CHERN-DOLD CHARACTER IN COMPLEX COBORDISMS AND THETA DIVISORS

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ABSTRACT. We show that the smooth theta divisors of general principally polarised abelian varieties can be chosen as irreducible algebraic representatives of the coefficients of the Chern-Dold character in complex cobordisms and describe the action of the Landweber-Novikov operations on them. We introduce a quantisation of the complex cobordism theory with the dual Landweber-Novikov algebra as the deformation parameter space and show that the Chern-Dold character can be interpreted as the composition of quantisation and dequantisation maps. Some smooth real-analytic representatives of the cobordism classes of theta divisors are described in terms of the classical Weierstrass elliptic functions. The link with the Milnor-Hirzebruch problem about possible characteristic numbers of irreducible algebraic varieties is discussed.

1. Introduction

In the complex cobordism theory, going back to the foundational works of Milnor and Novikov [34], [41], a prominent role is played by the Chern-Dold character introduced by the first author in [6].

By definition, the Chern-Dold character ch_U is a natural multiplicative transformation of cohomology theories

$$ch_U : U^*(X) \to H^*(X, \Omega_U \otimes \mathbb{Q}),$$

where $U^*(X)$ is the complex cobordism ring of a CW-complex X and $\Omega_U = U^*(pt)$, where pt is a point, is the coborodism ring of the stably complex manifolds (or, in short, U-manifolds). It is uniquely defined by the condition that when $X = pt$

$$ch_U : \Omega_U \to H^*(pt, \Omega_U \otimes \mathbb{Q}) = \Omega_U \otimes \mathbb{Q} \tag{1}$$

is the canonical homomorphism of Ω_U to $\Omega_U \otimes \mathbb{Q}$. This implies that for any finite CW-complex X the transformation

$$ch_U \otimes \mathbb{Q} : U^*(X) \otimes \mathbb{Q} \to H^*(X, \Omega_U \otimes \mathbb{Q}) \tag{2}$$

is an isomorphism of Ω_U-modules.

The fundamental Milnor-Novikov result says that the coefficient ring of the theory $U^*(X)$ is the graded polynomial ring

$$\Omega_U = \mathbb{Z}[y_1, \dots, y_n, \dots], \deg y_n = -2n$$

of infinitely many generators y_n, $n \in \mathbb{N}$.

Let $u \in U^2(\mathbb{C}P^\infty)$ and $z \in H^2(\mathbb{C}P^\infty)$ be the first Chern classes of the universal line bundle over $\mathbb{C}P^\infty$ in the complex cobordisms and cohomology theory respectively. The Chern-Dold character is uniquely defined by its action

$$ch_U : u \to \beta(z), \quad \beta(z) := z + \sum_{n=1}^{\infty} [\mathcal{B}^{2n}] \frac{z^{n+1}}{(n+1)!}, \tag{3}$$

where $\mathcal{B}^{2n}$ are certain U-manifolds, characterised by their properties in [6].

The series $\beta(z)$ is the exponential of the commutative formal group

$$F(u, v) = u + v + \sum_{i,j} a_{i,j} u^i v^j$$

of the geometric complex cobordisms introduced by Novikov in [43], so that

$$F(\beta(z), \beta(w)) = \beta(z + w).$$

Quillen identified this group with Lazard's universal one-dimensional commutative formal group [46].

The inverse of this series is the logarithm of this formal group, which can be given explicitly by the Mischenko series [43, 7]:

$$\beta^{-1}(u) = u + \sum_{n=1}^{\infty} [\mathbb{C}P^n] \frac{u^{n+1}}{n+1}. \tag{4}$$

The question whether there are smooth irreducible algebraic representatives of the cobordism classes $[\mathcal{B}^{2n}]$ in the exponential of the formal group given by the Chern-Dold character was open for a long time since 1970 (see [6]).

In this paper we give the following answer to this question, presenting an explicit form of the series (3) as

$$\beta(z) = z + \sum_{n=1}^{\infty} [\Theta^n] \frac{z^{n+1}}{(n+1)!}, \tag{5}$$

where Θ^n is a smooth theta divisor of a general principally polarised abelian variety A^{n+1}, considered as the complex manifold of real dimension $2n$. The cobordism class of the theta divisor does not depend on the choice of such abelian variety provided Θ^n is smooth, which is true in general case [2].

Theorem 1.1. *The theta divisor Θ^n of a general principally polarised abelian variety A^{n+1} is a smooth irreducible projective variety, which can be taken as an algebraic representative of the coefficient $[\mathcal{B}^{2n}]$ in the Chern-Dold character.*

As a corollary we have the following representation of the cobordism class of any U-manifold M^{2n} in terms of the theta divisors:

$$[M^{2n}] = \sum_{\lambda : |\lambda| = n} c^{\nu}_{\lambda}(M^{2n}) \frac{[\Theta^{\lambda}]}{(\lambda+1)!}, \tag{6}$$

where the sum is over all partitions $\lambda = (i_1, \dots, i_k)$ of $|\lambda| = i_1 + \dots + i_k = n$,

$$\Theta^{\lambda} := \Theta^{i_1} \times \dots \times \Theta^{i_k}, \tag{7}$$

$(\lambda + 1)! := (i_1 + 1)! \dots (i_k + 1)!$ and $c^{\nu}_{\lambda}(M^{2n}) \in \mathbb{Z}$ are the Chern numbers of M^{2n} corresponding to the normal bundle $\nu(M^{2n})$ (see the next section for details).

As another corollary we have the following explicit expression of the exponential generating function of any Hirzebruch genus Φ of theta divisors:

$$\Phi(\Theta, z) := \sum_{n=1}^{\infty} \Phi(\Theta^n) \frac{z^{n+1}}{(n+1)!} = \frac{z}{Q(z)}, \tag{8}$$

where $Q(z) = 1 + \sum_{n \in \mathbb{N}} a_n z^n$ is the characteristic power series of Hirzebruch genus Φ (see Section 3 for details).

In particular, for the Todd genus we have $Q(z) = \frac{z}{1-e^{-z}}$, so

$$Td(\Theta, z) := \sum_{n=1}^{\infty} Td(\Theta^n) \frac{z^{n+1}}{(n+1)!} = 1 - e^{-z} = \sum_{n \in \mathbb{N}} (-1)^n \frac{z^{n+1}}{(n+1)!},$$

so that the Todd genus of the theta divisors is

$$Td(\Theta^n) = (-1)^n \tag{9}$$

(cf. [6]). Thus we have the following formula for the Todd genus for any U-manifold M^{2n}

$$Td(M^{2n}) = \sum_{\lambda : |\lambda| = n} c_\lambda^\nu (M^{2n}) \frac{(-1)^n}{(\lambda+1)!}. \tag{10}$$

Since the Todd genus is integer, this implies the divisibility condition on the Chern numbers $c_\lambda^\nu(M^{2n})$ of U-manifolds. All divisibility conditions for U-manifolds one can get applying to formula (6) the Landweber-Novikov operations and taking the Todd genus (see the discussion in the last section).

The action of the Landweber-Novikov operations on the theta divisors can be described explicitly in the following way.

Let $\lambda = (i_1, \dots, i_k)$ be a partition of $|\lambda| := i_1 + \dots + i_k$ with $(k) = (k, 0, \dots, 0)$ being a one-part partition. Let $S_\lambda[M]$ be the result of the action of the Landweber-Novikov operation S_λ on U-manifold M defined in terms of its stable normal bundle (see [30, 43]). Consider a smooth complete intersection

$$\Theta_k^{n-k} = \Theta^n \cap \Theta^n(a_1) \cap \dots \Theta^n(a_k) \tag{11}$$

of Θ^n with k general translates $\Theta^n(a_i)$, $a_i \in A^{n+1}$ of the theta divisor Θ^n.

Let $D = c_1(L) \in H^2(A^{n+1}, \mathbb{Z})$ be the first Chern class of the principal polarisation bundle L, which is the cohomology class Poincaré dual to the cycle defined by $\Theta^n \subset A^{n+1}$. Then Θ_k^{n-k} is a realisation of the homology class from $H_{2n-2k}(A^{n+1}, \mathbb{Z})$, which is Poincaré dual to $D^{k+1} \in H^{2k+2}(A^{n+1}, \mathbb{Z})$.

Note that for a general principally polarised abelian variety the cohomology class D^p generates the corresponding Hodge group $H^{2p}_{Hodge}(A^{n+1}) = H^{2p}(A^{n+1}, \mathbb{Q}) \cap H^{p,p}(A^{n+1})$ for all p, due to Mattuck [33], who proved the Hodge (p,p)-conjecture in this case (see section 17.4 in [5]).

Theorem 1.2. *Let λ be a partition with $|\lambda| < n$ and S_λ be the corresponding Landweber-Novikov operation, then the cobordism class $S_\lambda[\mathcal{B}^{2n}]$ has a smooth irreducible algebraic representative. More precisely, if λ is not a one-part partition, then $S_\lambda[\mathcal{B}^{2n}] = S_\lambda[\Theta^n] = 0$, while for $\lambda = (k)$, $k \leq n$ we have*

$$S_{(k)}[\mathcal{B}^{2n}] = S_{(k)}[\Theta^n] = [\Theta_k^{n-k}]. \tag{12}$$

As a corollary we have the following expression for $[\Theta_k^{n-k}]$ as a residue at zero

$$[\Theta_k^{n-k}] = \frac{(n+1)!}{2\pi i} \oint \beta(z)^{k+1} \frac{dz}{z^{n+2}}, \tag{13}$$

with $\beta(z)$ given by (5). Moreover, the cobordism class $[\Theta_k^{n-k}]$ is a polynomial of $[\Theta^1], \dots, [\Theta^{n-k}]$ with positive integer coefficients, which implies that the polynomial subring $\Theta_U \subset \Omega_U$ generated by the theta-divisors:

$$\Theta_U = \mathbb{Z}[t_1, t_2, \dots], \quad t_k = [\Theta^k],\ k \in \mathbb{N} \tag{14}$$

is invariant under the Landweber-Novikov operations (see section 4 below).

We use the Landweber-Novikov algebra, which is a graded Hopf algebra S over $\mathbb{Q}$ generated as vector space by S_λ, to define the *quantum complex cobordism theory* as the extraordinary cohomology theory $U^\star := U^* \otimes S^*$ with $U^\star(pt) = \Omega^\star := \Omega_U \otimes S^*$, where $S^* = Hom(S, \mathbb{Q})$ is the dual Landweber-Novikov algebra, considered here as the deformation parameter space.

From the results of Landweber [30] and Novikov [43] (see also [8]) it follows that there is a canonical isomorphism of algebras $\sigma : S^* \cong \Omega_U \otimes \mathbb{Q}$. We show that the image of the dual basis $S^\lambda \in S^*$ can be given explicitly as

$$\sigma(S^\lambda) = \frac{[\Theta^\lambda]}{(\lambda+1)!}, \tag{15}$$

where Θ^λ are the products of theta divisors (7) with the cobordism classes $[\Theta^\lambda]$ giving the canonical basis in our ring Θ_U.

Inspired by constructions from [9]), we introduce the quantisation map $q^\star : U^*(X) \to U^\star(X) = U^*(X) \otimes S^*$ as

$$q^\star(x) = x \otimes 1 + \sum_\lambda S_\lambda(x) \otimes S^\lambda \in U^\star(X), \quad x \in U^*(X), \tag{16}$$

where the sum here is over all non-empty partitions.

Define also a dequantisation map

$$\mu^\star : U^*(X) \otimes S^* \to H^*(X, \Omega_U \otimes Q), \quad \mu^\star = \mu \otimes \sigma, \tag{17}$$

where $\mu : U^*(X) \to H^*(X, \mathbb{Z})$ is the cycle realisation homomorphism, which is defined uniquely by the property $\mu(u) = z$ for the same $u \in U^2(\mathbb{C}P^\infty)$ and $z \in H^2(\mathbb{C}P^\infty, \mathbb{Z})$ as before.

This allows us to interpret Chern-Dold character in complex cobordisms as the composition

$$U^*(X) \xrightarrow{q^\star} U^*(X) \otimes S^* \xrightarrow{\mu^\star} H^*(X, \Omega_U \otimes Q).$$

Theorem 1.3. *The Chern-Dold character in the complex cobordisms is the composition of quantisation and dequantisation maps:*

$$ch_U = \mu^\star \circ q^\star. \tag{18}$$

For $X = pt$ this composition is the canonical embedding $\Omega_U \to \Omega_U \otimes \mathbb{Q}$, but even in this case this leads to a non-trivial formula (6) (see Section 4).

We should mention here very interesting work by Coates and Givental [13, 14], who considered an analogue of quantum cohomology with the corresponding Gromov-Witten invariants [55] taking values in complex cobordisms. Some important relations of complex cobordisms with conformal field theory and integrable systems were discussed also in [28, 29, 38].

We consider also the most degenerate case of abelian variety $A^{n+1} = \mathcal{E}^{n+1}$, where $\mathcal{E}$ is an elliptic curve. In that case the theta-divisor is singular, but we show that there is a smooth real-analytic representative of the same homology class in $H_{2n}(A^{n+1}, \mathbb{Z})$. More precisely, we have the following result (see more details in Section 5).

Theorem 1.4. *There is a smooth real-analytic U-manifold $\mathcal{M}_W^{2n} \subset \mathcal{E}^{n+1}$ given in terms of classical Weierstrass functions, which can be used as a representative of the cobordism class $[\Theta^n]$.*

For every $k > 1$ the cobordism class $k^{n+1}[\Theta^n]$ can be realised by an irreducible algebraic subvariety of $\mathcal{E}^{n+1}$.

The structure of the paper is following. We start with a review of the main notions and results in complex cobordism theory, including the Chern-Dold character and Riemann-Roch-Grothendieck-Hirzebruch theorem.

In the central section 3 we describe the topological properties of the smooth theta divisors and use them to express explicitly the Todd class and Chern-Dold character in complex cobordism theory. We introduce also the corresponding dual complex bordism classes and study their properties.

In section 4 we discuss the Landweber-Novikov algebra and use its dual algebra as the deformation parameter space for certain quantisation of the complex cobordism theory, giving a different interpretation of the Chern-Dold character. We describe the action of the Landweber-Novikov operations on the cobordism classes of the theta divisors in terms of the algebraic cycles in general abelian varieties.

In section 5 we present some real-analytic representatives of the cobordism classes of theta divisors written explicitly in terms of the classical Weierstrass sigma and zeta functions.

In the last section we discuss the link with the Milnor-Hirzebruch problem about description of possible characteristic numbers of the smooth irreducible algebraic varieties.

2. Complex bordisms and cobordisms

We present now a brief review of the complex cobordism theory, referring for the details to Stong's lecture notes [51], or, for more algebraic view, to Quillen's work [47]. For the theory of the characteristic classes in cohomology we refer to Milnor and Stasheff [36], in K-theory to Atiyah [3] and in cobordism theory to Conner and Floyd [15], for the relations with algebraic cobordisms to Panin et al [44] and to the survey [10].

Let M^m be a smooth closed real manifold. By *stable complex structure* (or, simply *U-structure*) on M^m we mean an isomorphism of real vector bundles $TM^m \oplus (2N - m)_{\mathbb{R}} \cong r\xi$, where TM^m is the tangent bundle of M^m, $(2N - m)_{\mathbb{R}}$ is trivial real $(2N - m)$-dimensional bundle over M^m, ξ is a complex vector bundle over M^m and $r\xi$ is its real form. A manifold M^m with a chosen U-structure is called *U-manifold.*

Note that a complex structure in the stable tangent bundle TM^m determines complex structure in the stable normal bundle νM^m.

Two closed smooth real m-dimensional manifolds M_1 and M_2 are called *bordant* if there exists a real $(m + 1)$-dimensional U-manifold W such that the boundary ∂W is a disjoint union of M_1 and M_2 and the restriction of the stable tangent bundle TW to M_i is stably equivalent to TM_i, $i = 1, 2$.

Similarly, two closed smooth real m-dimensional manifolds M_1 and M_2 are called *cobordant* if there exists a real $(m+1)$-dimensional U-manifold W such that the boundary ∂W is a disjoint union of M_1^m and M_2^m and the restriction of the stable normal bundle νW to M_i coincides with νM_i, $i = 1, 2$.

Define the following operations in the corresponding equivalence classes of U-manifolds. The sum of bordism classes of two closed U-manifolds M_1^m and M_2^m is defined as $[M_1^m]+[M_2^m] = [M_1^m \cup M_2^m]$, where $M_1^m \cup M_2^m$ is the disjoint union of M_1^m and M_2^m. Similarly define the product of the bordism classes of $M_1^{m_1}$ and $M_2^{m_2}$ by $[M_1^{m_1}][M_2^{m_2}] = [M_1^{m_1} \times M_2^{m_2}]$, where $M_1^{m_1} \times M_2^{m_2}$ is the corresponding direct product. This defines the commutative graded ring

$$\Omega^U = \sum_{m \geq 0} \Omega_m^U,$$

where Ω_m^U is the group of bordism classes of m-dimensional U-manifolds.

Similarly, we have the graded ring $\Omega_U = \sum_{m \geq 0} \Omega_U^{-m}$, where Ω_U^{-m} is the group of cobordism classes of m-dimensional U-manifolds.

From the correspondence between stable complex structures in tangent and normal bundles it follows that the groups Ω_m^U and Ω_U^{-m} and the rings Ω_U and Ω^U are isomorphic. This isomorphism can be extended to Poincare duality between complex bordisms and cobordisms for any U-manifold.

Let $\lambda = (i_1, \ldots, i_k)$, $i_1 \geq \cdots \geq i_k$ be a partition of $n = i_1 + \cdots + i_k$, and $p(n)$ be the number of such partitions. Using the standard splitting principle one can define the Chern classes $c_\lambda(TM) \in H^{2n}(M, \mathbb{Z})$ of a U-manifold M corresponding to the monomial symmetric functions $m_\lambda(t) = t_1^{i_1} \ldots t_k^{i_k} + \ldots$ (see [36]).

The *Chern number* $c_\lambda(M^{2n})$, $|\lambda| = n$ of U-manifold M^{2n} is defined as the value of the cohomology class $c_\lambda(TM^{2n})$ on the fundamental cycle $\langle M^{2n} \rangle$:

$$c_\lambda(M^{2n}) := (c_\lambda(TM^{2n}), \langle M^{2n} \rangle). \tag{19}$$

We have $p(n)$ Chern numbers $c_\lambda(M^{2n})$, which depend only on the bordism class of M^{2n}.

The following fundamental result is due to Milnor and Novikov.

Theorem 2.1. *(Milnor* [34]*, Novikov* [41]*) The graded complex bordism ring Ω^U is isomorphic to the graded polynomial ring $\mathbb{Z}[y_1, \ldots, y_n, \ldots]$ of infinitely many variables $y_n, n \in \mathbb{N}$, where $\deg y_n = 2n$. In particular, $\Omega_{2n-1}^U = 0$.*

Since Ω^U is torsion-free we have

Corollary 2.2. *Two closed $2n$-dimensional U-manifolds M_1 and M_2 are U-bordant if and only if all the corresponding Chern numbers are the same.*

The choice of suitable algebraic representatives of the bordism classes $y_k \in \Omega^U$, $k \in \mathbb{N}$ was discussed starting from the work of Milnor and Novikov, see the references and latest results in [48].

It will be more convenient for us to use the Chern numbers $c_\lambda^\nu(M^{2n})$ defined using the stable normal bundle νM^{2n}:

$$c_\lambda^\nu(M^{2n}) := (c_\lambda(\nu M^{2n}), \langle M^{2n} \rangle). \tag{20}$$

They can be expressed through the usual Chern numbers $c_\lambda(M^{2n})$ and contain the same information about U-manifold M^{2n}.

We will use the following convenient class of U-manifolds from [12].

Let M^{2n} be a smooth real manifold of dimension $2n$. A *complex framing* of M^{2n} is a choice of complex line bundle $\mathcal{L}$ on M^{2n}, such that the direct sum $TM^{2n} \oplus \mathcal{L}$ admits a structure of trivial complex vector bundle. Thus complex framing is a U-structure of very special type. The examples of such structures is given by the following natural construction.

Let X be a complex manifold of (complex) dimension $n+1$ with holomorphically trivial tangent bundle and L be a complex line bundle over X. Let S be a real-analytic section $S : X \to L$, transversal to the zero section and consider $M^{2n} = \{x \in X : S(x) = 0\} \subset X$, which is a smooth real-analytic submanifold of X. Then the line bundle $\mathcal{L} = i^*(L)$, where $i : M^{2n} \to X$ is the natural embedding, determines the complex framing on M.

In our main example X is a principally polarised abelian variety and L is the canonical line bundle, with holomorphic section given by the θ-function (see next section). A more explicit example of a real-analytic submanifold for X being a product of elliptic curves is discussed in section 5.

In complex cobordism theory there exists an analogue of the celebrated Riemann-Roch formula [6]. To explain it recall first its Hirzebruch version.

Let X be a CW-complex and $\xi \to X$ be a complex vector bundle over X. The *characteristic Todd class* $Td(\xi) \in H^*(X,\mathbb{Q})$ of ξ is uniquely defined by following properties:

- $Td(\xi_1 \oplus \xi_2) = Td(\xi_1)Td(\xi_2)$;
- $Td(\eta) = \frac{z}{1-\exp(-z)}$, where η is the standard line bundle over $\mathbb{C}P^n$ and $z = c_1(\eta) \in H^2(\mathbb{C}P^n;\mathbb{Z})$ for every n.

The Todd class $Td : K(X) \to H^*(X,\mathbb{Q})$ in K-theory and the classical Chern character $ch : K(X) \to H^*(X,\mathbb{Q})$ are related by the fundamental relation

$$Td(\eta)ch(c_1^K(\eta)) = c_1(\eta) \tag{21}$$

where $c_1^K(\eta) = 1 - \eta^*,\ \eta^* = Hom(\eta,\mathbb{C})$ is the first Chern class of the line bundle η in K-theory (see [6, 15]).

The *Todd genus* of a U-manifold M^{2n} is the characteristic number

$$Td(M^{2n}) = (Td(TM^{2n}), \langle M^{2n}\rangle),$$

where $\langle M^{2n}\rangle$ is the fundamental cycle of manifold M^{2n}. Todd genus defines the ring homomorphism $\Omega_U \to \mathbb{Z}$, which, due to Thom's results [52], is uniquely determined by the condition that $Td(\mathbb{C}P^n) = 1$ for all n.

Let M be a smooth complex algebraic variety and g_i be the complex dimension of the space of holomorphic forms of degree i on M. Then Todd genus $Td(M)$ of M is equal to its arithmetic genus, or holomorphic Euler characteristic: $Td(M) = \sum\limits_{i\geqslant 0}(-1)^i g_i$ (see [24, 25]).

More generally, let ξ be a holomorphic vector bundle over M, $ch(\xi)$ be its Chern character [36] and $H^i(M,\mathcal{O}(\xi))$ be the corresponding cohomology groups [25], then we have the following generalisations of the Riemann-Roch formula due to Hirzebruch and Grothendieck.

Theorem 2.3. *(Riemann-Roch-Hirzebruch* [24, 25]*)*

$$\sum_{i\geqslant 0}(-1)^i \dim_{\mathbb{C}} H^i(M,\mathcal{O}(\xi)) = (ch(\xi)\,Td(TM),\langle M\rangle).$$

Theorem 2.4. *(Riemann-Roch-Grothendieck* [20, 18]*)*

Let $f\colon X\to Y$ be a proper morphism of smooth algebraic varieties. Then for any $x\in K(X)$ we have

$$ch(f_!x)Td(TY) = f_*(ch(x)\,Td(TX)), \tag{22}$$

where $f_!\colon K(X)\to K(Y)$ and $f_*\colon H^*(X)\to H^*(Y)$ are pushforward (Gysin) homomorphisms in K-theory and cohomology respectively.

The Grothendieck's version reduces to Hirzebruch's one when Y is a point (see e.g. [18]). To describe its extension in the theory of complex cobordisms we recall that the characteristic Todd class $Td_U(\xi)$ of complex vector bundle over CW-complex X with values in $H^*(X,\Omega_U\otimes\mathbb{Q})$ is uniquely defined by the following properties (see [6]):

- For every two vector bundles ξ_1 and ξ_2 over X
$$Td_U(\xi_1\oplus\xi_2) = Td_U(\xi_1)Td_U(\xi_2),$$
- For any U-manifold M^{2n}
$$(Td_U(TM^{2n}),\langle M^{2n}\rangle) = [M^{2n}],$$
where TM^{2n} is the tangent bundle of M^{2n} and $[M^{2n}]$ is its bordism class.

Note that from the first condition the Todd class is uniquely defined by its value on the canonical line bundle η over $\mathbb{C}P^N$:

$$Td_U(\eta) = 1+\sum_{n\in\mathbb{N}} A_n z^n, \quad z = c_1(\eta)\in H^2(\mathbb{C}P^N,\mathbb{Z}), \tag{23}$$

where the coefficients $A_n\in\Omega_U^{-2n}\otimes\mathbb{Q}$ are determined by the second condition. One of the results of this paper is an explicit expression for these coefficients given in the next section.

For any complex vector bundle ξ over CW-complex X we have

$$Td_U(\xi) = 1+\sum_{\lambda} c_\lambda(\xi)A_\lambda, \tag{24}$$

where the sum is over all partitions $\lambda=(i_1,\dots,i_k)$, $c_\lambda(\xi)\in H^{2|\lambda|}(X,\mathbb{Z})$ are the corresponding Chern classes and $A_\lambda = A_{i_1}\dots A_{i_k}$.

The analogue of the fundamental relation (21) has the form

$$Td_U(\eta)ch_U(c_1^U(\eta)) = c_1(\eta) \tag{25}$$

where the cobordism class $c_1^U(\eta)\in U^2(\mathbb{C}P^n)$ is the first Chern class of the canonical bundle η in complex cobordisms [15], which is dual to the bordism class of the canonical embedding $\mathbb{C}P^{n-1}\subset\mathbb{C}P^n$. This relation is a key ingredient in the proof of the following analogue of Riemann-Roch formula in complex cobordisms [6].

For U-manifolds we have the Poincaré-Atiyah duality in complex cobordisms [51]:

$$D_U\colon U_k(M^{2n})\to U^{2n-k}(M^{2n}), \qquad D^U\colon U^k(M^{2n})\to U_{2n-k}(M^{2n}).$$

Let $f\colon M_1^{2n} \to M_2^{2m}$ be the mapping of two U-manifolds, $f_* : U_k(M^{2n}) \to U_k(M^{2m})$ be the standard bordism homomorphism, and

$$f_\sharp = (D_2)_U f_* (D_1)^U : U^k(M_1^{2n}) \to U^{2m-2n+k}(M_2^{2m}) \tag{26}$$

be the corresponding Gysin homomorphism in complex cobordisms [6]. In particular, when $a \in U^{2k}(M^{2n})$ is the cobordism class dual to the bordism class of a smooth U-submanifold $M^{2n-2k} \subset M^{2n}$, then for the mapping $f\colon M^{2n} \to M^0 = pt$ we have $f_\sharp a = [M^{2n-2k}]$.

Theorem 2.5. *(Riemann-Roch-Grothendieck-Hirzebruch* [6]*)*
Let $f\colon X \to Y$ be a mapping of closed U-manifolds, then for any $x \in U^k(X)$ we have

$$ch_U(f_\sharp x) Td_U(TY) = f_!(ch_U(x) Td_U(TX)), \tag{27}$$

where $f_!\colon H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$ is the Gysin homomorphism.

In particular, when $Y = pt$ is a point, we have $ch_U(f_\sharp x) = f_\sharp x$ by (1), $Td_U(TY) = 1$, so in the left hand side we have simply $f_\sharp x$. Since for the mapping $f : X \to pt$ we have $f_!(a) = (a, \langle X\rangle)$, the formula (27) reduces to

$$f_\sharp x = (ch_U(x) Td_U(TX), \langle X\rangle). \tag{28}$$

So we see that both Chern-Dold character ch_U and Todd class Td_U play a prominent role in these formulas.

We will show now that they both can be explicitly described in terms of the theta divisors (see Theorems 3.1 and 3.5 below).

3. Theta divisors, Todd class and Chern-Dold character in complex cobordisms

Let $A^{n+1} = \mathbb{C}^{n+1}/\Gamma$ be a principally polarised abelian variety with lattice Γ generated by the columns of the $(n+1) \times 2(n+1)$ matrix $(I,\ \tau)$ with complex symmetric $(n+1) \times (n+1)$ matrix τ having positive imaginary part [19]. It has a canonical line bundle L with one-dimensional space of sections generated by the classical Riemann θ-function

$$\theta(z, \tau) = \sum_{l \in \mathbb{Z}^{n+1}} \exp[\pi i(l, \tau l) + 2\pi i(l, z)],\ z \in \mathbb{C}^{n+1}. \tag{29}$$

The corresponding theta divisor $\Theta^n \subset A^{n+1}$ given by $\theta(z, \tau) = 0$ is known (after Andreotti and Mayer [2]) to be smooth for a general principally polarised abelian variety A^{n+1}. The topology of the smooth theta divisor does not depend on the choice of such abelian variety.

In particular, for $n = 1$ a generic abelian surface A^2 is the Jacobi variety of a smooth genus 2 curve $\mathcal{C}$ with theta divisor $\Theta^1 \cong \mathcal{C}$. For $n = 2$ the indecomposable A^3 is Jacobi variety of a smooth genus 3 curve $\mathcal{C}$; in that case $\Theta^2 \cong S^2(\mathcal{C})$ is smooth for all non-hyperelliptic curves $\mathcal{C}$, which must be then trigonal. For $n \geq 3$ the general case of A^{n+1} is not Jacobian, and the theta divisor is smooth outside a locus in the moduli space of the abelian varieties of complex codimension 1. For more detail on the geometry of theta divisors we refer to the survey [21] by Grushevsky and Hulek.

The line bundle L is ample with L^3 known (after Lefschetz [5]) to be very ample, so that the sections of L^3 determine the embedding of A^{n+1}

into corresponding projective space $\mathbb{P}^N$, $N = 3^{n+1} - 1$. The corresponding quadratic and cubic equations, defining the image in $\mathbb{P}^N$, were described by Birkenhake and Lange [4] (see also Ch. 7 in [5]). For the elliptic curves this reduces to the Hasse cubic equation $x^3 + y^3 + z^3 = 3\lambda xyz$.

Note that the line bundle L^2 is very ample only on the quotient $A^{n+1}/\mathbb{Z}_2$ by the involution $z \to -z$, which is known as *Kummer variety.* When $n = 1$ this is the famous Kummer quartic surface in $\mathbb{P}^3$ with 16 singular points, see e.g. [5].

Let $i : \Theta^n \to A^{n+1}$ be the natural embedding and $\mathcal{D} = c_1(\mathcal{L}) \in H^2(\Theta^n, \mathbb{Z})$ be the first Chern class of the line bundle $\mathcal{L} := i^*(L)$, which is also the normal bundle of $\Theta^n \subset A^{n+1}$. Then the total Chern class $c(\Theta^n) = \sum_{k=1}^n c_k(\Theta^n)$ satisfies

$$c(\Theta^n)(1 + \mathcal{D}) = 1 \tag{30}$$

since the tangent bundle of an abelian variety is trivial. This means that the Euler characteristic

$$\chi(\Theta^n) = c_n(\Theta^n) = (-1)^n \mathcal{D}^n = (-1)^n (n+1)!, \tag{31}$$

since $\mathcal{D}^n = (n+1)!$ (see next section). Alternatively, we can use formula (8) with $Q(z) = 1 + z$ corresponding to Euler characteristic $\Phi = \chi$:

$$\sum_{n=1}^{\infty} \chi(\Theta^n) \frac{z^{n+1}}{(n+1)!} = \frac{z}{1+z}.$$

The Betti numbers of the theta divisors Θ^n are not difficult to compute, see e.g. [27, 39]. Indeed, by the Lefschetz hyperplane theorem the embedding $i : \Theta^n \to A^{n+1}$ induces the isomorphisms

$$i_* : H_k(\Theta^n, \mathbb{Z}) \to H_k(A^{n+1}, \mathbb{Z}), \quad i_* : \pi_k(\Theta^n) \to \pi_k(A^{n+1})$$

for $k < n$, while for $k = n$ these homomorphisms are surjections [31, 35].

In particular, for $n \geq 2$ the theta divisor has the fundamental group

$$\pi_1(\Theta^n) = \pi_1(A^{n+1}) = \mathbb{Z}^{2n+2}$$

and $\pi_k(\Theta^n) = \pi_k(A^{n+1}) = 0$ for $1 < k < n$. When $n = 1$ $\Theta^1 \cong \mathcal{C} \subset J(\mathcal{C})$ is a genus 2 curve with non-commutative $\pi_1(\mathcal{C})$ and $i_* : \pi_1(\mathcal{C}) \to \mathbb{Z}^4$ being the abelianisation map and all other homotopy groups being trivial.

The manifolds with free abelian fundamental group were studied by Novikov [42] in relation with the famous problem of topological invariance of rational Pontryagin classes. The theta divisors give non-trivial examples of such manifolds with non-zero Pontryagin classes.

Using Poincare duality we obtain now all Betti numbers of Θ^n as

$$b_k(\Theta^n) = b_k(A^{n+1}) = \binom{2n+2}{k} = b_{2n-k}(\Theta^n), \ k < n,$$

except the middle one b_n, which can be found using the formula (31) for the Euler characteristic:

$$b_n(\Theta^n) = (n+1)! + \frac{n}{n+2}\binom{2n+2}{n+1} = (n+1)! + nC_{n+1}, \tag{32}$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number, see [49].

Since the cohomology groups of Θ^n have no torsion [27], this defines them uniquely, but the multiplication structure seems yet to be understood. Note that we can compute the signature $\tau(\Theta^n)$ of the corresponding quadratic form on the middle cohomology for even n using our general formula (8) (see also [12]). Indeed, the signature corresponds to Hirzebruch L-genus with $Q(z) = \frac{z}{\tanh z}$. Since

$$\frac{z}{Q(z)} = \tanh z = \sum_{k=0}^{\infty} 2^{2k+2}(2^{2k+2}-1)B_{2k+2}\frac{z^{2k+1}}{(2k+2)!}$$

from (8) we have that the signature of the theta divisor Θ^n for even n is

$$\tau(\Theta^n) = \frac{2^{n+2}(2^{n+2}-1)}{n+2}B_{n+2}, \tag{33}$$

where B_n are the classical Bernoulli numbers:

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},\ B_8 = -\frac{1}{30},\ B_{10} = \frac{5}{66}, \dots$$

In particular, for $n = 2$ we have $b_2(\Theta^2) = 16$ and $\tau(\Theta^2) = \frac{2^4(2^4-1)B_4}{4} = -2$. Note that the integrality of the right hand side of (33) is not obvious. The appearance of both Bernoulli and Catalan numbers looks quite intriguing and invites further study here.

We should mention here very interesting work by Nakayashiki and Smirnov on the computation of cohomology groups of the complement to (singular) theta divisor in hyperelliptic Jacobi variety [39, 40].

For all n the smooth theta divisor Θ^n is a projective variety of general type. Indeed, since the canonical class of abelian variety is zero, by the adjunction formula [19] the canonical bundle $K_{\Theta^n} = i^*(L) = \mathcal{L}$, which is ample. In particular, $\mathcal{L}$ is known to have n-dimensional space of sections generated by the partial derivatives $\partial_\xi \theta(z,\tau)$ of the theta function. By Bertini theorem the system of equations

$$\theta(z,\tau) = 0,\ \partial_{\xi_1}\theta(z,\tau) = 0, \dots, \partial_{\xi_k}\theta(z,\tau) = 0, \quad z \in A^{n+1} \tag{34}$$

with generic $\xi_1, \dots, \xi_k \in \mathbb{C}^{n+1}$ determine smooth complete intersections $\Theta_k^{n-k} \subset \Theta^n \subset A^{n+1}$. Alternatively, we can consider the intersection

$$\Theta_k^{n-k} = \Theta^n \cap \Theta^n(a_1) \cap \dots \Theta^n(a_k) \tag{35}$$

of Θ^n with k generic translates $\Theta^n(a_i)$, $a_i \in A^{n+1}$ of the theta divisor Θ^n.

The canonical bundle of $S^{n-k} = \Theta_k^{n-k} \subset A^{n+1}$ is $K_{S^{n-k}} = i^*_{S^{n-k}}(L^{k+1})$, where $i_{S^{n-k}}$ is the embedding of $S^{n-k} \subset A^{n+1}$. It is ample, so S^{n-k} is of general type as well. In particular, $S^0 = \Theta_n^0$ consists of $(n+1)!$ points and $S^1 = \Theta_{n-1}^1$ is a curve with Euler characteristic $\chi = -n(n+1)!$.

Note that the varieties Θ_k^{n-k} with $k \neq n$ are irreducible, since they are smooth and (by Lefschetz theorem) their Betti number $b_0 = 1$.

Theorem 3.1. *The characteristic Todd class of complex vector bundle ξ over CW-complex X is given by the formula*

$$Td_U(\xi) = \sum_\lambda c_\lambda(-\xi)\frac{[\Theta^\lambda]}{(\lambda+1)!}, \tag{36}$$

where the sum is over all partitions $\lambda = (i_1, \dots, i_k)$ *and* $\Theta^\lambda = \Theta^{i_1} \times \dots \times \Theta^{i_k}$ *as before. In particular, for any* U*-manifold* M *we have*

$$Td_U(TM) = \sum_\lambda c_\lambda(\nu M) \frac{[\Theta^\lambda]}{(\lambda+1)!}. \tag{37}$$

Proof. Denote the right hand side of formula (36) as $\mathbb{T}(\xi)$ and check that this characteristic class satisfies both defining properties of the Todd class.

The first property $\mathbb{T}(\xi_1 + \xi_2) = \mathbb{T}(\xi_1)\mathbb{T}(\xi_2)$ follows from the well-known formula for the Chern classes

$$c_\lambda(\xi_1 + \xi_2) = \sum_{\lambda=(\lambda_1,\lambda_2)} c_{\lambda_1}(\xi_1) c_{\lambda_2}(\xi_2).$$

The evaluation $(\mathbb{T}(TM), \langle M \rangle) \in \Omega_U \otimes \mathbb{Q}$ defines the homomorphism of $\Omega_U \otimes \mathbb{Q}$ into itself. We claim that it is the identity.

Lemma 3.2. *The Chern numbers (20) of the theta divisor* Θ^n *are*

$$c^\nu_\lambda(\Theta^n) = 0 \tag{38}$$

for any partition λ *of* n *different from the one-part partition* $\lambda = (n)$*, and*

$$c^\nu_{(n)}(\Theta^n) = (n+1)!. \tag{39}$$

Proof. Since the tangent bundle of abelian variety is trivial, the normal bundle $\nu\Theta^n$ is stably equivalent to the line bundle $\mathcal{L} = i^*(L)$, where $i : \Theta^n \to A^{n+1}$ is natural embedding and L is the principal polarisation line bundle on A^{n+1}. This immediately implies (38).

To prove condition (39) we need only to use the well-known fact that

$$D^g = g! \in H^{2g}(A^g, \mathbb{Z}) = \mathbb{Z}$$

where $D \in H^2(A^g, \mathbb{Z})$ is the Poincare dual cohomology class of the theta divisor $\Theta \subset A^g$ of any principally polarised abelian variety (see e.g. [5]). Geometrically, this means that the intersection of g generic shifts of theta divisor Θ of abelian variety A^g consists of $g!$ points. One can see this easily in the degenerate case when $X^g = \mathcal{E}^g$ is the product of g elliptic curves. □

Due to the results of Milnor and Novikov, since $c_{(n)}(\Theta^n) \neq 0$ the theta divisors can be chosen as multiplicative generators of the algebra $\Omega_U \otimes \mathbb{Q}$. Hence to prove that $(\mathbb{T}(TM), \langle M \rangle) = [M]$ it is enough to check this for all theta divisors, which immediately follows from the lemma. By uniqueness $\mathbb{T}(\xi) = Td_U(\xi)$, which completes the proof. □

Corollary 3.3. *The cobordism class of any* U*-manifold* M^{2n} *can be given by formula (6).*

Indeed, we have formula (6) by evaluating formula (37) on the fundamental cycle of M^{2n} and using the second property of the Todd class.

Recall now that the Todd class is uniquely defined by its value on the canonical line bundle η over $\mathbb{C}P^\infty$ by formula (23):

$$Td_U(\eta) = 1 + \sum_{n \in \mathbb{N}} A_n z^n, \quad z = c_1(\eta) \in H^2(\mathbb{C}P^\infty, \mathbb{Z})$$

where some coefficients $A_n \in \Omega_U^{-2n} \otimes \mathbb{Q}$. Now we can describe these coefficients in terms of theta divisors.

Corollary 3.4. *Todd class of η satisfies the relation*

$$Td_U(\eta)\sum_{n=0}^{\infty}\frac{[\Theta^n]}{(n+1)!}z^n = 1, \qquad z = c_1(\eta), \tag{40}$$

so

$$Td_U(\eta) = \left(\sum_{n=0}^{\infty}\frac{[\Theta^n]}{(n+1)!}z^n\right)^{-1}.$$

Indeed, this follows from the relation $Td_U(\eta)Td_U(-\eta) = 1$ and formula (36) applied to $\xi = -\eta$.

Now we are ready to prove Theorem 1.1.

Theorem 3.5. *The Chern-Dold character is uniquely defined by the formula*

$$ch_U(u) = z + \sum_{n=1}^{\infty}[\Theta^n]\frac{z^{n+1}}{(n+1)!}, \tag{41}$$

where $z = c_1(\eta) \in H^2(\mathbb{C}P^N, \mathbb{Z})$ and $u \in U^2(\mathbb{C}P^N)$ is the first Chern class of the canonical line bundle η over $\mathbb{C}P^N$ in the complex cobordisms.

Proof. We use the fundamental relation (25), which in these notations has the form

$$Td_U(\eta)ch_U(u) = z. \tag{42}$$

Comparing this formula with (40) we have formula (41) and the claim. □

We can prove now also formula (8) for the exponential generating function of any Hirzebruch genus Φ of all theta divisors.

The Hirzebruch genus in complex cobordisms [25] is a homomorphism $\Phi : \Omega_U \to \mathcal{A}$, where $\mathcal{A}$ is some algebra over $\mathbb{Q}$, determined by its characteristic power series

$$Q(z) = 1 + \sum_{n=1}^{\infty} a_n z^n, \quad a_n \in \mathcal{A}. \tag{43}$$

Theorem 3.6. *For any Hirzebruch genus Φ with characteristic power series $Q(z)$ we have the following relations with the Chern-Dold character and Todd class*

$$\Phi(ch_U(u)) = \frac{z}{Q(z)}, \quad \Phi(Td_U(\eta)) = Q(z). \tag{44}$$

In particular, the exponential generating function of $\Phi(\Theta^n)$ is

$$z + \sum_{n=1}^{\infty}\Phi(\Theta^n)\frac{z^{n+1}}{(n+1)!} = \frac{z}{Q(z)}. \tag{45}$$

Proof. Let $Q_a(z)$ be given by (43) and introduce $Q_b(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ as the inverse series $Q_b(z) = Q_a(z)^{-1}$:

$$Q_a(z)Q_b(z) \equiv 1.$$

By definition the Hirzebruch genus Φ_a of any U-manifold M^{2n} corresponding to $Q_a(z)$ is given by

$$\Phi_a(M^{2n}) := \sum_{\lambda:|\lambda|=n} a_\lambda c_\lambda(M^{2n}), \tag{46}$$

where for partition $\lambda = (i_1, \dots, i_k)$ we have $a_\lambda = a_{i_1} \dots a_{i_k}$ and $c_\lambda(M^{2n}) = (c_\lambda(TM^{2n}), < M^{2n} >)$. Then from the theory of symmetric functions (see e.g. [32]) it follows that

$$\Phi_b(M^{2n}) := \sum_{\lambda:|\lambda|=n} b_\lambda c_\lambda(M^{2n}) = \sum_{\lambda:|\lambda|=n} a_\lambda c_\lambda^\nu(M^{2n}),$$

where the characteristic numbers

$$c_\lambda^\nu(M^{2n}) = (c_\lambda(\nu M^{2n}), < M^{2n} >)$$

correspond to the normal bundle of M^{2n}. Applying this to the theta divisor Θ^n and using Lemma 3.2 we have

$$\Phi_b(\Theta^n) = a_n(n+1)!, \quad \Phi_a(\Theta^n) = b_n(n+1)!.$$

This means that

$$\Phi_a(ch_U(u)) = z + \sum_{n=1}^{\infty} \Phi_a(\Theta^n)\frac{z^{n+1}}{(n+1)!} = z + \sum_{n=1}^{\infty} b_n z^{n+1} = zQ_b(z) = \frac{z}{Q_a(z)}.$$

Now from (42) we have $\Phi_a(Td_U(\eta)) = Q_a(z)$, which completes the proof. □

Consider now the special case of $\mathcal{A} = \Omega \otimes \mathbb{Q}$ and $\Phi : \Omega \to \mathcal{A} = \Omega \otimes \mathbb{Q}$ being the natural embedding. Then from (44) we see that the corresponding series $Q(z) = z/\beta(z)$ with $\beta(z)$ given by (5).

Define the cobordism classes $v_n \in \Omega_U \otimes \mathbb{Q}$ as the coefficients of $Q_v(z)$ written in the form

$$Q_v(z) = 1 + \sum_{n=1}^{\infty}(-1)^n v_n \frac{z^n}{(n+1)!}, \tag{47}$$

with the series Q_v defined by

$$\left(1 + \sum_{n=1}^{\infty}(-1)^n v_n \frac{z^n}{(n+1)!}\right)\left(1 + \sum_{n=1}^{\infty}[\Theta^n]\frac{z^n}{(n+1)!}\right) \equiv 1. \tag{48}$$

In the theory of symmetric functions [32] this corresponds to the duality ω between elementary symmetric functions e_n and complete symmetric functions h_n (see formula (2.6) in [32]), where we substitute

$$e_n = \frac{\Theta^n}{(n+1)!}, \quad h_n = \frac{v_n}{(n+1)!}. \tag{49}$$

The determinantal formula for this duality (see e.g. page 28 in [32])

$$h_n = \det(e_{1-i+j})_{1 \le i,j \le n}$$

allows to express v_n as a polynomial of $t_k := [\Theta^k]$, $k = 1, \dots, n$ with rational coefficients. For example,

$$v_1 = t_1, \ v_2 = -t_2 + \frac{3}{2}t_1^2, \ v_3 = t_3 - 4t_1t_2 + 3t_1^3, \tag{50}$$

$$v_4 = -t_4 + 5t_1t_3 - 15t_1^2t_2 + \frac{10}{3}t_2^2 + \frac{15}{2}t_1^4, \tag{51}$$

$$v_5 = t_5 - 6t_1t_4 + 30t_1t_2^2 - 60t_1^3t_2 - 10t_2t_3 + \frac{45}{2}t_1^2t_3 + \frac{45}{2}t_1^5. \tag{52}$$

In fact, since the series in (48) are the exponential generating functions of $y_n = \frac{v_n}{n+1}$ and $x_n = \frac{[\Theta^n]}{n+1}$ respectively, we can apply the results about Hurwitz series [26, 45] to deduce that $y_n \in \mathbb{Z}[x_1, \dots, x_n]$ is a polynomial with integer coefficients (which is clearly not the case in the formulae for v_n above).

Let $v_\lambda = v_{i_1} \dots v_{i_k}$, $\lambda = (i_1, \dots, i_k)$ then the cobordism classes v_λ considered as elements of S^* form a basis dual to the Landweber-Novikov operations $\bar{S}_\lambda$ defined using the tangent bundles (see Novikov [43]). In particular, the (usual, tangent) characteristic numbers $c_\lambda(v_n) = 0$ for any $\lambda \neq (n)$, and

$$c_{(n)}(v_n) = (-1)^n (n+1)!.$$

Theorem 3.7. *For any U-manifold M^{2n} we have the following analogue of formula (6):*

$$[M^{2n}] = \sum_{\lambda:|\lambda|=n} (-1)^{|\lambda|} c_\lambda(M^{2n}) \frac{v_\lambda}{(\lambda+1)!}, \tag{53}$$

where instead of $c_\lambda^\nu(M^{2n})$ we use the characteristic numbers $c_\lambda(M^{2n})$ of the tangent bundle TM^{2n}.

The Hirzebruch genus $\Phi(v_n)$ with characteristic power series $Q(z)$ can be found from the generating function

$$1 + \sum_{n=1}^{\infty} (-1)^n \Phi(v_n) \frac{z^n}{(n+1)!} = Q(z). \tag{54}$$

The proof follows directly from Theorem 3.6 and formulae (46), (47).

In particular, for the Euler characteristic with $Q(z) = 1 + z$ we conclude that $\chi(v_n) = 0$ for all $n > 1$ with $\chi(v_1) = -2$. Similarly, for the Todd genus we have

$$1 + \sum_{n=1}^{\infty} (-1)^n Td(v_n) \frac{z^n}{(n+1)!} = \frac{z}{1 - e^{-z}} = \sum_{n=0}^{\infty} (-1)^n B_n \frac{z^n}{n!},$$

where B_n are the Bernoulli numbers. This implies that

$$Td(v_n) = (n+1) B_n \tag{55}$$

is zero for odd $n > 1$ and non-zero rational for even n. In particular, $Td(v_2) = \frac{1}{2}$, so the cobordism class v_2 cannot be represented by a U-manifold.

A natural question is what is the minimal integer $k_n \in \mathbb{N}$ such that $k_n v_n \in \Omega_U$ is a cobordism class of some U-manifold. The following result gives the answer to this question.

Let q_n be the denominator of the fraction $(n+1)B_n$ written in the simplest form, where for odd $n > 1$ with $B_n = 0$ we put $q_n = 1$.

Theorem 3.8. *The cobordism class*

$$q_n v_n = [V^{2n}] \in \Omega_U \tag{56}$$

for some U-manifold V^{2n}, and this is the smallest multiple of v_n with such property.

Proof. Let $k_n v_n = [M^{2n}] \in \Omega_U$, then the Todd genus $k_n(n+1)B_n$ must be an integer, which means that k_n is divisible by q_n, and thus $k_n = q_n$ must be minimal.

To prove that the cobordism class $q_n v_n = [V^{2n}]$ for some U-manifold V^{2n} we use the fundamental result of Hattori [22] and Stong [50, 51], saying that element $a \in \Omega_U \otimes \mathbb{Q}$ belongs to Ω_U if and only if the Todd genus of all the results $S_\lambda(a)$ of the Landweber-Novikov operations applied to a are integer. This is obviously true for the Todd genus of $a = q_n v_n$ since $Td(a) = q_n(n+1)B_n$. We have the following important result.

Lemma 3.9. *Every Landweber-Novikov operation S_λ, different from the identity, maps the cobordism class v_n to the subring $\Theta_U = \mathbb{Z}[t_1, t_2, \dots] \subset \Omega_U$ generated by the theta divisors $t_k = [\Theta^k]$, $k \in \mathbb{N}$.*

In particular, $S_{(1)}(v_1) = -2$, $S_{(1)}(v_n) = 0$ for $n > 1$ and for $n \geq 2$ $S_{(2)}(v_n) = -n(n+1)[\Theta^{n-2}]$.

We will use the following properties of the Landweber-Novikov operations:

$$S_{(k)}(\beta(z)) = \beta(z)^{k+1}, \tag{57}$$

where $\beta(z)$ is given by (5), and $S_\lambda(\beta(z)) = 0$ for all non one-part partitions (see the next section for more details).

We will prove Lemma by induction in the length of λ. For length one partition $\lambda = (k)$ we apply the operation $S_{(k)}$ to the relation (48), which we rewrite as $Q_v(z)\beta(z) = z$, to have

$$S_{(k)}(Q_v(z)\beta(z)) = S_{(k)}Q_v(z)\beta(z) + Q_v(z)\beta(z)^{k+1} = 0,$$

where we have used (57). Since $Q_v(z)\beta(z) = z$ this implies that

$$S_{(k)}(Q_v(z)) = \sum_{n=1}^{\infty}(-1)^n S_{(k)}(v_n)\frac{z^n}{(n+1)!} = -z\beta(z)^{k-1}.$$

In particular, for $k = 1$ we have $S_{(1)}(Q_v(z)) = -z$, so $S_{(1)}(v_n) = 0$ for $n > 1$ and $S_{(1)}(v_1) = -2$. For $k = 2$ we have $S_{(2)}(Q_v(z)) = -z\beta(z)$, which implies that for $n \geq 2$ $S_{(2)}(v_n) = -n(n+1)[\Theta^{n-2}]$. When $k > 2$ we can write

$$S_{(k)}(Q_v(z)) = -z\beta(z)^{k-1} = -zS_{(k-2)}(\beta(z))$$

and use the induction in k to conclude the proof for one-part partitions.

The general case follows from the multiplicative property (60) of the Landweber-Novikov operations and the fact that $S_\lambda(\beta(z)) = 0$ for all partitions λ of length more than one.

Now the claim of the Theorem follows from Lemma and Stong and Hattori results, since the Todd genus of any U-manifold is integer. □

Remark 3.10. *It is natural to introduce also the following realisation of the power sums in complex cobordisms $p_n = \frac{(-1)^{n-1}}{(n-1)!}w_n$, where $w_n \in \Omega_U \otimes \mathbb{Q}$ are defined by the formula*

$$\beta(z) = z\exp\left(\sum_{n=0}^{\infty} w_n \frac{z^n}{n!}\right) \tag{58}$$

(see [32] *and formulae (47), (49) above). Applying the Landweber-Novikov operation $S_{(1)}$ to both sides and using (57) we have*

$$\beta(z) = \sum_{n=1}^{\infty}[\Theta^{n-1}]\frac{z^n}{n!} = \sum_{n=1}^{\infty} S_{(1)}(w_n)\frac{z^n}{n!}, \tag{59}$$

which implies that $S_{(1)}(w_n) = [\Theta^{n-1}]$ is the class of theta divisor. It would be interesting to find an analogue of Theorem 3.8 for the classes w_n.

4. The Landweber-Novikov algebra and quantisation of complex cobordisms

The Landweber-Novikov algebra introduced in [30, 43] is an important subalgebra of all cohomological operations in complex cobordisms with additive basis given by the Landweber-Novikov operations $S_\lambda, \lambda \in \mathcal{P}$, where $\mathcal{P}$ is the set of all partitions $\lambda = (i_1, \dots, i_k)$ (see [32]).

Recall that the cobordism class $\alpha \in U^2(X)$ is called *geometric* if it belongs to the image of the natural homomorphism $H^2(X, \mathbb{Z}) \to U^2(X)$ (see [43]). The action of the Landweber-Novikov operations on any geometric cobordism class α is defined as follows: for any one-part partition $\lambda = (k)$, $k \in \mathbb{N}$ the Landweber-Novikov operation $S_{(k)}(\alpha) = \alpha^{k+1}$, while for all other partitions $S_\lambda(\alpha) = 0$ (see Lemma 5.6 in [43]). Together with the rule

$$S_\lambda(xy) = \sum_{\lambda=(\lambda_1,\lambda_2)} S_{\lambda_1}(x) \otimes S_{\lambda_2}(y), \quad x, y \in U^*(X) \tag{60}$$

this uniquely defines the operations $S_\lambda : U^*(X) \to U^*(X)$ for any CW-complex X.

The action of S_λ on the cobordism class of U-manifold M^{2n} can be given as

$$S_\lambda([M^{2n}]) = p_\sharp(c^U_\lambda(\nu M^{2n})), \tag{61}$$

where $p_\sharp : U^*(M^{2n}) \to \Omega_U$ is the Gysin homomorphism for $p : M^{2n} \to pt$ and $c^U_\lambda(\nu M^{2n}) \in U^{2|\lambda|}(M^{2n})$ are the Conner-Floyd characteristic classes of the normal bundle of M^{2n} [15].

In particular, when $|\lambda| = n$ we have $c^U_\lambda(\nu M^{2n}) \in U^{2n}(M^{2n}) = \mathbb{Z}$ and

$$S_\lambda([M^{2n}]) = c^\nu_\lambda(M^{2n}) \tag{62}$$

are the characteristic numbers (20).

In this paper we consider the Landweber-Novikov algebra as the graded algebra S over $\mathbb{Q}$ with a special basis $S_\lambda, \lambda \in \mathcal{P}$, where the degree of S_λ equals to $2|\lambda| = 2(i_1 + \dots + i_k)$ and the identity is defined as $S_\emptyset$. It is also a Hopf algebra with the diagonal given by

$$\triangle S_\lambda = \sum_{\lambda=(\lambda_1,\lambda_2)} S_{\lambda_1} \otimes S_{\lambda_2}. \tag{63}$$

The diagonal is symmetric, so the dual graded Hopf algebra $S^* = Hom(S, \mathbb{Q})$ is commutative.

The following result, which can be extracted from [30, 43], plays an important role in constructions from [8].

Proposition 4.1. *There is a canonical isomorphism of algebras*

$$\sigma : S^* \cong \Omega_U \otimes \mathbb{Q}. \tag{64}$$

Proof. Let $\mu : U^*(X) \to H^*(X, \mathbb{Z})$ be the *cycle realisation homomorphism*, which is defined uniquely by the property $\mu(u) = z$, where as before $u \in U^2(\mathbb{C}P^\infty)$ and $z \in H^2(\mathbb{C}P^\infty, \mathbb{Z})$ be the first Chern classes of the universal line bundle η over $\mathbb{C}P^\infty$ in the complex cobordisms and cohomology theory respectively. In particular, when $X = pt$ is a point we have the *augmentation ring homomorphism* $\mu : \Omega_U \to \mathbb{Z}$, which is 1 on the identity and 0 on elements with non-zero grading.

We have the canonical graded pairing between Landweber-Novikov algebra S and $\Omega_U \otimes \mathbb{Q}$:

$$\phi : S \otimes (\Omega_U \otimes \mathbb{Q}) \to \mathbb{Q}, \quad S_\lambda \otimes [M] \to \mu(S_\lambda([M])). \tag{65}$$

which is known to be non-degenerate [43]. This implies the canonical isomorphism $\sigma : S^* \cong \Omega_U \otimes \mathbb{Q}$, which is also the algebra isomorphism as it follows from (60) and (63). $\square$

A natural question is what is the basis $S^\lambda \in \Omega_U \otimes \mathbb{Q}$ dual to the Landweber-Novikov operations S_λ, $\lambda \in \mathcal{P}$. We can now give an explicit answer to this question.

Theorem 4.2. *The dual basis to Landweber-Novikov operations S_λ, $\lambda \in \mathcal{P}$ in $\Omega_U \otimes \mathbb{Q}$ is given by*

$$\sigma(S^\lambda) = \frac{[\Theta^\lambda]}{(\lambda + 1)!}, \tag{66}$$

where Θ^λ is the product of theta divisors (7).

Proof. Indeed, from formula (62) and Lemma 3.2 we have $S_\lambda([\Theta^n]) = 0$ for any partition λ of n different from $\lambda = (n)$, and $S_{(n)}([\Theta^n]) = (n+1)!$. Now the claim follows from the multiplicativity property of the Landweber-Novikov operations. $\square$

We can now describe the action of the Landweber-Novikov operations on the theta divisors and to prove Theorem 1.2.

Let $[\Theta_k^{n-k}]$ be the cobordism class of the intersection divisor (11).

Theorem 4.3. *For the one-part partitions $\lambda = (k)$, $k \leq n$ we have*

$$S_{(k)}([\Theta^n]) = [\Theta_k^{n-k}], \tag{67}$$

while for all other partitions $S_\lambda[\Theta^n] = 0$.

Proof. Recall that the normal bundle $\nu\Theta^n$ can be identified with $\mathcal{L} = i^*(L)$, where $i : \Theta^n \to A^{n+1}$ and L is the principal polarisation bundle of A^{n+1}. Since $\mathcal{L}$ is of rank one from formula (61) and properties of Conner-Floyd characteristic classes it immediately follows that $S_\lambda([\Theta^n]) = 0$ if λ is not a one-part partition.

For the proof of formula (67) we need the following properties of the Gysin homomorphism $f_\sharp : U^*(M_1) \to U^*(M_2)$ for any mapping of U-manifolds $f : M_1 \to M_2$ (see e.g. Proposition D.3.6 in [11]):

$$f_\sharp(x f^*(y)) = f_\sharp(x) y, \quad x \in U^*(M_1),\, y \in U^*(M_2) \tag{68}$$

and for $f : M_1 \to M_2$ and $g : M_2 \to M_3$ we have $(gf)_\sharp = g_\sharp f_\sharp$.

Let $[1] \in U^0(\Theta^n)$ be the cobordism class Poincaré-Atiyah dual to the identity map of Θ^n to itself, $[D] \in U^2(A^{n+1})$ be the cobordism class dual to the bordism class $\{\Theta^n\} \in U_{2n}(A^{n+1})$ defined by the embedding $i : \Theta^n \to A^{n+1}$, then by construction $i_\sharp([1]) = [D]$. Note that $[D] = c_1^U(L)$ is the first Conner-Floyd class of the line bundle L.

Now for the one-part partition $\lambda = (k)$, $k \leq n$ we have

$$c^U_{(k)}(\nu\Theta^n) = c^U_{(k)}(\mathcal{L}) = c^U_{(k)}(i^*(L)) = i^*(c_1^U(L)^k) = i^*[D]^k.$$

By (61) and the properties of Gysin homomorphisms

$$S_{(k)}([\Theta^n]) = p_\sharp(i^*[D]^k) = f_\sharp i_\sharp(i^*[D]^k),$$

where f and $p = f \circ i$ are the mappings to a point of A^{n+1} and Θ^n respectively. Since [1] is the identity in the ring $U^*(\Theta^n)$ we have using property (68) with $x = [1]$ and $y = [D]^k$ that

$$i_\sharp(i^*[D]^k) = i_\sharp([1] \cdot i^*[D]^k) = i_\sharp([1])[D]^k = [D]^{k+1},$$

and thus $S_{(k)}([\Theta^n]) = f_\sharp([D]^{k+1})$. By the definition of the Gysin homomorphism the cobordism class $f_\sharp([D]^{k+1})$ is dual to the transversal intersection of $k+1$ copies of cycles dual to $[D]$, which can be realised as the intersection of generally shifted theta-divisors (11). This means that $f_\sharp([D]^{k+1}) = [\Theta_k^{n-k}]$, which proves the theorem. □

Let us introduce now the *quantum complex cobordism theory* as the extraordinary cohomology theory $U^\star := U^* \otimes S^*$ with $U^\star(pt) = \Omega^\star := \Omega_U \otimes S^*$, where the tensor product is considered over $\mathbb{Z}$. The algebra S^* is used here as the deformation parameter space.

Consider the following *quantisation map* $q^\star : U^*(X) \to U^*(X) \otimes S^*$ inspired by [9]:

$$q^\star(x) = x \otimes 1 + \sum_\lambda S_\lambda(x) \otimes S^\lambda. \tag{69}$$

In particular, if $X = pt$ then $U^*(X) = \Omega_U$ and for any U-manifold M^{2n} we have

$$q^\star(M^{2n}) = [M^{2n}]\otimes 1 + \sum_{\lambda:|\lambda|<n} S_\lambda([M^{2n}])\otimes S^\lambda + 1\otimes \sum_{\lambda:|\lambda|=n} c^\nu_\lambda(M^{2n})\otimes S^\lambda. \tag{70}$$

Here we have used that when $|\lambda| = n$ we have $S_\lambda([M^{2n}]) = c^\nu_\lambda(M^{2n})$ (see Novikov [43]).

Introduce also the following quantum analogue of cycle realisation homomorphism as

$$\mu^\star : U^*(X) \otimes S^* \to H^*(X, \Omega_U \otimes \mathbb{Q}), \quad \mu^\star = \mu \otimes \sigma, \tag{71}$$

where we used the natural isomorphism $H^*(X, \mathbb{Q}) \otimes \Omega_U \cong H^*(X, \Omega_U \otimes \mathbb{Q})$. It can be viewed as a kind of "dequantisation" map.

We claim that the Chern-Dold character in complex cobordisms is the composition of the quantisation and dequantisation maps:

$$U^*(X) \xrightarrow{q^\star} U^*(X) \otimes S^* \xrightarrow{\mu^\star} H^*(X, \Omega_U \otimes \mathbb{Q}).$$

Theorem 4.4. *The quantisation map defined by (69) is the algebra homomorphism. The Chern-Dold character* $ch_U : U^*(X) \to H^*(X, \Omega_U \otimes \mathbb{Q})$ *is the composition*

$$ch_U = \mu^\star \circ q^\star. \tag{72}$$

Proof. The algebra homomorphism property $q^\star(xy) = q^\star(x)q^\star(y)$ follows from the multiplicative property (60) of the Landweber-Novikov operations.

To prove the rest it is enough to check (72) only for $u = c_1^U(\eta) \in U^2(\mathbb{C}P^\infty)$. In that case by definition

$$q^\star(u) = u \otimes 1 + \sum_\lambda S_\lambda(u) \otimes S^\lambda = u \otimes 1 + \sum_{n\in\mathbb{N}} S_{(n)}(u) \otimes S^{(n)},$$

since u is geometric. But $S_{(n)}(u) = u^{n+1}$, so $q^\star(u) = u \otimes 1 + \sum_{n\in\mathbb{N}} u^{n+1} \otimes S^{(n)}$. Now we use that $\mu(u) = z = c_1(\eta)$ and the previous theorem saying that $\sigma(S^{(n)}) = \frac{[\Theta^n]}{(n+1)!}$ to conclude that

$$\mu^\star \circ q^\star(u) = z + \sum_{n\in\mathbb{N}} z^{n+1} \frac{[\Theta^n]}{(n+1)!} = ch_U(u)$$

due to Theorem 3.5. □

Corollary 4.5. *The Chern-Dold character in complex cobordisms can be written as*

$$ch_U(x) = \mu(x) + \sum_{\lambda} \mu(S_\lambda x) \frac{[\Theta^\lambda]}{(\lambda+1)!}, \quad x \in U^*(X) \tag{73}$$

where $\mu : U^*(X) \to H^*(X, \mathbb{Z})$ is the cycle realisation homomorphism.

For $X = pt$ and $x = [M^{2n}] \in \Omega_U$ we have

$$ch_U([M^{2n}]) = \sum_{\lambda:|\lambda|=n} c_\lambda^\nu(M^{2n}) \frac{[\Theta^\lambda]}{(\lambda+1)!} = [M^{2n}] \tag{74}$$

in agreement with (1) and (6).

Note that the composition of the quantisation and dequantisation maps for $X = pt$ is non surprisingly the identical map, but leads to a non-trivial formula (74).

According to general construction of Dold [17] every extraordinary cohomology theory h^* has its analogue of Chern character, which is the transformation of cohomology theories

$$ch_h : h^*(X) \to H^*(X, \Omega_h \otimes \mathbb{Q}), \quad \Omega_h = h^*(pt)$$

with characteristic property that for $X = pt$ it is the canonical homomorphism $\Omega_h \to \Omega_h \otimes \mathbb{Q}$.

In our case of quantum complex cobordism theory $U^\star$ the corresponding *quantum Chern-Dold character* $ch_U^\star : U^\star(X) = U^* \otimes S^* \to H^*(X, \Omega^\star)$ has the following explicit form

$$ch_U^\star(x \otimes s) = ch_U(x) \otimes s + \sum_\lambda S_\lambda(ch_U(x)) \otimes S^\lambda s, \quad x \in U^*(X),\ s \in S^*. \tag{75}$$

Let us deduce now our formula (13).

Theorem 4.6. *The cobordism class of Θ_k^{n-k} can be given as the residue integral at zero*

$$[\Theta_k^{n-k}] = \frac{(n+1)!}{2\pi i} \oint \beta(z)^{k+1} \frac{dz}{z^{n+2}},$$

where $\beta(z)$ is given by (5).

Proof. We use the fact that the Chern-Dold character $ch_U : U^*(X) \to H^*(X, \Omega_U \otimes \mathbb{Q})$ commutes with the action of the Landweber-Novikov algebra, which is trivial on the cohomology $H^*(X, \mathbb{Q})$ (cf. [6]).

Applying Landweber-Novikov operation S_λ to both sides of the relation (41) we have $S_\lambda ch_U(u) = ch_U(S_\lambda u) = \beta(z)^{k+1}$ if $\lambda = (k)$ for some $k \in \mathbb{N}$ and 0 otherwise. On the right hand side we have

$$\sum_{n=k}^{\infty} S_{(k)}[\Theta^n]\frac{z^{n+1}}{(n+1)!} = \sum_{n=k}^{\infty}[\Theta_k^{n-k}]\frac{z^{n+1}}{(n+1)!},$$

and zero for any $\lambda \neq (k)$. Comparison proves the claim and formula (13). □

Corollary 4.7. *The cobordism class* $[\Theta_k^{n-k}]$ *is a polynomial of* $[\Theta^1], \dots, [\Theta^{n-k}]$ *with positive integer coefficients.*

The polynomial subring $\Theta_U \subset \Omega_U$ *generated by the theta divisors:*

$$\Theta_U = \mathbb{Z}[t_1, \dots, t_n, \dots],\ t_k = [\Theta^k],\ k \in \mathbb{N},$$

is invariant under the Landweber-Novikov operations.

Indeed, since $\beta(z)$ is an exponential generating function, this follows from the properties of Hurwitz series [26], see also problems 174-177 in [45], Vol. 2, Ch. 8. In particular,

$$[\Theta_1^{n-1}] = \sum_{k=0}^{n-1}\binom{n+1}{k+1}[\Theta^k][\Theta^{n-k-1}], \quad [\Theta_{n-1}^1] = \frac{n(n+1)}{2}n!\,[\Theta^1].$$

The following important interpretation of the Landweber-Novikov algebra was found by Buchstaber and Shokurov [8].

Consider the group $Diff_1$ of the formal diffeomorphisms of the line given by

$$f(x) = x + \sum_{k\in\mathbb{N}} \alpha_k x^{k+1}, \quad x, \alpha_k \in \mathbb{R}.$$

Its Lie algebra $\mathfrak{diff}_1$ is the Lie algebra of the corresponding formal vector fields.

Theorem 4.8. *(Buchstaber and Shokurov* [8]*) The real version* $S\otimes\mathbb{R}$ *of the Landweber-Novikov algebra is isomorphic to the universal enveloping algebra of the Lie algebra* $\mathfrak{diff}_1$*, which can be identified with the algebra of the left-invariant differential operators on the group* $Diff_1$*.*

In particular, as an algebra S is generated by two Landweber-Novikov operations $S_{(1)}$ and $S_{(2)}$, corresponding to two vector fields on the group $Diff_1$, which can be written in coordinates α_k, $k \in \mathbb{N}$ as

$$S_{(1)} = \frac{\partial}{\partial\alpha_1} + \sum_{k=2}^{\infty} k\alpha_{k-1}\frac{\partial}{\partial\alpha_k},\ S_{(2)} = \frac{\partial}{\partial\alpha_2} + \sum_{k=3}^{\infty}(k-1)\alpha_{k-2}\frac{\partial}{\partial\alpha_k}. \quad (76)$$

The algebra A^U of all cohomological operations in complex cobordisms was introduced in [43] and is known as *Novikov algebra.* It is $\mathbb{Z}$-graded algebra, which is a free left Ω_U-module with generators S_λ and the commutation relations

$$S_\lambda \cdot x = \sum_{\lambda=(\lambda_1,\lambda_2)} S_{\lambda_1}(x)S_{\lambda_2}, \quad x \in \Omega_U \quad (77)$$

(see Lemma 5.4 in Novikov [43]).

Let us introduce two new families of elements of Novikov algebra

$$T_{\lambda,\mu} = [\Theta^\lambda]S_\mu, \quad V_{\lambda,\mu} = [V^\lambda]S_\mu,\ \lambda, \mu \in \mathcal{P}, \quad (78)$$

where for partition $\lambda = (\lambda_1, \dots, \lambda_k)$ we define $V^\lambda := V^{2\lambda_1} \times \dots \times V^{2\lambda_k}$ with V^{2n} being U-manifolds (56) from the previous section.

From our results it follows that they give additive bases in two proper subalgebras of Novikov algebra, generating this algebra over $\mathbb{Q}$. Our Theorem 1.2, Corollary 4.7 and Lemma 3.9 can be used to describe the multiplication in these bases, which could be useful to study the representations of Novikov algebra.

The important elements of Novikov algebra, motivated by the Conner and Floyd results [15], describing the K-theory in terms complex cobordisms, are the *Adams-Novikov operations* Ψ_U^k (see [43]). They are defined uniquely as the multiplicative operations in complex cobordisms, which satisfy the relation

$$\Psi_U^k(u) = \frac{1}{k}\beta(k\beta^{-1}(u)), \quad u = c_1^U(\eta) \in U^2(\mathbb{C}P^\infty). \tag{79}$$

For any cobordism class $[M^{2n}]$ of U-manifold M^{2n} we have

$$\Psi_U^k([M^{2n}]) = k^n[M^{2n}]. \tag{80}$$

Let $\Theta^n(k)$ be a smooth zero locus of generic section of the k-th tensor power L^k of the line bundle L, defining the principal polarisation of an abelian variety A^{n+1}. Note that for $k \geq 2$ the zero locus of generic section of L^k is smooth for any abelian variety by Bertini theorem. We would like to mention that the subvarieties $\Theta^n(k) \subset A^{n+1}$ have appeared in [37] as the spectral varieties of certain commutative algebras of differential operators.

The cohomology and the corresponding cobordism class $[\Theta^n(k)]$ do not depend on the choice of such section. As for the theta divisor, by Lefschetz hyperplane theorem the corresponding Betti numbers are

$$b_j(\Theta^n(k)) = b_j(A^{n+1}) = \binom{2n+2}{j} = b_{2n-j}(\Theta^n(k)), \ j < n,$$

except the middle one b_n, which can be found using the formula for the Euler characteristic $\chi(\Theta^n(k)) = (-1)^n k^{n+1}(n+1)!$:

$$b_n(\Theta^n(k)) = k^{n+1}(n+1)! + \frac{n}{n+2}\binom{2n+2}{n+1}. \tag{81}$$

The signature of $\tau(\Theta^n(k) = k^{n+1}\tau(\Theta^n)$, so for even n we have

$$\tau(\Theta^n(k)) = \frac{2^{n+2}(2^{n+2}-1)k^{n+1}}{n+2}B_{n+2}, \tag{82}$$

where B_n are the Bernoulli numbers.

Theorem 4.9. *The cobordism class* $[\Theta^n(k)]$ *can be expressed as*

$$[\Theta^n(k)] = k^{n+1}[\Theta^n] = k\Psi_U^k([\Theta^n]). \tag{83}$$

Proof. The normal bundle to $\Theta^n(k)$ can be identified with $\mathcal{L}^k = i^*L^k$ induced from L^k by the embedding $i : \Theta^n(k) \to A^{n+1}$. According to formula (6) we have $[\Theta^n(k)] = c^\nu_{(n)}(\Theta^n(k))\frac{[\Theta^n]}{(n+1)!}$, so we need only to compute $c^\nu_{(n)}(\Theta^n(k))$.

The characteristic class $c_{(n)}(\nu\Theta^n(k)) = c_1^n(\mathcal{L}^k) = i^*c_1^n(L^k) = k^n i^*c_1^n(L)$, so we have $c^\nu_{(n)}(\Theta^n(k)) = (c_{(n)}(\nu\Theta^n(k)), \langle\Theta^n(k)\rangle) = k^n(i^*c_1^n(L), \langle\Theta^n(k)\rangle)$ $= k^n(c_1^n(L), i_*\langle\Theta^n(k)\rangle) = k^{n+1}(c_1^{n+1}(L), \langle A^{n+1}\rangle) = k^{n+1}(n+1)!$, which together with (80) proves the claim. □

5. Real-analytic elliptic representatives

Consider now the most degenerate case of the abelian variety, when A^{n+1} is the product of $n+1$ copies of an elliptic curve $\mathcal{E} = \mathbb{C}/\Gamma$, where Γ is the lattice with periods $2\omega_1, 2\omega_2$ (in the classical notations from [54]).

Let L be the canonical line bundle on it with the holomorphic section

$$S_0(u) = \sigma(u_1)\dots\sigma(u_{n+1}), \quad u = (u_1, \dots, u_{n+1}) \in \mathcal{E}^{n+1},$$

where σ is the classical Weierstrass sigma function [54]. The corresponding theta divisor defined by $S_0(u) = 0$ is the union of $n+1$ coordinate hypersurfaces given by $u_i = 0$. It is singular, but one might try to find a smooth algebraic representative in the same homology class. We claim that this is not possible.[1]

Consider a slightly more general case of the product $A^{n+1} = \mathcal{E}_1 \times \dots \times \mathcal{E}_{n+1}$ of $n+1$ elliptic curves.

Theorem 5.1. *There are no smooth algebraic representatives in homology class of the theta divisor of this product.*

Proof. In this case the theta divisor is the union $\Theta^n = \Theta_1 \cup \dots \cup \Theta_{n+1}$, where each Θ_i is the pullback to A^{n+1} of a point on $\mathcal{E}_i$.

Now let $p : A^{n+1} \to \mathcal{E}_1 \times \dots \times \mathcal{E}_n$ be the projection onto the product of all but last factor. So the fibres of p are just copies of $\mathcal{E}_{n+1}$. Let F be the class of a fibre, then we have that the intersection number $\Theta^n \cdot F = 1$.

Now suppose there is a smooth codimension 1 subvariety D in A^{n+1} which is homologous to Θ^n, then we must have $D \cdot F = 1$ as well. This means that D meets a general fibre transversely in 1 point. So the restriction of the projection p to D gives a birational morphism $p' : D \to \mathcal{E}_1 \times \dots \times \mathcal{E}_n$. But if $f : Y \to Z$ is a birational morphism between smooth projective varieties, then either it is an isomorphism, or it has an exceptional divisor, which is covered by rational curves (see Proposition 4 in [1]).

However in our case, D is a subset of the abelian variety A^{n+1}, and an abelian variety cannot contain any rational curves. So p' must be an isomorphism, in particular D is an abelian variety of dimension n.

By the adjunction formula the normal bundle of D is trivial, so that the intersection number $D^{n+1} = 0$. On the other hand, it is easy to see that the corresponding self-intersection number of the homology class of the theta divisor Θ^n is $(n+1)!$, which is a contradiction. □

Remark 5.2. *The same arguments work for a product $A^{n+1} = A^n \times \mathcal{E}$ of some principally polarised abelian variety A^n and elliptic curve $\mathcal{E}$. Most likely, the claim is true for any product $A^{n+1} = A_1^k \times A_2^l$ of two principally polarised abelian varieties, but this is still to be proved.*

Consider now the classical Weierstrass zeta function $\zeta(z)$ with simple poles at the lattice points and the transformation properties

$$\zeta(z + 2\omega_1) = \zeta(z) + 2\eta_1, \quad \zeta(z + 2\omega_2) = \zeta(z) + 2\eta_2,$$

where $\eta_i = \zeta(\omega_i),\ i = 1, 2$, see [54].

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[1] We are very grateful to Artie Prendergast-Smith for providing a proof of this for us.

Introduce the following non-holomorphic function (inspired by the theory of periodic vortices [23, 53])

$$\xi(z) = \zeta(z) + az + b\bar{z}, \tag{84}$$

where a, b is the unique solution of the following linear system

$$a\omega_1 + b\overline{\omega}_1 + \eta_1 = 0, \quad a\omega_2 + b\overline{\omega}_2 + \eta_2 = 0, \tag{85}$$

or, explicitly

$$a = -\frac{\eta_1\overline{\omega}_2 - \eta_2\overline{\omega}_1}{\omega_1\overline{\omega}_2 - \omega_2\overline{\omega}_1} = -\frac{\eta_1\overline{\omega}_2 - \eta_2\overline{\omega}_1}{2\Im\left(\omega_1\overline{\omega}_2\right)}, \quad b = \frac{\eta_1\omega_2 - \eta_2\omega_1}{\omega_1\overline{\omega}_2 - \omega_2\overline{\omega}_1} = \frac{\pi}{4\Im\left(\omega_1\overline{\omega}_2\right)},$$

where we have used the Legendre identity [54]

$$\eta_1\omega_2 - \eta_2\omega_1 = \frac{\pi i}{2}.$$

Lemma 5.3. *The function ξ is an odd doubly-periodic, complex-valued, real-analytic, harmonic function with the asymptotic behaviour $\xi(z) = 1/z + \mathcal{O}(z)$ at zero and with zeros at all 3 half periods $\omega_1, \omega_2, \omega_3 = \omega_1 + \omega_3$.*

In the lemniscatic case with $\omega_1 = \omega \in \mathbb{R}$, $\omega_2 = i\omega$ we have

$$\xi(z) = \zeta(z) - \frac{\pi}{4\omega^2}\bar{z}, \tag{86}$$

which has zeros precisely at the 3 half-periods.

Proof. The property $\xi(-z) = -\xi(z)$ follows from the same property of $\zeta(z)$. The double-periodicity follows from the transformation properties of function ζ :

$$\xi(z + 2\omega_1) = \xi(z) + 2\eta_1 + 2a\omega_1 + 2b\overline{\omega}_1 = \xi(z),$$
$$\xi(z + 2\omega_2) = \xi(z) + 2\eta_2 + 2a\omega_2 + 2b\overline{\omega}_2 = \xi(z)$$

due to (85). Combining these two properties we have $\xi(\omega_i) = -\xi(-\omega_i) = -\xi(\omega_i)$, so $\xi(\omega_i) = 0$ for all half-periods.

In the lemniscatic case with $\omega_1 = \omega \in \mathbb{R}$, $\omega_2 = i\omega$ we have $\zeta(iz) = -i\zeta(z)$ and thus $\eta_2 = -i\eta_1$, $\eta_1 \in \mathbb{R}$. Corresponding $\wp$-function satisfies equation

$$(\wp')^2 = 4\wp(\wp^2 - e^2), \quad e = \wp(\omega) = \frac{\Gamma^4(1/4)}{32\pi\omega^2},$$

where Γ is the classical Euler's Γ-function.

From Legendre identity we have $i\omega\eta_1 - \omega\eta_2 = 2i\omega\eta_1 = \frac{\pi i}{2}$, so in this case

$$\eta_1 = \frac{\pi}{4\omega}, \ \eta_2 = -i\frac{\pi}{4\omega},$$

which gives $a = 0, \ b = -\frac{\pi}{4\omega^2}$ and the relation (86).

Note that the function $\xi(z)$ satisfies the equation $\partial\bar{\partial}\xi(z, \bar{z}) = 0$ and thus is indeed a complex-valued harmonic function. The zeros of such functions were extensively studied, see [16] and references therein.

The number of the zeros of such functions depends on the position of zero in relation with the caustic defined as the image $\xi(\Sigma) \subset \mathbb{C}$ of the critical set

$$\Sigma := \{z \in \mathbb{C} : J_\xi(z, \bar{z}) = 0\},$$

where J_ξ is the Jacobian of the map $\xi : \mathbb{R}^2 \to \mathbb{R}^2$. The real Jacobian of the harmonic function $f(z, \bar{z}) = g(z) + h(\bar{z})$ with holomorphic g, h is

$$J_f(z, \bar{z}) = |g'(z)|^2 - |h'(z)|^2.$$

Thus in our case the critical set is

$$\Sigma := \{z \in \mathbb{C} : |\wp(z)| = \frac{\pi}{4\omega^2}\}.$$

Note that this level of the real function $F(z,\bar{z}) = |\wp(z)|$ is non-singular. Indeed, if $\wp'(z) = 0$ then z is a half-period and thus $\wp(z) = 0$, or $\wp(z) = \pm e$. Since $3.6 < \Gamma(1/4) < 3.7$ we have

$$e = |\wp(z)| = \frac{\Gamma^4(1/4)}{32\pi\omega^2} > \frac{\pi}{4\omega^2}.$$

Thus by the general theory [16] the equation $\xi(z) = c$ has one solution if c lies outside the caustic and 3 solutions if c is inside the caustic. Since $c = 0$ is inside the equation $\xi(z) = 0$ has 3 solutions, which are precisely the half-periods.

Note that the corresponding Jacobians at the half-periods $\omega_1 = \omega$, $\omega_2 = i\omega$ are

$$J(z,\bar{z}) = |\wp(\omega)|^2 - \left(\frac{\pi}{4\omega^2}\right)^2 = \left(\frac{\Gamma^4(1/4)}{32\pi\omega^2}\right) - \left(\frac{\pi}{4\omega^2}\right)^2 > 0,$$

while at $\omega_3 = \omega + i\omega$ we have

$$J(z,\bar{z}) = |\wp(\omega_3)|^2 - \left(\frac{\pi}{4\omega^2}\right)^2 = -\left(\frac{\pi}{4\omega^2}\right)^2 < 0,$$

so the sum of the indices is 1, as it is expected since the degree of the map $\xi : \mathcal{E} \to \mathbb{C}P^1$ is 1. □

Let $I \subset [n+1] := \{1, 2, \ldots, n+1\}$ be a finite subset and denote

$$\xi_I(u) := \prod_{i \in I} \xi(u_i), \quad u \in \mathcal{E}^{n+1},$$

where for empty subset $I = \emptyset$ we set $\xi_I(u) \equiv 0$.

Consider now the following family of the non-holomorphic (but real-analytic) sections of the complex line bundle L given by

$$S(u,a) = S_0(u) + S_0(u) \sum_{I,J \subset [n+1], I \cap J = \emptyset} a_{IJ}(\xi_I(u) + \xi_J(u)) \qquad (87)$$

with arbitrary coefficients $a_{IJ} = a_{JI} \in \mathbb{C}$, assuming that both subsets I and J cannot be empty simultaneously.

Theorem 5.4. *For generic coefficients a_{IJ} the zero locus of this section*

$$\mathcal{M}_W(a) = \{u \in \mathcal{E}^{n+1} : S(u,a) = 0\} \subset \mathcal{E}^{n+1}$$

is a smooth connected real-analytic U-manifold, which can be used as a representative of the cobordism class of the coefficient $[\mathcal{B}^n]$ in the Chern-Dold character.

Proof. Consider the set

$$\mathcal{M} = \{(u,a) : u \in \mathcal{E}^{n+1},\, a = (a_{IJ}) \in \mathbb{C}^N, S(u,a) = 0\} \subset \mathcal{E}^{n+1} \times \mathbb{C}^N.$$

We claim that this is a smooth submanifold of this product. Indeed, assume that

$$\frac{\partial S}{\partial a_{IJ}} = S_0(u)(\xi_I(u) + \xi_J(u)) = 0$$

for all pairs of non-intersecting subsets $I, J \subset [n+1]$. Then, in particular, we have

$$\prod_{i=1}^{n+1} \sigma(u_i) = 0, \ \prod_{i=1}^{n+1} \sigma(u_i)\xi(u_i) = 0,$$

so some of the coordinates of the potential singularities equal to 0 and some to a half-period. Let

$$I = \{i \in [n+1] : \sigma(u_i) = 0\}, \ J = \{j \in [n+1] : \xi(u_i) = 0\}, \ I \cap J = \emptyset,$$

then the corresponding $\frac{\partial S}{\partial a_{IJ}} = S_0(u)(\xi_I(u) + \xi_J(u)) = S_0(u)\xi_I(u) \neq 0$, since $\sigma(z)\xi(z) = 0$ only at half-periods. The contradiction means that the submanifold $\mathcal{M}$ is indeed non-singular. Now the smoothness of $\mathcal{M}_W(a)$ follows from Sard's Lemma, saying that the set of critical values of the natural projection $\pi : \mathcal{M} \to \mathbb{C}^N$ has measure zero.

Since both $\mathcal{M}_W(a)$ and Θ^n are the smooth zero loci of two sections of the same line bundle, they have the same cobordism class. Now the claim follows from Theorem 1.1. □

Note that Theorem 4.9 implies that any multiple of the theta divisor $k^{n+1}[\Theta^n] = [\Theta^n(k)]$, $k \geq 2$ does admit an explicit smooth algebraic realisation inside $\mathcal{E}^{n+1}$ as zero locus of a generic section of the line bundle L^k.

For example, for $k = 2$ the space of sections of L^2 has dimension 2^{n+1} and is generated by

$$\Phi_\varepsilon(u) = \prod_{k=1}^{n+1} \phi_{\varepsilon_k}(u_k), \ u = (u_1, \dots, u_{n+1}) \in \mathcal{E}^{n+1}, \ \varepsilon = (\varepsilon_1, \dots, \varepsilon_{n+1}) \in \mathbb{Z}_2^{n+1},$$

$$\phi_0(z) = \sigma^2(z), \quad \phi_1(z) = \sigma^2(z+\omega)e^{-2\eta z}, \tag{88}$$

where ω is any half-period and $\eta = \zeta(\omega)$. Indeed, $\sigma(z)$ satisfies the following transformation properties under the shift by the periods [54]

$$\sigma(z + 2\omega_k) = -\sigma(z)e^{2\eta_k(z+\omega_k)}, \ k = 1, 2$$

which imply the transformation properties

$$\phi_\epsilon(z + 2\omega_k) = \phi_\epsilon(z)e^{4\eta_k(z+\omega_k)}, \ \epsilon = 0, 1, \ k = 1, 2.$$

For $\phi_0(z) = \sigma^2(z)$ this is straightforward, while for $\phi_1(z)$ this follows from the Legendre identity. From the results of the previous section we obtain

Proposition 5.5. *For generic coefficients c_ε, $\varepsilon \in \mathbb{Z}_2^{n+1}$ the zero locus*

$$\Phi_c(u) := \sum_{\varepsilon \in \mathbb{Z}_2^{n+1}} c_\varepsilon \Phi_\varepsilon(u) = 0, \quad u = (u_1, \dots, u_{n+1}) \in \mathcal{E}^{n+1} \tag{89}$$

is a smooth irreducible algebraic variety, representing the cobordism class $2^{n+1}[\Theta^n] = [\Theta^n(2)]$.

6. Discussion: Milnor-Hirzebruch problem

The Milnor-Hirzebruch problem was first posed by Hirzebruch in his ICM-1958 talk [24]. Its algebraic version can be formulated in our notations as follows:

Which sets of $p(n)$ integers c_λ, $\lambda \in \mathcal{P}_n$ can be realised as the Chern numbers $c_\lambda(M^n)$ of some smooth irreducible complex algebraic variety M^n?

In this version it still remains largely open, although some arithmetic restrictions are known since the work of Milnor and Hirzebruch.

In particular, in the (complex) dimension $n = 1, 2, 3$ we have the following congruences for the usual Chern numbers of any almost complex manifold (see Hirzebruch [24], section 7):

$$n = 1: \quad c_1 \equiv 0 \mod 2, \qquad n = 2: \quad c_2 + c_1^2 \equiv 0 \mod 12,$$

$$n = 3: \quad c_1 c_2 \equiv 0 \mod 24, \quad c_3 \equiv c_1^3 \equiv 0 \mod 2.$$

$$n = 4: \quad -c_4 + c_1 c_3 + 3c_2^2 + 4c_1^2 c_2 - c_1^4 \equiv 0 \mod 720, \quad c_1^2 c_2 + 2c_1^4 \equiv 0 \mod 12,$$

$$-2c_4 + c_1 c_3 \equiv 0 \mod 4.$$

Since the total Chern class of Θ^n satisfies the relation (30) with $\mathcal{D}^n = (n+1)!$ all the characteristic numbers of Θ^n equal $\pm(n+1)!$.

In particular, for $n = 1$ we have $c_1 = -2$, for $n = 2 : c_1^2 = c_2 = 6$, for

$$n = 3 : c_1^3 = -c_1 c_2 = c_3 = -24,$$

$$n = 4 : c_1^4 = c_1^2 c_2 = c_1 c_3 = c_2^2 = c_4 = 120.$$

We see that the first Hirzebruch congruence in each case is sharp for the theta divisors, which means that it cannot be improved in the algebraic setting. This is related to the fact that the Todd genus $Td(\Theta^n) = (-1)^n$.

Recall that the divisibility conditions in terms of characteristic classes in K-theory were described by Hattori [22] and Stong [50, 51].

We can get now all divisibility conditions on the Chern numbers $c_\lambda^\nu(M^{2n})$ of U-manifolds in the following more effective way. We apply the Landweber-Novikov operations S_λ, $|\lambda| < n$ to our formula (6)

$$[M^{2n}] = \sum_{\lambda : |\lambda| = n} c_\lambda^\nu(M^{2n}) \frac{[\Theta^\lambda]}{(\lambda + 1)!}$$

and use that, according to Theorem 1.2, $S_\lambda[\Theta^n] = 0$ unless $\lambda = (k)$, $k < n$ when $S_{(k)}[\Theta^n] = [\Theta_k^{n-k}]$, which is a polynomial of $t_j = [\Theta^j], j = 1, \dots, n-k$ with positive integer coefficients. Substituting now $t_j = (-1)^j$ and demanding the result to be integer, we get all divisibility conditions on the Chern numbers $c_\lambda^\nu(M^{2n})$ of U-manifolds.

It is natural to ask if they can be improved for irreducible algebraic varieties. We plan to discuss this in the light of our results elsewhere.

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