let X^∞ denote the fiberwise one point compactification of $X \times \mathbf{R}^N$. Then we can take a fiberwise Pontrjagin–Thom collapse construction and obtain a map $X^\infty \to E^\infty$. Since $X^\infty \cong \Sigma^N(X_+)$ and $E^\infty \cong \Sigma^N(E_+)$, we obtain a stable map from X_+ to E_+, the transfer.

In the equivariant case, we study the particular case of regular coverings, which may be identified with orbit projections from free group actions. Thus, let X be a finite G-complex, where G is a finite group; and let $N \lhd G$ be a normal subgroup. Then the orbit space X_N may also be viewed as a G-space (in fact a G/N-space), and we wish to construct the transfer from X_N to X as a stable G-map. The construction proceeds as above, by embedding X in a G-vector bundle over X_N. However, we may not necessarily choose that bundle to be of the form $X_N \times \mathbf{R}^K$, where $\mathbf{R}^K$ has trivial G-action. What we can do is embed X in $X \times V$, where V is some representation of G. The construction now proceeds as in the nonequivariant case, to produce a G-map $S^V \wedge (X/N_+) \to S^N \wedge (X_+)$. Adams performed this construction in [2], and showed that one could in fact produce a similar transfer in the based context, where the N action on X is free off the basepoint. Both these constructions require suspensions by non-trivial representations, and confirm that we have stabilized in the correct way.

Remark. There is an enlightening way to view this transfer in a special case. Recall that on the level of chain complexes of free left $\mathbf{Z}[G]$-complexes, there is a transfer $\mathbf{Z} \otimes_{\mathbf{Z}[G]} C_* \to C_*$, defined by multiplication by $\sum_{g \in G} g$. In particular, for a free cyclic $\mathbf{Z}[G]$-module, we have the inclusion $\mathbf{Z} \to \mathbf{Z}[G]$, $1 \to \sum_{g \in G} g$. On the other hand, the usual intuition about the functor Q is that it is the homotopy theoretic analogue of the free abelian group functor. Thus, the ordinary transfer associated to $G \to *$ is a stable map $S^0 \to Q(G_+)$, which has degree one on each factor in the decomposition $Q(G_+) \cong \prod_{g \in G} Q(g_+)$. However, as it stands, this map cannot be made G-equivariant. For, (using the non-equivariant version of Q), $Q(G_+)^G = Q((G_+)^G) = Q(+) \cong *$, so every G equivariant map from S^0 to $Q(G_+)$ is homotopically trivial. The point is that although we have a homotopy equivalence $Q(G_+) \cong \prod_{g \in G} Q(g_+) \cong F(G_+, Q(S^0))$, the equivalence cannot be made equivariant when G acts on $F(G_+, Q(S^0))$ via its left multiplication action on G. However, it is not hard to check that $Q^G(G_+)$ is G-homotopy equivalent to $F(G_+, Q(S^0))$, and hence a suitable transfer is defined from S^0, with trivial action, to $Q^G(G_+)$. Thus, the non-trivial suspensions have the effect of allowing the functor Q to better mimic the behavior of the free abelian group functor in the presence of group action.

Another very useful construction in stable homotopy theory is the Snaith decomposition. Let $X^{\wedge n}$ denote the n-fold smash product of X with itself, with Σ_n acting in the evident way. Then, if X is connected, the Snaith decomposition asserts that $Q(Q(X)) \cong \prod_n Q(E\Sigma_{n+} \wedge_{\Sigma_n} X^{\wedge n})$. The fact that May's construction CX is a model for QX shows that there is a filtration by subspectra of $\Sigma^\infty QX$ whose subquotients are $\Sigma^\infty E\Sigma_{n+} \wedge_{\Sigma_n} X^{\wedge n}$, and Snaith proved that the filtration is actually split. The equivariant analogue proceeds as follows. Let $E_G\Sigma_n$ denote the $G \times \Sigma_n$-space $E_{\mathscr{F}}(G \times \Sigma_n)$, where $\mathscr{F}$ is the family of subgroups of $G \times \Sigma_n$, which intersect Σ_n trivially. Hausschild's configuration space construction, and the analysis of the equivariant configuration spaces, show that subquotients in the corresponding filtration of $Q^G X$ are $Q^G(E_G\Sigma_{n+} \wedge_{\Sigma_n} X^{\wedge n})$. The corresponding Snaith decomposition (see [35] for a proof) now asserts that if X^K is connected for all $K \subseteq G$, $Q^G(Q^G X)$ is G-homotopy equivalent to the product $\prod_n Q^G(E_G\Sigma_{n+} \wedge_{\Sigma_n} X^{\wedge n})$. This splitting is important in applications [17], [18].

(D) *Interpretation of π_*^G in terms of fixed point sets*

From definition (a^G), we see that elements of $\pi_n^G(X)$ are elements in $[S^n, Q^G(X)]^G$. Since G acts trivially on S^n, this is the same thing as $\pi_n((Q^G X)^G)$, so the equivariant stable homotopy groups of X are identified with the ordinary homotopy groups of a fixed point space. We have already obtained an identification of the homotopy groups of $(Q^G X)^G$; we will now identify it as a space. For simplicity, we deal only with the case $G = \mathbf{Z}/2\mathbf{Z}$.

The first observation is that we have a map $\rho: Q^G(X)^G \to Q(X^G)$ defined by $(f: S^V \to S^V \wedge X) \to (f^G: S^{V^G} \to S^{V^G} \wedge X^G)$. It is easy to check that $(S^V \wedge X)^G = S^{V^G} \wedge X^G)$, and of course S^{V^G} is itself a sphere. One can check that since this map amounts to a restriction map of function spaces, it is in fact a fibration. Further, it admits a section since we have a G-decomposition $V \cong V^G \oplus \bar{V}^G$, and can therefore construct an inclusion $s: \varinjlim_k \Omega^{kV^G} S^{kV^G}(X^G) \to \varinjlim_k \Omega^{kV} S^{kV}(X^G) \to Q^G(X)$, which is easily checked to split the restriction map. We now wish to examine the fiber of this restriction map. By inspection, the fiber of the map $F(S^V, S^V \wedge X)^G \to F(S^{V^G}, S^{V^G} \wedge X^G)$, $f \to f^G$, can be identified with the function space $F(S^V/S^{V^G}, S^V \wedge X)^G$. As a G-complex, S^V/S^{V^G} is free off the basepoint, so by the based version of 1.8, $F(S^V/S^{V^G}, S^V \wedge X)^G \cong F(S^V/S^{V^G}, S^V \wedge X \wedge EG_+)^G$, where EG denotes a contractible space on which G acts freely. $S^V \wedge X \wedge EG_+$ is now free off the basepoint, so any equivariant map (and homotopy from S^V to $S^V \wedge X \wedge EG_+$ automatically factors through S^V/S^{V^G}, so we have $F(S^V/S^{V^G}, S^V \wedge X \wedge EG_+)^G \cong F(S^V, S^V \wedge X \wedge EG_+)^G$. The concluson is, after passing to limits, that the fiber of ρ may be identified with $Q^G(X \wedge EG_+)^G$, and the above splitting shows that $Q^G(X)^G \cong Q^G(X \wedge EG_+)^G \times Q(X^G)$. One uses the fact that all spaces in question are infinite loop spaces and that the section s is an infinite loop map. We now identify $Q^G(X \wedge EG_+)^G$.

PROPOSITION 2.2. *Let Z be a free (off the basepoint) G-complex. Then $Q^G(Z)^G \cong Q(Z_G)$*

Proof. On the homotopy group level, and if $Z = Z^0_+$, where Z^0 is a free G-complex, this can be obtained from model (c^G) from the observation that a framed G-manifold over Z^0 is the same thing (via passing to universal covers) as a framed manifold over Z^0_G. We use the transfer to obtain the result. Define $j: Q(Z_G) \to Q^G(Z)^G$ to be the composite $Q(Z_G) \to Q^G(Z_G)G \xrightarrow{\tau} Q^G(Z)^G$, where the first map is obtained by viewing Z_G as a G-space with trivial action, and applying the section s to the given map, and τ is Adams' transfer applied to the map $Z \to Z_G$, which satisfies Adams' hypotheses since the action on Z is free (off the basepoint). One can now apply an induction over the cells of Z, and check the case G_+, which is straightforward. The case of a general complex which is free off the basepoint follows directly. □

The following corollary is a theorem of tom Dieck [21].

COROLLARY 2.3. *For the case $G = \mathbf{Z}/2\mathbf{Z}$, $Q^G(X)^G \cong Q(EG_+ \wedge_G X) \times Q(X^G)$. More generally, $(Q^G X)^G \cong \prod_{K \leq G} Q(EW_G(K)_+ \wedge_{W_G(K)} X^K)$, where K ranges over conjugacy classes of subgroups of G.*

Proof. The case $G = \mathbf{Z}/2\mathbf{Z}$ is immediate from the above analysis. The general case is just an elaboration of this case. □

Recall that in the non-equivariant situation, a cofiber sequence $X \to Y \to Y/X$ induces a fibration up to homotopy $QX \to QY \xrightarrow{p} QY/X$, i.e. the natural $Q(X) \to F(p, *)$ is a weak equivalence, where $F(p, *)$ denotes the homotopy fiber of the map p at the basepoint $*$ of Y/X. A similar result holds in the equivariant setting.

PROPOSITION 2.4. *Let $X \to Y \to Y/X$ be a cofibration sequence of G-spaces. Then the natural map $Q^G X \to F(p, *)$ is a weak equivalence of G-spaces, where $F(p, *)$ denotes the G-homotopy fiber of p at $*$. We say $Q^G(X) \to Q^G(Y) \to Q^G(Y/X)$ is a G-fibration up to homotopy.*

Proof. From the definition of G weak equivalences, it will suffice to show that $Q^G(X)^K \to F(p, *)^K$ is a weak equivalence for each K. From the formula for fixed point spaces 2.3, it suffices to show that for any subgroup $K \subseteq G$, the sequence

$$Q\left(EW_G(K)_+ \underset{W_G(K)}{\wedge} X^K\right) \to Q\left(EW_G(K)_+ \underset{W_G(K)}{\wedge} Y^K\right) \to Q\left(EW_G(K)_+ \underset{W_G(K)}{\wedge} (Y/X)^K\right)$$

is a fibration up to homotopy. But by the remarks about G cofibrations in §1, $X^K \to Y^K$ is a cofibration, and clearly the functor $EW_G(K)_+ \bigwedge_{W_G(K)}$ takes cofibration sequences of $W_G(K)$-complexes to cofiber sequences of spaces, so the non-equivariant result implies the equivariant case. □

This means that the functor π_*^G is a homology theory on the category of G-spaces, in the sense that it assigns long exact sequences to cofibration sequences, and π_*^G is canonically decomposed into a sum of copies of the non-equivariant theory π_*^s applied to Borel constructions applied to fixed point sets of subgroups $K \subseteq G$. It is interesting to consider the corresponding equivariant *cohomology* theory, $\pi_G^n(X) \cong \{X, S^n\}^G$. It behaves well on free complexes, as does π_*^G.

PROPOSITION 2.5. $\pi_G^n(X) \cong \pi_s^n(X_G)$ *when X is free off the basepoint.*

Proof. A straightforward induction reduces to the case $X = G_+$, for which it is immediate. □

We note further that, for example, when $G = \mathbf{Z}/2\mathbf{Z}$, we have natural maps $\pi_G^n(X) \to \pi_s^n(X^G)$ by restriction to the fixed point set. Further, the fiber of the restriction map

$$F(X, Q^G(S^n))^G \to F(X^G, Q(S^n))$$

which induces this map on homotopy groups can be identified with $F(X, Q^G(EG_+ \wedge S^n))^G$, so we have a fibration sequence

$$F(X, Q^G(EG_+ \wedge S^n))^G \to F(X, Q^G(S^n))^G \to F(X^G, Q(S^n)).$$

However, in contrast to the case of π_n^G, this is definitely *not* split in general. For instance, examining the case $X = S^V$, where V is the sign representation, will convince the reader that this is the case.

3. EQUIVARIANT SPECTRA AND COHOMOLOGY THEORIES

(*A*) *G-spectra and spectra with G-action*

We recall that a spectrum is a family of based spaces $\{X_i\}$ together with homeomorphisms $S_i: X_i \to \Omega X_{i+1}$. A prespectrum is a family of based spaces with closed inclusions $\Sigma X_i \to X_{i+1}$. Maps of spectra and prespectra are families of maps $\{f_i\}$ which strictly commute with the given structure. There is a canonical construction of a spectrum from a prespectrum, which replaces X_i by $\varinjlim_k \Omega^k X_{i+k}$. For any $j \in \mathbf{Z}$ and spectrum $\underset{\sim}{X} = \{X_i\}$, we define $\pi_j(\underset{\sim}{X}) = \pi_{i+j}(X_i)$, whenever this is defined. A map of spectra is said to be a weak equivalence if it induces isomorphisms on all homotopy groups.

Definition 3.1. A spectrum with G-action is a family of based G-spaces $\{X_i\}$ with G-homeomorphisms $X_i \to \Omega X_{i+1}$, where ΩX_{i+1} is given a G action by $(g\varphi)(t) = g(\varphi(t))$, for any loop φ. There is an analogous notion of prespectrum with G-action, and one may associate a spectrum with G-action to any prespectrum with G-action.

Examples:

(a) Any spectrum or prespectrum with trivial trivial action.
(b) Let M be any G-module. Then by taking the functorial simplicial construction of Eilenberg–MacLane spaces [39], we obtain a spectrum with G-action whose n-th space is $K(M, n)$.
(c) Let $X_i = Q^G(S^i)$, where G acts trivially on S^i, equipped with the evident maps $Q^G(S^i) \to \Omega Q^G(S^{i+1})$.

Spectra with G-action occur very commonly in practice. For instance, the infinite loop space machines of May and Segal produce spectra with G-action when applied to symmetric monoidal categories with symmetric monoidal G-action. We have seen, though, that a more elaborate stabilization procedure is appropriate when considering G-spaces.

Definition 3.2. A G-spectrum is a family of spaces X_i together with G-equivalences $X_i \to \Omega^V X_{i+1}$, where V denotes the regular representation of G, and $\Omega^V X = F(S^V, X)$, with conjugation action on $F(S^V, X)$ by G. A G-prespectrum is defined in the evident way. G-maps of G-spectra and G-prespectra are families of G-maps strictly commuting with the structure.

Remark. We have chosen an economical (in terms of bookkeeping) definition of G-spectra. It does not, however, make explicit all the information contained in the definition. For instance, let W be any finite dimensional representation of G. Then W can be embedded in V^k for some k, since V contains a copy of every irreducible representation. Since $\Omega^{W_1 \oplus W_2} = \Omega^{W_1}\Omega^{W_2}$, and since $W \subseteq V^k$ admits a complementary summand $W^\perp$, we see that $X_i = \Omega^{V^k} X_{i+k} = \Omega^W(\Omega^{W^\perp} X_{i+k})$, so the G-spectrum gives rise to "W-deloopings" by any representation W of G. In particular, if we take W to be an n-dimensional trivial representation, we see that we obtain a spectrum with G-action as part of the data. A more honest definition would be one which defines spectra as families of spaces $\{X_V\}$, for any representation V of G. To make this precise, one either makes explicit choices for all the representations of G, or defines a spectrum as a rule which assigns a space to every finite dimensional G-subspace of a given infinite-dimensional G-space which contains every irreducible representation infinitely often, subject to certain compatibility hypotheses. This second plan, which is useful even in the non-equivariant situation when dealing, for instance, with associativity questions concerning smash products, was carried out by Lewis, May, and Steinberger [35].

The conclusion of this remark is that to every G-spectrum $\underset{\sim}{X}$ we associate a spectrum with G-action $\rho\underset{\sim}{X}$, and ρ is in fact a functor from the category of G-spectra to the category of spectra with G-action. Note that spectra with G-action admit fixed point and orbit spectra, by applying the fixed point and orbit functors to the individual spaces making up the spectrum with G-action.

Definition 3.3. Let $\underset{\sim}{X}$ be a G spectrum. Then we define the fixed point and orbit spectra of $\underset{\sim}{X}$, $\underset{\sim}{X}^G$ and $\underset{\sim}{X}_G$, to be the fixed point and orbit spectra of $\rho\underset{\sim}{X}$, respectively.

One can ask which spectra with G-action are in the image of ρ. This is a difficult question in general; in our examples above, (a) and (b) are generally not, and (c) is by construction. However, one can say the following.

PROPOSITION 3.4. *For any spectrum with G-action $\underset{\sim}{X}$, there is a G-spectrum $\underset{\sim}{X}^\wedge$ and a G-map $\underset{\sim}{X} \to \rho\underset{\sim}{X}^\wedge$ which is a quasi-equivalence for each of the spaces defining $\underset{\sim}{X}$ and $\rho\underset{\sim}{X}^\wedge$. (quasi-equivalences were defined in* §1).

Proof. Let V be the regular representation of G, and write $V \cong \varepsilon \oplus \bar{V}$, where ε is a trivial one-dimensional representation. We define $X_k^\wedge$ to be $\varinjlim_{i,l} \Omega^l \Omega^{l\bar{V}} S^{(k+l)\bar{V}} X_{i+k}$. This clearly gives the desired spectrum with G-action. □

Let X be a based G-space, and suppose $\underset{\sim}{W}$ and $\underset{\sim}{Z}$ are spectra with G-action and G-spectra, respectively. One readily constructs a spectrum with G-action $X \wedge \underset{\sim}{W}$ and a G-spectrum $X \wedge \underset{\sim}{Z}$. We now obtain equivariant homology theories h_*^W and h_*^Z by setting $h_i^W(X) = \pi_i((X \wedge \underset{\sim}{W})^G)$ and $h_i^Z(X) = \pi_i((X \wedge \underset{\sim}{Z})^G)$. These are both homology theories in the G-space X, in the sense that cofibration sequences produce long exact sequences of groups. h_i^W does not, however, admit duality and transfer for a sufficiently large class of complexes, while h_i^Z does. It is immediate that h_i^Z and h_i^W admit suspension isomorphisms $h_i^W(X) \cong h_{i+1}^W(\Sigma X)$ and $h_i^Z(X) \cong h_{i+1}^Z(\Sigma X)$, but h_i^Z admits an additional kind of suspension isomorphism, as follows. Note first that, if we let G act trivially on S^i, $\pi_i((X \wedge \underset{\sim}{Z})^G)$ can be viewed as $\varinjlim_k [S^i \wedge X^{kV}, (X \wedge \underset{\sim}{Z})_k]^G$, where V denotes the regular representation. If α denotes any other representation, we may by analogy consider the groups $\varinjlim_k [S^\alpha \wedge S^{kV}, (X \wedge \underset{\sim}{Z})_k]^G$, and denote this by $\pi_\alpha^G(X \wedge \underset{\sim}{Z})$. Thus, for each representation, not only the trivial ones, we obtain a homotopy group. This is formalized as follows. For any finite group G, let $RO[G]$ denote the group completion of the monoid of isomorphism classes of finite dimensional real representations. $RO[G]$ is a finitely generated abelian group, and by an extension of the assignment $\alpha \to \pi_\alpha^G$, one obtains an $RO[G]$-graded group. Consequently, the G-homology theory associated to a G-spectrum can actually taken to be an $RO[G]$-graded group. If α is any finite dimensional real representation and $[\alpha]$ denotes its class in $RO[G]$, we obtain a suspension isomorphism $h_\beta^Z(X) \xrightarrow{\sim} h_{\beta+[\alpha]}^Z(S^\alpha \wedge X)$, and this is the additional kind of suspension isomorphism mentioned above. We will refer to an $RO[G]$-graded homology theory with suspension ismorphisms as a fully equivariant G-homology theory.

Remark. The above discussion sweeps a few details under the rug. Adams [2] pointed out the difficulties involved in constructing an $RO[G]$-graded theory; one must remember that when one writes S^α, one must have a fixed model of α in mind, not just its isomorphism class, otherwise one may not have a preferred choice of suspension isomorphism. However, the difficulties can be circumvented (see e.g. [35]), and indeed Adams [2] suggested strategies for doing this. One interesting feature that arises in the following. Note that any element in $h_\alpha^Z(X) = \pi_\alpha^G(X \wedge \underline{Z})$ can be precomposed by any equivariant self map $S^{kV} \wedge S^\alpha \to S^{kV} \wedge S^\alpha$, i.e. an element of $\pi_0^G(S^0) \cong A[G]$, so $h_*^Z(X)$ is always an $A[G]$-module. What now occurs is that diagrams involving suspension isomorphisms which in the non-equivariant case commute up to sign in this case commute up to multiplication by a unit in $A[G]$.

(B) *Permutative categories and the recognition principle*

We recall that a connective spectrum is in fact determined by the zero-th space of the spectrum together with structure it carries as a result of being "infinitely deloopable". This structure consists of the H-space structure it carries as a result of being a loop space together with a complicated set of higher coherence homotopies. May [40] has encoded all these homotopies into a single map $CX \to X$, where CX is the configuration space model for QX. There are several diagrams involving CX which are required to commute, making it into a "monad". We have Hausschild's equivariant configuration space model $C_G X$, discussed in §2. It is possible to prove a recognition principle, i.e. a theorem which reconstructs a G-spectrum from its zero-th G-space and a G-map $C_G X \to X$, although to the author's knowledge this has not been published yet. Another formulation of the recognition principle is Segal's principle, which observes that the zero-th space of a connective spectrum is a Γ-space, where Γ is a category defined by Segal [46], and recovers the entire spectrum from this structure. There is a notion of a G-Γ-space, and it is my understanding that in this form the recognition principle has been proved by Matumoto. Rather than discuss these constructions in detail, we explore what some of the necessary conditions on the direct sum operation on a G-symmetric monoidal category to produce a G-infinite loop space are. We hope this will give a feel for the issues involved.

Recall that a symmetric monoidal category is a category $\underline{C}$ with sum operation $\oplus: \underline{C} \times \underline{C} \to \underline{C}$, together with isomorphisms of functors $\oplus \circ T \simeq \oplus$ and $\oplus \circ (\oplus \times \mathrm{Id}) \cong \oplus \circ (\mathrm{Id} \times \oplus)$, satisfying certain coherence diagrams. Such categories are the information needed by the infinite loop space machines of May and Segal [40], [46] to construct spectra. The category may be replaced by one in which the associativity isomorphism is the identity map, so $\oplus \circ (\oplus \times \mathrm{Id}) = \oplus \circ (\mathrm{Id} \times \oplus)$, but one cannot assume this for the commutativity isomorphism. The information given by the commutativity isomorphism can be summarized as follows. There are $n!$ distinct functors $\underline{C}^n \to \underline{C}$, obtained by taking sums using all the different reorderings of $\underline{C}^n$; the commutativity isomorphism gives rise to a choice of isomorphism between any two of these functors. This can be stated formally by giving a Σ_n-equivariant functor $\theta: \underline{\Sigma_n} \times \underline{C}^n \to \underline{C}$, where $\underline{\Sigma_n}$ denotes the category whose objects are the elements of Σ_n, with a unique morphism between any pair of objects, and where Σ_n acts on $\underline{\Sigma_n}$ by right multiplication on $\underline{C}^n$ by the evident permutation action, and trivially on the target $\underline{C}$. Suppose now we have a representation $f: G \to \Sigma_n$, and that G acts on the category $\underline{C}$. Then $g \in G$ acts on the left of $\underline{\Sigma_n}$ by left multiplication by $f(g)$, and G acts diagonally on $\underline{C}^n$. We obtain an action of $G \times \Sigma_n$ on $\underline{\Sigma_n}$, and $G \times \Sigma_n$ acts on $\underline{C}$ via the G-action composed with the projection $G \times \Sigma_n \to G$. Each $f: G \to \Sigma_n$ gives rise to a different G-structure on the category $\underline{\Sigma_n}$, and the natural G-equivariant condition is that one should have $G \times \Sigma_n$-equivariant functions $\underline{\Sigma_n} \times \underline{C}^n \to \underline{C}$ for each such f. The existence of these functors for each f is thus a necessary condition for being the zero-th space of a G-spectrum. There are of course also necessary coherences among these functors which we will not discuss here. In the case of $G = \mathbf{Z}/2\mathbf{Z}$, this means in particular that in addition to the equivariance of the sum functor $\underset{\sim}{C} \times \underset{\sim}{C} \to \underset{\sim}{C}$, there must also be an equivariant functor $\underset{\sim}{C} \times \underset{\sim}{C} \to \underset{\sim}{C}$, where $\underset{\sim}{C} \times \underset{\sim}{C}$ is acted on by $g(x, y) = (gy, gx)$, if $G = \{1, g\}$.

This last observation gives rise to real restrictions on the categories in question. For instance, if G acts trivially on $\underset{\sim}{C}$, the above requirement shows that the sum map $\underset{\sim}{C} \times \underset{\sim}{C} \to \underset{\sim}{C}$ is equivariant, when G acts on $\underset{\sim}{C} \times \underset{\sim}{C}$ by $g(x, y) = (y, x)$. This means that the sum map is strictly commutative, so the resulting spectrum turns out to be a product of Eilenberg-MacLane spectra.

(*C*) ***The tom Dieck filtration***

In paragraph (D) of §2, we discussed a theorem of tom Dieck [21] to the effect that if X is a finite G-complex, then the fixed point set of the G-action on $Q^G X$ has the homotopy type of the product $\prod_{K \leq G} Q(EW_G(K)_+ \wedge_{W_G(K)} X^K)$, where the product ranges over conjugacy classes of subgroups of G. On the spectrum level, this tells us that the fixed point spectrum of $\Sigma^\infty X$ is $\bigvee_{K \leq G} \Sigma^\infty EW_G(K)_+ \wedge_{W_G(K)} X^K$. One could ask whether there is a similar decomposition of the fixed point spectrum for a general G-spectrum. In order to understand the generalization, we develop an alternative way to describe the decomposition in the case of a suspension spectrum. For simplicity, we deal with the case $G = \mathbf{Z}/2\mathbf{Z}$.

As above, let EG denote the infinite sphere, with antipodal G-action. Of course, we have the equivariant map $EG_+ \to S^0$, and we let $\widetilde{EG}$ denote the mapping cone of this map. As a space, it is the unreduced suspension of EG. Thus, $\widetilde{EG}$ is contractible, but its fixed point set consists of two points, i.e. it is S^0. From the general remarks in paragraph (A), the sequence $Q^G(EG_+) \to Q^G(S^0) \to Q^G(\widetilde{EG})$ is a G-fibration up to homotopy, consequently the induced sequence on fixed point sets is a fibration up to homotopy. Further, 2.2 asserts that the fixed point space of $Q^G(EG_+)$ is $Q(BG_+)$. We expect, then, to be able to identify the fixed point space of $Q^G(\widetilde{EG})$ with $Q(S^0)$.

PROPOSITION 3.5. *Let X and Y be based G-spaces with X a G-complex. Then the space $F(X, \widetilde{EG} \wedge Y)^G$ equivariant maps from X to $\widetilde{EG} \wedge Y$ is weakly homotopy equivalent to the space of maps $F(X^G, Y^G)$, and the equivalence is realized by the restriction map associated to $X^G \to X$.*

Proof. This is just 1.4 □

COROLLARY 3.6. $Q^G(\widetilde{EG} \wedge X)^G \cong Q(X^G)$.

Proof. 3.5 shows that the mapping space $F(S^V, S^V \wedge \widetilde{EG} \wedge X)^G$ is equivalent to $F(S^{V^G}, S^{V^G} \wedge X^G)$. Passing to direct limits now gives the result. □

Thus, for suspension spectra, the cofibration sequence $\Sigma^\infty EG_+ \to S^0 \to \Sigma^\infty \widetilde{EG}$ gives rise to the tom Dieck formula for fixed point spectra. But with this formulation, we may smash the sequence with any G-spectrum $\underset{\sim}{X}$ and obtain a cofiber sequence $\Sigma^\infty EG_+ \wedge \underset{\sim}{X} \to \underset{\sim}{X} \to \Sigma^\infty \widetilde{EG} \wedge X$. This is the analogue of the tom Dieck formula for a general G-spectrum. Let us describe the fixed point sets of $\Sigma^\infty EG_+ \wedge \underset{\sim}{X}$ and $\Sigma^\infty \widetilde{EG} \wedge \underset{\sim}{X}$. Let $\rho \underset{\sim}{X}$ denote the associated spectrum with G-action to $\underset{\sim}{X}$.

PROPOSITION 3.8. *Let Z be any free (off the basepoint) G-complex. Then the fixed point spectrum of $Z \wedge \underset{\sim}{X}$ has the homotopy type of $Z \wedge_G \rho \underset{\sim}{X}$.*

Proof. One observes that the transfer map gives a natural transformation which is readily verified to be an equivalence when $Z = G_+$. An induction over skeleta now gives the result. □

In particular, we have a spectral sequence with $E^2_{p,q}$-term $H_p(G, \pi_q(Z \wedge \underset{\sim}{X}))$ converging to the homotopy groups of the fixed point spectrum of $Z \wedge \underset{\sim}{X}$, i.e. $\pi^G_{p+q}(Z \wedge \underset{\sim}{X})$.

To analyze the fixed point spectrum of $\Sigma^\infty \widetilde{EG} \wedge \underset{\sim}{X}$, we first note that the fixed point spectrum of $\underset{\sim}{X}$ is obtained by taking the fixed point spectrum of $\rho\underset{\sim}{X}$, i.e. fixed point sets of the deloopings of X_0 corresponding to trivial representations. It does not involve the fixed point sets of the deloopings by other representations, say of the regular representation V. However, we may construct maps $\Sigma X^G_k \to X^G_{k+1}$, where X_k is the delooping corresponding to kV. One simply composes the structure map $\Sigma^V X_k \to X_{k+1}$ with the inclusion $\Sigma^\varepsilon X_k \to \Sigma^V X_k$, induced by $\varepsilon \subseteq V$, where ε denotes the one-dimensional trivial representation, and takes the induced map on fixed point sets. The colimit of the adjoint direct system $\Omega^k X^G_k$ we call $\Phi^G X$, the "geometric fixed point spectrum."

3.7. Under the standing assumption $G = Z/2Z$, the fixed point spectrum of $\widetilde{EG} \wedge \underset{\sim}{X}$ has the homotopy type of $\Phi^G X$.

Proof. This follows from the same argument as in the suspension case. □

In the case of a more general group, where the tom Dieck splitting contains more factors, one obtains a filtration on the fixed point spectrum of $\underset{\sim}{X}$ by choosing an ordering $\leq$ on the set of conjugacy classes of subgroups so that if K is subconjugate to H, then $K \leq H$. One then takes the K-fixed point set of a direct limit system $\{\Omega^{lk} X_k\}$, where l is the index of K in G, to describe the subquotient in the filtration corresponding to the subgroup K.

An unfortunate fact one must deal with is that the fixed point spectrum sequence associated wth $EG_+ \wedge \underset{\sim}{X} \to \underset{\sim}{X} \to \widetilde{EG} \wedge \underset{\sim}{X}$ is not necessarily split, as is the case when we have a suspension spectrum. This can be seen even in the case of a spectrum $\Sigma^{-V} X$, where X is a G-complex and V is a non-trivial representation. The notation $\Sigma^{-V} X$ denotes the G-spectrum obtained from $\Sigma^\infty X$ by applying Ω^V levelwise. This has come up recently in the work of Bökstedt, Hsiang, and Madsen [12] on the K-theory analogue of the Novikov conjecture, where they study the topological Hochshild homology of the Eilenberg-MacLane spectrum $K(\mathbf{Z}, 0)$ as a $\mathbf{Z}/p^n\mathbf{Z}$-spectrum. We remark that Φ^G is a functor from G-spectra to spectra. If X is any G-complex, and $\Sigma^\infty X$ denotes its equivariant suspension spectrum, then $\Phi^G(\Sigma^\infty X) \simeq \Sigma^\infty(X^G)$. Thus, it isolates one particular functor in the tom Dieck splitting of $(\Sigma^\infty X)^G$. Φ^G also preserves smash products and homotopy colimits.

(*D*) *Equivariant spectra and homotopy fixed point sets*

Recall from § 1 the discussion of the homotopy fixed point set X^{hG} of a G-complex X. Let $\underset{\sim}{X}$ be any spectrum with G-action. As usual, we let EG denote a contractible space on which G acts freely. We may then form the function spectrum with G-action, whose k-th space is $F(EG_+, X_k)$, and the fixed point spectrum of this spectrum with G-action is called the homotopy fixed point spectrum of $\underset{\sim}{X}$, $\underset{\sim}{X}^{hG}$. Of course, its k-th space is X^{hG}_k. Similarly, if $\underset{\sim}{X}$ is a G-spectrum, we may form a function G-spectrum by applying $F(EG_+, -)$ levelwise. Its fixed point spectrum will be denoted by $\underset{\sim}{X}^{hG}$. For a spectrum with G-action $\underset{\sim}{X}$, let $\underset{\sim}{X}^\wedge$ denote the G-spectrum constructed in 3.3, so $\underset{\sim}{X} \to \rho\underset{\sim}{X}^\wedge$ is an equivalence of spectra, where ρ is the forgetful functor assigning to a G-spectrum a spectrum with G-action. Note that $\underset{\sim}{X} \to \rho\underset{\sim}{X}^\wedge$ is of course not necessarily a G-equivalence.

PROPOSITION 3.9. *For X any spectrum with G-action, the natural map $\underset{\sim}{X}^{hG} \to (\rho\underset{\sim}{X}^\wedge)^{hG}$ is a weak equivalence of spectra. Further, $(\rho\underset{\sim}{X}^\wedge)^{hG} \cong (\underset{\sim}{X}^\wedge)^{hG}$.*

Proof. The first fact follows directly from the fact that $\underset{\sim}{X} \to \rho \underset{\sim}{X}^{\wedge}$ is an equivalence of spectra, and 1.6. The second fact follows from the remark that, for a G-spectrum $\underset{\sim}{X}$, the fixed point spectrum $\underset{\sim}{X}$ and the fixed point spectrum of $\rho\underset{\sim}{X}$ are by definition identical. □

PROPOSITION 3.10. *Suppose $\underset{\sim}{X}$ is a spectrum with trivial G-action. Then $\underset{\sim}{X}^{\wedge hG}$ is equivalent to the function spectrum $F(BG_+, \underset{\sim}{X})$, where $\underset{\sim}{X}$ is viewed as an ordinary spectrum.*

Proof. Immediate from 3.9 and 1.7. □

These two remarks show that if we let $\underset{\sim}{S}^0$ denote the G-suspension spectrum of S^0, then $\underset{\sim}{S}^{0hG} \cong F(BG_+, S^0)$. In general, for a spectrum with G-action $\underset{\sim}{X}$, we have a map $\underset{\sim}{X}^G \to \underset{\sim}{X}^{hG}$. The point of the above discussion is that since $\underset{\sim}{X}^{hG} \cong (\underset{\sim}{X}^{\wedge})^{hG}$, we actually have a factorization $\underset{\sim}{X}^G \to (\underset{\sim}{X}^{\wedge})^G \to \underset{\sim}{X}^{hG}$, so in a sense $(\underset{\sim}{X}^{\wedge})^G$ is a better approximation to $\underset{\sim}{X}^{hG}$ than is $\underset{\sim}{X}^G$. We obtain a dramatic illustration of this by considering the case of the spectrum with trivial G-action S^0. In this case, $(S^0)^G \cong S^0$, and $(\underset{\sim}{S}^{0\wedge})^G \cong \bigvee_K \Sigma^\infty BW_G(K)_+$, and the solution to the Segal conjecture [14], [15] shows that the map $(\underset{\sim}{S}^{0\wedge})^G \to (\underset{\sim}{S}^0)^{hG}$ is close to being an equivalence.

4. A SURVEY OF THE CURRENT STATE OF VARIOUS G-HOMOLOGY THEORIES

(A) *Ordinary homology, Borel homology*

In order to introduce the equivariant analogues of ordinary singular homology theory, we will need some definitions. Let G be a finite group, and let $\mathcal{O}_G$ denote the category of based finite G-sets and G-maps. Note that we do not use only the isomorphisms, so if $K \subseteq H \subseteq G$, there is a G-map $G/K \to G/H$ given by projection. The following definition is due to Bredon [13].

Definition 4.1. A covariant coefficient system for G is a covariant functor from $\mathcal{O}_G$ to abelian groups, which carries sums (one point unions) to products of abelian groups.

It will be convenient to think in terms of simplicial G-sets instead of G-spaces. For $X_.$ a simplicial finite G-set, and a covariant coefficient system $\underset{\sim}{F}: \mathcal{O}_G \to \underline{Ab}$, we obtain a simplicial abelian group $\underset{\sim}{F}X$ by applying $\underset{\sim}{F}$ levelwise. The homology groups of the resulting chain complex will be written $H_i(X; \underset{\sim}{F})$. This construction has the property that a cofibration sequence $X_. \to Y_. \to Z_.$ of simplicial finite G-sets induces a long exact sequence on the homology groups $H_*(- ; \underset{\sim}{F})$. This follows from the restriction in 4.1, since at level k, Y_k may be identified with the one point union of based sets $X_k \vee Z_k$, so we obtain a short exact sequence on the associated chain complexes.

Examples:

(i) $\underset{\sim}{F}(X)$ is the free abelian group on X. This does not depend on the G-action on X, and gives ordinary singular homology as its associated homology theory.

(ii) $\underset{\sim}{F}(X)$ is the free abelian group on the orbit space X_G. The homology theory assigns to X the homology of the orbit space of X.

(iii) $\underset{\sim}{F}(X)$ is the free abelian group on X^G, or more generally on X^K for $X \subseteq G$. The homology theory assigns to X the homology of the fixed point set.

(iv) $\underset{\sim}{F}(X)$ is the free abelian group on the *singular locus* $\delta(X) = \bigcup_{K \subseteq G, K \neq \{e\}} X^K$. The homology theory assigns to X the homology of the singular locus of X.

We have referred to the groups $H_*(X; \underline{F})$ as homology theories, and they are homology theories in the weak sense mentioned above. They are not necessarily, however, fully equivariant homology theories, since they do not necessarily admit suspension isomorphisms by non-trivial representations. Lewis, May, and McClure [36] have determined which coefficient systems admit such isomorphisms, and hence give rise to fully equivariant theories. We describe their results.

For G a finite group, we let $\mathcal{M}_G$ denote the category whose objects are finite based G-sets, and so that the morphisms from X to Y are the G-linear homomorphisms from $\mathbf{Z}[X]$ to $\mathbf{Z}[Y]$, where $\mathbf{Z}[—]$ denotes the free abelian group functor, with basepoint identified to zero. Note that this is a category which can be described in a fairly compact way – the objects are one point unions of G-sets of the form G/K_+, so the groups $\mathrm{Hom}_G(\mathbf{Z}[G/K], \mathbf{Z}[G/H])$ contain enough information to describe it completely. Note that there is an inclusion functor $\mathcal{O}_G \to \mathcal{M}_G$; moreover, if X and Y are two objects of $\mathcal{O}_G$ (and hence of $\mathcal{M}_G$), then any integral linear combination of morphisms from X to Y in $\mathcal{O}_G$ defines a morphism from X to Y in $\mathcal{M}_G$. However, not all morphisms in $\mathcal{M}_G$ are linear combinations of morphisms in $\mathcal{O}_G$. For instance, let $G = \mathbf{Z}/2\mathbf{Z}$, let $X = S^0$ with trivial G-action, and let $Y = G_+$, with G acting by left multiplication. Then in $\mathcal{M}_G$, we have a G-map $X \to Y$, given on the non-basepoint p of S^0 by $p \to 1 + T$, where $G = \{1, T\}$. This "transfer" morphism is not a linear combination of morphisms in $\mathcal{O}_G$, since the only morphism from X to Y in $\mathcal{O}_G$ sends all of X to the basepoint. The result of Lewis, May, and McClure [36] now reads as follows.

THEOREM 4.2. *Let G be a finite group, and let $\underline{F}$ be a covariant coefficient system. Then $\underline{F}$ extends to a fully equivariant G-homology theory if and only if $\underline{F}$ extends to a functor from $\mathcal{M}_G$ to abelian groups. F is thus a "Mackey functor."*

Remark. This theorem points out the key role played by transfers in this theory. All the morphisms in $\mathcal{M}_G$ can be written as iterated composites of linear combinations of morphisms in $\mathcal{O}_G$ with transfers similar to the one mentioned above for $\mathbf{Z}/2\mathbf{Z}$, so the result can be interpreted as saying that the existence of sufficient "transfer data" ensures the existence of delooping corresponding to non-trivial representations. Also, their result gives a similar characterization for compact Lie groups.

Let us examine the problem of extending to $\mathcal{M}_G$ the coefficient systems we gave as examples above. In example (i), it is immediate that $\underline{F}$ extends over $\mathcal{M}_G$. This corresponds to the fact that if V is any representation of G, then $H_{i+\dim V}(S^V \wedge X) \cong H_i(X)$, in a natural way. Of course, the isomorphism is not equivariant, but that is not required. The corresponding $RO[G]$-graded homology theory thus is canonically isomorphic to a theory which is pulled back from a $\mathbf{Z}$-graded theory along the augmentation map $RO[G] \to \mathbf{Z}$, given by $[V] \to \dim V$. In example (ii), it is also easy to see that $\underline{F}$ extends over $\mathcal{M}_G$. Indeed, it is the composite $\mathcal{O}_G \to \mathcal{M}_G \xrightarrow{\alpha} \underline{G\text{-mod}} \xrightarrow{\otimes} \underline{Ab}$, where $\underline{G\text{-mod}}$ denotes the category of left $\mathbf{Z}[G]$-modules, $\otimes$ is the functor $M \to \mathbf{Z} \otimes_{\mathbf{Z}[G]} M$, and α associates to a based finite G-set X the free abelian group on X, with the basepoint set to 0. The coefficient systems in (iii) and (iv) do not extend over $\mathcal{M}_G$. We show this for (iii). The composite $S^0 \xrightarrow{\tau} G_+ \xrightarrow{\pi} S^0$ in $\mathcal{M}_G$, where $\tau(p) = 1 + T$ as above, is multiplication by 2. If $\underline{F}$ were to extend to $\mathcal{M}_G$, we would have that the composite $\mathbf{Z} \to 0 \to \mathbf{Z}$ is multiplication by 2.

We also observe that Borel homology extends to an $RO[G]$-graded homology theory. Let G be finite as usual, and let G be a free contractible G-complex. For a based G-complex X, we define the (reduced) Borel homology of X with coefficients in an abelian group A to be

$H_*(EG \times_G X, EG \times_G *; A)$. Let V be any representation of G. Then $EG \times_G (S^V \wedge X)/EG \times_G *$ is equivalent to the cofiber of the map $T(\xi) \xrightarrow{i} T(\hat{\xi})$, where T denotes Thom complex, ξ and $\hat{\xi}$ are the vector bundles

$$\begin{array}{ccc} V \to EG \underset{G}{\times} V & & V \to EG \underset{G}{\times} X \times V \\ \downarrow & \text{and} & \downarrow \\ BG & & EG \underset{G}{\times} X \end{array}$$

respectively, and i is the inclusion given by $* \to X$. The Thom isomorphism now will give that, in the case of an orientable representation, $H^{\text{Borel}}_{*+\dim V}(S^V \wedge X; A) \cong H^{\text{Borel}}_*(X; A)$. This yields the appropriate isomorphisms, and hence the result. Borel homology, and particularly its dual cohomology theory, are extremely useful invariants of group actions. For instance, if $G = (\mathbf{Z}/p\mathbf{Z})^k$, one can recover information about the fixed point set of a G-action on a finite complex as follows. The $\mathbf{F}_p$-Borel cohomology of S^0, with trivial G-action, is a graded ring, isomorphic to $H^*(BG; \mathbf{F}_p)$. Further, $H^*_{\text{Borel}}(X; \mathbf{F}_p)$ is always a module over $H^*_{\text{Borel}}(S^0)$. There is now the following theorem.

THEOREM 4.3 [31]. *Let X be a finite G-complex. Then the inclusion map $X^G \to X$ induces an isomorphism on Borel cohomology, localized by inverting all non-zero elements in $H^1_{\text{Borel}}(S^0)$ if $p = 2$, and by inverting all non-nilpotent elements in $H^2_{\text{Borel}}(S^0)$ if p is odd. Thus, one can recover the sum of the mod p Betti numbers of the fixed point set from the Borel cohomology of X, as an $H^*_{\text{Borel}}(S^0)$-module.*

Remark. The G-spectra representing Borel homology as a G-homology theory and Borel cohomology as a G-cohomology theory are quite different.

(B) Equivariant K-theory

Recall that if X is a space, a complex vector bundle over X is a space E with projection map $E \xrightarrow{\pi} X$, so that there exists an open covering $\{U_\alpha\}_{\alpha \in A}$ of X and isomorphisms $\varphi_\alpha : \pi^{-1} U_\alpha \xrightarrow{\sim} U_\alpha \times \mathbf{C}^n$ over U_α, so that $\varphi_\beta \circ \varphi_\alpha^{-1} : U_\alpha \cap U_\beta \times \mathbf{C}^n \to U_\alpha \cap U_\beta \times \mathbf{C}^n$ is of the form $(u, v) \to (u, \vartheta(u)(v))$, where $\vartheta: U_\alpha \cap U_\beta \to GL_n(\mathbf{C})$ is a continuous map. We refer to [6] or [9] for a complete discussion. Similarly, if X is a G-space, a G-vector bundle is a G-space E, with G-map $E \xrightarrow{\pi} X$, so that there is a G-invariant open covering $\{U_\alpha\}_{\alpha \in A}$ and G-isomorphisms $\varphi_\alpha : \pi^{-1} U_\alpha \cong U_\alpha \times V$, where V is a complex representation of G, satisfying the above mentioned properties. Of course, ϑ must now be a G-map, where G is acting on $\text{Aut}(V)$ by conjugation. See [6] or [9] for a discussion of G-vector bundles.

Definition 4.4. For X a finite G-complex, we define $KU^0_G(X)$ to be the group completion of the monoid of isomorphism classes of G-vector bundles over X. This construction is contravariantly functorial in X, via pullback of G-vector bundles. If X is a finite based G-complex, then we define the reduced KU_G group, $\widetilde{KU}^0_G(X)$, to be the kernel of the natural map $KU^0_G(X) \to KU^0_G(\text{point})$.

Recall from [6] that in the non-equivariant theory, one obtains a periodicity isomorphism $KU^0(X) \xrightarrow{\sim} \widetilde{KU}^0(X \times D^2/X \times S^1)$, and by factoring out $KU^0(\text{point})$ and $\widetilde{KU}^0(X \times D^2/* \times S^1)$, respectively, an isomorphism $\widetilde{KU}^0(X) \xrightarrow{\sim} \widetilde{KU}^0(X \times D^2/* \times D^2 \cup X \times S^1) \cong \widetilde{KU}^0(S^2 \wedge X)$. The complex $X \times D^2/X \times S^1$ is of course the Thom complex of the one-dimensional trivial complex line bundle over X, and the above mentioned isomorphism is a special case of a Thom isomorphism theorem for the KU^0-groups. Atiyah in [8] proved a similar Thom isomorphism theorem in the equivariant case.

THEOREM 4.5. *Let X be a finite G-complex, let $\xi \to X$ be a G-vector bundle over X, and let $T\xi$ denote the Thom complex of ξ, a based finite G-complex. There is an isomorphism $KU_G^0(X) \xrightarrow{\cong} \widetilde{KU}_G^0(T\xi)$. In particular, if X is a based finite G-complex and V is any complex representation of G, then we obtain an isomorphism $\widetilde{KU}_G^0(X) \cong \widetilde{KU}_G^0(S^V \wedge X)$.*

The proof of this theorem uses the theory of elliptic partial differential operators, with G-action. In the non-equivariant theory, one uses this periodicity to show that one obtains a $\mathbf{Z}/2\mathbf{Z}$-graded generalized cohomology theory KU^*. The same procedure applied in the equivariant case gives a G-cohomology theory on based finite G-complexes. The Thom isomorphism further shows that one obtains suspension isomorphisms for all representations, and moreover that the grading factors through the composite $RO[G] \xrightarrow{\dim} \mathbf{Z} \to \mathbf{Z}/2\mathbf{Z}$, so the theory is in effect $\mathbf{Z}/2\mathbf{Z}$-graded. There is also a $\mathbf{Z}/8\mathbf{Z}$-graded theory KO_G^*, whose properties are more involved. See the paper of Atiyah–Segal [9] for details.

We have the following result, which in particular computes the coefficients in equivariant K-theory.

PROPOSITION 4.6.

(a) *KU_G^* is a graded ring valued cohomology theory; in particular, $KU_G^0(S^0)$ is a ring, and it is isomorphic to $R[G]$, the complex representation ring of G. The groups $KU_G^*(X)$ are thus $R[G]$-modules for all X. Also, $KU_G^1(S^0) = 0$.*

(b) *$KU_G^*(G/K^+) \cong R[K]$, where $K \subseteq G$ is a subgroup of $GR[K]$ is here an $R[G]$-module via the restriction ring homomorphism $R[G] \to R[K]$.*

(c) *If X is a finite based G-complex, free off the basepoint, then $KU_G^*(X) \cong KU^*(X_G)$.*

KU_G^*-theory is represented by a G-spectrum. We describe its zero-th space. Recall that the zero-th space for the KU-spectrum is $BU \times \mathbf{Z}$, where BU is the union of the spases $BU(n)$, and $BU(n)$ is the Grassmannian of n-planes in $\mathbf{C}^\infty$. Let $\mathbf{C}_G^\infty$ denote an infinite sum of copies of the regular representation of G, and let $BU_G(n)$ denote the space of n-planes in $\mathbf{C}_G^\infty$. G acts on $BU_G(n)$, and there are equivariant inclusions $BU_G(n) \to BU_G(n+1)$. The union, called BU_G, is the zero-th component of the zero-th space in the spectrum representing KU_G.

We now show how KU_G^* gives a quick calculation of $KU^*(\mathbf{R}P_+^\infty)$. This is just the method of Atiyah-Segal [9]. Let $G = \mathbf{Z}/2\mathbf{Z}$, so $\mathbf{R}P^\infty = BG$. (Note that $\mathbf{R}P^\infty$ is not a finite complex, so KU^*-theory is not actually defined by vector bundles, but one must take the representable version, defined using maps into $BU \times \mathbf{Z}$. A similar strategy works equivariantly, using G-maps into the G-spectrum described above.) Standard $\varprojlim^1$-arguments ($\varprojlim^1$ vanishes here) show that $KU^*(\mathbf{R}P_+^\infty) \cong \varprojlim KU^*(\mathbf{R}P_+^n)$. Let S^n denote the n-sphere with antipodal G-action. S_+^n is free off the basepoint, so $KU_G^*(S_+^n) \cong KU^*(\mathbf{R}P_+^n)$, and we must describe $\varprojlim KU_G^*(S_+^n)$. There is a natural map $S_+^n \to S^0$, which collapses S^n to the non-basepoint, so we obtain a homomorphism $KU_G^*(S^0) \xrightarrow{\alpha} \varprojlim_n KU_G^*(S_+^n)$. This map does not, as it stands, turn out to be an isomorphism. However, one can 2-adically complete all the groups in question. This corresponds geometrically to an equivariant 2-adic completion procedure applied to the spectrum representing KU_G^*. We denote the 2-adic completion of an abelian group by $\hat{2}$. Since $KU_G^*(S^0) \cong R[G]$, we obtain a homomorphism

$$R[G]_2^\wedge \to \varprojlim_n KU_G^*(S_+^n)_2^\wedge ,$$

which we will show is an isomorphism. Note that if we let E_n be the mapping cone of the map $S^n_+ \to S^0$, we obtain a long exact sequence

$$\cdots \to KU^i_G(E_n)^\wedge_2 \to KU^i_G(S^0)_2 \to KU^i_G(S^n_+)^\wedge_2 \to KU^{i+1}_G(E_n)^\wedge_2 \to \cdots$$

and hence ($\varprojlim^1$ terms can be checked to vanish, so $\varprojlim$ is exact here) a long exact sequence

$$\cdots \to \varprojlim_n KU^i_G(E_0)^\wedge_2 \to KU^i_G(S^0)^\wedge_2 \to \varprojlim_n KU^i_G(S^n_+)^\wedge_2 \to \cdots.$$

Since $KU^*_G(S^0)^\wedge_2$ is understood, if we can show that $\varprojlim KU^*_G(E_n)^\wedge_2 = 0$, we will have our result. Now, we observe that E_n is just the one point compactification of the representation $n\sigma$, where σ is the one-dimensional sign representation of G. This is the same thing as the Thom complex of $n\sigma$, viewed as a G-vector bundle over a point. Conseqently, when n is even,

$$\begin{cases} KU^0_G(E_n)^\wedge_2 \cong R[G]^\wedge_2 \\ KU^1_G(E_n)^\wedge_2 \cong 0 \end{cases}$$

by the Thom isomorphism theorem given above. This immediately gives $\varprojlim KU^1_G(E_n)^\wedge_2 = 0$, and we must describe the homomorphism $KU^0_G(E_{2n+2})^\wedge_2 \to KU^0_G(E_{2n})^\wedge_2$. Both sides are cyclic $R[G]^\wedge_2$-modules, so we must only describe the image of the element 1 in $R[G]^\wedge_2$, i.e. the "Thom class". But the calculations in Atiyah [8] show that this image is $1 - \rho$, where ρ denotes the class of the complex sign representation in $R[G]^\wedge_2$. Consequently, $\varprojlim_n KU^*_G(E_n)$ is isomorphic to the inverse limit of the inverse system

$$(*) \qquad \cdots \xrightarrow{\times(1-\rho)} RU[G]^\wedge_2 \xrightarrow{\times(1-\rho)} RU[G]^\wedge_2 \xrightarrow{\times(1-\rho)} \cdots \xrightarrow{\times(1-\rho)} RU[G]^\wedge_2 .$$

We examine $RU[G]$. There are two isomorphism classes of irreducible complex representations of G, namely the trivial representation ε and the sign representation ρ, so $RU[G] \cong \mathbf{Z}\varepsilon + \mathbf{Z}\rho$. ε acts as identity element in the ring structure, and $\rho^2 = \varepsilon$. $(1-\rho)^2$ is therefore equal to $2(1-\rho)$, so $(1-\rho)^n = 2^{n-1}(1-\rho)$. It is now clear that the inverse system $(*)$ has trivial inverse limit, so we show that $\varprojlim KU^*_G(E_n)^\wedge_2 = 0$. This leads one to the conclusion that

$$\begin{cases} KU^0(\mathbf{R}P^\infty_+)^\wedge_2 \cong R[G]^\wedge_2 \\ KU^1(\mathbf{R}P^\infty_+) \cong 0 \end{cases} \quad \text{and}$$

Atiyah [7] proved a result holding for any finite group, which generalizes this.

THEOREM 4.7. *Let $I(G)$ be the kernel of the restriction map $R[G] \to R[\{e\}]$, and let $R[G]^\wedge$ denote the completion of $RU[G]$ at $I(G)$. Then*

$$\begin{cases} KU^0(BG_+) \cong R[G]^\wedge \\ KU^1(BG_+) \cong 0 \end{cases}$$

for all finite groups G.

Another interesting application of equivariant stable homotopy theory is the dual (homological) version of Atiyah's result. In its cleanest form, it is given by J. P. C. Greenlees [25] in a recent preprint, but similar results were obtained earlier by G. Wilson [49] and K. Knapp [32]. An advantage of Greenlees' point of view is that it gives a very direct proof of Atiyah's theorem.

Greenlees' result reads as follows. For any ring A and ideal I, one can define the Grothendieck local cohomology groups $H^*_I(A)$ (see [26] for this construction). If A has

Krull dimension d, then these groups vanish for $* > d$. It is not hard to check that for any finite group G, $R[G]$ has Krull dimension 1.

THEOREM 4.8 (see [25]). *Let KU^G_* denote the generalized G-homology theory associated to the equivariant spectrum representing KU^*_G. Then*

$$\begin{cases} KU^G_0(EG_+) \cong H^0_{I(G)}(R[G]) & \text{and} \\ KU^G_1(EG_+) \cong H^1_{I(G)}(R[G]). \end{cases}$$

Moreover, $KU_*(BG_+) \cong KU^G_*(EG_+)$, so we obtain an explicit algebraic description of $KU_*(BG_+)$.

In fact, $H^*_{I(G)}(R[G])$ is typically rather straightforward to compute. For instance, for a p-group G, $H^0_{I(G)}(R[G]) \cong \mathbf{Z}$ and $H^1_{I(G)}(R[G]) \cong \bar{R}[G] \otimes \mathbf{Z}/p^\infty$, where $\bar{R}[G]$ denotes the $\mathbf{Z}$-module $R[G]/\mathbf{Z}$.

Another recent development is the description by Adams, Haeberly, Jackowski, and May of $KU^*_G(E_{\mathscr{F}}G_+)$, where $\mathscr{F}$ is any family of subgroups of G closed under downward inclusion and conjugation. The case $\mathscr{F} = \{\{e\}\}$ is Atiyah's theorem. The description is in terms of the completion of the representation ring at an appropriate ideal associated to the family $\mathscr{F}$, and appears in [4].

(C) ***Stable homotopy theory***

An important application of equivariant stable homotopy theory has been the solution of Segal's Burnside ring conjecture. Since this has been discussed in detail elsewhere ([15], [16]), we content our selves to state the result. Given a finite group G, let $A[G]$ denote the Burnside ring of G. Let $I(G) \subseteq A[G]$ be the kernel of the restriction map $A[G] \to A[\{e\}]$, and let $A[G]^\wedge$ denote the completion of $A[G]$ at $I(G)$. Note that for any G-space X, $\pi^G_*(X)$ is an $A[G]$ module. The theorem now goes as follows.

THEOREM 4.9 (W. H. Lin, May–McClure, Adams–Gunawardena–Miller, Carlsson). *Let X be a finite G-complex. Recall the notation $(\ -\ )^{hG}$ for G-spectra from §3. Then $\pi_i((\Sigma^\infty X)^{hG})$ is isomorphic to the $I(G)$-adic completion of $\pi^G_i(\Sigma^\infty X)$. In particular, if X has trivial G-action, the non-equivariant spectrum $F(BG_+, \Sigma^\infty X)$ has homotopy groups given by the $I(G)$-adic completion of $\pi^G_* X$, so $\pi_0(F(BG_+, \Sigma^\infty S^0)) \cong A[G]^\wedge$. This last statement is Segal's original conjecture.*

Remarks.

(i) When G is a finite p-group, $I(G)$-adic completion is essentially p-adic completion.
(ii) May–McClure [41] reduced the general case to the case of p-groups, W. H. Lin [37] proved the case $G = \mathbf{Z}/2\mathbf{Z}$, J. Gunawardena [27] proved the case $G = \mathbf{Z}/p\mathbf{Z}$, Adams–Gunawardena–Miller [3] proved the case $G = (\mathbf{Z}/p\mathbf{Z})^k$, and Carlsson [14] reduced the general p-group case to $(\mathbf{Z}/p\mathbf{Z})^k$.
(iii) In the proof of Atiyah's theorem, one makes great use of the fact that one can describe $KU^*(S^0)$ explicitly in building the induction to a general p-group from the cyclic case. The formal properties of equivariant stable homotopy theory are the replacement for the unavailable information concerning $\pi_*(S^0)$, and play a much more serious role in the proof of this theorem.
(iv) Adams, Haeberly, Jackowski, and May [5] have generalized the theorem significantly to prove similar results about function G-spectra with domain $E_{\mathscr{F}}G_+$, where $\mathscr{F}$ is a family of subgroups of G closed under downward inclusion and conjugation.

(v) A number of authors (Nishida [42], Feshbach [24], Lee [34], and Bauer [11]) have studied the analogous question for compact Lie groups, but information is still incomplete.

(vi) The theorem can be applied to study homotopy fixed point problems for *spaces* rather than spectra, such as the Sullivan conjecture [17]. See also [18].

(D) Equivariant bordism theories

The definitions of bordism groups of manifolds with additional structure on their tangent bundles (oriented, almost complex, framed, etc.) extends to the equivariant setting without change. Some of these groups have been studied by Stong [47], Wasserman [48], tom Dieck [20], and others, and one has a fair amount of information about them. Non-equivariant bordism theories yield homology theories of a space X by considering bordism theories of manifolds with reference maps to X. Also, the bordism groups can be identified with the homotopy groups of a Thom spectrum associated to the kind of structure required of the manifolds and bordisms, via the Pontrjagin-Thom construction. The G-homology theories one obtains via the naive generalization of these constructions do not yield fully equivariant theories, because of a failure of transversality in the equivariant setting. We give a brief sketch of how transversality fails.

Recall that if $f: M \to N$ is a smooth map of manifolds, f is transverse to a submanifold $P \subseteq N$ if for every point $x \in M$ so that $f(x) \in P$, the projection of the Jacobian map $Df(x)$ to the normal bundle of P in M at $f(x)$ has maximal rank. In particular, f is transverse to a point $n \in N$ if and only if $Df(x)$ has maximal rank for all $x \in f^{-1}(n)$. Under these circumstances, $f^{-1}(n)$ is a submanifold of M. Thom's transversality theorem asserts that via a small perturbation of f, one may construct a map homotopic to f which is transverse to P, if M is compact. The local verification of this amounts to the observation that for any map $\mathbf{R}^n \to \mathbf{R}^m \oplus \mathbf{R}^p$, with $f(0) = 0$, there is a neighborhood of 0 in $\mathbf{R}^n$ and a map $\tilde{f}$ arbitrarily close to f so that $\tilde{f}|U$ is transverse to $\mathbf{R}^p$. In particular, $\pi \circ D\tilde{f}(0)$, where π is projection from $\mathbf{R}^m \oplus \mathbf{R}^p$ to $\mathbf{R}^m$, has maximal rank. This local property fails in the equivariant setting. In fact, if $G = \mathbf{Z}/2\mathbf{Z}$, $\mathbf{R}^n$ is $\mathbf{R}^2$ with trivial G-action, $\mathbf{R}^p$ is $\mathbf{R}$ with trivial G-action, $\mathbf{R}^m$ is $\mathbf{R}$ with the sign action of G, and $f: \mathbf{R}^n \to \mathbf{R}^p$ is any surjective map, then any G-map must carry $\mathbf{R}^n$ into $\mathbf{R}^p$ and cannot, therefore, be of maximal rank when projected into $\mathbf{R}^m$. The problem, as stated in Hausschild [28], is that there must be "enough room" in the tangent bundle of M to allow a surjective G-bundle map to the tangent bundle of N. Hausschild shows, in [28], that with a sufficiently large tangent bundle of M, one can prove an equivariant version of Thom's theorem. This theorem allows one to construct G-homology theories with suspension isomorphisms for all representations of G.

We outline this construction, as described, e.g., in tom Dieck [20]. For simplicity, suppose we deal with unoriented bordism. Let $X \subseteq Y$ be G-spaces. We define $\Omega_n^G(Y, X)$ as in Stong [47], to be a bordism group of compact smooth G-manifolds with boundary $(M, \partial M)$ together with reference map $f: M \to Y$ so that $f(\partial M) \subseteq X$. Let V be the regular representation of G, and let $D(V)$ and $S(V)$ denote the unit disc and unit sphere in V under an invariant metric. Taking products with $D(V)$ gives homomorphisms $\Omega_k^G(D(lV) \times X, S(lV) \times X) \to \Omega_{k+|G|}^G(D((l+1)V) \times X, S((l+1)V) \times X)$, and we let $\tilde{\Omega}_n^G(X) = \varinjlim_l \Omega_{n+l|G|}^G(D(lV) \times X, S(lV) \times X)$. This will be called the G-equivariant stable bordism group of X, and yields a fully equivariant G-homology theory. In addition, this theory is represented by a G-spectrum MO^G.

One can ask how much of paragraphs (B) and (C), concerned with stable KU-theory and stable homotopy theory, carries over for these bordism theories. That is, does one obtain a calculation for $\Omega^*(BG_+)$ and $\Omega_U^*(BG_+)$ as one did for KU^* and π_S^*? Recall from §3 that given a spectrum with G-action $\underset{\sim}{Z}$, there is a "universal" G-spectrum $\underset{\sim}{Z}^{\wedge}$, equipped with an equivariant map of spectra with G-action $Z \to \rho Z^{\wedge}$, which is an equivalence of spectra non-equivariantly. The G-spectrum for Π_G^* is obtained by applying this procedure to S^0, equipped with trivial action. Moreover, equivariant K-theory is obtained by allowing G to act on the complex vector spaces, and MO^G and MU^G are obtained from MO and MU by permitting the group to act on the manifolds which represent elements in MO_* and MU_*. Since $KU_G^*(S^0)$ and $\pi_G^*(S^0)$ are both very closely related to $KU^*(BG_+)$ and $\pi_S^*(BG_+)$, one might hope that this would also be true for equivariant bordism. One might hope, for instance, that if G is a p-group, $\Omega^*(BG_+)_p^{\wedge}$ would be isomorphic to $\tilde{\Omega}_G^*(S^0)_p^{\wedge}$. Unfortunately, one can see that this fails, since one has quite explicit control over the bordism groups for lens spaces. Thus, the modifications to a homology theory to make it fully equivariant are not generally enough to describe homotopy fixed point set problems, although they are in the case of K-theory and stable cohomotopy.

A related problem has been studied by Hopkins, Kuhn, and Ravenel [33]. They attempt to describe $K(n)_*(BG_+)$ and $K(n)^*(BG_+)$ for G a finite group, where $K(n)$ denotes the n-th Morava K-theory [44]. They do not completely succeed, but they manage to compute the Euler characteristic. (Morava K-theory is $Z/2(p^n - 1)$-graded.) The answer is described in terms of n-tuples of commuting elements in G, and hence depends explicitly on the algebraic structure of G. The answer is so explicit that it suggests that one could identify $K(n)^*(BG_+)$ with an appropriately defined G-equivariant Morava K-group of a point. A problem with this idea is the way Morava K-theory is defined. As defined by Baas [10], using ideas of Sullivan, the n-th Morava K-theory is defined using bordism groups of manifolds with singularity of a particular kind. Unfortunately, the restriction on the singularity is that it be isomorphic to the cone on a family of bordism representatives of particular classes in the complex bordism of a point. This kind of restriction doesnot seem to be sufficiently natural and functorial to suggest a reasonable definition of equivariant Morava K-theory. Thus, it seems that what is needed is a more intrinsic definition of the types of singularities occurring in the definition, so that there will be a natural notion of what the G-manifolds and G-singularities defining the theory should be. Alternatively, one could attempt to get a sufficiently explicit computation of the equivariant stable complex G-bordism of a point, so that one can carry out the Baas construction on an appropriate family of generators in the equivariant complex bordism of a point. Recent work of Madsen [38] and Costenoble [19] have established an analogue of the Conner–Floyd isomorphism for equivariant bordism. Progress in these directions should give a better understanding of the chromatic filtration of stable homotopy theory.

5. PROBLEMS

(I) Recall that if $\underset{\sim}{X}$ is a G-spectrum, then the orbit spectrum $\underset{\sim}{X}_G$ is obtained by taking the orbit space levelwise in the associated spectrum with G-action $\rho\underset{\sim}{X}$. The fixed point spectrum of a suspension G-spectrum is well understood by tom Dieck's result; is it possible to give a reasonable description of the orbit spectrum?

(II) Define and compute equivariant Morava K-theory spectra. See §4 for a brief discussion.

(III) Formulate a conjecture about $\Omega_U^*(BG)$, for G a finite group.

(IV) Bökstedt has defined a "topological Hochshild homology" spectrum $THH(\underset{\sim}{A})$ for a ring spectrum $\underset{\sim}{A}$. Let $\underset{\sim}{K}$ denote the Eilenberg–MacLane spectrum associated to the integers. $THH(\underset{\sim}{K})$ is a key tool in the work of Bökstedt, Hsiang, and Madsen [12] on the K-theory analogue of the Novikov conjecture. $THH(\underset{\sim}{K})$ is equipped with an action by $\mathbf{Z}/p^n\mathbf{Z}$ for all n, and is in fact a $\mathbf{Z}/p^n\mathbf{Z}$-spectrum. Can one give a description of $THH(\underline{K})^{\mathbf{Z}/p^n\mathbf{Z}}$?

(V) Describe the function spectrum $F(BG, S^0)$, when G is a compact Lie group. A fair amount of work has been done on this problem, (see [24], [34], [11]), but results are incomplete.

(VI) Develop the equivariant theory for profinite groups, in an attempt to analyze homotopy fixed point sets for profinite groups.

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