Equivariant Structure on Smash Powers

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Preface

Introduction

Since the early 1990's there have been several useful symmetric monoidal model structures on the underlying category of a point set category of spectra. Topological Hochschild homology (THH) is constructed from smash powers of ring-spectra, that is, monoids in the category of spectra. Already in the 1980's such a construction was made. Building on ideas of Goodwillie and Waldhausen, Bökstedt introduced a class of ring-spectra for which he could define THH via a homotopy-invariant ad hoc construction of smash powers [Bö], and he determined the homotopy types of THH applied to $\mathbb{Z}$ and to $\mathbb{Z}/p\mathbb{Z}$. When symmetric monoidal model categories of spectra appeared (e.g. $\mathbb{S}$-modules in the sense of Elmendorf-Kriz-Mandell-May [EKMM], symmetric spectra [HSS] and orthogonal spectra [MMSS],) it turned out that the Hochschild complex in these model categories provided a construction of THH. For $\mathbb{S}$-modules this was shown already in [EKMM], for symmetric spectra it was shown by Shipley [Sh]. The case of orthogonal spectra was treated thesis [Kr] of Kro.

It was discovered by Bökstedt, Hsiang and Madsen that the action of the circle group S^1 on THH contains crucial information about algebraic K-theory [BHM]. However it was surprisingly hard to provide symmetric monoidal model structures of equivariant spectra, and the equivariant homotopy type of the Hochschild complex depends on the chosen point set category of spectra as remarked in [EKMM, IX.3.9]. For a commutative ring-spectrum A there is an alternative description of the Hochschild complex as the categorical tensor $A \otimes S^1$ in the category of commutative ring spectra. In order for this to be homotopically meaningful we need a convenient model structure on the category of commutative ring spectra [Sh04]. The action of S^1 on topological Hochschild homology needed for the construction of Bökstedt, Hsiang and Madsen's Topological Cyclic homology (TC) [BHM] has been addressed by Kro in his thesis [Kr], but he works in the category of ring-spectra as opposed to the category of commutative ring-spectra. Model structures on commutative orthogonal ring-spectra with action of a compact Lie group are also important in the norm construction of Hopkins, Hill and Ravenel in [HHR].

Recently iterated Topological Hochschild homology and its relation to the chromatic filtration has been studied together with its versions of "higher Topological Cyclic homology" (e.g. [BCD], [CDD], [Schl] and [BM]). In the present work we study the categorical tensor $A \otimes X$ of a commutative orthogonal ring-spectrum and a space X.

When a compact Lie group G acts on X, then by functoriality it also acts on $A \otimes X$. We study properties of this action and use it as a basis for a new and relatively simple construction of higher TC.

Motivated by the construction of higher TC we introduce, following [Sh04] various more or less convenient model structures on the category of orthogonal spectra with action of a compact Lie group G. We call the model structure we find most convenient the $\mathbb{S}$-model structure. Working with the $\mathbb{S}$-model structure, the categorical tensor $A \otimes X$ of a commutative orthogonal ring spectrum A and a G-space X is a model for the Loday functor $\Lambda_X A$ from [BCD]. Thus, in this setting, the constructions of higher versions of TC in [BCD], also called covering homology, can be transferred to $A \otimes X$.

Organization

In Chapter 1 we collect the results about unstable equivariant spaces that we are going to need. In particular we recall Illman's Triangulation Theorem and inspired by Shipley [Sh04] we provide mixed model structures on G-spaces for G a compact Lie group depending on pairs of families of subgroups of G.

In Chapter 2 we present mixed model structures on the category of orthogonal G-spectra. We follow the by now standard way of passing from so called level model structures on orthogonal spectra to stable model structures. By focusing more on semi-free orthogonal spectra than on free ones we gain flexibility in the choice of level model structures. We observe that there is a bijection between the set of isomorphism classes of n-dimensional orthogonal representation of G and the set of conjugacy classes of subgroups P of $G \times \mathbf{O}_{\mathbb{R}^n}$ with the property that the projection to the first factor induces an embedding of P in G. This observation leads us to work with compatible families of subgroups of the groups $G \times \mathbf{O}_{\mathbb{R}^n}$ instead of universes of G-representations, which in turn leads level model structures on orthogonal G-spectra different from the ones usually obtained from universes.

In Chapter 3 we study fixed point spectra of finite smash powers of orthogonal spectra. First we recall some general results about filtrations of smash powers of cell complexes. Next we recollect some results about geometric fixed point spectra. In particular we review results of Kro stating that geometric fixed points commute with restriction to subgroups. The main result 3.2.14 of this chapter is quite technical. It is both used in the construction of model structures on the category of commutative orthogonal G-ring-spectra and in the identification of geometric fixed points of a smash-power as a smash-power 3.2.16.

In Chapter 4 we introduce the Loday functor as the categorical tensor of a commutative orthogonal ring-spectrum and a space. This is a kind of smash-power with a topological space as exponent. We use the results of Chapter 3 to equip the Loday functor with the structure needed to construct covering homology, that is, a generalized version of topological cyclic homology. We end the chapter by comparing to the construction of covering homology in [BCD].

Chapter 5 consists of general results about category theory needed in the main part.

Chapter 6 is a recollection of facts about topological model categories together with a result about assembling model structures on certain categories to a model structure on a functor category. This result is used in the construction of level model structures on the category of orthogonal G-spectra.

Notation

We use the following notation:

(i) $\mathbb{N} = \{0, 1, \dots\}$, $\mathbb{Z} = \{\dots, -1, 0, 1, 2 \dots\}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are the sets of natural, integer, rational, real and complex numbers, with the usual algebraic structure.

(ii) For $n \geq 0$, $\mathbb{R}^n$ is n-dimensional Euclidean space (with the dot product.) For $1 \leq i \leq n$, $e_i \in \mathbb{R}^n$ forms the standard basis. We choose $\mathbb{R}^{m+n}$ together with the canonical inclusions of R^m onto the first coordinates and of $\mathbb{R}^n$ as the last coordinates as the direct sum $\mathbb{R}^m \oplus \mathbb{R}^n$.

(iii) The phase "let V be a Euclidean space" means "let $V = \mathbb{R}^n$ for some $n \geq 0$".

(iv) The one-point compactification S^V of a real inner product space V with its induced action by the orthogonal group $\mathbf{O}_V$ is denoted S^V, and is referred to as the V-*sphere*. Given $x \in V$ we consider x as an element $x \in S^V$ through the inclusion of V in its one-point compactification S^V. Given $x, y \in S^V$ we allow ourselves to write $x + y$ with the convention that $\infty + x = \infty = x + \infty$.

(v) The category $\mathcal{U}$ is the category of compactly generated weak Hausdorff spaces and continuous functions (see e.g., [St]), and $\mathcal{T}$ is the category of based spaces in $\mathcal{U}$. Unless we explicitly state otherwise, a *space* is an object in $\mathcal{T}$ and maps between spaces are assumed to be continuous (and basepoint preserving). If $X \in \mathcal{U}$ is an unbased space, then $X_+ \in \mathcal{T}$ is the space obtained by adding a disjoint basepoint.

(vi) When we use the symbol G to denote a group it will always be a compact Lie group. Subgroups of compact Lie groups are always assumed to be closed.

Chapter 1

Unstable equivariant homotopy theory

In this section we will recall results from (unstable) equivariant homotopy theory. We begin with a recollection on model structures on G-spaces and will continue with some consequences of the results of Illman [Ill83]. We work in the pointed setting $\mathcal{T}$ (the category of based, compactly generated, weak Hausdorff spaces) as this is the more important case for us, but all results could be stated in the unbased category $\mathcal{U}$ as well.

1.1 G-Spaces

Let G be a compact Lie group, and let $G\mathcal{T}$ be the category of spaces with a continuous action of G. Considering G as a one object category, $G\mathcal{T}$ is the category of continuous functors from G to $\mathcal{T}$. We let I_G be the set of G-maps given by the standard inclusions $(S^{n-1} \times G/H)_+ \to (D^n \times G/H)_+$ for $n \geq 1$ and H a closed subgroup of G. A map of G-spaces is a *genuine cofibration* if it is a retract of a relative I_G-cell complex. Limits and colimits in $G\mathcal{T}$ are formed in $\mathcal{T}$ and then given the induced G-action.

Definition 1.1.1. The continuous functor $\mathcal{T} \to G\mathcal{T}$ equipping a space with the trivial G-action has both a left and a right adjoint. The left adjoint

$$(-)_G \colon G\mathcal{T} \to \mathcal{T},$$

assigns to a G-space X its *orbit space* X_G, and the right adjoint

$$(-)^G \colon G\mathcal{T} \to \mathcal{T},$$

assigns the subspace X^G of X of *G-fixed points*.

The fact that they are adjoints, implies in particular that $(-)_G$ preserves colimits and $(-)^G$ preserves limits, but even more is true (cf. [MM, III.1.6], or [Mal, Proposition 1.2]):

Lemma 1.1.2. *The functor $(-)^G$ preserves coproducts, pushouts of diagrams one leg of which is a closed inclusion, and colimits along sequences of closed inclusions. For X and Y in $G\mathcal{T}$, we have $(X \wedge Y)^G = X^G \wedge Y^G$.*

For subgroups H of G, we can use the forgetful functors induced by the inclusion of one object categories $i\colon H \to G$ to define fixed point functors

$$G\mathcal{T} \xrightarrow{i^*} H\mathcal{T} \xrightarrow{(-)^H} \mathcal{T},$$

and analogously for orbit spaces. It is often convenient to factor these functors in a different way, so that not all of the group action is forgotten: Let N be a closed normal subgroup of G. For a G-space X, the quotient group G/N acts on the N-fixed points X^N, and we can redefine the functor $(-)^N\colon G\mathcal{T} \to (G/N)\mathcal{T}$. The slight double use of notation is remedied by the fact that the following diagram of functors commutes:

$$\begin{array}{ccc} G\mathcal{T} & \xrightarrow{i_1^*} & N\mathcal{T} \\ \downarrow{\scriptstyle (-)^N} & & \downarrow{\scriptstyle (-)^N} \\ (G/N)\mathcal{T} & \xrightarrow{i_2^*} & \mathcal{T}, \end{array}$$

where $i_1\colon N \to G$ and $i_2\colon \{e\} \to G/N$ are the inclusions. Similarly we can consider the N-orbit functors $G\mathcal{T} \to (G/N)\mathcal{T}$.

Recall that an h-cofibration is a map $i\colon A \to X$ in $G\mathcal{T}$ so that the canonical map $Mi = X \coprod_A A \wedge I_+ \to X \wedge I_+$ has a retract. The following technicality proves to be helpful in several places:

Lemma 1.1.3. *The fixed point functors $(-)^H$ preserve h-cofibrations.*

Proof. By Lemma 1.1.2 there are homeomorphisms $(X \wedge I_+)^H \cong X^H \wedge I_+$ and $(Mi)^H \cong M(i^H)$, so that $(Mi)^H \to (X \wedge I_+)^H$ has a retraction. □

The same arguments also apply to the case where H is a normal subgroup and we consider $(-)^H$ as a functor to $G/H\mathcal{T}$.

The left adjoint of the restriction $i^*\colon G\mathcal{T} \to H\mathcal{T}$ is given by inducing up:

Definition 1.1.4. For an H-space $Y \in H\mathcal{T}$ the smash product $G_+ \wedge Y$ has an action of $G \times H$, with G acting from the left on G_+ and H acting diagonally, from the right on G_+ and from the left on Y, that is, $(g,h)(a \wedge y) = gah^{-1} \wedge hy$ for $(g,h) \in G \times H$ and $a \wedge y \in G_+ \wedge Y$. The *induced G-space* is the H-orbit G-space

$$G_+ \wedge_H Y = (G_+ \wedge Y)_H.$$

While the inducing up functor is generally not symmetric monoidal, there is an important compatibility property with the smash product of spaces which one checks by inspection:

Lemma 1.1.5. *If X is an H-space and Y a G-space there is a natural G-equivariant homeomorphism*

$$(G_+ \wedge_H X) \wedge Y \cong G_+ \wedge_H (X \wedge i^* Y), \qquad ([g,x],y) \leftrightarrow [g,(x,g^{-1}y)].$$

Orbits and Fixed Points for semi-direct Products

Throughout our work, actions of groups that are semi-direct products appear in various places. We will recall a few elementary properties, before investigating the more complicated interactions of the orbit and fixed point functors that play a role in computing the fixed points of smash powers. We restate the definition, in order to fix some notation:

Definition 1.1.6. Let $(G, \cdot, e) \in \mathcal{U}$ be a compact Lie group acting on another compact Lie group $(O, \odot, E) \in \mathcal{U}$ via a group homomorphism $\phi\colon G \to \mathrm{Aut}(O)$. The *semi-direct product* $G \ltimes (O, \phi)$ is the product space $G \times O$ equipped with the multiplication defined by

$$\begin{array}{rcl}(G \ltimes (O,\phi)) \times (G \ltimes (O,\phi)) & \to & (G \ltimes (O,\phi)) \\ (g,A),(h,B) & \mapsto & (g \cdot h, A \odot \phi(g)(B)).\end{array}$$

When ϕ is implicit in the context we write $G \ltimes O$ instead of $G \ltimes (O, \phi)$.

For the rest of this section G and O are compact Lie groups with a group homomorphism $\phi\colon G \to \mathrm{Aut}(O)$.

Remark 1.1.7. Specifying an action of the semi-direct product $G \ltimes O$ on a space Z is the same as a giving actions of G and O on Z such that the map $O \to \mathcal{T}(Z, Z)$ defining the action of O on Z is a G-map.

Remark 1.1.8. If G acts on O through inner automorphisms, that is, if there is a homomorphism $\psi\colon A \to O$ such that $\phi(g)(A) = \psi(g) \odot A \odot \psi(g^{-1})$, we write $G \ltimes_\psi O$ instead of $G \ltimes (O, \phi)$. In this situation then there is an isomorphism $G \ltimes_\psi O \to G \times O$ of topological groups taking (g, A) to $(g, A \odot \psi(g))$.

It is usually cumbersome to explicitly write the signs "$\cdot$" and "$\odot$" for the multiplications of G and O, so we often omit them. The following properties are elementary:

Lemma 1.1.9. *(i) Mapping $A \in O$ to $(e, A) \in G \ltimes O$ embeds O as a (closed) normal subgroup.*

(ii) Mapping $g \in G$ to $(g, E) \in G \ltimes O$ embeds G as a closed subgroup.

(iii) For $A \in O$ and $g \in G$, the following elements of the semi-direct product are equal:

$$(e, \phi(g)(A)) = (g, E)(e, A)(g, E)^{-1}$$

(iv) G is normal in $G \ltimes O$ if and only if ϕ is trivial. In this situation the semi-direct product is actually the direct product.

(v) The projection $\mathrm{pr}_1\colon G \ltimes O \to G$ to the first factor is a group homomorphism.

Motivated by the first two points in the lemma, we identify $A \in O$ with $(e, A) \in G \ltimes O$ and $g \in G$ with $(g, E) \in G \ltimes O$. Under this identification, the third point of the lemma reads $gAg^{-1} = \phi(g)(A)$.

Lemma 1.1.10. *Let Z be an $G \ltimes G$-space (cf. Remark 1.1.7.) Given $z \in Z$, write* Stab_z *for the stabilizator subgroup of z with respect to the corresponding action of $G \ltimes O$ on Z. The O-orbit Oz is free if and only if the composition*

$$\mathrm{Stab}_z \subseteq G \ltimes O \xrightarrow{\mathrm{pr}_1} G$$

is injective.

Proof. The O-orbit Oz is free if and only if the stabilizer subgroup Stab_z^O of O is trivial. Under the embedding of O in $G \ltimes O$, this stabilizer subgroup corresponds to the kernel $\mathrm{Stab}_z \cap (\{e\} \ltimes O) = \mathrm{Stab}_z \cap \mathrm{pr}_1^{-1}(e)$ of the composite homomorphism in question. □

For any space Z with an action of the semi-direct product, its orbit space Z_O inherits an action of G. We want to investigate in how far taking such O-orbits commutes with taking fixed points with respect to subgroups of G.

Proposition 1.1.11. *Let Z be an $G \ltimes G$-space (cf. Remark 1.1.7.) Assume that O acts freely on Z (away from the basepoint).*
Then the canonical map from the quotient of the fixed points into the fixed points of the quotient

$$(Z^G)_{O^G} \to (Z_O)^G \tag{1.1.12}$$

is injective.

Proof. Assume $[z_1]$ and $[z_2]$ in $(Z^G)_{O^G}$ map to the same element in the target, i.e., $z_1 = Az_2$, for some A in O. Then for any g in G we have

$$Az_2 = z_1 = gz_1 = g(Az_2) = (\phi(g)A)(gz_2) = (\phi(g)A)z_2.$$

But since the O action on Z was free, this implies that $A = \phi(g)A$ for all $g \in G$. Thus $A \in O^G$ and $[z_1] = [z_2]$. □

Surjectivity can not be guaranteed in this generality, as the following example illustrates:

Example 1.1.13. Let $Z = S(\mathbb{C})_+$ the unit circle in $\mathbb{C}$, with some disjoint basepoint. Let $O = \mathbb{Z}/4$ and $G = \mathbb{Z}/2$ such that the action of the non trivial element in G maps an element of O to its inverse. In particular $G \ltimes O$ is the dihedral group D_4. Then O acts freely on $S(\mathbb{C})$ through rotations by 90 degrees, G acts by complex conjugation, and one checks that the actions compatibly fit together into an action of D_4. Take a closer look at source and target of the map $(Z^G)_{O^G} \to (Z_O)^G$:
Note that O^G is the subgroup of self inverse elements $\mathbb{Z}/2 \subset \mathbb{Z}/4$, i.e., generated by the rotation by 180 degrees. The G-fixed point space $(S(\mathbb{C}))^G$ has 2 points which are in the same O^G-orbit, i.e., the source of $(Z^G)_{O^G} \to (Z_O)^G$ contains only one point. On the other hand, taking orbits first, we see that Z_O is isomorphic to $S(\mathbb{C})$, with the action of G again given by complex conjugation, i.e., target of $(Z^G)_{O^G} \to (Z_O)^G$ consists of 2 points, such that the map can not be surjective.

The intuition behind the failure in surjectivity is that there are "diagonal" copies of G in $G \ltimes O$, and points with isotropy type of such a diagonal copy contribute to the target of $(Z^G)_{O^G} \to (Z_O)^G$ but not to the source. Motivated by this, we will give a formal sufficient condition for the surjectivity of $(Z^G)_{O^G} \to (Z_O)^G$. First, consider the simple case of only one $G \ltimes O$-orbit, and take a closer look at the target space:

Lemma 1.1.14. *Let Z be a $G \ltimes O$-space consisting of a single O-free $G \ltimes O$-orbit, i.e., $Z = G \ltimes O/P$ for some closed subgroup P of $G \ltimes O$ with the projection $\mathrm{pr}_1 : P \to G$ to the first factor injective. Then $(Z_O)^G$ contains at most one element, and is non-empty if and only if the projection $\mathrm{pr}_1 : P \to G$ is an isomorphism.*

Proof. The projection $\mathrm{pr}_1 : P \to G$ to the first factor is injective by 1.1.10. Hence $Z_O \cong G/\mathrm{pr}_1\, P$ as G-spaces, and the latter has exactly one G-fixed point if and only if $\mathrm{pr}_1(P) = G$, and is empty otherwise. □

On the other hand, the following elementary fact gives a characterization of the source:

Lemma 1.1.15. *Let H be a topological group with subgroups P and G. Then the space $(H/P)^G$ is non-empty, if and only if G is subconjugate to P.*
More precisely, $(H/P)^G$ is a quotient of the subspace of all those elements $h \in H$, that conjugate G into P.

Proof. Let hP be a point in the orbit space. Then hP is G-fixed, if and only if for all $g \in G$ we have $ghP = hP$, or equivalently $h^{-1}gh \in P$. □

Proposition 1.1.16. *Let Z be a genuinely cofibrant $G \ltimes O$-space. Suppose O acts freely on Z. Then the map $(Z^G)_{O^G} \to (Z_O)^G$ is an isomorphism if for every stabilizer subgroup P of an orbit appearing in the cell-decomposition of Z the following two statements are logically equivalent:*

(i) the projection to the first factor induces an isomorphism $\mathrm{pr}_1(P) \cong G$

(ii) $G \ltimes \{E\} \subset G \ltimes O$ is subconjugate to P.

Proof. Note that both taking fixed points and taking orbits preserves the cell-complex construction by 1.1.2. Hence the natural map $(Z^G)_{O^G} \to (Z_O)^G$ induces a natural isomorphism of cell diagrams, hence an isomorphism on the transfinite composition. □

Corollary 1.1.17. *Suppose $G \ltimes O$ has the property that for every subgroup P of $G \ltimes O$ the projection to the first factor induces an isomorphism $\mathrm{pr}_1(P) \cong G$ if and only if $G \ltimes \{E\} \subset G \ltimes O$ is subconjugate to P. Then for any genuinely cofibrant $G \ltimes O$-space the natural map $(Z^G)_{O^G} \to (Z_O)^G$ is an isomorphism.*

Example 1.1.18. Note that 1.1.13 gives a non-example of this. In particular, the semi-direct product D_4 has a subgroup P of order 2 generated by $(1, 2) \in \mathbb{Z}/2 \ltimes \mathbb{Z}/4$ which acts on $S(\mathbb{C})$ via the reflection with respect to the imaginary axis. Note that since this element is in the center of D_4 the subgroup it generates can not be conjugate to G. In particular, since $S(\mathbb{C})$ has two P-fixed points, an equivariant cell decomposition will have to use a cell of type $G \ltimes O/P_+ \wedge S_0$.

The main example we want to apply Corollary 1.1.17 to is the following:

Example 1.1.19. Let G be a discrete group, X a discrete free G-space and $\varphi\colon G \to \mathbf{O}_V$ a finite dimensional G-representation. Then G acts on $\mathbf{O}_V$ through the compositon $G \xrightarrow{\varphi} \mathbf{O}_V \to \mathrm{Aut}(\mathbf{O}_V)$ of φ and the conjugation homomorphism letting $\mathbf{O}_V$ act on itself by inner automorphisms. Let in the above notation $O = \prod_X \mathbf{O}_V$ with G acting by conjugation: $g \in G$ acts on $\{M_x\}_{x\in X} \in O$ by taking it to $\{\varphi(g)M_{g^{-1}x}\varphi(g^{-1})\}_{x\in X}$. The conditions of Corollary 1.1.17 are satisfied by the semi-direct product $G \ltimes \prod_X \mathbf{O}_V$:

Indeed, consider a subgroup P with $\mathrm{pr}_1(P) \cong G$ and let

$$\begin{array}{rccl} \psi\colon & G & \to & P \subset G \ltimes O \\ & g & \mapsto & (g, \{A_x^g\}_{x\in X}) \end{array}$$

be the inverse of pr_1. Looking at the multiplication in $G \ltimes O$, the fact that ψ is a group homomorphism, i.e., $\psi(gh) = (gh, \{A_x^{gh}\}_{x\in X})$ the formula

$$A_x^g \varphi(g) A_{g^{-1}x}^h \varphi(g^{-1}) = A_x^{gh} \qquad \forall g, h \in G,\ x \in X \tag{1.1.20}$$

Now chose a system of representatives R for the G-orbits in X. Let $B = \{B_x\} \in O$ be the element given by

$$B_x = B_{hr} := \varphi(h) A_r^{h^{-1}} \varphi(h^{-1}),$$

where $x = hr$ is the unique presentation of x with $h \in G$ and $r \in R$. Combining the formulas above gives us that for any $(g, \{A_x^g\}) \in P$, we have:

$$(e, \{B_x\})(g, \{A_x^g\}) = (g, \{B_x A_x^g\})$$

and

$$(g, E)(e, \{B_x\}) = (g, \{\varphi(g) B_{g^{-1}x} \varphi(g^{-1})\}).$$

If $x = hr$, then $g^{-1}x = g^{-1}hr$ and $B_{g^{-1}x} = \varphi(g^{-1}h) A_r^{h^{-1}g} \varphi(h^{-1}g)$. By equation 1.1.20 we have $A_r^{h^{-1}g} = A_r^{h^{-1}} \varphi(h^{-1}) A_{hr}^g \varphi(h)$ so

$$\varphi(g) B_{g^{-1}x} \varphi(g^{-1}) = \varphi(h) A_r^{h^{-1}g} \varphi(h^{-1}) = \varphi(h) A_r^{h^{-1}} \varphi(h^{-1}) A_{hr}^g = B_x A_x^g.$$

Hence P is subconjugate to $G \ltimes \{E\}$ via $B \in O$. Because pr_1 is surjective this implies that P is conjugate to $G \ltimes \{E\}$ and therefore taking O-orbits commutes with taking G-fixed points.

Proposition 1.1.21. *Let G be a discrete group, X a free discrete G-space and let V be a finite dimensional G-representation. Let as in Example 1.1.19 O be the group $\prod_X \mathbf{O}_V$ with G acting by conjugation. For every O-free genuinely cofibrant $G \ltimes O$-space Z and* *for every subgroup H of G, taking O-orbits commutes with taking H-fixed points in the sense that the canonical map*

$$(Z^H)_{O^H} \to [Z_O]^H ,$$

is an isomorphism.

Proof. Note that for subgroups $H \subset G$, any free G-set is a free H-set, and any genuinely cofibrant $G \ltimes \prod_X \mathbf{O}_V$-space is also genuinely cofibrant as a $H \ltimes \prod_X \mathbf{O}_V$-space by 1.2.4. We can then apply Corollary 1.1.17 with the help of Example 1.1.19 for all choices of H. □

1.2 Illman's Triangulation Theorem

In several places we will need to check cofibrancy with respect to the genuine cofibrations. By the general theory for model categories it will usually suffice to understand the class of I_G-cell complexes in $G\mathcal{T}$. From here on for the rest of the section, we restrict to the case of G a compact Lie group. For convenience, we will recall the statements and the relevant definition from [Ill83] before we give some corollaries.

Definition 1.2.1. Let X be a G-space. Given an orbit $[x] \in X_G$, define the *G-isotropy type* of $[x]$ as the conjugacy class of the stabilizer subgroup $\mathrm{Stab}_x = \{g \in G \mid gx = x\}$ of G. Since the stabilizer subgroups of elements in the same orbit are all conjugate, this indeed only depends on the element $[x] \in X_G$.

Theorem 1.2.2 ([Ill83, 5.5, 6.1, 7.1])**.** *Let X be space with action of a compact Lie group G and a triangulation $t\colon K \to X_G$ of the orbit space X_G, such that the G-isotropy type is constant on open simplices, i.e., for each open simplex $\mathring{s}$ of K the G-isotropy type is constant on $t(\mathring{s}) \subset X_G$. Then X admits a structure of a G-equivariant CW-complex. In particular X is an I_G-cell complex.*

Furthermore, if M is a smooth G-manifold (with or without boundary), then the orbit space M_G does *admit a triangulation such that the G-isotropy type is constant on open simplices, and consequently M admits a structure of a G-equivariant CW-complex.*

Remark 1.2.3. In particular, for every representation V of a compact Lie group G, its one-point compactification S^V is a G-CW-complex.

Corollary 1.2.4. *Let G be a compact Lie group and H a closed subgroup. Then G is an H-CW complex and an I_G-cell complex is an I_H-cell complex.*

Proof. By induction on the cell structure, and since smashing with a space preserves colimits, it suffices to show that for any closed subgroup $K \subset G$ the orbit space G/K is an I_H-cell complex. However, G/K is a smooth G-manifold, and thus also a smooth H-manifold. Theorem 1.2.2 now implies that G/K is an I_H-cell complex. □

The following result is elementary:

Lemma 1.2.5. *Let Y be a retract of an I_G-cell complex X. The cells of X are of the form $D^n \times G/L$, where L is the isotropy group of an element of X. Conversely, if L is the isotropy group of an element of Y, then there is at least one cell of X isomorphic to a cell of the form $D^n \times G/L$.*

Corollary 1.2.6. *Let H and K closed subgroups of G. The product $G/H \times G/K$ is again an I_G-cell complex, and the only orbit types that appear are of the form G/L, with L subconjugate to both H and K.*

Proof. Note that $G/H \times G/K$ is isomorphic to $(G \times G)/(H \times K)$ and embed G into $G \times G$ as the diagonal (closed) subgroup, so that Corollary 1.2.4 gives that $G/H \times G/K$ is an I_G-cell complex. For the statement about orbit types, we check what kind of stabilizer subgroups can appear in the product $G/H \times G/K$ and use Lemma 1.2.5. In fact, if L is the stabilizer of $([g_1], [g_2])$, then we have that $Lg_1 \subset g_1 H$ or equivalently that L is subconjugate to H. The analogous argument for K finishes the proof. □

1.3 Mixed Model Structures

In this section G is a compact Lie group.

Definition 1.3.1. A set $\mathscr{A}$ of closed subgroups of G is a *family*, if it is closed under taking subgroups and conjugates.

Definition 1.3.2. Let $\mathscr{A}$ be a family of subgroups of G. A map $f\colon X \to Y$ in $G\mathcal{T}$ is an $\mathscr{A}$-equivalence if $f^H\colon X^H \to Y^H$ is a weak equivalence in $\mathcal{T}$ for every H in $\mathscr{A}$.

It is often convenient to have homotopical properties of h-cofibrations at hand when working with topological model categories. The following lemma sums up what we will use in the G-equivariant context:

Lemma 1.3.3. *Let $\mathscr{A}$ be a family of subgroups of G.*

(i) sequential colimits of $\mathscr{A}$-equivalences along h-cofibrations are $\mathscr{A}$-equivalences

(ii) pushouts of $\mathscr{A}$-equivalences along h-cofibrations between well based G-spaces are $\mathscr{A}$-equivalences

(iii) the cube lemma 6.1.6 holds for $\mathscr{A}$-equivalences and h-cofibrations between well-based G-spaces

(iv) the cube lemma holds for G-homotopy equivalences and h-cofibrations.

Proof. All points follow directly from the analogous statement about weak equivalences in $\mathcal{T}$ 6.1.28, and Lemma 1.1.2. □

Let $\mathscr{B} \subseteq \mathscr{A}$ be families of subgroups of G. We consider $\mathscr{A}$ and $\mathscr{B}$ as orbit categories, that is, as $\mathcal{T}$-categories with $\mathscr{A}(K, H) = G\mathcal{T}(G/H_+, G/K_+)$. The inclusion $i\colon \mathscr{B} \to \mathscr{A}$ induces a functor $i^*\colon \mathscr{A}\mathcal{T} \to \mathscr{B}\mathcal{T}$ with a left adjoint functor $L\colon \mathscr{B}\mathcal{T} \to \mathscr{A}\mathcal{T}$. Explicitly

L can be constructed by left Kan extension. With respect to the *projective level model structures* provided by Theorem 6.2.1 the functors L and i^* form a Quillen adjoint pair.

For $H \in \mathscr{A} - \mathscr{B}$, we let $l_H\colon C_H \to i^*\mathscr{A}(H,-) = \mathscr{A}(H, i(-))$ be the cofibrant replacement of $i^*\mathscr{A}(H,-)$ in the projective level model structure on $\mathscr{B}\mathcal{T}$, and we let $\lambda_H\colon LC_H \to \mathscr{A}(H,-)$ be the adjoint morphism of l_H in $\mathscr{A}\mathcal{T}$. Using the mapping cylinder like in the proof of Ken Brown's Lemma [HirL, Lemma 7.7.1], we factor λ_H as the composition $LC_H \xrightarrow{s_H} M_H \xrightarrow{r_H} \mathscr{A}(H,-)$ of a cofibration s_H and the left inverse r_H of an acyclic cofibration in $\mathscr{A}\mathcal{T}$.

Definition 1.3.4. The set S consists of the morphisms s_H for $H \in \mathscr{A} - \mathscr{B}$, and we W is the class of *$\mathscr{B}$-equivalences* in $\mathscr{A}\mathcal{T}$, that is, morphisms f in $\mathscr{A}\mathcal{T}$ with the property that i^*f is a *weak equivalence in $\mathscr{B}\mathcal{T}$. A map $g\colon z \to w$ in $\mathscr{A}\mathcal{T}$ is an* S-fibration *if it is a fibration in $\mathscr{A}\mathcal{T}$ with the property that for every morphism $s\colon a \to b$ of S, the square*

$$\begin{array}{ccc} \mathscr{A}\mathcal{T}(b,z) & \xrightarrow{g_*} & \mathscr{A}\mathcal{T}(b,w) \\ {\scriptstyle s^*}\downarrow & & \downarrow{\scriptstyle s^*} \\ \mathscr{A}\mathcal{T}(a,z) & \xrightarrow{g_*} & \mathscr{A}\mathcal{T}(a,w) \end{array}$$

is a homotopy pullback square in $\mathcal{T}$.

Note that every member of S is a $\mathscr{B}$-equivalence.

Proposition 1.3.5. *If $g\colon z \to w$ is both a $\mathscr{B}$-equivalence in $\mathscr{A}\mathcal{T}$ and an S-fibration, then it is a weak equivalence in the projective level model structure on $\mathscr{A}\mathcal{T}$.*

Proof. It suffices to show that if $H \in \mathscr{A} - \mathscr{B}$, then $\mathscr{A}\mathcal{T}(\mathscr{A}(H,-), g) \cong g(H)$ is a weak equivalence in $\mathcal{T}$. Since g is a fibration in $\mathscr{A}\mathcal{T}$ and a $\mathscr{B}$-equivalence, the map i^*g is an acyclic fibration in $\mathscr{B}\mathcal{T}$ so $\mathscr{A}\mathcal{T}(LC_H, g) \cong \mathscr{B}\mathcal{T}(C_H, i^*g)$ is an acyclic fibration in $\mathcal{T}$.

Since g is an S-fibration, for every $s_H\colon LC_H \to M_H$ in S, the square

$$\begin{array}{ccc} \mathscr{A}\mathcal{T}(M_H,z) & \xrightarrow{\mathscr{A}\mathcal{T}(s_H,z)} & \mathscr{A}\mathcal{T}(LC_H,z) \\ {\scriptstyle \mathscr{A}\mathcal{T}(M_H,g)}\downarrow & & \downarrow{\scriptstyle \mathscr{A}\mathcal{T}(LC_H,g)} \\ \mathscr{A}\mathcal{T}(M_H,w) & \xrightarrow{\mathscr{A}\mathcal{T}(s_H,w)} & \mathscr{A}\mathcal{T}(LC_H,w) \end{array}$$

is a homotopy pullback. Since r_H is the left inverse of an acyclic cofibration in $\mathscr{A}\mathcal{T}$, also the square

$$\begin{array}{ccc} \mathscr{A}\mathcal{T}(\mathscr{A}(H,-),z) & \xrightarrow{\mathscr{A}\mathcal{T}(\lambda_H,z)} & \mathscr{A}\mathcal{T}(LC_H,z) \\ {\scriptstyle \mathscr{A}\mathcal{T}(\mathscr{A}\mathcal{T}(H,-),g)}\downarrow & & \downarrow{\scriptstyle \mathscr{A}\mathcal{T}(LC_H,g)} \\ \mathscr{A}\mathcal{T}(\mathscr{A}(H,-),w) & \xrightarrow{\mathscr{A}\mathcal{T}(\lambda_H,w)} & \mathscr{A}\mathcal{T}(LC_H,w) \end{array}$$

is a homotopy pullback. Since i^*g is an acyclic fibration, the right hand vertical map $\mathscr{A}\mathcal{T}(LC_H, g) \cong \mathscr{B}\mathcal{T}(C_H, i^*g)$ a weak equivalence. Thus also the left hand vertical map $\mathscr{A}\mathcal{T}(\mathscr{A}(H,-), g) \cong g(H)$ is a weak equivalence in $\mathcal{T}$. □

Applying Bousfield localization [HirL, Theorem 4.1.1] we obtain an S-local model structure on $\mathscr{A}\mathcal{T}$.

Definition 1.3.6. The *$\mathscr{B}$-model structure* on $\mathscr{A}\mathcal{T}$ is the Bousfield localization of the projective level model structure on $\mathscr{A}\mathcal{T}$ with respect to the set S. We write J_S for the generating set of acyclic cofibrations given by the union of the set of generating acyclic cofibrations for the projective level model structure on $\mathscr{A}\mathcal{T}$ and the set consisting of morphisms of the form $i\Box s$ for $i \in I$ a generating cofibration for $\mathcal{T}$ and $s \in S$.

Note that the cofibrations for the $\mathscr{B}$-model structure on $\mathscr{A}\mathcal{T}$ are the original cofibrations of $\mathscr{A}\mathcal{T}$ and that this model structure is left proper and cellular.

Definition 1.3.7. Let $\mathscr{B} \subseteq \mathscr{A}$ be non-empty families of subgroups of G. The set $I_{\mathscr{A}}$ is the set of G-maps of the form $(i \times G/H)_+$ for $i \in I$ a generating cofibration for $\mathcal{T}$ and $H \in \mathscr{A}$. The set $J_{\mathscr{B},\mathscr{A}}$ is the union of the set of $J_{\mathscr{A}}$ of G-maps of the form $(i \times G/H)_+$ for $j \in J$ a generating acyclic cofibration for $\mathcal{T}$ and $H \in \mathscr{A}$ with the set consisting of maps of the form $i\Box k_H$ for $i \in I$ a generating cofibration for $\mathcal{T}$ and $k_H = s_H(G/e)$ for $H \in \mathscr{A} - \mathscr{B}$ and s_H in the set S from 1.3.4.

Note that if $\mathscr{A}$ is equal to the family of all closed subgroups of G, then $I_{\mathscr{A}} = I_G$

Theorem 1.3.8. *Let $\mathscr{B} \subseteq \mathscr{A}$ be non-empty families of subgroups of G. The set $I_{\mathscr{A}}$ is a set of generating cofibrations and the set $J_{\mathscr{B},\mathscr{A}}$ is a set of generating acyclic cofibrations for a left proper cellular monoidal model structure on $G\mathcal{T}$ of pointed G-spaces with the class of $\mathscr{B}$-equivalences as weak equivalences.*

Proof. The functor $\Phi\colon G\mathcal{T} \to \mathscr{A}\mathcal{T}$ with $\Phi(X)(H) = X^H$ is right adjoint to the functor $\Lambda\colon \mathscr{A}\mathcal{T} \to G\mathcal{T}$ with $\Lambda(Y) = Y(G/e)$. Via this adjunction we can follow Mandell and May [MM, Theorem III.1.8] and pull the $\mathscr{B}$-model structure on $\mathscr{A}\mathcal{T}$ back to a model structure on $G\mathcal{T}$, where a G-map $f\colon X \to Y$ is a cofibration, weak equivalence or fibration if and only if $\Phi(f)$ is so in the $\mathscr{B}$-model structure on $\mathscr{A}\mathcal{T}$. This is the $(\mathscr{B},\mathscr{A})$-mixed model structure on $G\mathcal{T}$. The statement about generating (acyclic) cofibrations is a consequence of the fact that Λ takes a set of generating (acyclic) cofibrations for the $\mathscr{B}$-model structure on $\mathscr{A}\mathcal{T}$ to a set of generating (acyclic) cofibrations for the $(\mathscr{B},\mathscr{A})$-model structure on $G\mathcal{T}$. □

Definition 1.3.9. The *$(\mathscr{B},\mathscr{A})$-model structure*, on the category $G\mathcal{T}$ of pointed G-spaces is the model structure specified in Theorem 1.3.8.

Note that in the particular case where $\mathscr{B} = \mathscr{A}$, the fibrations in the $(\mathscr{B},\mathscr{A})$-model structure are the $\mathscr{A}$-fibrations, that is, the maps $f\colon X \to Y$ with the property that $f^H\colon X^H \to Y^H$ is a fibration for every H in $\mathscr{A}$.

Definition 1.3.10. Let $\mathscr{A}$ be the family of all closed subgroups of G. The *genuine model structure* on $G\mathcal{T}$ is the $(\mathscr{A},\mathscr{A})$-model structure. The *genuine cofibrations*, *genuine equivalences* and *genuine fibrations* are the cofibrations, weak equivalences and fibrations in the genuine model structure.

Definition 1.3.11. The cofibrant replacement of S^0 in the $(\mathscr{A}, \mathscr{A})$-model structure is denoted $E\mathscr{A}_+$. The fiber over the non base point of the map $E\mathscr{A}_+ \to S^0$ is an unpointed space denoted $E\mathscr{A}$.

Lemma 1.3.12. *Let $\mathscr{B} \subseteq \mathscr{A}$ be non-empty families of subgroups of a compact Lie group G and let $i\colon H \to G$ be the inclusion of a subgroup. If we let $i^*\mathscr{B}$ be the family of subgroups $P\mathrm{Cat}H$ of H obtained by intersecting members P of $\mathscr{B}$ with H and likewise let $i^*\mathscr{A}$ be the family of intersections members of $\mathscr{A}$ with H, then the forgetful functor $i_H^*\colon G\mathcal{T} \to H\mathcal{T}$ is a right Quillen functor of mixed model structures with respect to $(\mathscr{B}, \mathscr{A})$ and $(i^*\mathscr{B}, i^*\mathscr{A})$ respectively.*

Chapter 2

Equivariant Orthogonal Spectra

2.1 Orthogonal spectra as functors

In this section we give a short review of one definition of orthogonal spectra as functors. Later, when we work with model structures, this functoriality is convenient. An orthogonal spectrum is defined to be a functor from a $\mathcal{T}$-category **O** to spaces. This category appeared first in [MMSS, Example 4.4.].

Definition 2.1.1. The $\mathcal{U}$-category $\mathcal{L}$ of linear isometric embeddings has as objects the class of inner product spaces, and as morphism spaces the spaces of linear isometric embeddings.

The following is essentially [MM, Definition I.5.9].

Definition 2.1.2. The $\mathcal{T}$-category $\mathcal{O}$ has as objects the class of inner product spaces. Given inner product spaces V and W, the morphism space $\mathcal{O}(V,W)$ is the subspace of $\mathcal{L}(V,W)_+ \wedge S^W$ consisting of points of the form $f \wedge x$ for $f\colon V \to W$ a linear isometric embedding and x contained in the subspace of S^W given by the one-point compactification of the orthogonal complement of $f(V)$ in W. The composition $(f,x)\circ(g,y)$ of two non-trivial morphisms $f \wedge x \in \mathcal{O}(V,W)$ and $g \wedge y \in \mathcal{O}(U,V)$ is $fg \wedge (x + fy)$.

The category $\mathcal{O}$ has a symmetric monoidal product

$$\begin{array}{rcl} \mathcal{O}(V,W) \times \mathcal{O}(V',W') & \xrightarrow{\oplus} & \mathcal{O}(V \oplus V', W \oplus W') \\ (f \wedge x, f' \wedge x') & \mapsto & (f \oplus f') \wedge (x + x'). \end{array}$$

There is a $\mathcal{T}$-functor $p\colon \mathcal{O} \to \mathcal{L}_+$ which is the identity on objects and with $p(f \wedge x) = f$ when $f \wedge x$ is different form the base point in $\mathcal{O}(V,W)$.

Definition 2.1.3. The *untwisting map*

$$\tau_{V,W}\colon \mathcal{O}(V,W) \wedge S^V \to \mathcal{L}(V,W)_+ \wedge S^W$$

takes $f \wedge x \wedge y \in \mathcal{O}(V,W) \wedge S^V$, where $f \in \mathcal{L}(V,W)$, x is in the orthogonal complement of $f(V)$ in W and $y \in V$, to $f \wedge (x + f(y)) \in \mathcal{L}(V,W)_+ \wedge S^W$.

Lemma 2.1.4. *The untwisting map*

$$\tau_{V,W} \colon \mathcal{O}(V,W) \wedge S^V \xrightarrow{\cong} \mathcal{L}(V,W)_+ \wedge S^W$$

is a homeomorphism.

Lemma 2.1.5. *For all morphisms $\varphi \colon V' \to V$ and $\psi \colon W \to W'$ the following diagram commutes:*

$$\begin{array}{ccccccc}
\mathcal{O}(V,W) \wedge S^V & \xrightarrow{\tau_{V,W}} & \mathcal{L}(V,W) \wedge S^W & \xrightarrow{\varphi^*} & \mathcal{L}(V',W) \wedge S^W & \xrightarrow{\tau^{-1}_{V',W}} & \mathcal{O}(V',W) \wedge S^V \\
\downarrow{\psi_*} & & & & & & \downarrow{\psi_*} \\
\mathcal{O}(V,W') \wedge S^V & \xrightarrow{\tau_{V,W'}} & \mathcal{L}(V,W') \wedge S^{W'} & \xrightarrow{\varphi^*} & \mathcal{L}(V',W') \wedge S^{W'} & \xrightarrow{\tau^{-1}_{V',W'}} & \mathcal{O}(V',W') \wedge S^V.
\end{array}$$

Definition 2.1.6. The symmetric monoidal category $\mathbf{O}$ is the full subcategory of $\mathcal{O}$ with the coordinate spaces $\mathbb{R}^n$ for $n \in \mathbb{N}$ as objects and the symmetric monoidal category $\mathbf{L}$ is the full subcategory of $\mathcal{L}$ with the coordinate spaces $\mathbb{R}^n$ for $n \in \mathbb{N}$ as objects.

Note that for $m \leq n$, the morphism space $\mathbf{L}(\mathbb{R}^m, \mathbb{R}^n)$ is homeomorphic to $\mathbf{O}_{\mathbb{R}^n}/\mathbf{O}_{\mathbb{R}^{n-m}}$.

Remark 2.1.7. The standard basis gives an isomorphism $\mathbf{O}_{\mathbb{R}^n} \cong \mathbf{O}_n$ between the group $\mathbf{O}_{\mathbb{R}^n} = \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ of isometric automorphisms of $\mathbb{R}^n$ and the group $\mathbf{O}_n$ of orthogonal $n \times n$-matrices.

Definition 2.1.8. An orthogonal spectrum X is a $\mathcal{T}$-functor $X \colon \mathbf{O} \to \mathcal{T}$. Morphisms of orthogonal spectra are natural transformations. We write $\mathbf{O}\mathcal{T}$ for the category of orthogonal spectra.

Given an orthogonal spectrum X and an object V of $\mathbf{O}$, we write X_V for the value of the functor X at V. Let us stress that in this paper, an orthogonal spectrum is only defined on the inner product spaces $\mathbb{R}^n$ with the dot product as inner product. Choosing an equivalence $\mathcal{O} \to \mathbf{O}$ of categories we can extend an orthogonal spectrum to a functor defined on $\mathcal{O}$. A more canonical way of extending orthogonal spectra to $\mathcal{O}$ is by letting $X(V) = \mathcal{L}(\mathbb{R}^n, V)_+ \wedge_{\mathbf{O}_{\mathbb{R}^n}} X_{\mathbb{R}^n}$ when V is an n-dimensional inner product space. We have chosen only to evaluate orthogonal spectra on coordinate spaces since this leads us to be more explicit about morphisms in certain constructions.

Example 2.1.9 (The sphere spectrum)**.** The *sphere spectrum* $\mathbb{S} \colon \mathbf{O} \to \mathcal{T}$ is the representable functor $\mathbb{S} = \mathbf{O}(0, -)$ with $\mathbb{S}_V = \mathbf{O}(0, V) \cong S^V$.

Since the category $\mathbf{O}\mathcal{T}$ is a category of $\mathcal{T}$-functors into $\mathcal{T}$ it is enriched and tensored and cotensored over $\mathcal{T}$. The tensor $K \wedge X$ of a space K and an orthogonal spectrum X is the functor given as the composition

$$\mathbf{O} \xrightarrow{X} \mathcal{T} \xrightarrow{K \wedge -} \mathcal{T}.$$

The cotensor $F(K,X)$ is the functor given as the composition

$$\mathbf{O} \xrightarrow{X} \mathcal{T} \xrightarrow{F(K,-)} \mathcal{T},$$

where $F(K,-)$ takes a space L to the space $F(K,L)$ of functions from K to L. Applying the adjunction isomorphism for the smash product and function space objectwise we obtain adjunction isomorphisms

$$\mathcal{T}(K, \mathbf{O}\mathcal{T}(X,Y)) \cong \mathbf{O}\mathcal{T}(K \wedge X, Y) \cong \mathbf{O}\mathcal{T}(X, F(K,Y))$$

that are natural in the space K and in the orthogonal spectra X and Y.

Example 2.1.10. The tensor $K \wedge \mathbb{S}$ of a space K with the sphere spectrum is the suspension spectrum on K. The Yoneda homeomorphism $\mathbf{O}\mathcal{T}(\mathbb{S}, X) \cong X_0$ and the above adjunction isomorphism for the tensor gives a natural homeomorphism

$$\mathcal{T}(K, X_0) \cong \mathcal{T}(K, \mathbf{O}\mathcal{T}(\mathbb{S}, X)) \cong \mathbf{O}\mathcal{T}(K \wedge \mathbb{S}, X).$$

Thus the functor $\Sigma^\infty := - \wedge \mathbb{S} \colon \mathcal{T} \to \mathbf{O}\mathcal{T}$ is left adjoint to the evaluation functor $\mathrm{ev}_0 \colon \mathbf{O}\mathcal{T} \to \mathcal{T}$ with $\mathrm{ev}_0(X) = X_0$.

Since the category of orthogonal spectra is the category of $\mathcal{T}$-functors from a symmetric monoidal $\mathcal{T}$-category to $\mathcal{T}$, it can be equipped with a symmetric monoidal structure. This is discussed in detail in [MMSS, §21] based on the more general statement of [D, §3,4].

Let X and Y be orthogonal spectra. Their *external smash product* is the functor

$$\begin{array}{rrcl} X\bar{\wedge}Y\colon & \mathbf{O} \times \mathbf{O} & \to & \mathcal{T}. \\ & (V,W) & \mapsto & X_V \wedge Y_W \end{array}$$

The *(internal) smash product* of X and Y is the $\mathcal{T}$-enriched left Kan extension

$$X \wedge Y := \mathrm{Lan}_{\oplus}(X\bar{\wedge}Y) \colon \mathbf{O} \to \mathcal{T}$$

of $X\bar{\wedge}Y$ along the monoidal product $\oplus \colon \mathbf{O} \times \mathbf{O} \to \mathbf{O}$. In particular, by [K, 4.25], we can write the smash product as the coend

$$(X \wedge Y)_U = \int^{(V,W) \in \mathbf{O} \times \mathbf{O}} \mathbf{O}(V \oplus W, U) \wedge X_V \wedge Y_W.$$

The internal hom-object $\hom_{\mathbf{O}\mathcal{T}}(X,Y)$ is given by

$$\hom_{\mathbf{O}\mathcal{T}}(X,Y)_V = \mathbf{O}\mathcal{T}(X, Y(V \oplus -)).$$

The fact that the definition given here indeed gives a closed symmetric monoidal structure on $\mathbf{O}\mathcal{T}$ can be checked by applying the enriched Kan-extension to the coherence diagrams for $\mathbf{O}$, using the fact that the Kan-extension is natural in all its inputs, together with the fact that $\mathcal{T}$ itself was closed symmetric monoidal.

2.2 Equivariant Orthogonal Spectra

In this section we construct model structures on orthogonal spectra with action of a compact Lie group G. For the rest of this chapter G will be a fixed but arbitrary compact Lie group.

Definition 2.2.1. An *orthogonal G-spectrum X* is an orthogonal spectrum with continous action of G, that is, with a continous monoid homomorphism $G \to \mathbf{O}\mathcal{T}(X, X)$. Morphisms of orthogonal G-spectra are required to respect the action of G. We write $G\mathbf{O}\mathcal{T}$ for the category of orthogonal G-spectra.

Similar ways of looking at G-spectra have come up before, for example in the context of Γ-spaces [Shi].

Example 2.2.2 (The Sphere G-Spectrum)**.** Since the automorphism group of 0 in $\mathbf{O}$ is trivial, G can only act trivially on the sphere spectrum $\mathbb{S} = \mathbf{O}(0, -)$.

Example 2.2.3 (Free orthogonal G-spectra)**.** Let K be a pointed G-space. Given an object W of $\mathbf{O}$ and a homomorphism $\varphi\colon G \to \mathbf{O}_W$, the *free orthogonal G-spectrum* on φ and K is the orthogonal G-spectrum $\mathcal{F}_W K = \mathbf{O}(W, -) \wedge K$, where G acts diagonally on $(\mathcal{F}_W K)_V = \mathbf{O}(W, V) \wedge K$.

Considering G as a topological category with one object, the category $G\mathbf{O}\mathcal{T}$ is the category of $\mathcal{T}$-functors from G_+ to $\mathbf{O}\mathcal{T}$. Since the category $\mathbf{O}\mathcal{T}$ is enriched, tensored and cotensored over $\mathcal{T}$, the category $G\mathbf{O}\mathcal{T}$ is enriched, tensored and cotensored over $G\mathcal{T}$. Moreover, since $\mathbf{O}\mathcal{T}$ is closed symmetric monoidal, so is $G\mathbf{O}\mathcal{T}$. Explicitly, the enrichment in $G\mathcal{T}$ is given by considering the space $\mathbf{O}\mathcal{T}(X, Y)$ of morphisms between two orthogonal G-spectra as a G-space by letting G act by conjugation, that is, $g \in G$ takes $f \in \mathbf{O}\mathcal{T}(X, Y)$ to the composition

$$X \xrightarrow{g^{-1}} X \xrightarrow{f} Y \xrightarrow{g} Y.$$

We will denote this morphism G-space $\mathbf{O}\mathcal{T}_G(X, Y)$. The G-action on the smash product $X \wedge Y$ is through the diagonal embedding of G in $G \times G$. The internal function spectrum $\hom_{G\mathbf{O}\mathcal{T}}(X, Y)$ is given by the underlying internal function spectrum $\hom_{\mathbf{O}\mathcal{T}}(X, Y)$ with G-action by conjugation as above.

For X an orthogonal spectrum, the adjunction isomorphisms for the tensor $\wedge$ and for the suspension spectrum $\Sigma^\infty X$ give natural isomorphisms

$$\mathcal{T}(G_+, \mathbf{O}\mathcal{T}(X, X)) \cong \mathbf{O}\mathcal{T}(G_+ \wedge X, X) \cong \mathbf{O}\mathcal{T}(\Sigma^\infty G_+ \wedge X, X).$$

Remark 2.2.4. The action map $\mu\colon G_+ \to \mathbf{O}\mathcal{T}(X, X)$ is adjoint to a map $\bar{\mu}\colon \Sigma^\infty G_+ \wedge X \to X$. The fact that μ is a map of monoids exactly translates to X being a module over the orthogonal ring spectrum $\mathbb{S}_{[G]} := \Sigma^\infty G_+$ via $\bar{\mu}$. Then morphisms in $\mathbf{O}\mathcal{T}_G$ correspond to $\mathbb{S}$-module morphisms between $\mathbb{S}_{[G]}$-modules, whereas morphisms in $G\mathbf{O}\mathcal{T}$ correspond to $\mathbb{S}_{[G]}$-module maps.

By naturality the $\mathcal{T}$-isomorphisms for tensors and cotensors in $\mathbf{O}\mathcal{T}$ over $\mathcal{T}$ can be considered as $G\mathcal{T}$-natural isomorphisms

$$\mathcal{T}_G(D, \mathbf{O}\mathcal{T}_G(X,Y)) \cong \mathbf{O}\mathcal{T}_G(D \wedge X, Y) \cong \mathbf{O}\mathcal{T}_G(X, F(D,Y))$$

for tensors $\wedge$ and cotensors F in $\mathbf{O}\mathcal{T}_G$ over $\mathcal{T}_G$. Analogously to Example 5.2.29, taking the G-fixed points of the spaces above yields

$$G\mathcal{T}(D, \mathbf{O}\mathcal{T}_G(X,Y)) \cong G\mathbf{O}\mathcal{T}(D \wedge X, Y) \cong G\mathbf{O}\mathcal{T}(X, F(D,Y)).$$

Definition 2.2.5. Let V be an Euclidean space and let $Y \in G\mathbf{O}\mathcal{T}$ be an orthogonal G-spectrum.

(i) The restriction of Y to V is $Y_V \in G\mathbf{O}_V\mathcal{T}$, and we consider it as an object Y_V of $(G \times \mathbf{O}_V)\mathcal{T}$.

(ii) If $\varphi\colon G \to \mathbf{O}_V$ is a continous homomorphism, then through the homomorphism $i_\varphi\colon G \to G \times \mathbf{O}_V$ with $i_\varphi(g) = (g, \varphi(g))$ we obtain the G-space $i_\varphi^* Y_V$. Below we omit i_φ^* from the notation and refer to $i_\varphi^* Y_V$ as *the G-space Y_V*.

Example 2.2.6 (The Sphere G-Spectrum)**.** Recall from 2.2.2 that G must act trivially on $\mathbb{S}$. Given a homomorphism $\varphi\colon G \to \mathbf{O}_V$, the G-space $\mathbb{S}_V = S^V$ is the one-point compactification S^V of the representation V.

Example 2.2.7 (Free orthogonal G-spectra)**.** Let K be a pointed G-space and let $G \to \mathbf{O}_W$ be a homomorphism of Lie groups. Given a homomorphism $G \to \mathbf{O}_V$, the G-space $(\mathcal{F}_W K)_V = \mathbf{O}(W,V) \wedge K$ has diagonal G-action of the $G \times G$-action obtained from the action of G on K and the action of G on $\mathbf{O}(W,V)$ by conjugation.

Remark 2.2.8. Given any universe $\mathcal{U}$ of G-representations in the sense of [MM, Definition II.1.1], the category of orthogonal G-spectra considered here is equivalent to the category in [MM, Definition II.2.1] of spectra defined on representations isomorphic to finite dimensional G-invariant subspaces of $\mathcal{U}$, as explained by Mandell and May in [MM, Theorem V.1.5].

2.3 Semi-Free Equivariant Spectra

Definition 2.3.1. Let V be an object of $\mathbf{O}$. The functor $\mathcal{G}_V\colon \mathbf{O}_V\mathcal{T} \to \mathbf{O}\mathcal{T}$ is the $\mathcal{T}$-functor taking a pointed $\mathbf{O}_V$-space K to the *semi-free orthogonal spectrum* $\mathcal{G}_V K$ with

$$(\mathcal{G}_V K)_W = \mathbf{O}(V,W) \wedge_{\mathbf{O}_V} K$$

and with functoriality in W induced by composition in $\mathbf{O}$.

Notice that if K is of the form $K = \mathbf{O}_{V+} \wedge C$ for a space C, then $\mathcal{G}_V K$ is naturally isomorphic to $\mathcal{F}_V C$. The following result holds since $\mathcal{G}_V$ is an explicit construction of the left Kan extension of $K\colon \mathbf{O}_V \to \mathcal{T}$ along the inclusion $\mathbf{O}_V \to \mathbf{O}$.

Lemma 2.3.2. *The $\mathcal{T}$-functor $\mathcal{G}_V : \mathbf{O}_V\mathcal{T} \to \mathbf{O}\mathcal{T}$ is left adjoint to the evaluation $\mathcal{T}$-functor $\mathrm{ev}'_V : \mathbf{O}\mathcal{T} \to \mathbf{O}_V\mathcal{T}$ taking an orthogonal spectrum X to the $\mathbf{O}_V$-space X_V.*

Let $\mathbf{O}'$ be the subcategory of $\mathbf{O}$ with $\mathbf{O}'(V,V) = \mathbf{O}_V$ and $\mathbf{O}'(V,W) = \emptyset$ for $V \neq W$. The functor $\mathrm{ev}' : \mathbf{O}\mathcal{T} \to \mathbf{O}'\mathcal{T}$ has a left adjoint $\mathcal{G}$ with $\mathcal{G}Y \cong \coprod_V \mathcal{G}_V Y_V$ for $Y \in \mathbf{O}'\mathcal{T}$. Given an orthogonal spectrum X, we write $c_X : \mathcal{G}^{\mathrm{ev}}_{\prime} X \to X$ for the counit of this adjunction.

Lemma 2.3.3. *For every orthogonal spectrum X, the diagram*

$$\mathcal{G}\,\mathrm{ev}'\,\mathcal{G}\,\mathrm{ev}'\,X \underset{\mathcal{G}\,\mathrm{ev}'\,c_X}{\overset{c_{\mathcal{G}\,\mathrm{ev}'\,X}}{\rightrightarrows}} \mathcal{G}\,\mathrm{ev}'\,X \xrightarrow{c_X} X$$

is a coequalizer diagram.

Proof. After applying ev' the diagram becomes a split coequalizer diagram in $\mathbf{O}'\mathcal{T}$. □

One of the most important properties of semi-free spectra is that it is easy to calculate their smash products with other spectra and in particular with each other. The following proposition makes this precise, and generalizes Lemma [MMSS, 1.8]. It is analogous to Lemma [S, I.4.5-6] in the case of symmetric spectra.

Proposition 2.3.4. *Let V and W be objects of $\mathbf{O}$, $K \in \mathbf{O}_V\mathcal{T}$ and $L \in \mathbf{O}_W\mathcal{T}$. The adjoint of the canonical $\mathbf{O}_V \times \mathbf{O}_W$-map*

$$K \wedge L \cong (\mathcal{G}_V K)_V \wedge (\mathcal{G}_W L)_W \to (\mathcal{G}_V K \wedge \mathcal{G}_W L)_{V\oplus W}$$

is a natural isomorphism

$$\mathcal{G}_{V\oplus W}(\mathbf{O}_{V\oplus W+} \wedge_{\mathbf{O}_V \times \mathbf{O}_W} K \wedge L) \xrightarrow{\cong} \mathcal{G}_V K \wedge \mathcal{G}_W L.$$

Proof. Using the monoidal product $\mathbf{O}(V,V') \wedge \mathbf{O}(W,W') \to \mathbf{O}(V\oplus W, V'\oplus W')$ of $\mathbf{O}$ we obtain a map

$$\begin{aligned} (\mathcal{G}_V K)_{V'} \wedge (\mathcal{G}_W L)_{W'} &\cong \mathbf{O}(V,V') \wedge \mathbf{O}(W,W') \wedge_{\mathbf{O}_V \times \mathbf{O}_W} K \wedge L \\ &\to \mathbf{O}(V\oplus W, V'\oplus W') \wedge_{\mathbf{O}_V \times \mathbf{O}_W} K \wedge L \\ &\cong \mathcal{G}_{V\oplus W}(\mathbf{O}_{V\oplus W+} \wedge_{\mathbf{O}_V \times \mathbf{O}_W} K \wedge L)_{V'\oplus W'}. \end{aligned}$$

By the universal property of the smash-product, this map induces a map

$$\mathcal{G}_V K \wedge \mathcal{G}_W L \to \mathcal{G}_{V\oplus W}(\mathbf{O}_{V\oplus W+} \wedge_{\mathbf{O}_V \times \mathbf{O}_W} K \wedge L).$$

It is a consequence of the universal defining properties of the smash-product and of $\mathcal{G}_{V\oplus W}$ that this map is inverse to the map in the statement. □

When studying smash-powers, we need to know how the above isomorphism interacts with the twist isomorphism $\tau : X \wedge Y \to Y \wedge X$ of orthogonal spectra. We need the following elementary result:

Lemma 2.3.5. *Let V and W be objects of $\mathbf{O}$, $K \in \mathbf{O}_V\mathcal{T}$ and $L \in \mathbf{O}_W\mathcal{T}$. Let*

$$t\colon \mathbf{O}_{V\oplus W+} = \mathbf{O}(V \oplus W, V \oplus W) \to \mathbf{O}(W \oplus V, V \oplus W)$$

be induced by precomposition with the twist isomorphism $\tau\colon W \oplus V \to V \oplus W$. The diagram

$$\begin{array}{ccc} \mathbf{O}_{V\oplus W+} \wedge_{\mathbf{O}_V\times\mathbf{O}_W} K \wedge L & \xrightarrow{\cong} & (\mathcal{G}_V K \wedge \mathcal{G}_W L)_{V\oplus W} \\ \downarrow{\scriptstyle t\wedge\tau} & & \downarrow{\scriptstyle \tau_{V\oplus W}} \\ \mathbf{O}(W \oplus V, V \oplus W) \wedge_{\mathbf{O}_W\times\mathbf{O}_V} L \wedge K & \xrightarrow{\cong} & (\mathcal{G}_W L \wedge \mathcal{G}_V K)_{V\oplus W}, \end{array}$$

commutes, where the upper horizontal map is induced up from the canonical $\mathbf{O}_V \times \mathbf{O}_W$ used in Proposition 2.3.4 and where the lower horizontal map is the composition of the maps

$$\begin{aligned} &\mathbf{O}(W \oplus V, V \oplus W) \wedge_{\mathbf{O}_W\times\mathbf{O}_V} L \wedge K \to \\ &\mathbf{O}(W \oplus V, V \oplus W) \wedge_{\mathbf{O}_W\times\mathbf{O}_V} (\mathcal{G}_W L \wedge \mathcal{G}_V K)_{W\oplus V} \to \\ &(\mathcal{G}_W L \wedge \mathcal{G}_V K)_{V\oplus W}. \end{aligned}$$

Proof. This is a consequence of the fact that given orthogonal spectra X and Y, the diagram

$$\begin{array}{ccccc} X_V \wedge Y_W & \longrightarrow & (X \wedge Y)_{V\oplus W} & \xrightarrow{\tau_{V\oplus W}} & (Y \wedge X)_{V\oplus W} \\ \downarrow{\scriptstyle \tau} & & & & \downarrow{\scriptstyle (Y\wedge X)_\tau} \\ Y_W \wedge X_V & & \longrightarrow & & (Y \wedge X)_{W\oplus V} \end{array}$$

commutes. ◻

Corollary 2.3.6. *Let V be an object of $\mathbf{O}$, let $K \in \mathbf{O}_V\mathcal{T}$ and let $t\colon \mathbf{O}_{V\oplus V} \to \mathbf{O}_{V\oplus V}$ given by right multiplication with the isomorphism twisting the two factors of V. The diagram*

$$\begin{array}{ccc} \mathcal{G}_{V\oplus V}(\mathbf{O}_{V\oplus V+} \wedge_{\mathbf{O}_V\times\mathbf{O}_V} K \wedge K) & \longrightarrow & \mathcal{G}_V K \wedge \mathcal{G}_V K \\ \downarrow{\scriptstyle \mathcal{G}_{V\oplus V}(t\wedge\tau)} & & \downarrow{\scriptstyle \tau} \\ \mathcal{G}_{V\oplus V}(\mathbf{O}_{V\oplus V+} \wedge_{\mathbf{O}_V\times\mathbf{O}_V} K \wedge K) & \longrightarrow & \mathcal{G}_V K \wedge \mathcal{G}_V K \end{array}$$

commutes.

Proof. Letting $W = V$ and $L = K$, the upper- and the lower row in the commutative diagram of Lemma 2.3.5 are identical. By Proposition 2.3.4 that commutative diagram gives the asserted commutative diagram. ◻

Notice that the right vertical map in 2.3.6 is of the form $\mathcal{G}_{V\oplus V}(t \wedge \tau)$: the two copies of V in $\mathcal{G}_{V\oplus V}$ are *not* permuted.

Let X be a finite discrete G-set, let V be an Euclidean space and let K be an $\mathbf{O}_V$-space. Then G acts on $K^{\wedge X}$ and on $\prod_x \mathbf{O}_V$ by permuting factors. The group $G \times \mathbf{O}_{V\oplus X}$ acts on the space $\mathbf{O}_{V\oplus X+} \wedge_{\prod_x \mathbf{O}_V} K^{\wedge X}$ by letting G act on $\mathbf{O}_{V\oplus X}$ by right multiplication of permutation of summands and letting $\mathbf{O}_{V\oplus X}$ act on itself by left multiplication. Considering $V^{\oplus X}$ as an Euclidean space, we obtain an orthogonal G-spectrum $\mathcal{G}_{V^{\oplus X}}(\mathbf{O}_{V^{\oplus X}+} \wedge_{\prod_x \mathbf{O}_V} K^{\wedge X})$. Notice that G acts trivially on the Euclidean space $V^{\oplus X}$ in $\mathcal{G}_{V^{\oplus X}}$.

Since permutations of a finite set X are generated by transpositions, iterated application of the 2.3.6 gives:

Proposition 2.3.7. *Let X be a finite discrete G-set, V an Euclidean space and K an $\mathbf{O}_V$-space. There is a natural isomorphism of G-spectra*

$$\mathcal{G}_{V^{\oplus X}}(\mathbf{O}_{V^{\oplus X}+} \wedge_{\prod_X \mathbf{O}_V} K^{\wedge X}) \xrightarrow{\cong} (\mathcal{G}_V K)^{\wedge X},$$

where G acts on the spectrum $(\mathcal{G}_V K)^{\wedge X}$ by permuting factors.

Proposition 2.3.8. *Let X be an orthogonal spectrum and let K be an $\mathbf{O}_V$-space. The structure map $K \wedge X_W \cong (\mathcal{G}_V K)_V \wedge X_W \to (\mathcal{G}_V K \wedge X)_{V\oplus W}$ for the smash-product induces a natural isomorphism*

$$\mathbf{O}_{V\oplus W+} \wedge_{\mathbf{O}_V \times \mathbf{O}_W} K \wedge X_W \cong (\mathcal{G}_V K \wedge X)_{V\oplus W}$$

of $\mathbf{O}_{V\oplus W}$-spaces. If the dimension of V' is smaller than the dimension of V, then $(\mathcal{G}_V K \wedge X)_{V'}$ is a one-point space.

Proof. Considered as functors of X both sides commutes with colimits. By 2.3.3 we can represent X as a colimit of semi-free spectra so we can reduce to the case where $X = \mathcal{G}_U L$ is semi-free. In this case the asserted isomorphism is a consequence of Proposition 2.3.4 since $\mathbf{O}_{V\oplus W+} \wedge_{\mathbf{O}_W} \mathbf{O}(U, W)$ is isomorphic to $\mathbf{O}(V \oplus U, V \oplus W)$. □

By functoriality and naturality, the $\mathcal{T}$-functors $\mathcal{G}_V$ and ev'_V associated to the Euclidean space V promote to adjoint $G\mathcal{T}$-functors $\mathcal{G}_V\colon G\mathbf{O}_V\mathcal{T} \to G\mathbf{O}\mathcal{T}$ and $\mathrm{ev}'_V\colon G\mathbf{O}\mathcal{T} \to G\mathbf{O}_V\mathcal{T}$. Since the isomorphisms in 2.3.4 and 2.3.8 are natural, they also hold for semi-free orthogonal G-spectra.

Often actions will occur through the semi-direct product $G \ltimes_\varphi \mathbf{O}_V$ of the inner automorphism of a representation $\varphi\colon G \to \mathbf{O}_V$. Recall that $G \ltimes_\varphi \mathbf{O}_V$ has $G \times \mathbf{O}_V$ as underlying set and multiplication given by the rule

$$(g, \alpha)(h, \beta) = (gh, \alpha\varphi(g)\beta\varphi(g^{-1})).$$

Definition 2.3.9. Let V be an Euclidean space, let $\varphi\colon G \to \mathbf{O}_V$ be a group homomorphism and let $f\colon G \times \mathbf{O}_V \to G \ltimes_\varphi \mathbf{O}_V$ be the isomorphism $f(g, \alpha) = (g, \alpha\varphi(g^{-1}))$. We define the semi-free orthogonal G-spectrum $\mathcal{G}_V^\varphi K$ by

$$\mathcal{G}_V^\varphi K = \mathcal{G}_V f^* K.$$

Note that there is an isomorphism of $G \times \mathbf{O}_V$-spaces of the form

$$f^*K \cong \mathbf{O}(V,V) \wedge_{\mathbf{O}_V} K, \quad k \mapsto \mathrm{id}_V \wedge k,$$

where $G \times \mathbf{O}_V$ acts on $\mathbf{O}(V,V) \wedge_{\mathbf{O}_V} K$ by the rule $(g,\alpha)\cdot(\gamma \wedge k) = \alpha\gamma\varphi(g^{-1}) \wedge gk$.

Lemma 2.3.10. *Let V be an Euclidean space and let K be a $G \times \mathbf{O}_V$-space. The unit of adjunction gives an isomorphism*

$$\mathbf{O}\mathcal{T}(\mathcal{G}_V K, \mathcal{G}_V K) \cong \mathbf{O}_V\mathcal{T}(K, \mathrm{ev}'_V \mathcal{G}_V K) \cong \mathbf{O}_V\mathcal{T}(K,K)$$

of topological monoids. Since G acts on K, this isomorphism provides an action of G on $\mathcal{G}_V K$. The levelwise orbit spectrum $[\mathcal{G}_V K]_G$ is naturally isomorphic to $\mathcal{G}_V(K_G)$.

Proof. The claimed isomorphism of monoids comes from the unit isomorphism $K \cong \mathrm{ev}'_V \mathcal{G}_V K$. To check the statement about orbits we note that on levels W we get the isomorphisms

$$[\mathcal{G}_V K_W]_G = [\mathcal{O}(V,W) \wedge_{\mathbf{O}_V} K]_G \cong [\mathcal{O}(V,W) \wedge K]_{\mathbf{O}_V \times G} \cong \mathcal{O}(V,W) \wedge_{\mathbf{O}_V} (K_G).$$

□

2.4 Families of Representations

Traditionally a G-universe is used to define the various stable model structures of orthogonal G-spectra. However we will find it convenient to have the flexibility of choosing compatible universes consisting of an indexing category of H-representations for each subgroup H of G. The G-typical families of representations introduced in this section form a way to encode this.

Let H be a subgroup of G and let $V \in H\mathbf{L}$ be a representation of H, that is, V is an object of $\mathbf{L}$ together with a homomorphism $\varphi \colon H \to \mathbf{O}_V \subseteq \mathbf{L}(V,V)$. As explained below, the subgroup H and the homomorphism φ can be recovered from the subgroup P_V of $G \times \mathbf{O}_V$ consisting of pairs $(h, \varphi(h))$ for h in H.

Given an object V of $\mathbf{L}$ and a subgroup P of $G \times \mathbf{O}_V$, let $\mathrm{pr}_1 \colon P \to G$ be the restriction of the projection $G \times \mathbf{O}_V \to G$ to P through the inclusion $P \subseteq G \times \mathbf{O}_V$.

Definition 2.4.1. Let V be an object of $\mathbf{L}$ and let P be a subgroup of $G \times \mathbf{O}_V$ such that $\mathrm{pr}_1 \colon P \to G$ is injective with image $H = \mathrm{pr}_1(P)$. The H-representation $V(P) = (V, \varphi)$ has underlying vectorspace V and H-action $\varphi \colon H \to \mathbf{O}_V$ given as the composition

$$H \xrightarrow{\mathrm{pr}_1^{-1}} P \subseteq G \times \mathbf{O}_V \xrightarrow{\mathrm{pr}_2} \mathbf{O}_V.$$

There is an isomorphism

$$G \times \mathbf{O}_V/P \xrightarrow{\cong} \mathbf{O}_{V(P)}, \quad (g,\alpha)P \mapsto \alpha\,\varphi(g^{-1}) \tag{2.4.2}$$

of $G \times \mathbf{O}_V$-spaces, where $\varphi \colon G \to \mathbf{O}_V$ is the action of G and $\mathbf{O}_{V(P)}$ is the space $\mathbf{O}_V$ with $G \times \mathbf{O}_V$ acting via multiplication by elements of $\mathbf{O}_V$ from the left and multiplication from the right by $\varphi(g^{-1})$ for elements g of G.

Lemma 2.4.3. *For each inner product space V the assignment $P \mapsto V(P)$ is a bijection between the set of closed subgroups P of $G \times \mathbf{O}_V$ with the property that $\mathrm{pr}_1 \colon P \to G$ is injective and the set of representations (V, φ) of (closed) subgroups H of G with V as underlying vector space.*

Proof. Let B be the set of pairs (H, φ) where H is a subgroup of G and $\varphi \colon H \to \mathbf{O}_V$ is a continous homomorphism. There is an obvious bijection between B and the set of representations (V, φ) of any subgroup of G with underlying vector space V.

Let A be the set of closed subgroups P of $G \times \mathbf{O}_V$ with the property that $\mathrm{pr}_1 \colon P \to G$ is injective. Given $(H, \varphi) \in B$, the inclusion $i \colon H \to G$ together with $\varphi \colon H \to \mathbf{O}_V$ gives an injective homomorphism $(i, \varphi) \colon H \to G \times \mathbf{O}_V$, and thus the pair (H, φ) yields a closed subgroup, namely the image $P = (i, \varphi)(H)$ of $G \times \mathbf{O}_V$ with the property that $\mathrm{pr}_1 \colon P \to G$ is injective. This gives a function $f \colon B \to A$.

Conversely, if $P \in A$, we obtain a pair $(H, \varphi) \in B$ by letting $H = \mathrm{pr}_1(P)$ and letting φ be the composition of $\mathrm{pr}_1^{-1} \colon H \to P$ and the homomorphism $P \to \mathbf{O}_V$ obtained from the projection $G \times \mathbf{O}_V \xrightarrow{\mathrm{pr}_2} \mathbf{O}_V$. This gives a function $g \colon A \to B$, and f and g are inverse bijections. □

Definition 2.4.4. Given Euclidean spaces V and W and subgroups $P \subseteq G \times \mathbf{O}_V$ and $Q \subseteq G \times \mathbf{O}_W$ with the restriction of pr_1 to P and Q injective, we define $P \oplus Q \subseteq G \times \mathbf{O}_{V \oplus W}$ to be the inverse image of $P \times Q$ under the diagonal inclusion $G \times \mathbf{O}_V \times \mathbf{O}_W \subseteq G \times \mathbf{O}_V \times G \times \mathbf{O}_W$, where we consider $\mathbf{O}_V \times \mathbf{O}_W$ as a subgroup of $\mathbf{O}_{V \oplus W}$ in the usual way.

Notice that in the above situation $V(P \oplus Q)$ is the representation $V(P) \oplus V(Q)$ of $\mathrm{pr}_1(P) \cap \mathrm{pr}_1(Q)$.

Definition 2.4.5. A *G-typical family of representations* consists of a sequence $\mathscr{H} = (\mathscr{H}^V)_V$ indexed over the objects of $\mathbf{L}$ such that for each Euclidean space V and W:

(i) $\mathscr{H}^V$ is a family of subgroups of $G \times \mathbf{O}_V$

(ii) for each $P \in \mathscr{H}^V$ the composition $\mathrm{pr}_1 \colon P \to G$ of the inclusion $P \subseteq G \times \mathbf{O}_V$ and the projection $G \times \mathbf{O}_V \to G$ is injective.

(iii) There is a non-zero U such that $\mathscr{H}^U$ contains $G \times \{\mathrm{id}_U\}$.

(iv) $\mathscr{H}$ is closed under sum in the sense that if $P \in \mathscr{H}^V$ and $Q \in \mathscr{H}^W$, then the isotropy groups of elements of the $G \times \mathbf{O}_{V \oplus W}$-space

$$\mathbf{O}_{V \oplus W} \times_{\mathbf{O}_V \times \mathbf{O}_W} ((G \times \mathbf{O}_V)/P \times (G \times \mathbf{O}_W)/Q)$$

are in $\mathscr{H}^{V \oplus W}$. Here the group G acts on $(G \times \mathbf{O}_V)/P \times (G \times \mathbf{O}_W)/Q$ through the diagonal embedding $G \to G \times G$.

(v) For every P in $\mathscr{H}^V$, there exist $Q \in \mathscr{H}^W$ and $R \in \mathscr{H}^{V \oplus W}$ with $\mathrm{pr}_1(Q) = \mathrm{pr}_1(P)$ and $\mathrm{pr}_1(R) = G$ such that $P \oplus Q = (\mathrm{pr}_1(P) \times \mathbf{O}_W) \cap R$.

In the above definition condition (i) allows us to use standard methods from equivariant homotopy theory. Condition (ii) gives a firm connection to representations of subgroups of G. Condition (iii) is needed for the stable model structure on orthogonal G-spectra to be stable in the sense of model categories [H, Definition 7.1.1]. Condition (v) is of a more technical nature related to cofinality of index categories in homotopy colimits used in a construction of fibrant replacement. It is a consequence of (v) that $\mathscr{H}$ is closed under direct sum in the sense that if P and Q are in $\mathscr{H}$, then so is $P \oplus Q$.

Given $g \in G$ and a subgroup H of G we write $c_g \colon H \to gHg^{-1}$ for the conjugation isomorphism with $c_g(h) = ghg^{-1}$.

Proposition 2.4.6. *Let $\mathscr{H}$ be a G-typical family of representations. The set $\mathcal{V}$ of isomorphism classes of representations of the form $V(P)$ for P in $\mathscr{H}$ is closed under conjugation and restriction, that is, if $g \in G$ and $i \colon K \to gHg^{-1}$ is an inclusion of subgroups of G and if an H-representation W is in $\mathcal{V}$, then the K-representation $i^*(c_g^{-1})^*W$ is in $\mathcal{V}$.*

Proof. Choose $P \in \mathscr{H}$ with an isomorphism $W \cong V(P)$. The representation $(c_g^{-1})^*W$ of gHg^{-1} is then isomorphic to $(c_g^{-1})^*V(P) \cong V(c_g(P))$, where $c_g(P)$ consists of the elements of the form $(g, \mathrm{id})p(g^{-1}, \mathrm{id})$ for $g \in G$ and $p \in P$. The subgroup c_gP of $G \times \mathbf{O}_V$ is in $\mathscr{H}$ because $\mathscr{H}$ is closed under conjugation. Since $i \colon K \to gHg^{-1}$ is an inclusion of subgroups of G, we have $i^*V(c_gP) = V(Q)$ for the subgroup $Q = \mathrm{pr}_1^{-1}(K)$ of c_gP. □

Note that if $P \in \mathscr{H}$ has $\mathrm{pr}_1(P) = H$ and $Q = (g, \alpha)P(g, \alpha)^{-1}$ for some $\alpha \in \mathbf{O}_V$ and $g \in G$, then α is an isomorphism $\alpha \colon V(P) \to c_g^*V(Q)$ of H-representations.

Definition 2.4.7. We say that $\mathscr{H}$ is *closed under retracts* if it has the property that if V is a retract of an H-representation $V(P)$ for some P in $\mathscr{H}$, then V is isomorphic to $V(Q)$ for some Q in $\mathscr{H}$.

Note that every G-typical family of representations $\mathscr{H} = (\mathscr{H}^V)_V$ has a *closure* $\overline{\mathscr{H}}$, that is, a smallest G-typical family of representations closed under retracts containing it.

Example 2.4.8.

(i) There is a maximal G-typical family of representations consisting of all subgroups P of $G \times \mathbf{O}_V$ satisfying that $\mathrm{pr}_1 \colon P \to G$ is injective. We need to explain why those satisfy the sum-axiom (iv): Every element of the $G \times \mathbf{O}_{V\oplus W}$-space

$$\mathbf{O}_{V\oplus W} \times_{\mathbf{O}_V \times \mathbf{O}_W} ((G \times \mathbf{O}_V)/P \times (G \times \mathbf{O}_W)/Q)$$

is represented by an element of the form $x = (\alpha, [g_V, I_V], [g_W, I_W])$ for some $g_V, g_W \in G$ and $\alpha \in \mathbf{O}_{V\oplus W}$. If (g, β) is in the isotropy group $(G \times \mathbf{O}_{V\oplus W})_x$ of x for some $g \in G$ and $\beta \in \mathbf{O}_{V\oplus W}$, then β is of the form $\beta = \beta_V \oplus \beta_W$, and $(g, \beta_V) \in g_V P g_V^{-1}$ and $(g, \beta_W) \in g_W Q g_W^{-1}$. Since $\mathrm{pr}_1 \colon P \to G$ is injective, this implies that $\mathrm{pr}_1 \colon (G \times \mathbf{O}_{V\oplus W})_x \to G$ is injective.

Every representation of any subgroup of G is in this family. This G-typical family of representations is closed under retracts.

(ii) The argument of the above example also shows that there is a minimal G-typical family of representations which is closed under retracts consisting of all subgroups of G considered as subgroups of $G \times \mathbf{O}_V$. The representations in this G-typical family of representations are the trivial representations.

(iii) There is a G-typical family of representation with the representations in $\mathscr{H}$ given by the representations of subgroups of G containing the one-dimensional trivial representation. That is, all representations of a subgroup H of G containing a non-zero H-invariant element. This G-typical family of representations is not closed under retracts. This kind of G-typical families are important for the construction of "positive" model structures on the category of commutative orthogonal G-ring-spectra.

2.5 Level model structure

In this section $\mathscr{H}$ is a fixed but arbitrary G-typical family of representations. Given V in $\mathbf{O}$, we can consider the $(\mathscr{H}^V, \mathscr{H}^V)$-model structure on $(G \times \mathbf{O}_V)\mathcal{T}$.

Definition 2.5.1. An *$\mathscr{H}$-model structure* $\mathcal{M}$ consists of a sequence $\mathcal{M} = (\mathcal{M}_V)_{V \in \mathbf{L}}$ of cofibrantly generated model structures, where each $\mathcal{M}_V$ is a $G\mathcal{T}$-model structure on the category $(G \times \mathbf{O}_V)\mathcal{T}$, considering $(G \times \mathbf{O}_V)\mathcal{T}$ as enriched and tensored over $G\mathcal{T}$ via the isomorphism $(G \times \mathbf{O}_V)\mathcal{T} \cong \mathbf{O}_V(G\mathcal{T})$. Writing J_V for the set of generating acyclic cofibrations of $\mathcal{M}_V$, the sequence $\mathcal{M}$ of model categories is required to satisfy the following four conditions:

(i) All cofibrations in $\mathcal{M}_V$ are hurewicz cofibrations.

(ii) For every V and W in $\mathbf{L}$ we have that for every generating acyclic cofibration $j \in J_V$ the map $(\mathcal{G}_V j)_W$ is a weak equivalence in $\mathcal{M}_W$.

(iii) The class of weak equivalences in $\mathcal{M}_V$ is the class of $\mathscr{H}^V$-equivalences.

(iv) For every $V \in \mathbf{O}$, the class of cofibrations in the $(\mathscr{H}^V, \mathscr{H}^V)$-model structure on $(G \times \mathbf{O}_V)\mathcal{T}$ is contained in the class of cofibrations in $\mathcal{M}_V$.

In the above definition condition (i) is of a technical kind used to ensure that homotopy groups have associated cofibration sequences. Condition (ii) is needed to obtain a level $\mathcal{M}$-model structure on the category of orthogonal spectral Condition (iv) is used to identify the stably fibrant orthogonal spectra as a kind of Ω-spectra. Together with the other conditions, condition (iii) determines the stable $\mathcal{M}$-model structure on orthogonal spectra. Condition (iv) implies that every fibration f in $\mathcal{M}_V$ is an $\mathscr{H}^V$-fibration in the sense that f^P is a fibration for every P in $\mathscr{H}$.

Definition 2.5.2. Let $\mathcal{M}$ be an $\mathscr{H}$-model structure. Writing I_V for the set of generating cofibrations for M_V we say that $\mathcal{M}$ *satisfies the pushout-product axiom* if it is so that given $i \in I_V$ and $j \in I_W$, the map $\mathbf{O}_{V \oplus W+} \wedge_{\mathbf{O}_V \times \mathbf{O}_W} i \square j$ is a cofibration in $\mathcal{M}_{V \oplus W}$, and if in addition one of the former maps is a weak equivalence, so is the latter.

Definition 2.5.3. Let $\mathcal{M}$ be an $\mathscr{H}$-model structure and let $f\colon X \to Y$ be a morphism of orthogonal G-spectra.

(i) f is a *level* $\mathcal{M}$*-equivalence* if f_V is an $\mathcal{M}_V$-equivalence for all V in $\mathbf{L}$.

(ii) f is a *level* $\mathcal{M}$*-fibration* if f_V is a fibration in $\mathcal{M}_V$ for all V in $\mathbf{L}$.

(iii) f is an $\mathcal{M}$*-cofibration* if it satisfies the left lifting property with respect to all maps that are both level $\mathcal{M}$-equivalences and level $\mathcal{M}$-fibrations.

Definition 2.5.4. Given an $\mathscr{H}$-model structure $\mathcal{M}$, we write $\mathcal{G}I_{\mathcal{M}} = \cup_V \mathcal{G}I_V$, where $\mathcal{G}I_V$ is the set of all maps of the form $\mathcal{G}_V i$ for i in the set I_V of generating cofibrations for $\mathcal{M}_V$. Similarly we use the notation $\mathcal{G}J_{\mathcal{M}} = \cup_V \mathcal{G}J_V$, where $\mathcal{G}J_V$ is the set of all maps of the form $\mathcal{G}_V j$ for j in the set J_V of generating acyclic cofibrations for $\mathcal{M}_V$.

Theorem 2.5.5. *If $\mathcal{M}$ is an $\mathscr{H}$-model structure, then the $\mathcal{M}$-cofibrations, level $\mathcal{M}$-equivalences and level $\mathcal{M}$-fibrations give a left proper cofibrantly generated G-topological model structure on the category $G\mathbf{O}\mathcal{T}$ of orthogonal G-spectra. The set $\mathcal{G}I_{\mathcal{M}}$ is a set of generating cofibrations, and the set $\mathcal{G}J_{\mathcal{M}}$ is a set of generating acyclic cofibrations for this model structure.*

Proof. We use the Assembling Theorem 6.2.7. Proposition 6.2.9 shows that the resulting structure is G-topological. To apply Theorem 6.2.7 we need to know that the maps in $\mathcal{G}J_V$ are actually level equivalences. However, if $j \in J_V$, then by part (ii) of Definition 2.5.1 $(\mathcal{G}_V j)_W$ is a weak equivalence in $\mathcal{M}_W$.

In order to see that the level $\mathcal{M}$-model structure on $G\mathbf{O}\mathcal{T}$ is left proper we note that (i) of Definition 2.5.1 implies that if $f\colon A \to X$ is a level $\mathcal{M}$-cofibration in $G\mathbf{O}\mathcal{T}$, then $f_V\colon A_V \to X_V$ is a hurewicz cofibration for all V. Now left propernes is a consequence of part (iii) of Definition 2.5.1. □

Definition 2.5.6. The model structure of Theorem 2.5.5 is the *level* $\mathcal{M}$*-model structure* on $G\mathbf{O}\mathcal{T}$.

Theorem 2.5.7. *If an $\mathscr{H}$-model structure $\mathcal{M}$ satisfies the pushout-product axiom, and $\mathscr{H}^0$ contains all subgroups of $G \times \mathbf{O}_0$, then the level $\mathcal{M}$-model structure on $G\mathbf{O}\mathcal{T}$ is monoidal.*

Proof. Since $\mathscr{H}^0$ contains all subgroups of $G \times \mathbf{O}_0$ the zero sphere S^0 is cofibrant in $\mathcal{M}_0$, the unit $\mathbb{S} = \mathcal{G}_0 S^0$ for the monoidal product is cofibrant. Thus we only need to verify the pushout product axiom. It is a direct consequence of the natural isomorphism

$$\mathcal{G}_V i \Box \mathcal{G}_W j = \mathcal{G}_{V\oplus W}(\mathbf{O}_{V\oplus W+} \wedge_{\mathbf{O}_V \times \mathbf{O}_W} i \Box j)$$

and Definition 2.5.2. □

2.6 Mixing pairs

Given a G-typical family of representations $\mathscr{H}$ and a sequence $\mathscr{G} = (\mathscr{G}^V)_{V\in\mathbf{L}}$ of families of subgroups of $G \times \mathbf{O}_V$ with $\mathscr{H}^V$ contained in $\mathscr{G}^V$ for all $V \in \mathbf{L}$, we will consider the $(\mathscr{H}^V, \mathscr{G}^V)$-model structure on $(G \times \mathbf{O}_V)\mathcal{T}$ from Theorem 1.3.8. Recall that the closure $\overline{\mathscr{H}}$ of a G-typical family $\mathscr{H}$ is its closure under retracts.

Definition 2.6.1. A *G-mixing pair* $(\mathscr{H}, \mathscr{G})$ consists of a G-typical family of representations $\mathscr{H}$ and a sequence $\mathscr{G} = (\mathscr{G}^V)_{V\in\mathbf{L}}$ of families $\mathscr{G}^V$ of subgroups of $G\times\mathbf{O}_V$ satisfying the following conditions for each Euclidean space V and W:

(i) $\mathscr{H}^V$ is contained in $\mathscr{G}^V$

(ii) $\overline{\mathscr{H}}^V \cap \mathscr{G}^V$ is contained in $\mathscr{H}^V$

(iii) $\mathscr{G}$ is closed under sum in the sense that if $P \in \mathscr{G}^V$ and $Q \in \mathscr{G}^W$, then the isotropy groups of elements of the $G \times \mathbf{O}_{V\oplus W}$-space

$$\mathbf{O}_{V\oplus W} \times_{\mathbf{O}_V\times\mathbf{O}_W} ((G \times \mathbf{O}_V)/P \times (G \times \mathbf{O}_W)/Q)$$

are in $\mathscr{G}^{V\oplus W}$. Here the group G acts on $(G \times \mathbf{O}_V)/P \times (G \times \mathbf{O}_W)/Q$ through the diagonal embedding $G \to G \times G$.

Example 2.6.2. For every G-typical family $\mathscr{H}$ of representations, $(\mathscr{H}, \mathscr{H})$ is a G-mixing pair.

Example 2.6.3. If $\mathscr{H}$ is closed under retracts and $\mathscr{G}^V$ is the family of all subgroups of $G \times \mathbf{O}_V$, then $(\mathscr{H}, \mathscr{G})$ is a G-mixing pair.

Example 2.6.4 (Positive mixing pair)**.** The *positive mixing pair* for G is defined as follows: For $V \neq 0$, the family $\mathscr{G}^V$ consists of all subgroups of $G\times\mathbf{O}_V$, and $\mathscr{H}^V$ consists of the subgroups P of $G \times \mathbf{O}_V$ with the property that $\mathrm{pr}_1 \colon P \to G$ is injective. When $V = 0$, the families $\mathscr{G}^V$ and $\mathscr{H}^V$ are empty.

Recall from Theorem 1.3.8 that for a G-mixing pair $(\mathscr{H}, \mathscr{G})$, the $(\mathscr{H}^V, \mathscr{G}^V)$-model structure on $G \times \mathbf{O}_V$ is cofibrantly generated, with the set $I_{\mathscr{G}^V}$ of maps of the form $(i \times (G \times \mathbf{O}_V)/P)_+$ for $i \in I$ a generating cofibration for $\mathcal{U}$ and $P \in \mathscr{G}^V$, as set of generating cofibrations. The set $J_{\mathscr{H}^V,\mathscr{G}^V}$ of generating acyclic cofibrations is the union of the set of $J_{\mathscr{G}^V}$ of maps of the form $(j \times (G \times \mathbf{O}_V)/P)_+$ for $j \in J$ a generating acyclic cofibration for $\mathcal{U}$ and $P \in \mathscr{G}^V$ and the set consisting of maps of the form $i\square k_P$ for $i \in I$ a generating cofibration for $\mathcal{U}$ and a specific $\mathscr{H}^V$-equivalence $k_P = s_P(G/e)$ for $P \in \mathscr{G}^V - \mathscr{H}^V$.

Definition 2.6.5. The set I_V is the set $I_{\mathscr{G}^V}$ of generating cofibrations for the $(\mathscr{H}^V, \mathscr{G}^V)$-model structure on $G \times \mathbf{O}_V$ and the set J_V is the set $J_{\mathscr{H}^V,\mathscr{G}^V}$ of generating acyclic cofibrations for the $(\mathscr{H}^V, \mathscr{G}^V)$-model structure on $G \times \mathbf{O}_V$.

Recall that a map $f\colon X \to Y$ of orthogonal spectra is a level $\mathscr{H}$-equivalence if and only if for each Euclidean space V, the map f_V is an $\mathscr{H}^V$-equivalence. We can formulate this in terms of the genuine model structure on $G\mathcal{T}$ 1.3.10. by noting that f_V is an $\mathscr{H}^V$-equivalence if and only if for every $P \in \mathscr{H}^V$, the map f_V is an a genuine equivalence of P-spaces. Writing $H = \mathrm{pr}_1(P)$, the map f_V is a genuine equivalence of P-spaces if and only if the map $f_{V(P)}$ is an genuine equivalence of H-spaces.

Recall that given a subgroup P of a group A and an element a of A, the conjugation homomorphism $c_a\colon P \to aPa^{-1}$ is defined by $c_a(p) = apa^{-1}$.

Lemma 2.6.6. *Let P and B be subgroups of a compact Lie group A, let $a \in A$ and let X be a B-space. If $a^{-1}Pa$ is contained in B, then the map $f\colon c^*_{a^{-1}}X \to PaB_+ \wedge_B X$ with $f(x) = [a \wedge x]$ is a isomorphism of P-spaces. If $a^{-1}Pa$ is not contained in B, then the space of P-fixed points of $PaB_+ \wedge_B X$ is the one-point space. In particular, the P-fixed subspace of $A_+ \wedge_B X$ is the one point space unless P is subconjugate to B.*

Proof. If $(PaB/B)^P$ is non-empty, then $paB = aB$ for all $p \in P$, so $a^{-1}Pa \subseteq B$. Thus, if $a^{-1}Pa$ is not contained in B, then the space of P-fixed points of $PaB_+ \wedge_B X$ is the one-point space. Now suppose $a^{-1}Pa$ is contained in B. Then

$$f(a^{-1}pax) = [a \wedge a^{-1}pax] = [aa^{-1}pa \wedge x] = [pa \wedge x] = p[a \wedge x] = pf(x),$$

so f is a P-map. Multiplication with a^{-1} gives an inverse

$$PaB_+ \wedge_B X \to c^*_{a^{-1}}(a^{-1}PaB_+ \wedge_B X) = c^*_{a^{-1}}(B_+ \wedge_B X) \cong c^*_{a^{-1}}X$$

to f. The statement about the P-fixed subspace of $A_+ \wedge_B X$ now follows from the fact that A is the disjoint union of subspaces of the form PaB. □

Proposition 2.6.7. *Let $(\mathscr{H},\mathscr{G})$ be a G-mixing pair. For each Euclidean space V let $\mathcal{M}_V$ be the $(\mathscr{H}^V,\mathscr{G}^V)$-model structures on $(G \times \mathbf{O}_V)\mathcal{T}$. The model structures $\mathcal{M}_V$ form an $\mathscr{H}$-model structure $\mathcal{M}$ with the property that if $i \in I_V$ and $j \in I_W$, then $\mathbf{O}_{V\oplus W+} \wedge_{\mathbf{O}_V \times \mathbf{O}_W} i \Box j$ is a cofibration in $\mathcal{M}_{V\oplus W}$. If $\mathscr{H}$ is closed under retracts, then $\mathcal{M}$ satisfies the $\mathscr{H}$-pushout-product axiom, and $\mathbb{S}$ is cofibrant in the level $\mathcal{M}$-model structure on $G\mathbf{O}\mathcal{T}$. Letting $I_\mathcal{M} = I_V$ from 2.6.5 and $J_\mathcal{M} = J_V$ from 2.6.5, the sets $\mathcal{G}I_\mathcal{M}$ and $\mathcal{G}J_\mathcal{M}$ are sets of generating cofibrations and generating acyclic cofibrations respectively for the level $\mathcal{M}$-model structure.*

Proof. By design, the weak equivalences in $\mathcal{M}_V$ are the $\mathscr{H}^V$-equivalences. The generating cofibrations for the $(\mathscr{H}^V,\mathscr{G}^V)$-model structure are Hurewicz cofibrations, and thus so are all cofibrations, and (i) of Definition 2.5.1 holds.

We need to work harder to verify part (ii) of Definition 2.5.1, that is, that the maps in $\mathcal{G}J$ are actually $\mathscr{H}$-level equivalences. So let $j\colon X \to Y$ be a generating acyclic $(\mathscr{H}^V,\mathscr{G}^V)$-cofibration. Then $(\mathcal{G}_V j)_W$ is the map

$$\mathbf{O}(V,W) \wedge_{\mathbf{O}_V} j\colon \mathbf{O}(V,W) \wedge_{\mathbf{O}_V} X \to \mathbf{O}(V,W) \wedge_{\mathbf{O}_V} Y.$$

We have to show that this is an $\mathscr{H}^W$-equivalence. Suppose there exists an isometric embedding $\varphi\colon V \to W$. Otherwise $\mathbf{O}(V,W)$ is the one-point space and $(\mathcal{G}_V j)_W$ is obviously a weak equivalence. Let $A = G \times \mathbf{O}_W$ and $B = G \times \mathbf{O}_{\varphi^\perp} \times \mathbf{O}_{\varphi(V)}$. Using φ we identify $\mathbf{O}(V,W) \wedge_{\mathbf{O}_V} j$ with the map

$$A_+ \wedge_B (S^{\varphi^\perp} \wedge X) \xrightarrow{A_+ \wedge_B (\mathrm{id} \wedge j)} A_+ \wedge_B (S^{\varphi^\perp} \wedge Y).$$

Given $P \in \mathscr{H}^W$ Lemma 2.6.6 says that the P-fixed points of the above spaces consist of just the base point unless P is subconjugate to B.

Consider A as a $P \times B^{\mathrm{op}}$-space with $(p,b) \in P \times B^{\mathrm{op}}$ acting on $a \in A$ by the rule $(p,b)a = pab$. By Illmans's Theorem 1.2.2, the space A is an $P \times B^{\mathrm{op}}$ CW-complex. By Lemma 1.2.5 the cells of this $P \times B^{\mathrm{op}}$ CW-complex are isomorphic to products $PaB \times D^n$ of an orbit in A and a disc. Lemma 2.6.6 would imply that $((PaB \times D^n) \wedge_B (\mathrm{id} \wedge j))^P$ and $((PaB \times S^{n-1}) \wedge_B (\mathrm{id} \wedge j))^P$ are weak equivalences if we knew that $(\mathrm{id} \wedge j)^{a^{-1}Pa}$ were a weak equivalence, whenever $a^{-1}Pa \subseteq B$. An induction on the cells using the Cube Lemma [H, Lemma 5.2.6] would then give that $(A_+ \wedge_B (\mathrm{id} \wedge j))^P$ is a weak equivalence.

In order to finish the verification of part (ii) of Definition 2.5.1 it thus suffices to show that if $P \in \mathscr{H}^W$ is a subgroup of B, then $(\mathrm{id} \wedge j)^P$ is a weak equivalence. Let P_1 be the image of P under the projection

$$B = G \times \mathbf{O}_{\varphi^\perp} \times \mathbf{O}_{\varphi(V)} \to G \times \mathbf{O}_{\varphi(V)} \cong G \times \mathbf{O}_V.$$

Then by Proposition 1.1.2 we have $(\mathrm{id} \wedge j)^P = \mathrm{id} \wedge j^{P_1}$, and since X and Y are cofibrant in the mixed $(\mathscr{H}^V, \mathscr{G}^V)$-model structure, it suffices to show that j^{P_1} is a weak equivalence. Note that $V(P_1)$ is a subrepresentation of $V(P)$, so the group P_1 is in $\overline{\mathscr{H}}^V$. If $P_1 \in \mathscr{H}^V$, then j^{P_1} is a weak equivalence in $\mathcal{T}$. Otherwise P_1 is in the complement of $\mathscr{H}^V$ in $\overline{\mathscr{H}}^V$. In part (ii) of Definition 2.6.1 we require that $\overline{\mathscr{H}}^V \cap \mathscr{G}^V \subseteq \mathscr{H}^V$, so this implies that P_1 is not in $\mathscr{G}^V$. Since both source and target of j are cofibrant in the mixed $(\mathscr{H}^V, \mathscr{G}^V)$-model structure this implies that both source and target of j^{P_1} are the one-point space. In particular j^{P_1} is a weak equivalence.

Next we show that if $i \in I_V$ and $j \in I_W$, then $\mathbf{O}_{V \oplus W+} \wedge_{\mathbf{O}_V \times \mathbf{O}_W} i \Box j$ is a cofibration. Let

$$i = (\left[S^{n-1} \to D^n\right] \times (G \times \mathbf{O}_V)/P)_+ \quad \text{and} \quad j = (\left[S^{m-1} \to D^m\right] \times (G \times \mathbf{O}_W)/Q)_+$$

be maps in I_V and I_W respectively. Then $\mathbf{O}_{V \oplus W+} \wedge_{\mathbf{O}_V \times \mathbf{O}_W} i \Box j$ is isomorphic to:

$$(\left[S^{n+m-1} \to D^{n+m}\right] \times \mathbf{O}_{V \oplus W} \times_{\mathbf{O}_V \times \mathbf{O}_W} (G \times \mathbf{O}_V)/P \times (G \times \mathbf{O}_W)/Q)_+.$$

Since as a left adjoint, taking the product with a space preserves colimits, it suffices by Illman's Theorem 1.2.2 to note that in part (iii) of Definition 2.6.1 we require that the isotropy groups of the smooth $G \times \mathbf{O}_{V \oplus W}$ manifold

$$\mathbf{O}_{V \oplus W} \times_{\mathbf{O}_V \times \mathbf{O}_W} ((G \times \mathbf{O}_V)/P \times (G \times \mathbf{O}_W)/Q)$$

are in $\mathscr{G}^{V\oplus W}$.

Suppose that $\mathscr{H}$ is closed under retracts. We show that if X_V is a cofibrant object in $\mathcal{M}_V$ and f is an $\mathscr{H}^V$-equivalence of cofibrant objects of $\mathcal{M}_W$, then the map $\mathbf{O}_{V\oplus W+} \wedge_{\mathbf{O}_V \times \mathbf{O}_W} X_V \wedge f$ is a weak equivalence in $\mathcal{M}_{V\oplus W}$. Since the generating acyclic cofibrations for $\mathcal{M}_W$ are weak equivalences of cofibrant objects and source and target of the generating cofibrations are cofibrant this implies the statement about acyclic cofibrations in the pushout-product axiom. Let $A = G \times \mathbf{O}_{V\oplus W}$ and let $B = G \times \mathbf{O}_V \times \mathbf{O}_W$. Given $P \in \mathscr{H}^{V\oplus W}$ it suffices by cell induction to show that $(PaB_+ \wedge_B (X_V \wedge f))^P$ is a weak equivalence. As above, we may without loss of generality assume that $a^{-1}Pa$ is contained in B. Let P_1 be the image of $a^{-1}Pa$ under the projection

$$B = G \times \mathbf{O}_V \times \mathbf{O}_W \to G \times \mathbf{O}_V$$

and let P_2 be the image of $a^{-1}Pa$ under the projection

$$B = G \times \mathbf{O}_V \times \mathbf{O}_W \to G \times \mathbf{O}_W.$$

By Lemma 2.6.6 $(PaB_+ \wedge_B (X_V \wedge f))^P$ can be identified with $X_V^{P_1} \wedge f^{P_2}$. Since X is cofibrant in $\mathcal{M}$, the space $X_V^{P_1}$ is cofibrant, and likewise the P_2-fixed points of the source and target of f are cofibrant. Thus it suffices to show that f^{P_2} is a weak equivalence. However, since $V(P_2)$ is a subrepresentation of $V(P)$ and $\mathscr{H}$ is closed under retracts, we have that $P_2 \in \mathscr{H}^W$, and thus f^{P_2} is a weak equivalence because f is an $\mathscr{H}^W$-equivalence.

Finally, we verify that if $\mathscr{H}$ is closed under retracts, then $\mathbb{S} = \mathcal{G}_0 S^0$ is cofibrant. For this it is enough to know that S^0 is cofibrant in $\mathcal{M}_0$. However, since the zero representation is a retract of every representation, there exists $P \in \mathscr{H}^0$ with $V(P)$ equal to the zero-dimensional of G. Since $P \subset G \times \mathbf{O}_0$ this implies that $P = G \times \mathbf{O}_0$, and thus $\mathscr{H}^0$ consists of all subgroups of $G \times \mathbf{O}_0$. Since $\mathscr{H}^0 \subseteq \mathscr{G}^0$ this implies that S^0 is cofibrant in $\mathcal{M}_0$. □

Definition 2.6.8. Let $(\mathscr{H}, \mathscr{G})$ be a G-mixing pair. The level $\mathcal{M}$ model structure on $G\mathbf{O}\mathcal{T}$ with $\mathcal{M}$ as in Proposition 2.6.7 is the *level mixed $(\mathscr{H}, \mathscr{G})$-model structure.* Let $f \colon X \to Y$ be a morphism of orthogonal G-spectra.

(i) f is a *level $(\mathscr{H}, \mathscr{G})$-fibration* if for each V in $\mathbf{L}$ the map $f_V : X_V \to Y_V$ is a fibration in the $(\mathscr{H}^V, \mathscr{G}^V)$-model structure on $(G \times \mathbf{O}_V)\mathcal{T}$.

(ii) f is an *$(\mathscr{H}, \mathscr{G})$-cofibration* if f satisfies the left lifting property with respect to all maps that are both level $\mathscr{H}$-equivalences and level $(\mathscr{H}, \mathscr{G})$-fibrations.

Note that unless $\mathscr{H}$ is closed under retracts, the sphere spectrum is not cofibrant in the level $(\mathscr{H}, \mathscr{G})$-model structure.

Lemma 2.6.9. *Let $P \in \mathscr{H}^V$ with $\mathrm{pr}_1(P) = H$ and let C be a cofibrant replacement of the sphere spectrum $\mathbb{S}$ in the $(\mathscr{H}, \mathscr{G})$-model structure. The H-space $C_{V(P)}$ is H-homotopy equivalent to the representation sphere $S^{V(P)}$.*

Proof. The map $C_V \to S^V$ is an $\mathscr{H}^V$-equivalence. Since $P \in \mathscr{H}^V$, it is a weak equivalence of cofibrant P-spaces. Thus it is a P-homotopy equivalence. Finally, H acts through the isomorphism $\mathrm{pr}_1 \colon P \to H$. □

Note that since the mixed $(\mathscr{H}^V, \mathscr{G}^V)$-model structures are left proper and cellular in the sense of Hirschhorn [HirL, Definition 12.1.1], the level $(\mathscr{H}, \mathscr{G})$-model structure is also left proper and cellular. The following theorem is a recollection of results in this section.

Theorem 2.6.10. *Let $(\mathscr{H}, \mathscr{G})$ be a G-mixing pair. The level $(\mathscr{H}, \mathscr{G})$-model structure on $G\mathbf{O}\mathcal{T}$ is a left proper and cellular G-topological model structure. The weak equivalences in this model structure are the level $\mathscr{H}$-equivalences. The fibrations are the level $(\mathscr{H}, \mathscr{G})$-fibrations and the cofibrations are the $(\mathscr{H}, \mathscr{G})$-cofibrations. If $\mathscr{H}$ is closed under retracts, then the level $(\mathscr{H}, \mathscr{G})$-model structure on $G\mathbf{O}\mathcal{T}$ is monoidal.*

We have seen that f is a level $\mathscr{H}$-equivalence if and only if for all V in $\mathbf{L}$ and all $P \in \mathscr{H}^V$, the map $f_{V(P)}$ is an $\mathscr{F}^P$-equivalence of $H = \mathrm{pr}_1(P)$-spaces for the family $\mathscr{F}^P = \{\mathrm{pr}_1(Q) \cap \mathrm{pr}_1(P) \,|\, Q \in \mathscr{H}^V\}$ of subgroups of H.

Note that if $(\mathscr{H}, \mathscr{G})$ and $(\mathscr{H}, \overline{\mathscr{G}})$ are G-mixing pairs with $\mathscr{G} \subseteq \overline{\mathscr{G}}$, then the identity functor is a left Quillen functor from the level $(\mathscr{H}, \overline{\mathscr{G}})$-model structure to the level $(\mathscr{H}, \mathscr{G})$-model structure on $G\mathbf{O}\mathcal{T}$. In fact this is the left Quillen functor in a Quillen equivalence.

Given a G-mixing pair $(\mathscr{H}, \mathscr{G})$ and a subgroup H of G with inclusion homomorphism $i_H \colon H \to G$ we let $(i_H^*\mathscr{H}, i_H^*\mathscr{G})$ be the H-mixing pair with $i_H^*\mathscr{H}^V$ consisting of the subgroups of $H \times \mathbf{O}_V$ obtained by intersecting members of $\mathscr{H}^V$ with $H \times \mathbf{O}_V$. Similarly $i_H^*\mathscr{G}^V$ consists of the subgroups of $H \times \mathbf{O}_V$ obtained by intersecting members of $\mathscr{G}^V$ with $H \times \mathbf{O}_V$.

Lemma 2.6.11. *Given a G-mixing pair $(\mathscr{H}, \mathscr{G})$ and a subgroup H of G with inclusion homomorphism $i_H \colon H \to G$ the functor $i_H^* \colon G\mathbf{O}\mathcal{T} \to H\mathbf{O}\mathcal{T}$ is a right Quillen functor of level mixed model structures with respect to $(\mathscr{H}, \mathscr{G})$ and $(i_H^*\mathscr{H}, i_H^*\mathscr{G})$ respectively.*

Proof. This is a direct consequence of Lemma 1.3.12. □

2.7 Stable equivalences

In this section we work with a fixed $\mathscr{H}$–model stucture $\mathcal{M}$ satisfying the pushout-product axiom.

Given an Euclidean space V and a subgroup P of $G \times \mathbf{O}_V$, we define the $G \times \mathbf{O}_V$-space

$$\widetilde{S}^P := (G \times \mathbf{O}_V)_+ \wedge_P S^V.$$

Here P acts on S^V through the action of $G \times \mathbf{O}_V$ where G acts trivially and $\mathbf{O}_V$ acts via functoriality of the one-point compactification.

We let

$$\lambda_P \colon \mathcal{G}_V \widetilde{S}^P \to \mathbb{S}$$

be the adjoint to action map

$$\widetilde{S}^P = (G \times \mathbf{O}_V)_+ \wedge_P S^V \to S^V = \mathbb{S}_V.$$

Given an orthogonal G-spectrum X, we let

$$\lambda_P^X : \mathcal{G}_V \widetilde{S}^P \wedge X \to X$$

be the composition of $\lambda_P \wedge X$ and the structure isomorphism $\mathbb{S} \wedge X \cong X$. Note that for X of the form $X = \mathcal{G}_W C$ for C a $G \times \mathbf{O}_W$-space, the map λ_P^X is the composition of the isomorphism

$$\mathcal{G}_V \widetilde{S}^P \wedge \mathcal{G}_W C \cong \mathcal{G}_{V\oplus W}(\mathbf{O}_{V\oplus W+} \wedge_{\mathbf{O}_V \times \mathbf{O}_W} \widetilde{S}^P \wedge C)$$

and the map

$$\mathcal{G}_{V\oplus W}(\mathbf{O}_{V\oplus W+} \wedge_{\mathbf{O}_V \times \mathbf{O}_W} \widetilde{S}^P \wedge C) \to \mathcal{G}_W(C).$$

Using a shear map, we obtain an isomorphism of $(G \times \mathbf{O}_V)$-spaces

$$\widetilde{S}^P = (G \times \mathbf{O}_V)_+ \wedge_P S^V \cong (G \times \mathbf{O}_V)/P_+ \wedge S^V, \quad (g, A) \wedge x \mapsto (g, A) \wedge Ax$$

where $G \times \mathbf{O}_V$ acts on S^V in the same way as described above.

In the situation where the projection $\mathrm{pr}_1 : P \to G$ is an isomorphism, we can interpret the $G \times \mathbf{O}_V$-space as follows: let $\varphi\colon G \to \mathbf{O}_V$ be the composition of the inverse of $\mathrm{pr}_1 : P \to G$, the inclusion of P in $G \times \mathbf{O}_V$ and the projection $\mathrm{pr}_2 : G \times \mathbf{O}_V \to \mathbf{O}_V$. Let $\mathbf{O}_{V(P)}$ be the space $\mathbf{O}_V$ with $G \times \mathbf{O}_V$ acting via multiplication by elements of $\mathbf{O}_V$ from the left and multiplication from the right by the inverse of the action elements of G. There is an isomorphism $(G \times \mathbf{O}_V)_+ \wedge_P S^V \cong \mathbf{O}_{V(P)+} \wedge S^{V(P)}$, of $G \times \mathbf{O}_V$-spaces taking an element of $(G \times \mathbf{O}_V)_+ \wedge_P S^V$ represented by $(g, \alpha) \wedge x \in (G \times \mathbf{O}_V)_+ \wedge S^V$ to $\alpha\varphi(g^{-1}) \wedge \varphi(g)x \in \mathbf{O}_V \wedge S^V$. In 2.2.3 this orthogonal G-spectrum $\mathcal{G}_V \widetilde{S}^P$ is denoted $\mathcal{F}_V S^{V(P)}$ and in [MM] it is denoted $F_{V(P)} S^{V(P)}$, or $F_W S^W$ for W any representation of G.

Definition 2.7.1. Let P be a subgroup of $G \times \mathbf{O}_V$ and let X be an orthogonal G-spectrum.

(i) The *negative P-shifted V-loop spectrum* of X is the orthogonal G-spectrum

$$R_P X := \hom_{\mathbf{O}\mathcal{T}}(\mathcal{G}_V \widetilde{S}^P, X)$$

(ii) The *positive P-shifted V-loop spectrum* of X is the orthogonal G-spectrum

$$R_{+P} := \mathcal{G}_V \widetilde{S}^P \wedge X$$

Definition 2.7.2. An orthogonal G-spectrum X is an $\mathcal{M}$-Ω-*spectrum* if it is fibrant in the level $\mathcal{M}$-model structure and for every subgroup P of $G \times \mathbf{O}_V$ in $\mathscr{H}^V$ with $\mathrm{pr}_1(P) = G$, the map $\hom_{\mathbf{O}\mathcal{T}}(\lambda_P, X)\colon X \to R_P X$ induced by $\lambda_P \colon \mathcal{G}_V \widetilde{S}^P \to \mathbb{S}$ is a level $\mathcal{M}$-equivalence.

Definition 2.7.3. A morphism $f\colon X \to Y$ of cofibrant G-orthogonal spectra in the level $\mathcal{M}$-model structure is an $\mathcal{M}$-*stable equivalence* if for every $\mathcal{M}$-Ω spectrum E, the map $G\mathbf{O}\mathcal{T}(f, E)\colon G\mathbf{O}\mathcal{T}(Y, E) \to G\mathbf{O}\mathcal{T}(X, E)$ is an weak equivalence in $\mathcal{T}$. More generally, a morphism of arbitrary orthogonal G-spectra is an $\mathcal{M}$-stable equivalence if the induced map of cofibrant replacements is an $\mathcal{M}$-stable equivalence.

The proof of [MMSS, 8.11] gives the following result.

Lemma 2.7.4. *Every $\mathcal{M}$-stable equivalence between $\mathcal{M}$-Ω-spectra is a level $\mathcal{M}$-equivalence.*

The proof of [MM, III.3.4.] gives the following result.

Lemma 2.7.5. *Let $\mathscr{F}$ be the G-typical family of representations consisting of the trivial representations of G. A morphism between $\mathcal{M}$-Ω-spectra is a level $\mathscr{H}$-equivalence if and only if it is a level $\mathscr{F}$-equivalence.*

Proposition 2.7.6. *If P is in $\mathscr{H}$ with $\mathrm{pr}_1\colon P \to G$ an isomorphism, then λ_P^X is a $\mathcal{M}$-stable equivalence for every cofibrant spectrum X in the level $\mathcal{M}$-model structure.*

Proof. We have to show that for all $\mathcal{M}$-Ω-spectra E the map

$$G\mathbf{O}\mathcal{T}(\lambda_P^X, E) = G\mathbf{O}\mathcal{T}(X, \mathrm{hom}_{\mathbf{O}\mathcal{T}}(\lambda_P, E))$$

is a weak equivalence. However $\mathrm{hom}_{\mathbf{O}\mathcal{T}}(\lambda_P, E)$ is a level equivalence of level fibrant objects. Since X is cofibrant, the map $G\mathbf{O}\mathcal{T}(X, \mathrm{hom}_{\mathbf{O}\mathcal{T}}(\lambda_P, E))$ is a weak equivalence. □

Definition 2.7.7. Let V be an Euclidean space, let P be a subgroup of $G \times \mathbf{O}_V$ such that $\mathrm{pr}_1\colon P \to G$ is an isomorphism and let X be an orthogonal G-spectrum.

(i) The P-*loop spectrum* of X is the orthogonal G-spectrum

$$\Omega^P X := \mathrm{hom}_{\mathbf{O}\mathcal{T}}(\mathcal{G}_0 S^{V(P)}, X).$$

(ii) The P-*suspension spectrum* of X is the orthogonal G-spectrum

$$\Sigma^P X := \mathcal{G}_0 S^{V(P)} \wedge X.$$

Note that the category of orthogonal G-spectra is enriched over the category of G-spaces, and that $\Sigma^P X = S^{V(P)} \otimes X$ is the tensor of $S^{V(P)}$ and X, and that $\Omega^P X$ is the cotensor of $S^{V(P)}$ and X. Since the $\mathcal{M}$-level model structures on the category of orthogonal G-spectra is a $G\mathcal{T}$-model structure we can conclude that (Σ^P, Ω^P) is a Quillen adjoint pair of endofunctors of the category of orthogonal G-spectra.

Lemma 2.7.8. *A map $f\colon X \to Y$ of orthogonal G-spectra is an $\mathcal{M}$-stable equivalence if and only if for every $\mathcal{M}$-Ω spectrum E, the induced map $f^*\colon [Y, E] \to [X, E]$ of morphism sets in the level $\mathcal{M}$-homotopy category is a bijection.*

Proof. One direction is easy since for $\mathcal{M}$-level cofibrant Y and fibrant E, we have that $[Y,E]$ is the set of components of $G\mathbf{O}\mathcal{T}(Y,E)$. Conversely, suppose that for every $\mathcal{M}$-Ω spectrum E, the induced map $f^*\colon [Y,E]\to[X,E]$ of morphisms sets in the level $\mathcal{M}$-homotopy category is a bijection. Since $\mathcal{M}$-level equivalences are $\mathcal{M}$-stable equivalences we may without loss of generality assume that X and Y are cofibrant in the level $\mathcal{M}$-model structure. Given an $\mathcal{M}$-Ω spectrum E' we show that the map

$$\mathrm{hom}_{\mathbf{O}\mathcal{T}}(f,E')\colon \mathrm{hom}_{\mathbf{O}\mathcal{T}}(Y,E')\to \mathrm{hom}_{\mathbf{O}\mathcal{T}}(X,E')$$

is a level equivalence. Since X and Y and cofibrant, this implies by evaluating at level 0 and taking G-fixed points that

$$G\mathbf{O}\mathcal{T}(f,E')\colon G\mathbf{O}\mathcal{T}(Y,E')\to G\mathbf{O}\mathcal{T}(X,E')$$

is a weak equivalence. In order to show that $\mathrm{hom}_{\mathbf{O}\mathcal{T}}(f,E')$ is a level equivalence, it suffices to show that for every source or target C of a generating cofibration for the level $\mathcal{M}$-model structure, the morphism $[C,\mathrm{hom}_{\mathbf{O}\mathcal{T}}(f,E')]$ in the level $\mathcal{M}$-homotopy category is an isomorphism. However, this map is isomorphic to the adjoint $[f,\mathrm{hom}_{\mathbf{O}\mathcal{T}}(C,E')]$, and since C is cofibrant, the spectrum $\mathrm{hom}_{\mathbf{O}\mathcal{T}}(C,E')$ is an $\mathcal{M}$-Ω-spectrum. □

Lemma 2.7.9. *Let V be an Euclidean space. For every member P of $\mathscr{H}^V$ the $G\times\mathbf{O}_V$-space $(G\times\mathbf{O}_V/P)_+$ is cofibrant in the $\mathcal{M}_V$-model structure.*

Proof. Since the isotropy groups of the $G\times\mathbf{O}_V$-CW-complex $(G\times\mathbf{O}_V/P)_+$ are subconjugate to P it is cofibrant in the mixed $(\mathscr{H}^V,\mathscr{H}^V)$-model structure. By part (iv) of 2.5.1 this implies that it is cofibrant in the $\mathcal{M}_V$-model structure. □

Corollary 2.7.10. *If $P\in\mathscr{H}^V$, then $\widetilde{S}^P$ is cofibrant in $\mathcal{M}_V$.*

Definition 2.7.11. Let P be a subgroup of $G\times\mathbf{O}_V$ such that $\mathrm{pr}_1\colon P\to G$ is an isomorphism and let X be an orthogonal G-spectrum.

(i) The *negative P-shift* of X is the orthogonal G-spectrum

$$s_{-P}X := \mathrm{hom}_{\mathbf{O}\mathcal{T}}(\mathcal{G}_V(G\times\mathbf{O}_V/P)_+,X)$$

(ii) The *positive P-shift* of X is the orthogonal G-spectrum

$$s_{+P}X := \mathcal{G}_V(G\times\mathbf{O}_V/P)_+\wedge X$$

Using we get the following: Using we get the following:

Lemma 2.7.12. *Let P be a subgroup of $G\times\mathbf{O}_V$ such that $\mathrm{pr}_1\colon P\to G$ is an isomorphism and let X be an orthogonal G-spectrum.*

(i) There are isomophisms

$$s_{-P}X\cong \mathrm{hom}_{\mathbf{O}\mathcal{T}}(\mathcal{G}_V\mathbf{O}_{V(P)_+},X)\qquad\text{and}\qquad s_{+P}X\cong\mathcal{G}_V\mathbf{O}_{V(P)_+}\wedge X.$$

In particular $(s_{-P}X)_W\cong X_{W\oplus V(P)}$.

(ii) *The P-shifted V-loop spectrum $R_P X$ (cf. 2.7.1) is naturally G isomorphic to both $\Omega^P s_{-P} X$ and $s_{-P}\Omega^P X$.*

Proof. Part (i) is a consequence of the isomorphism $G \times \mathbf{O}_V/P \cong \mathbf{O}_{V(P)}$ from 2.4.2. Part (ii) is a direct consequence of Lemma 2.7.12 and the isomorphisms

$$\widetilde{S}^P \cong S^{V(P)} \wedge \mathbf{O}_{V(P)_+} \cong \mathbf{O}_{V(P)_+} \wedge S^{V(P)}.$$

□

Lemma 2.7.13. *For every G-typical family of representations $\mathscr{H}$ and every subgroup P of $G \times \mathbf{O}_V$ in $\mathscr{H}$ with $\mathrm{pr}_1(P) = G$, the functor R_P preserves level $\mathscr{H}$-equivalences.*

Proof. This is a consequence of the fact that in the level $(\mathscr{H}, \mathscr{H})$-model structure $\mathcal{G}_V \widetilde{S}^P$ is cofibrant and every object is fibrant. □

Proposition 2.7.14. *Let X be a level $\mathcal{M}$-fibrant orthogonal G-spectrum. If $P \in \mathscr{H}$ with $\mathrm{pr}_1(P) = G$, then the map $i = \mathrm{hom}_{\mathbf{O}\mathcal{T}}(\lambda_P, X)\colon X \to R_P X$ is an $\mathcal{M}$-stable equivalence.*

Proof. If E is an $\mathcal{M}$-Ω-spectrum, that is, $i\colon E \to R_P E$ is a level $\mathcal{M}$-equivalence, then so is and $R_P E$. Let X be a $\mathcal{M}$-level fibrant orthogonal G-spectrum. There is a commutative diagram of morphism sets in the level $\mathcal{M}$-homotopy category:

$$\begin{array}{ccc} [X, E] & \xrightarrow{i_*} & [X, R_P E] \\ \uparrow{\scriptstyle i^*} & \searrow^{R_P} & \uparrow{\scriptstyle i^*} \\ [R_P X, E] & \xrightarrow{i_*} & [R_P X, R_P E], \end{array}$$

where both of the functions labeled i_* are bijections. It follows that

$$R_P\colon [X, E] \to [R_P X, R_P E]$$

is a bijection. Hence

$$i^*\colon [R_P X, E] \to [X, E]$$

is a bijection. Now we apply Lemma 2.7.8. □

Corollary 2.7.15. *If P is in $\mathscr{H}^V$ and $\mathrm{pr}_1\colon P \to G$ is an isomorphism, then for every orthogonal G-spectrum X, the map $i = \mathrm{hom}_{\mathbf{O}\mathcal{T}}(\lambda_P, X)\colon X \to R_P X$ is an $\mathcal{M}$-stable equivalence.*

Proof. Let $X \to Y$ be a level $\mathcal{M}$-equivalence with Y level $\mathcal{M}$-fibrant. Thus $X \to Y$ is a level $\mathscr{H}$-equivalence, and by 2.7.13 the morphism $R_P X \to R_P Y$ is also a level $\mathscr{H}$-equivalence. Since level $\mathscr{H}$-equivalences are stable $\mathcal{M}$-equivalences the result is a consequence of $Y \to R_P Y$ being an $\mathcal{M}$-stable equivalence by 2.7.14 and the commutative square

$$\begin{array}{ccc} X & \longrightarrow & R_P X \\ {\scriptstyle \simeq}\downarrow & & \downarrow{\scriptstyle \simeq} \\ Y & \xrightarrow{\sim} & R_P Y. \end{array}$$

□

2.8 Homotopy Groups

We fix a G-typical family of representations $\mathscr{H}$ and an $\mathscr{H}$-model structure $\mathcal{M}$. We introduce a $\mathcal{T}$-functor

$$\widetilde{\lambda}_G\colon (G\mathbf{L})^{\mathrm{op}} \to G\mathbf{O}\mathcal{T} \cong \mathbf{O}G\mathcal{T}$$

whose value at an object V of $G\mathbf{L}$ is the functor $\widetilde{\lambda}_G(V)\colon \mathbf{O} \to G\mathcal{T}$ taking an Euclidean space W to the G-space $\widetilde{\lambda}_G(V)_W = \mathbf{O}(V,W) \wedge S^V$ with G acting diagonally on $\mathbf{O}(V,W)$ and S^V.

Recall from Definition 2.1.3 the untwisting G-isomorphism

$$\tau_{V,W}\colon \mathbf{O}(V,W) \wedge S^V \to \mathbf{L}(V,W) \wedge S^W$$

taking an element in $\mathbf{O}(V,W) \wedge S^V$ of the form $(f\colon V \to W, w \in f(V)^\perp, v \in V)$ to the element $(f, w + f(v))$ in $\mathbf{L}(V,W) \wedge S^W$. The functoriality of $\widetilde{\lambda}_G$ is described via a $\mathcal{T}$-functor $l\colon (G\mathbf{L})^{\mathrm{op}} \wedge \mathbf{O} \to G\mathcal{T}$ defined on objects by $l(V,W) = \widetilde{\lambda}_G(V)_W = \mathbf{O}(V,W) \wedge S^V$. The map

$$l\colon (G\mathbf{L})^{\mathrm{op}}(V,V') \wedge \mathbf{O}(W,W') \to G\mathcal{T}(\mathbf{O}(V,W) \wedge S^V, \mathbf{O}(V',W') \wedge S^{V'})$$

is adjoint to the G-map

$$G\mathbf{L}(V',V) \wedge \mathbf{O}(W,W') \wedge \mathbf{O}(V,W) \wedge S^V \to \mathbf{O}(V',W') \wedge S^{V'}$$

given as the composition

$$\begin{aligned} G\mathbf{L}(V',V) \wedge \mathbf{O}(W,W') \wedge \mathbf{O}(V,W) \wedge S^V &\to G\mathbf{L}(V',V) \wedge \mathbf{O}(V,W') \wedge S^V \\ &\to G\mathbf{L}(V',V) \wedge \mathbf{L}(V,W') \wedge S^{W'} \\ &\to \mathbf{L}(V',W') \wedge S^{W'} \\ &\to \mathbf{O}(V',W') \wedge S^{V'}, \end{aligned}$$

where the first map is induced by composition in $\mathbf{O}$, the second map is induced by the untwisting map $\tau_{V,W'}$, the third map is induced by composition in $\mathbf{L}$ and the last map is the inverse of $\tau_{V',W'}$. Given

$$f \wedge (g,x) \wedge (h,y) \wedge z \in G\mathbf{L}(V',V) \wedge \mathbf{O}(W,W') \wedge \mathbf{O}(V,W) \wedge S^V$$

with $f\colon V' \to V$, $g\colon W \to W'$ and $h\colon V \to W$ embeddings and with $x \in W'$ in the orthogonal complement of $g(W)$, $y \in W$ in the orthogonal complement of $h(V)$ and $z \in V$ we have

$$l(f \wedge (g,x) \wedge (h,y) \wedge z) = (ghf, w) \wedge v,$$

where $v \in V'$ and $w \in W'$ are uniquely determined by requiring $ghf(v)+w = x+gy+ghz$. It is a consequence of Lemma 2.1.5 that this defines a functor $l\colon (G\mathbf{L})^{\mathrm{op}} \wedge \mathbf{O} \to G\mathcal{T}$. We write $\widetilde{\lambda}\colon (G\mathbf{L})^{\mathrm{op}} \to G\mathbf{O}\mathcal{T}$ for the adjoint functor.

Lemma 2.8.1. *If $P \in \mathscr{H}^V$ has $\mathrm{pr}_1 P = G$, then $\lambda_P = \widetilde{\lambda}(0 \to V(P))$.*

We say that an element P in $\mathscr{H}^V$ is *$\mathscr{H}$-irreducible* if $\mathrm{pr}_1\colon P \to G$ is an isomorphism and P is not of the form $Q \oplus R$ for $Q \in \mathscr{H}^W$ and $R \in \mathscr{H}^{W'}$ with both W and W' non-zero Euclidean spaces. Observe that if V is a non-zero Euclidean space, then every member $\mathscr{H}^V$ is a direct sum of $\mathscr{H}$-irreducible elements of $\mathscr{H}$.

Let B be a set containing one representative for each conjugacy class of $\mathscr{H}$-irreducible elements of $\mathscr{H}$. We write B^* for the free monoid on the set B. Given two words $b = P_1 \dots P_m$ and $b' = P'_1 \dots P'_n$ in B^*, we write $b \leq b'$ if there is an increasing sequence $i_1 < \dots < i_m$ such that $P_j = P'_{i_j}$ for $j = 1, \dots, m$, that is, if b can be obtained from b' by removing some letters. This is a partial order, and we consider $B^* = (B^*, \leq)$ as a category.

The morphisms λ_b for $b \in B$ give us a functor $\lambda^*_B\colon (B^*)^{\mathrm{op}} \to G\mathbf{O}\mathcal{T}$ taking $b = P_1 \dots P_m$ to $\lambda_B(b) = \bigwedge_{i=1}^m \mathcal{G}_{V(P_i)} \widetilde{S}^{P_i}$. Consider the free commutative monoid $\mathbb{N}\{B\}$ as a filtered partially ordered set with $\sum n_b b \leq \sum m_b b$ if and only if $n_b \leq m_b$ for all $b \in B$. Choosing a total order on B, there is an order-preserving function $\mathbb{N}\{B\} \to B^*$ taking $n_1 P_1 + \dots n_r P_r$ with $P_1 < \dots < P_r$ to $P_1^{n_1} \dots P_r^{n_r}$. Given $b = n_1 P_1 + \dots n_r P_r \in \mathbb{N}\{B\}$ with $P_1 < \dots < P_r$ we define $V(b) = V(P_1)^{\oplus n_1} \oplus \dots \oplus V(P_r)^{\oplus n_r}$, and we define $\Omega^b = (\Omega^{P_r})^{\circ n_r} \circ \dots \circ (\Omega^{P_1})^{\circ n_1}$

Definition 2.8.2. The functor $\lambda_B\colon \mathbb{N}\{B\}^{\mathrm{op}} \to G\mathbf{O}\mathcal{T}$ is the composition of the functors $\mathbb{N}\{B\}^{\mathrm{op}} \to (B^*)^{\mathrm{op}}$ and $\lambda^*_B\colon (B^*)^{\mathrm{op}} \to G\mathbf{O}\mathcal{T}$.

Definition 2.8.3. Given an orthogonal G-spectrum X, the orthogonal G-spectrum $Q_{\mathscr{H}}X$ whose value on an Euclidean space V is the homotopy colimit

$$\operatorname*{hocolim}_{b \in \mathbb{N}\{B\}} \hom_{\mathbf{O}\mathcal{T}}(\lambda_B(b), X)_V \cong \operatorname*{hocolim}_{b \in \mathbb{N}\{B\}} \Omega^b X_{V(b) \oplus V}$$

in the category of G-spaces. The spectrum QX is the fibrant replacement of $Q_{\mathscr{H}}X$ in the level $\mathcal{M}$-model structure.

Lemma 2.8.4. *The natural map $X \to QX$ given by the inclusion of $X \cong \hom_{\mathbf{O}\mathcal{T}}(\lambda_B(0), X)$ in the homotopy colimit defining QX is an $\mathcal{M}$-stable equivalence.*

Proof. Let $\widetilde{X}$ be a cofibrant replacement of the functor $b \mapsto \hom_{\mathbf{O}\mathcal{T}}(\lambda_B(b), X)$ in the model structure on the category of $\mathbb{N}\{B\}$-diagrams in the level $\mathcal{M}$-model structure on $G\mathbf{O}\mathcal{T}$. In particular, given a morphism $\beta\colon b \to b'$ in $\mathbb{N}\{B\}$, the map $\widetilde{X}^\beta\colon \widetilde{X}^b \to \widetilde{X}^{b'}$ is a map between cofibrant objects and $\operatorname{hocolim}_{b \in \mathbb{N}\{B\}} \widetilde{X}^b$ is a cofibrant replacement of QX.

Let E be an $\mathcal{M}$-Ω-spectrum and let $\beta\colon b \to b'$ be a morphism in $\mathbb{N}\{B\}$. By Proposition 2.7.15 the map $\widetilde{X}^\beta$ is an $\mathcal{M}$-stable equivalence. Given an Euclidean space V and P in $\mathscr{H}^V$, the G-orthogonal spectrum $\mathcal{G}_V(G \times \mathbf{O}_V/P_+)$ is cofibrant in the level $\mathcal{M}$-model structure, and thus

$$\hom_{\mathbf{O}\mathcal{T}}(\widetilde{X}^\beta, E)^P_V \cong G\mathbf{O}\mathcal{T}(\mathcal{G}_V(G \times \mathbf{O}_V/P_+) \wedge \widetilde{X}^\beta, E)$$

is a weak equivalence. That is, $\hom_{\mathbf{O}\mathcal{T}}(\widetilde{X}^\beta, E)$ is a level $\mathcal{M}$-equivalence. Now, since $\operatorname{hocolim}_{\beta \in \mathbb{N}\{B\}} \widetilde{X}^\beta$ is a cofibrant replacement of QX, the isomorphism

$$\hom_{\mathbf{O}\mathcal{T}}(\operatorname*{hocolim}_{\beta \in \mathbb{N}\{B\}} \widetilde{X}^\beta, E) \cong \operatorname*{holim}_{\beta \in \mathbb{N}\{B\}^{\mathrm{op}}} \hom_{\mathbf{O}\mathcal{T}}(\widetilde{X}^\beta, E)$$

shows that $X \to \operatorname{hocolim}_{\beta \in \mathbb{N}\{B\}} \widetilde{X}^{\beta}$ is an $\mathcal{M}$-stable equivalence. □

Proposition 2.8.5. *For every orthogonal G-spectrum X, the orthogonal G-spectrum QX is an $\mathcal{M}$-Ω-spectrum.*

Proof. It suffices to show that for every $P \in \mathscr{H}$ with $\mathrm{pr}_1 \colon P \to G$ an isomorphism the map

$$\hom_{\mathbf{O}\mathcal{T}}(\lambda_P, QX) \colon QX \to R_P QX$$

is a level $\mathcal{M}$-equivalence. Since spheres are compact, the canonical map $R_P QX \to QR_P X$ is a level $\mathcal{M}$-equivalence. Thus it suffices to show that the map

$$Q \hom_{\mathbf{O}\mathcal{T}}(\lambda_P, X) \colon QX \to QR_P X$$

is a level $\mathcal{M}$-equivalence. Given b in $\mathbb{N}\{B\}$, we write $X^b = \hom_{\mathbf{O}\mathcal{T}}(\lambda_B(b), X)$, and given a morphism β in $\mathbb{N}\{B\}$, we let $X^{\beta} = \hom_{\mathbf{O}\mathcal{T}}(\lambda_B(\beta), X)$. We have to show for every Euclidean space W the map $i_X \colon X \to R_P X$ induces a $\mathscr{H}^W$-equivalence

$$\operatorname*{hocolim}_{b \in \mathbb{N}\{B\}} X^b_W \to \operatorname*{hocolim}_{b \in \mathbb{N}\{B\}} (R_P X)^b_W.$$

Choose an isomorphism $V(P) \cong \bigoplus_{i=1}^k V_{p_i}$ for some not necessarily distinct $p_1, \dots, p_k \in B$ and let $p = \sum_{i=1}^k p_i \in \mathbb{N}\{B\}$. We let $\mathbb{N}\{P\} \subseteq \mathbb{N}\{B\}$ be the partially ordered subset consisting of elements of the form np for $n \in \mathbb{N}$, and we let $B_P \subseteq B$ be the complement of $\{p_1, \dots, p_k\}$ in B. The sum in $\mathbb{N}\{B\}$ induces a cofinal inclusion $\mathbb{N}\{P\} \times \mathbb{N}\{B_P\} \to \mathbb{N}\{B\}$ of partially ordered sets. Therefore it suffices to show that the map $i_X \colon X \to R_P X$ induces an $\mathscr{H}^W$-equivalence

$$\operatorname*{hocolim}_{n \in \mathbb{N}} X^{np}_W \to \operatorname*{hocolim}_{n \in \mathbb{N}} (R_P X)^{np}_W.$$

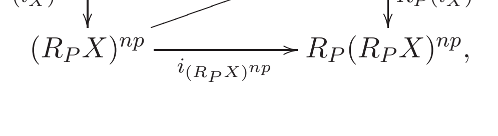
We thus have to show that for every $k \geq 0$ and every Q in $\mathscr{H}^W$ (and for all choices of base point in $(\operatorname{hocolim}_{n \in \mathbb{N}} X^{np}_W)^Q$), the homomorphism

$$\pi_k(\operatorname*{hocolim}_{n \in \mathbb{N}} X^{np}_W)^Q \to \pi_k(\operatorname*{hocolim}_{n \in \mathbb{N}} (R_P X)^{np}_W)^Q$$

is an isomorphism. By compactness of $S^k \wedge (G \times \mathbf{O}_W/Q)_+$, this is an isomorphism if and only if the homomorphism

$$\operatorname*{colim}_{n \in \mathbb{N}} \pi_k(X^{np}_W)^Q \to \operatorname*{colim}_{n \in \mathbb{N}} \pi_k((R_P X)^{np}_W)^Q$$

is an isomorphism. We consider diagrams of the form



where the diagonal arrow is induced by the morphism shifting a factor of $\mathcal{G}_V S^P$ from the front to the back. The outer square in the above diagram commutes, as does the upper triangle, but the lower triangle does not. However, by Schur's Lemma, the space $G\mathbf{L}(V(P), V(P) \oplus V(P))$ is path connected, so the lower triangle commutes up to homotopy. From the commutative diagram

$$\begin{array}{ccc} \pi_k(X_W^{np})^Q & \xrightarrow{\;i_{X^{np}}\;} & \pi_k(R_P X_W^{np})^Q \\ {\scriptstyle (i_X)^{np}}\downarrow & \nearrow & \downarrow{\scriptstyle R_P(i_X)^{np}} \\ \pi_k((R_P X)_W^{np})^Q & \xrightarrow[\;i_{(R_P X)^{np}}\;]{} & \pi_k(R_P(R_P X)_W^{np})^Q, \end{array}$$

we conclude that the homomorphisms

$$\operatorname*{colim}_{n \in \mathbb{N}} \pi_k(X_W^{np})^Q \to \operatorname*{colim}_{n \in \mathbb{N}} \pi_k((R_P X)_W^{np})^Q$$

are isomorphisms. (For all choices of base point in one of the spaces $(X_W^{np})^Q$.)

□

Corollary 2.8.6. *A map $X \to Y$ of orthogonal G-spectra is an $\mathcal{M}$-stable equivalence if and only if $QX \to QY$ is a level $\mathcal{M}$-equivalence.*

Proof. If $X \to Y$ is an $\mathcal{M}$-stable equivalence, then by Lemma 2.8.4 and Proposition 2.8.5 the induced map $QX \to QY$ is a stable equivalence of $\mathcal{M}$-Ω-spectra, and by Lemma 2.7.4 it is a level $\mathcal{M}$-equivalence. Conversely, if $QX \to QY$ is a level $\mathcal{M}$-equivalence, then it is also a stable equivalence, and $X \to Y$ is an $\mathcal{M}$-stable equivalence by Lemma 2.8.4. □

Corollary 2.8.7. *All $\mathscr{H}$-model categories $\mathcal{M}$ have the same class of stable $\mathcal{M}$-equivalences as the $(\overline{\mathscr{H}}, \overline{\mathscr{H}})$-model structure. In particular, the class of $\mathcal{M}$-stable equivalences only depends on the closure $\overline{\mathscr{H}}$ of the G-typical family of representations $\mathscr{H}$.*

Proof. Let $\overline{B}$ be a set containing one representative for each conjugacy class of $\overline{\mathscr{H}}$-irreducible elements of $\overline{\mathscr{H}}$. Then $\mathbb{N}\{B\}$ is cofinal in $\mathbb{N}\{\overline{B}\}$, so for every Euclidean space V, the canonical map $Q_{\mathscr{H}} X \to Q_{\overline{\mathscr{H}}} X$ is a level $\overline{\mathscr{H}}$-equivalence, and these are both level $\mathcal{M}$-equivalent to QX. □

In view of the above result, we say that a morphism of orthogonal G-spectra is a $\mathscr{H}$-stable equivalence if it is a $\mathcal{M}$-stable equivalence for some (and hence every) $\mathscr{H}$-model structure $\mathcal{M}$.

Corollary 2.8.8. *If $X \to Y \to Z$ is a fibration sequence in the level $(\mathscr{H}, \mathscr{H})$-model structure, then so is $QX \to QY \to QZ$.*

Proof. Let $b \in \mathbb{N}\{B\}$. If $X \to Y \to Z$ is a fibration sequence in the level $(\mathscr{H}, \mathscr{H})$-model structure, then so is the sequence $\hom_{\mathbf{O}\mathcal{T}}(\lambda_B(b), X) \to \hom_{\mathbf{O}\mathcal{T}}(\lambda_B(b), Y) \to \hom_{\mathbf{O}\mathcal{T}}(\lambda_B(b), Z)$. From [HirH, Theorem 14.19] we conclude that $QX \to QY \to QZ$ is a fibration sequence in the level $(\mathscr{H}, \mathscr{H})$-model structure. □

Definition 2.8.9. Let k be an integer, let X be an orthogonal G-spectrum and let H be a subgroup of G. The homotopy group $\pi_k^H(X, \mathscr{H})$ is defined as the following abelian group:

$$\pi_k^H(X, \mathscr{H}) := \begin{cases} \pi_k(QX)_0^H & k \geq 0 \\ \pi_0(QX)_{\mathbb{R}^k}^H & k < 0. \end{cases}$$

By compactness, there are isomorphisms

$$\pi_k^H(X, \mathscr{H}) \cong \begin{cases} \operatorname{colim}_{b \in \mathbb{N}\{B\}} \pi_k(\Omega^{V_b} X_{V_b})^H & k \geq 0 \\ \operatorname{colim}_{b \in \mathbb{N}\{B\}} \pi_0(\Omega^{V_b} X_{V_b \oplus \mathbb{R}^k})^H & k < 0. \end{cases}$$

Proposition 2.8.10. *A map $f \colon X \to Y$ of orthogonal G-spectra is an $\mathscr{H}$-stable equivalence if and only if the induced homomorphism $\pi_k^H(f, \mathscr{H})$ is an isomorphism for every subgroup H of G and for every integer k.*

Proof. Let $\mathscr{F}$ be the G-typical family of representations consisting of the trivial representations of G. If the induced homomorphism $\pi_k^H(f, \mathscr{H})$ is an isomorphism for every subgroup H of G and for every integer k, then the map $Qf \colon QX \to QY$ a level $\mathscr{F}$-equivalence. Since it is a map of $\mathcal{M}$-Ω-spectra, Lemma 2.7.5 implies that it is a level $\mathcal{M}$-equivalence. Thus f is an $\mathcal{M}$-stable equivalence.

Conversely, if f is an $\mathcal{M}$-stable equivalence, then Qf is an level $\mathcal{M}$-equivalence, and also a level $\mathscr{F}$-equivalence. Thus the induced homomorphism $\pi_k^H(f, \mathscr{H})$ is an isomorphism for every subgroup H of G and for every integer k. □

The infinite dimensional G-representation $\mathcal{U} = \operatorname{colim}_{n \in \mathbb{N}} \bigoplus_{b \in B} V_b^n$ is a universe of G-representations in the sense of Mandell and May who define equivariant stable homotopy groups as follows:

$$\pi_k^H(X, \mathcal{U}) = \begin{cases} \operatorname{colim}_{U \subseteq \mathcal{U}} \pi_k(\Omega^U X_U)^H & k \geq 0 \\ \operatorname{colim}_{U \subseteq \mathcal{U}} \pi_0(\Omega^U X_{U \oplus \mathbb{R}^k})^H & k < 0. \end{cases}$$

Here the colimits are taken over the partially ordered set of finite dimensional representations in $\mathcal{U}$. Since $\mathbb{N}\{B\}$ is cofinal in this partially ordered set, the groups $\pi_k^H(X, \mathscr{H})$ and $\pi_k^H(X, \mathcal{U})$ are isomorphic. If $i \colon H \to G$ is the inclusion of a subgroup, then by part (v) of 2.4.5, the groups $\pi_k^H(X, \mathscr{H})$ and $\pi_k^H(i^*X, i^*\mathscr{H})$ are isomorphic. We could have used the universe $\mathcal{U}$ in the construction of QX. Working with $\mathbb{N}\{B\}$ we are keep our presentation close to the one of [MMSS].

Most of the following crucial result is taken directly from [MMSS, Theorem 7.4].

Theorem 2.8.11. *Let $\mathscr{H}$ be a G-typical family of representations.*

(i) For every G-CW complex A, the functor $- \wedge A$ preserves $\mathscr{H}$-stable equivalences of orthogonal G-spectra.

(ii) A morphism f of orthogonal G-spectra is an $\mathscr{H}$-stable equivalence if and only if its suspension Σf is an $\mathscr{H}$-stable equivalence. Moreover, the natural map $\eta \colon X \to \Omega^V \Sigma^V X$ is an $\mathscr{H}$-stable equivalence for all orthogonal G-spectra X for all V in $\mathscr{H}$.

(iii) *The homotopy groups of a wedge of orthogonal G-spectra are the direct sums of the homotopy groups of the wedge summands, hence a wedge of $\mathscr{H}$-stable equivalences is an $\mathscr{H}$-stable equivalences.*

(iv) *Cobase changes of maps that are $\mathscr{H}$-stable equivalences and levelwise h-cofibrations are $\mathscr{H}$-stable equivalences.*

(v) *The generalized cobase change and cube lemmas (6.1.4,6.1.6) hold for all orthogonal G-spectra, levelwise h-cofibrations and $\mathscr{H}$-stable equivalences.*

(vi) *If X is the colimit of a sequence of h-cofibrations $X_n \to X_{n+1}$, each of which is an $\mathscr{H}$-stable equivalence, then the map from the initial term X_0 into X is an $\mathscr{H}$-stable equivalence.*

(vii) *For every morphism $f\colon X \to Y$ of orthogonal G-spectra, there are natural long exact sequences*

$$\cdots \to \pi_k^H(Ff, \mathscr{H}) \to \pi_k^H(X, \mathscr{H}) \to \pi_k^H(Y, \mathscr{H}) \to \pi_{k-1}^H(Ff, \mathscr{H}) \to \cdots$$

and

$$\cdots \to \pi_k^H(X, \mathscr{H}) \to \pi_k^H(Y, \mathscr{H}) \to \pi_k^H(Cf, \mathscr{H}) \to \pi_{k-1}^H(X, \mathscr{H}) \to \cdots,$$

where Ff and Cf denote the levelwise homotopy fiber and cofiber of f, that is, in each level it is given be the homotopy fiber and cofiber. The natural map $Ff \to \Omega Cf$ is an $\mathscr{H}$-stable equivalence.

Proof. Part (i) is [MM, Theorem III.3.11]. For part (ii), note that a map f induced an isomorphism on homotopy groups if and only Ωf is so. Thus f is an $\mathscr{H}$-stable equivalence if and only if Ωf is so. Now, the arguments used in the proof of Proposition 2.8.5 give the statements of (ii). For the rest of the statements, the arguments in the proof of [MMSS, Theorem 7.4] carry over to our situation. □

Corollary 2.8.12. *Let V a Euclidean space and let $P \in \mathscr{H}^V$. A map $f\colon X \to Y$ is an $\mathscr{H}$-stable equivalence if and only if its $V(P)$-suspension $\Sigma^{V(P)} f\colon \Sigma^{V(P)} X \to \Sigma^{V(P)} Y$ is an $\mathscr{H}$-stable equivalence.*

Proof. Note that $S^{V(P)}$ is a G-CW-complex. Thus (i) of 2.8.11 implies that is f is an $\mathscr{H}$-stable equivalence, then so is $\Sigma^{V(P)} f$. Conversely, if $\Sigma^{V(P)} f$ is an $\mathscr{H}$-stable equivalence, then so is $\Omega^{V(P)} \Sigma^{V(P)} f$. Now use (ii) of 2.8.11. □

2.9 The stable model structure

Let $\mathcal{M}$ be an $\mathscr{H}$-model structure.

Definition 2.9.1. Let $f\colon X \to Y$ be a map of orthogonal G-spectra. We say that f is:

(i) a stably $\mathcal{M}$-acyclic cofibration if it is an $\mathscr{H}$-stable equivalence and a level $\mathcal{M}$-cofibration;

(ii) an $\mathcal{M}$-stable fibration if it satisfies the right lifting property with respect to stably $\mathcal{M}$-acyclic cofibrations;

(iii) stably $\mathcal{M}$-acyclic fibration if it is an $\mathscr{H}$-stable equivalence and a level $\mathcal{M}$-fibration.

Recall from Definition 2.5.4 that the level $\mathcal{M}$-model structure has the set $\mathcal{G}I$ of generating cofibrations and the set $\mathcal{G}J$ of generating acyclic cofibrations.

Definition 2.9.2. We let $S(\mathcal{M})$ be the set consisting of the morphisms

$$G/H_+ \wedge \lambda_P^X \colon G/H_+ \wedge \mathcal{G}_V \widetilde{S}^P \wedge X \to G/H_+ \wedge X$$

where H is a subgroup of G, where $P \in \mathscr{H}^V$ for some $V \in \mathbf{O}$ and where $X = \mathcal{G}_W C$ for C a cofibrant replacement of either a source or a target of a generating cofibration in one of the categories $\mathcal{M}_W$.

Note that the map λ_P^X is the composition of an isomorphism

$$\mathcal{G}_V \widetilde{S}^P \wedge X \xrightarrow{\cong} \mathcal{G}_{V \oplus W}(\mathbf{O}_{V\oplus W+} \wedge_{\mathbf{O}_V \times \mathbf{O}_W} \widetilde{S}^P \wedge C)$$

and a map of the form

$$\mathcal{G}_{V \oplus W}(\mathbf{O}_{V\oplus W+} \wedge_{\mathbf{O}_V \times \mathbf{O}_W} \widetilde{S}^P \wedge C) \to \mathcal{G}_W(C).$$

Definition 2.9.3. Given λ in $S(\mathcal{M})$, we let $M\lambda$ be the mapping cylinder of λ. Then λ factors as the composite of a level $\mathcal{M}$-cofibration

$$k_\lambda \colon \mathcal{G}_{V \oplus W}(\mathbf{O}_{V\oplus W+} \wedge_{\mathbf{O}_V \times \mathbf{O}_W} \widetilde{S}^P \wedge C) \to M\lambda$$

and a deformation retraction

$$r_\lambda \colon M\lambda \to \mathcal{G}_W(C).$$

We let K_λ be the set of maps of the form $k_\lambda \square i$, where $i \in I$ is a generating cofibration for $\mathcal{T}$. Let K be the union of $\mathcal{G}J$ and the sets K_λ for $\lambda \in S$.

Definition 2.9.4. An $S(\mathcal{M})$-*fibration* is a level $\mathcal{M}$-fibration, say $g \colon Z \to W$, of orthogonal G-spectra with the property that for every morphism $\lambda \colon A \to B$ of $S(\mathcal{M})$, the square

$$\begin{array}{ccc} G\mathbf{O}\mathcal{T}(B,Z) & \xrightarrow{g_*} & G\mathbf{O}\mathcal{T}(B,W) \\ \downarrow{\scriptstyle \lambda^*} & & \downarrow{\scriptstyle \lambda^*} \\ G\mathbf{O}\mathcal{T}(A,Z) & \xrightarrow{g_*} & G\mathbf{O}\mathcal{T}(A,W) \end{array}$$

is a homotopy pullback square in $\mathcal{T}$.

Example 2.9.5. Let us take a look at the $S(\mathcal{M})$-fibrations in the situation where $\mathcal{M}$ is the level $(\mathscr{H},\mathscr{G})$-model structure for a G-mixing pair $(\mathscr{H},\mathscr{G})$. The set $\mathcal{G}I$ of generating cofibrations consists of maps of the form

$$\mathcal{G}_V(i \wedge (G \times \mathbf{O}_V)/P_+)$$

for $i\colon S^{n-1}_+ \to D^n_+$ a generating cofibration for $\mathcal{T}$ and P a member of $\mathscr{G}^V$. Since we work with model categories enriched over $\mathcal{T}$, the $S(\mathcal{M})$-fibrations do not change if replace $S(\mathcal{M})$ by the set of maps of the form $\lambda_W^{\mathcal{G}_V C}$ where C is a transitive $(G \times \mathbf{O}_V)$-spaces of the form $(G \times \mathbf{O}_V)/P_+$ for $P \in \mathscr{G}^V$. In this situation we write $K_{\mathscr{G},\mathscr{H}}$ for the set K of generating stably acyclic cofibrations.

The arguments proving [MMSS, Proposition 9.5] give:

Proposition 2.9.6. *A map $p\colon E \to B$ of orthogonal G-spectra satisfies the right lifting property with respect to K if and only if it is an $S(\mathcal{M})$-fibration.*

Corollary 2.9.7. *The map $F \to *$ satisfies the right lifting property with respect to K if and only if F is an $\mathcal{M}$-Ω-spectrum.*

Corollary 2.9.8. *If $p\colon E \to B$ is an $\mathscr{H}$-stable equivalence that satisfies the right lifting property with respect to K, then p is a both a level $\mathscr{H}$-equivalence and a level $\mathcal{M}$-fibration.*

Proof. Since $\mathcal{G}J$ is contained in K, we only need to prove that p is a level $\mathscr{H}$-equivalence. Let $F = p^{-1}(*)$ be the fiber over the basepoint. Since p satisfies the right lifting property with respect to K, so does the map $F \to *$, and thus by 2.9.7 the orthogonal G-spectrum F is an $\mathcal{M}$-Ω-spectrum. Since p is an $\mathscr{H}$-stable equivalence, the corollaries 2.8.8 and 2.8.6 imply that $F \to *$ is also an $\mathscr{H}$-stable equivalence. By the level long exact sequences of homotopy groups, for each $p_V\colon E_V \to B_V$ and each P in $\mathscr{H}^V$ and each $k \geq 1$, the group homomorphism $\pi_k^P(p_V)$ is an isomorphism. To see that $\pi_0^P(p_V)$ is a bijection, note that $\pi_0^P(p_V) = \pi_0^H(p_{V(P)})$ for a subgroup $H = \mathrm{pr}_1(P)$ of G. By part (v) of 2.4.5 we may assume that $\mathrm{pr}_1(P) = G$. By part (iii) of 2.4.5 we may choose a Euclidean space W and $Q \in \mathscr{H}^W$ so that $V(Q)$ is a trivial representation of G. The homotopy pullback diagram of Definition 2.9.4 associated to the map $\lambda = \lambda_Q^{\mathcal{G}_{\tilde{S}}P}$ is of the form

$$\begin{array}{ccc} E_{V(P)} & \xrightarrow{p_{V(P)}} & B_{V(P)} \\ \downarrow{\scriptstyle\lambda^*} & & \downarrow{\scriptstyle\lambda^*} \\ \Omega^{V(Q)} E_{V(P)\oplus V(Q)} & \xrightarrow{\Omega^{V(Q)} p_{V(P)\oplus V(Q)}} & \Omega^{V(Q)} B_{V(P)\oplus V(Q)}. \end{array}$$

The map $\Omega^{V(Q)} p_{V(P)\oplus V(Q)}$ depends only on basepoint components and is a weak equivalence of G-spaces. Therefore $p_{V(P)}$ is a weak equivalence of G-spaces. In particular it is a weak equivalence of H-spaces and $\mathscr{H}^V$-equivalence as required. ☐

The following theorem can be proved as in Section 9 of [MMSS].

Theorem 2.9.9. *The category $G\mathbf{O}\mathcal{T}$ of orthogonal G-spectra is a cofibrantly generated proper G-topological model category with respect to the $\mathscr{H}$-stable equivalences, $\mathcal{M}$-stable fibrations and level $\mathcal{M}$-cofibrations.*

Remark 2.9.10. Actually it is a compactly generated model category, and thus also cellular.

In the main part of this paper we work with the following $\mathbb{S}$-model structure taken from Shipley's paper [Sh04].

Definition 2.9.11 (The $\mathbb{S}$-model structure)**.** The $\mathbb{S}$-*model structure* on $G\mathbf{O}\mathcal{T}$ is the model category obtained in Theorem 2.9.9 from the positive mixing pair $(\mathscr{H},\mathscr{G})$. For $V \neq 0$, the family $\mathscr{G}^V$ consists of all subgroups of $G \times \mathbf{O}_V$, and $\mathscr{H}^V$ consists of the subgroups P of $G \times \mathbf{O}_V$ with the property that $\mathrm{pr}_1 \colon P \to G$ is injective. When $V = 0$, the families $\mathscr{G}^V$ and $\mathscr{H}^V$ are empty.

We call the cofibrations and fibrations in this model structure $\mathbb{S}$-cofibrations and $\mathbb{S}$-fibrations respectively. Moreover we use the notation $\mathbb{S}I = \mathcal{G}I$ and $\mathbb{S}J = \mathcal{G}K$ for the sets of generating cofibrations and generating acyclic cofibrations respectively for this model structure. We shall also write stable equivalence or π_*-isomorphism instead of $\mathscr{H}$-stable equivalence.

Lemma 2.9.12. *If P is in $\mathscr{H}$, then the endofunctors s_{+P} and s_{-P} of $G\mathbf{O}\mathcal{T}$ from 2.7.12 form a Quillen adjoint pair with respect to the stable $\mathcal{M}$-model structure.*

Proof. The functors s_{+P} and s_{-P} form a Quillen adjoint pair with respect to the level $\mathcal{M}$-model structure. Therefore it suffices to show that $s_{+P}(\lambda_Q^X)$ is a stable equivalence for all elements λ_Q^X of $S(\mathcal{M})$. However $Y = \mathcal{G}_V(G\times\mathbf{O}_V/P)_+\wedge X$ is cofibrant and $s_{+P}(\lambda_Q^X) = \lambda_Q^Y$, so by Proposition 2.7.6 the morphism $s_{+P}(\lambda_Q^X)$ is a stable equivalence. ☐

Lemma 2.9.13. *For every cofibrant G-space A, the endofunctor $X \mapsto A \wedge X$ is a left Quillen functor on $G\mathbf{O}\mathcal{T}$ with respect to both the level $\mathcal{M}$-model structure and the $\mathcal{M}$-stable model structure.*

Proof. Use the isomorphisms $A \wedge \mathcal{G}_V(f) \cong \mathcal{G}_V(A \wedge f)$ and $A \wedge \lambda_P^C \cong \lambda_P^{A\wedge C}$ together with Proposition 2.7.6 and the assumption that $\mathcal{M}_V$ is a $G\mathcal{T}$-model structure. ☐

Theorem 2.9.14. *Let V be an Euclidean space and let $P \in \mathscr{H}^V$ with $\mathrm{pr}_1 \colon P \to G$ an isomorphism. The pairs (s_{+P}, s_{-P}) and (Σ^V, Ω^V) are Quillen equivalences of $G\mathbf{O}\mathcal{T}$ in the $\mathcal{M}$-stable model structure.*

Proof. By definition, for every $\mathcal{M}$-Ω-spectrum E, the map

$$\mathrm{hom}_{\mathbf{O}\mathcal{T}}(\lambda_P, E) \colon E \to R_P E$$

is a level equivalence. In particular it is a stable equivalence, and thus the right derived functor of R_P is naturally isomorphic to the identity. Since by 2.7.12 $R_P \cong s_{-P}\Omega^V \cong \Omega^V s_{-P}$, the right derived functors of s_{-P} and Ω^V are inverse equivalences of categories. Thus both Ω^V and s_{-P} are right adjoint functors in Quillen equivalences. ☐

Corollary 2.9.15. *The stable $\mathcal{M}$-model structure is a stable model structure in the sense that the homotopy category is a triangulated category (cf. [H, Chapter 7].)*

2.10 Ring- and Module Spectra

Let us fix a G-mixing pair $(\mathscr{H},\mathscr{G})$. We use Theorem [SS, 4.1] to lift the $(\mathscr{H},\mathscr{G})$-model structure to categories of modules and algebras. First we need to verify the monoid axiom.

Proposition 2.10.1. *Cofibrant spectra in the $(\mathscr{H},\mathscr{G})$-model structure are flat, in the sense that for any $(\mathscr{H},\mathscr{G})$-cofibrant spectrum X, the functor $X\wedge -$ preserves stable equivalences.*

Proof. Since smashing with any spectrum preserves level h-cofiber sequences, and by the long exact sequence for homotopy groups 2.8.11 (vii), it suffices to show that if Z is an orthogonal spectrum with $\pi_*(Z) = 0$, then also $\pi_*(X\wedge Z) = 0$. Since smashing with Z preserves the cell complex construction, we can further reduce to the case where X is either the source or the target of one of the generating $(\mathscr{H},\mathscr{G})$-cofibrations, i.e., X is of the form $\mathcal{G}_V\left[(G \times \mathbf{O}_V)/P_+\wedge S^k_+\right]$ or $\mathcal{G}_V\left[(G \times \mathbf{O}_V)/P_+\wedge D^k_+\right]$ for P in $\mathscr{G}^V$. Since, for every space K, the spectrum $\mathcal{G}_V\left[(G \times \mathbf{O}_V)/P_+\wedge K_+\right]$ is equal to $\mathcal{G}_V\left[(G \times \mathbf{O}_V)/P_+\right]\wedge K_+$ and since we know from 2.8.11 (i) that smashing with a cofibrant G-space preserves $\mathscr{H}$-stable equivalences it suffices to show that if Z has trivial homotopy groups, then also the spectrum $\mathcal{G}_V\left[(G \times \mathbf{O}_V)/P_+\right]\wedge Z$ has trivial homotopy groups.

In order to simplify notation, we let $\mathbf{G}_V = G \times \mathbf{O}_V$. Recall that

$$\begin{aligned}\left(\mathcal{G}_V\left[\mathbf{G}_V/P_+\right]\wedge Z\right)_{V\oplus W} &= \mathbf{O}_{V\oplus W+}\wedge_{\mathbf{O}_V\times\mathbf{O}_W}\left(\mathbf{G}_V/P_+\wedge Z_W\right)\\ &\cong \mathbf{G}_{V\oplus W}/P_+\wedge_{\mathbf{O}_W} Z_W,\end{aligned}$$

where the structure maps are the composites:

$$\begin{array}{ccc}\mathbf{G}_{V\oplus W}/P_+\wedge_{\mathbf{O}_W} Z_W\wedge S^U & \xrightarrow{\mathrm{id}\wedge\sigma} & \mathbf{G}_{V\oplus W}/P_+\wedge_{\mathbf{O}_W} Z_{W\oplus U}\\ & & \big\downarrow{\scriptstyle p\circ\mathrm{inc}}\\ & & \mathbf{G}_{V\oplus W\oplus U}/P_+\wedge_{\mathbf{O}_{W\oplus U}} Z_{W\oplus U},\end{array}$$

where σ is the structure map of Z, the map inc is induced by the standard inclusion $\mathbf{O}_{V\oplus W} \to \mathbf{O}_{V\oplus W\oplus U}$ and p is the projection from $\mathbf{O}_W$-orbits to $\mathbf{O}_{W\oplus U}$-orbits.

Recall that $\{V_b\}_{b\in B}$ is a totally ordered set of representatives of $\mathscr{H}$-irreducible representations of G. Let us represent elements of the free commutative monoid $\mathbb{N}\{B\}$ as functions $f\colon B \to \mathbb{N}$ with finite support. We will take the liberty to use the symbol W to denote both an element $W = f \in \mathbb{N}\{B\}$ and the G-representation $V(f) = \bigoplus_{P\in B} V(P)^{\oplus f(P)}$ associated to f. The homotopy groups $\pi_k^H(\mathcal{G}_V\left[\mathbf{G}_V/P_+\right]\wedge Z)$ are therefore isomorphic to the colimit:

$$\operatorname*{colim}_{W\in\mathbb{N}\{B\}} \pi_k^H(\Omega^W(\mathbf{G}_{V\oplus W}/P_+ \wedge_{\mathbf{O}_W} Z_W))^H.$$

The projection

$$\mathbf{G}_{V\oplus W}/P_+ \wedge_{\mathbf{O}_W} Z_{W\oplus U} \xrightarrow{p\circ\mathrm{inc}} \mathbf{G}_{V\oplus W\oplus U}/P_+ \wedge_{\mathbf{O}_{W\oplus U}} Z_{W\oplus U}$$

is the identity when U is zero, so it induces a surjective homomorphism from

$$\operatorname*{colim}_{W\in\mathbb{N}\{B\}} \operatorname*{colim}_{U\in\mathbb{N}\{B\}} \pi_k^H(\Omega^{W\oplus U}(\mathbf{G}_{V\oplus W}/P_+ \wedge_{\mathbf{O}_W} Z_{W\oplus U}))^H$$

to

$$\operatorname*{colim}_{W\in\mathbb{N}\{B\}} \pi_k^H(\Omega^W(\mathbf{G}_{V\oplus W}/P_+ \wedge_{\mathbf{O}_W} Z_W))^H.$$

Let us write $s_{-W}Z = s_{-P_W}Z$ for $P_W \in \mathscr{H}$ with $W = V(P_W)$. The group $G \times \mathbf{O}_W$ acts on the orthogonal spectrum $s_{-W}Z$ since $(s_{-W}Z)_U = Z_{W\oplus U}$ and with structure maps inherited from Z. The colimit

$$\operatorname*{colim}_{U\in\mathbb{N}\{B\}} \pi_k^H(\Omega^{W\oplus U}(\mathbf{G}_{V\oplus W}/P_+ \wedge_{\mathbf{O}_W} Z_{W\oplus U}))^H$$

is the homotopy groups of the orthogonal spectrum $\mathbf{G}_{V\oplus W}/P_+ \wedge_{\mathbf{O}_W} s_{-W}Z$. Thus if we show that spectra of this type have trivial homotopy groups, we are done.
Since G acts on $W = V(P_W)$ and P is contained in $\mathbf{O}_W$, we can consider the space $\mathbf{G}_{V\oplus W}/P$ as a $G \times \mathbf{O}_W$-space. Since $\mathbf{O}_{V(P_W)} \cong (G \times \mathbf{O}_W)/P_W$, the isotropy groups of this $G \times \mathbf{O}_W$-space are all subconjugate to P_W. Therefore $\mathbf{G}_{V\oplus W}/P$ has $G \times \mathbf{O}_W$-cells of the form $D^p_+ \wedge (G \times \mathbf{O}_W)/Q_+$ for $Q \subseteq P_W$. Writing $H = \mathrm{pr}_1(Q)$ and $i\colon H \to G$ for the inclusion, there are isomorphisms

$$(G \times \mathbf{O}_W)/Q_+ \cong G_+ \wedge_H \mathbf{O}_{V(Q)} \cong G/H_+ \wedge \mathbf{O}_{V(P_W)}$$

of $G \times \mathbf{O}_W$-spaces, and there is an isomorphism

$$(G \times \mathbf{O}_W)/Q_+ \wedge_{\mathbf{O}_W} s_{-W}Z \cong G/H_+ \wedge (s_{-W}Z).$$

Suppose that $* = X_0 \to X_1 \to \ldots \to X_l$ is the cellular filtration of $\mathbf{G}_{V\oplus W}/P_+$. Both levelwise smash product with spaces and taking levelwise $\mathbf{O}_W$-orbits commute with (levelwise) colimits hence with the cell complex construction. From the cell structure we get the gluing diagrams of spectra of the form:

$$\begin{array}{ccc}
S^{p-1}_+ \wedge (G \times \mathbf{O}_W)/Q_+ \wedge_{\mathbf{O}_W} s_{-W}Z & \longrightarrow & X_i \wedge_{\mathbf{O}_W} s_{-W}Z \\
\downarrow & \ulcorner & \downarrow \\
D^p_+ \wedge (G \times \mathbf{O}_W)/Q_+ \wedge_{\mathbf{O}_W} s_{-W}Z & \longrightarrow & X_{i+1} \wedge_{\mathbf{O}_W} s_{-W}Z,
\end{array}$$

that is, of the form

$$\begin{array}{ccc}
S^{p-1}_+ \wedge G/H_+ \wedge (s_{-W}Z) & \longrightarrow & X_i \wedge_{\mathbf{O}_W} s_{-W}Z \\
\downarrow & \ulcorner & \downarrow \\
D^p_+ \wedge G/H_+ \wedge (s_{-W}Z) & \longrightarrow & X_{i+1} \wedge_{\mathbf{O}_W} s_{-W}Z.
\end{array}$$

Then as Z was $\mathscr{H}$-acyclic, and by 2.7.15 so is $R_{P_W}Z \cong \Omega^{P_W} s_{-W}Z$. Since Σ^{P_W} and $s_{-W} = s_{-P_W}$ commute (i) and (ii) of Theorem 2.8.11 imply that $s_{-W}Z$ is $\mathscr{H}$-acyclic. Thus the two spectra in the left column of the square are $\mathscr{H}$-acyclic by part (i) of Theorem 2.8.11. Assume that $X_i \wedge_{\mathbf{O}_W} s_{-W}Z$ is stably $\mathscr{H}$-acyclic, so the top horizontal maps are $\mathscr{H}$-stable equivalences, hence so are the bottom maps since the left vertical maps are h-cofibrations. Thus $X_{i+1} \wedge s_{-W}Z$ is $\mathscr{H}$-acyclic, and it follows by induction that $X_l \wedge s_{-W}Z$ is $\mathscr{H}$-acyclic. □

Following [MMSS, Proposition 12.5] we obtain the following crucial result:

Proposition 2.10.2. *If $i\colon A \to X$ is an acyclic cofibration in the stable $(\mathscr{H},\mathscr{G})$-model structure and Y is any orthogonal G-spectrum, then the map $i \wedge Y\colon A \wedge Y \to X \wedge Y$ is a stable equivalence.*

Proof. Let $Z = X/A$. There is a Hurewicz cofiber sequence $A \wedge Y \to X \wedge Y \to Z \wedge Y$. By part (vii) of 2.8.11 it suffices to show that $Z \wedge Y$ is acyclic. Let $j\colon Y' \to Y$ be a cofibrant approximation of Y. By Proposition 2.10.1, the map $Z \wedge j$ is a stable equivalence. Thus, it is enough to show that $Z \wedge Y$ is acyclic when Z is acyclic and both Z and Y are cofibrant. Now Proposition 2.10.1 implies that the map $* = * \wedge Y \to Z \wedge Y$ is a stable equivalence because Y is cofibrant and $* \to Z$ is a stable equivalence. □

Proposition 2.10.3. *The stable $(\mathscr{H},\mathscr{G})$-model structure is monoidal.*

Proof. It follows from Proposition 2.6.7 that if $i\colon A \to B$ and $j\colon X \to Y$ are cofibrations, then also $i\Box j$ is a cofibration. Now, suppose that j is an acyclic cofibration. Then the cofiber Y/X of j is acyclic, and there is a cofibration sequence

$$A \wedge Y \cup_{A \wedge X} B \wedge X \xrightarrow{i\Box j} B \wedge Y \to B/A \wedge Y/X.$$

By stability it suffices to show that the spectrum $B/A \wedge Y/X$ is acyclic. However, since B/A is cofibrant and Y/X is acyclic, this is a consequence of Proposition 2.10.1. We have now shown that the push-out-product axiom holds in the stable $(\mathscr{H},\mathscr{G})$-model structure. The criterion for being a monoidal model category concerning cofibrant replacement of the sphere spectrum is a direct consequence of 2.10.2. □

Now [SS, Prop. 4.1] gives the following:

Theorem 2.10.4. *Let R be an orthogonal ring G-spectrum.*

(i) The category of left R-modules is a compactly generated proper model category with respect to the $\mathscr{H}$-stable equivalences and the underlying $(\mathscr{H},\mathscr{G})$-stable fibrations. The sets of generating cofibrations and acyclic cofibrations are $R \wedge FI$ and $R \wedge K$.

(ii) If R is cofibrant in $G\mathbf{O}\mathcal{T}$, then the forgetful functor from R-modules to orthogonal spectra preserves cofibrations. Hence every cofibrant R-module is cofibrant as an orthogonal G-spectrum.

(iii) *Let R be commutative. The model structure of (i) is monoidal and satisfies the monoid axiom.*

(iv) *Let R be commutative. The category of R-algebras is a compactly generated right proper model category with respect to the stable $\mathscr{H}$-equivalences and the underlying $(\mathscr{H},\mathscr{G})$-fibrations. The sets of generating cofibrations and acyclic cofibrations are $R \wedge \mathbb{A}FI$ and $R \wedge \mathbb{A}K$.*

(v) *Let R be commutative. Every cofibration of R-algebras whose source is cofibrant as an R-module is also a cofibration of R-modules. In particular, every cofibrant R-algebra is cofibrant as an R-module.*

All the above model structures are Quillen equivalent to the one obtained from the $(\overline{\mathscr{H}},\overline{\mathscr{H}})$-model structure, that is the classical one from [MM, III.7.6] with respect to the G-universe $\mathfrak{U} = \operatorname{colim}_{n\in\mathbb{N}} \bigoplus_{b\in B} V_b^n$, via the identity functor.

We will deal with the lift to commutative algebras in a separate section (4.2).

Chapter 3

Filtering smash powers

In this chapter we work with the positive model structure on orthogonal G-spectra from Definition 2.9.11, and we let $(\mathscr{H}, \mathscr{G})$ be the positive mixing pair from Example 2.6.4.

3.1 Fixed Points of Cells

We start our study of fixed points of smash powers of orthogonal spectra by considering what happens in the simplest cases. As a matter of fact, the basics of this study is essential for even setting up our model structure for commutative equivariant ring spectra, but it turns out that a slightly deeper investigation reveals structure that will be used critically in later chapters. Since we want to work with the $\mathbb{S}$-model structure (4.2.14) for commutative orthogonal ring spectra we will have to extend the classical results to a bigger class of (cofibrant) spectra throughout the following sections.

For ease of reference, we allow ourselves to introduce the following notation for smash powers: if X is a finite set with group of automorphisms Σ_X and L an orthogonal spectrum, then

$$\bigwedge_X L := L \wedge \ldots \wedge L, \quad \text{smash factor indexed by } X,$$

considered as a Σ_X-spectrum. This notation will reappear crucially in 4.3.7 where we will extend the functoriality of X in the case when L is a commutative orthogonal ring spectrum.

3.1.1 Cellular Filtrations

For reference we list some easily checked facts about "cellular filtrations" which hold for any closed symmetric category $\mathcal{C}$ with all small colimits (most of the facts do not need all this structure). In particular we show:

Theorem 3.1.2. *Let $(\mathcal{C}, \wedge)$ be closed symmetric monoidal with all small colimits. Let I and J be sets of morphisms in $\mathcal{C}$ and let $f\colon A \to X$ and $g\colon B \to Y$ be relative I- and*

J-cellular, respectively. Then their pushout product $f \square g$ is relative $(I \square J)$-cellular. In particular, if λ and μ are partially ordered indexing sets for cells of f and g, respectively, then $\lambda \times \mu$ is a partially ordered indexing set for cells of $f \square g$.

It seems hard to actually find a proof of the above theorem explicitly spelled out in the literature. Since we are going to have to work with such filtrations in more detail later, we give them below.

Pushouts and Pushout Products

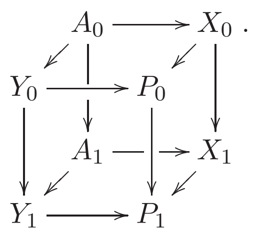
Lemma 3.1.3. *Consider the following commutative diagram in $\mathcal{C}$:*

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ D & \longrightarrow & P & \longrightarrow & Q. \end{array}$$

(i) If both the left and the right subsquare of the diagram are pushout diagrams, then so is the outer rectangle.

(ii) If both the left subsquare and the outer rectangle are pushout diagrams, then so is the right subsquare.

Lemma 3.1.4. *Let $\mathcal{C}$ have all pushouts. Consider a commutative cube in $\mathcal{C}$, where either the top and bottom faces or the left and right faces are pushouts:*



(3.1.5)

Then the induced square

$$\begin{array}{ccc} X_0 \coprod_{A_0} A_1 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ P_0 \coprod_{Y_0} Y_1 & \longrightarrow & P_1 \end{array}$$

is again pushout.

Lemma 3.1.6. *Consider two pushout squares*

$$\begin{array}{ccc} A_b & \xrightarrow{g_b} & X_b \\ \downarrow & \ulcorner & \downarrow \\ A_f & \xrightarrow[g_f]{} & X_f \end{array} \qquad \qquad \begin{array}{ccc} B_b & \xrightarrow{h_b} & Y_b \\ \downarrow & \ulcorner & \downarrow \\ B_f & \xrightarrow[h_f]{} & Y_f. \end{array}$$

Their row-wise pushout product is also a pushout square

$$\begin{array}{ccc} (A_b{\wedge}Y_b)\coprod_{A_b\wedge B_b}(X_b{\wedge}B_b) & \xrightarrow{g_b\square h_b} & X_b{\wedge}Y_b \\ \downarrow & \ulcorner & \downarrow \\ A_f{\wedge}Y_f)\coprod_{A_f\wedge B_f}(X_f{\wedge}B_f) & \xrightarrow[g_f\square h_f]{} & X_f{\wedge}Y_f. \end{array}$$

Relative Cellular Maps

We will use Lemma 3.1.6 to recognize a relative cellular structure on the $\square$-product of relative cellular maps. Recall the following definition (e.g., [H, 2.1.9]):

Definition 3.1.7. Let I be a class of morphisms of $\mathcal{C}$. Then a morphism $f\colon A \to X$ in $\mathcal{C}$ is *relative I-cellular*, if it is a transfinite composition of pushouts of coproducts of elements of I.

Remark 3.1.8. Let $f\colon A \to X$ be a relative I-cellular map, and let $A = X_0 \to X_1 \to \dots$ be a λ-sequence that exhibits this structure, i.e., λ an ordinal and for any $\alpha \leq \lambda$ we have pushout diagrams

$$\begin{array}{ccc} S_\alpha & \xrightarrow{\sigma_\alpha} & \operatorname*{colim}_{\beta<\alpha} X_\alpha \\ {\scriptstyle i_\alpha}\downarrow & \ulcorner & \downarrow{\scriptstyle f_\alpha} \\ D_\alpha & \longrightarrow & X_\alpha, \end{array}$$

where i_α is a coproduct $\coprod_{c\in C_\alpha} i_c$, with all the maps i_c in I, and C_α empty whenever α is a limit ordinal. Then the union of the C_α is partially ordered, with $i_c \in C_\alpha$ smaller than $i_d \in C_\beta$ if and only if $\alpha < \beta$. In this situation, we say that $\bigcup_{\alpha\leq\lambda} C_\alpha$ indexes the attached cells of f in the λ-sequence.

The following lemma helps with keeping the "length" of the transfinite composition in check when the domains of the morphisms in I are sufficiently small:

Lemma 3.1.9. *Let $f\colon A \to X$ be an I-cell complex and assume that the domains of the maps in I are κ-small. Then there is a κ-sequence of maps exhibiting f as relative I-cellular.*

Proof. Assume that f is the transfinite composition of a λ-sequence $\{X_\alpha\}_{\alpha\leq\lambda}$ that exhibits the a cellular structure, i.e., for $\alpha < \lambda$ there are pushout diagrams

$$\begin{array}{ccc} S_\alpha & \xrightarrow{\sigma_\alpha} & \operatorname*{colim}_{\beta<\alpha} X_\alpha \\ {\scriptstyle i_\alpha}\downarrow & \ulcorner & \downarrow{\scriptstyle f_\alpha} \\ D_\alpha & \longrightarrow & X_\alpha, \end{array}$$

such that i_α is the identity of the initial object for α a limit ordinal, and a coproduct of maps in I otherwise. For $\gamma \leq \kappa$, define sets $C_\gamma^<$ and C_γ as well as commutative diagrams

$$\begin{array}{ccccc} \operatorname*{colim}_{\delta<\gamma} X_\delta & \longrightarrow & \operatorname*{colim}_{\delta<\gamma} Y_\delta & & \\ \downarrow & & \downarrow & \searrow & \\ X_\gamma & \longrightarrow & Y_\gamma, & \longrightarrow & X \end{array} \tag{3.1.10}$$

by transfinite induction: Let $C_0 := 0$ and $Y_0 := X_0 = A$. Continuing, for μ a limit ordinal let C_μ be empty.Otherwise define the set

$$C_\gamma^< := \{\alpha \leq \lambda, \sigma_\alpha \text{ factors through } Y_{\gamma-1}\}.$$

Furthermore, let $C_\gamma := C_\gamma^< \setminus \bigcup_{\delta<\gamma} C_\delta^<$. Finally, define Y_γ as the pushout

$$\begin{array}{ccc} \coprod_{\alpha\in C_\gamma} S_\alpha & \longrightarrow & \operatorname*{colim}_{\delta<\gamma} Y_\delta \\ \downarrow & \ulcorner & \downarrow \\ \coprod_{\alpha\in C_\gamma} D_\alpha & \longrightarrow & Y_\gamma \end{array}$$

Define a map $Y_\gamma \to X$ on the attached cells $S_\alpha \to D_\alpha$ by going through the X_α. Note that $\sigma_\gamma \colon S_\gamma \to X$ factors through $\operatorname{colim}_{\delta<\gamma} X_\delta$, hence we get a map $X_\gamma \to Y_\gamma$ which fits into the diagram 3.1.10. Finally, note that since S_α is κ-small, all attaching maps σ_α for $\alpha \leq \lambda$ factor through some X_γ, hence through Y_γ. the union $\bigcup_{\gamma\leq\kappa} C_\gamma$ contains all $\alpha \leq \lambda$. Therefore there are canonical maps in both directions between the colimits

$$\operatorname*{colim}_{\alpha\leq\lambda} X_\alpha \cong \operatorname*{colim}_{\gamma\leq\kappa} Y_\gamma,$$

which are isomorphisms by cofinality. □

Remark 3.1.11. Note that attaching cells via coproducts, gives a partial order on the set of cells. Every such partially ordered set can be linearly ordered as in [H, 2.1.11], which corresponds to giving a λ-sequence in which the cells are attached one at a time. Lemma 3.1.9 gives us a much more convenient way to revert this process, than simply forgetting the extra information. Returning to a closed symmetric monoidal category $(\mathcal{C}, \wedge)$, observe that taking coproducts interacts distributive with the smash product, hence also with the $\square$-product. We therefore allow ourselves to switch freely between attaching cells one at a time or in bigger groups via the coproduct.

Proof of Theorem 3.1.2. We assume without loss of generality (cf. 3.1.11) that λ and μ are ordinals linearly indexing the cells of f and g, respectively. That is for each $\alpha \leq \lambda$

we have a pushout diagram

$$\begin{array}{ccc} S_\alpha & \xrightarrow{i_\alpha \in I} & D_\alpha \\ \downarrow & \ulcorner & \downarrow \\ \operatorname*{colim}_{\gamma<\alpha} X_\gamma & \xrightarrow[f_\alpha]{} & X_\alpha, \end{array}$$

such that the λ-sequence $A = X_0 \to X_\lambda = X$ is the map f, and and analogous for g. Chose the product partial order on $\lambda \times \mu$, i.e., $(\gamma, \delta) < (\alpha, \beta)$ if and only if $\gamma < \alpha$ and $\delta < \beta$. Let $E\colon \lambda \times \mu \to \mathcal{C}$ be the sequence defined by the pushout diagrams

$$\begin{array}{ccc} A \wedge B & \longrightarrow & X_\alpha \wedge B \\ \downarrow & \ulcorner & \downarrow \\ A \wedge Y_\beta & \longrightarrow & E_{\alpha,\beta}, \end{array}$$

and note that $E_{\lambda,\mu}$ is the source of $f \square g$. We claim that the desired filtration is then given by $\{F_{\alpha,\beta}\}$, the (pointwise) pushout of $\lambda \times \mu$-sequences in the diagram:

$$\begin{array}{ccc} E_{\alpha,\beta} & \longrightarrow & X_\alpha \wedge Y_\beta \\ \downarrow & \ulcorner & \downarrow \\ E_{\lambda,\mu} & \longrightarrow & F_{\alpha,\beta}, \end{array} \tag{3.1.12}$$

where $E_{\lambda,\mu}$ is the constant sequence. To prove the claim, note that the transformation $E \to X_{(-)} \wedge Y_{(-)}$ of sequences factors through the sequence P, given pointwise as the pushout

$$\begin{array}{ccccc} \operatorname*{colim}_{\gamma<\alpha,\delta<\beta} X_\gamma \wedge Y_\delta & \longrightarrow & \operatorname*{colim}_{\delta<\beta} X_\alpha \wedge Y_\delta & & \\ \downarrow & \ulcorner & \downarrow & & \\ \operatorname*{colim}_{\gamma<\alpha} X_\gamma \wedge Y_\beta & \longrightarrow & P_{(\alpha,\beta)} & \xrightarrow{f_\alpha \square g_\beta} & X_\alpha \wedge Y_\beta. \end{array} \tag{3.1.13}$$

That is, $P_{\alpha,\beta}$ is the source of the map $f_\alpha \square g_\beta$. We apply the cobase change as in 3.1.12 to this factorization to get the diagram

$$\begin{array}{ccccc} E & \longrightarrow & P & \xrightarrow{f_{(-)} \square g_{(-)}} & X_{(-)} \wedge Y_{(-)} \\ \downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow \\ E_{\lambda,\mu} & \longrightarrow & P \amalg_E E_{\lambda,\mu} & \longrightarrow & F. \end{array} \tag{3.1.14}$$

Now comparing the colimits pointwise, cofinality lets us identify $P_{\alpha,\beta} \amalg_E E_{\lambda,\mu}$ as the following pushout:

$$\begin{array}{ccccc} \operatorname*{colim}_{\gamma<\alpha,\delta<\beta} F_{\gamma,\delta} & \longrightarrow & \operatorname*{colim}_{\delta<\beta} F_{\alpha,\delta} & & \\ \downarrow & \ulcorner & \downarrow & & \\ \operatorname*{colim}_{\gamma<\alpha} F_{\gamma,\beta} & \longrightarrow & P_{\alpha,\beta} \amalg_E E_{\lambda,\mu} & \longrightarrow & F_{\alpha,\beta}. \end{array}$$

In particular we have

$$P \amalg_E E_{\lambda,\mu} \cong \operatorname*{colim}_{(\gamma,\delta)<(\alpha,\beta)} F_{\alpha,\beta},$$

and the map

$$\operatorname*{colim}_{(\gamma,\delta)<(\alpha,\beta)} F_{\gamma,\delta} \to F_{\alpha,beta}$$

is a cobase change of $f_\alpha \square g_\beta$. Therefore to show that F is indeed a filtration by $I\square J$-cells, it suffices by 3.1.14 to show that $f_\alpha \square g_\beta$ is the attaching of a $I\square J$-cell. This is a consequence of Lemma 3.1.6, which implies that there are pushout diagrams

$$\begin{array}{ccc} S_{(\alpha,\beta)} & \xrightarrow{i_\alpha \square j_\beta \in I\square J} & D_{(\alpha,\beta)} \\ \downarrow & \ulcorner & \downarrow \\ P_{(\alpha,\beta)} & \xrightarrow{f_\alpha \square g_\beta} & X_\alpha \wedge Y_\beta. \end{array}$$

Note that as in Remark 3.1.11 we can extend the partial order on $\lambda \times \mu$ to a linear one, finishing the proof. □

Remark 3.1.15. In a lot of cases of interest, for example $\mathcal{C} = \mathcal{T}$, with I and J the sets of generating (acyclic) cofibrations, we will actually have that $I\square J \subset J(-\text{cell})$, such that the above proposition also gives $f\square g$ the structure of a relative J-cellular map.

Remark 3.1.16. In categories where we can think of the maps in I, J and $I\square J$ as inclusions, and of the filtered colimits as unions of subobjects, the intuition behind the filtration in the theorem simplifies significantly. In particular the cellular maps then give a λ-sequence of inclusions of subobjects

$$A \hookrightarrow X_1 \hookrightarrow \ldots \hookrightarrow X,$$

and similar for $B \hookrightarrow Y$. Note that $f\square g$ is the inclusion

$$f\square g \colon X\wedge B \cup_{A\wedge B} A\wedge Y = X\wedge B \cup A\wedge Y \hookrightarrow X\wedge Y,$$

and the filtration given by the theorem is through objects

$$F_{\alpha,\beta} = X\wedge B \cup X_\alpha \wedge Y_\beta \cup A\wedge Y.$$

Corollary 3.1.17. *The monoidal product $X\wedge Y$ of an I-cellular X object with a J-cellular object Y is $I\square J$-cellular.*

Corollary 3.1.18. *In the situation of Theorem 3.1.2, the map $f \wedge g$ is relative $(I\Box J)\cup K$-cellular, where K is the set of maps $I\wedge B\cup A\wedge J$. In particular, if A and B are themselves respectively I-cellular and J-cellular, then $f\wedge g$ is even $I\Box J$-cellular.*

Proof. Use the theorem on the maps $\star \to A \to X$ and $\star \to B \to Y$ which are respectively $I \cup \{\star \to A\}$-cellular and $J \cup \{\star \to B\}$-cellular. Note that the indexing of the filtrations is shifted, and the new filtration factors through $F_{1,1} = A\wedge B$. All the later cells are then of type $(I\Box J) \cup K$. ☐

Corollary 3.1.19. *Since the $\Box$-product is associative, Theorem 3.1.2 immediately gives specific filtrations for iterated $\Box$-products of maps. The indexing set for the cells of the iterated $\Box$ is always given by the product of the indexing sets with some (linear) order that is compatible with the product partial order.*

3.1.20 Fixed Point Spectra

For a closed normal subgroup N of G, consider the short exact sequence E of compact Lie-groups:

$$E: \quad 1 \to N \to G \xrightarrow{\epsilon} J \to 1,$$

where ϵ denotes the projection on N-orbits.
Similar to the case of equivariant spaces 1.1.2, where N-fixed points of G-spaces inherit J-actions, can consider the categorical fixed point functor $G\mathbf{O}\mathcal{T} \to J\mathbf{O}\mathcal{T}$ taking X to its N-fixed spectrum X^N.

Remark 3.1.21. Note that for subgroups H that are not necessarily normal, we can first restrict the G-actions to the normalizer N_H of H in G before taking fixed points, to get a functor

$$G\mathbf{O}\mathcal{T} \to N_H\mathbf{O}\mathcal{T} \to W_H\mathbf{O}\mathcal{T},$$

to spectra with actions of the Weyl group $W_H := N_H/H$.

Let $\mathscr{H}$ be a G-typical family of representations as in 2.4.5. We now explain an alternative "geometric" way of forming N-fixed points. It depends on $\mathscr{H}$ and makes use of an intermediate category:

Definition 3.1.22. In the above situation denote by $\mathbf{O}_E^{\mathscr{H}}$ the J-category with G-representations of the form $V(P)$ for $P \in \mathscr{H}$ as objects and with morphism spaces

$$\mathbf{O}_E^{\mathscr{H}}(V,W) := \mathbf{O}(V,W)^N$$

given by the N-fixed points of the morphism space with G acting by conjugation.

Definition 3.1.23. Let $\mathcal{O}_J$ be the $J\mathcal{T}$-category consisting of J-objects in $\mathbf{O}$. The $J\mathcal{T}$-functor $\phi\colon \mathbf{O}_E^{\mathscr{H}} \to \mathcal{O}_J$ is given on objects by

$$\begin{array}{rcl} \phi\ :\ \mathbf{O}_E^{\mathscr{H}} & \to & \mathcal{O}_J \\ V & \mapsto & V^N \end{array}$$

and on morphisms by

$$\begin{array}{rcl} \mathbf{O}(V,W)^N & \to & \mathcal{O}_J(V^N,W^N). \\ (g,t) & \mapsto & (g^N,t) \end{array}$$

Here, the isometry g as above indeed maps the N-fixed points of V into those of W. Similarly t is indeed in W^N.

Given an orthogonal G-spectrum X and $V \in \mathbf{O}$, we write $tV \in \mathcal{O}_J$ for V considered as an object of $\mathcal{O}_J$ with trivial action of J.

Definition 3.1.24. Let X be an orthogonal G-spectrum. The *geometric N-fixed point spectrum of* X is the orthogonal J-spectrum $\Phi^N_{\mathscr{H}} X$ defined by letting $\Phi^N_{\mathscr{H}} X_V$ be the coend $\int^{W \in \mathbf{O}^{\mathscr{H}}_E} \mathcal{O}_J(W^N, tV) \wedge X^N_W$. The maps $\phi\colon \mathbf{O}^{\mathscr{H}}_E(W,V) \to \mathbf{O}^{\mathscr{H}}_E(W^N,V^N)$ induce a map

$$X^N_V \cong \int^{W \in \mathbf{O}^{\mathscr{H}}_E} \mathbf{O}^{\mathscr{H}}_E(W,V) \wedge X^N_W \to \int^{W \in \mathbf{O}^{\mathscr{H}}_E} \mathcal{O}_J(W^N,V^N) \wedge X^N_W.$$

Specializing to the situation where V is a trivial G-representation, we obtain a map $\gamma\colon X^N \to \Phi^N_{\mathscr{H}} X$ of orthogonal J-spectra.

Note that if U is in $\mathcal{O}_J$, the J-space $\Phi^N_{\mathscr{H}} X_U$ is the coend $\int^{W \in \mathbf{O}^{\mathscr{H}}_E} \mathcal{O}_J(W^N,U) \wedge X^N_W$. Before going into more detail, we will list properties of the fixed point functors and study their interaction with free spectra, which form the basis for computations on fixed points of smash powers.

Proposition 3.1.25. *[MM, III.1.6] The geometric fixed point functor $\Phi^N_{\mathscr{H}}$ preserves coproducts, pushouts along h-cofibrations, sequential colimits along h-cofibrations and tensors with spaces.*

Proof. The functor from $G\mathbf{O}\mathcal{T}$ to the category of J-functors from $\mathbf{O}^{\mathscr{H}}_E$ to $J\mathcal{T}$ taking X to $W \to X^N_W$ preserves these by 1.1.2. So does the continuous prolongation functor $Y \mapsto \int^{W \in \mathbf{O}^{\mathscr{H}}_E} \mathcal{O}_J(W^N,(-)^N) \wedge Y_W$ from $\mathbf{O}^{\mathscr{H}}_E$-spaces to $\mathcal{O}_J$-spaces. □

Proposition 3.1.26. *[MM, IV.4.5] For any finite dimensional G-representation V in $\mathscr{H}$ and any G-space K, the map*

$$K^N \to \mathcal{O}_J(V^N,V^N) \wedge K^N \cong \int^{W \in \mathbf{O}^{\mathscr{H}}_E} \mathcal{O}_J(W^N,V^N) \wedge \mathbf{O}^{\mathscr{H}}_E(V,W) \wedge K^N = (\Phi^N_{\mathscr{H}} \mathcal{F}_V K)_{V^N}$$

induced by id_{V^N} in $\mathcal{O}_J(V^N,V^N)$ is adjoint to a natural isomorphism

$$\mathcal{F}_{V^N} K^N \to \Phi^N_{\mathscr{H}} \mathcal{F}_V K$$

of free J-spectra.

Proof. Recall that $\mathcal{F}_V K$ is given in level W by $\mathbf{O}(V,W)\wedge K$ where G acts diagonally on the smash product, by conjugation on the morphism space and on K via its G-space structure. Thus $(\mathcal{F}_V K)^N_W = \mathbf{O}(V,W)^N \wedge K^N$. The result now follows from the isomorphism $\int^{\mathbf{O}_E^{\mathscr{H}}} \mathcal{O}_J(\phi(-),U) \wedge \mathbf{O}_E^{\mathscr{H}}(V,-) \cong \mathcal{O}_J(\phi(V),U) = \mathbf{O}(V^N,U)$ and the fact that coends commute with smash products. □

Let $\mathscr{H}^N$ be the J-typical family of representations corresponding to representations of the form $V(P)^N$ for P in $\mathscr{H}$.

Corollary 3.1.27. *cf. [MM, IV.4.5] The geometric fixed point functor $\Phi^N_{\mathscr{H}}$ takes $(\mathscr{H},\mathscr{H})$-cofibrations to $(\mathscr{H}^N,\mathscr{H}^N)$-cofibrations and stably acyclic $(\mathscr{H},\mathscr{H})$-cofibrations to stably acyclic $(\mathscr{H}^N,\mathscr{H}^N)$-cofibrations.*

Proof. Recall the generating cofibrations and acyclic cofibrations from 2.9.5. By 3.1.26 $\Phi^N_{\mathscr{H}}$ sends $\mathcal{G}I_{\mathscr{H}}$ cells to $\mathcal{G}J_{\mathscr{H}^N}$ cells. Since it also preserves the mapping cylinder construction and hence sends the maps $k_{V,W}$ in the definition of the generating acyclic cofibrations in $G\mathbf{O}\mathcal{T}$ to their counterparts in $J\mathbf{O}\mathcal{T}$. Note that $k_{V,W}\Box i$ is mapped to $k_{V^N,W^N}\Box i^N$, which is K_J-cellular by Theorem 3.1.2. By 3.1.25, $\Phi^N_{\mathscr{H}}$ preserves the cell complex construction. □

Given orthogonal G-spectra X and Y and $P, P' \in \mathscr{H}$, the monoidal structure of $\mathbf{O}$ and the G-maps $X_{V(P)} \wedge Y_{V(P')} \to (X\wedge Y)_{V(P)\oplus V(P')}$ give maps

$$\begin{array}{c} \mathbf{O}(V(P)^N,V)\wedge X^N_{V(P)} \wedge \mathbf{O}(V(P')^N,V')\wedge Y^N_{V(P')} \\ \Big\downarrow \\ \mathbf{O}((V(P)\oplus V(P'))^N, V\oplus V')\wedge (X\wedge Y)^N_{V(P)\oplus V(P')}, \end{array}$$

and these maps induce a map $\Phi^N_{\mathscr{H}}X \wedge \Phi^N_{\mathscr{H}}Y \to \Phi^N_{\mathscr{H}}(X\wedge Y)$.

Proposition 3.1.28. *[MM, IV.4.7] The functor $\Phi^N_{\mathscr{H}}$ is lax monoidal and for $(\mathscr{H},\mathscr{H})$-cofibrant orthogonal G-spectra X and Y the natural (equivariant) morphism of J-spectra*

$$\alpha\colon \Phi^N_{\mathscr{H}}X\wedge\Phi^N_{\mathscr{H}}Y \to \Phi^N_{\mathscr{H}}(X\wedge Y),$$

is an isomorphism.

Proof. (Cf. [MM, 4.7]) The natural map α is J-equivariant by definition. It is an isomorphism for X and Y free G-spectra by 3.1.26. This implies that α induces a bijection of sets along which we identify as $\Phi^N_{\mathscr{H}}(\mathcal{G}I_{\mathscr{H}}\Box\mathcal{G}I_{\mathscr{H}})$ and $(\Phi^N_{\mathscr{H}}\mathcal{G}I_{\mathscr{H}})\Box(\Phi^N_{\mathscr{H}}\mathcal{G}I_{\mathscr{H}})$. Abbreviate this set as $\hat{I}$. Let now X and Y be $\mathcal{G}I_{\mathscr{H}}$-cellular, and chose specific cellular filtrations with indexing sets C respectively D for the attached cells. Then by Proposition 3.1.25, $\Phi^N_{\mathscr{H}}X$ is $\Phi^N_{\mathscr{H}}(\mathcal{G}I_{\mathscr{H}})$-cellular with the cells still indexed by the same set C, and similar for $\Phi^N_{\mathscr{H}}Y$. Theorem 3.1.2 then gives explicit filtrations of $\Phi^N_{\mathscr{H}}X\wedge\Phi^N_{\mathscr{H}}Y$ and $(\Phi^N_{\mathscr{H}}X\wedge Y)$ as $\hat{I}$-cellular objects with the same indexing set $C\times D$, and α exactly transports one filtration diagram into the other. Since retracts are preserved by any functor, this proves the proposition. □

The examples of G-spectra where we are most interested in calculating geometric fixed points, are smash powers of orthogonal spectra. We begin with studying the fixed points of free and semi-free spectra before we move to more general spectra 3.2.1, and general cofibrant spectra in Section 4.4.

3.1.29 Free Cells

Let us begin by investigating what happens in the case of free spectra. For X a finite discrete set with an action of a discrete group G, the smash power $\bigwedge_X$ sends an orthogonal spectrum A to its X-fold smash power $\bigwedge_X A = A^{\wedge X}$. Using the smash-power $K \mapsto K^{\wedge X}$ of pointed spaces, this smash power can be constructed just as we defined the smash product. Since $K^{\wedge X}$ is a quotient of the mapping space $F(X, K)$ we do not need a total order on X in this definition. The action of G on X induces an action by permuting the smash factors using the symmetry isomorphism for the smash product in $\mathbf{O}\mathcal{T}$.

Lemma 3.1.30. *For X a finite discrete G-set, $K \in \mathcal{T}$ a space and $\mathcal{F}_V K$ a free orthogonal spectrum, the structure map of the smash power $(\mathcal{F}_V K)^{\wedge X}$ gives a map*

$$K^X \cong ((\mathcal{F}_V K)_V)^{\wedge X} \to ((\mathcal{F}_V K)^{\wedge X})_{V^{\oplus X}}$$

adjoint to a natural isomorphism of G-spectra

$$\mathcal{F}^G_{V^{\oplus X}} K^{\wedge X} \xrightarrow{\cong} (\mathcal{F}_V K)^{\wedge X}.$$

Here G acts on the vector space $V^{\oplus X}$ by permuting summands and on the spectrum $(\mathcal{F}_V K)^{\wedge X}$ and the space $K^{\wedge X}$ by permuting smash factors.

Proof. Let τ be the various twist maps. The proof can be reduced to showing that for pointed spaces K and L, the diagram

$$\begin{array}{ccc} \mathcal{F}_{V\oplus W}(K \wedge L) & \xrightarrow{\cong} & \mathcal{F}_V K \wedge \mathcal{F}_W L \\ \downarrow{\scriptstyle \mathcal{F}_\tau(\tau)} & & \downarrow{\scriptstyle \tau} \\ \mathcal{F}_{W\oplus V}(L \wedge K) & \xrightarrow{\cong} & \mathcal{F}_W L \wedge \mathcal{F}_V K \end{array}$$

commutes. Here the horizontal isomorphisms are provided by Proposition 2.3.4 by noting that $\mathcal{F}_V K \cong \mathcal{G}_V(\mathbf{O}_{V+} \wedge K)$. However, by the universal property of $\mathcal{F}$, this is a consequence of the fact that given orthogonal spectra X and Y, the following diagram commutes:

$$\begin{array}{ccccc} X_V \wedge Y_W & \longrightarrow & (X \wedge Y)_{V\oplus W} & \xrightarrow{\tau_{V\oplus W}} & (Y \wedge X)_{V\oplus W} \\ \downarrow{\scriptstyle \tau} & & & & \downarrow{\scriptstyle (X\wedge Y)_\tau} \\ Y_W \wedge X_V & \longrightarrow & (Y \wedge X)_{W\oplus V} & \xrightarrow{\text{id}} & (Y \wedge X)_{W\oplus V}, \end{array}$$

where the non decorated arrows are structural morphisms for the smash product. □

In combination with 3.1.26 we get:

Proposition 3.1.31. *For X a finite discrete G-set, N a normal subgroup of G, $K \in \mathcal{T}$ a space and $A = \mathcal{F}_V K$ a free orthogonal spectrum, the maps of Proposition 3.1.26 and Lemma 3.1.30 induce natural isomorphism of J-spectra*

$$\Phi^N_{\mathscr{H}}(A^{\wedge X}) \xleftarrow{\cong} \mathcal{F}^J_{V^{\oplus X_N}} K^{\wedge X_N} \xrightarrow{\cong} \bigwedge_{X_N} A,$$

where X_N is the orbit J-space of X factoring out the N-action.

Proof. The only thing left to do is that the N-fixed point spaces of $V^{\oplus X}$ and $K^{\wedge X}$ are indeed J-isomorphic to $V^{\oplus X_N}$ respectively $K^{\wedge X_N}$. □

Remark 3.1.32. This proposition serves as a good starting point for calculating geometric fixed points of smash powers of general $(\mathscr{H}, \mathscr{H})$-cofibrant orthogonal spectra. This has for example been studied in the special case of G a cyclic group by Kro in [Kr, 3.10.1] and Hill, Hopkins and Ravenel in [HHR, B.96]. However, we are ultimately interested in smash powers of commutative ring spectra, and $(\mathscr{H}, \mathscr{H})$-cofibrant approximation does not preserve strictly commutative ring spectra. To remedy this, we first study smash powers of semi-free spectra and later generalize to $(\mathscr{H}, \mathscr{G})$-cofibrant spectra.

3.1.33 Semi-Free Cells

Fixed Points of Semi-Free Cells

The result for semi-free spectra analogous to Proposition 3.1.31, is somewhat more involved.

Theorem 3.1.34. *Let V be an inner product space, let P be a subgroup of $\mathbf{O}_V$, let X be a discrete set with action of a discrete group G and let K be a pointed space. Let N be a normal subgroup of G with factor group J and let $A = \mathcal{G}_V(G \times \mathbf{O}_V/P_+) \wedge K$. If the permutation representation $V^{\oplus X}$ is in $\mathscr{H}$, then the composition*

$$A^{\wedge X_N} \to (A^{\wedge X})^N \to \Phi^N_{\mathscr{H}}(A^{\wedge X})$$

of the map induced by the diagonal embedding of $A^{\wedge X_N}$ in $A^{\wedge X}$ and the map γ from Definition 3.1.24 is an isomorphism of J-spectra.

Proof. First note that for every orthogonal G-spectrum X and G-space L, there is an isomorphism $\Phi^N_{\mathscr{H}}(X \wedge L) \cong \Phi^N_{\mathscr{H}}(X) \wedge L^N$. Thus we can without loss of generality assume that $K = S^0$. By 2.3.7 there is an isomorphism

$$A^{\wedge X} \cong \mathcal{G}_{V^{\oplus X}}(\mathbf{O}_{V^{\oplus X}+} \wedge_{\prod_X \mathbf{O}_V} (\mathbf{O}_V/P_+)^{\wedge X}) \cong \mathcal{G}_{V^{\oplus X}}(\mathbf{O}_{V^{\oplus X}}/\prod_X P_+).$$

Now, by 1.1.21, there is a natural isomorphism

$$\mathbf{O}(V^{\oplus X}, W)^N/(\prod_X P)^N \to (A^{\wedge X})^N_W$$

we can finish the proof by proceeding as in the proof of 3.1.26 and observing that orbits commute with coends. □

This finishes the proof of Theorem 3.1.34. As mentioned before, 3.1.34 will form the basis for the identification of the geometric fixed points of smash powers of general $\mathbb{S}$-cofibrant spectra in 3.2.16. We need one more crucial inputs, namely Kro's observation on the interaction of the geometric fixed point functor with induced spectra from [Kr, 3.8.10]. Since we will in particular need analogous results for the functors restricting to H-spectra for H a subgroup of G to lift our results into to the case of G a compact Lie group, we will go into more details in the next section.

3.1.35 Fixed Points and Change of Groups

For the whole section, let G be a compact Lie-group and H a closed subgroup of G with inclusion map $i\colon H \to G$. We consider the restriction functor

$$i^*\colon G\mathbf{O}\mathcal{T} \to H\mathbf{O}\mathcal{T},$$

and its left adjoint induction functor

$$G_+ \wedge_H (-)\colon H\mathbf{O}\mathcal{T} \to G\mathbf{O}\mathcal{T}.$$

The following lemma will be helpful later, it is a direct consequence of 1.1.5.

Lemma 3.1.36. *For an orthogonal H-spectrum X and an orthogonal G-spectrum Y, there is a natural isomorphism:*

$$(G_+ \wedge_H X) \wedge Y \cong G_+ \wedge_H (X \wedge i^* Y).$$

We recollect material from [Kr, 3.8.2], both for completeness and to adapt notation to our conventions. Afterwards we expand the results to the restriction functor.

At first we need to give Kro's Definition of the change of sequence functors for the categories $\mathbf{O}_E^{\mathscr{H}}\mathcal{T}$ (cf. [Kr, 3.8.7]). The main point of difficulty stems from the fact that, contrary to the case of G-spectra, the change of universe does not necessarily give an equivalence of categories.

Let $j\colon E_0 \to E$ be the morphism of short exact sequences of compact Lie groups:

$$\begin{array}{lccccccccc} E_0\colon & 1 & \longrightarrow & N & \longrightarrow & H & \longrightarrow & J_0 & \longrightarrow & 1\,, \\ \downarrow{\scriptstyle j} & & & \| & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i_1} & & \\ E\colon & 1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & J & \longrightarrow & 1 \end{array} \tag{3.1.37}$$

where N is normal in G and hence in H, and J and J_0 denote the respective quotients. Now for $\mathscr{H}$ a G-typical family of representations, let $i^*\mathscr{H}$ be the induced H-typical family with $i^*\mathscr{H}^V$ consisting elements of $\mathscr{H}^V$ contained in $H \times \mathbf{O}_V$.

There is a J_0-functor $i^*\colon i_1^*\mathbf{O}_E^{\mathscr{H}} \to \mathbf{O}_{E_0}^{i^*\mathscr{H}}$ taking a G-representation V to its restricted H-representation i^*V. On morphism spaces it is the identity $i_1^*\mathbf{O}(V,W)^N = \mathbf{O}(i^*V, i^*W)^N$ of subspaces of $\mathbf{O}(V,W)$.

Definition 3.1.38. The *change of sequence functor* $j^* : \mathbf{O}_E^{\mathscr{H}}\mathcal{T} \to \mathbf{O}_{E_0}^{i^*\mathscr{H}}\mathcal{T}$ is given by sending an $\mathbf{O}_E^{\mathscr{H}}$-space X to the $\mathbf{O}_{E_0}^{i^*\mathscr{H}}$-space given by

$$(j^*X)_{i^*V} := i_1^* X_V.$$

The structure maps are transported directly, using $\mathbf{O}(i^*V, i^*W)^N = i_1^*\mathbf{O}(V,W)^N$ for G-representations V and W.

Lemma 3.1.39. *[Kr, 3.8.2] Let L be an H-space. The natural morphism*

$$J_+ \wedge_{J_0} L^N \to (G_+ \wedge_H L)^N$$

is a homeomorphism of J-spaces.

Definition 3.1.40. Let Y be a $\mathbf{O}_{E_0}^{i^*\mathscr{H}}$-space, then the *induced $\mathbf{O}_E^{\mathscr{H}}$-space $J_+ \wedge_{J_0} Y$* is given by

$$(J_+ \wedge_{J_0} Y)_V := J_+ \wedge_{J_0} Y_{i^*V}$$

The (J-equivariant) structure maps are given as the composite

$$\begin{aligned} \mathbf{O}(V,W)^N \wedge (J_+ \wedge_{J_0} Y_{i^*V}) &\cong J_+ \wedge_{J_0} (i_1^*\mathbf{O}(V,W)^N \wedge Y_{i^*V}) \\ &= J_+ \wedge_{J_0} (\mathbf{O}(i^*V, i^*W)^N \wedge Y_{i^*V}) \\ &\to J_+ \wedge_{J_0} Y_{i^*W}, \end{aligned}$$

where is isomorphism is from 3.1.36

Induced Spectra and (Geometric) Fixed Points

We give a recollection of [Kr, 3.8.3] before giving further results in the same spirit needed in particular when dealing with infinite groups. Fix a G-typical family of representations $\mathscr{H}$ that is closed under retract. Let as in (3.1.37) $j\colon E_0 \to E$ be a morphism of short exact sequences of compact Lie groups:

$$\begin{array}{ccccccccccc} E_0\colon & & 1 & \longrightarrow & N & \longrightarrow & H & \longrightarrow & J_0 & \longrightarrow & 1\,, \\ \downarrow j & & & & \| & & \downarrow i & & \downarrow i_1 & & \\ E\colon & & 1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & J & \longrightarrow & 1 \end{array}$$

The following condition is an important prerequisite already in Kro's exposition, and will be important for us as well:

Condition 1. For every orthogonal N-representation W in $i^*\mathscr{H}$ there exists an orthogonal G-representation U in $\mathscr{H}$ and an N-linear isometric embedding $W \to U$ inducing an isomorphism on N-fixed spaces, i.e., $U^N \cong W^N$.

Lemma 3.1.41. *Let $F\colon (\mathbf{O}_{E_0}^{i^*\mathscr{H}})^{\mathrm{op}} \wedge \mathbf{O}_{E_0}^{i^*\mathscr{H}} \to \mathcal{T}$ be given by $F(W,W') = \mathbf{O}(W^N, V)$. Given morphisms $\gamma\colon W_0 \to W_1$ and $f_0\colon W_0 \to W_0'$ in $\mathbf{O}_{E_0}^{i^*\mathscr{H}}$ there exist morphisms $g'\colon W_0' \to W_1'$ and $g_1\colon W_1 \to W_1'$ in $\mathbf{O}_{E_0}^{i^*\mathscr{H}}$ so that $g_1\gamma = g'g_0$ and so that the map $F(g_1, W)$ is a homeomorphism for every object W of $\mathbf{O}_{E_0}^{i^*\mathscr{H}}$.*

Proof. Recall that the morphism $\gamma\colon W_0 \to W_1$ in $\mathbf{O}_{E_0}^{i\mathscr{H}}$ is of the form $(\widetilde{\gamma}, x_\gamma)$ where $\widetilde{\gamma}\colon W_0 \to W_1$ is a linear isometry and x_γ is in the one-point compactification of the orthogonal complement of $\widetilde{\gamma}(W_0)$ in W_1. Consider the pushout of underlying real vector spaces

$$\begin{array}{ccc} W_0 & \xrightarrow{\widetilde{\gamma}} & W_1 \\ {\scriptstyle \widetilde{f}_0}\downarrow & & \downarrow{\scriptstyle \widetilde{g}_1} \\ W_0' & \xrightarrow{\widetilde{g}'} & W_1'. \end{array}$$

Note that there is exactly one inner product on W_1' so that this becomes a diagram of linear isometries. Since $i^*\mathscr{H}$ is closed under retract $g_1 = (\widetilde{g}_1, \widetilde{g}'(x_{f_0}))$ and $g' = (\widetilde{g}', \widetilde{g}_1(x_\gamma))$ are morphisms in $\mathbf{O}_{E_0}^{i^*\mathscr{H}}$ with the desired property. ∎

Lemma 3.1.42. *Suppose Condition 1 is satisfied. For every G-spectrum X and every Euclidean space U, the inclusion $i^*\colon i_1^*\mathbf{O}_E^{\mathscr{H}} \to \mathbf{O}_{E_0}^{i^*\mathscr{H}}$ induces a homeomorphism.*

$$\int^{W \in i_1^*\mathbf{O}_E^{\mathscr{H}}} \mathbf{O}(W^N, U) \wedge X_W^N \xrightarrow{i} \int^{W \in \mathbf{O}_{E_0}^{i^*\mathscr{H}}} \mathbf{O}(W^N, U) \wedge X_W^N.$$

Proof. Using the zero section $\mathbf{L}(V, W) \to \mathbf{O}(V, W)$ Condition 1 implies that for every object W_0 of $\mathbf{O}_{E_0}^{i^*\mathscr{H}}$ there exista an object V of $i_1^*\mathbf{O}_E^{\mathscr{H}}$ and a morphism $f\colon W_0 \to V$ so that $\mathbf{O}(V^N, U) \to \mathbf{O}(W_0^N, U)$ is a homeomorphism for every Euclidean space U. Let $F\colon (\mathbf{O}_{E_0}^{i^*\mathscr{H}})^{\mathrm{op}} \wedge \mathbf{O}_{E_0}^{i^*\mathscr{H}} \to \mathcal{T}$ be the functor $F(W, W') = \mathbf{O}(W^N, U) \wedge (X_{W'})^N$. By Lemma 3.1.41 and Lemma 5.2.44 the inclusion $i^*\colon i_1^*\mathbf{O}_E^{\mathscr{H}} \to \mathbf{O}_{E_0}^{i^*\mathscr{H}}$ is F-cofinal. Thus Proposition 5.2.45 implies that the map i is a homeomorphism. ∎

Lemma 3.1.43. *If N is of finite index in G, then Condition 1 is satisfied.*

Proof. As Kro already states in [Kr, 3.8.11], it suffices to look at the irreducible subrepresentations of V one at a time. If V is the trivial representation of N, there is nothing to do. If V is an irreducible and non trivial representation of N, then both V^N and $(G \times_N V)^N$ are the zero vector space, so the induced representation (which is still finite dimensional) extends V in the desired way. ∎

Remark 3.1.44. We prove that Condition 1 holds for all finite subgroups of tori in 4.4.20. Since the extension problem is transitive, this covers a big class of configurations for N and G. Note that the case where N is finite suffices for all applications of the theory we give, so a proof that Condition 1 holds for N a maximal torus of a compact Lie group G would immediately be very fruitful. (cf. 4.4.25).

Proposition 3.1.45. *[Kr, 3.8.10] Suppose Condition 1 holds. Then for orthogonal H-spectra X, there is a natural isomorphism of J-spectra*

$$J_+ \wedge_{J_0} (\Phi_{i^*\mathscr{H}}^N X) \xrightarrow{\cong} \Phi_{\mathscr{H}}^N (G_+ \wedge_H X).$$

Proof. By 3.1.42, there is an isomorphism

$$(J_+ \wedge_{J_0} \Phi^N_{i^*\mathscr{H}} X)_V = J_+ \wedge_{J_0} \int^{W \in \mathbf{O}^{i^*\mathscr{H}}_{E_0}} i_1^* \mathbf{O}(W^N, V) \wedge X^N_W$$
$$\cong J_+ \wedge_{J_0} \int^{W \in i_1^* \mathbf{O}^{\mathscr{H}}_E} i_1^* \mathbf{O}(W^N, V) \wedge X^N_W.$$

By 1.1.5 applied to the coequalizer diagram representing the coend, there is an isomorphism

$$J_+ \wedge_{J_0} \int^{W \in i_1^* \mathbf{O}^{\mathscr{H}}_E} i_1^* \mathbf{O}(W^N, V) \wedge X^N_W \cong \int^{W \in \mathbf{O}^{\mathscr{H}}_E} \mathbf{O}(W^N, V) \wedge (J_+ \wedge_{J_0} X^N_W).$$

Finally, 3.1.39 gives an isomorphism

$$\int^{W \in \mathbf{O}^{\mathscr{H}}_E} \mathbf{O}(W^N, V) \wedge (J_+ \wedge_{J_0} X^N_W) \cong \int^{W \in \mathbf{O}^{\mathscr{H}}_E} \mathbf{O}(W^N, V) \wedge (G_+ \wedge_H X_W)^N$$
$$= \Phi^N_{\mathscr{H}}(G_+ \wedge_H X)$$

□

Restricted spectra and (Geometric) Fixed Points

Let as in (3.1.37) E and E_0 be exact sequences of groups. For orthogonal G-spectra Y, taking categorical N-fixed points commutes with the restriction to H-spectra, i.e., there is a natural isomorphism of J_0-spectra

$$i_1{}^*(Y^N) \cong (i^* Y)^N.$$

The following proposition is similar in spirit to Kro's 3.1.45 from above. We will use it when passing from finite groups to compact Lie groups 4.4.25.

Proposition 3.1.46. *If Suppose Condition 1 holds, then taking geometric N-fixed points commutes with the restriction to H-spectra, i.e., there is a natural isomorphism of J_0-spectra*

$$i_1{}^* \Phi^N_{\mathscr{H}} Y \xrightarrow{\cong} \Phi^N_{i^*\mathscr{H}}(i^* Y).$$

Proof. Since the functor $i_1^* \colon J\mathcal{T} \to J_0\mathcal{T}$ preserves colimits, there is an isomorphism

$$(i_1^* \Phi^N_{\mathscr{H}} Y)_V = i_1^* \int^{W \in \mathbf{O}^{\mathscr{H}}_E} \mathbf{O}(W^N, V) \wedge Y^N_W$$
$$\cong \int^{W \in i_1^* \mathbf{O}^{\mathscr{H}}_E} i_1^* \mathbf{O}(W^N, V) \wedge i_1^* Y^N_W.$$

Using the isomorphism

$$i_1^* Y_W^N \cong (i^* Y_{i^*W})^N$$

and Lemma 3.1.42 we obtain an isomorphism

$$\int^{W \in i_1^* \mathbf{O}_E^{\mathscr{H}}} i_1^* \mathbf{O}(W^N, V) \wedge i_1^* Y_W^N \cong \int^{W \in \mathbf{O}_{E_0}^{i^*\mathscr{H}}} \mathbf{O}(W^N, V) \wedge Y_W^N = (\Phi^N_{i^*\mathscr{H}} Y)_V.$$

□

3.2 Cellular filtrations of smash powers

While much of the contents of this section is well known, we need some additional control of how smash powers of orthogonal spectra can be assembled. In Section 3.2.13 we construct the cellular structures that form the technical heart of our constructions. We generalize Kro's approach from [Kr, 2.2], correcting some minor mistakes along the way. In particular we drop the assumption that all λ-sequences are $\mathbb{N}$-sequences in order to be able to attach cells one at a time, and work in general categories. This allows us to apply the theory in a lot of different contexts, cf. 4.2.2, 3.1.27 and 3.1.28, but also for a potential extension of our results to multiplicative norm constructions (cf. Remark 4.4.19).

3.2.1 Induced Regular Cells

We go back to finite discrete groups G for this subsection. The class of semi-free G-spectra is too big to fully control the geometric fixed point functor. Our studies of the smash powers of semi-free non equivariant spectra has given us a specific example of a class where such control is possible. Now we will define classes of *regular spectra*, and *induced regular spectra* which the smash powers are examples of.

Definition 3.2.2. Let $\varphi\colon G \to \mathbf{O}_V$ be a G-representation and let P be a G-invariant subgroup of $\mathbf{O}_V$. Given a finite free G-set X, we consider the group $G \ltimes \prod_X P$, where G acts on the product by the action corresponding to conjugation of functions from X to P. Let $\psi\colon G \to \mathbf{O}_{V\oplus X}$ be the direct sum representation. Given a $G \ltimes \prod_X P$-space L, we say that the induced $G \ltimes_\psi \mathbf{O}_{V^{\oplus X}}$-space $\mathbf{O}_{V^{\oplus X}+} \wedge_{\prod_X P} L$ is $V^{\oplus X}$*-regular.*
Given a G-representation $\rho\colon G \to \mathbf{O}_W$ containing the direct sum representation $\psi\colon G \to \mathbf{O}_{V^{\oplus X}}$, we say that a $G \ltimes_\rho \mathbf{O}_W$-space K is W*-regular* if it is isomorphic to an $\mathbf{O}_W \ltimes G$-space of the form $\mathbf{O}_W \wedge_{\mathbf{O}_{V^{\oplus X}}} K'$ for a $V^{\oplus X}$-regular space K'.
A semi-free G-spectrum is called $\mathscr{H}$*-regular* if it is of the form $\mathcal{G}^\rho_W K$ for $\rho\colon G \to \mathbf{O}_W$ a G-representation in $\mathscr{H}$ and a W-regular $\mathbf{O}_W \ltimes_\rho G$-space K.

Remark 3.2.3. Free spectra are regular, since we can choose $V = 0$.

Note that since $\prod_X P \subseteq \mathbf{O}_W$ is G-invariant, there are inclusions of N-fixed subgroups $(\prod_X P)^N \subset (\mathbf{O}_W)^N$ for all subgroups N of G.

Remark 3.2.4. Let $Q = \prod_X P$ and suppose that the image of $G \ltimes Q$ in $G \times \mathbf{O}_W$ under the isomorphism $G \times \mathbf{O}_W \cong G \ltimes_\rho \mathbf{O}_W$ is a member of $\mathscr{G}$ and that the complement of the sub representation $V^{\oplus X}$ of W is a representation in $\mathscr{H}$. Since all the involved functors preserve colimits and since inducing up preserves $(\mathscr{H},\mathscr{G})$-cofibrations by (iii) of 2.6.1 the regular semi-free spectrum $\mathcal{G}^\rho_W K \cong \mathcal{G}_W(\mathbf{O}(W,W)_+ \wedge_Q L)$ is $(\mathscr{H},\mathscr{G})$-cofibrant if L is genuinely $G \ltimes Q$-cofibrant. More generally, if $i \colon L \to L'$ is a genuine $G \ltimes Q$-cofibration, then $\mathcal{G}_V(\mathbf{O}(W,V)_+ \wedge_Q i)$ is an $(\mathscr{H},\mathscr{G})$-cofibration.

Example 3.2.5. The most important example of a regular semi-free G-spectrum is the smash power

$$(\mathcal{G}^\varphi_V K)^{\wedge X} \cong \mathcal{G}^\psi_{V^{\oplus X}}[\mathbf{O}_{V^{\oplus X}+} \wedge_{\prod_X \mathbf{O}_V} K^{\wedge X}],$$

where X is a finite free G-set.

The following proposition is proved completely analogous to 3.1.34, using that Proposition 1.1.21 is slightly more general than was needed there.

Proposition 3.2.6. *Let N be a normal subgroup of G with quotient group denoted by J. Let $\rho \colon G \to \mathbf{O}_W$ be a G-representation in $\mathscr{H}$, let $\mathcal{G}^\rho_W K$ be a regular semi-free G-spectrum and let $\overline{\rho} \colon J \to \mathbf{O}_{W^N}$ be the induced representation of J. Then there is a natural isomorphism of J-spectra*

$$\mathcal{G}^{\overline{\rho}}_{W^N} K^N \cong \Phi^N_{\mathscr{H}}(\mathcal{G}^\rho_W K).$$

Combining this result with Proposition 3.1.45 for induced spectra motivates the following definition.

Definition 3.2.7. A semi-free G-spectrum A is called *induced $\mathscr{H}$-regular* if there is an isomorphism of G-spectra

$$A \cong G_+ \wedge_H B,$$

for H a subgroup of G and B an $\mathscr{H}$-regular semi-free H-spectrum.

Note that by Lemma 3.1.43 Condition 1 holds in the case of finite groups, so we get the following characterizations of geometric fixed points for induced regular spectra which is analogous to Kro's Lemma 3.10.8:

Theorem 3.2.8. *Let G be a finite group and H and N subgroups with N normal. The geometric fixed points of an induced regular semi-free G-spectrum $G_+ \wedge_H \mathcal{G}_W K$ are given by the natural isomorphism:*

$$\Phi^N_{\mathscr{H}}(G_+ \wedge_H \mathcal{G}_W K) \cong \begin{cases} G/N_+ \wedge_{H/N} [\mathcal{G}_{W^N} K^N] & \text{if } N \subset H \\ * & \text{otherwise.} \end{cases}$$

Proof. The first part is a direct consequence of 3.2.6 and 3.1.45. For the second part, note that already for H-spaces A we have $(G_+ \wedge_H A)^N \cong \star$ if N is not contained in H, hence the functor $(G_+ \wedge_H \mathcal{G}_W K)^N_V \cong \star$ for all V (cf. [Kr, 3.10.8]). □

Note that induction $G_+ \wedge_H -$ takes level $(i^*\mathscr{H}, i^*\mathscr{G})$-cofibrations to level $(\mathscr{H}, \mathscr{G})$-cofibrations. Thus we get the following:

Lemma 3.2.9. *For V an H-representation, X a finite free H set, P an H-invariant subgroup of $\mathbf{O}_V$, for $Q = \Pi_X P$ and i a genuine cofibration of $H \ltimes Q$-spaces, the map*

$$f := G_+ \wedge_H \mathcal{G}_W^H[\mathbf{O}_{W+} \wedge_Q i]$$

of G-spectra is an $(\mathscr{H}, \mathscr{G})$-cofibration for every G-representation W containing $V^{\oplus X}$. Denote the class of $(\mathscr{H}, \mathscr{G})$-cofibrations of this form by $\mathbb{S}^{\mathscr{G}}_{\text{reg}}$.

Remark 3.2.10. For $\mathbb{S}^{\mathscr{G}}_{\text{reg}}$-cell complexes, Theorem 3.2.8 allows us to easily compute the geometric fixed points via a cell induction. This could be used to define a class of cofibrations very much in the spirit of Kro's *induced cells* ([Kr, 3.4.4]) and *orbit cells* ([Kr, 3.4.6]), but more general than both. Since we are not going to construct model structures or even replacement functors for any of these classes, we will not go into this generality. Instead we will focus on the type of cells that will appear in the cell structure for the smash powers.

Finally we give a name for the cells that we will use in the equivariant filtration theorem:

Definition 3.2.11. Let $j : e \to G$ be the inclusion of the trivial subgroup. An $(\mathscr{H}, \mathscr{G})$-cofibration f of orthogonal G-spectra is an *induced regular cell* if it arises from a generating $(j^*\mathscr{H}, j^*\mathscr{G})$-cofibration i of orthogonal spectra via

$$f = G_+ \wedge_H i^{\square H},$$

for H a normal subgroup of G, and $\square$ the pushout product construction from 5.2.30. We denote the *class of all induced regular cells* by $\mathrm{Ind}^{\text{reg}}_{\mathscr{G}}$.

The significance of the pushout product will become more obvious in the next section, when we give details on equivariant filtrations (cf. in particular Lemma 3.1.6). The following remark, however, is immediate:

Remark 3.2.12. Both source and target of an induced regular cell are induced regular in the sense of 3.2.7. For the target this is obvious, since if i is the map

$$\mathcal{G}_V \left(\mathbf{O}_V / P_+ \wedge [S^{n-1} \to D^n] \right),$$

then the formula 2.3.4 together with the fact that inducing up preserves colimits yield that $i^{\square H}$ is isomorphic to the map

$$\imath^{\square H} \cong \mathcal{G}_{V \oplus H} \left(\mathbf{O}_{V \oplus H} / \Pi_H P_+ \wedge [S^{|H|n-1} \to D^{|H|n}]_+ \right),$$

where $|H|$ is the order of H. In particular it is represented by the inclusion of the boundary sphere of the $H \ltimes \Pi_H P$ space $D^{|H|n}$, where $\Pi_H P$ acts trivially, and H acts by permuting coordinates blockwise.

3.2.13 Equivariant cellular filtrations of smash powers

We finally give the equivariant cellular structure for smash powers of orthogonal spectra. All our applications are to the positive structure and for concreteness is this the version we write out.

If Q is a Lie group, G a finite group and X a finite G-set, then the Lie group $Q^X = \mathbf{Map}(X, Q)$ has a G-action by precomposition: $gf(x) = f(g^{-1}x)$ for $f\colon X \to Q$, $g \in G$ and $x \in X$.

Given a subgroup $P_x \subseteq Q \times \mathbf{O}_{V_x}$ for each $x \in X$, we consider $P = \prod_{x\in X} P_x$ as a subgroup of $Q^X \times \mathbf{O}_V$ for $V = \bigoplus_{x\in X} V_x$. We let $\mathscr{H}(G, X)$ be the smallest $G \ltimes Q^X$-typical family of representations so that if every P_x is in $\mathscr{H}$ and H is any subgroup of G, then $H \ltimes P \subseteq G \ltimes (Q^X \times \mathbf{O}_V) = (G \ltimes Q^X) \times \mathbf{O}_V$ is a member of $\mathscr{H}(G, X)$. (Here we use the trivial action of G on V. Later, when some of the factors V_x are identical, we will also make use of the non-trivial actions of subgroups of G on V given by permutation of identical factors.) We let $\mathscr{G}(G, X)$ be the smallest collection of families of subgroups of $(G \ltimes Q^X) \times \mathbf{O}_V$ for V in $\mathbf{O}$ so that $(\mathscr{H}(G, X), \mathscr{G}(G, X))$ is a $G \ltimes Q^X$-mixing pair, and so that if every P_x is in $\mathscr{G}$ and H is any subgroup of G, then $H \ltimes P \subseteq G \ltimes (Q^X \times \mathbf{O}_V) = (G \ltimes Q^X) \times \mathbf{O}_V$ is a member of $\mathscr{G}(G, X)$.

We will give a filtration of the map

$$(\star \to L)^{\square X} \cong (\star \to L^{\wedge X})$$

by $\mathscr{G}(G, X)$-cofibrations using Theorem 3.1.2.

When X is G-free it will follow from the construction that all the attaching maps are $\mathscr{G}(G, X)$-cofibrations between induced regular $\mathscr{G}(G, X)$)-cofibrant $G \ltimes Q^X$-spectra. Our methods are inspired by [Kr, 3.10.1], where a similar filtration is given for the case that $X = G = C_q$ a finite cyclic group.

If λ is an ordinal, and X a set, we define a partial order on the product λ^X: for $\alpha = \{\alpha_x\}_{x\in X}, \beta = \{\beta_x\}_{x\in X} \in \lambda^X$ we say that $\beta \leq \alpha$ if for all $x \in X$ we have that $\beta_x \leq \alpha_x$. If a group G acts on X it acts on λ^X by letting $(g\alpha)_x = \alpha_{g^{-1}x}$. This induces a partial ordering on the set $(\lambda^X)_G$ of G-orbits by declaring for $u, v \in (\lambda^X)_G$ that $u \leq v$ if there exist $\alpha \in u$ and $\beta \in v$ such that $\alpha \leq \beta$.

As in Remark 3.1.16, since the maps in $\mathcal{G}I_{\mathscr{G}}$ are levelwise inclusions, we lose no generality by assuming that a given relative $\mathcal{G}I_{\mathscr{G}}$-cell complex is an inclusion.

Consider a λ-sequence $K = L_0 \subseteq L_1 \subseteq \ldots \subseteq L$ of Q-spectra, exhibiting $K \subseteq L$ as relative $\mathcal{G}I_{\mathscr{G}}$-cell complex, i.e., if $a \in \lambda$ is a limit ordinal, then $L_a = \bigcup_{b<a} L_b$, and if $a \in \lambda$ is not a limit ordinal, there is an object V_a of $\mathbf{O}$, a member P_a of $\mathscr{G}^{V_a}$ and a pushout diagram $\mathbf{D}_a$ of Q-spectra

$$\begin{array}{ccc} \mathcal{G}_{V_a}(S^{n_a-1} \times (Q \times \mathbf{O}_{V_a})/P_a)_+ & \xrightarrow{i_a} & \mathcal{G}_{V_a}(D^{n_a} \times (Q \times \mathbf{O}_{V_a})/P_a)_+ \\ \downarrow & \ulcorner & \downarrow \\ \bigcup_{b<a} L_b & \longrightarrow & L_a, \end{array}$$

with the inclusion i_a an element in the generating cofibrations $\mathcal{G}I_{\mathscr{G}}$.

Theorem 3.2.14. *Let Q be a compact Lie group and let $K \subseteq L$ be a relative $\mathcal{G}I_{\mathscr{G}}$-cellular inclusion of orthogonal Q-spectra with cells indexed by the ordinal λ. Let G be a finite group and let X be a finite G-set. Then the smash power*

$$K^{\wedge X} \subseteq L^{\wedge X}$$

is a relative $\mathcal{G}I_{\mathscr{G}(G,X)}$-cell complex with $(\lambda^X)_G$ indexing the $\mathcal{G}I_{\mathscr{G}(G,X)}$-cells in the following sense: Let $\alpha \in \lambda^X$ with orbit $u = [\alpha]$ and stabilizer group $G_\alpha \subseteq G$ and let $L_u^{\wedge X}$ be the $G \ltimes Q^X$-subspectrum of $L^{\wedge X}$ defined as the union $L_u^{\wedge X} = \bigcup_{\beta \in u} L_\beta^{\wedge X}$ with $L_\beta^{\wedge X} = \bigwedge_{x \in X} L_{\beta_x}$ so that

$$\operatorname*{colim}_{u \in (\lambda^X)_G} L_u^{\wedge X} \cong \bigcup_{u \in (\lambda^X)_G} L_u^{\wedge X} = L^{\wedge X}.$$

If α_x is a limit ordinal for some $x \in X$, then $L_u^{\wedge X} = \bigcup\limits_{v<u} L_v^{\wedge X}$. Otherwise there is an Euclidean vector space V, a closed subgroup P of $(G \ltimes Q^X) \times \mathbf{O}_V$ in $\mathscr{G}(G,X)^V$ and a pushout diagram of $G \ltimes Q^X$-spectra of the form

$$\begin{array}{ccc}
\mathcal{G}_V(S^{n-1} \times ((G \ltimes Q^X) \times \mathbf{O}_V)/P)_+ & \xrightarrow[\subseteq]{k_\alpha} & \mathcal{G}_V(D^n \times ((G \ltimes Q^X) \times \mathbf{O}_V)/P)_+ \\
\downarrow & \ulcorner & \downarrow \\
\bigcup\limits_{v<u} L_v^{\wedge X} & \xrightarrow{\subseteq} & L_u^{\wedge X}.
\end{array}$$

In particular, extending the partial order $(\lambda^X)_G$ to a total order λ', the smash power $K^{\wedge X} \subseteq L^{\wedge X}$ is a $\mathcal{G}I_{\mathscr{G}(G,X)}$-cell complex with indexed by λ'.

Furthermore:

(i) The $\square$-product $(K \subseteq L)^{\square X}$ is in this notation given by $\bigcup_u L_u^{\wedge X} \subseteq L^{\wedge X}$, where u varies over the orbits in $(\lambda^X)_G$ whose representatives take the value 0 at least once, and both the inclusions $K^{\wedge X} \subseteq \bigcup_u L_u^{\wedge X} \subseteq L^{\wedge X}$ are relative $\mathcal{G}I_{\mathscr{G}(G,X)}$-cell complexes.

(ii) If X is a finite free G-set, then the top inclusion k_α of the pushout of $G \ltimes Q^X$-spectra is isomorphic to one on the form $G_+ \wedge_{G_\alpha} (k^{\square G_\alpha})$ with $k \in \mathcal{G}I_{\mathscr{G}}$.

Proof. Let $K = L_0 \subseteq L_1 \subseteq \ldots \subseteq L$ be the λ-sequence exhibiting L as $\mathcal{G}I_{\mathscr{G}}$-cellular, using the names V_a, P_a, n_a, i_a and $\mathbf{D}_a$ for each $a \in \lambda$ as in the comment just above the statement of the theorem. Let $\alpha \in \lambda^X$. If α_x is a limit ordinal for any $x \in X$, then $L_u^{\wedge X} = \bigcup\limits_{v<u} L_v^{\wedge X}$. Suppose now that non of the α_x are limit ordinals. Taking the row-wise $\square$-product over $x \in X$ of all the diagrams $\mathbf{D}_{\alpha_x}$ as displayed just before the start of the theorem, we get by 3.1.6 a pushout diagram of $G_\alpha \ltimes Q^X$-spectra whose top

map is $\square_{x\in X} i_{\alpha_x}$, which we by 2.3.4 may identify with the following pushout diagram $\mathbf{D}_\alpha$

$$\begin{array}{ccc} \mathcal{G}_{V_\alpha}(S^{n_\alpha-1}\times(Q^X\times\mathbf{O}_{V_\alpha})/P_\alpha)_+ & \xrightarrow[\subseteq]{i_\alpha} & \mathcal{G}_{V_\alpha}(D^{n_\alpha}\times(Q^X\times\mathbf{O}_{V_\alpha})/P_\alpha)_+ \\ \downarrow & \ulcorner & \downarrow e_\alpha \\ \bigcup_{\gamma<\alpha} L_\gamma^{\wedge X} & \xrightarrow{\subseteq} & L_\alpha^{\wedge X}, \end{array}$$

where $V_\alpha = \bigoplus_{x\in X} V_{\alpha_x}$, $n_\alpha = \sum_{x\in X} n_{\alpha_x}$ and $P_\alpha = \prod_{x\in X} P_{\alpha_x}$ considered as a subgroup of $Q^X\times\mathbf{O}_{V_\alpha}$ via the inclusion

$$(Q^X\times\prod_{x\in X}\mathbf{O}_{V_{\alpha_x}})\subseteq Q^X\times\mathbf{O}_{V_\alpha},$$

and where G_α acts by permuting the X-coordinates on each of these ingredients of $\mathbf{D}_\alpha$. Via the shear isomorphism $G_\alpha\ltimes\mathbf{O}_{V_\alpha}\cong G_\alpha\times\mathbf{O}_{V_\alpha}$ we change the G_α-action so that G_α acts trivially on $\mathbf{O}_{V_\alpha}$, and so that the above diagram is a pushout of $(G_\alpha\ltimes Q^X)\times\mathbf{O}_{V_\alpha}$-spaces.

If $\alpha\in\lambda^X$ and $g\in G$, then conjugation yields an isomorphism $c_g\colon G_\alpha\cong G_{g\alpha}$, $c_g(h)=ghg^{-1}$. Furthermore, $\alpha\mapsto\mathbf{D}_\alpha$ is natural in the sense that acting by g (i.e., permuting the coordinates in each of the vertices of the square) defines an isomorphism between $\mathbf{D}_\alpha$ and $\mathbf{D}_{g\alpha}$, and even a G_α-isomorphism $\mathbf{D}_g\colon\mathbf{D}_\alpha\cong c_g^*\mathbf{D}_{g\alpha}$.

Fixing the orbit u, we write $\mathbf{D}_\alpha^u$ for what we just called $\mathbf{D}_\alpha$, and for $\gamma\in\lambda^X$ with $[\gamma]<[\alpha]$ we let $\mathbf{D}_\gamma^u$ be the pushout diagram

$$\begin{array}{ccc} pt & \longrightarrow & pt \\ \downarrow & \ulcorner & \downarrow \\ L_\gamma^{\wedge X} & \longrightarrow & L_\gamma^{\wedge X}. \end{array}$$

The assignment $\gamma\mapsto\mathbf{D}_\gamma^u$ is functorial on the partially ordered set consisting of those $\gamma\in\lambda^X$ with $[\gamma]\le u$. Taking the colimit of the pushout diagrams $\mathbf{D}_\gamma^u$ we obtain a pushout diagram

$$\begin{array}{ccc} \bigvee_{\beta\in u}\left[\mathcal{G}_{V_\beta}(S^{n_\beta-1}\times(Q^X\times\mathbf{O}_{V_\beta})/P_\beta)_+\right] & \xrightarrow{\vee i_\beta} & \bigvee_{\beta\in u}\left[\mathcal{G}_{V_\beta}(D^{n_\beta}\times(Q^X\times\mathbf{O}_{V_\beta})/P_\beta)_+\right] \\ \downarrow & \ulcorner & \downarrow \sum e_\beta \\ \bigcup_{v<u}(L^{\wedge X})_v & \longrightarrow & (L^{\wedge X})_u \end{array}$$

of $G\ltimes Q^X$-spectra where $g\in G$ acts by taking factors of the form $S^{n_\beta-1}$ and D^{n_β} to $S^{n_{g\beta}-1}$ and $D^{n_{g\beta}}$ respectively and by the permutation action on Q^X.

Fixing one α in the orbit u and using the isomorphisms translating between the different i_βs and our i_α we can rewrite this diagram on the form

$$\begin{array}{ccc} G_+\wedge_{G_\alpha}\Big[\mathcal{G}^{G_\alpha}_{V_\alpha}(S^{n_\alpha-1}\times(Q^X\times\mathbf{O}_{V_\alpha})/P_\alpha)_+\Big] & \xrightarrow{k_\alpha} & G_+\wedge_{G_\alpha}\Big[\mathcal{G}^{G_\alpha}_{V_\alpha}(D^{n_\alpha}\times(Q^X\times\mathbf{O}_{V_\alpha})/P_\alpha)_+\Big] \\ \downarrow & \ulcorner & \downarrow \\ \bigcup_{v<u} L_v^{\wedge X} & \longrightarrow & L_u^{\wedge X}, \end{array}$$

where the top map is $k_\alpha = G_+\wedge_{G_\alpha} i_\alpha$

We claim that the inclusion k_α is a $\mathscr{G}(G,X)$-cofibration of $G\ltimes Q^X$-spectra. To see this consider the inclusion $S^{n_\alpha-1}\subseteq D^{n_\alpha}$ of G_α-spaces. This is a genuine G_α-cofibration (as can be seen by viewing D^{n_α} as a cone on the barycentric subdivision of an $(n_\alpha-1)$-simplex), and so i_α is a relative cell complex with cells of the form

$$\mathcal{G}_{V_\alpha}((S^{m-1}\times G_\alpha/H_\alpha)\times(Q^X\times\mathbf{O}_{V_\alpha})/P_\alpha)_+\subseteq\mathcal{G}_{V_\alpha}((D^{m}\times G_\alpha/H_\alpha)\times(Q^X\times\mathbf{O}_{V_\alpha})/P_\alpha)_+$$

for H_α a subgroup of G_α, and thus k_α is a relative cell complex with cells of the form

$$\mathcal{G}_{V_\alpha}((S^{m-1}\times G/H)\times(Q^X\times\mathbf{O}_{V_\alpha})/P_\alpha)_+\subseteq\mathcal{G}_{V_\alpha}((D^{m}\times G/H)\times(Q^X\times\mathbf{O}_{V_\alpha})/P_\alpha)_+$$

for H a subgroup of G_α. Thus it suffices to show that this is a $\mathcal{G}I_{\mathscr{G}(G,X)}$-cell. This is seen by first noting that since P_α is a G_α-invariant subgroup of $Q^X\times\mathbf{O}_{V_\alpha}$, there is an isomorphism

$$G_\alpha/H\times(Q^X\times\mathbf{O}_{V_\alpha})/P_\alpha\to(G_\alpha\ltimes(Q^X\times\mathbf{O}_{V_\alpha}))/(H\ltimes P_\alpha)$$

of $(G_\alpha\ltimes Q^X)\times\mathbf{O}_{V_\alpha}$-spaces sending $(gH,(A,q)P_\alpha)$ to $(g,q,A)(H\ltimes P_\alpha)$. Here we use that since G_α acts trivially on $\mathbf{O}_{V_\alpha}$, the groups $(G_\alpha\ltimes Q^X)\times\mathbf{O}_{V_\alpha}$ and $G_\alpha\ltimes(Q^X\times\mathbf{O}_{V_\alpha})$ are identical. The projection $(G\ltimes Q^X)\times\mathbf{O}_{V_\alpha}\to G$ induces an isomorphism between

$$((G\ltimes Q^X)\times\mathbf{O}_{V_\alpha})\times_{(G_\alpha\ltimes Q^X)\times\mathbf{O}_{V_\alpha}}(G_\alpha/H\times(Q^X\times\mathbf{O}_{V_\alpha})/P_\alpha)$$

and

$$G\times_{G_\alpha}(G_\alpha/H\times(Q^X\times\mathbf{O}_{V_\alpha})/P_\alpha)\cong G/H\times(Q^X\times\mathbf{O}_{V_\alpha})/P_\alpha$$

of $(G\ltimes Q^X)\times\mathbf{O}_{V_\alpha}$-spaces. Thus inducing up the action of $(G_\alpha\ltimes Q^X)\times\mathbf{O}_{V_\alpha}$ to an action of $(G\ltimes Q^X)\times\mathbf{O}_{V_\alpha}$ we obtain an isomorphism

$$G/H\times(Q^X\times\mathbf{O}_{V_\alpha})/P_\alpha\to(G\ltimes(Q^X\times\mathbf{O}_{V_\alpha}))/(H\ltimes P_\alpha)$$

of $(G\ltimes Q^X)\times\mathbf{O}_{V_\alpha}$-spaces. By design, the subgroup $H\ltimes P_\alpha$ is in $\mathscr{G}(G,X)$. We conclude that k_α is a relative $\mathcal{G}I_{\mathscr{G}(G,X)}$-cell complex.

The statement about the $\Box$-product is immediate from the construction.

Lastly, we treat the special case when X is free as a G-set. Recall that the inclusion i_α in the diagram $\mathbf{D}_\alpha$ was given by the iterated $\Box$-product $\Box_{x\in X} i_{\alpha_x}$, with the G_α-action permuting $\square$-factors that are identical. Since G acts freely on X, so does G_α. Therefore, choosing any system of representatives R for the G_α-orbits of X and letting $k \in \mathcal{G}I_{\mathscr{G}}$ be the inclusion

$$\mathcal{G}^Q_{V_R}(S^{n_R-1} \times (Q \times \mathbf{O}_{V_R})/P_R)_+ \subseteq \mathcal{G}^Q_{V_R}(D^{n_R} \times (Q \times \mathbf{O}_{V_R})/P_R)_+,$$

where $V_R = \bigoplus_{r\in R} V_r$, $n_R = \sum_{r\in R} n_r$ and $P_R = \prod_{r\in R} P_r$, we get a G_α-equivariant isomorphism $i_\alpha \cong k^{\square G_\alpha}$. $\square$

Example 3.2.15. Consider the case when $G = \Sigma_n$ acting on $X = \{1, \dots, n\}$, and $K \subseteq L$ is obtained by attaching a single $i\colon s \to t \in \mathcal{G}I_{\mathscr{G}}$. Then we can think of the map $\star \to K$ as another single cell as in Corollary 3.1.18, i.e., think of $\lambda = \{0 < 1 < 2\}$ with $L_0 \subseteq L_1 \subseteq L_2$ being $* \subseteq K \subseteq L$. Then all $\square$-product summands containing $*$ are trivial, so we consider the αs with only 1 and 2 as values. The stabilizer group G_α is isomorphic to $\Sigma_m \times \Sigma_{n-m}$ where $m = |\alpha^{-1}(2)|$, and we get a pushout diagram of the form

$$\begin{array}{ccc} A\wedge K^{\wedge\alpha^{-1}(1)} & \xrightarrow{i^{\square\alpha^{-1}(2)}\wedge K^{\wedge\alpha^{-1}(1)}} & L^{\wedge\alpha^{-1}(2)}\wedge K^{\wedge\alpha^{-1}(1)} \\ \downarrow & & \downarrow \\ \bigcup_{\gamma<\alpha} L^{\wedge X}_\gamma & \xrightarrow{\subseteq} & L^{\wedge X}_\alpha, \end{array}$$

where A is the source of $i^{\square\alpha^{-1}(2)}$. Hence, the the cell attached to reach $L^{\wedge X}_{[\alpha]}$ is of the form

$$\Sigma_n \wedge_{\Sigma_m\times\Sigma_{n-m}} i^{\square m} \wedge K^{\wedge n-m}.$$

Varying m from 1 to n we run through all the conjugacy classes $[\alpha]$ of such αs and get the n cells needed to build $L^{\wedge X}$ from $K^{\wedge X}$.

Together with Theorem 3.2.8, we can now calculate the geometric fixed points of smash powers of $\mathbb{S}$-cofibrant spectra. Let as above

$$1 \to N \to G \xrightarrow{\epsilon} J \to 1,$$

be a short exact sequence of finite discrete groups, and X a finite free G-set.

Theorem 3.2.16. *If L is an $\mathbb{S}$-cofibrant orthogonal spectrum and X a finite free G-set, then there the diagonal map $\Delta(X, L)$ in 4.4.3 is a natural isomorphism of J-spectra*

$$\Delta(X, L)\colon L^{\wedge X_N} \xrightarrow{\cong} \Phi^N(L^{\wedge X}).$$

Proof. This is again quite similar in spirit to the analog theorem [Kr, 3.10.7]. It suffices to look at $\mathbb{S}I$-cellular L, since retracts are preserved by any functor. We keep the notation from the proof of Theorem 3.2.14 as far as possible. We let σ be the set of J-orbits of $\lambda^{\times X}$ and we let σ' be the J-orbits of $\lambda^{\times X_N}$. The projection $X \to X_N$ induces a "diagonal" J-map

$$\begin{array}{rcl} \epsilon^*\colon \quad \lambda^{\times X_N} & \to & \lambda^{\times X}, \\ \{\kappa_{[x]}\}_{[x]\in X_N} & \mapsto & \{\kappa_{[x]}\}_{x\in X} \end{array}$$

which descends to orbits to give a map $\epsilon^*\colon \sigma' \to \sigma$. Note that in the cell structure for $L^{\wedge X}$, cells that are not indexed by $\epsilon^*[\sigma']$ do not contribute to the geometric fixed points: By induction over the cellular filtration, assume that for all $[\beta] < [\alpha]$ in σ we have

$$\Phi^N(\bigcup_{[\delta]\le[\beta]} L^{\wedge X}_{[\delta]}) \cong \Phi^N(\bigcup_{[\epsilon^*\kappa]\le\beta} L^{\wedge X}_{[\epsilon^*\kappa]}),$$

then the same is true for β replaced by α: In the case $\alpha \in \epsilon^*\sigma'$ there is nothing to do, otherwise note that for $[\alpha] \notin \epsilon^*\sigma'$, the group N is not contained in Stab_α, hence in the attaching diagram $D_{[\alpha]}$, the top row has trivial geometric fixed points by 3.2.8. Since taking geometric fixed points commutes with the cell-complex construction, we therefore get colimit diagrams for $\Phi^N L^{\wedge X}$ and $L^{\wedge X_N}$ of exactly the same shape, with attaching diagrams indexed by $\epsilon^*\sigma' \cong \sigma'$. Then again by induction $[\kappa] \in \lambda^{X_N}$ we show that the J-spectra $L^{\wedge X_N}_{[\kappa]}$ and $\Phi^N(L^{\wedge X}_{\epsilon^*[\kappa]})$ are isomorphic. Herefore compare the attaching diagrams:

$$\begin{array}{ccccc} \bigcup_{[\gamma]<[\kappa]} \Phi^N(L^{\wedge X})_{[\epsilon^*\gamma]} & \longleftarrow & \Phi^N\left(G_+\wedge_H[\mathcal{G}_V(S^{n-1}\times \mathbf{O}_V/P)_+]^{\square H}\right) & \longrightarrow & \Phi^N\left(G_+\wedge_H[\mathcal{G}_V(D^n\times \mathbf{O}_V/P)_+]^{\wedge H}\right) \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ \bigcup_{[\gamma]<[\kappa]} (L^{\wedge X_N})_{[\gamma]} & \longleftarrow & J_+\wedge_{J_0}[\mathcal{G}_V(S^{n-1}\times \mathbf{O}_V/P)_+]^{\square J_0} & \longrightarrow & J_+\wedge_{J_0}[\mathcal{G}_V(D^n\times \mathbf{O}_V/P)_+]^{\wedge J_0} \end{array}$$

where the vertical maps are isomorphisms by 3.2.8, hence induce an isomorphism on pushouts. Naturality is after the discussion so far only obvious for cellular morphisms. For all morphisms between $\mathbb{S}$-cofibrant orthogonal spectra and for all isomorphisms of finite free G-sets naturality follows from the construction in 4.4.3, cf. Remark 4.4.7. $\square$

Chapter 4

Smash Powers

4.1 Introduction

4.1.1 Hochschild Homology

In [HH], Hochschild defined a cohomology theory for bimodules of algebras over a field, mirroring the definition of group homology in terms of the bar complex. Hochschild cohomology and its dual, Hochschild homology, have proven valuable tools in a wide variety of mathematical disciplines, spanning from algebra and topology to mathematical physics and functional analysis.

For future comparison, we will very briefly recall some basics, not touching on most of the rich theory that follows. For further details, the reader should consult Loday's book [L] which provides a very readable and thorough introduction.

Hochschild Homology, since it was defined by Hochschild for the case of algebras over fields in 1944 ([HH]), was adapted to various more general contexts and has proven a valuable tool both in algebra, topology and geometry. We will be very brief in recalling some basics, not touching on most of the rich theory that follows. A very readable and thorough introduction can for example be found in Loday's book on Cyclic Homology [L], where in particular everything that follows here can be extracted from. We shall focus on the commutative setting, and will always assume that the coefficients are the commutative algebra itself. Let for the whole section R denote a commutative unital ground ring, and denote the tensor product over R simply by $\otimes$.

Definition 4.1.2. Let A be a commutative R-algebra. *Hochschild Homology of A* is the homology $HH(A)$ of the simplicial commutative R-algebra $Z(A)$

$$\begin{array}{c} \vdots \\ \downarrow \\ A \otimes A \otimes A \\ \downarrow\uparrow\downarrow\uparrow\downarrow \\ A \otimes A \\ \downarrow\uparrow\downarrow \\ A \end{array}$$

with $Z(A)_n = A^{\otimes n+1}$ and with face maps $b_i\colon Z(A)_n \to Z(A)_{n-1}$ given by

$$b_i(a_0,\dots,a_n) := (a_0,\dots,a_i.a_{i+1},\dots,a_n),$$

for $i < n$ and

$$b_n(a_0,\dots,a_n) := (a_n.a_0, a_1 \dots, a_{n-1}).$$

The degeneracy operators $s_i\colon Z(A)_n \to Z(A)_{n+1}$ are given by $s_i(a_0,\dots,a_n) = (a_0,\dots,a_i,1,a_{i+1},\dots,a_n)$

Loday realized that this definition could be seen as a special case of a functorial construction on R-algebras (cf. [L89, 4.2]):

Definition 4.1.3. Let Fin be the category of finite sets and R_{CAlg} the category of commutative R-algebras. Define the *algebraic Loday Functor*

$$\bigotimes\nolimits_{(-)}(-)\colon \mathrm{Fin} \times R_{\mathrm{CAlg}} \to R_{\mathrm{CAlg}},$$

on objects as the iterated tensor product

$$\bigotimes\nolimits_X(A),$$

that is, taking $X, A)$ to the coproduct of X copies of A in the category R_{CAlg}. The Loday Functor takes a morphism $(f,\phi) : (X,A) \to (Y,B)$ to the A-linear map given by

$$(f,\phi)(a_x)_{x\in X} := \left(\prod_{f(x)=y} \phi(a_x)\right)_{y\in Y},$$

where the product over an empty indexing set is to be understood as the unit in R.

Remark 4.1.4. Note that we really use that A is a *commutative* R-algebra when defining the functor from Fin. If we would restrict to the category of finite sets and isomorphisms, the same formulas would give a functor with input mere R-modules. Extending to non surjective maps requires a unit map of some sort, and non injective but at least monotonous maps between ordered sets would only require an associative multiplication.

It is this formula that we emulate in the topological setting of commutative orthogonal ring spectra, where we construct a continuous analog to the Loday functor. One application of this topological Loday functor, is a convenient definition of topological Hochschild homology in the same spirit as the following definition:
The algebraic Loday functor gives an explicit example of functors for which we can define Hochschild homology:

Definition 4.1.5. Let $F\colon \mathrm{Fin} \to R_{\mathrm{Alg}}$ be a functor. Define its Hochschild homology $HH(F)$ as the homology of the simplicial algebra

$$\Delta^{\mathrm{op}} \xrightarrow{S^1} \mathrm{Fin} \xrightarrow{F} R_{\mathrm{Alg}}.$$

Then immediately, inspection of the defining chain complex yields that Hochschild homology of commutative algebras is the same as Hochschild homology of the Loday functor:

$$HH(A) = HH(\Lambda_{(-)}A). \tag{4.1.6}$$

4.2 Commutative Ring Spectra

In this section, we define a convenient model structure on the category of commutative orthogonal ring G-spectra. We mainly work in the $\mathbb{S}$-model structure on $G\mathbf{O}\mathcal{T}$ 2.9.11

Lemma 4.2.1. *Let Y be an orthogonal G-spectrum and let X be the semi-free orthogonal G-spectrum $\mathcal{G}_V\,(G \times \mathbf{O}_V/P)_+ \wedge K$ from 2.3.1, for K a based CW-complex, $V \neq 0$ in $\mathbf{O}$ and $P \subseteq G \times \mathbf{O}_V$. Then the map*

$$q\colon \left(E\Sigma_{i+} \wedge_{\Sigma_i} X^{\wedge i}\right) \wedge Y \to X^{\wedge i}_{\Sigma_i} \wedge Y$$

induced by $E\Sigma_{i+} \to *$ is a π_*-isomorphism. Indeed, if W in $\mathbf{O}$ is of dimension at least i times the dimension of V the map q_W is a G-equivalence.

Proof. We prove that q is a G-equivalence in all levels $W = V^{\oplus i} \oplus V'$. Let $\Gamma := G \times (\Sigma_i \ltimes P^{\times i}) \times \mathbf{O}_{V'}$ and let $p\colon \Gamma \to \Sigma_i$ be the projection onto the second factor. The space $p^*E\Sigma_{i+}$ is Γ-homotopy equivalent to $E\mathscr{F}$, where $\mathscr{F}$ is the family of subgroups of Γ whose image under $p\colon \Gamma \to \Sigma_i$ is trivial. Since $V \neq 0$, the homomorphism $(\Sigma_i \ltimes \mathbf{O}_V^{\times i}) \to \mathbf{O}_{V^{\oplus i}}$ sending $(\sigma, (\alpha_1, \ldots, \alpha_i))$ to $(\alpha_1 \oplus \cdots \oplus \alpha_i)\sigma$ is injective. We let $\Sigma_i \ltimes (G \times \mathbf{O}_V)^{\times i}$ act on $G^{\times i} \times \mathbf{O}_{V^{\oplus i}}$ via this homomorphism on the second factor and via the standard action of $\Sigma_i \ltimes G^{\times i}$ on $G^{\times i}$ on the second factor. Observe that this action is free. In particular $G^{\times i} \times \mathbf{O}_W$ is a free $(\Sigma_i \ltimes P^{\times i}) \times \mathbf{O}_{V'}$-space and, by 1.2.2 it is an $\mathscr{F}$-complex.

The target and source of q_W may be identified as the $\Sigma_i \times \mathbf{O}_{V'}$-orbits of $\Sigma_i \times \mathbf{O}_{V'} \times G \times \mathbf{O}_W$-spaces of the form

$$J \wedge \mathbf{O}_{W+} \wedge_{\mathbf{O}_V^{\times i}} ((G \times \mathbf{O}_V)/P)_+^{\wedge i} \wedge K^{\wedge i} \wedge Y_{V'}$$

where $J = S^0$ for the target and $J = E\Sigma_{i+}$ for the source. This $\Sigma_i \times \mathbf{O}_{V'} \times G \times \mathbf{O}_W$-space is isomorphic to

$$J \wedge G^{\times i} \times \mathbf{O}_{W+} \wedge_{(G\times\mathbf{O}_V)^{\times i}} (G \times \mathbf{O}_V)^{\times i}/P^{\times i}{}_+ \wedge K^{\wedge i} \wedge Y_{V'}$$

and to

$$(J \wedge G^{\times i} \times \mathbf{O}_W)_+ \wedge_{P^{\times i}} K^{\wedge i} \wedge Y_{V'}.$$

Under these isomorphisms

$$j \wedge \alpha \wedge (g_1, \ldots, g_i) \wedge k \wedge y \in (J \wedge G^{\times i} \times \mathbf{O}_W)_+ \wedge_{P^{\times i}} K^{\wedge i} \wedge Y_{V'}$$

corresponds to the element $j \wedge \alpha \wedge g_1P \wedge \cdots \wedge g_iP \wedge k \wedge y$ in

$$J \wedge \mathbf{O}_{W+} \wedge_{\mathbf{O}_V^{\times i}} ((G \times \mathbf{O}_V)/P)_+^{\wedge i} \wedge K^{\wedge i} \wedge Y_{V'}.$$

Using this identification, we can rewrite the source and target of q_W as

$$(J \wedge G^{\times i} \times \mathbf{O}_W)_+ \wedge_{(\Sigma_i \ltimes P^{\times i}) \times \mathbf{O}_{V'}} K^{\wedge i} \wedge Y_{V'}.$$

Since $G^{\times i} \times \mathbf{O}_W$ is an $\mathscr{F}$-complex, the map $E\mathscr{F}_+ \wedge G^{\times i} \times \mathbf{O}_W \to S^0 \wedge G^{\times i} \times \mathbf{O}_W$ is a Γ-homotopy equivalence, and thus q_W is a G-homotopy equivalence.

□

Lemma 4.2.2. *Let $(\mathscr{H},\mathscr{G})$ be a G-mixing pair (2.6.1) with $\mathscr{G}^0$ empty. Suppose that for every W in $\mathbf{O}$ and every P in $\mathscr{G}^W$ and every $n \geq 0$, the orthogonal G-spectrum $Y = S^n_+ \wedge \mathcal{G}_W((G \times \mathbf{O}_W/P)_+)$ has the following property: for every $j \geq 1$, the spectrum $Y^{\wedge j}_{\Sigma_j}$ is $\mathscr{G}$-cofibrant. Then for every $\mathscr{G}$-cofibrant X the quotient map*

$$q : E\Sigma_{i+}\wedge_{\Sigma_i} X^{\wedge i} \to X^{\wedge i}_{\Sigma_i}$$

is a π_*-isomorphism.

Proof. We proceed by induction on i and a G-equivariant cellular filtration of X. For the induction start $i = 1$, the statement is trivially true. Hence let us assume it holds for all $j < i$ and for a $\mathscr{G}$-cofibrant spectrum A such that X is built from A by attaching a single cell $\mathcal{G}_W\left[(S^{n-1} \subseteq D^n)_+\wedge(G \times \mathbf{O}_W/P)_+\right]$. Then, as explained in Remark 3.2.15, Theorem 3.2.14 states that $X^{\wedge i}$ is built from $A^{\wedge i}$ by attaching induced cells of the form

$$\Sigma_{i+}\wedge_{\Sigma_j\times\Sigma_{i-j}} \left(\mathcal{G}_W(S^{n-1} \subseteq D^n)_+\wedge(G \times \mathbf{O}_W/P)_+\right)^{\square j} \wedge A^{\wedge(i-j)},$$

with cofiber $\Sigma_{i+}\wedge_{\Sigma_j\times\Sigma_{i-j}} \left(\mathcal{G}_W S^n_+\wedge(G \times \mathbf{O}_W/P)_+\right)^{\wedge j} \wedge A^{\wedge(i-j)}$. Note that since $\mathscr{G}^0$ is empty we have $W \neq 0$. For ease of notation, we let $Y = S^n_+\wedge\mathcal{G}_W(G \times \mathbf{O}_W/P)_+$. Thus, for the induction step, it suffices to show that the projection

$$E\Sigma_{i+}\wedge_{\Sigma_j\times\Sigma_{i-j}} Y^{\wedge j}\wedge A^{\wedge(i-j)} \to \left[Y^{\wedge j}\wedge A^{\wedge(i-j)}\right]_{\Sigma_j\times\Sigma_{i-j}}$$

is a π_*-isomorphism. We use that $E\Sigma_i$ is $\Sigma_j \times \Sigma_{i-j}$-equivariantly homotopy equivalent to $E\Sigma_j \times E\Sigma_{i-j}$ to factor this projection as:

$$\begin{array}{c} E\Sigma_{j+}\wedge_{\Sigma_j} Y^{\wedge j}\wedge E\Sigma_{i-j+}\wedge_{\Sigma_{i-j}} A^{\wedge i-j} \\ \Big\downarrow \hat{q}\wedge \mathrm{id} \\ Y^{\wedge j}_{\Sigma_j}\wedge E\Sigma_{i-j+}\wedge_{\Sigma_{i-j}} A^{\wedge i-j} \\ \Big\downarrow \mathrm{id}\wedge\hat{q}\wedge \mathrm{id} \\ Y^{\wedge j}_{\Sigma_j}\wedge A^{\wedge i-j}_{\Sigma_{i-j}} \end{array}$$

Here the first map is a level homotopy equivalence by Lemma 4.2.1. For $j > 0$ the second map is a π_*-isomorphism by the induction hypothesis using that $Y^{\wedge j}_{\Sigma_j}$ is $\mathscr{G}$-cofibrant. For $j = 0$ the second map is a π_*-isomorphism by assumption on A. □

Definition 4.2.3. The functor $\mathbb{E}\colon G\mathbf{O}\mathcal{T} \to \mathbb{S}_{\mathrm{cAlg}}$ from orthogonal G-spectra to commutative orthogonal G-ring spectra is left adjoint to the forgetful functor. Explicitly $\mathbb{E}Y = \bigvee_i Y^{\wedge i}_{\Sigma_i}$.

Proposition 4.2.4. *For $V \neq 0$, let $\mathscr{G}^V$ be the family of all closed subgroups of $G \times \mathbf{O}_V$, and let $\mathscr{G}^0$ be empty. The family $(\mathscr{G}^V)_V$ satisfies the assumptions in Lemma 4.2.2. That is, if P is in $\mathscr{G}^V$ for some $0 \neq V \in \mathbf{O}$ and $Y = S^n_+ \wedge \mathcal{G}_V((G \times \mathbf{O}_V/P)_+)$, then the orthogonal spectrum $Y^{\wedge i}_{\Sigma_i}$ is $\mathscr{G}$-cofibrant. As a consequence the inclusion $\mathbb{S} \to \mathbb{E}Y$ is a $\mathscr{G}$-cofibration.*

Proof. Then the Σ_i-orbits of the i-fold smash power of Y are isomorphic to the spectrum

$$\mathcal{G}_{V^{\oplus i}}(((S^n)^{\times i} \times G^{\times i} \times \mathbf{O}_{V^{\oplus i}}/\Sigma_i \ltimes P^{\times i})_+),$$

where $G \times \mathbf{O}_{V^{\oplus i}}$ acts on $((S^n)^{\times i} G^{\times i} \times \mathbf{O}_{V^{\oplus i}}/\Sigma_i \ltimes P^{\times i})$ through the diagonal inclusion $G \to G^{\times i}$ and multiplication on the left with $P^{\times i}$ acting trivially on $(S^n)^{\times i}$. By Illman's Theorem 1.2.2 the manifold $(S^n)^{\times i} \times G^{\times i} \times \mathbf{O}_{V^{\oplus i}}$ is a $G \times \mathbf{O}_{V^{\oplus i}} \times \Sigma_i \ltimes P^{\times i}$-CW-complex, and thus the orbit space $(G^{\times i} \times \mathbf{O}_{V^{\oplus i}}/\Sigma_i \ltimes P^{\times i})$ is a $G \times \mathbf{O}_{V^{\oplus i}}$-CW-complex.

□

Corollary 4.2.5. *Let P be a subgroup of $G \times \mathbf{O}_V$ for some $0 \neq V \in \mathbf{O}$ and let $Y = K_+ \wedge \mathcal{G}_V((G \times \mathbf{O}_V/P)_+)$ for some G-manifold K. Then the functor $\mathbb{E}Y \wedge (-)$ on orthogonal spectra preserves stable equivalences.*

Proof. Given a stable equivalence of orthogonal spectra $f : Z \to Z'$ it is enough to show that each summand $Y_{\Sigma_i} \wedge f$ is a weak equivalence. By Lemma 4.2.1 it is enough to show that $E\Sigma_{i+} \wedge_{\Sigma_i} (Y^{\wedge i} \wedge f)$ is a stable equivalence. Since $E\Sigma_i$ is a free Σ_i-cell complex it is enough to show that $Y^{\wedge i} \wedge f$ is a stable equivalence, which is a consequence of 2.10.2. □

Corollary 4.2.6. *The functor $\mathbb{E}$ preserves stable equivalences between $\mathbb{S}$-cofibrant orthogonal spectra. In particular, each map in $\mathbb{E}\mathbb{S}J$ is a stable equivalence.*

Proof. Iterated use of the pushout product axiom for the $\mathbb{S}$-model structure 2.9.9 implies that the i-fold smash power of an acyclic cofibration between $\mathbb{S}$-cofibrant spectra is an acyclic cofibration. Both $E\Sigma_{i+} \wedge_{\Sigma_i} (-)$ and taking wedges preserve π_*-isomorphisms, that is, stable equivalences. Hence Ken Brown's Lemma gives the result. □

We will need to calculate realizations of simplicial objects and some other specific colimits in the category of commutative orthogonal ring spectra. Recall Proposition 4.3.4 and the following lemma from [MMSS], that allow us to do so in the underlying category of spectra:

Lemma 4.2.7 ([MMSS, 15.11])**.** *Let $\{R_i \to R_{i+1}\}$ be a sequence of maps of commutative orthogonal ring G-spectra that are h-cofibrations of orthogonal G-spectra. Then the underlying orthogonal G-spectrum of the colimit of the sequence in commutative orthogonal ring G-spectra is the colimit of the sequence computed in the category of orthogonal G-spectra.*

The following Proposition is inspired by Lemma 15.9 in [MMSS]. It deals with the other part of the cofibration hypothesis for $\mathbb{E}\mathbb{S}I$ and brings us closer to the convenience property 4.2.15.

Proposition 4.2.8. *Let $f \colon X \to Y$ be a wedge of maps in $\mathbb{S}I$ and let $\mathbb{E}X \to R$ be a map of commutative orthogonal ring spectra. Then, considered as a map in $G\mathbf{O}\mathcal{T}$, the cobase change $j \colon R \to R \wedge_{\mathbb{E}X} \mathbb{E}Y$ is an h-cofibration. If smashing with R additionally preserves $\mathbb{S}$-cofibrations, j is even an $\mathbb{S}$-cofibration.*

Proof. As in the proof of [MMSS, 15.9], we identify the inclusion $S^n_+ \to D^{n+1}_+$ of spaces with the realization of the inclusion of the S^n_+ summand in the 0-simplices of the two-sided bar construction $B_*(S^n_+, S^n_+, S^0)$ of the monoid S^n_+ with respect to the monoidal structure given by wedge sum: $S^n_+ \vee S^n_+ \to S^n_+$ (cf. 6.1.36). Its q simplices are given by a wedge of $q+1$ copies of S^n_+ with S^0, with degeneracy maps the inclusions of wedge summands and face maps induced from folding maps $S^n_+ \vee S^n_+ \to S^n_+$, respectively the collapse map $S^n_+ \vee S^0 \to S^0$ for the last face in each simplicial level. Note that both the smash product with an $\mathbf{O}_V$-orbit and the semi-free functors $\mathcal{G}_V$ preserve colimits and tensors, hence the simplicial realization. Thus we can express f analogously as the inclusion a summand of the 0-simplices of the simplicial orthogonal spectrum $B_*(X, X, T)$, where $X = \bigvee_i \mathcal{G}_{V_i} S^{n_i}_+ \wedge (G \times \mathbf{O}_{V_i}/P_i)_+$ and $T = \bigvee_i \mathcal{G}_{V_i} S^0 \wedge (\mathbf{O}_{G\times V_i}/P_i)_+$. Applying $\mathbb{E}$ takes coproducts to smash products, fold maps to multiplication maps, and inclusions of the basepoints to unit maps of commutative orthogonal ring spectra. It also preserves tensors over $\mathcal{U}$, i.e., sends $X \wedge A_+$ to $X \otimes A$, hence it sends the realization of $B_*(X, X, T)$ to the realization of the bar construction $\mathbf{B}_*(\mathbb{E}X, \mathbb{E}X, \mathbb{E}T)$. Finally, since we can compute geometric realizations in terms of the underlying spectra (4.3.4), and since smashing with R commutes with this realization, we can identify

$$R \wedge_{\mathbb{E}X} \mathbb{E}Y \cong R \wedge_{\mathbb{E}X} \mathbf{B}(\mathbb{E}X, \mathbb{E}X, \mathbb{E}T) \cong \mathbf{B}(R, \mathbb{E}X, \mathbb{E}T). \tag{4.2.9}$$

We look at $\mathbf{B}_*(R, \mathbb{E}X, \mathbb{E}T)$ in more detail: R includes into the 0-simplices $R \wedge \mathbb{E}T$ as a wedge summand, i.e., via an h-cofibration. All the other wedge summands are of the form

$$R \wedge \left(\mathcal{G}_{V_i} S^0 \wedge (G \times \mathbf{O}_{V_i}/P_i)_+\right)^{\wedge k}_{\Sigma_k},$$

hence they are $\mathbb{S}$-cofibrant if smashing with R preserves $\mathbb{S}$-cofibrations by 4.2.4. Then in particular $R \wedge \mathbb{E}T$ is $\mathbb{S}$-cofibrant and the inclusion of R is an $\mathbb{S}$-cofibration.
The degeneracy maps are given by inclusions

$$R \wedge (\mathbb{E}X)^{\wedge q} \wedge \mathbb{E}T = R \wedge (\mathbb{E}X)^{\wedge r} \wedge \mathbb{S} \wedge (\mathbb{E}X)^{\wedge q-r} \wedge \mathbb{E}T \longrightarrow R \wedge (\mathbb{E}X)^{\wedge q+1} \wedge \mathbb{E}T.$$

Therefore the inclusion of degenerate simplices (6.1.42) is in each level q given by the map

$$R \wedge (\mathbb{S} \to \mathbb{E}X)^{\square q+1} \wedge \mathbb{E}T,$$

which is an h-cofibration because $\mathbb{S} \to \mathbb{E}X$ is an inclusion of a wedge summand. Furthermore 4.2.4 states that $\mathbb{S} \to \mathbb{E}X$ is an $\mathbb{S}$-cofibration. Hence by the pushout product axiom, the inclusion of degenerate simplices is an $\mathbb{S}$-cofibration if smashing with R preserves $\mathbb{S}$-cofibrations. Hence the bar construction is h-proper, and even $\mathbb{S}$-proper for the stronger assumption on R. The result then follows using Proposition 6.1.46. □

Remark 4.2.10. Note that the sphere spectrum $\mathbb{S}$ is an important specific example of a ring spectrum that preserves $\mathbb{S}$-cofibrations under the smash product.

Corollary 4.2.11 ([MMSS, 15.9]). *The set $\mathbb{E}\mathbb{S}I$ of maps of commutative orthogonal ring spectra satisfies the cofibration hypothesis 6.1.30. Since it consists of $\mathbb{E}\mathbb{S}I$-cell complexes, so does $\mathbb{E}\mathbb{S}J$.*

Lemma 4.2.12 ([MMSS, 15.12]). *Let $i\colon R \to R'$ be an $\mathbb{S}$-cofibration of commutative orthogonal ring spectra. Then the functor $(-)\wedge_R R'$ on commutative R-algebras preserves stable equivalences.*

Proof. Assume inductively that i is a cobase change of a wedge of maps in $\mathbb{ES}I$. Then as in 4.2.9 we can identify $(-)\wedge_R R'$ with an appropriate $\mathbf{B}(-, \mathbb{E}X, \mathbb{E}T)$. This functor preserves stable equivalences by 6.1.47, since the bar construction is h-proper. □

Finally we get the analog of [MMSS, 15.4], using the same proof as in the classical case (cf. [MMSS, p. 490]):

Proposition 4.2.13. *Every relative $\mathbb{ES}J$-cell complex is a stable equivalence.*

This once more allows us to use Lemma [SS, 2.3], and we obtain the $\mathbb{S}$-model structure for commutative orthogonal ring spectra:

Theorem 4.2.14. *The underlying $\mathbb{S}$-fibrations and stable equivalences give a compactly generated proper topological model structure on the category of commutative orthogonal ring G-spectra. The generating (acyclic) cofibrations are given by the sets $\mathbb{ES}I$ and $\mathbb{ES}J$, respectively.*
Again the identity functor gives a Quillen equivalence to the classical model structure from [MMSS, 15.1].

We call cofibrant objects in this model structure simply $\mathbb{S}$-cofibrant, inspired by the following Theorem, which is implied by the second statement of Proposition 4.2.8, and provides the main motivation for the constructions in this section:

Theorem 4.2.15. *The $\mathbb{S}$-model structure on commutative orthogonal ring spectra is "convenient" in the sense that if A is a commutative orthogonal ring G-spectrum that is $\mathbb{S}$-cofibrant, it is already $\mathbb{S}$-cofibrant as an orthogonal G-spectrum.*

Even slightly more is true:

Theorem 4.2.16. *Let $f\colon R \to R'$ be a map of commutative orthogonal ring G-spectra, that is a cofibration in the model structure of Theorem 4.2.14. If the smash product with R preserves $\mathbb{S}$-cofibrations of orthogonal G-spectra, then f is an underlying $\mathbb{S}$-cofibration.*

Proof. Reduce to the case of a $\mathbb{ES}I$-cell complex. Induction on the cellular filtration and the stronger second statement of Proposition 4.2.8 give the result. □

Theorem 4.2.17. *For a commutative orthogonal G-ring spectrum R, the $\mathbb{S}$-model structure induces a compactly generated proper topological model structure on commutative R-algebras. This R-model structure is convenient with respect to the R-model structure on R-modules from Theorem 2.10.4(i).*
The identity functor on commutative R-algebras induces a Quillen equivalence to the classical model structure one would get by applying [DS, 3.10] to the structure from [MM, III.8.1].

Proof. We can use [DS, 3.10]. An analog of Theorem 4.2.15 is then immediate, since the free commutative R-algebra functor $\mathbb{E}_R$ satisfies $\mathbb{E}_R(-) \cong R \wedge \mathbb{E}(-)$, and thus any cofibration of commutative R-algebras is an underlying R-cofibration. □

4.3 Tensors in Commutative Orthogonal Ring Spectra

The functor category $G\mathbf{O}\mathcal{T}$ is tensored and cotensored over $G\mathcal{T}$. Since the functor adding a disjoint basepoint is monoidal, it is also tensored and cotensored over the category $G\mathcal{U}$. That the associated categories of modules, algebras and commutative algebras, i.e., orthogonal ring spectra, commutative ring spectra as well as modules and algebras over such are topological, is an entirely categorical argument given in [EKMM, VII 2.10] and [MMSS, 5.1]. Using [EKMM, II.7.2] and [MMSS, 5.2] this can be used to show the following:

Proposition 4.3.1. *Let R be an orthogonal ring spectrum and let G be a compact Lie group.*

(i) The category of R-modules with action of G is topologically bicomplete with limits, colimits, tensors and cotensors calculated in the category $G\mathbf{O}\mathcal{T}$.

(ii) If R is commutative (e.g., $R = \mathbb{S}$), then the category of (commutative) R-algebras with action of G is topologically bicomplete with limits and cotensors created in R-modules with action of G.

For the case of tensors, the result that will be most useful to us is the analog to [EKMM, VII.3.4]. In the following let always R denote a commutative orthogonal ring spectrum as above, and let $\mathcal{C}$ denote either one of the categories of G-objects in R-modules, R-algebras or commutative R-algebras. We refer to 6.1.37 for the notion of geometric realization.

Theorem 4.3.2. *Let A be an object of $\mathcal{C}$ and X_* a simplicial space. There is a natural isomorphism*

$$A \otimes_{\mathcal{C}} |X_*| \cong |A \otimes_{\mathcal{C}} X_*|$$

in $\mathcal{C}$. Here the realization on the right side is to be understood in the category of orthogonal spectra.

This is a consequence of the following two results, which are the orthogonal spectrum versions of [EKMM, VII.3.2, VII.3.3]:

Proposition 4.3.3. *[EKMM, VII.3.2] Let A be an object of $\mathcal{C}$ and X_* a simplicial space. There is a natural isomorphism*

$$A \otimes_{\mathcal{C}} |X_*| \cong |A \otimes_{\mathcal{C}} X_*|_{\mathcal{C}},$$

where the realization on the right side is in $\mathcal{C}$.

Proposition 4.3.4. *[EKMM, VII.3.3] Let A_* be a simplicial object of $\mathcal{C}$, then we have a natural isomorphism*

$$|A_*|_{\mathcal{C}} \cong |A_*|.$$

Proof. We should first describe how the realization $|A_*| = |A_*|_{G\mathbf{O}\mathcal{T}}$ is an object of $\mathcal{C}$ again. We treat the case of commutative orthogonal ring spectra over $R = \mathbb{S}$, all the others are similar. For the unit morphism $\mathbb{S} \to |A_*|$ view $\mathbb{S}$ as $|\mathbb{S}_*|$, the realization of the constant simplicial spectrum, and use that since the simplicial structure maps of A_* are ring maps, the collection of unit maps of the A_q gives a map of simplicial ring spectra $\mathbb{S}_* \to A_*$ which induces a map on realizations. For the multiplication, first recall that the geometric realization is defined as a coend (6.1.37). Since coends and tensors in orthogonal spectra are defined levelwise, we have a natural isomorphism $|A_*|_V \cong |(A_*)_V|$ in each level V. Recall the coend definition of the smash product of orthogonal spectra and the fact that for simplicial spaces the realization commutes with both the smash product and the inducing up functor. The Fubini theorem for coends ([McL, IX.8]) then implies a natural isomorphism of spectra

$$|A_*|\wedge|A_*| \cong |A_*\wedge A_*|.$$

Here the latter smash product is calculated separately in each simplicial level. Hence the multiplication maps of the A_q induce multiplication maps on the realization. It is tedious, but not too hard to verify the associativity, unitality, commutativity and coherence conditions, and we omit more details here.
Going back to the proof, we continue as in case of (commutative) $\mathbb{S}$-algebras in the EKMM setting. For (commutative) orthogonal G-ring spectra A and B and a G-space X we claim that a morphism $A\otimes_{\mathcal{C}}X \to B$ of G-ring spectra determines and is determined by a morphism of G-spectra

$$A \otimes X = A\wedge X_+ \to B,$$

such that for all points $x \in X$ the map

$$A \cong A\wedge S^0 \xrightarrow{A\wedge i_x} A\wedge X_+ \longrightarrow B$$

is a map of G-ring spectra. To see this, let $\mathcal{C}_G$ be the $G\mathcal{T}$-category with G acting by conjugation on morphism spaces, and take a look at the defining adjunctions

$$\mathcal{C}(A \otimes_{\mathcal{C}} X, B) \cong G\mathcal{U}(X, \mathcal{C}_G(A,B)) \longrightarrow G\mathcal{U}(X, \mathbf{O}\mathcal{T}_G(A,B)) \cong G\mathbf{O}\mathcal{T}(A\wedge X_+, B),$$

where the middle arrow is induced by the faithful (!) forgetful functor $\mathcal{C}_G \to \mathbf{O}\mathcal{T}_G$. Note that this is the point where the enrichment over $G\mathcal{U}$ instead of $G\mathcal{T}$ proves handy, since otherwise we would have to be very careful with trivial maps here.
Given a simplicial object A_* in $\mathcal{C}$ a morphism $\hat{g} \in \mathcal{C}(|A_*|, B)$ is completely determined by a morphism of spectra $\hat{f} \in G\mathbf{O}\mathcal{T}(|A_*|, B)$, which by 6.1.38 is adjoint to a morphism of simplicial spectra $f \in sG\mathbf{O}\mathcal{T}(A_*, \Delta_{\mathbf{O}\mathcal{T}}B)$. For each simplicial level q, the morphism f_q is adjoint to a morphism of G-spectra $\hat{f}_q\colon A_q \otimes_{\mathbf{O}\mathcal{T}} \Delta^q \to B$ that is pointwise a morphism of ring spectra. As we saw above, these exactly correspond to algebra morphisms $\hat{h}_q\colon A_q \otimes_{\mathcal{C}} \Delta^q \to B$, whose adjoints fit together into a map of simplicial ring spectra $h \in s\mathcal{C}(A, \Delta_{\mathcal{C}}B)$. Altogether we defined a natural isomorphism

$$\mathcal{C}(|A_*|, B) \cong s\mathcal{C}(A_*, \Delta_{\mathcal{C}}B),$$

i.e., we showed that $|A_*|$ and $|A_*|_{\mathcal{C}}$ have the same right adjoint. This proves the proposition. □

The immense usefulness of Theorem 4.3.2 stems from the fact that, for discrete spaces X, tensors $A \otimes_{\mathcal{C}} X$ are easily computable in any topologically enriched category:

Proposition 4.3.5. *Let $\mathcal{D}$ be a category which is enriched and tensored over $G\mathcal{U}$. Let A be an object of $\mathcal{D}$ and X a discrete space in $G\mathcal{U}$. Then there is a continuous natural isomorphism*

$$A \otimes X \cong \coprod_X A.$$

Proof. We use the defining universal properties of tensors and coproducts. Let B be some object of $\mathcal{D}$, we have natural isomorphisms

$$\mathcal{D}(A \otimes X, B) \cong G\mathcal{U}(X, \mathcal{D}(A, B)) \cong \prod_{x \in X} \mathcal{D}(A, B) \cong \mathcal{D}(\coprod_X A, B).$$

Hence the topological Yoneda lemma gives the desired continuous natural isomorphism. □

In the case we are most interested in, using Lemma 5.1.17, we get

Corollary 4.3.6. *Let A be a commutative orthogonal ring spectrum, X a discrete space in $G\mathcal{U}$. There is a (continuous) natural isomorphism*

$$A \otimes X \cong \bigwedge_X A,$$

between the tensor of A with X and the X-fold smash power of A, i.e., the X-fold smash product of A with itself.

This is the main point of motivation for the translation from the algebraic case which we present in the next subsection.

4.3.7 The Loday Functor

Remember from 5.1.17 that the coproduct in the category of commutative monoids in a symmetric monoidal category $(\mathcal{C}, \otimes, e)$ is nothing but $\otimes$ itself. In particular, with the considerations in 5.1.10 we get

Corollary 4.3.8. *Let $(\mathcal{C}, \otimes, e)$ be a symmetric monoidal category. Then the category of commutative monoids in $\mathcal{C}$ is tensored over the category of finite sets, sending a commutative monoid M and a finite set S to $\bigotimes_S M = M^{\otimes S}$.*

As discussed in 5.1.10, $M^{\otimes\{1,\dots,n\}} = (\dots(M \otimes M) \otimes \dots) \otimes M$, while the functoriality in S given by sending a function of finite sets $f\colon S \to T$ to the composite

$$\bigotimes_S M \cong \bigotimes_{t \in T} \bigotimes_{f^{-1}(t)} M \to \bigotimes_T M,$$

where the isomorphism is a (uniquely given by f) structure isomorphism in $(\mathcal{C}, \otimes, e)$ and the morphism is the tensor of the maps $\bigotimes_{f^{-1}(t)} M \to M$ induced by multiplication (the unit if $f^{-1}(t)$ is empty).

In particular, if R is a commutative orthogonal ring spectrum (i.e., a commutative $\mathbb{S}$-algebra in $\mathbf{O}\mathcal{T}$), A a commutative R-algebra, and X a finite set, considered as a discrete space. Then Corollary 4.3.8 gives a natural isomorphism

$$A \otimes_{R_{\mathrm{CAlg}}} X \cong A \wedge_R \ldots \wedge_R A$$

($|X|$-fold smash). Recall from 5.1.10 that the right hand side is functorial in bijections even if A is only an R-module, and in injections if A is an R-module under R.

Since tensors commute with colimits we get that if X is a set considered as a discrete space, then $A \otimes_{R_{\mathrm{CAlg}}} X$ is the filtered colimit of $S \mapsto A \otimes_{R_{\mathrm{CAlg}}} S$, where S varies over the finite subsets of X. A priori, this colimit is in R_{CAlg}, but is created in R_{Mod}. Note that, since the colimit is over inclusions, this makes no use of the multiplication, and applies to any R-mudule under R.

Definition 4.3.9. The *discrete Loday functor* applied to an R-module M under R and set X is given by

$$\bigwedge_X^R M = \operatorname*{colim}_{U \subseteq X} \bigwedge_U M$$

(functorial in injections). The *Loday functor* applied to a commutative R-algebra A and space X is given by

$$\bigwedge_X^R A = A \otimes_{R_{\mathrm{CAlg}}} X.$$

As commented above, if A is a commutative R-algebra and X a set, then the two potential interpretations of the discrete Loday functor as modules under R are naturally isomorphic.

Finally note that Proposition 4.3.2 implies the following lemma, which makes it possible to realize $\bigwedge_{|X|}^R A$ concretely as a "smash power indexed over X" for simplicial sets X:

Lemma 4.3.10. *Given a simplicial space Y and A a commutative R-algebra, there is a natural isomorphism*

$$\bigwedge_{|Y|}^R (A) \cong |\bigwedge_Y^R (A)|,$$

where the realization on the right is in orthogonal spectra.

We will mostly be interested in the case where R is the sphere spectrum $\mathbb{S}$, and for simplicity of notation we restrict ourselves to that case in the following and omit R from the notation.

A similar construction has been carried out in in [BCD, Section 4]. The construction we give here is much simpler than the one presented in op. cit., so it will be crucial to study its properties in detail, so as to make sure, that we indeed capture all the desired information. In particular the equivariant homotopy type, when X is equipped with an action of some (compact Lie-) group G, will require some care. We admit that the choice in [BCD] of Γ-spaces as a model for connective spectra was not optimal, but with very minor rewriting the paper makes sense when based on symmetric spectra or simplicial functors. This rewriting is necessary for applying [BCD] to commutative ring spectra that do not have strictly commutative models in Γ-spaces [Law].

Remark 4.3.11. For a morphism $\varphi : R \to S$ of commutative orthogonal ring spectra, we get adjoint pairs of functors between the categories of R-, respectively S-modules, -algebras and commutative S-algebras. All the left adjoints are given by *induction*, i.e., using $S \wedge_R (-)$, which is strong monoidal and preserves tensors, hence commutes with all versions of the Loday functor from above. The respective right adjoint functors do in general not exhibit similar properties.

Note that since all group actions are through isomorphisms, a (continuous) action of a (topological) group G on X induces (continuous) actions on the targets of the Loday functor by precomposition as follows

$$G \to \mathcal{A}(X, X) \to \mathcal{C}(\bigwedge_X^R A, \bigwedge_X^R A),$$

where ($\mathcal{A}$, $\mathcal{C}$) is any of the pairs of categories for which we defined the Loday functor above and $A \in \mathcal{C}$. In this light, we can for each of the above definitions and any (topological) group G consider equivariant analogs

$$\bigwedge_{(-)}^R (-) \colon [G, \mathcal{A}] \times \mathcal{C} \to [G, \mathcal{C}].$$

As already discussed in the Introduction, it is crucial to investigate the equivariant properties of these Loday functors, to make sure that they are usable for our applications. The next sections will be devoted to this topic.

4.4 Fixed Points of Smash Powers

4.4.1 Fixed Points and the Loday Functor

The following proposition gives the important naturality property, which we will need when generalizing the above result to infinite and non-discrete spaces. It is inspired by Kro's proposed "diagonal map" $L^q \to \Phi^{C_r} L^{rq}$ in the case of finite cyclic groups. The definition given in [Kr, 3.10.4], however, seems to mix up the left and right adjoints involved. We will instead define a natural zig-zag of maps, where the arrow in the wrong direction becomes an isomorphism for $\mathbb{S}$-cofibrant input.
In this section we work with the G-typical family $\mathscr{H}$ of all representations of subgroups of G. Recall the categories $\mathcal{O}_J$ and $\mathcal{O}_E^{\mathscr{H}}$ from Section 3.1.20, and let $\mathcal{O}_G$ be the $G\mathcal{T}$-category of G-objects in $\mathbf{O}$. In this section we write $\mathcal{O}_E$ instead of $\mathcal{O}_E^{\mathscr{H}}$ since $\mathscr{H}$ is fixed. Before we give the construction, let $\mathcal{O}_G^{\mathrm{reg}} \subset \mathcal{O}_G$ and $\mathcal{O}_J^{\mathrm{reg}} \subset \mathcal{O}_J$ be the full subcategories of objects of the form $V^{\oplus X}$ and $V^{\oplus X_N}$ respectively for V an object of $\mathbf{O}$.

Let $\mathcal{O}_E^{\mathrm{reg}}$ be the N-fixed category of $\mathcal{O}_G^{\mathrm{reg}}$, and note that the following diagram of functors commutes:

$$\begin{array}{ccc} \mathcal{O}_E & \xleftarrow{\;i\;} & \mathcal{O}_E^{\mathrm{reg}} \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi^{\mathrm{reg}}} \\ \mathcal{O}_J & \xleftarrow[\;j\;]{} & \mathcal{O}_J^{\mathrm{reg}}, \end{array}$$

where ϕ^{reg} is the restriction of ϕ sending a regular representation $V^{\oplus X}$ to $V^{\oplus N_N} \cong (V^{\oplus X})^N$. The diagram then implies the following natural isomorphisms for the restriction functors and their left adjoints:

$$\mathbb{P}_j \mathbb{P}_\phi^{\mathrm{reg}} \cong \mathbb{P}_\phi \mathbb{P}_i \qquad \mathbb{U}_\phi^{\mathrm{reg}} \mathbb{U}_j \cong \mathbb{U}_i \mathbb{U}_\phi. \tag{4.4.2}$$

Let Fix^N be the functor from the category of orthogonal G-spectra to the category of $J\mathcal{T}$-functors from $\mathcal{O}_E$ to $J\mathcal{T}$ taking a G-spectrum X to $V \mapsto (\mathrm{Fix}^N X)_V = (X_V)^N$. Then $\mathbb{P}_\phi \mathrm{Fix}^N$ is isomorphic to the geometric fixed point functor $\Phi^N = \Phi^N_{\mathscr{H}}$ for the G-typical family $\mathscr{H}$ consisting of all G-representations.

Proposition 4.4.3. *For G a finite group and X a finite free G-space, and L any orthogonal spectrum there is a natural* diagonal zig-zag *of J-spectra*

$$L^{\wedge X_N} \xleftarrow{\;\varepsilon_j\;} \mathbb{P}_j \mathbb{U}_j L^{\wedge X_N} \longrightarrow \mathbb{P}_\phi \mathbb{P}_i \mathbb{U}_i \mathrm{Fix}^N(L^{\wedge X}) \xrightarrow{\mathbb{P}_\phi(\varepsilon_i)} \Phi^N(L^{\wedge X})\,.$$

For $\mathbb{S}$-cofibrant spectra L, the first map is an isomorphism such that the pointed composite $\Delta(X,L)$ exists and is the natural isomorphism from 3.2.16, which we therefore call the diagonal isomorphism*.*

Proof. The maps ε_j and ε_i are the counits of the adjoint pair $(\mathbb{P}_j, \mathbb{U}_j)$, respectively $(\mathbb{P}_i, \mathbb{U}_i)$ and are hence natural. The second map requires more work. First note that by (4.4.2) its target is naturally isomorphic to $\mathbb{P}_j\mathbb{P}_\phi^{\mathrm{reg}}\mathbb{U}_i\mathrm{Fix}^N(L^{\wedge X})$, so it suffices to define a natural map of regular J-spectra

$$\mathbb{U}_j L^{\wedge X_N} \to \mathbb{P}_\phi^{\mathrm{reg}}\mathbb{U}_i\mathrm{Fix}^N(L^{\wedge X}),$$

to which we then apply $\mathbb{P}_j$. We define the map levelwise, so let $U = V^{\oplus X_N}$ be a regular J-representation. By the coend definition of the smash products $L^{\wedge X}$ and $L^{\wedge X_N}$, as well as the evaluation of $\mathbb{P}_\phi^{\mathrm{reg}}$, it suffices to give morphisms

$$\bigwedge_{\bigoplus V_{[x]}=U} L_{V_{[x]}} \longrightarrow \int^{W\in\mathcal{O}_E^{\mathrm{reg}}} \mathcal{O}_J^{\mathrm{reg}}(W^N, U^N) \wedge \left[\int^{\bigoplus W_x\cong W} \mathbf{O}_W \wedge_{\prod_X \mathbf{O}_{W_x}} \bigwedge_{x\in X} L_{W_x}\right]^N. \tag{4.4.4}$$

But since U is regular, $W = V^{\oplus X}$ gives a preferred point in the first coend, and for each partition $\bigoplus V_{[x]} \cong U$ the choice $W_x = V_{[x]}$ gives a preferred point in the second coend, so that we can map $\bigwedge_{[x]\in X_N} L_{V_{[x]}}$ to the copy of $\bigwedge_{x\in X} L_{V_{[x]}}$ indexed by the identities of W^N and W, via the diagonal map

$$\begin{array}{ccc} \bigwedge_{[x]\in X_N} L_{V_{[x]}} & \longrightarrow & \bigwedge_{x\in X} L_{W_x}. \\ \{l_{[x]} \in L_{V_{[x]}}\}_{[x]\in X_N} & \mapsto & \{l_{[x]} \in L_{W_x}\})_{x\in X} \end{array}$$

Note that the map is obviously J-equivariant and maps into the coend of the fixed points, hence into the fixed points of the coend. It is compatible with the structure maps of L and natural with respect to all maps $L \to K$ of orthogonal spectra.
To see that the instance of ε_j is an isomorphism for $\mathbb{S}$-cofibrant L, note smash-powers of semi-free orthogonal spectra are semi-free J-spectra of the form $\mathcal{G}_K^V$ with V regular and that ε_j is an isomorphism for any semi-free J-spectrum of this form. By Theorem 3.2.14, a cell induction then gives the result. Similarly it suffices to show that for semi-free spectra the zig-zag yields the isomorphism from 3.1.34. For R a euclidean vector space and $L = \mathcal{G}_R K$, the maps (4.4.4) are represented by the map of $\prod_{X_N} \mathbf{O}_R$-spaces out of $K^{\wedge X_N}$ determined on the level $U = R^{\wedge X_N}$. Since this map is adjoint to the unit map

$$\eta_\phi \colon \mathcal{G}_U^E \left(\mathbf{O}_{R^{\wedge X}} \wedge_{\prod_X \mathbf{O}_R} K^{\wedge X}\right)^N \to \mathbb{U}_\phi\mathbb{P}_\phi\mathcal{G}_U^E \left(\mathbf{O}_{R^{\wedge X}} \wedge_{\prod_X \mathbf{O}_R} K^{\wedge X}\right)^N,$$

the result is implied by the Yoneda lemma and our description of the isomorphism (3.1.34). □

Corollary 4.4.5. *Let L and L' be $\mathbb{S}$-cofibrant orthogonal spectra, then the natural map α from 3.1.28 is an isomorphism*

$$\Phi^N(\Lambda_X L) \wedge \Phi^N(\Lambda_X L') \cong \Phi^N(\Lambda_X L \wedge \Lambda_X L').$$

Proof. Since $\Lambda_X L \wedge \Lambda_X L' \cong \Lambda_X(L \wedge L')$ via a G-equivariant shuffle permutation the map α is up to isomorphism the map

$$L^{\wedge X_N} \wedge L'^{\wedge X_N} \cong (L \wedge L')^{\wedge X_N}.$$

□

Corollary 4.4.6. *Let L be an $\mathbb{S}$-cofibrant orthogonal spectrum and let B be an $\mathrm{Ind}^{\mathrm{reg}}$-cellular G-spectrum, then the natural map α from 3.2.8 is an isomorphism*

$$\Phi^N(\Lambda_X L) \wedge \Phi^N(B) \cong \Phi^N(\Lambda_X L \wedge B).$$

Proof. First assume B is itself of the form $B \cong G_+ \wedge_H (L')^{\wedge H}$. If N is not contained in H, both source and target of α are trivial. Otherwise we have

$$L^{\wedge X} \wedge B \cong G_+ \wedge_H (L^{\wedge X} \wedge L'^{\wedge H})$$

by 3.1.36. Hence Theorem 3.2.8 gives that α is J-isomorphic to a map

$$L^{X_N} \wedge G/N_+ \wedge_{H/N} L'^{\wedge N} \cong G/N_+ \wedge_{H/N} (L^{X_N} \wedge L'^{N}),$$

which is another instance of 3.1.36 The general result follows by a cell induction over the cells of B. □

Remark 4.4.7. Since the Loday functor for mere spectra is in the X-variable only defined with respect to finite sets and isomorphisms between them, the best thing one can hope for is that the diagonal zig-zag of 4.4.3 is natural in the X variable with respect to isomorphisms of finite free G-sets. Then indeed, comparing the appropriate shuffle permutations through which G and J act, naturality in X is immediate for semi-free spectra, and can be followed through the coends in the proof of 4.4.3 with little more effort.

Corollary 4.4.8. *The diagonal isomorphism respects decompositions of X into G-orbits, i.e., for X a finite free G-set and L an $\mathbb{S}$-cofibrant orthogonal spectrum, we get a commutative diagram of natural isomorphisms*

$$\begin{array}{ccccc}
(L^{\wedge G/N})^{\wedge X_G} & \xleftarrow{\cong} & L^{\wedge X_N} & \xrightarrow{\cong} & (L^{\wedge X_G})^{\wedge G/N} \\
\downarrow{\scriptstyle \Delta(G,L)^{\wedge X_G}} & & & & \\
(\Phi^N L^{\wedge G})^{\wedge X_G} & & \downarrow{\scriptstyle \Delta(X,L)} & & \downarrow{\scriptstyle \Delta(G,L^{\wedge X_G})} \\
\downarrow{\scriptstyle \alpha} & & & & \\
\Phi^N((L^{\wedge G})^{\wedge X_G}) & \xleftarrow{\cong} & \Phi^N L^{\wedge X} & \xrightarrow{\cong} & \Phi^N((L^{\wedge X_G})^{\wedge G})
\end{array}$$

with the lower left vertical map an iterated version of the isomorphism α from 3.1.28.

Proof. Note that for L $\mathbb{S}$-cofibrant, a map out of $L^{\wedge X_N}$ is completely determined by its values on X_N-regular levels by the universal properties of the semi-free spectra appearing in the cell decomposition of Theorem 3.2.14. On these levels the right rectangle commutes by definition of the diagonal zig-zag. For the left rectangle, comparing the universal property defining α from 3.1.28 to the definition of the middle map in the diagonal zig-zag 4.4.3 gives the result. $\square$

Varying the Input Categories

In 4.3.7, we have defined several versions of the Loday functor, with the input categories varying between finite discrete sets together with general orthogonal spectra, to general spaces and commutative orthogonal ring spectra. In the above discussion we have restricted the first input further, to finite free G-sets, to study the equivariant structure induced on the output. Inspired by Remark 4.4.7, we will from now view the diagonal isomorphism as a natural transformation of functors

$$\Delta(\cdot,-)\colon \quad \Lambda_{(\cdot)_N}(-) \quad\longrightarrow\quad \Phi^N(\Lambda_{(\cdot)}(-)),$$

and study how we can vary the input categories.

Suppose that G is a finite group. We begin with checking naturality of the diagonal isomorphism with respect to injections of finite free G-sets. As usual, we have to adapt the cofibrancy condition.

Definition 4.4.9. An orthogonal spectrum L is $\mathbb{S}$-*cofibrant under* $\mathbb{S}$ if it is equipped with a $\mathbb{S}$-cofibration $\mathbb{S} \to L$.

Example 4.4.10. Every $\mathbb{S}$-cofibrant commutative orthogonal ring spectrum is $\mathbb{S}$-cofibrant under $\mathbb{S}$ via its unit map by 4.2.8, which in particular says that the unit maps for the commutative ring spectra appearing in the generating $\mathbb{S}$-cofibrations is an $\mathbb{S}$-cofibration.

Lemma 4.4.11. *The diagonal map is natural with respect to finite free G-sets and equivariant inclusions, and spectra $\mathbb{S}$-cofibrant under $\mathbb{S}$.*

Proof. Let L be cofibrant under $\mathbb{S}$, and let $X \to Y$ be an equivariant inclusion of finite free G-sets. There is an $\mathbb{S}I$-cellular structure for L such that the designated map $\mathbb{S} \to L$ is an inclusion of a subcomplex. Then $L^{\wedge X} \to L^{\wedge Y}$ is an inclusion of an equivariant subcomplex as in 3.2.14. Similarly $L^{\wedge X_N}$ is an equivariant subcomplex of $L^{\wedge Y_N}$ and it follows as from the proof of 3.2.16 that the diagonal isomorphism respects the inclusion of subcomplexes. $\square$

Moving towards infinite free G-sets X, the Loday functor is defined in 4.3.9 as via the colimit along inclusions of finite free G-subsets of X.

$$\Lambda_X(L) := \operatorname*{colim}_{F\subset X \text{ finite}} \Lambda_F L.$$

Note that by the proof of the previous lemma, this is a filtered colimit along h-cofibrations, so that it is preserved by Φ^N.

Lemma 4.4.12. *The diagonal isomorphism exists and is natural with respect to free G-sets and equivariant inclusions, and spectra $\mathbb{S}$-cofibrant under $\mathbb{S}$.*

Proof. We begin with the existence. Let L be cofibrant under $\mathbb{S}$, and let X be a free G-sets. The finite subsets of X_N are orbits of finite subsets of X hence there is a natural map $\Lambda_{X_N} L \to \Phi^N(\Lambda_X L)$ which is the colimit of isomorphisms hence an isomorphism itself. Naturality follows since equivariant inclusions induce inclusions of indexing categories for the colimits. □

As an alternative proof, we can use Corollary 4.4.8: Let $f\colon X \to Y = X \cup Z$ be an equivariant inclusion of free G-sets. Then f respects the orbit decomposition, i.e.,

$$X \cong_G \bigcup_{X_G} G, \qquad Y \cong_G \bigcup_{X_G} G \cup \bigcup_{Z_G} G,$$

and f corresponds to the obvious inclusion. Hence $\Lambda_f L$ is isomorphic to the map

$$\Lambda_G(\Lambda_{X_G} L) \cong \Lambda_G(\Lambda_{X_G} \wedge \Lambda_{Z_G}\mathbb{S}) \to \Lambda_{Y_G}(\Lambda_{Y_G} L),$$

i.e., it is the smash power of a map of $\mathbb{S}$-cofibrant spectra, so 4.4.3 gives the result. In particular, we even get that the map $\Lambda_{X_G} L \to \Lambda_{Y_G} L$ is a (non equivariant) $\mathbb{S}$-cofibration, thus the induced map of smash powers is an $\mathbb{S}$-cofibration of G-spaces by 3.2.9. To see this, filter Y_G through finite sets Y_i and let $X_i = f^{-1}Y_i$, $Z_i = Y_i \backslash f(X_i)$. As in 3.2.14, each of the maps $L^{\wedge X_i} \cong L^{\wedge X_i} \wedge \mathbb{S}^{\wedge Z_i} \to L^{\wedge Y_i}$ is an $\mathbb{S}$-cofibration, hence so is their colimit.
Finally we move towards $\mathbb{S}$-cofibrant commutative orthogonal ring spectra, where we want to work with the definition of the Loday functor in terms of the categorical tensor with spaces, i.e., for X a space and A an $\mathbb{S}$-cofibrant commutative orthogonal ring spectrum, $\Lambda_X A = A \otimes X$, where the tensor is in the category of commutative orthogonal ring spectra (cf. 4.3). As discussed in 4.3.5, the tensor specializes to the smash power for discrete inputs X, so all the results from above still apply. Note that we can now extend the naturality results to not necessarily injective maps:

Lemma 4.4.13. *The diagonal isomorphism from 4.4.12 is natural with respect to free G-sets and equivariant maps, and $\mathbb{S}$-cofibrant commutative orthogonal ring spectra.*

Proof. Let $f\colon X \to Y$ be an equivariant map between free G-sets. Similar to the above discussion, we filter X and Y by finite free G-sets X_i and Y_i and consider f as a colimit of maps $f_i\colon X_i \to Y_i$, where the transformations $\lambda_{f_i} A \to f_j$ for $i \leq j$ are along $\mathbb{S}$-cofibrations. Thus it suffices to check naturality for not necessarily injective equivariant maps between finite free G-sets. Here we can once again use the splitting into orbits, and the fact that the diagonal map is natural with respect to all morphisms between $\mathbb{S}$-cofibrant spectra. As in Corollary 4.4.8, we see that for $X \to Y$ an equivariant morphism of free G-sets, the map on smash powers $\Lambda_X A \to \Lambda_Y A$ is the G-fold smash power of the map $\Lambda_{X_G} A \to \Lambda_{Y_G} A$, hence the result follows. □

We can finally move on towards non discrete spaces. We begin with spaces that are geometric realizations of simplicial sets, since there we have Proposition 4.3.2, which makes computing the Loday functor much easier and in particular allows the following extension of the diagonal isomorphism:

Proposition 4.4.14. *For free G-simplicial sets X_* and equivariant maps between them, and $\mathbb{S}$-cofibrant commutative orthogonal ring spectra A, the diagonal map exists and is a natural isomorphism*

$$\Lambda_{|(X_*)|_N} A \cong \Phi^N(\Lambda_{|X_*|} A).$$

Proof. By 4.3.2, for a free G-simplicial set X_*, and an $\mathbb{S}$-cofibrant commutative orthogonal ring spectrum A, the tensor $A \otimes |X_*|$ is naturally isomorphic to the realization of the simplicial orthogonal spectrum

$$q \mapsto (A \otimes X_q) \cong A^{\wedge X_q} \cong \Lambda_{X_q} A.$$

By 6.1.44, the geometric realization of this spectrum is the colimit along the skeleton filtration, which is along levelwise h-cofibrations since the simplicial spectrum is h-good (cf. 6.1.49, 6.1.50 and 6.1.46): Every simplicial degeneracy map s_i is an injection of free G-sets, hence as in the comment before Lemma 4.4.12, it induces an $\mathbb{S}$- hence h-cofibration under the Loday functor. In particular, taking the geometric fixed points commutes with the geometric realization, since Fix^N does. Therefore the diagonal maps $\Delta(X_q, A)$ for each simplicial level induce an isomorphism on realizations. It is natural since maps of free G-simplicial sets are levelwise maps of free G-sets. □

Remark 4.4.15. For X the realization of a simplicial set, the diagonal map constructed in 4.4.14, does not depend on the simplicial model. Given two simplicial sets X_* and Y_* such that $|X_*| \cong |Y_*|$, there is a zig-zag of maps of simplicial sets between them, that realizes to the isomorphism, e.g., via the singular complex of $|Y_*|$. Thus we can use the naturality for simplicial maps.

We can continue this to work towards general cofibrant free G-spaces X, in particular since the generating naïve cofibrations are given by the inclusions $G\wedge(S^{n-1} \to D^n)$, which are the realizations of simplicial maps $G\wedge(\partial\Delta^n \to \Delta^n)$, we can write every cofibrant G-space as a colimit of pushouts of maps between spaces that are realizations of free G-simplicial sets. Note that tensoring with A of course preserves this colimit, but also maps it to a colimit along $\mathbb{S}$-cofibrations since the $\mathbb{S}$-model structure on commutative orthogonal ring spectra satisfies the pushout product axiom. In particular particular we can take geometric fixed points before going to the colimit, hence use the lemma for the simplicial case and a cell induction to get the following:

Lemma 4.4.16. *For naïvely cofibrant G-spaces X and equivariant maps between them, and $\mathbb{S}$-cofibrant commutative orthogonal ring spectra A, the diagonal isomorphism exists and is natural with respect to morphisms $X \to Y$ that are realizations of simplicial maps.*

Remark 4.4.17. Note that every equivariant map between naïvely cofibrant G-spaces is homotopy equivalent to the realization of a simplicial map via the unit of the adjunction between spaces and simplicial maps and the general Whitehead theorem for the naïve model structure. Since the Loday functor is continuous in both variables, this implies that the diagonal isomorphism of 4.4.16 is natural with respect to all continuous equivariant maps *up to homotopy equivalence*. The homotopy equivalence can be seen as one of orthogonal spectra which at each time is a morphism of ring spectra (cf. proof of 4.3.4).

This concludes our study of the case of a fixed finite group, since we have reached the other end of the generality of the definition of the Loday functor from 4.3.7. So far we have not touched upon functoriality of the Loday functor or naturality of the diagonal map for *changing the group*, so let now $\phi\colon H \to G$ be a homomorphism of topological groups. As usual we can look at the restriction functor $\phi^*\colon G\mathcal{T} \to H\mathcal{T}$, and it is immediate, that it commutes with the Loday functor in the sense that for orthogonal ring spectra A, there is a natural isomorphism of commutative H-ring spectra

$$\Lambda_{\phi^* X} A \cong \phi^* \Lambda_X A,$$

where on the right side ϕ^* is the restriction functor on commutative ring spectra. Note, that since ϕ^* does not send free G-sets to free H-sets if ϕ is not injective, we have in general no control over the diagonal map. Therefore we will only consider the case where G is a finite group and ϕ is the inclusion $i\colon H \to G$ of a subgroup. For $N \subset H$, this leads to the following:

Proposition 4.4.18. *The restriction functor preserves the diagonal map.*

Proof. Recall from Lemma 3.1.43 that we placed a representation theoretic condition (1) on the subgroup N of G. When G is finite Lemma 3.1.43 gives that Condition 1 is satisfied. Hence as in the proof of Proposition 3.1.46 the restriction commutes with $\mathbb{P}_\phi$. Therefore all the functors in the definition of the diagonal zig-zag commute with the restriction, and so does the whole zig-zag. The restriction functor preserves the colimit along the inclusions of finite subsets as well as the geometric realization and the cell complex construction, which we used to extend the diagonal map from the case of spectra above. □

Remark 4.4.19. The relation of the Loday functor with the induction of an H-set X along the inclusion $i\colon H \to G$ is more subtle. Intuitively the Loday functor itself should be viewed as the $\wedge$-induction of a spectrum with action of the trivial group to a spectrum with G-action. If one wants to start with an H-spectrum instead, this generalizes the study of multiplicative norm constructions, where X is assumed to be a discrete subgroup H of G. These norm functors have been famously put to use in the recent proof of the Kervaire-invariant problem by Hill, Hopkins and Ravenel. An introduction can be found in [HHR, A.3,4], or [S11, 8,9], both of which only became available very shortly before the third author's thesis (which very much is at the core of this paper) was finished. The third author learned about the interplay from Stefan Schwede, during a visit in Bonn in November 2010, where he presented the results of his thesis. As the study of multiplicative norms of course has to address some of the same questions we discussed here, we point out some similarities and differences to [HHR]. Due to the fact that the works are independent, the notation and viewpoint are rather different. First of all one should note the difference in model structures. We worked with the $\mathbb{S}$-model structure instead of the classical q-model structure on commutative orthogonal ring spectra, in order to get around the q-cofibrant replacement (cf. 3.1.32). [HHR] address this problem by proving that the symmetric powers appearing in the generating q-cofibrations are "very flat" (cf. [HHR, B.13,63]), which allows them to construct a natural weak equivalence calculating geometric fixed points. Their method has the advantage that it is more easily applicable to the general multiplicative norm case they aim to study. Our method on the other hand allows us to recognize the diagonal map as a natural *isomorphism*, strengthening their statement. Note that the "Slice Cells" discussed in [HHR, 4.1] are special cases of generating $\mathbb{S}I_G$-cofibrations in our language, and from this viewpoint the filtration given in [HHR, A.4.3] and our Theorem 3.2.14 achieve similar goals – an equivariant filtration of the smash power – with different methods. Finally note that the change of the indexing of the smash power away from a non discrete set we worked for in this section, is only addressed in the side note [HHR, A.35]. Since all groups discussed there are finite, this is not a major point in [HHR], but as we are going to move towards tori and more general compact Lie groups now, the details become important.

Infinite Groups

We now leave the realm of finite groups and move back to the case of compact Lie groups that is the main focus of our results. In particular to deal with (higher) topological Hochschild homology and Cyclic homology, we are interested in the case where G is a torus. The first thing we should address is that Condition 1 holds in these cases:

Lemma 4.4.20. *Condition 1 holds when G is the n-torus and H is the kernel of an isogeny of G, in particular for $G \cong S^1$ and H a finite subgroup.*

Proof. Let $G = S^1 \times \ldots \times S^1 = \mathbb{T}^n$. Observe that the group structure on $\mathbb{T}^n$ is inherited from $(\mathbb{R}^n, +)$ via the isomorphisms

$$\begin{array}{rcl} \mathbb{R}/\mathbb{Z} & \cong & S^1 \\ a+b & \mapsto & [a][b] \\ k \cdot a & \mapsto & [a]^k \text{ for } k \in \mathbb{Z} \end{array}$$

Given an isogeny α of $\mathbb{T}^n$, its kernel H is a finite subgroup. Note that there is a matrix $A = (a_{i,j}) \in M_n(\mathbb{Z}) \cap Gl_n(\mathbb{Q})$ such that $\alpha : \mathbb{T}^n \to T^n$ is induced by $(A\cdot) : \mathbb{R}^n \to \mathbb{R}^n$. Under this correspondence, the kernel of α is isomorphic to the projection of $A^{-1}\mathbb{Z}^n \subset \mathbb{Q}^n \subset \mathbb{R}^n$ to $\mathbb{R}^n/\mathbb{Z}^n$. Note that $A^{-1}\mathbb{Z}^n$ is finitely generated by the columns $c_1, \ldots c_n$ of A^{-1}. The orders ρ_i of these generators in $\mathbb{R}^n/\mathbb{Z}^n$ are given by the least common multiple of the denominators of entries $c_{i,j}$ of the respective columns c_i. Let W be an H-representation via $\phi : H \to \mathbf{O}_W$. We can restrict ourselves to irreducible representations W, hence assume that $\dim W \leq 2$. (The orthogonal matrices $\phi(l_i)$ commute and hence can be brought to normal form simultaneously). If $W = \mathbb{R}$ is the trivial L-representation, we can define $V := W$ as the trivial G-representation and are done. If W is one dimensional, define $V := \mathbb{C} \cong_{\mathbb{R}} \mathbb{R}^2$ with the standard metric and let $W \to V$ be the embedding as the real line. We prolong ϕ to $\mathbf{O}_V$, by mapping the non-trivial element of $\mathbf{O}_W$ to the rotation of order 2. For W of dimension 2, note that no element of H can be mapped to an element with negative determinant of $\mathbf{O}_W$ – if it does it has two different eigenspaces which have to be preserved by all other elements of $\phi(H)$, hence W could not have been irreducible. The generators of H have to map to elements of $\mathbf{O}_{\mathbb{C}}$ of order that divides their own. In particular $\phi(c_i)$ acts on $\mathbb{C}$ by multiplication with a root of unity $\zeta_{\rho_i}^{k_i}$, such that $\phi(c_i)^x$ is well defined for $x \in \mathbb{R}$. Define an action of $\mathbb{T}^n$ on $V = \mathbb{C}$ by:

$$\mathbb{R}^n/\mathbb{Z}^n \times V \xrightarrow{A\cdot\times V} \mathbb{R}^n/A\mathbb{Z}^n \times V \xrightarrow{\quad\mu\quad} V$$

$$[(x_1, \dots, x_n)], z \longmapsto \prod_i \phi(c_i)^{x_i} \cdot z$$

The definition of the lower map obviously gives an action of $(\mathbb{R}^n, +)$ on V, since the $\phi(c_i)$ commute. To see that it descends to an action of $\mathbb{T}^n$, we have to show that it is trivial on $\mathbb{Z}^n$. Let e_j be one of the standard base vectors of $\mathbb{R}^n$. Then multiplication with A takes e_j to the coordinate vector $(a_{1,j}, \dots, a_{n,j})$. We need to check that $\prod_i \phi(c_i)^{a_{i,j}} = 1$ in $\mathbb{C}$:
In $H \subset \mathbb{R}^n/\mathbb{Z}^n$ we can form the $\mathbb{Z}$-linear combination $\Sigma_i a_{i,j} c_i$. Since the c_i were the columns of A^{-1} this gives back the standard basis vector e_j which is congruent to 0 modulo $\mathbb{Z}^n$, i.e., the $\mathbb{Z}$-linear combination $\Sigma_i a_{i,j} l_i$ is equal to the unit in L. Since ϕ is a group homomorphism this implies that $\prod_i \phi(l_i)^{a_{i,j}} = 1$ in $\mathbf{O}_{\mathbb{C}}$, i.e., $\prod_i \phi(c_i)^{a_{i,j}} = 1$ in $\mathbb{C}$ as desired. This action extends the action of W by construction, since a generator c_i of H maps to e_i under the multiplication with A, hence acts via $\phi(c_i)$ on V. The fixed points of V under the $\mathbb{T}^n$ action are trivial, since no $\phi(c_i)$ that is not trivial fixes a point except the origin. ☐

We will need some more properties of tori, we begin with the one dimensional case.

Lemma 4.4.21. *Let C_n be a finite (cyclic) subgroup of order n in S^1. There is a simplicial C_n-set $(S_n)_*$ with a C_n-homeomorphism $|(S_n)_*| \cong S^1$.*

Proof. This is easily done by hand, or by applying edgewise subdivision to the standard simplicial model of S^1. ☐

Lemma 4.4.22. *Let H be the kernel of an isogeny α of the torus $\mathbb{T}^n$. There is a simplicial H-set T_* with an H-homeomorphism $|T_*| \cong \mathbb{T}^n$.*

Proof. We combine the methods of the two lemmas above. For the intuition that underlies the following, it is best to think of $\mathbb{R}^n$ as the n-fold product of the infinite simplicial complex $\mathbb{R}$, which has vertices lying on the points in $\mathbb{Z}$, and edges between them. We identify the action of H on this complex, and then produce a finer complex where the action is simplicial. As above associate to α an integer matrix $A \in M_n(\mathbb{Z}) \cap Gl_n(\mathbb{Q})$. In the notation above, the action of an element H on an element $[r] \in R^N/Z^N$ corresponds to adding a linear combinations of columns of A^{-1} to r. All the columns of A^{-1} are in $\det(A) \cdot \mathbb{Z}^n$ by Cramer's rule. Hence the difference of r to its image under the action of any $h \in H$ is in $\det(A) \cdot \mathbb{Z}^N$ as well. Thus let T_* be the n-fold product of simplicial sets

$$T_* = S_* \times \ldots \times S_*,$$

where S_* is a $C_{\det A}$-equivariant simplicial model of S^1. The action of H on the realization corresponds to a simplicial action on the resulting simplicial complex, hence T_* is the desired H-simplicial set. □

Finally, we can state the result about the diagonal map for our main case of interest:

Theorem 4.4.23. *Let $G = \mathbb{T}^n$ and N the kernel of an isogeny (such that Condition 1 holds.) For an $\mathbb{S}$-cofibrant commutative orthogonal ring spectrum A and a naïvely cofibrant G-space X, there is an isomorphism of G/N-spectra*

$$\Lambda_{X_N} A \to \Phi^N \Lambda_X A.$$

The isomorphism is natural, and restricts to the diagonal isomorphism under the restriction of the G-action to any finite subgroup of G/N.

Proof. We begin with constructing the map and check equivariance afterwards. Restricting to an arbitrary finite subgroup K of G that contains N, Proposition 4.4.14 states that there is a diagonal isomorphism

$$\Lambda_{X_N} A \to \Phi^N \Lambda_X A$$

of K/N-spectra. For K_1 and K_2 two such subgroups, the two maps they define restrict to the same map of $K_1 \cap K_2/N$-spectra, since the diagonal maps are preserved under restriction (4.4.18). In particular all of these maps restrict to the same underlying map of spectra. Note that for normal subgroups $N \subset G$ an element $[g]$ in the quotient group G/N has finite order, if and only if the subgroup generated by g intersects N non trivially, and in particular if and only if the subgroup generated by $\{g\} \cup N$ contains N as a subgroup of finite index. This implies that we constructed a map between spectra with J-action, which is equivariant with respect to the action of all points in J that have finite order. Since G/N is isomorphic to G, we know that the points of finite order are exactly the rational points in $G/N \cong \mathbb{T}^n$. Since the rational points are dense in $\mathbb{T}^N$, and the actions on J-spectra are continuous, the map is indeed J-equivariant. □

Since every element of a compact Lie group lies in a maximal torus, and in particular contains points of finite order in every one of its neighborhoods, this argument can be used in more general settings, as soon as one knows that Condition 1 holds and a property analogous to 4.4.22 is satisfied:

Condition 2. Let H be a finite subgroup of a compact Lie group G. There exists a simplicial free H-set G_* with an H-equivariant isomorphism

$$|G_*| \cong G.$$

Remark 4.4.24. As we have seen above, the tori $\mathbb{T}^n$ satisfy Condition 2 with respect to all kernels of isogenies. To the knowledge of the authors, the general case when G is a compact Lie group and H a closed subgroup is unknown. Note that Illman's triangulation theorem 1.2.2 constructs an H-equivariant triangulation of the smooth H-manifold G, but since the usual methods that produce simplicial sets from simplicial complexes fail for Illmans equivariant simplices ([Ill83, §3]), this is not enough.

Theorem 4.4.25. *Let G be a compact Lie group, and N a normal subgroup. Assume for all subgroups $i\colon K \to G$ containing N, such that N has finite index in K that Condition 1 holds and that G satisfies Condition 2 with respect to K. For an $\mathbb{S}$-cofibrant commutative orthogonal ring spectrum A and a naïvely cofibrant G-space X, there is an isomorphism of G/N-spectra*

$$\Lambda_{X_N} A \to \Phi^N \Lambda_X A.$$

The isomorphism is natural, and restricts to the diagonal map under the restriction of the action to any finite subgroup of G/N.

Before we end this section, let us briefly say something about the change of base rings. Recall that we defined R-model structures for the categories of R-modules for R a commutative orthogonal ring spectrum. In particular, the generating R-cofibrations were given by smashing R with the generating $\mathbb{S}$-cofibrations. Recall that the category of R-modules is symmetric monoidal with respect to the monoidal product $\wedge_R$, defined via the coequalizer diagram 5.1.16, and we defined the R-Loday functor using this product in 4.3.7. We would like to state a result analogous to Theorem 3.2.16, but there is an obstruction:

For an arbitrary, or even for an $\mathbb{S}$-cofibrant commutative orthogonal ring spectrum R, there is a priori no reason why the geometric fixed points of R equipped with the trivial G-action should be isomorphic to R itself. Note that the sphere spectrum of course has this property by 3.1.26, since $\mathbb{S} \cong \mathcal{F}^0_{S^0}$. Excluding this case, we still get the following

Theorem 4.4.26. *Let G be a compact Lie group, and N a normal subgroup. Assume for all subgroups $i\colon K \to G$ containing N, such that N has finite index in K that Condition 1 holds and that G satisfies Condition 2 with respect to K. Let R be an $\mathbb{S}$-cofibrant orthogonal ring spectrum such that $\Phi^N R \cong R$. There is a natural isomorphism*

$$\Lambda^R_{X_N} A \cong \Phi^N(\Lambda^R_X A),$$

if A is an R-cofibrant commutative R-algebra and X a naïvely cofibrant G-space, or if A is a cofibrant R-module and X is a finite free G-set.

Proof. For the spectrum case, since L is R-cofibrant, it is isomorphic to $R \wedge K$ with K an $\mathbb{S}I$-cellular spectrum. Applying The smash power $L^{\wedge_R X}$ is then isomorphic to $R \wedge K^{\wedge X}$. Here G acts trivially on R and in the usual way on the smash power of K. Then by 3.2.3 and 4.4.6 the result follows. For the algebra case we can follow the discussion for $R = \mathbb{S}$ above. □

4.5 Homotopical Properties

We finally turn to investigating the homotopy theoretical properties of the Loday functor. For one, this this will allow us to establish the comparison result to the BCD model. On the other hand it is very good to know, that the $\mathbb{S}$-model structures and in particular the induced regular cells are sufficiently well behaved to allow for the standard tools of equivariant stable homotopy theory to apply, without having to resort to q-cofibrant replacements.

4.5.1 Homotopy Groups

The main point we want to establish is the characterization of geometric fixed points for smash powers of 4.5.12. But before we can give precise statements, we have to once again recall definitions from [MM, V.4], the first one being homotopy groups for $\mathbf{O}_E^{\mathscr{H}}$-spectra, geometric homotopy groups, and several homomorphisms between the different homotopy groups associated to a G-spectrum. Here $\mathscr{H}$ is the maximal G-typical family of representations. Let as usual

$$E: \qquad 1 \longrightarrow N \longrightarrow G \stackrel{\epsilon}{\longrightarrow} J \longrightarrow 1$$

be an exact sequence of compact Lie groups.

Definition 4.5.2. Let $\nu\colon \mathcal{O}_J \to \mathbf{O}_E^{\mathscr{H}}$ be given on objects by

$$\begin{array}{rccc} \nu\colon & \mathcal{O}_J & \to & \mathbf{O}_E^{\mathscr{H}} \\ & W & \mapsto & \epsilon^* W \end{array}$$

and on morphisms by

$$\begin{array}{rcl} \mathcal{O}_J(V, V') & \to & \mathbf{O}_E^{\mathscr{H}}(\epsilon^* V, \epsilon^* V'), \\ (f, s) & \mapsto & (\epsilon^* f, \epsilon^* s) = (f, s) \end{array}$$

where $f\colon V \to V'$ is an isometric embedding and s an element in the orthogonal complement of $f(V)$.

Recall from 4.4.1 that for a $J\mathcal{T}$-functor $X\colon \mathcal{O}_E \to J\mathcal{T}$, the G-spectrum $\mathrm{Fix}^N X$ is given by $(\mathrm{Fix}^N X)_V = (X_V)^N$.

Definition 4.5.3. [MM, 4.8] Let Y a $\mathbf{O}_E^{\mathscr{H}}$-spectrum and X an orthogonal G-spectrum. Let $H/N = K \subset J$, with H a subgroup of G containing N.

(i) Define the *homotopy groups of* Y via

$$\pi_q^K Y := \begin{cases} \mathrm{colim}_{V \in \mathbf{O}_E^{\mathscr{H}}} \pi_q(\Omega^{V^N} Y_V)^K & \text{if } q \geq 0 \\ \mathrm{colim}_{\mathbb{R}^q \subset V \in \mathbf{O}_E^{\mathscr{H}}} \pi_0(\Omega^{V^N - \mathbb{R}^q} Y_V)^K & \text{if } q \leq 0 \end{cases}$$

(ii) Define a natural homomorphism

$$\psi\colon \pi_*^K(X^N) \to \pi_*^H(X),$$

by restricting the defining colimit system of $\pi_*^H(X)$ to N-fixed V as above, using that for these

$$(\Omega^V X_V^N)^K \cong (\Omega^V X_V)^H.$$

(iii) Define a natural homomorphism

$$\omega\colon \pi_*^H(X) \to \pi_*^K(\mathrm{Fix}^N X),$$

by sending an element of $\pi_q^H(X)$ represented by an H-equivariant map $f\colon S^q \wedge S^V \to X_V$ to the element of $\pi_q^K(\mathrm{Fix}^N X)$ represented by the K-equivariant map $f^N\colon S^q \wedge S^{V^N} \to X_V^N$ for $q \geq 0$ and similar for $q \leq 0$.

(iv) A morphism of $\mathbf{O}_E^{\mathscr{H}}$-spectra is a *$\pi_*$-isomorphism* if it induces isomorphisms on all homotopy groups π_*^K, a morphism of orthogonal G-spectra is a *$\pi_*\mathrm{Fix}^N$-isomorphism* if it induces isomorphisms on all geometric homotopy groups $\pi_*^K\mathrm{Fix}^N$.

Note that ψ is a natural isomorphism if X is a G-Ω-spectrum, in particular we can use the fixed point spectra of a fibrant approximation of X to calculate the equivariant homotopy groups of X. This implies the following characterization

Lemma 4.5.4. *Let $\mathscr{A}$ be a closed family of normal subgroups of G. A morphism f of G-Ω-spectra is an $\pi_*^{\mathscr{A}}$-isomorphism if and only if $\mathrm{Fix}^H f$ is a non equivariant π_*-isomorphism for all $H \in \mathscr{A}$.*

For orthogonal J-spectra Z the homomorphism

$$\zeta\colon \pi_*^K(Z) = \pi_*^K(\mathbb{U}_\nu \mathbb{U}_\phi Z) \to \pi_*^K(\mathbb{U}_\phi Z)$$

is an isomorphism by [MM, V.4.10]. This enables us to make the following definition:

Definition 4.5.5. For orthogonal G-spectra X, define the natural homomorphism $\eta_*\colon \pi_*^K\mathrm{Fix}^N X \to \pi_*^K(\Phi^N X)$ as the composition

$$\pi_*^K\mathrm{Fix}^N X \xrightarrow{\eta_\phi} \pi_*^K(\mathbb{U}_\phi\mathbb{P}_\phi\mathrm{Fix}^N X) \xrightarrow{(\omega\circ\psi)^{-1}} \pi_*^K(\mathbb{P}_\phi\mathrm{Fix}^N X) = \pi_*^K(\Phi^N X),$$

where η_ϕ is the unit of the adjoint pair $(\mathbb{P}_\phi, \mathbb{U}_\phi)$ from 3.1.23.

The following lemma is the first point where we have to adapt the argument to fit our more general cells:

Proposition 4.5.6. *cf. [MM, V.4.12] The map $\eta_*\colon \pi_*^K(\mathrm{Fix}^N X) \to \pi_*^K(\Phi^N X)$ is an isomorphism for $(\mathrm{Ind}^{\mathrm{reg}} \cup \mathcal{F}I_G)$-cellular orthogonal G-spectra X.*

Proof. As in the classical case, because Φ^N preserves cofiber sequences, wedges, and colimits of sequences of h-cofibrations, it suffices to check that η_* is an isomorphism on all objects of the form

$$X := G_+ \wedge_H (\mathcal{G}_V(L)^{\wedge H}),$$

with L a genuine $\mathbf{O}_V$-cell complex. Note that if N is not a subgroup of H, then $\mathrm{Fix}^N X$ is trivial, hence there is nothing to prove. Otherwise, as in the proof of Proposition 3.1.34 we get

$$\mathrm{Fix}^N X = J_+ \wedge_{J_1} \mathcal{G}_{V^{\oplus H}} \left[\mathbf{O}_{V^{\oplus H}} \wedge_{\prod_{J_1} \mathbf{O}_V} L^{\wedge J_1} \right],$$

where $J_1 = G/H$. Writing down the defining colimit systems, we see that η_* is the map

$$\operatorname*{colim}_{W \in \epsilon^* \mathcal{O}_J} \operatorname*{colim}_{U^N = W} \pi_q \left(\Omega^W \left[J_+ \wedge_{J_1} \mathbf{O}_E^{\mathscr{H}}(V^{\oplus H}, U) \wedge_{\prod_{J_1} \mathbf{O}_V} L^{\wedge J_1} \right] \right)^K$$

$$\longrightarrow \operatorname*{colim}_{W \in \epsilon^* \mathcal{O}_J} \pi_q \left(\Omega^W \left[J_+ \wedge_{J_1} \mathcal{O}_J(V^{\oplus J_1}, W) \wedge_{\prod_{J_1} \mathbf{O}_V} L^{\wedge J_1} \right] \right)^K.$$

Hence it suffices to prove that

$$p\colon \operatorname*{hocolim}_{U^N = W} \mathbf{O}_E^{\mathscr{H}}(V^{\oplus H}, U) \to \mathcal{O}_J(V^{\oplus J_1}, W)$$

is a $\Pi_{J_1} \mathbf{O}_V \rtimes J_1$-homotopy equivalence, where the map is induced by the restriction to the N-fixed space $V^{\oplus J_1} \subset V^{\oplus H}$ (cf. 3.1.23).
From the definition of $\mathbf{O}_E^{\mathscr{H}}$ (3.1.22), recall that $\mathbf{O}_E^{\mathscr{H}}(V^{\oplus H}, U) = \mathbf{O}(V^{\oplus H}, U)^N$. Note that any N-equivariant isometry has to preserve fixed spaces and isotypical factors. Hence for $W \oplus U' \cong U$ and $V^{\oplus J_1} \oplus V' \cong V^{\oplus H}$ orthogonal decompositions where N acts trivially on W and U' contains no summands with trivial N-action, there is an isomorphism

$$\mathbf{O}_E^{\mathscr{H}}(V^{\oplus H}, U) \cong \mathcal{O}_J(V^{\oplus J_1}, W) \times \mathbf{O}(V', U')^N.$$

Here $\mathbf{O}(V', U')$ is a sphere bundle over $\mathcal{L}(V', U')$ whose dimension depends linearly on the dimension of U' and the map p is induced from the projection to the first factor. Thus it suffices to prove that $\mathcal{L}(V', \operatorname{colim} U')^N \to *$ is a $\Pi_{J_1} \mathbf{O}_V \rtimes J_1$-homotopy equivalence. Since V' was the orthogonal complement of $V^{\oplus J_1}$, the $\Pi_{J_1} \mathbf{O}_V$ action is trivial, so [LMS, II.1.5] gives the desired result. □

We continue following [MM, V.4]: Recall the definition of the universal $\mathscr{A}$-space $E\mathscr{A}$ for a closed family of subgroups of G from 1.3.11.

Definition 4.5.7. For N a normal subgroup, let $\mathscr{N}$ be the family of subgroups of G that do not contain N. Let $E\mathscr{N}$ be the universal $\mathscr{N}$-space, and let $\tilde{E}\mathscr{N}$ be the cofiber of the quotient map $E\mathscr{N}_+ \to S^0$ that collapses $E\mathscr{N}$ to the non basepoint. For orthogonal G-spectra X, the map $\lambda\colon S^0 \to \tilde{E}\mathscr{N}$ induces a natural map $\lambda\colon X \to X \wedge \tilde{E}\mathscr{N}$.

Note that $(\tilde{E}\mathscr{N})^H = S^0$ if H contains N, and $(\tilde{E}\mathscr{N})^H$ is contractible otherwise.

Lemma 4.5.8. *[MM, V.4.15] For orthogonal G-spectra X, the map*

$$\Phi^N\lambda\colon \Phi^N X \to \Phi^N(X\wedge\tilde{E}\mathscr{N})$$

is a natural isomorphism of orthogonal J-spectra.

The following lemma is another point where we need to be careful about the type of cofibrant objects:

Lemma 4.5.9. *cf. [MM, V.4.16] Let $K = H/N$, with $N \subset H$. For $\mathrm{Ind}^{\mathrm{reg}} \cup \mathcal{F}I_G$-cellular orthogonal G-spectra X, the map*

$$\omega\colon \pi_*^H(X\wedge\tilde{E}\mathscr{N}) \to \pi_*^K(\mathrm{Fix}^N(X\wedge\tilde{E}\mathscr{N}))$$

is an isomorphism.

Proof. Note that Proposition [LMS, 9.3], which is essential for the proof given in [MM, 4.16] also holds for the weaker assumption of genuine G-cell complexes instead of G-CW-complexes. In particular there are bijections

$$[A, B\wedge\tilde{E}\mathscr{N}]_G \cong [A^N, B\wedge\tilde{E}\mathscr{N}]_G \cong [A^N, B]_G$$

between sets of G-homotopy classes for A any representation sphere and B a level of an induced regular spectrum, which is a genuine G-cell complex by 3.2.9 and 1.2.4. Hence as in the classical case, the map

$$\omega\colon \operatorname*{colim}_V \pi_q \left[\Omega^V(X_V\wedge\tilde{E}\mathscr{N})\right]^H \to \operatorname*{colim}_V \pi_q \left[\Omega^{V^N}(X_V\wedge\tilde{E}\mathscr{N})^N\right]^K$$

is a colimit of isomorphisms. □

Recall the map γ from Section 3.1.24. Let $K = H/N$, with $N \subset H$. By [MM, V.4.11], for an orthogonal Ω-G-spectrum X, the map $\gamma_*\colon \pi_*^K(X^N) \to \pi_*^K(\Phi^N X)$ is the composite

$$\pi_*^K(X^N) \xrightarrow{\cong} \pi_*^H(X) \xrightarrow{\omega} \pi_*^K(\mathrm{Fix}^N X) \xrightarrow{\eta^*} \pi_*^K(\Phi^N X).$$

Finally we can state the main purpose of this excursion, with the classical proof applying verbatim, using our modified results above:

Proposition 4.5.10. *[MM, V.4.17] For $\mathrm{Ind}^{\mathrm{reg}}$-cellular orthogonal G-spectra X, the diagram*

$$R(X\wedge\tilde{E}\mathscr{N})^N \overset{\gamma}{\to} \Phi^N R(X\wedge\tilde{E}\mathscr{N}) \overset{\Phi^N(\xi\lambda)}{\leftarrow} \Phi^N X \tag{4.5.11}$$

displays a pair of natural π_*-isomorphisms of orthogonal J-spectra, where R is a fibrant replacement functor from the classical stable model structure.

Remark 4.5.12. The significance of this proposition stems from the fact, that there are alternative definitions for the geometric fixed points of a G-spectrum. Classically, one would take the leftmost J-spectrum in the zigzag (4.5.11) as the definition. The proposition then tells us that the homotopy type of the geometric fixed points of an $\mathrm{Ind}^{\mathrm{reg}}$-cellular G-spectrum calculated in terms of Definition 3.1.24 is "correct", even without first applying a q-cofibrant replacement functor. Note that in the spirit of Remark 3.2.10, we could use the same proofs to extend 4.5.6 and 4.5.9 and hence Proposition 4.5.10 and Proposition 4.5.14 below to $\mathbb{S}_{\mathrm{reg}}$-cellular spectra. However the added generality makes the notation in the proofs even more convoluted, and we only need the weaker result.

The importance of this remark stems from the fact that we want to be able to use the "fundamental cofibration sequence". It is the following homotopy-cofiber sequence of (non equivariant) orthogonal spectra:

$$[R(X \wedge E\mathscr{N}_+)]^N \longrightarrow [R(X)]^N \longrightarrow [R(X \wedge \tilde{E}\mathscr{N})]^N , \tag{4.5.13}$$

which arises from the defining cofiber sequence of $\tilde{E}\mathscr{N}$ by smashing with X, fibrant replacement and passing to categorical fixed points. We saw above, that the homotopy groups of the right spectrum are closely related to the homotopy groups of the geometric fixed points of X. Together with Lemma 4.5.4 from above this implies the following statement:

Proposition 4.5.14. *Let $\mathscr{A}$ be a family of subgroups of G and let X and Y be $\mathrm{Ind}^{\mathrm{reg}} \cup \mathcal{F}I_G$-cellular. Then for a morphism $f\colon X \to Y$, the following are equivalent:*

(i) The map f is a $\pi_*^{\mathscr{A}}$-isomorphism.

(ii) For all $H \in \mathscr{A}$ the map $\Phi^H f$ is a (non equivariant) π_*-isomorphism.

Proof. Note that since $\mathrm{Ind}^{\mathrm{reg}}$ consists of $\mathbb{S}$-cofibrations, X and Y are levelwise genuine G- hence N-complexes. We compare the maps induced on the homotopy cofiber sequences for $N \in \mathscr{A}$:

$$\begin{array}{ccccc} [R(X \wedge E\mathscr{N}_+)]^N & \longrightarrow & [R(X)]^N & \longrightarrow & [R(X \wedge \tilde{E}\mathscr{N})]^N \\ \downarrow & & \downarrow & & \downarrow \\ [R(Y \wedge E\mathscr{N}_+)]^N & \longrightarrow & [R(Y)]^N & \longrightarrow & [R(Y \wedge \tilde{E}\mathscr{N})]^N \end{array} \tag{4.5.15}$$

We use induction on the size of the family $\mathscr{A}$, which is possible since G is compact (cf. [tD, 1.25.15]). For the trivial family, the result is true. We claim first, that both (i) and (ii) imply that the left vertical map is a π_*-isomorphism, for (i) this is trivial by 2.7.4. For (ii) we use the induction hypothesis which implies that f is an $\mathscr{A} \cap \mathscr{N}$-equivalence. Since $\mathscr{N}$ is the family of subgroups not containing N, $\mathscr{A} \cap \mathscr{N}$ is the family $\mathscr{N}_N$ of all proper subgroups of N. Note that as an N-space, $E\mathscr{N}$ is a universal $\mathscr{N}_N$-space. Thus $R(X \wedge E\mathscr{N}_+) \to R(Y \wedge E\mathscr{N}_+)$ is a levelwise $\mathscr{N}_N$ equivalence between genuine $\mathscr{N}_N$-complexes, thus an N-homotopy equivalence and the claim follows. Finally Lemma 4.5.4 and Proposition 4.5.10 finish the proof, since they show that property (i) is equivalent to the second vertical map in (4.5.15) being a π_*-isomorphism for all N, and (ii) being equivalent to the third vertical map in (4.5.15) being a π_*-isomorphism. □

This has the following immediate consequences:

Corollary 4.5.16. *Let $G = \mathbb{T}^n$ be the n-torus and let $\mathscr{A}$ be the family of kernels of isogenies. Let X be a free cofibrant $\mathbb{T}^n$-space. Then the Loday functor $\Lambda_X(-)$ sends π_*-isomorphisms between $\mathbb{S}$-cofibrant commutative orthogonal ring spectra to $\pi_*^{\mathscr{A}}$-equivalences of commutative orthogonal $\mathbb{T}^n$-spectra.*

Proof. Since the $\mathbb{S}$-model structure on commutative orthogonal ring spectra is topological, we know that the tensor with a cofibrant space non equivariantly preserves π_*-isomorphisms between cofibrant commutative $\mathbb{S}$-cofibrations. By Theorem 4.4.23, it therefore also induces non equivariant π_*-isomorphisms in geometric fixed points with respect to all subgroups $H \in \mathscr{A}$, so Proposition 4.5.14 gives the result. □

Similarly for more general compact Lie groups Theorem 4.4.25 gives the following analog:

Corollary 4.5.17. *Let G be a compact Lie group and let $\mathscr{A}$ be a closed family of normal subgroups that is closed under extensions of finite index, such for all inclusions $i\colon H \to G$ with $H \in \mathscr{H}$ Condition 1 holds and G satisfies Condition 2 with respect to all H. Let X be a free cofibrant G-space. Then the Loday functor $\Lambda_X(-)$ sends π_*-isomorphisms between $\mathbb{S}$-cofibrant commutative orthogonal ring spectra to $\pi_*^{\mathscr{A}}$-equivalences of commutative orthogonal G-spectra.*

Remark 4.5.18. Analogous results hold for the cases of mere $\mathbb{S}$-cofibrant spectra and finite free G-sets X as well as spectra $\mathbb{S}$-cofibrant under $\mathbb{S}$ and infinite free G-sets X (cf. Subsection 4.4.1).

4.5.19 Comparison to the Bökstedt-model

We compare the Loday functor of Section 4.3.7 to the model constructed in Section 4 of [BCD]. Since op. cit. is written in the language of Γ-spaces, respectively simplicial functors, we cannot do so directly, but instead have to use comparison theorems such as the ones given in [MMSS, §0] and [SS03, §7]. As mentioned in the introduction to [SS03, §7] the corresponding comparisons of categories of commutative monoids do not extend over the whole range of the comparison. We restrict our comparison to the underlying spectra, respectively the equivariant structure. Since the weak equivalences considered between commutative monoids are in both contexts created in the underlying category, this seems satisfactory.
We begin with the non equivariant discussion. Recall the definition of a strong monoidal Quillen equivalence from [SS03, 3.6]. The discussion in the introduction of [MMSS] and

[SS03, 7.1] state that there is a diagram of strong monoidal Quillen equivalences

$$\mathcal{SF} \underset{}{\overset{\mathbb{PT}}{\rightleftarrows}} \mathcal{WT} \underset{\mathbb{U}}{\overset{\mathbb{P}}{\leftrightarrows}} \mathbf{O}\mathcal{T}, \tag{4.5.20}$$

where $\mathcal{SF}$ is the category of simplicial functors and $\mathcal{WT}$ is the category of continuous functors from finite CW-complexes to spaces, respectively, and the model structure on $\mathbf{O}\mathcal{T}$ is the classical stable using the naïve- or projective model structure on the categories $\mathbf{O}_V\mathcal{T}$. The functor $\mathbb{T}$ is geometric realization, and the instances of $\mathbb{P}$ and $\mathbb{U}$ are prolongation and restriction functors. We displayed the strong monoidal left adjoints as the top arrow.
Since these Quillen equivalences are not composable as such, we compare the two constructions of the Loday functor in $\mathcal{WT}$. However, since it would take us too far to recall the whole construction from [BCD], we will immediatel reduce our comparison to smash powers with the help of the following lemma. We denote the Bökstedt version of the Loday functor used in [BCD] by $\hat{\Lambda}$.

Lemma 4.5.21. *[BCD, 4.4.4] If A is cofibrant in $\mathcal{WT}$ and T is a finite set, then there is a chain of stable equivalences between $\hat{\Lambda}_T A$ and the T-fold smash product $\bigwedge_T A$.*

In particular the functor $\hat{\Lambda}_T$ models the T-fold derived smash product of A with itself. Since the Loday functor we defined for orthogonal spectra has the analogous property, and since by (4.5.20) the homotopy categories of $\mathcal{SF}$ and $\mathbf{O}\mathcal{T}$ are monoidally equivalent, this could already be seen as a successful comparison. We can say a little bit more: Recall that the identity functor on $\mathbf{O}\mathcal{T}$ gave strong monoidal Quillen equivalences between the absolute q-, positive q- and $\mathbb{S}$-model structures:

$$\mathbf{O}\mathcal{T} \underset{\mathrm{id}}{\overset{\mathrm{id}}{\leftrightarrows}} \mathbf{O}\mathcal{T}_+ \underset{\mathrm{id}}{\overset{\mathrm{id}}{\rightleftarrows}} \mathbf{O}\mathcal{T}_{\mathbb{S}}. \tag{4.5.22}$$

Since the identity functors are in particular strong monoidal, the standard methods for (monoidal) Quillen pairs (and [H, 1.3.13, (b)]) give:

Lemma 4.5.23. *For A a cofibrant simplicial functor and X a finite set, there is a chain of natural stable equivalences in $\mathcal{WT}$ connecting*

$$\mathbb{PT}(A^{\wedge X}) \simeq \mathbb{P}((\textit{ cof. } \mathbb{U} \textit{ fib. } \mathbb{PT}A)^{\wedge X}),$$

where fib. denotes a functorial fibrant replacement in $\mathcal{WT}$, and cof. denotes a functorial $\mathbb{S}$-cofibrant replacement in $\mathbf{O}\mathcal{T}$.

The comparison of the equivariant properties is similarly obstructed by the fact that there seem to be no usable comparison results on model category level between equivariant orthogonal spectra and equivariant simplicial functors. Recall that prolongation is a strong symmetric monoidal left Quillen functor from the positive model structure on symmetric spectra to the category of simplicial functors. Thus, given a commutative symmetric ring spectrum, we obtain a cofibrant commutative monoid in simplicial functors. In other words, the homotopy type of every commutative ring spectrum can be represented by a commutative monoid in simplicial functors. We therefore can compare equivariant homotopy categories, making use of our results on the equivariant structure of $\Lambda_X A$ surrounding Proposition 4.5.14 and the analogous result [BCD, 5.2.5]:

Lemma 4.5.24. *Let G be a finite abelian group, X a free G-simplicial set and A a commutative monoid in simplicial functors. The homotopy fiber of the map*

$$[\hat{\Lambda}_X A]^G \to \operatorname*{holim}_{0 \neq H \subset G} \left[\hat{\Lambda}_{X_H} A\right]^{G/H} \tag{4.5.25}$$

induced by the restriction maps is connected by a chain of natural maps that are stable equivalences to the homotopy orbit spectrum $[\hat{\Lambda}_X A]_{hG}$.

There are two important translations to be made here. The first is identifying the target of (4.5.25) with the geometric fixed points (defined in terms of $\tilde{E}\mathscr{N}$, cf. Remark 4.5.12) as in [HM, 2.1]. The second is the identification of the homotopy fiber $[\Lambda_X A \wedge E\mathscr{N}_+]^H$ of 4.5.13 with the homotopy orbits $[\Lambda_X A \wedge E\mathscr{N}_+]_H$ in the homotopy category via the Adams isomorphism as in [MM, VI.4.6]. Then the two (co-)fiber sequences in homotopy category exactly say, that $\Lambda_X A$ and $\hat{\Lambda}_X A$ have the same equivariant structure, i.e., the same homotopy type on all fixed points with respect to (finite) subgroups.

4.6 THH, TC and covering homology

We finish the discussion of smash powers by identifying the structure necessary to define higher dimensional versions of topological cyclic homology.

4.6.1 The one-dimensional situation

Topological Hochschild Homology

We follow Kro's approach from [Kr, §5] for the one dimensional theory, and use it as motivation and guideline for our treatment of the higher analogs in subsection 4.6.18. Even though smashing over the circle allows non-commutative algebras as input, this will cease to be true in higher dimensions, and we concern ourselves only with the commutative case.

We do not try to give more of an overview over the existing theory in other settings than was already attempted in the introduction, and instead redirect the interested reader to [Sh] for the case of symmetric spectra, [MSV] for $\mathbb{S}$-algebras in the sense of

[EKMM] or [Mad] and [DGM] for a general overview over the classical approach and applications towards TC and K-theory. As we already mentioned in the introduction, we will use the following simple definition:

Definition 4.6.2. Let A be an $\mathbb{S}$-cofibrant commutative orthogonal ring spectrum. Define the *topological Hochschild homology spectrum* $\mathrm{THH}(L)$ to be the commutative orthogonal S^1-ring spectrum

$$\mathrm{THH}(L) := \bigwedge_{S^1} A = A \otimes S^1.$$

A similar definition is possible for non-cofibrant commutative ring spectra, but for the definition to have homotopical meaning, some such assumption is necessary. As usual, we can always precompose with a cofibrant replacement functor for the $\mathbb{S}$-model structure. However, if we start with a cofibrant spectrum, we want not to cofibrantly replace again, so we do not put the $\mathbb{S}$-cofibrant replacement in the definition. We note once more that, contrary to the classical model structure, the cofibrant replacement takes place in the category of commutative orthogonal ring spectra, so that the tensor definition still makes sense.
Note that Theorem 4.3.2, together with the standard simplicial model for S^1, implies the following lemma:

Lemma 4.6.3. *For a commutative orthogonal ring spectrum A, the topological Hochschild homology spectrum* $\mathrm{THH}(A)$ *is isomorphic to the geometric realization of the simplicial commutative ring spectrum given by* $\mathrm{THH}(A)_q := A^{\wedge q+1}$*, with the simplicial structure maps given by*

$$d_i = \begin{cases} \mathrm{id}^{\wedge i} \wedge \mu \wedge \mathrm{id}^{q-i-1} & \text{for} \quad 0 \le i < q \\ (\mu \wedge \mathrm{id}^{\wedge q-1}) \circ T_{q,1} & \text{for} \quad i = q \end{cases}$$

$$s_i = \mathrm{id}^{\wedge i+1} \wedge \eta \wedge \mathrm{id}^{q-i}$$

where μ and η are the multiplication respectively unit map of A and $T_{q,1}$ is the action of the shuffle permutation mapping

$$A^{\wedge q+1} = A^{\wedge q} \wedge A \xrightarrow{T_{q,1}} A \wedge A^{\wedge q} = A^{\wedge q+1}.$$

Recall that, for A an orthogonal ring spectrum whose unit $\mathbb{S} \to A$ is a q-cofibration (defined by the left lifting property with respect to all levelwise acyclic fibrations), Kro [Kr, 5.2.1] defines topological Hochschild homology as an orthogonal spectrum in this way. In particular, by 4.3.4 we get that Definition 4.6.2 yields a commutative orthogonal ring spectrum $\mathrm{THH}(A)$ whose underlying spectrum is isomorphic to Kro's.
For a ring spectrum A whose unit $\mathbb{S} \to A$ is only a closed inclusion of spectra, Kro further defines a functor Γ in [Kr, 2.2.13], such that $\Gamma(A) \to A$ is a map of ring spectra which is a level acyclic fibration of orthogonal spectra. Denote the cofibrant replacement functor in the $\mathbb{S}$-model structure for commutative orthogonal ring spectra by $\mathcal{E}$. The following lemma shows that the homotopy type of the spectrum $\mathrm{THH}(A)$ does not depend on which of the two cofibrant replacements we chose:

Lemma 4.6.4. *For a commutative orthogonal ring spectrum A whose unit $\mathbb{S} \to A$ is a closed inclusion of spectra, there is a π_*-isomorphism*

$$\mathrm{THH}(\Gamma(A)) \to \mathrm{THH}(\mathcal{E}(A)).$$

Proof. Since acyclic $\mathbb{S}$-fibrations of commutative orthogonal ring spectra are in particular acyclic q-fibrations of underlying spectra, the lifting property of Γ gives a π_*-isomorphism $f\colon \Gamma(A) \to \mathcal{E}(A)$. We use the simplical spectrum from Lemma 4.6.3 to calculate THH. Both of the resulting simplicial spectra are h-proper (in the sense of ref to the appendix where it is defined - has no label now) by the same argument as in the proof of in 4.2.8. Since both $\Gamma(A)$ and $\mathcal{E}(A)$ are $\mathbb{S}$-cofibrant as spectra, f induces a π_*-isomorphism $f^{\wedge q}$ in each simplicial level by the pushout product axiom for the $\mathbb{S}$-model structure. Hence Proposition 6.1.47 implies, that the induced map on realizations is a π_*-isomorphism. □

We also need to know that the equivariant homotopy type agrees, which will be an immediate consequence of Lemma 4.6.11 below.

Definition 4.6.5. Let $G = S^1$ be the circle and let $\mathscr{A}$ be the closed family of finite (cyclic) subgroups. An morphism of orthogonal S^1-spectra is called a *cyclotomic* π_*-isomorphism if it is a $\pi_*^{\mathscr{A}}$-isomorphism, i.e., if it induces isomorphisms on π_*^C for all subgroups $C \in \mathscr{A}$.

Definition 4.6.6. For $C \in \mathscr{A}$, let $\rho_C\colon S^1 \cong S^1/C$ be the orientation preserving group isomorphism.

As usual, ρ_C^* denotes the restriction functor from S^1/C-equivariant orthogonal spectra to S^1-equivariant orthogonal spectra. The following result is similar to Kro's Theorem 5.2.5:

Theorem 4.6.7. *Let A be an $\mathbb{S}$-cofibrant commutative orthogonal ring spectrum, then the diagonal map of Theorem 4.4.23 gives an isomorphism of S^1-equivariant commutative orthogonal ring spectra*

$$\mathrm{THH}(A) \cong \rho_C^* \Phi^C \,\mathrm{THH}(A).$$

For $f\colon A \to B$ a π_*-isomorphism of $\mathbb{S}$-cofibrant commutative orthogonal ring spectra,

$$\mathrm{THH}(f)\colon\ \mathrm{THH}(A) \to \mathrm{THH}(B)$$

is a cyclotomic π_*-isomorphism of S^1-equivariant commutative orthogonal ring spectra.

Proof. The first part is immediate from Theorem 4.4.23 and the fact that $\rho^* \bigwedge_X A \cong \bigwedge_{\rho^* X} A$ as in the discussion before 4.4.18. The second part is the one-dimensional case of Corollary 4.5.16 □

This exhibits THH(A) as an especially strong example of a *cyclotomic spectrum.* We are going to give the explicit definition after the next construction:

Construction 4.6.8. Let $G = S^1$, and let C, D and E be finite subgroups such that $\rho_C(D) = E/C$. In particular we have that for an S^1-space X we have that $(X^C)^{E/C} = X^E$, and hence that $(\rho_C^*(X^C))^D = X^E$. The same formulas hold for categorical fixed points of an S^1-spectrum L.
In other setups of equivariant stable homotopy theory and cyclotomic specta, one can sometimes also identify

$$\rho_D^*\Phi^D\rho_C^*\Phi^C L = \rho_E^*\Phi^E L, \tag{4.6.9}$$

see for example [HM, Definition 2.2]. However, care needs to be taken when adapting this to the orthogonal case, since both the classical Definition of the geometric fixed point functor Φ is different from the one we used here (cf. Remark 4.5.12 and [HM, p.32]), and there are "spectrification" functors hidden in the classical notation. In our setting, the equal sign in (4.6.9) is certainly not warranted. However, by Proposition 3.1.46 there is a natural isomorphism

$$\rho_D^*\Phi^D\rho_C^*\Phi^C L \to \rho_D^*\rho_E^{D*}\Phi^{E/C}\Phi^C L,$$

where $\rho_E^D\colon S^1/D \to S^1/E$ is the isomorphism $\rho_D^{-1} \circ \rho_E$.

Using that a coend of fixed points maps into the fixed points of the coend one constructs a natural map

$$\Phi^E L \to \Phi^{E/C}\Phi^C L.$$

For induced regular semi-free spectra (3.2.7) this map is an isomorphism via the identifications of Theorem 3.2.8, hence the same is true for $\mathbb{S}_{\mathrm{reg}}$-cellular (3.2.10) and in particular q-cofibrant or $\mathrm{Ind}^{\mathrm{reg}}$-cellular spectra by a cell induction argument.

This allows us to formulate the following definition:

Definition 4.6.10. (cf. [Kr, 5.1.3]) An orthogonal S^1-spectrum L is cyclotomic, if for each finite subgroup $C \subset S^1$ there is a π_*-isomorphism

$$r_C\colon \rho_C^*\Phi^C L \to L,$$

such that the diagrams

$$\begin{array}{ccc} \rho_D^*\Phi^D\rho_C^*\Phi^C L & \longleftarrow & \rho_E^*\Phi^E L \\ {\scriptstyle \rho_D^*\Phi^D r_C}\downarrow & & \downarrow{\scriptstyle r_E} \\ \rho_D^*\Phi^D L & \xrightarrow{\;r_D\;} & L \end{array}$$

commute for all finite subgroups C, D and E such that $\rho_C(D) = E/C$. A *map of cyclotomic spectra* is a morphism of orthogonal S^1-spectra which commutes with the cyclotomic structure maps r_C for all finite $C \subset S^1$.

Note that by the naturality of the diagonal map (4.4.14), the functor $\mathrm{THH}(-) \cong \bigwedge_{S^1}(-)$ not only produces cyclotomic spectra, but also maps of cyclotomic spectra from morphisms between $\mathbb{S}$-cofibrant commutative ring spectra.
The following is a generalization of [Kr, 5.1.5]

Lemma 4.6.11. *Let L and L' be $\mathbb{S}_{\mathrm{reg}}$-cellular S^1-spectra (e.g., q-cofibrant or* $\mathrm{Ind}^{\mathrm{reg}}$*-cellular). A map $f\colon L \to L'$ of cyclotomic spectra is a cyclotomic π_*-isomorphism if and only if it is a non-equivariant π_*-isomorphism.*

Proof. The proof is immediate from Proposition 4.5.14 and the two out of three property for cyclotomic π_*-isomorphisms. □

Note that this looks weaker than [Kr, 5.1.3] on first glance, but Kro suppresses the cofibrancy hypothesis, and in particular only provides proofs for the q-cofibrant case. As a corollary we get that the cyclotomic homotopy type of the underlying spectrum of our version of THH for commutative ring spectra agrees with the one Kro constructs:

Corollary 4.6.12. *The non equivariant π_*-isomorphism* $\mathrm{THH}(\Gamma(A)) \to \mathrm{THH}(C(A))$ *in Lemma 4.6.4 is a cyclotomic π_*-isomorphism.*

Topological Cyclic Homology

We finally discuss TC. As in the previous paragraph, we do not try to give an overview over the existing theory or even recall results, but rather show how to adapt the definitions to the setting of orthogonal spectra in order to motivate the approach to the higher theory. Better places to read about the classical constructions are for example [BHM], [HM] and again [DGM] and [Mad]. We will again stay close to Kro's exposition from [Kr, 5.1].

Definition 4.6.13. (cf. [HM, 4.1]) Let $\mathbb{I}$ be the category with objects the natural numbers $\{1, 2, 3, \dots\}$. The morphisms of $\mathbb{I}$ are generated by the Restrictions $R_r\colon rm \to m$ and the Frobenii $F^r\colon mr \to m$, subject to the following set of relations:

$$\begin{aligned} R_1 = F^1 &= \mathrm{id}_m \\ R_r R_s &= R_{rs} \\ F^r F^s &= F^{sr} \\ R_r F^s &= F^s R_r. \end{aligned} \tag{4.6.14}$$

Note that we were careful about the ordering of products mr versus rm in $\mathbb{N}$. This is of course not of consequence here, but will become important when passing to higher dimensional analogs.

Construction 4.6.15. A cyclotomic spectrum L defines a functor $\mathbb{I} \to \mathbf{O}\mathcal{T}$ by mapping $n \in \mathbb{I}$ to the categorical fixed point spectrum L^{C_n}, where C_n is the cyclic subgroup with n elements of S^1. The actions of the Frobenius maps are then given by the inclusions of fixed points

$$F^r\colon L^{C_{mr}} = \left(L^{C_m}\right)^{C_r} \to L^{C_m},$$

whereas the Restriction maps make use of the map γ from Section 3.1:

$$R_r\colon L^{C_{rm}} = (L^{C_r})^{C_m} \xrightarrow{\gamma^{C_m}} (\Phi^{C_r} L)^{C_m} \xrightarrow{r_{C_r}} L^{C_m}.$$

We suppressed several instances of maps ρ from the notation to keep the formulas readable. The appropriate relations can then be checked using the definition of cyclotomicity, but since we will spend more time on these later (4.6.28), we omit the details for now.

Note that since the model structures on $\mathbf{O}\mathcal{T}$ we discuss are topological in the sense of 6.1.8, it is in particular simplicial via the Quillen equivalence between spaces and simplicial sets. Hence there is a concrete model in $\mathbf{O}\mathcal{T}$ for the homotopy limit (e.g., [HirL, 18.1.8]) in the following definition:

Definition 4.6.16. Let L be a cyclotomic spectrum, the *topological cyclic homology spectrum* $\mathrm{TC}(L)$ is the orthogonal spectrum

$$\mathrm{TC}(L) := \operatorname*{holim}_{n \in \mathbb{I}} T^{C_n}.$$

For a commutative orthogonal ring spectrum A, abbreviate $\mathrm{TC}(\mathrm{THH}(A))$ as $\mathrm{TC}(A)$, and call $\mathrm{TC}(A)$ the *topological cyclic homology spectrum* of A.

Remark 4.6.17. Note that talking about cyclotomic commutative ring spectra here does not gain a lot of benefits, even though our construction of $\mathrm{THH}(A)$ gives the structure for free: Since the homotopy limit involves some objectwise fibrant replacement, we can only hope for $\mathrm{TC}(L)$ to have the correct homotopy type if L is at least an Ω-spectrum. Since we cannot guarantee, that the *fibrant* replacement functor in the $\mathbb{S}$-model structure preserves the Φ^N, we have to use a q-fibrant replacement functor, which in general destroys the strict commutativity. By [Kr, 5.1.10], the q-fibrant replacement functor preserves cyclotomicity, so it seems most natural to use it here (at the price of losing commutativity). However, the functoriality of the Loday functor $\bigwedge_X(A)$ will allow us to identify the higher analog of the cyclotomic structure much easier than in the classical setup, so it is still worthwhile to use it even when dealing with TC and not just THH.

4.6.18 Covering Homology

In this final section, we identify the structure on $\bigwedge_G(A)$, that is used when defining higher topological cyclic homology or covering homology as in [CDD], respectively [BCD].

We fix a compact Lie group G such that for all kernels K of surjective isogonies of G Condition 1 holds and G satisfies Condition 2 with respect to K. The main example we have in mind is $G = \mathbb{T}^n$ the n-dimensional torus, with the family $\mathscr{A}$ of all kernels of isogenies, since this is gives the higher analog of topological cyclic homology.

Definition 4.6.19. Let A be an $\mathbb{S}$-cofibrant commutative orthogonal ring spectrum. Define the *iterated topological Hochschild homology spectrum* to be the commutative orthogonal S^1-spectrum

$$\bigwedge_{\mathbb{T}^n} A = A \otimes \mathbb{T}^n.$$

Note that by the defining adjunctions of the categorical tensor, $A \otimes \mathbb{T}^n$ is isomorphic to the n-fold iterated application of THH to A, and in particular one could study the

S^1-equivariant structure induced by each of these iterations separately. However the current setup allows us to investigate much more intricate diagonal phenomena since we have the whole $\mathbb{T}^n$-equivariant structure available. We begin with a proposed definition of the higher analog of cyclotomicity mentioned in [CDD, 2.1]:

Definition 4.6.20. Let G and $\mathscr{A}$ be as above. A morphism of orthogonal G-spectra is called a *cyclotomic π_*-isomorphism* if it is a $\pi_*^{\mathscr{A}}$-isomorphism.

Definition 4.6.21. Let $\alpha\colon G \to G$ be a surjective isogeny with kernel L_α. Let

$$\phi_\alpha\colon G/L_\alpha \to G$$

be the isomorphism sending the class gL_α to $\alpha(g)$ and let $\rho_\alpha\colon G \to G/L_\alpha$ be its inverse.

Remark 4.6.22. Note that the notation is coherent with Definition 4.6.6, where the isogeny α of S^1 associated to a cyclic subgroup of order n is of course the map induced by raising a complex number z to the nth-power z^n. Since in the one-dimensional case such (orientation preserving) isogenies and their kernels are in one to one correspondence, there is no loss of information in the indexing.

Construction 4.6.23. For α and β isogenies we can form their composite $\alpha \circ \beta = \alpha\beta$ and their kernels $L_\beta \subset L_{\alpha\beta}$ satisfy

$$\rho_\beta(L_\alpha) = L_{\alpha\beta}/L_\beta.$$

Hence as in 4.6.8 there is a natural map

$$\Phi^{L_{\alpha\beta}} A \to \Phi^{L_{\alpha\beta}/L_\beta}\Phi^{L_\beta} A, \tag{4.6.24}$$

which using Proposition 3.1.46 induces a natural map

$$\rho_{\alpha\beta}^*\Phi^{L_{\alpha\beta}} A \to \rho_\alpha^*\Phi^{L_\alpha}\rho_\beta^*\Phi^{L_\beta} A,$$

and both of these become isomorphisms for $\mathbb{S}_{\mathrm{reg}}$-cellular G-spectra A.

Definition 4.6.25. A cyclotomic orthogonal G-spectrum is an orthogonal G-spectrum A, together with cyclotomic π_*-isomorphisms

$$r_\alpha\colon \rho_\alpha^*\Phi^{L_\alpha} A \to A,$$

for all isogenies α whose kernel L_α is in $\mathscr{A}$, such that the diagrams

$$\begin{array}{ccc}
\rho_\alpha^*\Phi^{L_\alpha}\rho_\beta^*\Phi^{L_\beta} A & \longleftarrow & \rho_{\alpha\beta}^*\Phi^{L_{\alpha\beta}} A \\
\Big\downarrow{\scriptstyle \rho_\alpha^*\Phi^\alpha r_\beta} & & \Big\downarrow{\scriptstyle r_{\alpha\beta}} \\
\rho_\alpha^*\Phi^\alpha A & \xrightarrow{\;r_\alpha\;} & A
\end{array}$$

commute for all isogenies α and β with L_α, L_β and $L_{\alpha\beta}$ in $\mathscr{A}$.
A *map of cyclotomic spectra* is a morphism of orthogonal S^1-spectra which commutes with the cyclotomic structure maps r_α.

One can choose to fix a collection $\mathcal{I}$ of isogenies instead of the family $\mathscr{A}$ of kernels and get a similar definition. For example to get a complete analog of the one dimensional case from Definition 4.6.10, one should restrict to orientation preserving isogenies of S^1. The analog of Theorem 4.6.7 uses our results 4.5.17 and 4.4.25 on the equivariant structure of the smash powers over more general compact Lie groups:

Proposition 4.6.26. *Let A be an $\mathbb{S}$-cofibrant commutative orthogonal ring spectrum then the underlying G-spectrum of $\bigwedge_G A$ is cyclotomic and $\bigwedge_G(-)$ sends π_*-isomorphisms to cyclotomic maps that are cyclotomic π_*-isomorphisms.*

We want to identify the Restriction and Frobenius maps that are used to define the covering homology and in particular higher topological cyclic homology (cf. [CDD, 2.2]). Since the indexing via isogenies and, in particular, the maps ρ_α complicate the notation significantly, we start by defining relative versions, postponing the coordinate change ensued by changing back to G-spectra to Definition 4.6.29:

Definition 4.6.27. Let X be a G-space, let $N \subset H$ be normal subgroups of G, and let $\rho_H^N\colon\ G/N \to G/H$ be the projection. For A an $\mathbb{S}$-cofibrant commutative orthogonal ring spectrum, the G/H-spectrum $(\bigwedge_X A)^H$ becomes a G/N-spectrum via the restriction along ρ_H^N. The *Frobenius map* F_N^H is the morphism of G/N-spectra

$$F_N^H\colon \left(\bigwedge\nolimits_X A\right)^H \to \left(\bigwedge\nolimits_X A\right)^N$$

given by the inclusion of fixed points on on each level. The *Restriction map* R_N^H is the natural map of G/H-spectra

$$R_N^H\colon \left(\bigwedge\nolimits_X A\right)^H \cong \left(\left(\bigwedge\nolimits_X A\right)^N\right)^{H/N} \xrightarrow{\gamma^{H/N}} \left(\Phi^N \bigwedge\nolimits_X A\right)^{H/N} \cong \left(\bigwedge\nolimits_{X_N} A\right)^{H/N}$$

Note that F_N^H defines a natural transformation $(-)^H \to (-)^N$. The following is the first example of a higher (relative) version of the relations (4.6.14):

Proposition 4.6.28. *The following diagram is commutative for all $\mathbb{S}$-cofibrant orthogonal spectra A and normal subgroups $N \subset H \subset J$ of G:*

$$\begin{array}{ccc} (\bigwedge_X A)^J & \xrightarrow{R_N^J} & \left(\bigwedge_{X_N} A\right)^{J/N} \\ \downarrow{\scriptstyle F_H^J} & & \downarrow{\scriptstyle F_{H/N}^{J/N}} \\ (\bigwedge_X A)^H & \xrightarrow[R_N^H]{} & \left(\bigwedge_{X_N} A\right)^{H/N} \end{array}$$

Proof. We split the diagram into the following:

$$\begin{array}{ccccccc} (\bigwedge_X A)^J & \xrightarrow{\cong} & ((\bigwedge_X A)^N)^{J/N} & \xrightarrow{\gamma^{J/N}} & (\Phi^N \bigwedge_X A)^{J/N} & \xrightarrow{\cong} & \left(\bigwedge_{X_N} A\right)^{J/N} \\ \downarrow{\scriptstyle F_H^J} & & \downarrow{\scriptstyle F_{H/N}^{J/N}} & & \downarrow{\scriptstyle F_{H/N}^{J/N}} & & \downarrow{\scriptstyle F_{H/N}^{J/N}} \\ (\bigwedge_X A)^H & \xrightarrow[\cong]{} & ((\bigwedge_X A)^N)^{H/N} & \xrightarrow[\gamma^{H/N}]{} & (\Phi^N \bigwedge_X A)^{H/N} & \xrightarrow[\cong]{} & \left(\bigwedge_{X_N} A\right)^{H/N} \end{array}$$

The two right squares are commutative by the naturality of the Frobenius $F_{H/N}^{J/N}$. Commutativity of the left square can be checked levelwise: Let B be any J spectrum, and V a representation of $J/N/H/N \cong J/H$, then the diagram of inclusions of fixed points commutes:

$$\begin{array}{ccc} (B_V)^J & \xrightarrow{\cong} & \left(B_V^N\right)^{J/N} \\ \downarrow & & \downarrow \\ (B_V)^H & \xrightarrow{\cong} & \left(B_V^N\right)^{H/N} \end{array}$$

□

Switching back to the isogeny notation, yields the following:

Definition 4.6.29. Let α and β be isogenies of G as above, and view $(\bigwedge_G A)^{L_\alpha}$ as a G-spectrum via ρ_α. Then define the *Frobenius maps* F^α as the natural morphism of G-spectra

$$F^\alpha := \rho_\beta^* \circ F_{L_\beta}^{L_{\alpha\beta}} \circ \phi_{\alpha\beta}^* \colon (\bigwedge_G A)^{L_{\alpha\beta}} \to (\bigwedge_G A)^{L_\beta},$$

and the *Restriction maps* R_β as the natural morphism of G-spectra

$$R_\beta := \rho_\alpha^* \circ (\rho_\beta^*)^{L_\alpha} \circ R_{L_\beta}^{L_{\alpha\beta}} \circ \phi_{\alpha\beta}^* \colon (\bigwedge_G A)^{L_{\alpha\beta}} \to (\bigwedge_{G/L_\beta} A)^{L_\alpha} \cong (\bigwedge_G A)^{L_\alpha},$$

Corollary 4.6.30. *The Restriction and Frobenius maps satisfy the following relations:*

$$\begin{aligned} F^\alpha &= \mathrm{id} \quad \textit{for } \alpha \textit{ invertible}, \\ F^\beta F^\alpha &= F^{\alpha\beta}, \\ R_\beta R_\alpha &= R_{\beta\alpha}, \\ R_\beta F^\alpha &= F^\alpha R_\beta. \end{aligned}$$

Proof. For the first one, use that $\rho_H^N = \rho_H \circ \phi_N$. The second one is immediate in the relative version and the third one uses the isomorphism (4.6.24). The last relation follows from (4.6.28). Note that even though γ is the identity map between the categorical and geometric $\{e\}$-fixed points, R_α is not usually trivial for α invertible because of the coordinate changes ρ^* involved. □

For completeness we very briefly repeat the higher analogs of Definitions 4.6.13 and 4.6.16, for more details, see [CDD, 2.3]:
Denote by $\mathcal{C}$ be the category with one object and morphisms the isogenies $\alpha \in \mathcal{I}$, respectively with $L_\alpha \in \mathscr{A}$.

Definition 4.6.31. Let $\mathcal{Ar}_\mathcal{C}$ be the twisted arrow category of $\mathcal{C}$, i.e., the category with the isogenies as objects and morphisms $\alpha \to \beta$ given by diagrams

$$\begin{array}{ccc} \star & \xrightarrow{\gamma} & \star \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ \star & \xleftarrow[\delta]{} & \star \end{array}$$

with composition given by horizontal concatenation of diagrams. Note that every such morphism is represented by the equation $\alpha = \delta \circ \beta \circ \gamma$ and factors as

$$\begin{array}{ccccc} \star & \xrightarrow{\mathrm{id}} & \star & \xrightarrow{\gamma} & \star \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta\gamma} & & \downarrow{\scriptstyle \beta} \\ \star & \xleftarrow[\delta]{} & \star & \xleftarrow[\mathrm{id}]{} & \star. \end{array}$$

Construction 4.6.32. For A an $\mathbb{S}$-cofibrant commutative orthogonal ring spectrum, we define a functor $\mathcal{A}r_{\mathcal{C}} \to G\mathbf{O}\mathcal{T}$, by sending the isogeny α to the categorical fixed point spectrum $(\bigwedge_G A)^{L_\alpha}$, viewed as a G-spectrum via ρ_α^*. Morphisms $\delta\beta = \delta \circ \beta \circ \mathrm{id}$ are sent to the Frobenii F^δ and morphisms $\beta\gamma = \mathrm{id} \circ \beta \circ \gamma$ to the Restrictions R_γ. Functoriality is a consequence of the relations from Corollary 4.6.30.

Definition 4.6.33. Let A be an $\mathbb{S}$-cofibrant commutative orthogonal ring spectrum. The *covering homology* associated to $\mathcal{C}$ is the functor from $\mathcal{A}r_{\mathcal{C}}$ to commutative orthogonal G-ring spectra

$$\alpha \mapsto \rho_\alpha^*(\bigwedge_G A)^{L_\alpha}.$$

As in the the one dimensional case, the associated homotopy limit becomes homotopically meaningful only after applying a fibrant replacement to $\bigwedge_G(A)$ (cf. Remark 4.6.17). There is even more structure on $\bigwedge_G A$. In [CDD, 2.5,3.2], the authors define Verschiebung maps for the general case and higher differentials for the special case of the p-adic n-torus. Both the definitions and verifications of relations analogous to 4.6.30 rely heavily on a good understanding of the stable equivariant transfer:

Definition 4.6.34. [CDD, 2.4] Let G and $\mathcal{I}$ be as above. For α and β isogenies in $\mathcal{I}$, choose a finite dimensional orthogonal G-representation W and an open G-embedding

$$i\colon W \times G/L_\beta \to W \times G/L_{\alpha\beta}$$

over the projection $\rho_{\alpha\beta} \circ \phi_\beta \colon G/L_\beta \to G/L_{\alpha\beta}$. Applying the Thom construction to this embedding yields a G-equivariant map

$$tr_\alpha = tr_\beta^{\alpha\beta} \colon S^W \wedge (G/L_{\alpha\beta})_+ \to S^W \wedge (G/L_\beta)_+$$

called the transfer, which does not depend on the choice of W or the embedding i up to stable equivariant homotopy.

Proofs for both the existence and the properties of the transfer maps are spread throughout the literature, the exposition in the original paper [KP] and its follow-up [KP78] very helpful, since in particular the existence is treated nicely there, which is usually omitted in later accounts. The exposition in [LMS, IV] is very thorough. A list of properties needed for the treatment of covering homology is given in [CDD, Proposition 2.4] and since our notation agrees with the one used in op. cit., we omit further details. Again we start with a version of the definition of the Verschiebung maps, that omits the coordinate changes:

Definition 4.6.35. Let $N \subset H \subset G$ be a sequence of subgroups and let

$$tr_N^H \colon S^W \wedge (G/N)_+ \to S^W \wedge (G/H)_+$$

be a model for the transfer. Define for an orthogonal G-spectrum B the *Verschiebung map* V_N^H be the natural stable map terminology needs to be explained: shift seems to have disappeared: can it come back again? $F(?, X)$ is the cotensor induced by the transfer on fixed points in the following way

$$\begin{array}{ccc}
B^N & \xrightarrow{\simeq} & (F(S^W, \mathrm{sh}_W B))^N \\
 & & \downarrow{=} \\
 & & F(S^W, \mathrm{sh}_W B)^N \\
 & & \downarrow{\cong} \\
{\scriptstyle V_N^H}\downarrow & & F(S^W \wedge G/N_+, \mathrm{sh}_W B)^G \\
 & & \downarrow{(tr_N^H)_*} \\
 & & F(S^W \wedge G/H_+, \mathrm{sh}_W B)^G \\
 & & \downarrow{\cong} \\
B^H & \xrightarrow[\simeq]{} & F(S^W, \mathrm{sh}_W B)^H
\end{array}$$

The version including the instances of ρ is the following:

Definition 4.6.36. Let α and β be surjective isogenies of G as above, and view $(\bigwedge_G)^{L_\alpha}$ as a G-spectrum via ρ_α. Then define the *Verschiebung maps* V_β as the natural stable morphisms of G-spectra

$$V_\alpha := \rho_{\alpha\beta}^* \circ V_{L_\beta}^{L_{\alpha\beta}} \circ \phi_\beta^* \colon (\bigwedge_G A)^{L_\beta} \to (\bigwedge_G A)^{L_{\alpha\beta}}.$$

Again, there are various relations between the Verschiebung, Frobenius and Restriction maps, and the authors of [CDD] develop the theory nicely. Since their methods are sufficiently general to be applied to our setting, for instance, essentially the same proof as in [CDD] gives that

Proposition 4.6.37. *Restriction and Verschiebung commute, that is, for α and β in $\mathcal{I}$, $V_\gamma R_\alpha = R_\alpha V_\gamma$ in the homotopy category $HoG\mathbf{O}\mathcal{T}$.*

We close our exposition by mentioning that for the case of G a (p-adic) torus, there is a fourth kind of structure maps, the higher differentials, which are defined using a stable splitting of $S^1_+ \simeq S^0 \vee S^1$, which can again be defined in terms of the equivariant transfer above (cf. [CDD, 3]). An exhaustive list of relations between these can be found in [CDD, 3.22], but it would go too far to reformulate them here, since the indexing alone is intricate enough to require extensive study. However, we have already seen above that the setting of orthogonal spectra, the smash power 4.3.9 is well equipped for the study of covering homology and the higher structure surrounding it, while having the advantage of being in some ways more concrete than the version using the Bökstedt approach.

Chapter 5

Category Theory

5.1 Some Category Theory

5.1.1 Categories

We recall some of the basics of category theory. We assume that the reader is familiar with the notions in this chapter, but the explicit definitions allow for an easier transition to monoidal and enriched categories in the next sections. The canonical reference and source for these definitions is Chapter I of [McL], though we have allowed ourselves some reformulations for the sake of uniformity when switching to the enriched setting.

Example 5.1.2. For any functor, there is an identity natural transformation, and composition of natural transformation is associative. Therefore the set of functors $\mathcal{C} \to \mathcal{D}$, denoted by $\mathrm{Cat}(\mathcal{C}, \mathcal{D})$ is itself a category with morphisms the natural transformations. A category arising in this way this is called *functor category*.

5.1.3 Monoidal Categories

Again we repeat the basic definitions as far as they will be used in the enriched setting in the next section. Again the definitions are only slight reformulations of the ones in [McL, VII], adapted to our needs.

Definition 5.1.4. A *monoidal category* consists of the following data:

- An *underlying category* $\mathcal{C}$.
- A bifunctor (i.e., a functor out of the product category) $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called the monoidal product.
- A designated object $\mathbb{I}$ of $\mathcal{C}$, called the identity object.
- Natural isomorphisms $\lambda\colon (\mathbb{I} \otimes \mathrm{id}) \to \mathrm{id}$ and $\rho(\mathrm{id} \otimes \mathbb{I}) \to \mathrm{id}$ expressing that $\mathbb{I}$ is a left and right identity object for the monoidal product.
- A natural isomorphism $a\colon [(- \otimes -) \otimes -] \to [- \otimes (- \otimes -)]$ expressing that the monoidal product is associative.

The natural transformations have to satisfy the following two coherence conditions:

- For all objects A and B of $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} (A \otimes \mathbb{I}) \otimes B & \xrightarrow{a_{A,I,B}} & A \otimes (\mathbb{I} \otimes B) \\ {\scriptstyle \rho_A \otimes \mathrm{id}_B} \searrow & & \swarrow {\scriptstyle \mathrm{id}_A \otimes \lambda_B} \\ & A \otimes B & \end{array}$$

- For all objects A, B, C and D of $\mathcal{C}$, the following pentagon commutes:

$$\begin{array}{ccccc} & & (A \otimes (B \otimes C)) \otimes D & & \\ & {\scriptstyle a_{A,B,C} \otimes \mathrm{id}_D} \nearrow & & \searrow {\scriptstyle a_{A,B \otimes C,D}} & \\ ((A \otimes B) \otimes C) \otimes D & & & & A \otimes ((B \otimes C) \otimes D) \\ & {\scriptstyle a_{A \otimes B,C,D}} \searrow & & \swarrow {\scriptstyle id_A \otimes a_{B,C,D}} & \\ & (A \otimes B) \otimes (C \otimes D) & \xrightarrow[a_{A,B,C \otimes D}]{} & A \otimes (B \otimes (C \otimes D)) & \end{array}$$

Instead of the tupel $(\mathcal{C}, \otimes, \mathbb{I}, \lambda, \rho, a)$, we often just refer to the monoidal category as $(\mathcal{C}, \otimes, \mathbb{I})$ or even just to $\mathcal{C}$, when it is clear which monoidal structure is meant.

Definition 5.1.5. A *lax monoidal functor* $\mathcal{F}\colon (\mathcal{C}, \otimes, \mathbb{I}) \to (\mathcal{D}, \times, \mathbb{J})$ between monoidal categories consists of the following data:

- An underlying functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$.
- A natural transformation $\mu\colon [\mathcal{F} \times \mathcal{F}] \to \mathcal{F}(- \otimes -)$.
- A designated morphism $\iota\colon \mathbb{J} \to \mathcal{F}(\mathbb{I})$ in $\mathcal{D}$.

These have to satisfy the following coherence conditions:

- For all objects A, B and C of $\mathcal{C}$, the following diagram commutes in $\mathcal{D}$:

$$\begin{array}{ccc} (\mathcal{F}(A) \times \mathcal{F}(B)) \times \mathcal{F}(C) & \xrightarrow{a_{\mathcal{D}}} & \mathcal{F}(A) \times (\mathcal{F}(B) \times \mathcal{F}(C)) \\ {\scriptstyle \mu_{A,B} \times \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \mathrm{id} \times \mu_{B,C}} \\ \mathcal{F}(A \otimes B) \times \mathcal{F}(C) & & \mathcal{F}(A) \times (\mathcal{F}(B \otimes C)) \\ {\scriptstyle \mu_{A \otimes B,C}} \downarrow & & \downarrow {\scriptstyle \mu_{A,B \otimes C}} \\ \mathcal{F}((A \otimes B) \otimes C) & \xrightarrow[\mathcal{F}(a_{\mathcal{C}})]{} & \mathcal{F}(A \otimes (B \otimes C)) \end{array}$$

- For every object A of $\mathcal{C}$, the following diagrams commute in $\mathcal{D}$:

$$\begin{array}{ccc} \mathcal{F}(A) \times \mathbb{J} & \xrightarrow{\rho_{\mathcal{D}}} & \mathcal{F}(A) \\ {\scriptstyle \mathrm{id} \times \iota_{\mathcal{D}}} \downarrow & & \uparrow {\scriptstyle \mathcal{F}(\rho_{\mathcal{C}})} \\ \mathcal{F}(A) \times \mathcal{F}(\mathbb{I}) & \xrightarrow[\mu_{A,\mathbb{I}}]{} & \mathcal{F}(A \otimes \mathbb{I}) \end{array} \qquad \begin{array}{ccc} \mathbb{J} \times \mathcal{F}(A) & \xrightarrow{\lambda_{\mathcal{D}}} & \mathcal{F}(A) \\ {\scriptstyle \iota_{\mathcal{D}} \times \mathrm{id}} \downarrow & & \uparrow {\scriptstyle \mathcal{F}(\lambda_{\mathcal{C}})} \\ \mathcal{F}(\mathbb{I}) \times \mathcal{F}(A) & \xrightarrow[\mu_{\mathbb{I},A}]{} & \mathcal{F}(I \times A) \end{array}$$

A lax monoidal functor $(\mathcal{F}, \mu, \iota)$ is *strong monoidal* if μ and ι are (natural) isomorphisms, it is *strict monoidal* if they are the identity (transformation).
Again we often only refer to $\mathcal{F}$ as the monoidal functor, suppressing μ and ι in the notation, where they are not critical to the discussion.

Definition 5.1.6. A monoidal category $(\mathcal{C}, \otimes, \mathbb{I})$ is called cartesian, if $\otimes$ is the categorical product and $\mathbb{I}$ is a terminal object.

Some monoidal categories have additional extra structure:

Definition 5.1.7. A monoidal category $(\mathcal{C}, \otimes, \mathbb{I})$ is called *closed*, if for all objects A of $\mathcal{C}$, the functor $(- \otimes A)\colon \mathcal{C} \to \mathcal{C}$ has a right adjoint ([McL, IV.1]), denoted by $\mathbf{Hom}(A, -)$. Objects of $\mathcal{C}$ the form $\mathbf{Hom}(A, B)$ are called internal **Hom** objects, the counits of these adjunctions are usually called the evaluations $\mathbf{Hom}(A, B) \otimes A \to B$.

Lemma 5.1.8. *If* $(\mathcal{C}, \otimes, \mathbb{I})$ *is closed monoidal, then there is a natural isomorphism:*

$$\mathbf{Hom}(A \otimes B, C) \cong \mathbf{Hom}(A, \mathbf{Hom}(B, C)).$$

Construction 5.1.9. Note that for any locally small monoidal category $(\mathcal{C}, \otimes, \mathbb{I})$, we get a lax monoidal functor $\mathcal{C}(\mathbb{I}, -)\colon \mathcal{C} \to \mathrm{Set}$, where the category of sets has the cartesian monoidal structure. The unit morphism ι sends the terminal one-point set to the identity morphism of $\mathbb{I}$, whereas the natural transformation

$$\mu\colon \mathcal{C}(\mathbb{I}, A) \times \mathcal{C}(\mathbb{I}, B) \xrightarrow{\otimes} \mathcal{C}(\mathbb{I} \otimes \mathbb{I}, A \otimes B) \cong \mathcal{C}(\mathbb{I}, A \otimes B),$$

uses the isomorphism $\lambda_{\mathbb{I}}\colon \mathbb{I} \otimes \mathbb{I} \to \mathbb{I}$. This functor assigns to objects of $\mathcal{C}$ their *underlying sets*.
If $\mathcal{C}$ is additionally closed and A and B are objects of $\mathcal{C}$, then the adjunction

$$(- \otimes A)\colon \mathcal{C} \rightleftarrows \mathcal{C}\colon \mathbf{Hom}(A, -)$$

gives the following natural isomorphism:

$$\mathcal{C}(\mathbb{I} \otimes A, B) \cong \mathcal{C}(\mathbb{I}, \mathbf{Hom}(A, B))$$

Since $\mathbb{I} \otimes A$ is isomorphic to A via λ_A, this implies that

$$\mathcal{C}(A, B) \cong \mathcal{C}(\mathbb{I}, \mathbf{Hom}(A, B)),$$

i.e., the underlying set of the internal **Hom** object $\mathbf{Hom}(A, B)$ is indeed naturally isomorphic to the morphism set $\mathcal{C}(A, B)$.
Considerations in this spirit lead to the study of enriched categories. We will discuss these further in Section 5.2.

For any monoidal category, there are categories of *monoids* and (*left* or *right*) *modules* over such. Definitions can for example be found in [McL, VII.3,4], and will be omitted here.

5.1.10 Symmetric Monoidal Categories

We repeat more of the definitions from [McL, XI], adapted to our notation.

Definition 5.1.11. A *symmetric monoidal category* is a monoidal category $(\mathcal{C}, \otimes, \mathbb{I}, \lambda, \rho, a)$, together with a natural isomorphism τ

$$\tau : \otimes \to \otimes \circ \text{twist},$$

where twist is the bifunctor that permutes the two inputs.
This data has to satisfy additional coherence conditions:

- The composition of τ with itself is the identity, i.e.,

  $$\tau_{B,A} \circ \tau_{A,B} = \mathrm{id}_{A \otimes B},$$

  for all objects A and B of $\mathcal{C}$.
- Compatibility with the unit, i.e.,

  $$\rho = \lambda \circ \tau_{-,\mathbb{I}}.$$

- For all objects A, B, and C of $\mathcal{C}$, the following hexagon commutes:

$$\begin{array}{ccccc} (A \otimes B) \otimes C & \xrightarrow{a} & A \otimes (B \otimes C) & \xrightarrow{\tau} & (B \otimes C) \otimes A \\ {\scriptstyle \tau \otimes \mathrm{id}_C}\downarrow & & & & \downarrow{\scriptstyle a} \\ (B \otimes A) \otimes C & \xrightarrow[a]{} & B \otimes (A \otimes C) & \xrightarrow[\mathrm{id}_B \otimes \tau]{} & B \otimes (C \otimes A) \end{array}$$

Example 5.1.12. All cartesian or cocartesian monoidal categories are symmetric, making use of the universal properties of (co-) products and the projections to respectively inclusions of coproduct factors.

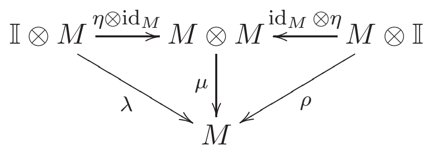
Since the following definition is widely used, but is only implicit in [McL], we give a few more details:

Definition 5.1.13. A *commutative monoid* in a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{I}, \tau)$ consists of the following data:

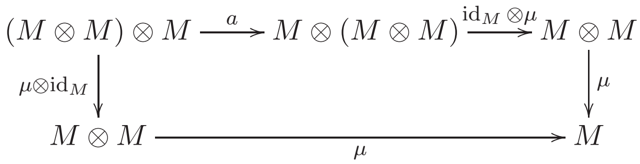
- An object M of $\mathcal{C}$.
- A morphism $\eta : \mathbb{I} \to M$ in $\mathcal{C}$, called the unit of M.
- A morphism $\mu : M \otimes M \to M$ in $\mathcal{C}$, called the multiplication of M.

Such that the following diagrams are commutative:

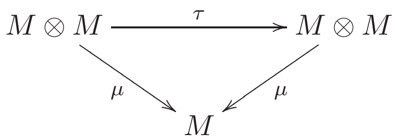
- *unit:*



- *associativity:*



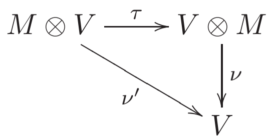
- *commutativity:*



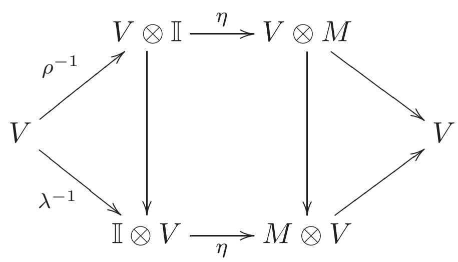
Again we can define categories of commutative monoids and modules over such as in the non-symmetric case. We work extensively with these in the case of $\mathcal{C}$ being the category $\mathbf{O}\mathcal{T}$ of orthogonal spectra. The following lemmas are well known, but it seems hard to find explicit references:

Lemma 5.1.14. *Let M be a commutative monoid, then the categories of left M-modules and right M-modules are isomorphic.*

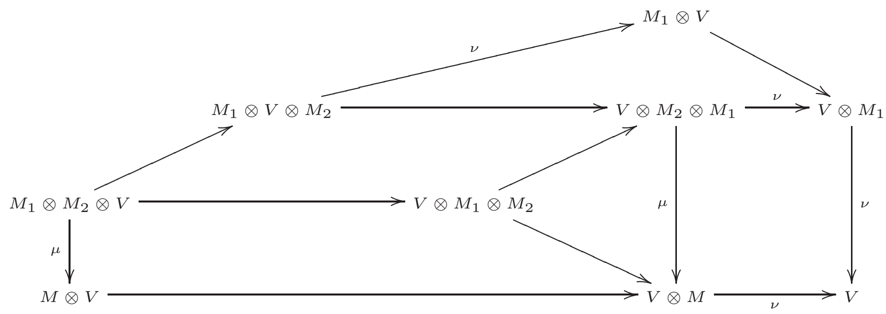
Proof. For V a right M module with action map $\nu : V \otimes M \to V$, define a left module structure on V in the following way:



There are coherence diagrams to be checked:



The left triangle commutes by the unit axiom for the symmetric monoidal structure, the middle square because the twist isomorphism is natural. Hence the lower composite is the identity since the upper one was. For associativity of the new multiplication map, check commutativity of the outermost two ways around the following diagram. We omit the categorical associativity isomorphisms, hence brackets, from the notation.



Here the center triangle commutes because M was commutative, and the lower right square does so because V was a right M-module. The center parallelogram is an instance of the hexagon coherence. The other subdiagrams both commute because of the naturality of the twist isomorphism.
An analogous argument gives a functor from left to right M-modules, and they are obviously inverses to each other. □

Lemma 5.1.15. *Let M be a commutative monoid in the closed symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{I}, \tau)$. Assume that $\mathcal{C}$ has equalizers and coequalizers. Then the category of (right) M-modules inherits a closed symmetric monoidal structure.*

Proof. For M-modules V and W define the monoidal product $V \otimes_M W$ of as the coequalizer

$$V \otimes M \otimes W \rightrightarrows V \otimes W \to V \otimes_M W, \tag{5.1.16}$$

where one of the arrows uses the action map on V, and the other the action on W precomposed with the twist $V \otimes \tau_{M,W}$. The internal **Hom** object $\mathbf{Hom}_M(V, W)$ is the equalizer

$$\mathbf{Hom}_M(V, W) \to \mathbf{Hom}(V, W) \rightrightarrows \mathbf{Hom}(V \otimes M, W),$$

where one of the arrows is induced by the action map of V, and the other one is induced by the adjoint of the action map of W, using the isomorphism $\mathbf{Hom}(V \otimes M, W) \cong \mathbf{Hom}(V, \mathbf{Hom}(M, W))$ (cf. 5.1.8). Checking coherence diagrams is then done using the universal properties of (co-)equalizers as well as the corresponding diagrams in $\mathcal{C}$ together with the same isomorphism 5.1.8. □

Lemma 5.1.17. *Let $(\mathcal{C}, \otimes, \mathbb{I})$ be a symmetric monoidal category. Then $\otimes$ is the coproduct in the category of commutative monoids in $\mathcal{C}$.*

Proof. The monoidal product of two commutative monoids M and N is again a commutative monoid using the unit map

$$\eta : \mathbb{I} \xrightarrow{\lambda^{-1}} \mathbb{I} \otimes \mathbb{I} \xrightarrow{\eta_M \otimes \eta_N} M \otimes N,$$

and the multiplication

$$\mu : M \otimes N \otimes M \otimes N \overset{\mathrm{id}\otimes\tau\otimes\mathrm{id}}{\longrightarrow} M \otimes M \otimes N \otimes N \overset{\mu_M\otimes\mu_N}{\longrightarrow} M \otimes N.$$

Then given maps of commutative monoids $M \to C$ and $N \to C$, we get a map

$$M \otimes N \to C \otimes C \xrightarrow{\mu_C} C.$$

On the other hand given a map of commutative monoids $M \otimes N \to C$, precomposition with the units of N and M, respectively, yields maps $M \to C$ and $N \to C$. These two constructions are obviously inverse to each other, hence $M \otimes N$ satisfies the universal property of the coproduct ([McL, III.3]). □

We will discuss monoids, (commutative) algebras and modules over such in various (symmetric) monoidal categories $(\mathcal{C}, \otimes, \mathbb{I})$. Often we use that the forgetful functors to $\mathcal{C}$ have left adjoints, and hence we recall how these adjoints are formed in general:

Lemma 5.1.18. *Let R be a monoid in $(\mathcal{C}, \otimes, \mathbb{I})$,*

- *the functor $- \otimes R$ is left adjoint to the forgetful functor from $\mathcal{C}$ to right R-modules.*
- *the functor $R \otimes -$ is left adjoint to the forgetful functor from $\mathcal{C}$ to left R-modules.*

In both cases the action of R simply uses the multiplication R. If R is commutative, and the category of R-modules has coproducts, then

- *the functor $\mathbb{A} := \coprod_{i\in\mathbb{N}} (-)^{\otimes_R i}$ is left adjoint to the forgetful functor from R-algebras to R-modules.*

If the category of R modules is cocomplete, then

- *the functor $\mathbb{E} := \coprod_{i\in\mathbb{N}} [(-)^{\otimes_R i}]_{\Sigma_i}$ is left adjoint to the forgetful functor from commutative R-algebras to R-modules.*

In both cases multiplication is by simply concatenating coproduct factors and the unit map is the inclusion of R as the factor indexed by 0.

Here $(-)^{\otimes_R i}$ denotes the i-fold tensor power over R. Then $[-]_{\Sigma_i}$ is taking the orbits of the action of Σ_i that permutes tensor factors, i.e., the action induces a functor from Σ_i viewed as a one-object category (cf. 5.2.17) and $[-]_{\Sigma_i}$ denotes its colimit. We could of course have given each of these functor in terms of the monads that the unit of the adjunction induces on $\mathcal{C}$.

Let Fin be the category of finite sets, and consider the skeleton $\overline{\mathrm{Fin}} \subseteq \mathrm{Fin}$ consisting of the finite sets $\mathbf{n} = \{1, 2, \ldots, n\}$ for $n \in \mathbf{N}$. Disjoint union is modelled permutatively by $\mathbf{n}+\mathbf{m} = \{1, \ldots, n+m\}$. Choose, once and for all, an inverse strongly symmetric monoidal equivalence $\mathrm{Fin} \to \overline{\mathrm{Fin}}$ (i.e., choose an ordering for each finite set; the associated natural isomorphisms are forced). For convenience we choose this equivalence so that finite sets of integers retain their order.

Let $\Sigma \subseteq \mathcal{I} \subseteq \overline{\mathrm{Fin}}$ be the permutative subcategories of respectively bijections and injections. Note that any injection $\phi\colon \mathbf{n} \to \mathbf{m}$ can uniquely be factored as the inclusion $\mathbf{n} \subseteq \mathbf{m}$ composed with a bijection $\mathbf{m} \cong \mathbf{m}$.

Let $\mathcal{C}$ be a symmetric monoidal category with monoidal product $\otimes$ and neutral element e. Let $e/\mathcal{C}$ be the category of objects under the neutral element e and $\mathbf{Com}_{\mathcal{C}}$ the category of symmetric monoids in $\mathcal{C}$.

Coherence for $\mathcal{C}$ amounts to saying that the assignment

$$(\mathbf{n}, c) \mapsto c^{\otimes \mathbf{n}} = (\dots (c \otimes c) \otimes \dots c) \otimes c$$

(n copies of c with parentheses moved as far left as possible) defines functors that are strong symmetric monoidal in each variable

$$\Sigma \times \mathcal{C} \to \mathcal{C}, \quad \mathcal{I} \times e/\mathcal{C} \to e/\mathcal{C}, \quad \mathrm{Fin} \times \mathbf{Com}_{\mathcal{C}} \to \mathbf{Com}_{\mathcal{C}}.$$

Since this is central to our discussion we spell out some of the details, but for clarity restrict ourselves to a permutative $\mathcal{C}$ (so that the units and associators are identities). If $\sigma \in \Sigma_n$ is a bijection, the commutator associated with σ is an isomorphism $c^{\otimes \sigma}\colon c^{\otimes \mathbf{n}} \cong c^{\otimes \mathbf{n}}$. This defines the first functor, which is strong symmetric monoidal in each factor via $c^{\otimes(\mathbf{m}+\mathbf{n})} = c^{\otimes \mathbf{m}} \otimes c^{\otimes \mathbf{n}}$ and the shuffle commutator $(c \otimes d)^{\otimes \mathbf{n}} \cong c^{\otimes \mathbf{n}} \otimes d^{\otimes \mathbf{n}}$. For the second functor, the object $(\mathbf{n}, f\colon e \to c)$ is sent to the composite $f^{\otimes \mathbf{n}}\colon e = e^{\otimes \mathbf{n}} \to c^{\otimes \mathbf{n}}$. On morphisms in $\mathcal{I}$ we only lack the inclusions $\mathbf{n} \subseteq \mathbf{m}$ which is sent to the morphism $\mathrm{id}^{\otimes \mathbf{n}} \otimes f^{\otimes\{1,\dots,m-n\}}\colon c^{\otimes \mathbf{n}} = c^{\mathbf{n}} \otimes e^{\otimes\{1,\dots,m-n\}} \to c^{\mathbf{n}} \otimes c^{\otimes\{1,\dots,m-n\}} = c^{\mathbf{m}}$ under e.

Using our chosen equivalence $\mathrm{Fin} \to \overline{\mathrm{Fin}}$ these functors extend to functors on the bijections/injections in Fin.

If c is a symmetric monoid in $\mathcal{C}$ with structure maps $\mu\colon c \otimes c \to c$ and $e \to c$, we define a morphism $c^{\otimes f}\colon c^{\otimes \mathbf{m}} \to c^{\otimes \mathbf{n}}$ for any function $f\colon \mathbf{m} \to \mathbf{n} \in \overline{\mathrm{Fin}}$ as the composite

$$c^{\otimes \mathbf{m}} \cong c^{\otimes f^{-1}(1)} \otimes \cdots \otimes c^{\otimes f^{-1}(n)} \to c^{\otimes \mathbf{n}},$$

where the isomorphism is the commutator and the second map is the tensor of the multiplications $\mu\colon c^{\otimes f^{-1}(j)} \to c$ (unit if $f^{-1}(j)$ is empty). Again, we extend to Fin.

Since small colimits can be chosen functorially, any category with finite coproducts is tensored over Fin. In particular, the coproduct in $\mathbf{Com}_{\mathcal{C}}$ is $\otimes$, which can easily cause a conflict of notation. Hence we retain the notation $c^{\otimes S}$ (or $\bigotimes_S c$) for this coproduct indexed over the finite set S, and let this be our *choice* of tensor.

In our applications, $\mathcal{C}$ is cocomplete and closed. Then we can extend $\mathrm{Fin} \times \mathbf{Com}_{\mathcal{C}} \to \mathbf{Com}_{\mathcal{C}}$ to a functor (strong monoidal in each variable)

$$\mathrm{Set} \times \mathbf{Com}_{\mathcal{C}} \to \mathbf{Com}_{\mathcal{C}}, \qquad S \mapsto \bigotimes_S c = \operatorname*{colim}_{T \subseteq S} \bigotimes_T c,$$

where T varies over the finite subsets of the set S, which agrees with the cotesor of $\mathbf{Com}_{\mathcal{C}}$ over Set.

5.2 Enriched Category Theory

5.2.1 Enriched Categories

Let $(\mathcal{V}, \otimes, \mathbb{I})$ be a monoidal category.

Definition 5.2.2. A *category* $\mathcal{C}$ *enriched over* $\mathcal{V}$, or a $\mathcal{V}$-category [K, 1.2] amounts to the following structure:

- A class $\mathrm{Ob}(\mathcal{C})$ of *objects of* $\mathcal{C}$.
- For every two objects A and B of $\mathcal{C}$ an object $\mathcal{C}(A, B)$ of $\mathcal{V}$ called the Hom-object of A and B.
- For every object A of $\mathcal{C}$, a distinguished morphism $\mathrm{id}_A\colon \mathbb{I} \to \mathcal{C}(A, A)$ in $\mathcal{V}$, called *the identity of* A.
- For every three objects A, B and C of $\mathcal{C}$ a morphism $\gamma\colon \mathcal{C}(B, C) \otimes \mathcal{C}(A, B) \to \mathcal{C}(A, C)$, called *the composition in* $\mathcal{C}$.

This data has to satisfy the following two conditions:

- For all objects A,B,C and D of $\mathcal{C}$, the following diagram commutes in $\mathcal{V}$:

$$\begin{array}{ccccc}
(\mathcal{C}(C,D) \otimes \mathcal{C}(B,C)) \otimes \mathcal{C}(A,B) & & \xrightarrow{\;a\;} & & \mathcal{C}(C,D) \otimes (\mathcal{C}(B,C) \otimes \mathcal{C}(A,B)) \\
\downarrow{\scriptstyle \gamma\otimes\mathrm{id}} & & & & \downarrow{\scriptstyle \mathrm{id}\otimes\gamma} \\
\mathcal{C}(B,D) \otimes \mathcal{C}(A,B) & & & & \mathcal{C}(C,D) \otimes \mathcal{C}(A,C) \\
& \searrow{\scriptstyle \gamma} & & \swarrow{\scriptstyle \gamma} & \\
& & \mathcal{C}(A,D) & &
\end{array}$$

- For all objects A and B in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccccc}
\mathbb{I} \otimes \mathcal{C}(A,B) & & & & \mathcal{C}(A,B) \otimes \mathbb{I} \\
\downarrow{\scriptstyle \mathrm{id}_B\otimes\mathrm{id}} & \searrow{\scriptstyle \lambda} & & \swarrow{\scriptstyle \rho} & \downarrow{\scriptstyle \mathrm{id}\otimes\mathrm{id}_A} \\
\mathcal{C}(B,B) \otimes \mathcal{C}(A,B) & \xrightarrow{\;\gamma\;} & \mathcal{C}(A,B) & \xleftarrow{\;\gamma\;} & \mathcal{C}(A,B) \otimes \mathcal{C}(A,A)
\end{array}$$

Example 5.2.3. Note that the usual notion of a category generalizes to this context. One checks easily that those are the same as categories enriched over the cartesian monoidal category Set of sets.

Example 5.2.4. The *trivial* $\mathcal{V}$*-category* $\star$ has one object C, and the morphism object $\star(C, C) = \mathbb{I}$.

Definition 5.2.5. A *functor enriched over* $\mathcal{V}$ from $\mathcal{D}$ to $\mathcal{C}$ consists of the following data:

- A function $\mathcal{F}\colon\ \mathrm{Ob}(\mathcal{D}) \to \mathrm{Ob}(\mathcal{C})$.

- For each object A and B of $\mathcal{D}$, a morphism $\mathcal{F}_{A,B}\colon \mathcal{D}(A,B) \to \mathcal{C}(\mathcal{F}(A), \mathcal{F}(B))$ in $\mathcal{V}$.

These have to satisfy the following coherence conditions:

- *(identity)* For all objects A of $\mathcal{D}$ the following diagram commutes in $\mathcal{V}$:

$$\begin{array}{ccc} & & \mathcal{D}(A,A) \\ & \overset{\mathrm{id}_A}{\nearrow} & \\ \mathbb{I} & & \Big\downarrow \mathcal{F}_{A,A} \\ & \underset{\mathrm{id}_{\mathcal{F}(A)}}{\searrow} & \\ & & \mathcal{C}(\mathcal{F}(A), \mathcal{F}(A)) \end{array}$$

- *(composition)* For all objects A, B and C of $\mathcal{D}$ the following diagram commutes in $\mathcal{V}$:

$$\begin{array}{ccc} \mathcal{D}(B,C) \otimes \mathcal{D}(A,B) & \xrightarrow{\quad\gamma\quad} & \mathcal{D}(A,C) \\ {\scriptstyle \mathcal{F}\otimes\mathcal{F}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathcal{F}} \\ \mathcal{C}(\mathcal{F}(B), \mathcal{F}(C)) \otimes \mathcal{C}(\mathcal{F}(A), \mathcal{F}(B)) & \xrightarrow[\gamma]{} & \mathcal{C}(\mathcal{F}(A), \mathcal{F}(C)) \end{array}$$

Definition 5.2.6. For two functors $\mathcal{F}, \mathcal{G}\colon \mathcal{D} \to \mathcal{C}$ enriched over $\mathcal{V}$, an *enriched natural transformation* $\alpha\colon\ \mathcal{F} \to \mathcal{G}$ consists of morphisms $\mathbb{I} \to \mathcal{C}(\mathcal{F}(A), \mathcal{G}(A))$ in $\mathcal{V}$ for all objects A of $\mathcal{D}$, such that the following coherence diagrams commute in $\mathcal{V}$ for all objects A and B of $\mathcal{D}$:

$$\begin{array}{ccccc} & \mathbb{I} \otimes \mathcal{D}(A,B) & \xrightarrow{\alpha_B \otimes \mathcal{F}} & \mathcal{C}(\mathcal{F}(B), \mathcal{G}(B)) \otimes \mathcal{C}(\mathcal{F}(A), \mathcal{F}(B)) & \\ \overset{\lambda^{-1}}{\nearrow} & & & & \overset{\gamma}{\searrow} \\ \mathcal{D}(A,B) & & & & \mathcal{C}(\mathcal{F}(A), \mathcal{G}(B)) \\ \underset{\rho^{-1}}{\searrow} & & & & \underset{\gamma}{\nearrow} \\ & \mathcal{D}(A,B) \otimes \mathbb{I} & \xrightarrow[\mathcal{G} \otimes \alpha_A]{} & \mathcal{C}(\mathcal{G}(A), \mathcal{G}(B)) \otimes \mathcal{C}(\mathcal{F}(A), \mathcal{G}(A)) & \end{array} \tag{5.2.7}$$

Remark 5.2.8. This definition gives the class $[\mathcal{D}, \mathcal{C}]_0(\mathcal{F}, \mathcal{G})$ of functors enriched over $\mathcal{V}$ the structure of a category (one checks that there is an identity transformation and that composition of natural transformations is associative). This makes the category $\mathcal{V}$-Cat of $\mathcal{V}$-enriched categories and $\mathcal{V}$-enriched functors into a 2-category, i.e., into a category enriched over Cat.

Remark 5.2.9. Under more assumptions, one can also define a $\mathcal{V}$-enriched functor category $[\mathcal{D}, \mathcal{C}]$. Let $\mathcal{V}$ be closed and complete and $\mathcal{D}$ be equivalent to a small category.

Then for two enriched functors $\mathcal{F}$ and $\mathcal{G}$, the following end exists and forms the morphism $\mathcal{V}$-space $[\mathcal{D}, \mathcal{C}](\mathcal{F}, \mathcal{G})$:

$$\int_{d\in\mathcal{D}} \mathcal{C}(\mathcal{F}(d), \mathcal{G}(d)) \to \prod_{d\in\mathcal{D}} \mathcal{C}(\mathcal{F}(d), \mathcal{G}(d)) \rightrightarrows \prod_{d,d'\in\mathcal{D}} \mathbf{Hom}(\mathcal{D}(d, d'), \mathcal{C}(\mathcal{F}(d), \mathcal{G}(d'))).$$

As indicated it can be expressed as the equalizer along two maps adjoint to the two ways around diagram (5.2.7) above. Composition and identities are then inherited from $\mathcal{C}$ (cf. [K, 2.1]).

Construction 5.2.10. Note that if we have a lax monoidal functor $(\mathbb{M}, \mu, \iota)\colon (\mathcal{V}, \otimes, \mathbb{I}) \to (\mathcal{W}, \boxtimes, \mathbb{J})$, any category $\mathcal{C}$ enriched over $\mathcal{V}$ gives a category enriched over $\mathcal{W}$, by just applying $\mathbb{M}$ to all the Hom objects. The identity morphisms are defined as the composites

$$\mathrm{id}'_A\colon \mathbb{J} \xrightarrow{\iota} \mathbb{M}[\mathbb{I}] \xrightarrow{\mathbb{M}[\mathrm{id}_A]} \mathbb{M}[\mathcal{C}(A, A)].$$

The composition is given by

$$\gamma'\colon\ \mathbb{M}[\mathcal{C}(B, C)] \boxtimes \mathbb{M}[\mathcal{C}(A, B)] \xrightarrow{\mu} \mathbb{M}[\mathcal{C}(B, C) \otimes \mathcal{C}(A, B)] \xrightarrow{\mathbb{M}[\gamma]} \mathbb{M}[\mathcal{C}(A, C)].$$

One checks that the coherence diagrams still commute.
Also, in the same way $\mathcal{V}$-enriched functors give $\mathcal{W}$-enriched functors and $\mathcal{V}$-enriched natural transformations give $\mathcal{W}$-enriched natural transformations via the lax monoidal functor $\mathbb{M}$. (One checks that $\mathbb{M}(\mathcal{F})$ still takes identities to identities and respects composition, and that the appropriate diagram for $\mathbb{M}\,\alpha$ still commutes using the structure maps of $\mathbb{M}$).

Remark 5.2.11. In the spirit of the above Remark 5.2.8, one can check that $\mathbb{M}$ induces a Cat-enriched, or 2-functor $\mathbb{M}\colon \mathcal{V}\text{-Cat} \to \mathcal{W}\text{-Cat}$, i.e., that $\mathbb{M}$ takes the identity $\mathcal{V}$-enriched natural transformations to the identity $\mathcal{W}$-enriched natural transformations, and that it respects composition of enriched natural transformations.

Example 5.2.12. In this way, if $\mathcal{V}$ is a locally small monoidal category, every category enriched over $\mathcal{V}$ has a canonical underlying "normal" category, i.e., one enriched over Set, with the same objects. The morphism sets are obtained by using the monoidal functor $\mathcal{V}(\mathbb{I}, -)$ from 5.1.9 in the way described above.

Remark 5.2.13. For $\mathcal{D}$ and $\mathcal{C}$ categories enriched over $\mathcal{V}$ as in 5.2.9, the underlying underlying Set-category of the functor $\mathcal{V}$-category $[\mathcal{D}, \mathcal{C}]$ is $[\mathcal{D}, \mathcal{C}]_0$, if the former exists.

Example 5.2.14. Let $(\mathcal{V}_0, \otimes, \mathbb{I})$ be a closed monoidal category. Then there is a $\mathcal{V}_0$-category $\mathcal{V}$ that restricts to $\mathcal{V}_0$ along the monoidal functor $\mathcal{V}(\mathbb{I}, -)$.
Define $\mathcal{V}$ as having the same objects as $\mathcal{V}_0$ and for morphism objects set $\mathcal{V}(A, B) = \mathbf{Hom}(A, B)$. Then composition is adjoint to iterated evaluation, and the axioms for an enriched category trivially hold. When discussing a specific category $\mathcal{V}_0$, we will often identify $\mathcal{V}$ and $\mathcal{V}_0$ and therefore say that $\mathcal{V}$ is enriched over itself, but there are also important cases where we explicitly keep the notation separate (e.g., 5.2.20).

Remark 5.2.15. When viewing $\mathcal{V}$ as enriched over itself in this sense, Lemma 5.1.8 can be reformulated to state that the adjunctions between $- \otimes A$ and $\mathbf{Hom}(A, -)$ are actually enriched, i.e., imply natural isomorphisms even on morphism objects.

Example 5.2.16. Let $\mathcal{V} = \mathcal{T}op$ the cartesian monoidal category of topological spaces. Then a category $\mathcal{C}$ enriched over $\mathcal{T}op$ is a usual category, with a choice of topology on each morphism set, such that the composition law gives continuous maps. More important for us is the closed monoidal variation $\mathcal{U}$, containing only the *compactly generated weak Hausdorff spaces.*
For another example let $\mathcal{V}$ be the category $\mathcal{T}$ *of based compactly generated weak Hausdorff spaces*, i.e., objects of $\mathcal{U}$ with a distinguished basepoint. We will usually drop the extra adjectives and just call these *spaces.*
Since $\mathcal{T}$ has products and coproducts, it is monoidal in several ways: with the cartesian product $\times$ and unit a one point space $\{*\}$, or, more importantly for us, with respect to the smash product $\wedge$ and unit S^0, the 0-sphere. The latter choice makes $\mathcal{T}$ closed monoidal, and we will denote the internal **Hom** spaces merely as $\mathcal{T}(-, -)$ in agreement with 5.2.14. The identity functor $(\mathcal{T}, \wedge, S^0) \to (\mathcal{T}, \times, \{*\})$ is lax monoidal, just as the functor $\mathcal{T} \to \mathcal{U}$ that forgets the basepoints. The monoidal structure maps are given by the projections $X \times Y \to X \wedge Y$ and the inclusion of $\{*\}$ as the non-basepoint of S^0. These functors give us a canonical way to view a category enriched over $(\mathcal{T}, \wedge, S^0)$ as one enriched over $(\mathcal{T}, \times, \{*\})$, or $\mathcal{U}$. The forgetful functor from $\mathcal{U}$ to Set preserves products and is therefore strict monoidal, indeed it is isomorphic to the functor described in 5.1.9. Hence a category enriched over either monoidal structure on $\mathcal{T}$ (or $\mathcal{U}$) is a category. In the other directions, including sets as discrete topological spaces and adding disjoint basepoints to spaces in $\mathcal{U}$ give left adjoints to the forgetful functors and are also (strong) monoidal. Hence together with 5.2.14 we can view $\mathcal{U}$ and $\mathcal{T}$ as enriched over either themselves or each other. Generally, categories enriched over any of the above are called *topological categories.* Enriched functors between both $\mathcal{T}op$- $\mathcal{U}$- and $\mathcal{T}$-categories are usually called the continuous functors.

Definition 5.2.17. For G a group, there is a *category* $\mathcal{G}$ *associated to* G. It consists of one object $\star$, and the morphism set $\mathcal{G}(\star, \star)$ is given as the group G. The neutral element of the group is the identity morphism and the group multiplication gives composition of morphisms. Often we use the group G and its associated category synonymously.

If G is a topological group, its associated category is canonically a topological category. If G is in $\mathcal{U}$, its associated category is canonically enriched over $\mathcal{U}$, and adding a disjoint basepoint, enriched over $\mathcal{T}$.

Definition 5.2.18. We denote the category of functors $G \to \text{Set}$ and natural transformations between them by GSet instead of $[G, \text{Set}]_0$, its objects are called G-sets. Note that a G-set is the same as a set with a (left) action of G, and a morphism of G-sets is a G-equivariant map.

Just like Set, the category GSet is a cartesian monoidal category with respect to the usual cartesian product of sets, which is given the diagonal G-action. The unit object is the trivial G-set consisting of only one point. Note that there are **two** obvious monoidal functors $G\text{Set} \to \text{Set}$. One is the forgetful functor, which is obviously product preserving, but this is *not* the functor described in 5.1.9. In fact, $G\text{Set}(\star, X)$ assigns to a G-set X its set of G-fixed points X^G, and this gives the second monoidal functor. We distinguish this in language by saying X *is* a set, but *has* X^G as its underlying set (of G-fixed points).

Definition 5.2.19. Let G be a group, a *G-category* is a category enriched over GSet. We call the elements of the morphism G-sets *morphisms*, whereas the elements of the underlying G-fixed point sets are called *G-maps*. As above, every G-category is also a category, and has an underlying G-fixed category.
A *G-functor* $F\colon \mathcal{D} \to \mathcal{C}$ between G-categories is an enriched functor of enriched categories, i.e., the induced maps on morphism G-sets

$$F\colon \mathcal{D}(X,Y) \to \mathcal{C}(FX,FY)$$

have to be G-equivariant.
Two types of natural transformations are important for us: A *natural G-transformation* $\alpha\colon F \to F'$ between two G-functors, is an enriched natural transformation of enriched functors, i.e., it consists of a G-map $\alpha_X \in \mathcal{C}(FX, F'X)^G$ for every object X of $\mathcal{D}$ such that the diagrams

$$\begin{array}{ccc} FX & \xrightarrow{Ff} & FY \\ {\scriptstyle \alpha_X}\downarrow & & \downarrow{\scriptstyle \alpha_Y} \\ F'X & \xrightarrow[F'f]{} & F'Y \end{array} ,$$

commute in $\mathcal{C}$ for all $f \in \mathcal{D}(X,Y)$. These are the morphisms in the functor category $[\mathcal{D}, \mathcal{C}]_0$.
On the other hand, there are the natural tranformations, given as collections of maps $\alpha_X \in \mathcal{C}(FX, F'X)$. On the set of these transformations G again acts by conjugation. Then as indicated in 5.2.13, the G-natural transformations are exactly the G-fixed natural transformations, so that the functor G-category $[\mathcal{D}, \mathcal{C}]$ has the functor category $[\mathcal{D}, \mathcal{C}]_0$ as its underlying (G-fixed) category.

The following combination of the above definitions will be important in our studies of equivariant orthogonal spectra. Let G be a (compactly generated weak Hausdorff) topological group, respectively the associated one object $\mathcal{T}$-category with morphism space G_+

Definition 5.2.20. The category of G-spaces $G\mathcal{T}$, consists of functors $G \to \mathcal{T}$ and natural transformations between them. In particular, objects of $G\mathcal{T}$ are spaces with a (left) action of G and morphisms are G-equivariant continuous maps.
Giving smash products the diagonal G-action, $G\mathcal{T}$ inherits a closed symmetric monoidal

structure from $\mathcal{T}$. Again this allows us to view $G\mathcal{T}$ as enriched over itself, and we shall use the notation $\mathcal{T}_G$ for the ensuing enriched category (5.2.14), as well as $\mathcal{T}_G(-,-)$ for the internal **Hom**-functor of $G\mathcal{T}$. Then $\mathcal{T}_G$ has G-spaces as objects, and morphisms are (not necessarily G-equivariant) continuous maps.

Definition 5.2.21. A category $\mathcal{C}_G$ is called a *topological G-category* if it is enriched over $G\mathcal{T}$. Such a $\mathcal{C}_G$ has a *G-fixed category* $G\mathcal{C}$ that is obtained by applying the fixed point functor to the morphism G-spaces.
The appropriate functors enriched over $G\mathcal{T}$ are called *continuous G-functors*. The appropriate enriched natural transformations are called *continuous natural G-transformations*. ([MM, p. 27] calls these natural G-maps between functors.) We will often drop the extra adjective "continuous" in the future.

Remark 5.2.22. Note that the fixed point functor $(-)^G \colon G\mathcal{T} \to \mathcal{T}$ has a left adjoint giving a space the trivial G-action. As $(-)^G$, this preserves (smash-) products and is therefore strict monoidal.

Monoidal functors starting in $G\mathcal{T}$ allow us to transport enrichments as in 5.2.10. Transportation along functors in the commutative diagram

$$\begin{array}{c} \\ \text{forget.}\uparrow \\ \\ (-)^G \downarrow \\ \end{array} \qquad \begin{array}{ccccc} \mathcal{T} & \longrightarrow & \mathcal{U} & \longrightarrow & \mathrm{Set} \\ \uparrow & & \uparrow & & \uparrow \\ G\mathcal{T} & \longrightarrow & G\mathcal{U} & \longrightarrow & G\mathrm{Set} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{T} & \longrightarrow & \mathcal{U} & \longrightarrow & \mathrm{Set}, \end{array} \tag{5.2.23}$$

as well as their left adjoints, and even variations only using subgroups of G (1.1.2)appears at various points when doing equivariant homotopy theory.

Example 5.2.24. As it is defined, the $G\mathcal{T}$-category $\mathcal{T}_G$ has the underlying G-fixed $\mathcal{T}$-category $G\mathcal{T}$, which is closed symmetric monoidal. Also, $\mathcal{T}_G$ is closed symmetric monoidal itself, when viewing it as a mere category using the upper way through diagram 5.2.23, using the same smash product and internal hom functor as in $\mathcal{T}$. One choice of internal **Hom**-functor for $\mathcal{T}_G$ is $\mathcal{T}_G(-,-)$, and we agree to use this choice.

5.2.25 Tensors and Cotensors

Detailed treatment of the concepts of *(indexed) limits and colimits* in $\mathcal{V}$-enriched categories can be found in Chapter 3 of [K]. We will mainly be concerned with the special case of tensors and cotensors, and for vonvenience we repeat the definition.

Definition 5.2.26. Let $\mathcal{C}$ be enriched over the closed symmetric monoidal category $\mathcal{V}$. Let V be an object of $\mathcal{V}$ and A an object of $\mathcal{C}$. Then their *tensor product* $V \otimes A$ is an object of C, such that for objects B in $\mathcal{C}$, there is a $\mathcal{V}$-natural isomorphism:

$$\mathcal{C}(V \otimes A, B) \cong \mathbf{Hom}(V, \mathcal{C}(A, B)),$$

where **Hom** denotes the internal **Hom**-object in $\mathcal{V}$.
Their *cotensor product* $\mathcal{C}(V, A)$ is an object of $\mathcal{C}$, such that again for objects B in $\mathcal{C}$, there is a $\mathcal{V}$-natural isomorphism:

$$\mathcal{C}(B, \mathcal{C}(V, A)) \cong \mathbf{Hom}(V, \mathcal{C}(B, A)).$$

If all such (co-)tensor products exist we call $\mathcal{C}$ *(co-)tensored*. If we consider $\mathcal{C}$ as enriched over different monoidal categories, we clarify the one used for (co-)tensors by saying it is *(co-)tensored over* $\mathcal{V}$.

Remark 5.2.27. Note that for $\mathcal{V} = \text{Set}$, being tensored and cotensored over Set is equivalent to having all small copowers $\coprod_X A$. Dually, being cotensored over Set is equivalent to having all small powers $\prod_X A$.

Example 5.2.28. Considering the closed symmetric monoidal category $\mathcal{V}$ as enriched over itself (cf. Example 5.2.14), it is both tensored and cotensored over itself, by the defining adjunction of the internal **Hom**-space 5.1.7.

Example 5.2.29. As mentioned in Remark 5.2.24, the category $\mathcal{T}_G$ is enriched over $G\mathcal{T}$, but also over itself, i.e., $\mathbf{Hom}(X, Y) = \mathcal{T}_G(X, Y)$. This immediately implies that $\mathcal{T}_G$ is both tensored and cotensored over both itself and $G\mathcal{T}$, where both are displayed by the same natural isomorphisms, considered either in $G\mathcal{T}$ or $\mathcal{T}_G$:

$$\mathcal{T}_G(D, \mathcal{T}_G(A, B)) \cong \mathcal{T}_G(D \wedge A, B) \cong \mathcal{T}_G(A, \mathcal{T}_G(D, B)).$$

Since $\mathcal{T}_G$ has $G\mathcal{T}$ as its underlying G-fixed category, this implies natural isomorphisms in $\mathcal{T}$:

$$G\mathcal{T}(S, \mathcal{T}_G(A, B)) \cong G\mathcal{T}(S \wedge A, B) \cong G\mathcal{T}(A, \mathcal{T}_G(S, B)).$$

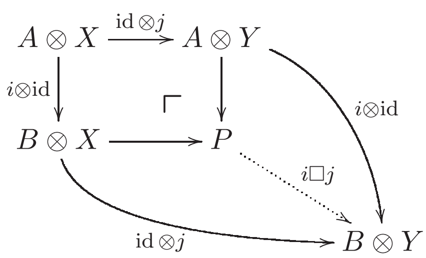
For S any object of $\mathcal{T}$, i.e., with trivial G-action, this reduces to:

$$\mathcal{T}(S, G\mathcal{T}(A, B)) \cong G\mathcal{T}(S \wedge A, B) \cong G\mathcal{T}(A, \mathcal{T}_G(S, B)),$$

which shows that $G\mathcal{T}$ is tensored and cotensored over $\mathcal{T}$.

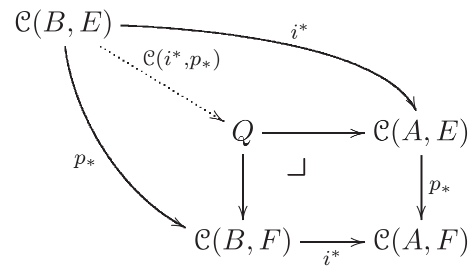
The following construction is important for the compatibility of an enrichment and the model structures on the involved categories, and also appears prominently in a lot of our constructions of cellular filtrations:

Definition 5.2.30. Let $(\mathcal{V}, \wedge, \mathbb{I})$ be a closed symmetric monoidal category. Let $\mathcal{C}$ be enriched and tensored over $\mathcal{V}$, and have pushouts. For $i : A \to B$ a morphism in $\mathcal{V}$, $j : X \to Y$ a morphism in the underlying category $\mathcal{C}_0$ of $\mathcal{C}$. Define the *pushout product* $i \square j$ to be the dotted map in $\mathcal{C}_0$ from the pushout in the diagram:



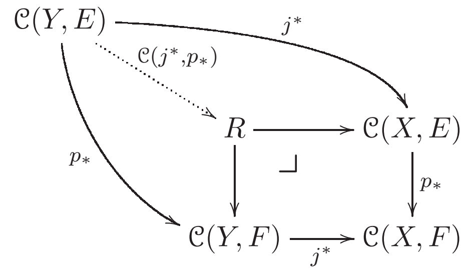
The dual construction is the following:

Definition 5.2.31. Let $(\mathcal{V}, \wedge, \mathbb{I})$ be a closed symmetric monoidal category. Let $\mathcal{C}$ be enriched and cotensored over $\mathcal{V}$, and have pushouts. For $i\colon A \to B$ a morphism in $\mathcal{V}$, $p\colon E \to F$ a morphism in the underlying category $\mathcal{C}_0$ of $\mathcal{C}$. Define the map $\mathcal{C}(i^*, p_*)$ in $\mathcal{C}_0$ to be the dotted map to the pullback in the diagram:



This again has an analog living in the category $\mathcal{V}$:

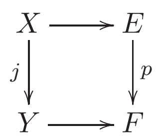
Definition 5.2.32. Let $(\mathcal{V}, \wedge, \mathbb{I})$ be a closed symmetric monoidal category having pullbacks. Let $\mathcal{C}$ be enriched over $\mathcal{V}$. For $j : X \to Y$ and $p : E \to F$ be morphisms in the underlying category $\mathcal{C}_0$ of $\mathcal{C}$. Define the map $\mathcal{C}(j^*, p_*)$ in $\mathcal{V}$ to be the dotted map to the pullback in the diagram in $\mathcal{V}$:



This construction can be used to characterize lifting properties in the enriched setting:

Lemma 5.2.33. *In the situation of Definition 5.2.32, the pair (j, p) has the lifting property in $\mathcal{C}_0$, if and only if the map of sets $\mathcal{V}(\mathbb{I}, \mathcal{C}(j^*, p_*)))$ is surjective.*

Proof. Recall that morphisms $X \to Y$ in $\mathcal{C}_0$ correspond to elements of $\mathcal{V}(\mathbb{I}, \mathcal{C}(X, Y)) = \mathcal{C}_0(X, Y)$ from 5.1.9. Then the universal property of the pullback gives that elements of $\mathcal{V}(\mathbb{I}, R)$ correspond exactly to commutative diagrams



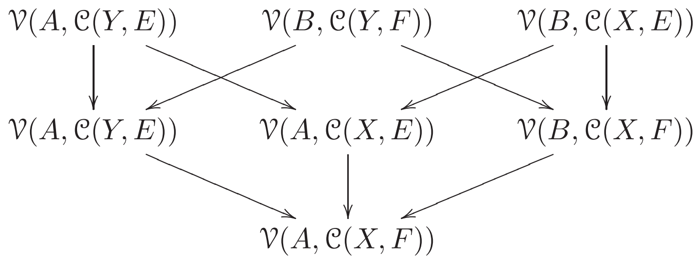
in $\mathcal{C}_0$. Then $\mathcal{V}(\mathbb{I}, \mathcal{C}(j^*, p_*)))$ sends maps $f\colon Y \to E$ in $\mathcal{C}_0$ to the diagram with $f \circ j$ as the top and $p \circ f$ as the bottom horizontal arrow, so that surjectivity indeed corresponds exactly to the existence of the lift. □

Given that all of the three above constructions are defined, there is the following crucial relation between them:

Lemma 5.2.34. *Let* $(\mathcal{V}, \wedge, \mathbb{I})$ *be closed symmetric monoidal and have small limits. Let* $\mathcal{C}$ *be enriched, tensored and cotensored over* $\mathcal{V}$ *and have pullbacks and pushouts. Let* $i : A \to B$ *a morphism in* $\mathcal{V}$ *and* $j : X \to Y$ *and* $p : E \to F$ *morphisms in the underlying category* $\mathcal{C}_0$ *of* $\mathcal{C}$*. Then the following maps in* $\mathcal{V}$ *are naturally isomorphic:*

$$\mathcal{C}((i\Box j)^*, p_*) \cong \mathcal{V}(i^*, \mathcal{C}(j^*, p_*)_*) \cong \mathcal{C}(j^*, \mathcal{C}(i^*, p_*))$$

Proof. Note that for the middle map we considered $\mathcal{V}$ as enriched over itself as in 5.2.14. By careful use of the universal properties of pushouts and pullbacks as well as the defining adjunctions for tensors and cotensors 5.2.26, one observes that all three maps are naturally isomorphic to the map from $\mathcal{V}(B, \mathcal{C}(Y, E))$ to the limit of



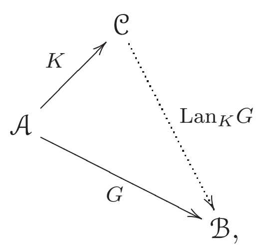
□

These two lemmas allow us to characterize lifting properties in $\mathcal{C}_0$ in terms of those in $\mathcal{V}$, which is of course of particular interest when $\mathcal{C}_0$ and $\mathcal{V}$ are model categories (cf. 6.1.8).

5.2.35 Kan Extensions

The discussion about enriched Kan extensions in [K, 4] is, due to its generality rather technical. As in the case of enriched (co-) limits, extra care has to be taken in several places. Since we do not need the full generality, we state a slightly simpler definition and list only the explicit properties we make use of, without going into much detail. We concentrate on the case of left Kan extensions, since the dual notion will not appear outside of pure existence statements.

Let $\mathcal{V}$ be closed symmetric monoidal and consider the solid arrow diagram of $\mathcal{V}$-categories and $\mathcal{V}$-functors:



where $\mathcal{A}$ is equivalent to a small $\mathcal{V}$-category and $\mathcal{B}$ is cotensored over $\mathcal{V}$.

Definition 5.2.36. In the above situation, a *left Kan extension* $\mathrm{Lan}_K G$ *of* G *along* K is a $\mathcal{V}$-functor $\mathcal{C} \to \mathcal{B}$, together with a $\mathcal{V}$-natural isomorphism

$$[\mathcal{C}, \mathcal{B}](\mathrm{Lan}_K G, S) \cong [\mathcal{A}, \mathcal{B}](G, S \circ K).$$

The image of the identity transformation for $S = \mathrm{Lan}_K G$ is a $\mathcal{V}$-natural transformation $\phi : G \to \mathrm{Lan}_K G \circ K$ and is called the *unit* of $\mathrm{Lan}_K G$.

It is important to note, that in a situation where $\mathcal{B}$ is not cotensored, this definition is not adequate, in that it does not describe the left Kan extension in the sense of Kelly, but rather a weaker notion. For counterexamples see the discussion after [K, 4.43].

The following proposition will give us the existence of left Kan extensions in all the cases that we will consider:

Proposition 5.2.37. *[K, 4.33] A $\mathcal{V}$-category $\mathcal{B}$ admits all left Kan extensions of the form* $\mathrm{Lan}_K G$*, where $K : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{A} \to \mathcal{B}$ and $\mathcal{A}$ is equivalent to a small $\mathcal{V}$-category, if and only if it is enriched cocomplete.*

To check the required cocompleteness, we will generally be able to use the following characterization, which is a combination of several statements in [K]:

Theorem 5.2.38. *Let $\mathcal{B}$ be enriched over $\mathcal{V}$:*

(i) $\mathcal{B}$ is cocomplete in the enriched sense, if and only if is tensored and admits all small conical (enriched) colimits.

(ii) $\mathcal{B}$ is complete in the enriched sense, if and only if is cotensored and admits all small conical (enriched) limits.

(iii) Assuming $\mathcal{B}$ is cotensored, it admits all small conical colimits, if and only if its underlying ordinary category $\mathcal{B}_0$ is complete.

(iv) Assuming $\mathcal{B}$ is tensored, it admits all small conical limits, if and only if its underlying ordinary category $\mathcal{B}_0$ is cocomplete.

In particular for tensored and cotensored $\mathcal{B}$, the conical (co-)limits are the ones created in $\mathcal{B}_0$.

Proof. The precise references in [K] are: Theorem 3.73 for (ii), dualize for (i). The discussion between 3.53 and 3.54 for conical (co-)-limits in $\mathcal{B}$ or $\mathcal{B}_0$, and the discussion between 3.33 and 3.34 for the connection to classical (co-)completeness, □

Since it is not always the enriched functor category from 5.2.9 that is of interest for us, we would also like a characterization of the left Kan extension in terms of the underlying category of enriched functors and enriched transformations. Luckily our assumption that $\mathcal{B}$ is cotensored allows us to use the following universal property from [K, 4.43] and the discussion that follows it:

Theorem 5.2.39. *If $\mathcal{B}$ is cotensored, a $\mathcal{V}$-functor L is a left Kan extension of G along K, if and only if there is a natural bijection of sets*

$$[\mathcal{C},\mathcal{B}]_0(\mathrm{Lan}_K G, S) \cong [\mathcal{A},\mathcal{B}]_0(G, S\circ K).$$

In particular a $\mathcal{V}$-functor L equipped with a $\mathcal{V}$-natural transformation $\phi : G \to L\circ K$ is a left Kan extension of G along K, if and only if any $\mathcal{V}$-natural transformation $\alpha : G \to L\circ K$ factors uniquely as $\alpha = \beta\circ\phi$.

Hence in the case of $\mathcal{B}$ tensored, cotensored and cocomplete, the two characterizations together with the existence result 5.2.37, allow us to state the following:

Proposition 5.2.40. *If $\mathcal{B}$ is cotensored, precomposition with K defines a $\mathcal{V}$-functor $K^*\colon [\mathcal{C},\mathcal{B}] \to [\mathcal{A},\mathcal{B}]$. The left Kan extension provides a left adjoint, both in the enriched sense, and on underlying ordinary categories.*

Finally, the following property helps to compute the Kan extensions in a lot of interesting special cases:

Proposition 5.2.41. *[K, 4.23] In the situation of Proposition 5.2.40, the $\mathcal{V}$-functor K is fully faithful if and only if the unit $\mathrm{id}_{[\mathcal{A},\mathcal{B}]} \to K^*\mathrm{Lan}_K -$ of the adjunction is a natural isomorphism.*

5.2.42 Cofinality for coends

Definition 5.2.43. Let $i\colon \mathcal{D}\to\mathcal{C}$ be a full and faithfull functor of $\mathcal{T}$-categories and let $F\colon \mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to\mathcal{T}$ be a $\mathcal{T}$-functor. We say that i is F-*cofinal* if the for all morphisms $\gamma\colon c_0\to c_1$ and $f_0\colon c_0\to c_0'$ in $\mathcal{C}$ so that $F(f_0,c)$ is a homeomorphism for every object c of $\mathcal{C}$, there exists an object d of $\mathcal{D}$ and morphisms $\delta\colon c_0'\to i(d)$ and $f_1\colon c_1\to i(d)$ so that

(i) $f_1\gamma = \delta f_0$, that is, the following square commutes:

$$\begin{array}{ccc} c_0 & \xrightarrow{\gamma} & c_1 \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ c_0' & \xrightarrow{\delta} & i(d). \end{array}$$

(ii) The map $F(f_1,c)$ is a homeomorphisms for every object c of $\mathcal{C}$.

Lemma 5.2.44. *Let $i\colon \mathcal{D}\to\mathcal{C}$ be a full and faithfull functor of $\mathcal{T}$-categories and let $F\colon \mathcal{C}^{\mathrm{op}}\wedge\mathcal{C}\to\mathcal{T}$ be a $\mathcal{T}$-functor. Suppose that*

(i) for all morphisms $\gamma\colon c_0\to c_1$ and $f_0\colon c_0\to c_0'$ in $\mathcal{C}$ so that $F(f_0,c)$ is a homeomorphism for every object c of $\mathcal{C}$, there exists morphisms $g'\colon c_0'\to c_1'$ and $g_1\colon c_1\to c_1'$ in $\mathcal{C}$ so that $g_1\gamma = g'f_0$ and so that the map $F(g_1,c)$ is a homeomorphisms for every object c of $\mathcal{C}$.

(ii) For every object c_0 *of* $\mathcal{C}$ *there exists an object* d *of* $\mathcal{D}$ *and a morphism* $f\colon c_0 \to i(d)$ *so that the map* $F(f, c)$ *is a homeomorphisms for every object* c *of* $\mathcal{C}$.

Then i *is* F*-cofinal.*

Proof. This is a directe concequence of the definition by letting $\delta = fg'$ and $f_1 = fg_1$. □

In the rest of this subsection we work with a $\mathcal{T}$-functor $F\colon \mathcal{C}^{\mathrm{op}} \wedge \mathcal{C} \to \mathcal{T}$ and a functor $i\colon \mathcal{D} \to \mathcal{C}$.

Note that if i is F-cofinal, then taking both of the maps in the F-cofinality condition be the identity on an object c of C, we can for each c choose a morphism $f\colon c \to i(d)$ so that $F(f, c')$ is a homeomorphism for every object c' of $\mathcal{C}$.

Proposition 5.2.45. *If* i *is* F*-cofinal, then the canonical map* $i\colon \int^{\mathcal{D}} i^*F \to \int^{\mathcal{C}} F$ *induced by the functor* i *is a homeomorphism.*

We prove the above result by constructing an inverse to the map i. Since $\int^{\mathcal{C}} F$ can be described as the coequalizer of

$$\bigvee_{c_0, c_1} F(c_1, c_0) \wedge \mathcal{C}(c_0, c_1) \rightrightarrows \bigvee_{c} F(c, c),$$

where the pointed sums are indexed over objects and pairs of objects of $\mathcal{C}$ respectively. The above maps take an element (c_0, c_1, x, γ), consisting of objects c_0 and c_1 of $\mathcal{C}$, a point $x \in F(c_1, c_0)$ and a morphism $\gamma\colon c_0 \to c_1$ in $\mathcal{C}$, to $F(\gamma, c_0)x$ and $F(c_1, \gamma)x$ respectively. Thus every element of $\int^{\mathcal{C}} F$ is represented by an element of the form (c, x), where c is an object of $\mathcal{C}$ and $x \in F(c, c)$.

In the rest of this section we suppose that i is F-cofinal.

Definition 5.2.46. For each object c of $\mathcal{C}$ we construct a continous map $\varphi_c\colon F(c, c) \to \int^{\mathcal{D}} i^*F$ as follows: Use F-cofinality to to choose a morphism $f\colon c \to i(d)$ with d an object of $\mathcal{D}$ so that $F(f, c')$ is a homeomorphism for every object c' of $\mathcal{C}$. Then $\varphi_c(x) \in \int^{\mathcal{D}} i^*F$ is defined to be the element represented by $(d, F(i(d), f)F(f, c)^{-1}x) \in F(i(d), i(d))$.

Lemma 5.2.47. *The map* φ_c *of Definition 5.2.46 is independent of the chosen* f.

Proof. Let $f'\colon c \to i(d')$ be another morphism with $F(f', c')$ a homeomorphism for every object c' of $\mathcal{C}$. We need to explain why

$$y = (d, F(i(d), f)F(f, c)^{-1}x)$$

and

$$y' = (d, F(i(d'), f')F(f', c)^{-1}x)$$

represent the same point of $\int^{\mathcal{D}} i^*F$. Use F-cofinality to choose a commutative square of the form

$$\begin{array}{ccc} c & \xrightarrow{f'} & i(d') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle ig} \\ i(d) & \xrightarrow{ig'} & i(d')' \end{array}$$

where $F(ig, c')$ is a homeomorphism for all c' in $\mathcal{C}$. Then $y = F(ig', i(d))F(ig', i(d))^{-1}y$ represents the same element as $F(i(d')', ig')F(ig', i(d))^{-1}y$ in $\int^{\mathcal{D}} i^*F$. However

$$F(i(d''), ig')F(ig', i(d))^{-1}y = F(i(d''), (ig')f)F((ig')f, c)^{-1}x,$$

and since $g'f = gf'$ this is equal to

$$F(i(d''), (ig)f')F((ig)f', c)^{-1}x = F(i(d''), ig)F(ig, i(d'))^{-1}y'.$$

Reasoning as above we see that this element represents the same element as y' in the coend $\int^{\mathcal{D}} i^*F$. $\square$

The maps φ_c assemble to a continuous map $\varphi\colon \bigvee_c F(c,c) \to \bigvee_d F(i(d), i(d))$ with $\varphi(c, x) = \varphi_c(x)$.

Lemma 5.2.48. *Given a $\mathcal{T}$-functor $F\colon \mathcal{C}^{\mathrm{op}} \wedge \mathcal{C} \to \mathcal{T}$ and an F-cofinal $i\colon \mathcal{D} \to \mathcal{C}$ we have that given $x \in F(c_1, c_0)$ and $\gamma \in \mathcal{C}(c_0, c_1)$ the point $\varphi(c_0, F(\gamma, c_0)x)$ is equal to the point $\varphi(c_1, F(c_1, \gamma)x)$ of $\int^{\mathcal{D}} i^*F$.*

The above lemma says that if c_0 and c_1 are objects of $\mathcal{C}$ and $\gamma \in \mathcal{C}(c_0, c_1)$, then the diagram

$$\begin{array}{ccc} F(c_1, c_0) & \xrightarrow{F(c_1,\gamma)} & F(c_1, c_1) \\ {\scriptstyle F(f,c_0)}\downarrow & & \downarrow{\scriptstyle \varphi_{c_1}} \\ F(c_0, c_0) & \xrightarrow{\varphi_{c_0}} & \int^{\mathcal{D}} i^*F \end{array}$$

commutes.

Proof. First we use F-cofinality to obtain a morphism $f_0\colon c_0 \to i(d_0)$ with $F(f_0, c)$ a homeomorphism for all objects c of $\mathcal{C}$. Next we use the F-cofinality condition to obtain a commutative square

$$\begin{array}{ccc} c_0 & \xrightarrow{\gamma} & c_1 \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ i(d_0) & \xrightarrow{i\delta} & i(d_1) \end{array}$$

with $F(f_1, c)$ a homeomorphism for all objects c of $\mathcal{C}$. By Lemma 5.2.47 $\varphi(c_0, F(\gamma, c_0)x)$ is represented by $(d_0, F(i(d_0), f_0)F(f_0, c_0)^{-1}F(\gamma, c_0)x)$ and $\varphi(c_1, F(c_1, \gamma)x)$ is represented by $(d_1, F(i(d_1), f_1)F(f_1, c_1)^{-1}F(c_1, \gamma)x)$. However the diagram

$$\begin{array}{ccccc} F(c_1, c_1) & \xleftarrow{F(c_1,\gamma)} & F(c_1, c_0) & \xrightarrow{F(\gamma,c_0)} & F(c_0, c_0) \\ \downarrow{\scriptstyle F(i(d_1),f_1)F(f_1,c_1)^{-1}} & & \downarrow{\scriptstyle F(i(d_1),f_0)F(f_1,c_0)^{-1}} & & \downarrow{\scriptstyle F(i(d_0),f_0)F(f_0,c_0)^{-1}} \\ F(i(d_1), i(d_1)) & \xleftarrow{F(i(d_1),i(\delta))} & F(i(d_1), i(d_0)) & \xrightarrow{F(i(\delta),i(d_0))} & F(i(d_0), i(d_0)) \end{array}$$

commutes. That is,

$$\begin{aligned}F(i(d_0), f_0)F(f_0, c_0)^{-1}F(\gamma, c_0)x &= F(i(d_0), f_0)F(f_0, c_0)^{-1}F(\gamma, c_0)F(f_1, c_0)F(f_1, c_0)^{-1}x \\ &= F(i(d_0), f_0)F(i\delta, c_0)F(f_1, c_0)^{-1}x \\ &= F(i\delta, i(d_0))F(i(d_1), f_0)F(f_1, c_0)^{-1}x,\end{aligned}$$

and this element represents the same element as

$$\begin{aligned}F(i(d_1), i\delta)F(i(d_1), f_0)F(f_1, c_0)^{-1}x &= F(i(d_1), f_1)F(i(d_1), \gamma)F(f_1, c_0)^{-1}x \\ &= F(f_1, i(d_1))^{-1}F(c_1, f_1)F(c_1, \gamma)x \\ &= F(i(d_1), f_1)F(f_1, c_1)^{-1}F(c_1, \gamma)x,\end{aligned}$$

that is, it represents $\varphi(c_1, F(c_1, \gamma)x)$ in $\int^{\mathcal{D}} i^*F$.

□

Proof of Proposition 5.2.45. By Lemma 5.2.48 the continuous map $\varphi\colon \bigvee_c F(c, c) \to \bigvee_d F(i(d), i(d))$ induces a unique map $\overline{\varphi}\colon \int^{\mathcal{C}} F \to \int^{\mathcal{D}} i^*F$. Lemma 5.2.47 implies that the composite $\overline{\varphi}i$ is the identity on $\int^{\mathcal{D}} i^*F$ since if $c = i(d)$, then we can choose f to be the identity on c. Conversely, if (c, x) represents a point in $\int^{\mathcal{C}} F$, then $i\varphi(c, x)$ is represented by $(d, F(i(d), f)F(f, c)^{-1}x)$ and this element represents the same element as (c, x) in $\int^{\mathcal{C}} F$. Thus $i\overline{\varphi}$ is the identity on $\int^{C} F$. □

Chapter 6

Model Categories

We assume that the reader is familiar with the basic theory of model categories, an introductory account can for example be found in [DS]. A more exhaustive source is [H] or [HirL].

6.1 Recollections

Almost all of the model structures we will discuss are *cofibrantly generated*, we recall the definition and state the main theorem we use to recognize such model structures from [H, 2.1.3]:

Definition 6.1.1. Let $\mathcal{C}$ be a model category. It is called *cofibrantly generated* if there are sets I and J of maps, such that:

(i) The domains of the maps of I are small with respect to $I-$cell,

(ii) The domains of the maps of J are small with respect to $J-$cell,

(iii) The class of fibrations is $J-$inj,

(iv) The class of acyclic fibrations is $I-$inj.

Theorem 6.1.2 (Recognition Theorem [H, 2.1.19])**.** *Suppose $\mathcal{C}$ is a category with all small colimits and limits. Suppose W is a subcategory of $\mathcal{C}$ and I and J are sets of maps of $\mathcal{C}$. Then there is a cofibrantly generated model structure on $\mathcal{C}$ with I as the set of generating cofibrations, J as the set of generating acyclic cofibrations, and W as the subcategory of weak equivalences if and only if the following conditions are satisfied:*

(i) The subcategory W has the two out of three property and is closed under retracts.

*(ii) The domains of I are small relative to $I-$*cell*.*

*(iii) The domains of J are small relative to $J-$*cell*.*

(iv) $J-\text{cell} \subset W \cap I-\text{cof}$.

(v) $I-\text{inj} \subset W \cap J-\text{inj}$.

(vi) Either $W \cap I\text{-cof} \subset J\text{-cof}$ *or* $W \cap J\text{-inj} \subset I\text{-inj}$.

The following lemmas are applicable in any model category. These are well known and often used without further mention in the literature, but since they lie at the heart of the homotopy theory we need, we recall the exact statements. Recall the following definition from [GJ, II 8.5]):

Definition 6.1.3. A *category of cofibrant objects* is a category $\mathcal{D}$ with all finite coproducts, with two classes of maps, called weak equivalences and cofibrations, such that the following axioms are satisfied:

(i) The weak equivalences satisfy the 2 out of 3 property.

(ii) The composite of two cofibrations is a cofibration. Any isomorphism is a cofibration.

(iii) Pushouts along cofibrations exist. Cobase changes of cofibrations (that are weak equivalences) are cofibrations (and weak equivalences).

(iv) All maps from the initial object are cofibrations.

(v) Any object X has a cylinder object $\mathrm{Cyl}(X)$, i.e., a factorization of the fold map $\nabla : X \coprod X \to X$ as

$$X \coprod X \xrightarrow{i} \mathrm{Cyl}(X) \xrightarrow{\sigma} X,$$

with i a cofibration and σ a weak equivalence.

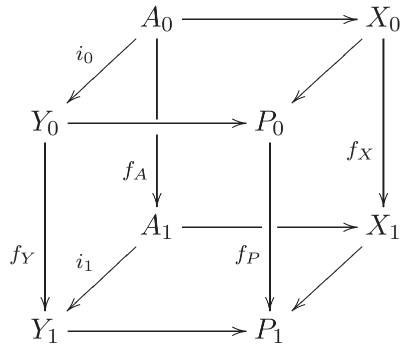
In any model category, the cofibrant objects form a *category of cofibrant objects*. This lets us apply the following two important lemmas:

Lemma 6.1.4 (Generalized Cobase Change Lemma (cf. [GJ, II.8.5]))**.** *Let* $\mathcal{C}$ *be a category of cofibrant objects. Suppose*

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & \ulcorner & \downarrow \\ Y & \xrightarrow[g]{} & P \end{array} \tag{6.1.5}$$

is a pushout diagram in $\mathcal{C}$*, such that* i *is a cofibration and* f *is a weak equivalence. Then* g *is also a weak equivalence.*

Lemma 6.1.6 (Generalized Cube Lemma (cf. [GJ, II.8.8]))**.** *Let $\mathcal{C}$ be a category of cofibrant objects. Suppose given a commutative cube*



(6.1.7)

in $\mathcal{C}$. Suppose further that the top and bottom faces are pushouts, that i_0 and i_1 are cofibrations and that the vertical maps f_A, f_X and f_Y are weak equivalences. Then the induced map of pushouts f_P is also a weak equivalence.

Note that not all examples of categories of cofibrant objects come from model structures, in particular we will want to apply Lemma 6.1.6 to cases where the cofibrations and weak equivalences come from different model structures on the same category in 6.1.20. In the case of topological model categories (cf. 6.1.8), May and Sigurdsson propose a more general treatment in [MS, 5.4], using so called well-grounded categories of weak equivalences. We will handpick some of the statements of [MS] in our Subsection 6.1.12 on topological model categories.

Definition 6.1.8. Let $(\mathcal{V}, \otimes, \mathbb{I})$ closed symmetric monoidal category, that is also a model category. Let $\mathcal{C}$ be a category enriched, tensored and cotensored over $\mathcal{V}$. Further let the underlying category $\mathcal{C}_0$ of $\mathcal{C}$ (cf. 5.2.12) have a model structure. Then this model structure on $\mathcal{C}$ is called *enriched over* $\mathcal{V}$, if the following two axioms hold:

(i) *Pushout product axiom* Let as in 5.2.30 i be a cofibration in $\mathcal{V}$ and let j be a cofibration in $\mathcal{C}_0$. Then the map $i \Box j$ in $\mathcal{C}_0$ is also a cofibration. If in addition either one of i or j is acyclic, so is $i \Box j$.

(ii) *Unit axiom* Let $q : \mathbb{I}^c \xrightarrow{\sim} \mathbb{I}$ be a cofibrant replacement of the unit object of $\mathcal{V}$. Then for every cofibrant object A in $\mathcal{C}$, the morphism

$$q \otimes \mathrm{id} : \mathbb{I}^c \otimes A \to \mathbb{I} \otimes A \cong A$$

is a weak equivalence.

If $\mathcal{C}$ is equal to $\mathcal{V}$, i.e., we consider $\mathcal{V}$ as enriched over itself (cf. 5.2.14), a model structure satisfying the above axioms is called *monoidal*.

Note that the unit axiom is redundant, if the unit object of $\mathcal{V}$ is itself cofibrant, since it is then implied by the pushout product axiom. For monoidal model categories, there is an additional important axiom:

Definition 6.1.9. A monoidal model category $(\mathcal{C}, \otimes)$ satisfies the *monoid axiom*, if every map in

$$(\{\text{acyclic cofibrations}\} \otimes \mathcal{C})-\text{cell}$$

is a weak equivalence.

The pushout product axiom has several adjoint formulations:

Lemma 6.1.10. *In the situation of Definition 6.1.8, the pushout product axiom is equivalent to both of the following formulations:*

- *Let p be a fibration in $\mathcal{C}_0$ and let j be a cofibration in $\mathcal{V}$, then $\mathcal{C}(j^*, p_*)$ is a fibration in $\mathcal{C}_0$, which is acyclic if either of p or j was.*
- *Let p be a fibration in $\mathcal{C}_0$ and let i be a cofibration in $\mathcal{C}_0$, then $\mathcal{C}(i^*, p_*)$ is a fibration in $\mathcal{C}$, which is acyclic if either of p or i was.*

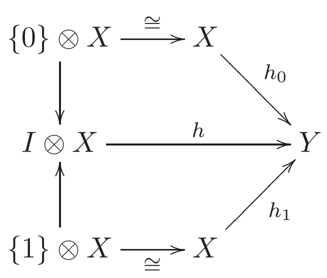
Proof. This is immediate from lemmas 5.2.33 and 5.2.34. □

Example 6.1.11. Taking $\mathcal{V}$ to be the categories of simplicial sets, spaces, symmetric spectra or G-spaces, yields, under the choice of the usual model structures, the well known notions of simplicial, topological, spectral and G-topological model categories.

In particular the example of topological and G-topological model categories will be very important for us. We discuss some of their distinct features in the following subsection.

6.1.12 Topological Model Categories

In this subsection we have to discuss two different categories of topological spaces. We distinguish between the category $\mathcal{U}$ of compactly generated weak Hausdorff spaces, and the category $\mathcal{T}$ of such spaces with a distinguished basepoint. Alternatively one can think of $\mathcal{T}$ as the under-category $* \to \mathcal{U}$ for $*$ any one-point object in $\mathcal{U}$.
Let I denote the unit interval in $\mathcal{U}$, as usual it comes equipped with the two inclusions of the endpoints. For any category $\mathcal{C}$ enriched and tensored over $\mathcal{U}$, we can then form *homotopies in* $\mathcal{C}_0$ in terms of the tensor with I:



Analogously for $\mathcal{C}$ enriched over $\mathcal{T}$, we can add a disjoint basepoints and use the tensor with I_+ to define *(based) homotopies*.

There are two classical model structures on $\mathcal{U}$ that are important for us, the Strøm- or h-model structure and the Quillen- or q-model structure. Especially the cofibrations of the former have very favorable properties, the defining one being the homotopy extension property:

Definition 6.1.13. Let $\mathcal{C}$ be enriched and tensored over $\mathcal{U}$. A map $i : A \to X$ in $\mathcal{C}_0$ is a *free h-cofibration* if it satisfies the *free homotopy extension property.* That is, for every map $f\colon X \to Y$ and homotopy $h : I \otimes A \to Y$ such that $h_0 = f \circ i$, there is a homotopy $H : I \otimes X \to Y$ such that $H_0 = f$ and $H \circ (i \otimes \mathrm{id}) = h$.

The universal test case for this property is the mapping cylinder $Y = Mi = X \cup_i (I \otimes A)$, with the obvious f and h. The exact statement is the following lemma.

Lemma 6.1.14. *Let $\mathcal{C}$ be enriched and tensored over $\mathcal{U}$ and have pushouts. A map $i : A \to X$ in $\mathcal{C}_0$ is a free h-cofibration if and only if the canonical map $Mi \to I \times X$ has a retraction.*

Remark 6.1.15. This implies a variety of closure properties of the class of free h-cofibrations. In particular any functor that preserves pushouts and the tensor with the interval also preserves h-cofibrations, since any functor preserves retractions.

Theorem 6.1.16. *[Str, Theorem 3], cf. [MS, 4.4.4] The homotopy equivalences, Hurevicz fibrations and free h-cofibrations give a proper model structure on $\mathcal{U}$.*

Note that Strøm originally works in the category of all topological spaces, but the intermediate objects for the factorizations he constructs are in $\mathcal{U}$ if source and target were. Properness is not mentioned in the original article, but is implied by the fact that all objects are fibrant and cofibrant.

Definition 6.1.17. Let f be a map in $\mathcal{U}$. Then f is a *weak equivalence* if it induces isomorphisms on all homotopy groups. Call f a *q-cofibration* if it has the left lifting property with respect to all Serre fibrations that are weak equivalences.

Remark 6.1.18. Recall that every Hurevicz fibration is a Serre fibration and every homotopy equivalence is a weak equivalence. Hence in particular any q-cofibration is a free h-cofibration.

Theorem 6.1.19. *[Q, II.3.1], cf. [H, 2.4.25] The weak equivalences, Serre fibrations and q-cofibrations give a proper model structure on $\mathcal{U}$.*

Again note that Quillen also works with general topological spaces, the transition to $\mathcal{U}$ is well documented in [H, 2.4]. Properness is proved using that every object is fibrant as well as the following lemma:

Lemma 6.1.20. *The category $\mathcal{U}$ is a category of cofibrant objects (6.1.3) with respect to the h-cofibrations and the weak equivalences. In particular the generalized cobase change (6.1.4) and cube lemma (6.1.6) hold for these choices.*

Moving to the context of based spaces, we can for example follow the discussion after Remark 1.1.7 of [H] to transport both model structures from $\mathcal{U}$ to $\mathcal{T}$. This proves satisfactory in case of the Quillen model structure:

Theorem 6.1.21. *The category $\mathcal{T}$ is a proper model category using those based maps that are q-cofibrations, Serre fibrations respectively weak equivalences in $\mathcal{U}$, i.e., when forgetting the basepoints. Similarly the underlying free h-cofibrations, Hurevicz fibrations and (free) homotopy equivalences give a proper model structure on $\mathcal{T}$.*

Remark 6.1.22. We will often make use of the fact that the Quillen model structures on $\mathcal{U}$ and $\mathcal{T}$ are cofibrantly generated. Generating sets of cofibrations and acyclic cofibrations are given in the pointed case by:

$$I := \{i : S^{n-1}_+ \to D^n_+, n \geq 0\} \text{ and}$$

$$J := \{i_0 : D^n_+ \to (D^n \times [0,1])_+, n \geq 0.\}$$

Remark 6.1.23. Note that not all spaces in $\mathcal{T}$ are cofibrant with respect to the second model structure in the above theorem. In particular the theorem only implies pointed analogs to the versions of the generalized cube and cobase change lemmas from 6.1.20 above for so called well based spaces:

Definition 6.1.24. An object X of $\mathcal{T}$ is called *well based* or *well pointed* if the inclusion of the basepoint is a free h-cofibration.

We need a stronger version of the cube lemma when we work in the $\mathcal{T}$-enriched setting:

Definition 6.1.25. Let $\mathcal{C}$ be enriched and tensored over $\mathcal{T}$. A map $i : A \to X$ in $\mathcal{C}_0$ is a *based h-cofibration* if it satisfies the *based homotopy extension property.* That is, for every map $f : X \to Y$ and based homotopy $h : I_+ \wedge A \to Y$ such that $h_0 = f \circ i$, there is a based homotopy $H : I_+ \wedge X \to Y$ such that $H_0 = f$ and $H \circ (\mathrm{id} \wedge i) = h$. In cases where no confusion is possible, we will usually omit the adjective based.

Remark 6.1.26. Again there is a recognition lemma analogous to 6.1.14 in terms of a reduced mapping cylinder, implying a similar closure property as in Lemma 6.1.14. Also note that all (free or based) h-cofibrations are closed inclusions (cf. [M, § 6, Ex 1,], [MMSS, 5.2 ff.]).

The following proposition is a combination of Proposition 9 in [Str] and the proposition on page 44 of [M]. Both are proved by explicitly constructing the required homotopies, respectively retractions.

Proposition 6.1.27. *Let $f : X \to Y$ be a map between well based spaces in $\mathcal{T}$. Then f is a based homotopy equivalence if and only if it is a free homotopy equivalence and it is a based h-cofibration if and only if it is a free h-cofibration.*

Note that being a weak equivalence in $\mathcal{T}$ and $\mathcal{U}$ is always equivalent, so we have the following corollary:

Corollary 6.1.28. *If all involved spaces are well based, then the generalized cube lemma and the generalized cobase change lemma hold for based h-cofibrations and homotopy equivalences. Also, they hold for h-cofibrations and weak equivalences if all the spaces* A_i *and* Y_i *in the diagrams 6.1.5 and 6.1.7 are well based.*

Finally we record the following property from [MMSS, 6.8(v)]:

Lemma 6.1.29. *Transfinite composition of h-cofibrations that are weak equivalences are weak equivalences.*

The following condition on sets of maps in a topological category has proven very helpful in several contexts. We use the formulation from [MMSS, 5.3], and hence use $\mathcal{T}$ for the enrichment. Let $\mathcal{A}$ and $\mathcal{C}$ be categories enriched over $\mathcal{T}$ that are (enriched) bicomplete and in particular tensored and cotensored. Let $\mathcal{A}$ be equipped with a continuous and faithful functor $F\colon \mathcal{A} \to \mathcal{C}$.

Condition 6.1.30. (Cofibration Hypothesis) Let I be a set of maps in $\mathcal{A}$. We say that I satisfies the *cofibration hypothesis* if it satisfies the following two conditions.

(i) Let $i : A \to B$ be a coproduct of maps in I. Then F takes any cobase change of i in $\mathcal{A}$ to an h-cofibration in $\mathcal{C}$.

(ii) The colimit of every sequences $\mathcal{A}$ that F takes to a sequence of h-cofibrations in $\mathcal{C}$ is preserved by F.

Remark 6.1.31. In particular F takes I-cell complexes in $\mathcal{A}$ to sequential colimits along h-cofibrations in $\mathcal{C}$.

The smallness conditions in the definition of a cofibrantly generated model category are as lax as possible. In many of the topological examples, we can actually be more strict, in order to get around having to deal with transfinite inductions as much as possible. A convenient condition is the following, again taken from [MMSS, 5.6, ff.], with $\mathcal{A}$ and $\mathcal{C}$ as above:

Definition 6.1.32. An object X of $\mathcal{A}$ is *compact* if

$$\mathcal{A}(X, \operatorname{colim} Y_n) \cong \operatorname{colim} \mathcal{A}(X, Y_n),$$

whenever $Y_n \to Y_{n+1}$ is a sequence of maps in $\mathcal{A}$ that are h-cofibrations in $\mathcal{C}$.

Definition 6.1.33. Let $\mathcal{A}$ be a model category. Then $\mathcal{A}$ is *compactly generated*, if it is cofibrantly generated with generating sets of (acyclic) cofibrations I and J, such that the domains of all maps in I or J are compact, and I and J both satisfy the cofibration hypothesis 6.1.30.

6.1.34 Simplicial Objects in Topological Categories

In this section, we recall some basic simplicial techniques. A convenient reference for a lot of the following discussion is [GJ, VII.3], but we need some rather specific technical lemmas which to the author's knowledge have not been formulated similarly before. We start by reminding the reader of the basic definitions:

Definition 6.1.35. The *simplicial category* $\mathbf{\Delta}$ has the finite ordinal numbers as objects and order preserving maps as morphisms between them.

To be more specific, we will denote objects of $\mathbf{\Delta}$ by $\mathbf{n}$, i.e.,

$$\mathbf{n} := \{0 < 1 < \ldots < n\}.$$

Recall the generating morphisms s_i and d_i in $\mathbf{\Delta}$ and the relations between them from [GJ, I.1.2].

Definition 6.1.36. Let $\mathcal{C}$ be a category. The *category $s\mathcal{C}$ of simplicial objects in* $\mathcal{C}$ is the functor category $[\mathbf{\Delta}^{\mathrm{op}}, \mathcal{C}]$.

Let from now on $\mathcal{C}$ be enriched, cocomplete and tensored over the category of simplicial sets.

Definition 6.1.37. The *geometric realization* $|X|_{\mathcal{C}}$ of a simplicial object $X \in s\mathcal{C}$ is the coend

$$|X|_{\mathcal{C}} := \int^{\mathbf{k}\in\Delta^{\mathrm{op}}} X_k \otimes \Delta^k,$$

where Δ^k is the simplicial n-simplex given by $\Delta^k_n = \mathbf{\Delta}(\mathbf{n}, \mathbf{k})$. With the obvious extension on morphisms, this defines a functor $|\cdot|_{\mathcal{C}} \colon s\mathcal{C} \to \mathcal{C}$.

We will often drop the subscript from $|\cdot|_{\mathcal{C}}$ when the category is clear. Note that any functor $\mathcal{C} \to \mathcal{C}'$ that preserves colimits and tensors preserves the geometric realization.

Definition 6.1.38. If $\mathcal{C}$ is also cotensored over simplicial sets, the geometric realization has a right adjoint given by the functor that assigns to an object Y of $\mathcal{C}$ the simplicial object $\mathcal{C}(\mathbf{\Delta}, Y)$ which is given in level $\mathbf{k}$ by

$$\mathcal{C}(\mathbf{\Delta}, Y)_k := \mathcal{C}(\Delta^k, Y).$$

Remark 6.1.39. The most important special case for our applications will be when the category $\mathcal{C}$ is actually enriched and tensored over $\mathcal{T}$. In this case, we can first transport the enrichment to $\mathcal{U}$ along the forgetful functor and then to simplicial sets via the singular set functor as in 5.2.10 since both of these are (lax) monoidal. Then the defining adjunctions immediately give an isomorphism

$$X_k \otimes_{s\mathrm{Set}} \Delta^k \cong X_k \otimes_{\mathcal{U}} |\Delta^k| \cong X_k \otimes_{\mathcal{T}} |\Delta^k|_+,$$

where the $|\Delta^k|$ denotes the topological k-simplex (with a disjoint basepoint on the right). Classical realization of simplicial sets is then a special case of the above by viewing sets as discrete objects of $\mathcal{T}$.

We want to filter the geometric realization, in analogy to the classical case where the geometric realization admits a filtration via its structure as a CW-complex. For this purpose consider for a natural number n the full subcategory $\mathbf{\Delta}_n$ of $\mathbf{\Delta}$ consisting of all objects $\mathbf{k}$ with $k \leq n$.

Definition 6.1.40. For $X \in s\mathcal{C}$ a simplicial object, define the *n-skeleton* $\operatorname{sk}_n |X|_{\mathcal{C}}$ as the coend

$$\operatorname{sk}_n |X|_{\mathcal{C}} := \int^{\mathbf{k} \in \Delta_n^{\mathrm{op}}} X_k \otimes \Delta^k.$$

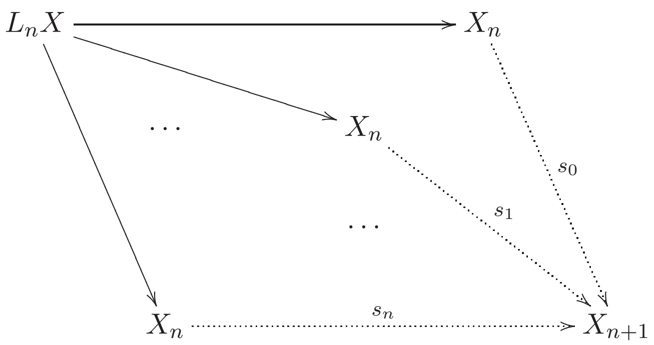
Again, we will often simplify notation and just write $\operatorname{sk} X$ instead of $\operatorname{sk} |X|_{\mathcal{C}}$ when the context does not allow confusion.

Lemma 6.1.41. *The inclusions of categories $\mathbf{\Delta}_n \to \mathbf{\Delta}_{n+1} \to \ldots \to \mathbf{\Delta}$ induce morphisms of coends and we get*

$$\operatorname*{colim}_n \operatorname{sk}_n X \cong |X|.$$

We define an analog to the degenerate simplices, or rather the latching spaces in the classical setting:

Definition 6.1.42. Let $X \in s\mathcal{C}$ be a simplicial object. The *latching object* L_nX comes together with a distinct map $L_nX \to X_n$ and is defined inductively as follows: Let L_0X be the initial object of $\mathcal{C}$. Assumw that L_nX and $L_nX \to X_n$ are already defined, and that the following diagram commutes:



(6.1.43)

We let $L_{n+1}X$ be the colimit of the following solid part of the above diagram. The map $L_{n+1}X \to X_{n+1}$ is induced by the simplicial degeneracy maps $s_i : X_n \to X_{n+1}$.

The importance of the latching objects lies in the following proposition:

Proposition 6.1.44. *[GJ, VII.3.8] Let $X \in s\mathcal{C}$ be a simplicial object. Then for all $n \geq 0$ there is a pushout diagram in $\mathcal{C}$:*

$$\begin{array}{ccc} X_n \otimes \partial\Delta^n \cup_{L_nX \otimes \partial\Delta^n} L_nX \otimes \Delta^n & \longrightarrow & \operatorname{sk}_{n-1} X \\ \downarrow & \ulcorner & \downarrow \\ X_n \otimes \Delta^n & \longrightarrow & \operatorname{sk}_n X, \end{array}$$

where the left vertical map is the pushout product of $L_nX \to X_n$ with the inclusion of the boundary $\partial\Delta^n \to \Delta^n$.

Definition 6.1.45. Let C be a class of morphisms in $\mathcal{C}$. We say that a simplicial object $X \in s\mathcal{C}$ is *C-proper*, if all the maps $L_nX \to X_n$ are in C.

We finally turn to the case of $\mathcal{C}$ being a topological model category, i.e., a model category enriched over $\mathcal{U}$ in the sense of 6.1.8.

Proposition 6.1.46. *Let $\mathcal{C}$ be enriched and tensored over $\mathcal{U}$ and let C be a class of morphisms in $\mathcal{I}$. Suppose that C is closed under cobase change and satisfies the pushout product axiom with respect to the Quillen model structure on $\mathcal{U}$ (e.g., if C is the class of cofibrations in a model category enriched over $\mathcal{U}$ in the sense of 6.1.8). Then for any C-proper simplicial object X in $s\mathcal{C}$, the skeleton filtration of $|X|_{\mathcal{C}}$ consists morphisms in C.*

$$\begin{array}{ccc}
X_n \otimes |\partial\Delta^n| \cup_{L_nX\otimes|\partial\Delta^n|} L_nX \otimes |\Delta^n| & \longrightarrow & \mathrm{sk}_{n-1} X \\
\downarrow & \ulcorner & \downarrow {\scriptstyle C-\mathrm{cof}} \\
X_n \otimes |\Delta^n| & \longrightarrow & \mathrm{sk}_n X,
\end{array}$$

The next proposition concerns interactions of simplicial objects with weak equivalences.

Proposition 6.1.47. *Let $\mathcal{C}$ be enriched and tensored over $\mathcal{U}$, with a class C of cofibrations and a class of weak equivalences, such that the cofibrant objects form a category of cofibrant objects in the sense of 6.1.3. Assume that the cofibrations and weak equivalences are compatible with the enrichment in the sense that the pushout product axiom 6.1.8(i) is satisfied. Let X and Y in $s\mathcal{C}$ be C-proper simplicial objects such that X_0 and Y_0 are cofibrant. If $f\colon X \to Y$ is a morphism of simplicial objects that is a weak equivalence in each simplicial degree, then the induced map of realizations*

$$|f|_{\mathcal{C}} : |X|_{\mathcal{C}} \to |Y|_{\mathcal{C}}$$

is a weak equivalence.

Proof. We begin with showing that all the X_n, Y_n and L_nX and L_nY are cofibrant. $L_1X = X_0$ is cofibrant by hypothesis, so assume inductively that $L_{n-1}X$ is cofibrant. Since X is C-proper, the solid arrow part of diagram 6.1.43 consists only of cofibrations, hence in particular X_n is cofibrant. Since $L_{n+1}X$ is an iterated pushout of X_n along cofibrations it is cofibrant itself. We continue by induction on the skeleton filtration of 6.1.46 to show that the maps $sk_nX \to \mathrm{sk}_n Y$ are weak equivalences. Note that the tensor with a cofibrant space preserves weak equivalences between cofibrant objects by [GJ, II.8.4]. Hence by the generalized cube lemma we only need to show that the maps

$$X_n \otimes |\partial\Delta^n| \cup_{L_nX\otimes|\partial\Delta^n|} L_nX \otimes |\Delta^n| \longrightarrow Y_n \otimes |\partial\Delta^n| \cup_{L_nY\otimes|\partial\Delta^n|} L_nY \otimes |\Delta^n|$$

are weak equivalences between cofibrant objects. Again using the generalized cube lemma on the defining diagram for the pushout product of $L_nX \to X$ and $\partial\Delta^n \to \Delta^n$, this reduces to showing that $L_nX \to L_nY$ is a weak equivalence. As above this is proven inductively, by comparing the diagrams 6.1.43 for X and Y and applying the generalized cube lemma to each of the iterated pushouts. □

Remark 6.1.48. A very obvious example for categories $\mathcal{C}$ which satisfy the requirements of the above proposition is given by a model category enriched over $\mathcal{U}$ in the sense of 6.1.8. However, we will in particular want to apply the proposition to (levelwise) h-cofibrations and π_*-isomorphisms of orthogonal spectra, so the more general formulation is necessary.

It can be hard to verify the properness of a simplicial object. Sometimes the following is easier to check:

Definition 6.1.49. Fix a class C of morphisms called in $\mathcal{C}$. We call a simplicial object $X \in s\mathcal{C}$ *C-good*, if for all n all the degeneracy maps $s_i\colon X_n \to X_{n+1}$ are in C.

In particular in $\mathcal{T}$ and $\mathcal{U}$, there is Lillig's Union Theorem [Li], which implies the following helpful statement:

Lemma 6.1.50. *For simplicial objects in the categories $\mathcal{T}$ or $\mathcal{U}$, h-proper and h-good are equivalent notions. Since colimits and tensors are computed levelwise, the same is true for levelwise h-cofibrations of (equivariant) orthogonal spectra.*

6.2 Assembling Model Structures

Given a model structure on a category $\mathcal{C}$, one often wants to give corresponding structures to categories of functors $\mathcal{D} \to \mathcal{C}$ for some diagram category $\mathcal{D}$. Theorems on the possibility and methods to do this are well studied in many cases, examples can be found in [HirL, 14.2.1] for cases of cofibrantly generated structures on $\mathcal{C}$, in [H, Chapter 5] for the case of $\mathcal{D}$ a Reedy category. More recently Angeltveit has studied the Reedy approach in an enriched setting ([A]). The result of this section is more in the direction of the former, in particular as a special case we will get an enriched version of Hirschhorns Theorem [HirL, 11.6.1]. However, the significant difference in our approach is, that we lift not just a single model structure on the target category, but rather assemble a new model structure from several given ones.
Hirschhorns method uses the evaluation functors that any diagram category is equipped with; we give a short recollection: Let $\mathcal{D}$ be small. Consider the trivial category $\star$ with one object $*$, and only one (identity) morphism. For each object d of $\mathcal{D}$, there is an embedding $\mathrm{inc}_d\colon * \to \mathcal{D}$ sending the object $*$ to d. Then the evaluation functor ev_d assigns to a functor $X\colon \mathcal{D} \to \mathcal{C}$ the precomposition $\mathrm{ev}_d X = X \circ \mathrm{inc}_d$ with the inclusion inc_d. We have adjoint pairs:

$$\mathcal{F}_d : \mathcal{C} \cong [\star, \mathcal{C}] \leftrightarrows [\mathcal{D}, \mathcal{C}] : \mathrm{ev}_d,$$

where $\mathcal{F}_d(-)$ is the left Kan extension. Then, given a cofibrantly generated model structure on $\mathcal{C}$ with generating sets of (acyclic) cofibrations I and J, we can form the sets

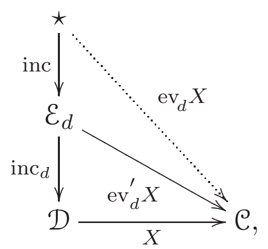
$$\mathcal{F}I := \bigcup_{d \in \mathcal{D}} \mathcal{F}_d I$$

and $\mathcal{F}J$ analogous.

Theorem 6.2.1. *[HirL, 11.6.1] Let $\mathcal{D}$ be a small category, and let $\mathcal{C}$ be a cofibrantly generated model category with generating cofibrations I and generating acyclic cofibrations J. Then the category $[\mathcal{D}, \mathcal{C}] = [\mathcal{D}, \mathcal{C}]_0$ of $\mathcal{D}$-diagrams in $\mathcal{C}$ is a cofibrantly generated model category in which a map $f\colon X \to Y$ is*

- *a weak equivalence if $\mathrm{ev}_d(f) : X_d \to Y_d$ is a weak equivalence in $\mathcal{C}$ for every object $d \in \mathcal{D}$,*
- *a fibration if $\mathrm{ev}_d(f) : X_d \to Y_d$ is a fibration in $\mathcal{C}$ for every object $d \in \mathcal{D}$, and*
- *an (acyclic) cofibration if it is a retract of a transfinite composition of cobase changes of maps in $\mathcal{F}I$ ($\mathcal{F}J$).*

Let us now move to an enriched setting. Let $(\mathcal{V}, \wedge, \mathbb{I})$ be a complete closed symmetric monoidal category, and let $\mathcal{C}$ and $\mathcal{D}$ be enriched over $\mathcal{V}$, such that $\mathcal{D}$ is $\mathcal{V}$-equivalent to a small category, hence the enriched functor category $[\mathcal{D}, \mathcal{C}]$ exists (5.2.9). Consider $\star$ as the trivial $\mathcal{V}$-category, i.e., as the $\mathcal{V}$-category with one object $*$ such that the morphism object $\star(*, *)$ is unit in $\mathcal{D}$. Then analogous to the discussion above, the inclusion of $\star$ at any object of $\mathcal{D}$ yields evaluation functors by precomposition. Under favorable conditions on $\mathcal{C}$, these have left adjoints which we again denote by $\mathcal{F}_d$ (e.g., if $\mathcal{C}$ is tensored, cotensored and enriched cocomplete, cf. 5.2.37). However this time, we want to consider an intermediate functor category: Given an object $d \in \mathcal{D}$, denote by $\mathcal{E}_d$ the full subcategory containing only that object. Then the inclusion of $\star$ at $d \in \mathcal{D}$ factors in the following way



(6.2.2)

and hence we have a factorization of evaluation functors

$$\mathcal{C} \xleftarrow{\cong} [\star, \mathcal{C}] \longleftarrow [\mathcal{E}_d, \mathcal{C}] \xleftarrow{\mathrm{ev}'_d} [\mathcal{D}, \mathcal{C}] \,, \qquad \mathrm{ev}_d\colon [\mathcal{D}, \mathcal{C}] \to [\star, \mathcal{C}] \tag{6.2.3}$$

Each of the functors in this factorization has an (enriched) left adjoint if and only if the appropriate left Kan extensions exist (5.2.37), and in that case we denote them in the following way:

$$\mathcal{C} \xrightarrow{\cong} [\star, \mathcal{C}] \xrightarrow{\mathcal{E}_d \otimes -} [\mathcal{E}_d, \mathcal{C}] \xrightarrow{\mathcal{G}_d} [\mathcal{D}, \mathcal{C}] \; . \qquad (6.2.4)$$

We call objects of the form $\mathcal{G}_d X$ *semi-free*, in analogy to the term *free* for objects $\mathcal{F}_d Y$. Note that the notation $\mathcal{E}_d \otimes -$ is not accidental, as it is in fact given by the categorical tensor with the endomorphism $\mathcal{D}$-object of d if it exists.

Assume that for each $d \in \mathcal{D}$, there is a cofibrantly generated model structure $\mathcal{M}_d$ on the underlying ordinary category $[\mathcal{E}_d, \mathcal{C}]_0$ of $[\mathcal{E}_d, \mathcal{C}]$, with generating (acyclic) cofibrations I_d and J_d, and classes of weak equivalences $\mathcal{W}_d$, respectively. Assume further that each $[\mathcal{E}_d, \mathcal{C}]$ is tensored and cotensored over $\mathcal{V}$, so that the semi-free functors $\mathcal{G}_d$ all exist. Define the sets of maps $\mathcal{G}I$ and $\mathcal{G}J$ in $[\mathcal{D}, \mathcal{C}]_0$ as

$$\mathcal{G}I := \bigcup_{d \in \mathcal{D}} \mathcal{G}_d I_d \qquad \qquad \mathcal{G}J := \bigcup_{d \in \mathcal{D}} \mathcal{G}_d J_d. \qquad (6.2.5)$$

Define the class $\mathcal{W}$ of maps in $[\mathcal{D}, \mathcal{C}]_0$ as

$$\mathcal{W} := \{ f \in [\mathcal{D}, \mathcal{C}]_0, \;\; \text{s.t. } \operatorname{ev}'_d(f) \in \mathcal{W}_d \; \forall \, d \in \mathcal{D} \}. \qquad (6.2.6)$$

Then the assembling theorem is the following

Theorem 6.2.7. *Let $\mathcal{V}$ be a complete closed symmetric monoidal category and let $\mathcal{C}$ and $\mathcal{D}$ be enriched over $\mathcal{V}$ such that $\mathcal{D}$ is equivalent to a small subcategory. Assume that each of the functor categories $[\mathcal{E}_d, \mathcal{C}]$ is tensored and cotensored over $\mathcal{V}$ and that we have a family of cofibrantly generated model structures $\{\mathcal{M}_d\}$ as above.*

Assume that the domains of the maps in $\mathcal{G}I$ are small relative to $\mathcal{G}I$-cell, the domains of the maps in $\mathcal{G}J$ are small with respect to $\mathcal{G}J$-cell and that $\mathcal{G}J$-cell $\subset \mathcal{W}$.

Then the underlying category $[\mathcal{D}, \mathcal{C}]_0$ of $[\mathcal{D}, \mathcal{C}]$ is a cofibrantly generated model category where a map $f \colon X \to Y$ is a fibration, if and only if each $\operatorname{ev}'_d f$ is a fibration in the model structure $\mathcal{M}_d$ on $[\mathcal{E}_d, \mathcal{C}]_0$, and a weak equivalence if and only if it is in $\mathcal{W}$. The generating cofibrations are given by $\mathcal{G}I$ and the generating acyclic cofibrations are given by $\mathcal{G}J$.

Proof. We check the conditions from the recognition theorem 6.1.2. First of all, enriched limits and colimits in $[\mathcal{D}, \mathcal{C}]$ are calculated pointwise by [K, 3.3], i.e., the (co-)limit of a diagram exists if and only if it does so after evaluating to the $[\mathcal{E}_d, \mathcal{C}]$ or equivalently to $[\star, \mathcal{C}]$. Since all the $[\mathcal{E}_d, \mathcal{C}]$ had model structures, they were in particular bicomplete. As they were also tensored and cotensored, they were enriched bicomplete hence so is $[\mathcal{D}, \mathcal{C}]$. The class $\mathcal{W}$ is a subcategory satisfying the 2 out of 3 axiom since it is defined by a levelwise property. By assumption, $\mathcal{G}J$-cell is in $\mathcal{W}$, and since as a left adjoint $\mathcal{G}_d$ preserves retracts and cell complexes, $\mathcal{G}_d J_d$-cell $\subset \mathcal{G}_d I_d$-cof, hence $\mathcal{G}J$-cell $\subset \mathcal{G}I$-cof. Since $\mathcal{G}_d$ is left adjoint to ev'_d, a map has the right lifting property with respect to $\mathcal{G}I$ if and only if for each d its evaluation is an acyclic fibration, in particular if and only if it is in $\mathcal{W}$ and has the lifting property with respect to $\mathcal{G}J$. ☐

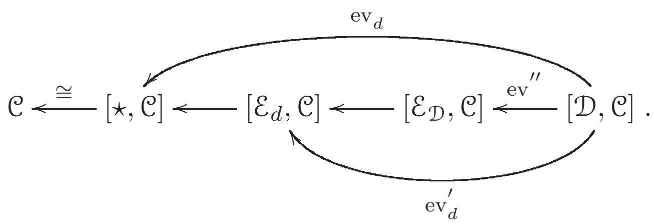
Remark 6.2.8. Similarly to the argument for the bicompleteness of $[\mathcal{D},\mathcal{C}]$, [K, 3.3] implies that the assumption, that each of the $[\mathcal{E}_d,\mathcal{C}]$ is tensored and cotensored, is immediately satisfied if $\mathcal{C}$ was so itself.

Proposition 6.2.9. *In the situation of Theorem 6.2.7 assume that $[\mathcal{D},\mathcal{C}]$ is itself tensored and cotensored over $\mathcal{V}$. If each of the model structures $\mathcal{M}_d$ satisfies the pushout product axiom (6.1.8(i)), then so does the assembled model structure on $[\mathcal{D},\mathcal{C}]_0$.*

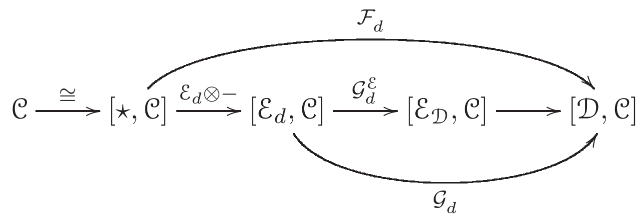
Proof. As in [HSS, Prop. 5.3.4], by the adjoint formulations in 6.1.10, it suffices to check the pushout product axiom for i a generating cofibration. But $\mathcal{G}_d$ commutes with tensors and pushouts, hence $j\Box\mathcal{G}_d i \cong \mathcal{G}_d(j\Box i)$. Since $\mathcal{G}_d$ also preserves cell complexes and retracts, $j\Box\mathcal{G}_d i$ is indeed a cofibration. The case of i or j being acyclic is exactly the same. $\square$

Remark 6.2.10. Hence if we can guarantee the analogous proposition for the Unit axiom, a family $\{M_d\}$ of enriched model assembles puzzles together to an enriched model structure on $[\mathcal{D},\mathcal{C}]$. In particular if the unit object of $\mathcal{V}$ is cofibrant this is trivial. A common other way to ensure this is demanding some sort of *cofibration hypothesis*, cf. 6.1.30 and a sufficiently general version of the cube lemma 6.1.6.

Depending on the setting, the condition $\mathcal{G}J$-cell $\subset \mathcal{W}$ in can be hard to verify. A way around this is using Schwede and Shipley 's lifting lemma [SS, 2.3] instead of the recognition Theorem 6.1.2. However, for that result to be applicable in our case, we require another layer of constructions:
In the situation of Theorem 6.2.7, consider the subcategory $\mathcal{E}_\mathcal{D}$ of $\mathcal{D}$, consisting of all objects but only the endomorphisms. More precisely, define $\mathcal{E}_\mathcal{D}(d,d) := \mathcal{D}(d,d)$ but let $\mathcal{E}(d,e)$ be initial in $\mathcal{D}$ for $d \neq e$. Then the inclusions 6.2.2 factor through $\mathcal{E}_\mathcal{D}$ and hence we get further factorizations of the evaluation functors from 6.2.3



Let



be the corresponding diagram of left adjoints 6.2.4. The induced functor pair $[\mathcal{D},\mathcal{C}]_0 \leftrightarrows [\mathcal{E}_\mathcal{D},\mathcal{C}]_0$ induces a monad T on $[\mathcal{E}_\mathcal{D},\mathcal{C}]_0$ (cf. [McL, IV.1]) and we claim that the associated

category of T-algebras is isomorphic to $[\mathcal{D}, \mathcal{C}]_0$. To prove this we check the prerequisites of Beck's Theorem in its weak form from [B, Theorem 1] (cf. [McL, Ex. VI.7. 1-3]). Indeed, since $[\mathcal{D}, \mathcal{C}]$ is enriched cocomplete with colimits calculated pointwise by [K, 3.3], $[\mathcal{D}, \mathcal{C}]_0$ has all coequalizers and they are preserved under evaluation to $[\mathcal{E}_\mathcal{D}, \mathcal{C}]_0$. Furthermore the evaluation *reflects* isomorphisms, since a $\mathcal{V}$-natural transformation $\{\alpha_d\}_{d\in\mathcal{D}}$ is an isomorphism if and only if each α_d is.
Further note that $[\mathcal{E}_\mathcal{D}, \mathcal{C}]_0 \cong \prod_{d\in\mathcal{D}}[\mathcal{E}_d, \mathcal{C}]_0$ since the $\mathcal{V}$-naturality condition 5.2.7 is void when $\mathcal{E}_\mathcal{D}(d, e)$ is initial. Hence given the family $\{\mathcal{M}_d\}_{d\in\mathcal{D}}$ we get the product model structure on $[\mathcal{E}_\mathcal{D}, \mathcal{C}]_0$:

Proposition 6.2.11. *In the situation of Theorem 6.2.7, there is a cofibrantly generated model structure on $[\mathcal{E}_\mathcal{D}, \mathcal{C}]_0$, where a map is a fibration, cofibration or weak equivalence if and only if it is one in $\mathcal{M}_d$, for all $d \in \mathcal{D}$. The generating sets of cofibrations and acyclic cofibrations are given by the sets $\mathcal{G}^\mathcal{E}I$ and $\mathcal{G}^\mathcal{E}J$, respectively, which are defined analogous to 6.2.5.*

Definition 6.2.12. In the situation of theorem 6.2.7, an object P of $[\mathcal{D}, \mathcal{C}]$ is a *path object* of an object X of $[\mathcal{D}, \mathcal{C}]$ if there is a factorization of the diagonal map

$$X \xrightarrow{w} P \xrightarrow{p} X \coprod X, \qquad p \circ w = \Delta,$$

with $w \in \mathcal{W}$ and p a pointwise fibration, i.e., a fibration in $\mathcal{M}_d$ after evaluating to $[\mathcal{E}_d, \mathcal{C}]_0$ for all $d \in \mathcal{D}$.

Then hypothesis (2) of [SS, 2.3] allows the following variation of Theorem 6.2.7

Theorem 6.2.13. *The assembling Theorem 6.2.7 still holds if we replace the assumption $\mathcal{G}J$-cell $\subset \mathcal{W}$, with the following:*
In each of the model structures $\mathcal{M}_d$, every object is fibrant and every object of $X \in [\mathcal{D}, \mathcal{C}]$ has a path object.

As promised we study an enriched version of Theorem 6.2.1:

Proposition 6.2.14. *Theorem 6.2.1 holds in the case of categories enriched over $\mathcal{V}$, if we additionally assume that the domains of the maps in $\mathcal{F}I$ are small relative to $\mathcal{F}I$-cell, the domains of the maps in $\mathcal{F}J$ are small with respect to $\mathcal{F}J$-cell and that $\mathcal{F}J$-cell consists of maps that are level weak equivalences.*

The proof works entirely analogous to the one of 6.2.7. Note that we can reformulate the extra assumptions slightly in the following way:

Lemma 6.2.15. *If tensoring with morphism objects of $\mathcal{D}$ preserves cofibrations and acyclic cofibrations, the extra assumptions in 6.2.14 are satisfied. In particular this is true if there is a model structure on $\mathcal{V}$, such that all the morphism objects of $\mathcal{D}$ are cofibrant, and the model structure on $\mathcal{C}_0$ satisfies the pushout product axiom.*

Proof. This is immediate once one checks that for objects d and e in $\mathcal{D}$, the composition $\mathrm{ev}_e \circ \mathcal{F}_d$ is isomorphic to tensoring with $\mathcal{D}(d, e)$. Since colimits in $[\mathcal{D}, \mathcal{C}]$ are calculated pointwise ([K, 3.3]), maps in $\mathcal{F}J$ cell are levelwise retracts of J-cell complexes, and the same for I. Then all three extra assumptions follow immediately from the axioms of a cofibrantly generated model category. □

Corollary 6.2.16. *Since in every model structure cofibrations and weak equivalences are preserved under coproducts, in the case* $\mathcal{V} = \mathrm{Set}$ *Theorem 6.2.14 reduces to 6.2.1.*

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