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THE COHOMOLOGY OF THE MOD 2 STEENROD ALGEBRA DOI:10.11582/2021.00077

ROBERT R. BRUNER AND JOHN ROGNES

Abstract. A minimal resolution of the mod 2 Steenrod algebra in the range $0 \leq s \leq 128$, $0 \leq t \leq 184$, together with chain maps for each cocycle in that range and for the squaring operation Sq^0 in the cohomology of the Steenrod algebra.

This article describes the archived dataset [8], available for download at the NIRD Research Data Archive `https://archive.sigma2.no`. Please refer to the dataset and this article by their digital object identifier `DOI:10.11582/2021.00077`.

Contents

1. Introduction

Let $\mathcal{A}$ denote the classical mod 2 Steenrod algebra over $\mathbb{F}_2$. This archive contains

(1) a minimal resolution of $\mathbb{F}_2$ over $\mathcal{A}$ in internal degrees $t \leq 184$ and cohomological degrees $s \leq 128$,
(2) chain maps lifting each member in the resulting basis for $\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ in this range, and
(3) a chain map which gives the Hopf algebra squaring operation

$$Sq^0 : \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \mathrm{Ext}_{\mathcal{A}}^{s,2t}(\mathbb{F}_2, \mathbb{F}_2).$$

This document describes the files containing this information. The resolution was produced by the first author's software `ext`, version 1.9.3. This is contained in the file `ext.1.9.3.tar.gz`. The remaining contents of the top level directory are a copyright notice, a listing (`ls-lR.txt`) of the files herein, and a directory `A`.

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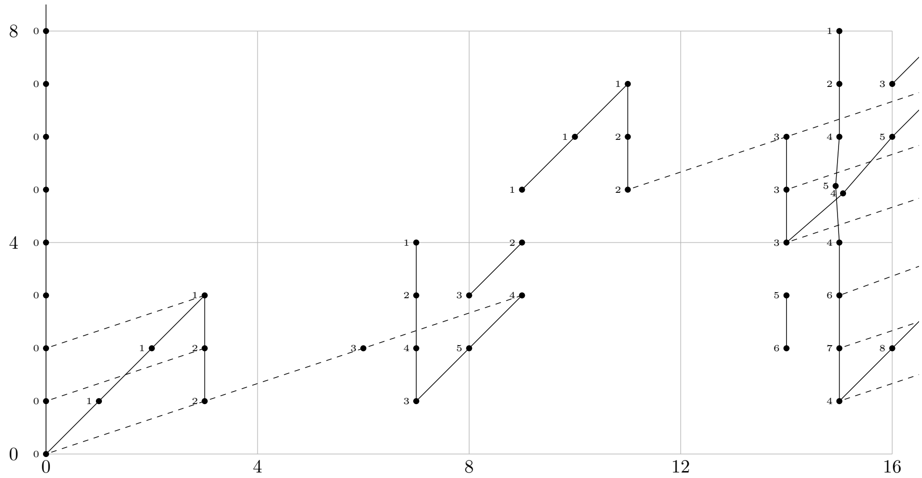


FIGURE 1. $\mathrm{Ext}_{\mathcal{A}}^{s,n+s}(\mathbb{F}_2,\mathbb{F}_2)$, $0 \le n \le 16$, $0 \le s \le 8$

The directory `A` has a subdirectory `S-184` which contains the data and a directory `src` containing the source code for a C program not included in `ext.1.9.3.tar.gz`. This program writes a script to create all possible cocycles.

The resolution together with the chain maps give $\mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_2,\mathbb{F}_2)$ as an algebra. It is worth pointing out that the minimal resolution and the chain maps contain vastly more information than this. Much secondary and higher order structure is available from this data, as well. The discussion of Toda brackets (Massey products) below, and the data contained in the file `himults` are examples: the file `himults` contains information about products by elements of cohomological degree 1 obtained without use of chain maps, while the Toda brackets files are produced from the chain maps without reference to any null-homotopies.

The package `ext` is designed to produce minimal resolutions

$$0 \longleftarrow M \xleftarrow{d_0} C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} \cdots \xleftarrow{d_S} C_S$$

and

$$0 \longleftarrow N \xleftarrow{d_0} D_0 \xleftarrow{d_1} D_1 \xleftarrow{d_2} \cdots \xleftarrow{d_S} D_S$$

for any finite $\mathcal{A}$-modules M and N and to lift cocycles $x \in \mathrm{Ext}_{\mathcal{A}}^{s_0,t_0}(M,N)$, represented as $\mathcal{A}$-homomorphisms $x : C_{s_0} \longrightarrow \Sigma^{t_0}N$, to chain maps $\{C_{s_0+s} \longrightarrow \Sigma^{t_0}D_s\}_s$. The exposition here is focused on the case $M = N = \mathbb{F}_2$, and the range $0 \le s \le 128$, $0 \le t \le 184$, but at various points it may be useful to remember the extra generality of the code.

2. The resolution

Let us write

$$0 \longleftarrow \mathbb{F}_2 \xleftarrow{d_0} C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} \cdots \xleftarrow{d_{128}} C_{128}$$

for our minimal resolution. It is significant that minimality allows us to identify $\mathrm{Hom}_{\mathcal{A}}^t(C_s,\mathbb{F}_2) = \mathrm{Hom}_{\mathcal{A}}(C_s,\Sigma^t\mathbb{F}_2)$ with $\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2,\mathbb{F}_2)$.

Since our resolution is limited to the range $0 \le s \le 128$ and $0 \le t \le 184$, it is simply an initial segment of a resolution. For brevity, we shall nonetheless call

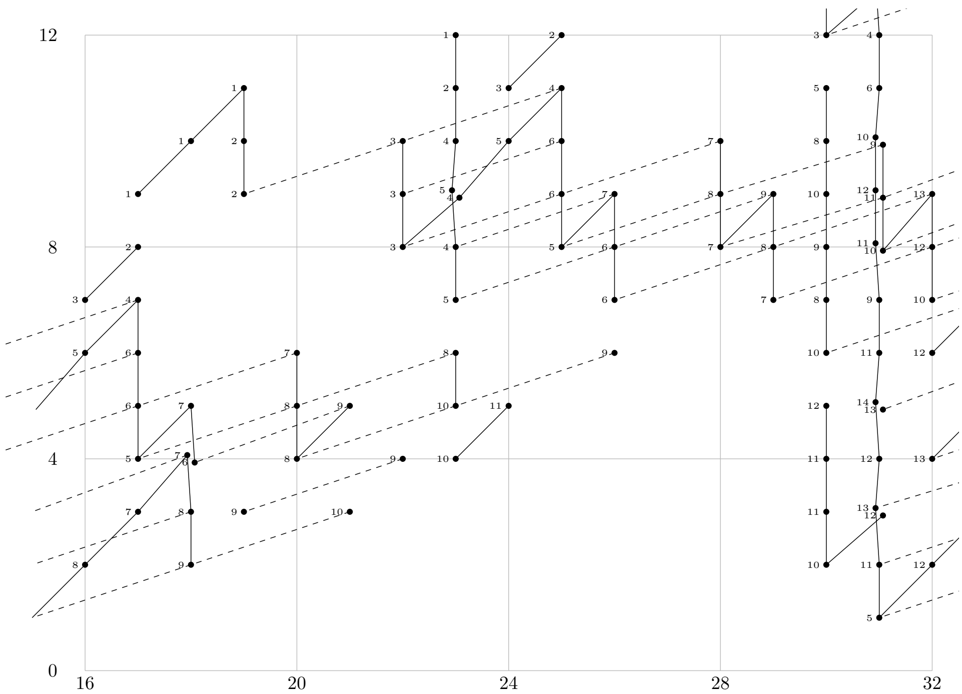


FIGURE 2. $\mathrm{Ext}_{\mathcal{A}}^{s,n+s}(\mathbb{F}_2, \mathbb{F}_2)$, $16 \le n \le 32$, $0 \le s \le 12$

it "the resolution" and write Ext for that part of $\mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ which lies in this range. The resolution is described by the following files.

(1) `Def`. This file defines the $\mathcal{A}$-module $\mathbb{F}_2$ as the module which is 1-dimensional over $\mathbb{F}_2$ with its sole generator in degree 0.
(2) `MAXFILT`. This contains the maximum cohomological degree, $S = 128$, through which the resolution is calculated.
(3) `Shape`. This file describes the $\mathcal{A}$-modules C_s. Its first entry, 128, gives the maximum cohomological degree S. This is followed by the 129 integers $\dim_{\mathcal{A}}(C_s)$ for $s = 0, 1, \ldots, 128$. This is followed by the internal degree of each of these generators, first for C_0, then for C_1, up to C_{128}. This data determines Ext as a bigraded $\mathbb{F}_2$ vector space.

   We write `s_g` or s_g for the cocycle dual to the *0-indexed* g^{th} generator of C_s. Thus, $0_0 \in \mathrm{Ext}_{\mathcal{A}}^{0,0}(\mathbb{F}_2, \mathbb{F}_2)$ is the unit $1 : C_0 \longrightarrow \mathbb{F}_2$ dual to the $\mathcal{A}$-module generator of C_0, while $1_i \in \mathrm{Ext}_{\mathcal{A}}^{1,2^i}(\mathbb{F}_2, \mathbb{F}_2)$ is the "Hopf map" $h_i : C_1 \longrightarrow \Sigma^{2^i}\mathbb{F}_2$ dual to the $\mathcal{A}$-module generator of C_1 which d_1 sends to Sq^{2^i}. When we need to refer to the $\mathcal{A}$-module generators of C_s, we shall write them as s_g^* or `s_g*`.
(4) `Diff.s` and `hDiff.s` for each s, $0 \le s \le 128$. These files contain the differentials d_s for $0 \le s \le 128$. The file `Diff.s` contains the elements $d_s(s_g^*)$, for $g = 0, 1, \ldots$ in that order. Each is preceded by its internal degree. The file `Diff.s` is written in a condensed format which takes the least space possible for the formats legible to the program `ext`. The files `hDiff.s` are

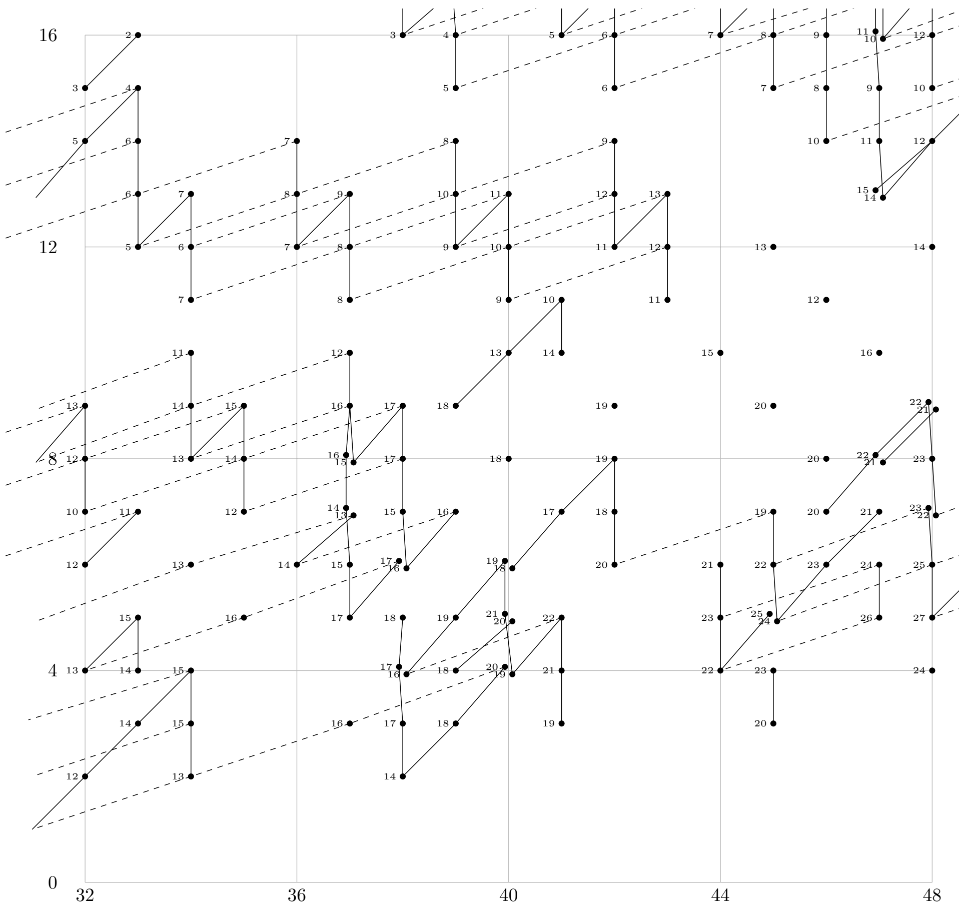


FIGURE 3. $\mathrm{Ext}_{\mathcal{A}}^{s,n+s}(\mathbb{F}_2, \mathbb{F}_2)$, $32 \leq n \leq 48$, $0 \leq s \leq 16$

provided as "humanly readable Diff files", and write the differentials using the Milnor basis. A couple of examples should suffice to show how to read them. The file `hDiff.1` starts

```
          8          184
1

1
0 1 1 i(1).

2

1
0 2 1 i(2).

4
```

```
1
0 4 2 i(4).

...
```

which should be read as saying that the $\mathbb{F}_2$-dimension of C_1 is 8, that the part of the resolution given here is complete through internal degree 184, and that

(a) generator 1_0^* lies in degree 1, and $d_1(1_0^*) = Sq^1(0_0^*)$,
(b) generator 1_1^* lies in degree 2, and $d_1(1_1^*) = Sq^2(0_0^*)$,
(c) generator 1_2^* lies in degree 4, and $d_1(1_2^*) = Sq^4(0_0^*)$, et cetera.

In the file `hDiff.2`, the fifth entry, i.e., the entry for $d_2(2_4^*)$, is

```
9

3
0 8 4 i(8)(2,2).
1 7 4 i(7)(4,1)(0,0,1).
3 1 1 i(1).
```

This says that 2_4^* has internal degree 9, and that

$$\begin{aligned} d_2(2_4^*) =&(Sq^8 + Sq^{(2,2)})(1_0^*) \\ &+ (Sq^7 + Sq^{(4,1)} + Sq^{(0,0,1)})(1_1^*) \\ &+ Sq^1(1_3^*)\,. \end{aligned}$$

Here, the initial 3 in the description of $d_2(2_4^*)$ says that $d_2(2_4^*)$ is the sum of three terms, and the subsequent lines describe those terms. The first line, "`0 8 4 i(8)(2,2).`", tells us that the first term is a multiple of 1_0^* with the degree 8 coefficient $Sq^8 + Sq^{(2,2)}$. The number 4 here is the $\mathbb{F}_2$-dimension of $\mathcal{A}$ in degree 8, and is used by the program `ext` to determine the amount of space which must be allocated. Note that elements of the Steenrod algebra are written in the Milnor basis, not the admissible basis.[1]

(5) `Ext-A-F2-F2-0-184.tex` and `Ext-A-F2-F2-0-184.pdf`. These are "charts" in the usual "Adams chart" format showing the resolution together with the action of the elements h_0, h_1 and h_2. The program `chart` included in `ext.1.9.3` allows one to make charts like this for any box $s_{lo} \le s \le s_{hi}$, $t_{lo} \le t \le t_{hi}$. It is also possible to simply use the `TikZ` command `clip` to extract desired sections from the TEX file we provide.

In these charts, each cocycle s_g is represented by a filled circle with the sequence number, g, written to its left, as in Figures 1 to 3. For example, in bidegree $(n, s) = (15, 5)$ we see that $5_4 = h_1 \cdot 4_3$ and $5_5 = h_0 \cdot 4_4$.

(6) `S-184.tex` and `S-184.pdf`. These give a stem-by-stem list of the results together with the products by the h_i.

(7) `Maxt` and `himults`. These are generated by the program `report` included in `ext.1.9.3` and are used by the `chart` and `vsumm` commands to create the Adams charts and the TEX summary. The file `Maxt` records, for each cohomological degree s, the degree t through which `Diff.s` is complete.

[1]The small "i" indicates that the notation is *internal* to the programs defining the Steenrod algebra. The main body of the code will compute minimal resolutions for any connected augmented $\mathbb{F}_2$-algebra, and the other notations for coefficients are generic notations for bitstrings which are independent of the algebra (see Section 4 for some discussion of formats "x" and "s").

The file `himults` records the h_i products using the observation cited in the first paragraph of Section 5 to extract this from the `Diff.s` files. The format of each entry is

```
s g s0 g0 i
```

meaning that the product $h_i \cdot (s_0)_{g_0}$ contains the term s_g. For example, in bidegree $(t-s, s) = (78, 8)$ we have the lines

```
8 60 7 33 4
8 60 7 34 4
8 60 7 56 1
8 60 7 58 1
8 61 7 33 4
8 61 7 34 4
8 61 7 48 3
8 61 7 56 1
8 61 7 57 1
8 61 7 60 0
8 62 7 2 6
8 62 7 35 4
8 62 7 61 0
```

saying that

$$
\begin{aligned}
h_0 \cdot 7_{60} &= 8_{61} \\
h_0 \cdot 7_{61} &= 8_{62} \\
h_1 \cdot 7_{56} &= 8_{60} + 8_{61} \\
h_1 \cdot 7_{57} &= 8_{61} \\
h_1 \cdot 7_{58} &= 8_{60} \\
h_3 \cdot 7_{48} &= 8_{61} \\
h_4 \cdot 7_{33} &= 8_{60} + 8_{61} \\
h_4 \cdot 7_{34} &= 8_{60} + 8_{61} \\
h_4 \cdot 7_{35} &= 8_{62} \\
h_6 \cdot 7_{2} &= 8_{62}\,,
\end{aligned}
$$

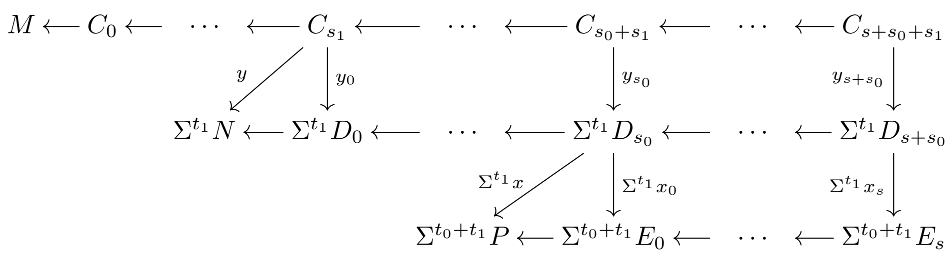
while the other products $h_i \cdot (s_0)_{g_0}$ in this bidegree are zero. This is used by `chart` to draw the lines representing h_0, h_1 and h_2 products.

3. Products

We compute products in Ext by composing chain maps. Suppose the classes $x \in \mathrm{Ext}_{\mathcal{A}}^{s_0,t_0}(N, P)$ and $y \in \mathrm{Ext}_{\mathcal{A}}^{s_1,t_1}(M, N)$ are represented by cocycles

$$x : D_{s_0} \longrightarrow \Sigma^{t_0} P \qquad \text{and} \qquad y : C_{s_1} \longrightarrow \Sigma^{t_1} N.$$

Then $\Sigma^{t_1} x \circ y_{s_0}$ is a cocycle representing the product xy, where $\{y_s\}_s$ is a chain map lifting y. The chain map with components $\Sigma^{t_1} x_s \circ y_{s+s_0}$ is a lift of this cocycle.



In our situation, $M = N = P = \mathbb{F}_2$ and $E_s = D_s = C_s$. In this case, the composite $\Sigma^{t_1} x \circ y_{s_0}$ is determined by recording those generators $(s_0 + s_1)^*_g$ of $C_{s_0+s_1}$ which are mapped nontrivially. When the cocycle x is $(s_0)_{g_0}$, dual to a generator $(s_0)^*_{g_0}$ of $C_{s_0} = D_{s_0}$, the list of such generators is the set of those whose image under y_{s_0} contains a term $1 \cdot (s_0)^*_{g_0}$, since all other terms will be sent to $0 \in \mathbb{F}_2$ by $(s_0)_{g_0}$. The program `collect` in the `ext` package gleans this information from the chain map files described in the next section and organizes it into the file `A/S-184/all.products`.

Each line in the file `all.products` has the form

```
s   g   (  s0    g0     F2)  s1_g1
```

which means that the chain map lifting the cocycle $(s_1)_{g_1}$, applied to the basis element s^*_g, contains the term $1 \cdot (s_0)^*_{g_0}$.

Proposition 3.1. *$(s_0)_{g_0} \cdot (s_1)_{g_1}$ is the sum of all such s_g.*

The file `all.products` is organized so that each paragraph lists, for a cocycle s_g, all such pairs $(s_0)_{g_0}$, $(s_1)_{g_1}$ with the $(s_0)_{g_0}$ in increasing order.[2]

To compute $(s_0)_{g_0} \cdot (s_1)_{g_1}$ it is best to start by noting which s_g span the bidegree containing the product, so as not to miss a term. For example, consider the products landing in $\mathrm{Ext}^{7,7+37}$. This bidegree is 2-dimensional, spanned by 7_{13} and 7_{14}. The entries in the file `all.products` for these two cocycles are

```
7   13   (  0    0     F2)  7_13
7   13   (  1    1     F2)  6_14
7   13   (  1    2     F2)  6_13
7   13   (  1    3     F2)  6_10
7   13   (  2    3     F2)  5_13
7   13   (  3    9     F2)  4_6
7   13   (  4    6     F2)  3_9
7   13   (  5   13     F2)  2_3
7   13   (  6   10     F2)  1_3
7   13   (  6   13     F2)  1_2
7   13   (  6   14     F2)  1_1

7   14   (  0    0     F2)  7_14
7   14   (  1    0     F2)  6_15
7   14   (  1    3     F2)  6_10
7   14   (  2    0     F2)  5_17
7   14   (  5   17     F2)  2_0
7   14   (  6   10     F2)  1_3
7   14   (  6   15     F2)  1_0
```

This says that

$$7_{13} = 0_0 \cdot 7_{13} = 1_1 \cdot 6_{14} = 1_2 \cdot 6_{13} = 2_3 \cdot 5_{13} = 3_9 \cdot 4_6$$
$$7_{14} = 0_0 \cdot 7_{14} = 1_0 \cdot 6_{15} = 2_0 \cdot 5_{17}$$
$$7_{13} + 7_{14} = 1_3 \cdot 6_{10}\,.$$

Eliminating redundant cases and using the traditional notations $h_i = 1_i$, $h_0^s = s_0$, $t = 6_{14}$, $n = 5_{13}$, $r = 6_{10}$, $c_1 = 3_9$, $f_0 = 4_6$, $x = 5_{17}$, these two paragraphs say that the two elements $h_1 t = h_2^2 n = c_1 f_0$ and $h_0^2 x$ span $\mathrm{Ext}^{7,7+37}$, that $h_3 r = h_1 t + h_0^2 x$,

[2] To be precise, the new program `collect.JR`, included in the directory `S-184`, sorts the entries in this manner. It will replace the old `collect` in the next release of the `ext` package. References to `collect` here should generally be interpreted to refer to the improved `collect.JR`.

and that all other products landing in this bidegree are 0. The h_0, h_1 and h_2 products in this bidegree can be seen in the chart shown in Figure 3. (The original definitions of f_0 did not distinguish between f_0 and $f_0 + h_1^3 h_4$. We eliminate this ambiguity by defining $f_0 = Sq^1(c_0)$. It is shown in [6] that $Sq^1(c_0) = 4_6$.)

Remark 3.2. Cocycles s_g whose paragraph in `all.products` has only a single entry, $0_0 \cdot s_g$, are clearly indecomposable. There are 912 of these in the range calculated, lying in filtrations 1 through 61. However, to identify all the indecomposables, simple textual work is insufficient: we must calculate $\mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal ideal of Ext. For example $\mathrm{Ext}^{7,7+133}$ is spanned by 7_{124}, 7_{125} and 7_{126} with products $h_3 \cdot 6_{97} = 7_{124} + 7_{125}$ and $h_1 \cdot 6_{107} = 7_{126}$. Thus, 7_{124} and 7_{125} project to the same nonzero element of $\mathfrak{m}/\mathfrak{m}^2$. Similarly, $h_4 \cdot 8_{114} = 9_{178} + 9_{179}$ with each term indecomposable, but equal to one another modulo decomposables.

4. Chain maps

A cocycle s_g of internal degree t is an $\mathcal{A}$-module homomorphism $s_g : C_s \longrightarrow \Sigma^t \mathbb{F}_2$. Using `ext` this can be lifted to a chain map of bidegree (s,t),

$$\{C_{s+s_0} \longrightarrow \Sigma^t C_{s_0} \mid 0 \le s_0 \le s + s_0 \le 128\}.$$

The data describing the cocycle s_g and our chain map lifting the cocycle are in the subdirectory `s_g` of the directory `S-184` containing the resolution. The relevant files are as follows.

(1) `maps` and subdirectories `A/S-184/s_g`. In `A/S-184`, the file `maps` is a list of all the cocycles s_g. Each of these has a subdirectory `A/S-184/s_g` which contains the cocycle's definition, the chain map lifting it, and data derived from this.

In order to study various aspects of the resolution it may be useful to focus on a smaller set of maps. The package `ext` has utilities, such as `collect`, which operate on such lists. For example, the invocation

```
./collect somemaps someproducts
```

would create a file named `someproducts` containing all products by maps listed in `somemaps`.

(2) `s_g/Def`. This file contains the definition of the cocycle s_g. If the internal degree of s_g is t, the file `s_g/Def` will contain

```
 s t F2 F2 s_g 1

g

1
0 0 1 x80
```

The reader who simply wants to use this data will not need the following discussion of cocycle definition files, but for completeness, here is the meaning of these entries.

Reading them in order, it says that $s_g : C_s \longrightarrow \Sigma^t \mathbb{F}_2$ is stored in the subdirectory `s_g` and maps exactly one generator of C_s nontrivially, namely generator number g, sending it to the unique nontrivial element of $\mathbb{F}_2$, which is the $\mathbb{F}_2$ basis element numbered 0. The format of a general *cochain* $x : C_s \longrightarrow \Sigma^t N$, where C_* is a resolution of M, is

```
 s t M N x n

g1
x(g1)
```

```
g2
x(g2)

...

gn
x(gn)
```

This specifies the values in N of x on the n generators numbered g_1 through g_n. Generators of C_s which are not mentioned are mapped to 0. This defines a *cocycle* iff it can be lifted to a chain map iff it can be lifted over the first stage, $d_1 : D_1 \longrightarrow D_0$, of a resolution of N. Thus, the process of lifting can be used to check that a cochain x is a cocycle. By minimality of the resolutions we produce, all cochains $x : C_s \longrightarrow \Sigma^t \mathbb{F}_2$ are cocycles.

For example, if N is 4-dimensional over $\mathbb{F}_2$, with basis elements 0 to 3 in degrees 0, 1, 1, and 2, respectively, and a cochain $x : C_5 \longrightarrow \Sigma^{12} N$ had nonzero values $x(5_3) = 1 + 2$ and $x(5_6) = 3$, the cochain definition file `x/Def` would be

```
 5 12 M N x 2

3

2
1 0 1 x80
2 0 1 s0.

6

1
3 0 1 i(0).
```

Here, "`0 1 x80`", "`0 1 s0.`" and "`0 1 i(0).`" are three different ways to write the unit $1 \in \mathcal{A}_0$. The initial "`0 1`" in each says that they describe an element of $\mathcal{A}_0$, which is 1-dimensional over $\mathbb{F}_2$. The first, "`x80`", uses hexadecimal notation to express the binary vector with a 1 in its first (and only) entry. The second, "`s0.`" lists the sequence number, 0, of the coordinates whose entries are 1 rather than 0. The third, "`i(0).`", writes the operation in Milnor basis form: $Sq^0 = 1 \in \mathcal{A}_0$. We could (and usually do) write the same information in the form

```
 5 12 M N x 2

3

2
1 0 1 x80
2 0 1 x80

6

1
3 0 1 x80
```

for simplicity and uniformity.

(3) `s_g/Map` and `s_g/Map.aug`. The `ext` package writes the chain map lifting the cocycle s_g in the file `s_g/Map`. This file is simply a list of entries of the form

```
s0 g0 0
```

meaning that s_g sends `s0_g0*` to 0, or

```
s0 g0
x
```

where x is the representation, in the manner discussed in item (4) of Section 2, of the nonzero image of `s0_g0*` under the chain map lifting s_g. The entries in the `Map` file do not have to occur in any particular order. There is a program, `checkmap s`, which determines the elements, if any, mapping to filtration $\leq s$ which are not yet present in the `Map` file. This has been used to verify the completeness of the data we present here.

The file `s_g/Map.aug` is extracted from the `Map` file by applying the augmentation $\mathcal{A} \longrightarrow \mathbb{F}_2$, i.e., by discarding all terms $a \cdot s_g^*$ in which $a \in \mathcal{A}$ has positive degree. The information in this file is used by `collect` to compile the file `all.products` described in the previous section.

(4) `s_g/brackets` and `s_g/brackets.sym`. These are discussed in the next section.

5. Toda brackets

Products $h_i \cdot s_g$ can be calculated directly from the resolution without computing either of the chain maps lifting h_i or s_g. Precisely, $h_i \cdot s_g$ is the sum of those $(s+1)_{g_1}$ such that $d_{s+1}((s+1)^*_{g_1})$ contains the term $Sq^{2^i} \cdot s_g^*$. In a similar way, having computed the chain maps, we are now able to evaluate all Toda brackets[3] of the form $\langle h_i, (s_0)_{g_0}, (s_1)_{g_1} \rangle$.

Proposition 5.1. *If $h_i \cdot (s_0)_{g_0} = 0$ and $(s_0)_{g_0} \cdot (s_1)_{g_1} = 0$ then the Toda bracket $\langle h_i, (s_0)_{g_0}, (s_1)_{g_1} \rangle$ contains the sum of all those s_g such that the chain map lifting $(s_1)_{g_1}$ applied to s_g^* contains a term $Sq^{2^i} \cdot (s_0)^*_{g_0}$.*

Conceptually, products by elements of cohomological degree 1 are visible in the resolution reduced modulo $\mathfrak{m}^2$, while brackets with first entry of cohomological degree 1 are visible in the chain maps reduced modulo $\mathfrak{m}^2$. This data is extracted from the `Map` file and placed in the files `brackets` and `brackets.sym`. The files contain the same information, but `brackets.sym` is easier for a human to read. Each entry in `s1_g1/brackets.sym` will have the form

```
s_g in < hi, g0, s1_g1 >
```

Since the filtration s of the bracket is $1 + s_0 + s_1 - 1 = s_0 + s_1$, we deduce that the middle entry is `s0_g0` with $s_0 = s - s_1$.

As with products, it is important to remember that the value of the bracket is the sum of all the s_g which appear, so that it is important to survey all the possible terms in the relevant bidegree before reaching conclusions.

We also hold the point of view that it is sufficient to produce one element of the bracket, with the other elements being obvious from the known indeterminacy. We also note that the `brackets.sym` file is simply recording information about the chain map, and that it is the responsibility of the user to check whether the bracket is defined.

For example, the first few entries in `1_0/brackets.sym` are

---

[3]We take the point of view that secondary products with respect to composition should be called *Toda brackets* while secondary products in a DGA should be called *Massey products*. By the usual device of considering the DGA of endomorphisms of a chain complex, the two are equivalent.

```
2_23 in < h7, 0, 1_0 >
2_8 in < h4, 0, 1_0 >
2_1 in < h0, 1, 1_0 >
2_5 in < h3, 0, 1_0 >
3_3 in < h3, 0, 1_0 >
3_3 in < h1, 3, 1_0 >
```

Since $h_0h_7 \neq 0$ and $h_0h_4 \neq 0$, the first two brackets are not defined. The third says that $h_1^2 = 2_1 \in \langle h_0, h_1, h_0\rangle$, a familiar consequence of Hirsch's formula $y(x \cup_1 x) \in \langle x, y, x\rangle$. The next two are also not defined. Following that we find $c_0 = 3_3 \in \langle h_1, h_2^2, h_0\rangle$.

6. Sq^0

In general, the cocommutative Hopf algebra Steenrod operations

$$Sq^i : \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \mathrm{Ext}_{\mathcal{A}}^{s+i,2t}(\mathbb{F}_2, \mathbb{F}_2)$$

are computationally quite expensive ([9, Problem Session] and [6]). However, the two extremes, $Sq^s : \mathrm{Ext}^{s,t} \longrightarrow \mathrm{Ext}^{2s,2t}$ and $Sq^0 : \mathrm{Ext}^{s,t} \longrightarrow \mathrm{Ext}^{s,2t}$ are easily calculated using `ext`. The first is simply the squaring operation $Sq^s(x) = x^2$ for $x \in \mathrm{Ext}^{s,t}$, which we have already discussed. At the other extreme, in [17, Proposition 11.10], it is shown that the operation Sq^0 can be calculated by $Sq^0([a_1|\dots|a_s]) = [a_1^2|\dots|a_s^2]$ in the cobar complex for the dual Steenrod algebra. This implies that if $\Phi\mathcal{A}_*$ is the double of the dual Steenrod algebra, in which the degrees of all the elements are doubled, then $Sq^0\colon \mathrm{Ext}_{\mathcal{A}_*}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \to \mathrm{Ext}_{\mathcal{A}_*}^{s,2t}(\mathbb{F}_2, \mathbb{F}_2)$ is induced by the degree-preserving Hopf algebra homomorphism $F\colon \Phi\mathcal{A}_* \to \mathcal{A}_*$ that sends $\Phi\xi_i$ to ξ_i^2 for each $i \geq 1$. Dually, it is induced by the degree-preserving Hopf algebra homomorphism $V\colon \mathcal{A} \longrightarrow \Phi\mathcal{A}$ that sends an "even" Milnor basis element $Sq^{(2r_1,\dots,2r_k)}$ to $\Phi Sq^{(r_1,\dots,r_k)}$, and other Milnor basis elements to 0. Restricting along this homomorphism gives

$$Sq^0\colon\ \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Ext}_{\Phi\mathcal{A}}^{s,2t}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \mathrm{Ext}_{\mathcal{A}}^{s,2t}(\mathbb{F}_2, \mathbb{F}_2).$$

A slight modification of the computer code that calculates chain maps can compute this: a program `startsq0` computes the restriction $V_{s-1}(d(x))$ for each generator $x = s_g^*$ in the minimal $\mathcal{A}$-module resolution (C_*, d) of $\mathbb{F}_2$, and the same program that computes lifts for chain maps then solves for an element $V_s(x)$ satisfying $d(V_s(x)) = V_{s-1}(d(x))$. We recover Sq^0 as $\mathrm{Hom}_{\mathcal{A}}(V_*, \mathbb{F}_2)$. This inductive calculation is begun by setting $V_0(0_0^*) = 0_0^*$, so that $Sq^0(1) = 1$. (This discussion is quoted from our proof of Proposition 11.26 in [7].)

The files involved are

(1) `dosq0` and `maps.sq0`. The first is a script to run the computation of the lifts, while the second tells which subdirectory contains the map Sq^0.
(2) `S-184/Sq0/Map` and `S-184/Sq0/Map.aug`. This contains the data defining the chain map V and its reduction modulo the augmentation ideal $\mathfrak{m} \subset \mathcal{A}$, respectively. This latter defines the dual of the map Sq^0.
(3) `all.sq0`. This contains the data in the `Sq0/Map.aug` file in the format of the `all.products` file. For example, its entries

```
0    0  (  0    0     F2)  Sq0
1    1  (  1    0     F2)  Sq0
1    2  (  1    1     F2)  Sq0
...
3    9  (  3    3     F2)  Sq0
...
3   19  (  3    9     F2)  Sq0
```

```
...
3   34  (  3   19      F2)  Sq0
...
3   55  (  3   34      F2)  Sq0
...
4   16  (  4    5      F2)  Sq0
4   19  (  4    6      F2)  Sq0
...
```

show that

$$
\begin{aligned}
Sq^0(1) &= Sq^0(0_0) = 0_0 = 1, \\
Sq^0(h_0) &= Sq^0(1_0) = 1_1 = h_1, \\
Sq^0(h_1) &= Sq^0(1_1) = 1_2 = h_2, \\
&\dots \\
Sq^0(c_0) &= Sq^0(3_3) = 3_9 = c_1, \\
&\dots \\
Sq^0(c_1) &= Sq^0(3_9) = 3_{19} = c_2, \\
&\dots \\
Sq^0(c_2) &= Sq^0(3_{19}) = 3_{34} = c_3, \\
&\dots \\
Sq^0(e_0) &= Sq^0(4_5) = 4_{16} = e_1 \text{ and} \\
Sq^0(f_0) &= Sq^0(4_6) = 4_{19} = f_1 \,.
\end{aligned}
$$

Later we find

```
6  102  (  6   33      F2)  Sq0
6  103  (  6   32      F2)  Sq0
6  103  (  6   33      F2)  Sq0
6  104  (  6   32      F2)  Sq0
```

which reminds us that these files are the dual, i.e., chain level, data. In Ext, these say

$$
\begin{aligned}
Sq^0(A_0 + A_0') &= Sq^0(6_{32}) = 6_{103} + 6_{104}, \\
Sq^0(A_0) &= Sq^0(6_{33}) = 6_{102} + 6_{103}, \quad \text{and hence} \\
Sq^0(A_0') &= Sq^0(6_{32} + 6_{33}) = 6_{102} + 6_{104} \,.
\end{aligned}
$$

This information is used in Section 8 to organize Ext into "families" linked by Sq^0.

7. A canonical basis for $\mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$

The traditional, and by now familiar, notation for the elements of Ext starts in a systematic fashion, with the elements

$$h_i, P^k h_1, P^k h_2, c_i, P^k c_0, d_i, P^k d_0, e_i, P^k e_0, f_i, g_i, i, j, k, \dots ,$$

but becomes somewhat chaotic as the calculations are extended into higher bidegrees. In the next section, we propose some ways of extending this notation in a methodical fashion, but they do not suffice to give names to elements of a basis for Ext even in the range we consider here, let alone for all of Ext.

In contrast, our s_g form a well-defined canonical basis for Ext, which we now describe. First, we totally order the terms $Sq^R s_g^*$ of $C_{s,t}$ by

$$Sq^R s_g^* < Sq^{R'} s_{g'}^*$$

iff

(1) $g < g'$, or
(2) $g = g'$ and $Sq^R < Sq^{R'}$, where the Milnor basis elements Sq^R are given reverse lexicographic order: $(r_1, r_2, \dots) < (r'_1, r'_2, \dots)$ iff for some k, $r_k < r'_k$ and $r_i = r'_i$ for all $i > k$. Thus,

$$\begin{aligned}(n) &< (n-3,1) < (n-6,2) < \cdots \\ &< (n-7,0,1) < (n-10,1,1) < \cdots < (n-14,0,2) < (n-17,1,2) < \cdots \\ &< (n-15,0,0,1) < (n-18,1,0,1) < \cdots < (n-22,0,1,1) < \cdots\end{aligned}$$

Each nonzero element $x \in C_{s,t}$ then has a *leading term* LT x, which is the lowest term in x.

In the totally ordered basis $\{Sq^R s_g^*\}$ of a given bidegree (s,t), the decomposable elements, those with $\deg(Sq^R) > 0$, form an initial segment which is followed by the generators s_g^* of bidegree (s,t).

We can now inductively define our canonical basis as follows. We start with the bases $\{0_0^*\}$ for C_0 and $\{\}$ for C_s with $s > 0$. We may inductively assume given the basis for C_s in degrees less than t, and for C_{s-1} in degrees less than or equal to t.

Step 1: Generating the image and kernel.
$\text{Im}_{s,t}$ will be a totally ordered list of pairs (x, dx) with the leading terms of the dx in strictly increasing order. $\text{Ker}_{s,t}$ will be a list of terms x. Both are initially empty.

Consider the terms $Sq^R s_g^*$ in order. Let $x = Sq^R s_g^*$ and compute $dx = Sq^R d(s_g^*)$. Then, while $dx \neq 0$, if $LT(dx) = LT(dy)$ for a pair $(y, dy) \in \text{Im}_{s,t}$, replace x by $x - y$ and dx by $dx - dy$. If not, add (x, dx) to $\text{Im}_{s,t}$ and proceed to the next decomposable term. If, instead $dx = 0$, add x to the end of the list $\text{Ker}_{s,t}$.

Note that the leading term of dx will be increased each time we replace dx by $dx - dy$ until it either becomes 0 or has a leading term not already found among the dy in $\text{Im}_{s,t}$.

Step 2: Adding new generators.
We may inductively assume given $\text{Ker}_{s-1,t}$. For each $x \in \text{Ker}_{s-1,t}$, in order, let $c = x$. Then, while $LT(x) = LT(dy)$ for some pair $(y, dy) \in \text{Im}_{s,t}$, replace x by $x - dy$. If this process terminates with $LT(x) \neq 0$, add a new generator s_g^* with $d(s_g^*) = c$, then add a new pair (z, x) to $\text{Im}_{s,t}$, where z is the difference of s_g^* and those y whose images dy were subtracted from c to get the final x with a new leading term. If the process terminates with $x = 0$, do nothing.

Remark 7.1. We could choose, at this second step, to let $d(s_g^*) = x$ and add the pair (s_g^*, x) to $\text{Im}_{s,t}$. The `ext` code prior to the year 2000 used that algorithm. Experience shows that the bases s_g obtained from the algorithm described here have $h_i \cdot s_g$ monomial far more frequently than those produced by the old algorithm. The Wayne State Research Report [4] used the older algorithm. The first difference visible in Ext charts lies in bidegree $(9, 9+23)$. In the new algorithm, $h_1 8_3 = 9_4$ and $h_0 8_4 = 9_5$. In the older algorithm, $h_0 8_4 = 9_5$ also, but $h_1 8_3 = 9_4 + 9_5$.

The change alters the resolution much earlier, and can be seen by doing hand calculations in low degrees. In the old algorithm, $d(2_1^*) = Sq^{(0,1)} \cdot 1_0^* + Sq^2 \cdot 1_1^*$, while the new algorithm gives $d(2_1^*) = Sq^3 \cdot 1_0^* + Sq^2 \cdot 1_1^*$.

8. Concordance

In this section, we present the relation between our s_g basis and the notation used by other works on the cohomology of the Steenrod algebra. In the process, we make a natural extension to the traditional notation using Sq^0.

The existing names are based on Tangora's calculation of the E_∞ term of the May spectral sequence in [19] and on Chen's Lambda algebra computation of $\mathrm{Ext}^s = \mathrm{Ext}^s_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ for $s \leq 5$ in [10].

There is some indeterminacy in the translation between our names and both the May spectral sequence names and Chen's Lambda algebra names.

In the case of the May spectral sequence, elements of the E_∞-term of the May spectral sequence only determine elements of Ext up to classes of higher May filtration. In Table 1 we indicate this by giving the indeterminacy of the May spectral sequence name in parentheses. For example, in bidegree $(5, 5+62)$, the May spectral sequence definition of $H_1(0)$ (written H_1 in [19]) defines the coset $5_{32} + \langle 5_{33}, 5_{34} \rangle$. We denote this, in our table, by writing $32(33, 34)$ in the column giving the sequence number of the element.

The indeterminacy in relating Chen's Lambda algebra element names to ours stems from the lack of a direct comparison between the two complexes. There are five such cases which we discuss in Remark 8.4. Except for T_0, which is beyond the range of Tangora's May spectral sequence calculation, this indeterminacy is the same as that due to the May spectral sequence. In the case of T_0, the indeterminacy is entirely due to the lack of a direct comparison between the Lambda algebra and our minimal resolution. For T_0, $Q_3(0)$ and $H_1(0)$, the indeterminacy could be eliminated if we knew certain h_i-multiples, as noted in Remark 8.4.

The preprint [13] of Isaksen-Wang-Xu studies the Adams spectral sequence through $t - s \leq 95$. They adopt some of Tangora's notation for Ext, augmented and regularized by the use of operators which they call M, Δ and Δ_1. The operator $Mx = \langle g_2, h_0^3, x \rangle$, though it is also sometimes used when this bracket is not defined. The operators Δ and Δ_1 are given by products with non-permanent cycles b_{03}^2 and b_{13}^2, respectively, in the E_2-term of the May spectral sequence. This is analogous to Tangora's definition[4] $Px = (b_{02})^2 x$. They are precursors of Toda brackets discussed in Section 9 in the sense that, when the brackets are defined, they often compute them. In fact, each could be interpreted as two of three distinct brackets which must sum to zero by the Jacobi identity. We discuss these brackets in Section 9.

In a few bidegrees they encounter classes with no name under this system. They adopt the notation $x_{n,s}$ for such a class if it is the unique such class in bidegree $(s, s+n)$. In bidegree $(10, 10+94)$ there are two such classes which they call $x_{94,10}$ and $y_{94,10}$.

8.1. Cohomological degrees up to 5.

Recall the theorems of Wang and Palmieri:

Theorem 8.1 ([20] and [18])**.** *For $s < 4$, the homomorphism $Sq^0 : \mathrm{Ext}^s \longrightarrow \mathrm{Ext}^s$ is injective. When $s = 4$, its kernel is $\langle h_0^4 \rangle$.*

This makes the "Sq^0-families" $\{x, Sq^0(x), (Sq^0)^2(x), \ldots\}$ especially useful in low cohomological degrees.

Remark 8.2. Because $h_1^4 = 0$, $Sq^0 : \mathrm{Ext}^5 \longrightarrow \mathrm{Ext}^5$ must send the nonzero elements h_0^5 and $h_0^4 h_i$, $i \geq 4$, to 0. In the range we have calculated, the only other element in its kernel is Ph_2, reflecting $Sq^0(Ph_2) = h_3 g = 0$.

Chen ([10, Theorem 1.2 and Theorem 1.3]) gives a complete description of Ext^s for $s \leq 5$, building on the work of Adams [1], Wang [20] and Lin [16].

[4] Tangora [19, pp. 32 and 48] notes that this is not always equal to the periodicity operator $Px = \langle h_3, h_0^3, x \rangle$. For example see [19, Note 3 on p. 48].

Theorem 8.3 ([10, Theorems 1.2 and 1.3])**.** *An $\mathbb{F}_2$-base for the indecomposable elements in cohomological degrees $s \leq 5$ is as follows. In each, i runs over all $i \geq 0$.*

Ext^1**:** h_i.
Ext^3**:** c_i.
Ext^4**:** d_i, e_i, f_i, g_{i+1}, p_i, $D_3(i)$ *and* p'_i.
Ext^5**:** Ph_1, Ph_2, n_i, x_i, $D_1(i)$, $H_1(i)$, $Q_3(i)$, K_i, J_i, T_i, V_i, V'_i *and* U_i.

Note that V'_0 and U_0 are in the 252 and 260 stems, respectively, so are beyond the range of our computation. For each of the other families in these lists, at least the first member of the family lies in the range of our calculations. Chen adopts the notation $D_1(i)$, et cetera, for the i^{th} member of a Sq^0-family in order to avoid double subscripts. We extend this practice into higher cohomological degrees as noted in the next section. In each family except $\{g_1 = g, g_2, \ldots\}$, the family starts with the 0^{th} element.

Since Chen's list is (mostly) in terms of Sq^0-families, we need only consider the first member of each family in order to establish the relation between his classes and ours. Thirteen of the families start in a bidegree which is 1-dimensional over $\mathbb{F}_2$, so that the correspondence is clear for them. The remaining families are f_i, n_i, $H_1(i)$, $Q_3(i)$ and T_i, and we consider each of these individually.

Remark 8.4.

(1) It is long established practice to define $f_0 = Sq^1 c_0$, because that allows us to take advantage of H_∞ ring spectrum relations and differentials. We do not know whether Chen's definition of f_0 in terms of the Lambda algebra equals this or $f_0 + h_1^3 h_4$.
(2) We choose n_0 to be the h_0-annihilated class in bidegree $(5, 5 + 31)$. According to Chen's unpublished preprint [11, Thm. 1.7] this agrees with his Lambda algebra definition of n_0.
(3) In two remaining cases, $H_1(0)$ and $Q_3(0)$, their May spectral sequence definition specifies a coset they must lie in. Chen's Lambda algebra classes with these names lie in the specified cosets. The precise elements in these cosets are determined by the classes $h_0 H_1(0)$, $h_4 H_1(0)$ and $h_3 Q_3(0)$. (That this suffices can be checked using `all.products`.) These three products are shown to be zero in Chen's [11, Thm. 1.7]. Thus, the May spectral sequence indeterminacy for $H_1(0)$ and $Q_3(0)$ is the indeterminacy reported in Table 1, but Chen's unpublished results allow the more precise correspondence $H_1(0) = 5_{32}$, $Q_3(0) = 5_{39}$, $H_1(1) = 5_{81}$ and $Q_3(1) = 5_{91}$.
(4) The remaining case, T_0, has only Chen's Lambda algebra definition from [10]. The precise indecomposable element in this bidegree is determined by the values of $h_1 T_0$ and $h_4 T_0$. As above, that these suffice can be checked using `all.products` and both are shown to be zero in Chen's [11, Thm. 1.7]. The indeterminacy reported in Table 1 is the set of decomposables, but Chen's unpublished results allow the more precise correspondence $T_0 = 5_{93}$.

8.2. **Cohomological degrees greater than** 5. In higher cohomological degrees, names for the elements of Ext come from Tangora's 1970 calculation [19] of the E_∞-term of the May spectral sequence. His Appendix 1 lists its indecomposables in the range $t - s \leq 70$ (omitting classes of the form $P^k a$). In three cases, known hidden extensions between the associated graded and Ext allow us to ignore these elements. These are $s = h_0 r$, $S_1 \in \langle h_1 x', h_0 R_1 \rangle$ and $g'_2 \in \langle h_1 B_{21}, h_0^2 B_4(0) \rangle$.

Some elements named by Tangora have Sq^0 which is decomposable. In those cases we keep Tangora's (often unsubscripted) notation. Others appear to be the start of Sq^0-families, with indecomposable members in the range of our calculation.

In these cases, we add a subscript 0 or suffix (0) to Tangora's name for the class, except for the g_i family, which starts with $g_1 = g$.

For unsubscripted elements that start Sq^0-families, like Tangora's m or A, we adopt the usual $a_{i+1} = Sq^0(a_i)$ notation for the subsequent elements of the family. For subscripted elements like the B_i or H_1 we use Chen's suffix notation, $Z(i+1) = Sq^0(Z(i))$, to avoid collision with other names used by Tangora. The suffix notation is also applied for C, C'' and G, due to the prior presence of classes C_0 and G_0.

Collisions would occur because subscripts in Tangora's subscripted capital letter classes do not indicate membership in a Sq^0-family. In particular, the B_1 through B_5, the C and C_0, et cetera, are not related by Sq^0. Nor are q in the 32-stem and q_1 in the 64-stem related in this manner. This is solved by Chen's $Z(i)$ notation.

If there is indeterminacy in the s_g name for the first member of a Sq^0-family, it is generally inherited by the subsequent members. $X_2(1)$, $E_1(1)$, $C_0(1)$, $G_{21}(1)$ and $B_4(1)$ are exceptions: Sq^0 annihilates the indeterminacy in the descriptions of $X_2(0)$, $E_1(0)$, $C_0(0)$, $G_{21}(0)$ and $B_4(0)$.

We have not listed all the indecomposable s_g which can be described using the Adams periodicity operators, but we have included those listed in Tangora's list of indecomposables. Similarly, we have not listed all the indecomposable s_g which can be described using Isaksen's "Mahowald operator" $M(x) = \langle g_2, h_0^3, x\rangle$, but we have noted that seven classes in Tangora's list, B_1, B_2, B_3, B_{21}, B_{22}, B_{23} and G_{11} could be so described using the modified operator $M'(x) = \langle h_0, h_0^2 g_2, x\rangle$. See Section 9 for further discussion of how to use our data to extend this into the full range of our calculation.

In a very few cases the indeterminacy reported in Table 1 is greater than the inherent indeterminacy in a May spectral sequence definition. These are the bidegrees $(t-s, s) = (63, 7)$, $(67, 9)$ and $(66, 10)$, where there are two indecomposables in the same bidegree, and possibly $(t-s, s) = (141, 5)$, where we do not know a May spectral sequence definition of the indecomposable class.

Table 1: Concordance between indecomposable s_g and other notations. Elements s_g for which we do not have a traditional name are omitted.

$t-s$	s	g	Tangora, Chen	Note
0	1	0	h_0	1
1	1	1	h_1	2
3	1	2	h_2	2
7	1	3	h_3	2
15	1	4	h_4	2
31	1	5	h_5	2
63	1	6	h_6	2
127	1	7	h_7	2
8	3	3	c_0	1
19	3	9	c_1	2
41	3	19	c_2	2
85	3	34	c_3	2
173	3	55	c_4	2
14	4	3	d_0	1
17	4	5	e_0	1

Table 1: Concordance between indecomposable s_g and other notations.

$t-s$	s	g	Tangora, Chen	Note
18	4	6	f_0	3
20	4	8	g_1	1
32	4	13	d_1	2
33	4	14	p_0	1
38	4	16	e_1	2
40	4	19	f_1	2
44	4	22	g_2	2
61	4	26	$D_3(0)$	1
68	4	32	d_2	2
69	4	33	p'_0	1
70	4	34	p_1	2
80	4	40	e_2	2
84	4	44	f_2	2
92	4	48	g_3	2
126	4	53	$D_3(1)$	2
140	4	65	d_3	2
142	4	67	p'_1	2
144	4	69	p_2	2
164	4	79	e_3	2
172	4	84	f_3	2
9	5	1	Ph_1	1
11	5	2	Ph_2	1
31	5	13	n_0	4
37	5	17	x_0	1
52	5	30	$D_1(0)$	1
62	5	$32(33, 34)$	$H_1(0)$	5
67	5	38	n_1	2
67	5	$39(38)$	$Q_3(0)$	6
79	5	50	x_1	2
109	5	75	$D_1(1)$	2
125	5	77	K_0	1
128	5	80	J_0	1
129	5	$81(82, 83)$	$H_1(1)$	2
139	5	90	n_2	2
139	5	$91(90)$	$Q_3(1)$	2
141	5	$93(94, 95)$	T_0	7
156	5	108	V_0	1
163	5	115	x_2	2
30	6	10	r_0	1
32	6	12	q	1
36	6	14	t_0	1
38	6	16	y	3

Table 1: Concordance between indecomposable s_g and other notations.

$t-s$	s	g	Tangora, Chen	Note
50	6	27	$C(0)$	1
54	6	30	$G(0)$	1
58	6	31	D_2	1
61	6	$32+33$	A_0'	8
61	6	33	A_0	8
64	6	38	A_0''	1
66	6	40	r_1	2
78	6	56	t_1	2
106	6	87	$C(1)$	2
114	6	92	$G(1)$	2
128	6	102	$A_1+h_2K_0=A_1'+h_0J_0$	9
134	6	110	A_1''	2
138	6	115	r_2	2
162	6	156	t_2	2
16	7	3	Pc_0	1
23	7	5	i	1
26	7	6	j	1
29	7	7	k	1
32	7	10	ℓ	1
35	7	12	m_0	1
46	7	20	$B_1(0)=M'h_1$	1
48	7	22(23)	$B_2(0)=M'h_2$	10
57	7	27	$Q_2(0)$	1
60	7	29	$B_3=M'h_4$	1
63	7	33(35)	$X_2(0)$	11
63	7	34(33, 35)	C'	11
66	7	40(41)	$G_0(0)$	12
77	7	56	$m_1+h_16_{53}$	2
99	7	85	$B_1(1)$	2
103	7	90(91)	$B_2(1)$	2
121	7	101	$Q_2(1)+h_6D_2$	2
133	7	124	$X_2(1)+h_36_{97}+h_16_{107}$	13
133	7	125	$X_2(1)+h_16_{107}$	13
139	7	137(138)	$G_0(1)$	2
161	7	184	$m_2+h_26_{149}+h_2h_7n$	2
22	8	3	Pd_0	1
25	8	5	Pe_0	1
46	8	20	N	1
62	8	32 + 33(34)	$E_1(0)$	14
62	8	33(34)	$C_0(0)$	14
68	8	43(44)	$G_{21}(0)$	15
69	8	46	$PD_3(0)$	16

Table 1: Concordance between indecomposable s_g and other notations.

$t-s$	s	g	Tangora, Chen	Note
132	8	$139+140$	$C_0(1)+h_2^2 6_{97}$	2
132	8	140	$E_1(1)$	2
144	8	176	$G_{21}(1)+h_1^3 h_7 d_0$	2
17	9	1	$P^2 h_1$	1
19	9	2	$P^2 h_2$	1
39	9	18	u	1
42	9	19	v	1
45	9	20	w	1
60	9	29(30)	$B_4(0)$	17
61	9	31	X_1	1
67	9	39(40)	$C''(0)$	18
67	9	40(39)	X_3	18
67	9	$39,40$	$C''(0), X_3$	18
129	9	145	$B_4(1)+h_2 8_{118}$	2
143	9	$197(199+200)$	$C''(1)$	2
41	10	14	z	1
53	10	18	x'	1
54	10	19(20)	R_1	19
56	10	22(21)	Q_1	20
59	10	24	$B_{21}=M'd_0$	1
62	10	$27(28,29)$	R	21
62	10	28(29)	$B_{22}\ni M'e_0$	21
64	10	32(33)	q_1	22
65	10	34	$B_{23}(0)=M'g_1$	1
66	10	$35+36$	B_5	23
66	10	35 or 36	D_2'	23
66	10	36	PD_2	16
69	10	40	PA_0	1
140	10	$196+197$	$B_{23}(1)+h_1 9_{178}$	24
24	11	3	$P^2 c_0$	1
34	11	7	Pj	1
67	11	35(36)	C_{11}	25
30	12	3	$P^2 d_0$	1
33	12	5	$P^2 e_0$	1
25	13	1	$P^3 h_1$	1
27	13	2	$P^3 h_2$	1
47	13	14	Q	26
47	13	$14+15$	$Q'=Q+Pu$	26
47	13	15	Pu	26
50	13	16	Pv	1
65	13	29(28)	R_2	27
68	13	30(31)	$G_{11}=M'i$	28

Table 1: Concordance between indecomposable s_g and other notations.

$t-s$	s	g	Tangora, Chen	Note
69	13	32	W_1	1
64	14	23	PQ_1	16
70	17	26(27)	R_1'	29
69	18	20	P^2x' (ill-defined)	30
$8k+1$	$4k+1$	1	P^kh_1	31
$8k+3$	$4k+1$	2	P^kh_2	31
$8k+8$	$4k+3$	3	P^kc_0	31
$8k+14$	$4k+4$	3	P^kd_0	31
$8k+17$	$4k+4$	5	P^ke_0	31
$8k+23$	$4k+7$	5	P^ki (for k even)	31
$8k+26$	$4k+7$	6	P^kj (for k even)	31
$8k+26$	$4k+7$	7	P^kj (for k odd)	31
$8k+39$	$4k+9$	18	P^ku (for k even)	31
$8k+39$	$4k+9$	15	P^ku (for k odd)	31
$8k+42$	$4k+9$	19	P^kv (for k even)	31
$8k+42$	$4k+9$	16	P^kv (for k odd)	31

Notes:

(1) There is a unique nonzero element in this bidegree.
(2) This is Sq^0 of an element we have already identified.
(3) We choose to let $f_0 = Sq^1(c_0)$ and $y = Sq^2(f_0)$, which are shown in [6] to be 4_6 and 6_{16}, respectively.
(4) This is the unique nonzero element in this bidegree whose h_0 multiple is 0.
(5) The May spectral sequence definition of $H_1 = H_1(0)$ has indeterminacy spanned by $5_{33} = h_1D_3$ and $5_{34} = h_0^3h_5^2$, so that $H_1(0)$ must be the indecomposable 5_{32} modulo them. The Lambda algebra class which Chen defines as $H_1(0)$ in [10] satisfies $h_0H_1(0) = 0$ and $h_4H_1(0) = 0$, according to Chen's preprint [11, Thm. 1.7], eliminating the possible summands 5_{34} and 5_{33}, respectively.
(6) The May spectral sequence definition of $Q_3 = Q_3(0)$ has indeterminacy spanned by $5_{38} = n_1$, so that $Q_3(0)$ must be 5_{39} modulo 5_{38}. The Lambda algebra class which Chen defines as $Q_3(0)$ in [10] satisfies $h_3Q_3(0) = 0$, according to Chen's preprint [11, Thm. 1.7], eliminating the possible summand 5_{38}.
(7) The indecomposable T_0 must be the indecomposable 5_{93} modulo the decomposables $5_{94} = h_7d_0$ and $5_{95} = h_1d_3$. If $T_0 = 5_{93} + \alpha 5_{94} + \beta 5_{95}$, then $h_1T_0 = \alpha 6_{126}$ while $h_4T_0 = \beta 6_{145}$. These products are both zero according to Chen's preprint [11, Thm. 1.7], so that $T_0 = 5_{93}$.
(8) Tangora [19] shows $h_0A = h_2D_2$, hence $A = 6_{33}$. He also shows $h_0^2A' = 0$, so that $A' = 6_{32} + 6_{33}$. We write them as A_0 and A_0', since Sq^0 is nonzero on both.
(9) $A_1 = Sq^0(A_0) = 6_{102} + 6_{103}$ and $A_1' = Sq^0(A_0') = 6_{102} + 6_{104}$, while $6_{103} = h_2K_0$ and $6_{104} = h_0J_0$.
(10) The May spectral sequence definition of B_2 has indeterminacy $7_{23} = h_0^2h_5e_0$, so that $B_2 \in \{7_{22}, 7_{22} + 7_{23}\}$. The value of $M'h_2$ is exactly the same set.

(11) Bidegree $(7, 7+63)$ is spanned by the two indecomposables 7_{33} and 7_{34} together with $7_{35} = h_0^6 h_6$. We have $h_2 7_{33} = 0$ and $h_2 7_{34} = 8_{41}$. In the May spectral sequence there are indecomposables C' and X_2. Tangora [19] reports that $h_2 C' = h_0 G_0$ is nonzero, and does not list $h_2 X_2$ as a nonzero value. Granting that $h_2 X_2 = 0$, it follows that $C' \equiv 7_{34}$ modulo $\langle 7_{33}, 7_{35} \rangle$ and $X_2 \equiv 7_{33}$ modulo 7_{35}. Since $Sq^0(7_{33})$ is indecomposable, we write $X_2 = X_2(0)$. We do not know whether $Sq^0(C')$ is indecomposable; if it is we should set $C' = C'(0)$ and note that $C'(1) \equiv X_2(1)$ modulo decomposables.

(12) Bidegree $(7, 7+66)$ is spanned by the indecomposable 7_{40} and $7_{41} = h_0 r_1$. Hence, the indecomposable $G_0 = G_0(0)$ must be 7_{40} modulo 7_{41}.

(13) Since $Sq^0(7_{34}) = 7_{124} + 7_{125} = h_3 6_{97}$, we see that 7_{124} and 7_{125} are congruent modulo decomposables, but are each indecomposable. Then $Sq^0(7_{33}) = 7_{125} + h_1 6_{107}$ shows that $7_{125} = Sq^0(7_{33}) + h_1 6_{107}$ and that $7_{124} = Sq^0(7_{33}) + h_1 6_{107} + h_3 6_{97}$.

(14) Bidegree $(8, 8+62)$ is spanned by the two indecomposables 8_{32} and 8_{33} together with $8_{34} = h_0^6 h_5^2$. Tangora [19] lists indecomposables C_0 and E_1, together with $h_0^6 h_5^2$. He reports $h_1 E_1 \neq 0$ and (implicitly) $h_1 C_0 = 0$. He also reports $h_2 C_0 \neq 0$ and (implicitly) $h_2 E_1 \equiv 0$ modulo $h_0^2 h_3 D_2$. This implies $E_1 \equiv 8_{32} + 8_{33}$ modulo 8_{34} and $C_0 \equiv 8_{33}$ modulo 8_{34}. We set $E_1(0) = E_1$ and $C_0(0) = C_0$ since $Sq^0(E_1) = 8_{140}$ and $Sq^0(C_0) = 8_{139} + 8_{140} + 8_{141} + 8_{142}$ are indecomposable.

(15) Bidegree $(8, 8+68)$ is spanned by the indecomposable 8_{43} and the decomposable $8_{44} = h_0 h_3 A_0'$. Hence, the indecomposable G_{21} must be 8_{43} modulo 8_{44}.

(16) This is the unique element in the bracket $\langle h_3, h_0^4, - \rangle$.

(17) The May spectral sequence definition of $B_4 = B_4(0)$ has indeterminacy spanned by $9_{30} = h_0^2 B_3$, so that $B_4(0)$ must be the indecomposable 9_{29} modulo 9_{30}

(18) Bidegree $(9, 9+67)$ is spanned by the two indecomposables 9_{39} and 9_{40}. We have $h_0 9_{39} = 0$, $h_0 9_{40} = 10_{37}$, $h_2 9_{39} = 10_{41}$ and $h_2 9_{40} = 0$. In the May spectral sequence, there are indecomposables C'' and X_3. The relations reported by Tangora [19] have $h_0 X_3 \neq 0$ and $h_2 C'' \neq 0$. It follows that $C'' \equiv 9_{39}$ modulo 9_{40} and $X_3 \equiv 9_{40}$ modulo 9_{39}, but, of course, $C'' \neq X_3$. Since $Sq^0(C'')$ is indecomposable, we write $C'' = C''(0)$. We do not know whether $Sq^0(X_3)$ is indecomposable; if it is we should set $X_3 = X_3(0)$ and note that $C''(1) \equiv X_3(1)$ modulo decomposables.

(19) Bidegree $(10, 10+54)$ is spanned by the indecomposable 10_{19} and the decomposable $10_{20} = h_0^2 h_5 i$, while Tangora's calculation has this bidegree spanned by $h_0^2 h_5 i$ and R_1. Hence, R_1 must be 10_{19} modulo 10_{20}.

(20) Bidegree $(10, 10+56)$ is spanned by the indecomposable 10_{22} and the decomposable $10_{21} = g_1 t_0$, while Tangora's calculation has this bidegree spanned by gt and Q_1. Hence, Q_1 must be 10_{22} modulo 10_{21}.

(21) Bidegree $(10, 10+62)$ is spanned by the decomposable $10_{29} = h_1 X_1 = PG$ together with the indecomposables 10_{27} and 10_{28}. In the May spectral sequence it is spanned by R, B_{22} and PG, in order of May filtration, so that B_{22} is defined modulo PG, while R is only defined modulo the other two. The relation $h_0 B_{22} = d_0 B_2$ holds in the May spectral sequence by [19]. Since $h_0 10_{28} = d_0 B_2$, while $h_0 10_{27}$ is not divisible by d_0 in Ext, B_{22} must be 10_{28} modulo 10_{29}. Since R is linearly independent of B_{22} and PG, it must be 10_{27} modulo $\langle 10_{28}, 10_{29} \rangle$.

(22) Tangora's q_1 in (10,10+64) is not $Sq^0(q)$, which is instead the decomposable $6_{46} = Sq^0(6_{12}) = h_2Q_3(0)$. The May spectral sequence definition of q_1 has indeterminacy $10_{33} = h_0^2h_3Q_2(0) = h_1^2E_1(0)$ so that q_1 is 10_{32} modulo 10_{33}.
(23) Bidegree $(10, 10+66)$ is spanned by the two indecomposables 10_{35} and 10_{36}. We have $h_0(10_{35} + 10_{36}) = h_1B_{23}(0) = h_2^2B_4(0)$, while $h_0^2 10_{35} = h_0^2 10_{36} = 12_{32} = h_1^2q_1$. In the May spectral sequence, there are indecomposables D_2' and B_5. The relations reported by Tangora [19] include $h_0B_5 = h_1B_{23} = h_2^2B_4$, which require $B_5 = 10_{35} + 10_{36}$. We then have $D_2' = 10_{35}$ or 10_{36}. In any case $\langle D_2', B_5\rangle = \langle 10_{35}, 10_{36}\rangle$.
(24) $B_{23}(1) = Sq^0(B_{23}(0)) = 10_{196} + 10_{197} + 10_{199}$ and $10_{199} = h_1 9_{178} = h_1 9_{179}$. Both 10_{196} and 10_{197} are indecomposable.
(25) Bidegree $(11, 11 + 67)$ is spanned by the indecomposable 11_{35} and $11_{36} = h_0^2X_3 = h_0h_3B_4(0) = ig_2 = r_0x_0$. Hence, the indecomposable C_{11} must be 11_{35} modulo 11_{36}.
(26) Bidegree $(13, 13 + 47)$ is spanned by the indecomposables 13_{14} and 13_{15}, while Tangora's calculation has this bidegree spanned by Pu and Q with $h_1Q \neq 0$. The brackets file shows that $Pu = \langle h_3, h_0^4, u\rangle = 13_{15}$. Since $h_1 13_{14} = h_1 13_{15}$, we must have $Q = 13_{14}$. Tangora defines $Q' = Q + Pu$.
(27) Bidegree $(13, 13+65)$ is spanned by the indecomposable 13_{29} and the decomposable $13_{28} = g_1w = r_0m$, while Tangora's calculation has this bidegree spanned by gw and R_2. Hence, R_2 must be 13_{29} modulo 13_{28}.
(28) Bidegree $(13, 13 + 68)$ is spanned by the indecomposable 13_{30} and the (highly) decomposable $13_{31} = h_0^5G_{21} = h_0h_5d_0i$. Hence, the indecomposable G_{11} must be 13_{30} modulo 13_{31}. This coset is $M'i$ since 13_{31} is in the indeterminacy of $M'i$.
(29) Bidegree $(17, 17+65)$ is spanned by the indecomposable 17_{26} and the decomposable $17_{27} = d_0^2v$, while Tangora's calculation has this bidegree spanned by $Pe_0w = d_0^2v$ and R_1'. Hence, R_1' must be 17_{26} modulo 17_{27}.
(30) Bidegree $(18, 18+69)$ is spanned by the indecomposable 18_{20} together with $18_{21} = P(gz)$. Tangora writes $P^2x' = \langle h_4, h_0^8, x'\rangle$ for an indecomposable in this bidegree, but this is not defined as a Toda bracket, since $h_0^8x' \neq 0$.
(31) By Adams periodicity $P^kh_1 = (4k+1)_1$, $P^kh_2 = (4k+1)_2$, $P^kc_0 = (4k+3)_3$, $P^kd_0 = (4k + 4)_3$ and $P^ke_0 = (4k + 4)_5$. Likewise, $P^ki = (4k + 7)_5$, $P^kj = (4k + 7)_6$, $P^ku = (4k + 9)_{18}$ and $P^kv = (4k + 9)_{19}$ for k even, and $P^kj = (4k + 7)_7$, $P^ku = (4k + 9)_{15}$ and $P^kv = (4k + 9)_{16}$ for k odd.

9. Operators

In this section, we describe the files `P.txt`, `P2.txt`, `P4.txt` and `MM.txt`, in which we collect the data needed to compute the Adams periodicity operators P^k, $k = 1, 2, 4$, and Isaksen's Mahowald operator M, then make some general remarks about such operators.

9.1. **Adams and Mahowald operators.** These are defined by

(1) $Px = \langle h_3, h_0^4, x\rangle$,
(2) $P^2x = \langle h_4, h_0^8, x\rangle$,
(3) $P^4x = \langle h_5, h_0^{16}, x\rangle$ and
(4) $Mx = \langle g_2, h_0^3, x\rangle$.

Isaksen's Mahowald operator $Mx = \langle g_2, h_0^3, x\rangle$ is not of the form that it is immediately evident in the `brackets.sym` files, but its variant $M'(x) = \langle h_0, h_0^2g_2, x\rangle$ is. Both contain $\langle h_0g_2, h_0^2, x\rangle$, when defined.

Recall from Proposition 5.1 that, if defined, the bracket $\langle h_i, (s_0)_{g_0}, (s_1)_{g_1}\rangle$ is the sum of those s_g with $s = s_0 + s_1$ such that the file `s1_g1/brackets.sym` contains a line

```
s_g in < hi, g0, s1_g1 >
```

Thus, the values of the operators P, P^2, P^4 and M' are recognized by the presence of lines

(1) `s_g in < h3, 0, s1_g1 >`
(2) `s_g in < h4, 0, s1_g1 >`
(3) `s_g in < h5, 0, s1_g1 >`
(4) `s_g in < h0, 21, s1_g1 >`

with $s = s_1 + 4$, $s = s_1 + 8$, $s = s_1 + 16$ or $s = s_1 + 6$, respectively. For M', we note that $6_{21} = h_0^2 g_2$.

The files `P.txt`, `P2.txt`, `P4.txt` and `MM.txt` collect this information from all the map files, together with information about products needed in determining the domain of definition and the indeterminacy. Each file starts with a short header describing the operator and the file's organization, then has three sections:

(a): values of the brackets,
(b): nonzero products which obstruct existence of the bracket, and
(c): nonzero products which give the indeterminacy.

In general, the indeterminacy in a bracket $\langle a, b, c\rangle$ is $a(\mathrm{Ext}) + (\mathrm{Ext})c$. However, the brackets P, P^2, P^4 and M' have indeterminacy $a(\mathrm{Ext})$. For the periodicity operators, this is because $\mathrm{Ext}^{4,12}$, $\mathrm{Ext}^{8,24}$ and $\mathrm{Ext}^{16,48}$ are zero, so that $(\mathrm{Ext})c = 0$ in the relevant bidegree. For $M' = \langle h_0, h_0^2 g_2, -\rangle$, it is because $\mathrm{Ext}^{6,51} = \langle h_0 h_5 d_0\rangle$, so that $(\mathrm{Ext})c$ is contained in $a(\mathrm{Ext}) = h_0(\mathrm{Ext})$ in the relevant bidegree.

For example, the file `P.txt` starts

```
% Adams operator P, in the range t\le184
%
% (a) Brackets Px = < h3, h0^4, x >.
% (b) Nonzero products h0^4 * x, obstructing existence.
% (c) Nonzero products h3 * y, giving indeterminacy.

% (a) Brackets Px = < h3, h0^4, x >.

5_1 in < h3, 0, 1_1 >
5_2 in < h3, 0, 1_2 >
5_5 in < h3, 0, 1_3 >

6_1 in < h3, 0, 2_1 >
```

From the first three lines in part (a) we see that, if they are defined, $Ph_1 = 5_1$, $Ph_2 = 5_2$ and $Ph_3 = 5_5 = h_0^4 h_4$, modulo their indeterminacy. To determine whether they are defined, we look at part (b), which starts

```
% (b) Nonzero products h0^4 * x, obstructing existence.

  5    0  (  4    0     F2)  1_0
  5    5  (  4    0     F2)  1_4
  5   14  (  4    0     F2)  1_5
  5   35  (  4    0     F2)  1_6
  5   79  (  4    0     F2)  1_7
  6    0  (  4    0     F2)  2_0
```

We see that there are no lines ending in `1_1`, `1_2` or `1_3`, so that h_0^4 annihilates all three of these. Thus Ph_1, Ph_2 and Ph_3 are all defined, but Ph_0, Ph_4, ..., are not.

Next, we consider the indeterminacy, which equals the h_3 multiples in the bidegree of the bracket. The bidegrees in question are $\mathrm{Ext}^{5,5+9} = \langle 5_1 \rangle$, $\mathrm{Ext}^{5,5+11} = \langle 5_2 \rangle$ and $\mathrm{Ext}^{5,5+15} = \langle 5_4, 5_5 \rangle$. We look at part (c) to see the h_3 multiples among these. We find

```
% (c) Nonzero products h3 * y, giving indeterminacy.

  2    4  (  1    3     F2)  1_0
  2    5  (  1    3     F2)  1_1
  2    6  (  1    3     F2)  1_3
  2   14  (  1    3     F2)  1_5
  2   19  (  1    3     F2)  1_6
  2   25  (  1    3     F2)  1_7
  ...
  4   85  (  1    3     F2)  3_54
  5   11  (  1    3     F2)  4_5
  5   18  (  1    3     F2)  4_12
  5   19  (  1    3     F2)  4_13
  ...
```

Since we do not find 5_1, 5_2, 5_4 or 5_5 among the h_3-multiples, the indeterminacy is 0.

For an example of a possible bracket which is, in fact, undefined, consider the entry

```
8_91 in < h3, 0, 4_48 >
```

in part (a). This does not mean that $P(4_{48}) = 8_{91}$ because in part (b) we find that $h_0^4 \cdot 4_{48} = 8_{77} \neq 0$:

```
  8   77  (  4    0     F2)  4_48
```

Finally, for an example with nontrivial indeterminacy, in part (a) we find an entry

```
6_5 in < h3, 0, 2_5 >
```

so that $6_5 \in P(2_5)$, but in part (c) we find

```
  6    5  (  1    3     F2)  5_1
```

showing that $6_5 = h_3 \cdot 5_1$ is in the indeterminacy. Hence $P(2_5) = \{0, 6_5\}$. (We have $6_5 = h_1^2 d_0 = h_3 P h_1 = c_0^2$.)

9.2. **General remarks.** We finish this section with some observations about operators like those we have just considered. Defining $Px = (b_{02})^2 x$ in the May spectral sequence is justified by the differential $d_4(b_{02}^2) = h_0^4 h_3$ ([19, Prop. 4.3]). However, this accounts for only part of the definition of a Massey product or Toda bracket. It is simplest[5] to discuss this in terms of Massey products in a commutative differential graded algebra over $\mathbb{F}_2$ like the E_r terms of the May spectral sequence.

Consider classes a, b and x satisfying $ab = bx = xa = 0$. The Jacobi identity says that

$$0 \in \langle a, b, x \rangle + \langle b, x, a \rangle + \langle x, a, b \rangle \, .$$

If we choose A, U and V such that $d(A) = ab$, $d(U) = bx$ and $d(V) = xa = ax$, then we have

(1) $Ax + aU \in \langle a, b, x \rangle$,
(2) $Ua + bV \in \langle b, x, a \rangle$ and
(3) $Ax + bV \in \langle b, a, x \rangle = \langle x, a, b \rangle$.

[5]An idealistic treatment would instead consider Toda brackets of chain maps, or Massey products in $\mathrm{End}(C_*)$, which is homotopy commutative but not commutative.

Approximating the bracket $\langle a, b, x\rangle$ by Ax, in those cases where Ax is a cycle, fails to distinguish between $\langle a, b, x\rangle$ and $\langle b, a, x\rangle = \langle x, a, b\rangle$. These differ by $\langle b, x, a\rangle = \langle a, x, b\rangle$.

This can lead to greater indeterminacy and to anomalies like Tangora's observation [19, Note 3, p. 48] that $h_5 i$ is annihilated by h_0^3, but $P(h_5 i)$, if defined to be $(b_{02})^2 h_5 i$, has $h_0^9 \cdot (b_{02})^2 h_5 i \neq 0$. In fact, consulting `P.txt` we see that $P(h_5 i) = P(8_{26}) = 0$ with zero indeterminacy.

By using only the precisely defined brackets we limit the indeterminacy and get the advantages of their good formal behavior.

Finally, let us point out two other operators which may be of use. They are

(1) the complex Bott periodicity operator $v_1(x) = \langle h_0, h_1, x\rangle$, which acts on h_1-annihilated classes such as the unit in $\mathrm{Ext}_{E(Q_0,Q_1)}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \pi_* ku$, and
(2) the mod 2 complex Bott periodicity operator $v_1'(x) = \langle h_1, h_0, x\rangle$, which acts on h_0-annihilated classes.

In the universal example, the Adams spectral sequence for $\pi_*(S \cup_2 e^1 \cup_\eta e^2)$, $v_1(0_0)$ is an h_1-annihilated class which supports an infinite h_0-tower, while $v_1'(0_0)$ is an h_0-annihilated class which supports h_1^2-multiplication. These are visible in the `brackets.sym` files in the form

```
s_g in < h0, 1, s1_g1 >
```

and

```
s_g in < h1, 0, s1_g1 >
```

respectively.

10. Validity

Several checks have been run to test the validity of the data.

(1) After the resolution was computed, a separate computation was done to check that $d^2 = 0$.
(2) A check of the $\mathbb{F}_2$-dimension of the kernel and image at each step, to ensure that the image at each step has the same dimension as the kernel at the previous step. This checks exactness when combined with the check that $d^2 = 0$.
(3) A check that the `Map` files are complete. If this were not true, an element s_g missing from `s1_g1/Map` might not be reported in `all.products` as a term in a product $(s_0)_{g_0} \cdot (s_1)_{g_1}$ even when it belongs there.
(4) A check that the `Map` files do define chain maps, i.e., that the maps m they specify do satisfy $dm = md$.

11. Machine processing of the data

Modern computer languages are adept at processing text. Nonetheless, the raw data which is used to produce `all.products` and `brackets.sym` is provided in the files `s_g/Map.aug` and `s_g/brackets`, since this raw data may be easier to process by a computer program. Examples should suffice to make clear the translation.

The entry which is reported as

```
  2    4  (  1    3     F2)  1_0
```

in `all.products` is derived from the line in `1_0/Map.aug` which says

```
2 4 3
```

as the rest of the data can be deduced from the map `1_0`.

Similarly, the entries

```
2_8 in < h4, 0, 1_0 >
2_1 in < h0, 1, 1_0 >
2_5 in < h3, 0, 1_0 >
```

in `1_0/brackets.sym` are derived from the lines in `1_0/brackets` which say

```
2 8 16 0
2 1 1 1
2 5 8 0
```

Here, the third entries in each line, 16, 1 and 8, are the internal degrees of the elements h_4, h_0 and h_3.

Finally, the entries in `Sq0/Map.aug` have the form

```
2 1 0
2 3 1
2 5 2
```

meaning that $Sq^0(2_0)$ contains 2_1, that $Sq^0(2_1)$ contains 2_3, and that $Sq^0(2_2)$ contains 2_5.

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Department of Mathematics, Wayne State University, USA
Email address: robert.bruner@wayne.edu

Department of Mathematics, University of Oslo, Norway
Email address: rognes@math.uio.no