On the homotopy groups of *BPL* and *PL/O*

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1. Introduction

Let *BO* and *BPL* be the classifying spaces for stable vector bundles and stable piecewise linear (*PL*) microbundles, respectively [21]. Define the space *PL/O* to be the fibre of the natural map $BO \longrightarrow BPL$. Hirsch and Mazur have shown that the homotopy exact sequence of the fibration $PL/O \longrightarrow BO \longrightarrow BPL$ breaks up into short exact sequences [15], [16]:

$$\text{(A)} \qquad 0 \longrightarrow \pi_{k+1}(BO) \longrightarrow \pi_{k+1}(BPL) \longrightarrow \pi_k(PL/O) \longrightarrow 0 \ .$$

Moreover, they have defined an isomorphism $\pi_k(PL/O) \xrightarrow{\cong} \Gamma_k$, where Γ_k is the group of concordance classes of smoothings on the k-sphere.

Kervaire and Milnor have studied another exact sequence involving the group Γ_k [18]. Let $bP_{k+1} \subset \Gamma_k$ be the subgroup consisting of those exotic spheres which bound π-manifolds. Let $J\colon \pi_k(O(N)) \longrightarrow \pi_{N+k}(S^N) \cong \pi_k^S$ be the Hopf-Whitehead homomorphism, $N > k+1$. Then there is an exact sequence

$$\text{(B)} \qquad 0 \longrightarrow bP_{k+1} \longrightarrow \Gamma_k \longrightarrow \pi_k^S/\mathrm{im}(J) \ .$$

To give this exact sequence a homotopy theoretic interpretation and relate it to (A), we recall that there is a classifying space *BF* for stable spherical fibrations modulo fibre homotopy equivalence [13]. There are natural maps $BO \longrightarrow BF$ and $BPL \longrightarrow BF$ with fibres *F/O* and *F/PL* respectively, and a commutative diagram, with rows and columns fibrations:

$$\begin{tikzcd}
 & F/O \arrow[rr] \arrow[dd] & & F/PL \arrow[dd] \\
PL/O \arrow[ur] \arrow[dr] & & & \\
 & BO \arrow[rr] \arrow[dr] & & BPL \arrow[dl] \\
 & & BF &
\end{tikzcd} \ .$$

Now, as is well-known, $\pi_{k+1}(BF) \cong \pi_k^S$ and the natural map $\pi_{k+1}(BO) \longrightarrow \pi_{k+1}(BF)$ coincides with the J-homomorphism above. Thus $\pi_k^S/\mathrm{im}(J) \subset \pi_k(F/O)$. Moreover, in the homotopy exact sequence

$$\text{(C)} \qquad \cdots \longrightarrow \pi_{k+1}(F/PL) \xrightarrow{\Theta} \pi_k(PL/O) \longrightarrow \pi_k(F/O) \longrightarrow \cdots$$

we have $\operatorname{im}(\Theta) = bP_{k+1} \subset \Gamma_k \cong \pi_k(PL/O)$ and the map $\pi_k(PL/O) \to \pi_k(F/O)$ can be identified with the Kervaire-Milnor map $\Gamma_k \to \pi_k^S/\operatorname{im}(J) \subset \pi_k(F/O)$ [26]. The space F/PL has been studied extensively by Sullivan. He makes use of the framed surgery arguments of [18], modified for PL bundles as in [11], together with Cerf's result that $\Gamma_4 = 0$, to show that $\pi_k(F/PL) \cong 0, \mathbf{Z}_2, 0, \mathbf{Z}$ for $k \equiv 1, 2, 3, 4$, mod 4, respectively. Thus $bP_{2n+1} = 0$, $bP_{4n+2} = \mathbf{Z}_2$ or 0, and bP_{4n} is cyclic. Moreover, the map $\Gamma_k \to \pi_k^S/\operatorname{im}(J)$ is surjective if $k \neq 4n + 2$, and has cokernel 0 or $\mathbf{Z}_2$ if $k = 4n + 2$ [18].

In this paper we obtain further results on the homotopy groups and maps in the diagram

1.1

$$\begin{tikzcd}
 & \pi_{4n}(F/PL) \arrow[r, equal] \arrow[d] & \pi_{4n}(F/PL) \arrow[d] \\
\pi_{4n}(BO) \arrow[r] \arrow[d, equal] & \pi_{4n}(BPL) \arrow[r] \arrow[d] & \pi_{4n-1}(PL/O) \arrow[d] \\
\pi_{4n}(BO) \arrow[r] & \pi_{4n}(BF) \arrow[r] & \pi_{4n-1}(F/O)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbf{Z} \arrow[r, equal] \arrow[d] & \mathbf{Z} \arrow[d, "\Theta"'] \\
\mathbf{Z} \arrow[r] \arrow[d, equal] & \pi_{4n}(BPL) \arrow[r] \arrow[d] & \Gamma_{4n-1} \arrow[d] \\
\mathbf{Z} \arrow[r, "J"] & \pi_{4n-1}^S \arrow[r] & \pi_{4n-1}^S/\operatorname{im}(J).
\end{tikzcd}$$

Let

1.2
$$\begin{aligned} j_n &= \operatorname{denom}(B_n/4n) \\ \theta_n &= \operatorname{num}(B_n/4n) \cdot a_n \cdot 2^{2n-2} \cdot (2^{2n-1} - 1) \end{aligned}$$

where B_n is the n^{th} Bernoulli number, and $a_n = 1$ if n is even, $a_n = 2$ if n is odd.

In § 4 we define a homomorphism $f \colon \Gamma_{4n-1} \to \mathbf{Z}_{\theta_n}$ such that the composition $f \circ \Theta \colon \mathbf{Z} \to \Gamma_{4n-1} \to \mathbf{Z}_{\theta_n}$ is the natural projection. Using the result of [18] that $\operatorname{im}(\Theta) = bP_{4n} \cong \mathbf{Z}_{\theta_n}$ or $\mathbf{Z}_{2\theta_n}$, we are able to deduce

THEOREM 1.3. *There is an isomorphism* $\Gamma_{4n-1} \cong bP_{4n} \oplus \pi_{4n-1}^S/\operatorname{im}(J)$.

The invariant f is closely related to the invariant $e \colon \pi_{4n-1}^S \to \mathbf{Z}_{j_n}$ studied by Adams and others [2]. Recall that Adams showed that $\operatorname{im}(J) \cong \mathbf{Z}_{j_n}$ or $\mathbf{Z}_{2j_n}$ and, since $e \circ J \colon \mathbf{Z} \to \pi_{4n-1}^S \to \mathbf{Z}_{j_n}$ is the natural projection, deduced that $\operatorname{im}(J)$ is a direct summand of π_{4n-1}^S, at least for odd n. Let $\operatorname{im}(J)' \subset \operatorname{im}(J)$ be the subgroup of elements of odd order. It is clear that $\operatorname{im}(J)' \subset \pi_{4n-1}^S$ is a direct summand for all n. In § 4 we prove

THEOREM 1.4.[1] *There is an isomorphism* $\pi_{4n}(BPL) \cong \mathbf{Z} \oplus \pi_{4n-1}^S/\operatorname{im}(J)'$, *for all* $n > 2$.

We also describe the maps in diagram 1.1 in terms of the invariants e and f and the direct sum decompositions of Theorems 1.3 and 1.4, at least for

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[1] Theorems 1.3 and 1.4 were conjectured by Novikov. [S. P. Novikov, *Homotopically equivalent smooth manifolds*, A.M.S. Translations, Ser. 2 (48), 271-396].

all n such that $\operatorname{im}(J) \cong \mathbf{Z}_{j_n}$.

The homomorphism $f\colon \Gamma_{4n-1} \rightarrow \mathbf{Z}_{\theta_n}$ is constructed by studying smooth $4n$-manifolds with boundary an exotic sphere. Using results on spin and U-cobordism, we prove

THEOREM 1.5. *Every exotic sphere* $\Sigma \in \Gamma_{4n-1}$ *bounds a spin manifold* M^{4n} (*resp. a U-manifold*) *with all Pontrjagin numbers, except possibly* p_n, *zero* (*resp. all Chern numbers, except* c_{2n}, *zero*).

The invariant $f(\Sigma) \in \mathbf{Z} \bmod \theta_n\mathbf{Z}$ is then defined to be essentially the index of such an M^{4n} modulo the indeterminacy resulting from the non-uniqueness of M^{4n}.

In § 2 we give the necessary preliminary results from K-theory and cobordism. In § 3, we study almost smooth manifolds, that is, manifolds with a smooth structure (in fact, a spin or U-structure) in the complement of a point. In particular, we show that the PL normal microbundle of such manifolds is orientable for K-theory. From this we deduce certain integrality conditions on the characteristic numbers, which for the decomposable numbers coincide with the Atiyah-Hirzebruch differentiable Riemann-Roch theorem [10]. Using the theorem of Stong that all relations among characteristic numbers of closed spin and U-manifolds are given by the Riemann-Roch theorem, we obtain Theorem 1.5.

The results in § 3 are closely related to the results of Conner and Floyd on manifolds with framed boundary [12]. In § 4 we prove the main theorems of the paper, assuming the following result, which is proved in § 5.

THEOREM 1.6. *Let* M^{4n} *be a spin manifold or U-manifold with boundary an exotic sphere* $\Sigma \in \Gamma_{4n-1}$, *and with decomposable characteristic numbers zero. Then* 8 *divides index* (M^{4n}).

Both Theorems 1.5 and 1.6 are necessary to obtain a sufficiently sharp invariant f. The proof of Theorem 1.6, for even n, depends crucially on properties of U-manifolds. This is the main reason for the simultaneous treatment of spin and U-manifolds throughout the paper.

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2. Preliminaries on K-theory and cobordism

In this section we collect for later use certain formulas and theorems from K-theory and cobordism. We also compute the smallest integer which can occur as the index of a closed, smooth, spin, or U-manifold with decomposable characteristic numbers zero. This is given in Corollary 2.4, which, along with Theorem 1.6 of the introduction, characterizes the indeterminacy of the index of the manifolds of Theorem 1.5 with a given boundary $\Sigma \in \Gamma_{4n-1}$.

Recall that for each of the structure groups $G = SO$, spin, U, SU, there are the cobordism groups Ω_k^G consisting of cobordism classes of k-manifolds M^k together with liftings $\tau_M\colon M^k \longrightarrow BG$ of the stable tangent bundle of M^k. The tangent maps induce a map

$$\tau\colon \Omega_k^G/\text{torsion} \longrightarrow H_k(BG, Q)$$

defined by $\tau(M^k) = (\tau_M)_*[M^k]$, where $[M^k]$ is the fundamental homology class. It is well-known that τ is an injection. The homology class $\tau(M^k)$ corresponds to the evaluation homomorphism on cohomology

$$e_M\colon H^k(BG, Q) \xrightarrow{\tau^*} H^k(M^k, Q) \xrightarrow{\langle\ ,[M]\rangle} Q\ .$$

For $G = SO$ or spin, $H^*(BG, Q) \cong Q[p_1, p_2, \cdots]$ is a polynomial algebra on universal Pontrjagin classes $p_j \in H^{4j}(BG, \mathbf{Z})$. For $G = U$, $H^*(BU, \mathbf{Z}) \cong \mathbf{Z}[c_1, c_2, \cdots]$, and for $G = SU$, $H^*(BSU, \mathbf{Z}) \cong \mathbf{Z}[c_1, c_2, \cdots]/(c_1)$ where $c_j \in H^{2j}(BU, \mathbf{Z})$ is the universal Chern class.

It is convenient to introduce variables x_i(dim $x_i = 2$) and write the total Chern class as $c = \sum_{j=0}^{\infty} c_j = \prod(1 + x_i)$. That is, c_j is the j^{th} elementary symmetric function in the x_i. Then the Chern character of a complex n-dimensional bundle ξ is defined by $\text{ch}(\xi) - n = \sum(e^{x_i} - 1)$. The Todd class is defined by $T(\xi) = \prod(x_i/(1 - e^{-x_i}))$. It is also convenient to introduce the characteristic classes $e_j(\xi)$, defined as the j^{th} elementary symmetric function in the variables $(e^{x_i} - 1)$.

Let $\lambda^j(\xi)$ be the j^{th} exterior power of the bundle ξ. Set $\lambda_t(\xi) = \sum_{j=0}^{\infty} \lambda^j(\xi) t^j$, and define K-theory operations $\gamma^j(\xi)$ by the formula $\gamma_t(\xi) = \sum_{j=0}^{\infty} \gamma^j(\xi) t^j = \lambda_{t/(1-t)}(\xi)$. The operation γ_t satisfies $\gamma_t(\xi + \eta) = \gamma_t(\xi) \otimes \gamma_t(\eta)$, and the γ^j form power series generators over the integers for all K-theory operations [6]. One sees by a computation that $\text{ch}\big(\gamma^j(\xi - n)\big) = e_j(\xi)$, where $e_j(\xi)$ is defined above [24].

If ξ is a real n-bundle, denote its complexification by ξ_c. Then the Pontrjagin classes are given by $p(\xi) = \sum p_j(\xi) = \sum(-1)^j c_{2j}(\xi_c)$. The Pontrjagin character is defined to be $\text{ph}(\xi) = \text{ch}(\xi_c)$. It is convenient to write formally $p(\xi) = \prod(1 + x_i^2)$ (dim $x_i^2 = 4$) and define the characteristic classes $f_j(\xi)$ to be the j^{th} elementary symmetric function in the variables $(e^{x_i} + e^{-x_i} - 2)$. Also, the $\hat{A}$ class and L class are defined by $\hat{A} = \prod\{(x_i/2)/\sinh(x_i/2)\}$ and $L = \prod(x_i/\mathrm{tgh}\, x_i)$.

The KO-theory operations $\gamma^i(\xi)$ are defined just as for complex bundles. One sees by a computation that $f_j(\xi)$ is a polynomial with integral coefficients in the $\mathrm{ph}(\gamma^i(\xi - n))$ [24].

Let ξ be a complex n-bundle over a space X. Denote the Thom space of ξ by $T(\xi)$. Recall that there is a canonical orientation class $U_c \in \widetilde{K}(T(\xi))$ [6]. That is, $i^* U_c =$ generator $\widetilde{K}(S^{2n}) \cong \mathbf{Z}$ where $i\colon S^{2n} \to T(\xi)$ is the inclusion of the fibre, compactified at infinity. If ξ is a real spin bundle with fibre dimension $8n$, then there is a canonical class $U \in \widetilde{KO}(T(\xi))$ with $i^* U =$ generator $\widetilde{KO}(S^{8n}) \cong \mathbf{Z}$ [8]. Moreover, these classes satisfy

$$\begin{aligned} \mathrm{ch}(U_c) &= \Phi(T^{-1}(\xi)) \\ \mathrm{ph}(U) &= \Phi(\hat{A}^{-1}(\xi)) \end{aligned}$$

where $\Phi\colon H^*(X, Q) \cong H^*(T(\xi), Q)$ is the Thom isomorphism in cohomology. A unified treatment of the orientations U and U_c can be found in [12].

The existence of these orientation classes is the main tool in the proof of the Atiyah-Hirzebruch, Riemann-Roch theorems for differentiable manifolds [10].

THEOREM. *Let ξ be a vector bundle over a spin manifold M (resp. a complex bundle over a U-manifold M). Then*

$$\langle \mathrm{ph}(\xi)\cdot\hat{A}(M), [M]\rangle \in a_n\mathbf{Z} \qquad [\textit{resp.}\ \langle \mathrm{ch}(\xi)\cdot T(M), [M]\rangle \in \mathbf{Z}]\ .$$

Here $\hat{A}(M)$ and $T(M)$ are the $\hat{A}$ and T classes of the tangent bundle τ_M of M. This theorem gives integrality conditions on the characteristic numbers of manifolds. Thus for spin manifolds M^{4n} consider the evaluation homomorphism

$$\overline{e_M}\colon H^{4n}(B\mathrm{spin}, Q) \longrightarrow H^{4n}(M^{4n}, Q) \longrightarrow H^{4n}(M^{4n}, Q/\mathbf{Z}) = Q/\mathbf{Z}\ .$$

Set $R^{\mathrm{spin}}_{4n} = \bigcap_{M^{4n}} \ker(\overline{e_M})$ where the intersection is taken over all $4n$ spin manifolds M^{4n}. Similarly, for U-manifolds M^{2n}, we have $\overline{e_M}\colon H^{2n}(BU, Q) \to H^{2n}(M^{2n}, Q) \to Q/\mathbf{Z}$. Set $R^U_{2n} = \bigcap_{M^{2n}} \ker(\overline{e_M})$.

Choosing integral polynomials in $\xi_i = \gamma^i(\tau_M - \dim M)$ in the Riemann-Roch theorem, and using the fact above that the characteristic class $f_j(\tau_M)$ is an integral polynomial in the $\mathrm{ph}(\gamma^i(\tau_M - \dim M))$, we see that

$$a_n\cdot R^{\mathrm{spin}}_{4n} \supseteqq \{(z\cdot\hat{A})_{4n} \mid z \in \mathbf{Z}[f_1, f_2, \cdots]\}$$

and similarly

$$R^U_{2n} \supseteqq \{(z\cdot T)_{2n} \mid z \in \mathbf{Z}[e_1, e_2, \cdots]\}\ .$$

Here, if $\alpha \in H^{**}(BG, Q)$, $(\alpha)_n$ denotes the homogeneous component of degree n. Stong has proved that equality holds in these inclusions [24], [25]. (See also [14]). Equivalently, let

$$B^{\text{spin}}_{4n} = \{\alpha \in H_{4n}(B\text{spin}, Q) \mid \langle (z\hat{A})_{4n}, \alpha\rangle \in a_n\mathbf{Z} \ \ \forall z \in \mathbf{Z}[f_1, f_2, \cdots]\}$$
$$B^{U}_{2n} = \{\alpha \in H_{2n}(BU, Q) \mid \langle (zT)_{2n}, \alpha\rangle \in \mathbf{Z} \ \ \forall z \in \mathbf{Z}[e_1, e_2, \cdots]\} .$$

Then

THEOREM (Stong). *$\tau\Omega^{\text{spin}}_{4n} = B^{\text{spin}}_{4n}$ and $\tau\Omega^{U}_{2n} = B^{U}_{2n}$ where τ is the inclusion*

$$\tau\colon \Omega^G_*/\text{torsion} \longrightarrow H_*(BG, Q).$$

Thus any collection of numbers $\{p_\omega = p_1^{i_1}\cdots p_r^{i_r} \mid \omega = (i_1, \cdots, i_r), \sum 4j\cdot i_j = 4n\}$ or $\{c_\omega \mid \omega = (i_1, \cdots, i_r), \sum 2j\cdot i_j = 2n\}$ which satisfy the integrality conditions of the Riemann-Roch theorem, are the characteristic numbers of some spin or U-manifold, respectively.

Now let $\varepsilon_k\colon Q[p_1, p_2, \cdots] \to Q$ and $\varepsilon_k\colon Q[c_1, c_2, \cdots] \to Q$ be the homomorphisms assigning to a polynomial its coefficient of p_k or c_k, respectively, with the convention that $p_0 = c_0 = 1$. Then the linear combinations of the decomposable characteristic numbers which are integral for all spin or U-manifolds, are given by

$$\tilde{R}^{\text{spin}}_{4n} = R^{\text{spin}}_{4n} \cap \ker(\varepsilon_n)$$
$$\tilde{R}^{U}_{2n} = R^{U}_{2n} \cap \ker(\varepsilon_n) .$$

It is a corollary of Stong's theorem that, given a collection of numbers $\{p_\omega \mid \omega \neq (n)\}$, there is a spin manifold with the $\{p_\omega\}$ as decomposable Pontrjagin numbers, provided that the linear combinations of the p_ω which belong to $\tilde{R}^{\text{spin}}_{4n}$ are integral. An analogous statement holds for $\tilde{R}^{U}_{2n}$ and a collection of numbers $\{c_\omega \mid \omega \neq (n)\}$. This is a very simple fact about vector spaces over Q, but since later it is a key step in the proof of Theorem 1.5, we include the argument.

Let $V = Q[p_1, \cdots, p_{n-1}]_{(4n)}$ be the vector space with basis the homogeneous monomials of degree $4n$ in the Pontrjagin classes $p_1, \cdots, p_{n-1}$. Consider the homomorphism

$$\tau\colon \Omega^{\text{spin}}_{4n} \longrightarrow V^* = \text{Hom}_Q(V, Q)$$

defined by the decomposable Pontrjagin numbers of manifolds. The image is a free, abelian subgroup of maximal rank. By Stong's theorem, the integral dual of image(τ) in V, that is, those elements of V which are integral on all $4n$ spin manifolds, is $\tilde{R}^{\text{spin}}_{4n}$. Since its rank is maximal, image(τ) can thus be described as the integral dual of $\tilde{R}^{\text{spin}}_{4n}$ in V^*. This is precisely the statement above.

We will need the following formulas in § 3.

LEMMA 2.1. (i) $(e_j)_{4n} = \dfrac{s_{j,n}}{(2n-1)!}c_{2n} + \cdots, \quad s_{j,n} \in \mathbf{Z}.$

(ii) $(f_j)_{4n} = \dfrac{t_{j,n}}{(2n-1)!}p_n + \cdots, \quad t_{j,n} \in \mathbf{Z}.$ *Also*

(iii) $(\hat{A})_{4n} = \dfrac{-\operatorname{num}(B_n/4n)}{(2n-1)!\,j_n}p_n + \cdots .$

(iv) $(T)_{4n} = \dfrac{(-1)^{n-1}\operatorname{num}(B_n/4n)}{(2n-1)!(j_n/2)}c_{2n} + \cdots .$

(v) $$(L)_{4n} = \frac{2^{2n+1}(2^{2n-1}-1)\operatorname{num}(B_n/4n)}{(2n-1)!\,j_n}p_n + \cdots$$
$$= \frac{8\theta_n}{a_n(2n-1)!\,j_n}p_n + \cdots .$$

PROOF. The integers j_n and θ_n are defined in 1.2. The dots, of course, indicate sums of decomposable terms. The last three formulas are well known [17]. The first two are easily proved by using

$$e_j(\xi) = \operatorname{ch}\bigl(\gamma^j(\xi - \dim \xi)\bigr)$$
$$f_j(\xi) = \text{polynomial in } \operatorname{ph}\bigl(\gamma^i(\xi - \dim \xi)\bigr)$$

and evaluating for $(\xi - \dim \xi) =$ generator $\widetilde{K}(S^{4n})$. In particular, since $e_1(\xi) = \operatorname{ch}(\xi - \dim \xi)$, $f_1(\xi) = \operatorname{ph}(\xi - \dim \xi)$, and $\langle \operatorname{ch}(\xi - \dim \xi), [S^{4n}]\rangle = 1$, we see that

$$(e_1)_{4n} = \frac{\pm c_{2n}}{(2n-1)!} + \cdots$$

and

$$(f_1)_{4n} = \frac{\pm p_n}{(2n-1)!} + \cdots .$$

We will also need further information on $\widetilde{R}^{\text{spin}}_{4n}$ and $\widetilde{R}^U_{4n}$.

LEMMA 2.2. *Let* $(z\cdot\hat{A})_{4n} \in a_n\cdot\widetilde{R}^{\text{spin}}_{4n}$. *Then the constant term* $\varepsilon_0(z) \in \mathbf{Z}$ *is divisible by* j_n. *Similarly, if* $(z\cdot T)_{4n} \in \widetilde{R}^U_{4n}$, *then* $(j_n/2)$ *divides* $\varepsilon_0(z)$.

PROOF. The coefficient of p_n in $(z\hat{A})_{4n}$ is

$$\varepsilon_0(z)\varepsilon_n(\hat{A}) + \varepsilon_n(z)\varepsilon_0(\hat{A}) = 0 .$$

By Lemma 2.1, (ii), (iii), this is

$$\frac{-\varepsilon_0(z)\operatorname{num}(B_n/4n)}{(2n-1)!\,j_n} + \frac{t}{(2n-1)!} = 0$$

for some $t \in \mathbf{Z}$. It follows immediately that j_n divides $\varepsilon_0(z)$ since $j_n = \operatorname{denom}(B_n/4n)$ and $\operatorname{num}(B_n/4n)$ are relatively prime. The proof of the second statement is analogous.

LEMMA 2.3. *Suppose M^{4n} is a spin manifold (resp. U-manifold) with decomposable characteristic numbers zero. Then $a_n(2n-1)!j_n$ divides $p_n(M)$ (resp. $(j_n/2)(2n-1)!$ divides $c_{2n}(M)$). Moreover, these results are best possible.*

PROOF. Since the decomposable numbers of M^{4n} are zero and $(f_j)_0 = 0$, we see that $(f_1\cdot\hat{A})_{4n} = (f_1)_{4n}$. Thus both $\langle (f_1)_{4n}, [M^{4n}]\rangle = \pm p_n(M)/(2n-1)!$, and $\langle (\hat{A})_{4n}, [M^{4n}]\rangle = \big(-\operatorname{num}(B_n/4n)/(2n-1)!j_n\big)p_n(M)$ are in $a_n\cdot\mathbf{Z}$. It follows that $a_n(2n-1)!j_n$ divides $p_n(M)$. The result for U-manifolds is similar. The best possible statement follows from Stong's theorem, since $\{p_\omega\} = \{0, \cdots, 0, p_n = a_n(2n-1)!j_n\}$ (resp. $\{c_\omega\} = \{0, \cdots, 0, c_{2n} = (j_n/2)(2n-1)!\}$) satisfy the necessary integrality conditions, hence are the characteristic numbers of some manifold.

COROLLARY 2.4. *Let M^{4n} be a spin manifold (resp. U-manifold) with decomposable characteristic numbers zero. Then $8\theta_n$ divides index (M^{4n}) (resp. $8\theta_n$ divides $a_n\cdot$index (M^{4n})).*

PROOF. This is immediate from Lemmas 2.1 (v) and 2.3 and the fact that, for a U-manifold, $p_n = -2c_{2n} +$ (decomposable terms).

Remark 2.5. Stong has also shown that, if M^{4n} is a U-manifold, all of whose Chern numbers divisible by c_1 vanish, and whose Pontrjagin numbers satisfy the integrality conditions of the spin Riemann-Roch theorem, then M^{4n} is U-cobordant to an SU-manifold [25].

We conclude this section with a useful definition of the Adams invariant $e\colon \pi^S_{4n-1}\to \mathbf{Z}_{j_n}$ [2]. Let $\alpha\in\pi^S_{4n-1}$, and let $S^{N+4n-1}\xrightarrow{\alpha}S^N\xrightarrow{i}Y\xrightarrow{j}S^{N+4n}\longrightarrow\cdots$ be the Puppe sequence with $N\equiv 0 \bmod 8$. Since $\widetilde{KO}\,(S^{N+4n-1}) = 0$, there is an element $U''\in\widetilde{KO}(Y)$ such that $i^*(U'') =$ generator $\widetilde{KO}(S^N)\cong\mathbf{Z}$. Then $\mathrm{ph}(U'') = g_N + \lambda a_n g_{N+4n}$ where $g_N, g_{N+4n}\in H^N(Y,\mathbf{Z}), H^{N+4n}(Y,\mathbf{Z})$ are generators and, $\lambda\in Q$. It is easy to show that the residue class $\bar\lambda\in Q/\mathbf{Z}$ is independent of the choice of U''. Then $e(\alpha) = \bar\lambda\in Q/\mathbf{Z}$ defines a homomorphism. Adams proved that $j_n\cdot\lambda\in\mathbf{Z}$, hence e can be interpreted as a homomorphism $e\colon\pi^S_{4n-1}\to\mathbf{Z}_{j_n}$.

If we work with complex K-theory, and choose $U''_c\in\tilde{K}(Y)$ such that $i^*(U''_c) =$ generator $\tilde{K}(S^N)\cong\mathbf{Z}$, then $\mathrm{ch}(U''_c) = g_N + \mu\cdot g_{N+4n}$ where $\bar\mu\in Q/\mathbf{Z}$ is independent of the choice of U''_c. Moreover, $(j_n/a_n)\cdot\mu\in\mathbf{Z}$, hence this defines a homomorphism $e'\colon\pi^S_{4n-1}\longrightarrow\mathbf{Z}_{j_n/a_n}$.

3. Almost smooth manifolds

In Lemma 3.1 below we describe the PL normal microbundle of almost smooth manifolds in terms of vector bundles and PL bundles over spheres.

Using this and the K-theory orientability of spin bundles and complex bundles, we construct an explicit K-theory orientation for almost smooth spin or U-manifolds. Computations of Chern characters then yield the integrality conditions of the Riemann-Roch theorem for the decomposable characteristic numbers. This is stated formally in Theorem 3.7. As a corollary of this and Stong's theorem, we obtain Theorem 1.5.

Let $\Sigma^k \in \Gamma_k$ be a homotopy sphere. Recall that homotopy spheres are π-manifolds, that is, have trivial stable normal bundles [18]. Embedding Σ^k in a sphere S^{N+k}, $N > k$, and choosing a framing of the normal bundle gives an element of the framed cobordism group $\Omega_k^{\text{framed}} \cong \pi_k^S$. By a result of Anderson, Brown, and Peterson, the natural homomorphism $\Omega_k^{\text{framed}} \longrightarrow \Omega_k^{\text{spin}}$ is zero, if $k \neq 8l + 1, 8l + 2$ [4]. Since the cobordism groups Ω_*^U have no torsion [20], we see that the homomorphism $\Omega_k^{\text{framed}} \longrightarrow \Omega_k^U$ is also zero. In particular, if $\Sigma \in \Gamma_{4n-1}$ and $\varphi\colon \Sigma \times R^N \subset S^{N+4n-1}$ is a framing, we can find a manifold $M^{4n} \subset D^{N+4n}$, $\partial M^{4n} = \Sigma$, with a spin or U-structure on its normal bundle which extends the framed structure φ on $\Sigma = \partial M^{4n}$.

Let ν_0 be the normal vector bundle of M^{4n}. By attaching a cone on Σ, we obtain an almost smooth PL manifold $\hat{M} = M \cup_\Sigma C\Sigma$.

There are two natural extensions of ν_0 to bundles over $\hat{M}$, namely:

(1) A vector bundle ξ obtained *via* the trivialization φ.

(2) The PL normal microbundle ν of $\hat{M}$.

If M^{4n} is a spin manifold, then ξ is a spin vector bundle. If M^{4n} is a U-manifold, then ξ is a complex vector bundle.

LEMMA 3.1. *There is an isomorphism of microbundles over* $\hat{M}$

$$\nu + e_N \cong \xi + d^*\sigma ,$$

where ν *is the normal microbundle,* e_N *is the trivial bundle,* ξ *is the vector bundle constructed above,* σ *is a PL bundle over* S^{4n}, *and* $d\colon \hat{M} \longrightarrow S^{4n}$ *is a map of degree one. Moreover, in the homomorphism*

$$\beta\colon \pi_{4n}(BPL) \longrightarrow \pi_{4n-1}(PL/O) \xrightarrow{\cong} \Gamma_{4n-1} ,$$

we have $\beta(\sigma) = -\Sigma$.

PROOF. Since $\xi|_M \cong \nu|_M$, it is clear that $\nu - \xi = d^*(\sigma - e_N)$ for some PL bundle σ over S^{4n}. From the exact sequence

$$\text{(A)} \qquad 0 \longrightarrow \pi_{4n}(BO) \longrightarrow \pi_{4n}(BPL) \longrightarrow \pi_{4n-1}(PL/O) \longrightarrow 0 ,$$

it follows that $\beta(\sigma) \in \pi_{4n-1}(PL/O) = \Gamma_{4n-1}$ is the obstruction to putting a vector bundle structure on the normal microbundle ν. The obstruction to smoothing $\hat{M}$, which is clearly $\Sigma \in \Gamma_{4n-1}$, is identified by smoothing theory with the obstruction to putting a vector bundle structure on the tangent microbundle of $\hat{M}$. Thus $\beta(\sigma) = -\Sigma$.

In fact, we can say more. The group $\pi_{4n-1}(PL)$ can be interpreted as concordance classes of framed exotic spheres $\sigma\colon \Sigma^{4n-1} \times R^N \subset S^{N+4n-1}$. The difference $\nu - \xi$ measures the obstruction to extending the framing $\varphi\colon \Sigma \times R^N \cong \xi|_\Sigma \subset S^{N+4n-1}$ over a disc $D^{4n} \subset D^{N+4n}$ with $\partial D^{4n} = \Sigma$. That is, stably $\nu - \xi = d^*\sigma$ where $\sigma \in \pi_{4n-1}(PL)$ is the concordance class of the framing $-\varphi\colon (-\Sigma) \times R^N \subset S^{N+4n-1}$.

Lemma 3.1 could be proved, without referring to the obstruction theory for smoothing, by a direct geometric argument describing the gluing functions for σ, ν, and ξ in terms of a PL isotopy of $\Sigma \subset S^{N+4n-1}$ to standard position.

Remark 3.2. Notice that we chose the framing $\varphi\colon \Sigma \times R^N \subset S^{N+4n-1}$, and then chose M^{4n}. To obtain Theorem 1.5 for U-manifolds, it will be important to choose φ such that $(j_n/2)e'(\varphi) = 0$ where φ is regarded as an element of the framed cobordism group π^S_{4n-1}, and $e'\colon \pi^S_{4n-1} \to \mathbf{Z}_{j_n/a_n}$ is defined in § 2. This is no restriction for odd n but holds for only half the possible framings for even n because then $\operatorname{image}(\pi_{4n-1}(U)) \subset \pi_{4n-1}(O) = \mathbf{Z}$ has index 2. The point is, while all $\Sigma \in \Gamma_{4n-1}$ do bound U-manifolds with decomposable Chern numbers zero, for even n not all possible framings of Σ bound such U-structures.

By Lemma 3.1 we have the stable equation $\nu = \xi + d^*\sigma$ over $\hat{M}$. Thus $T(\nu) = T(\xi + d^*\sigma)$. Now over $\hat{M} \times \hat{M}$ it is well known that $T(\xi \times d^*\sigma) = T(\xi) \wedge T(d^*\sigma)$ [7]. This gives rise to a diagram

3.3
$$\begin{tikzcd}
S^{2N+4n} \arrow[r, "\pi"] & T(\nu) \arrow[r, phantom, "="] \arrow[d, no head] & T(\xi + d^*\sigma) \arrow[r, "\Delta"] \arrow[d, no head] & T(\xi) \wedge T(d^*\sigma) \arrow[d, no head] \\
& \hat{M} \arrow[r, phantom, "="] & \hat{M} \arrow[r, "\Delta"] & \hat{M} \times \hat{M}
\end{tikzcd}$$

where π is a map of degree one determined by an embedding $\hat{M} \to S^{2N+4n}$. Specifically, π is the identity on the normal bundle ν and collapses $S^{2N+4n} - \nu$ to the point infinity in $T(\nu)$. Δ is the diagonal.

If M^{4n} is a spin manifold, there is the canonical orientation class $U' \in \widetilde{KO}(T(\xi))$, and if M^{4n} is a U-manifold, there is the complex orientation $U'_c \in \tilde{K}(T(\xi))$.

Next note that since σ is a bundle over S^{4n}, $T(\sigma)$ is a 2-cell complex. In fact, it is the cofibre of $J_{PL}(\sigma) \in \pi^S_{4n-1}$ where $J_{PL}\colon \pi_{4n}(BPL) \to \pi_{4n}(BF) \cong \pi^S_{4n-1}$ is the PL J-homomorphism. The proof of this for smooth bundles given in [2] works for PL bundles also. Let $U'' \in \widetilde{KO}(T(\sigma))$ and $U''_c \in \tilde{K}(T(\sigma))$ be orientations.

LEMMA 3.4. *Let $\hat{M}^{4n}$ be an almost smooth spin manifold or U-manifold.*

Then, in the first case, $U = \Delta^*(U' \cdot d^* U'') \in \widetilde{KO}(T(\nu))$ *is an orientation class for KO-theory. In the second case,* $U_c = \Delta^*(U_c' \cdot d^* U_c'')$ *is an orientation for K-theory.*

PROOF. By naturality, $d^* U'' \in \widetilde{KO}(T(d^*\sigma))$ and $d^*(U_c'') \in \tilde{K}(T(d^*\sigma))$ are orientations. By well-known multiplicative properties of *KO*- and *K*-theory, the products $U' \cdot d^* U''$ and $U_c' \cdot d^* U_c''$ are orientations in $\widetilde{KO}(T(\xi) \wedge T(d^*\sigma))$ and $\tilde{K}(T(\xi) \wedge T(d^*\sigma))$ respectively. Again by naturality, $\Delta^*(U' \cdot d^* U'') \in \widetilde{KO}(T(\nu))$ and $\Delta^*(U_c' \cdot d^* U_c'') \in \tilde{K}(T(\nu))$ are orientations, where, of course, Δ is the diagonal as in diagram 3.3.

We will use these orientation classes to deduce integrality theorems for the characteristic numbers of $\hat{M}$. First,

LEMMA 3.5. *Let* ξ *be the vector bundle over* $\hat{M}^{4n}$ *and* σ *the associated element of* $\pi_{4n}(BPL)$ *constructed above. Then, if* M^{4n} *is a spin manifold,*

$$\langle \hat{A}^{-1}(\xi)_{4n}, [\hat{M}] \rangle \equiv a_n \cdot e(J_{PL}(-\sigma)) \qquad \text{in } Q/a_n \cdot \mathbf{Z}\ .$$

If M^{4n} *is a U-manifold,*

$$\langle T^{-1}(\xi)_{4n}, [\hat{M}] \rangle \equiv e'(J_{PL}(-\sigma)) \qquad \text{in } Q/\mathbf{Z}\ .$$

PROOF. Let $U \in \widetilde{KO}(T(\nu))$ be the orientation class constructed above. We compute

$$\begin{aligned} \mathrm{ph}(U) &= \mathrm{ph}\, \Delta^*(U' \cdot d^* U'') = \Delta^*(\mathrm{ph}\, U' \cdot \mathrm{ph}\, d^* U'') \\ &= \Delta^*(\phi' \hat{A}^{-1}(\xi) \cdot d^* \phi''(1 + \lambda a_n g_n)) \end{aligned}$$

where $g_n \in H^{4n}(S^{4n}, \mathbf{Z})$ is the generator, and ϕ', ϕ'' are respectively the Thom isomorphisms in cohomology for the bundles ξ, σ. Since $d\colon \hat{M} \to S^{4n}$ and $\pi\colon S^{2N+4n} \to T(\nu)$ are maps of degree one, we have

$$\begin{aligned} \langle \mathrm{ph}\, \pi^* U, [S^{2N+4n}] \rangle &= \langle \mathrm{ph}\, U, [T(\nu)] \rangle = \langle \phi^{-1}\, \mathrm{ph}\, U, [\hat{M}] \rangle \\ &= \langle \hat{A}^{-1}(\xi) \cdot (1 + \lambda a_n d^* g_n), [\hat{M}] \rangle \\ &= \langle \hat{A}^{-1}(\xi)_{4n}, [\hat{M}] \rangle + a_n \cdot \lambda \in a_n \cdot \mathbf{Z}\ . \end{aligned}$$

Since, by definition, $e(J_{PL}(\sigma)) \equiv \lambda$ in $Q/\mathbf{Z}$, the first statement follows. The proof of the second statement is identical, with ch replacing ph, and $T^{-1}(\xi)$ replacing $\hat{A}^{-1}(\xi)$.

Remark 3.6. This is a special case of a result proved by Conner and Floyd in their work on manifolds with framed boundary [12].

THEOREM 3.7. *Let* $\hat{M}^{4n}$ *be an almost smooth manifold, and let* τ *be its tangent microbundle. If* M^{4n} *is a spin manifold and* $(z\hat{A})_{4n} \in a_n \cdot \tilde{R}_{4n}^{\mathrm{spin}}$, *then* $\langle z(\tau)\hat{A}(\tau), [\hat{M}] \rangle \in a_n \mathbf{Z}$. *If* M^{4n} *is a U-manifold with* $\nu = \xi + d^*\sigma$ *where* ξ *is a complex vector bundle and* $(j_n/2)e'(J_{PL}(\sigma)) \in \mathbf{Z}$, *and* $(zT)_{4n} \in \tilde{R}_{4n}^{U}$, *then*

$\langle z(\tau)T(\tau), [\hat{M}]\rangle \in \mathbf{Z}$.

PROOF. Since $(z\hat{A})_{4n} \in a_n \cdot \tilde{R}^{\text{spin}}_{4n}$, the coefficient of p_n in $(z\hat{A})_{4n}$ is zero. It follows from Lemma 3.1 that $z(\tau)\hat{A}(\tau) = z(-\xi)\hat{A}(-\xi) = z(-\xi)\hat{A}^{-1}(\xi)$, for the lower Pontrjagin classes of τ and $(-\xi)$ coincide. According to remarks in § 2 on the characteristic classes f_j, $z(-\xi) = \text{ph}(x)$ for some $x \in KO(\hat{M})$. There is the product pairing $KO(\hat{M}) \otimes \widetilde{KO}(T(\nu)) \to \widetilde{KO}(T(\nu))$, and we compute $\text{ph}(xU) = \text{ph}(x)\cdot\text{ph}(U) = z(-\xi)\cdot\phi(\hat{A}^{-1}(\xi)\cdot(1 + a_n\cdot e(J_{PL}(\sigma))d^*g_n))$. Thus

$$\begin{aligned}\langle \text{ph}\,\pi^*(x\cdot U), [S^{2N+4n}]\rangle &= \langle \phi^{-1}\,\text{ph}(xU), [\hat{M}]\rangle \\ &= \langle z(-\xi)\hat{A}^{-1}(\xi), [\hat{M}]\rangle + \varepsilon_0(z)\cdot a_n\cdot e(J_{PL}(\sigma)) \in a_n\cdot\mathbf{Z}.\end{aligned}$$

By Lemma 2.2, j_n divides $\varepsilon_0(z)$ hence $\varepsilon_0(z)\cdot a_n\cdot e(J_{PL}(\sigma)) \in a_n\cdot\mathbf{Z}$, and the first statement follows. The proof of the second is nearly identical, but at the last step requires the additional hypothesis on σ.

The following theorem, which is Theorem 1.5 of the introduction, now drops out and is the main objective of the preceding work.

THEOREM 3.8. *Let* $\Sigma \in \Gamma_{4n-1}$. *Then* $\Sigma = \partial M^{4n}$ *where* M^{4n} *can be chosen to be either a* spin *manifold with decomposable Pontrjagin numbers zero or a U-manifold with decomposable Chern numbers zero.*

PROOF. Let $\Sigma = \partial M'$ where M' is some spin manifold. By Theorem 3.7 and the properties of $\tilde{R}^{\text{spin}}_{4n}$ which follow from Stong's theorem, as discussed in § 2, there is a closed spin manifold M'' with the same decomposable numbers as M'. Then $M = M'\#(-M'')$ satisfies the conditions of the theorem where $\#$ means "connected sum." The proof for U-manifolds is similar but, in the original choice $\Sigma = \partial M'$, one must choose M' to be a U-manifold satisfying the extra condition in Theorem 3.7. This is possible by Remark 3.2.

Remark 3.9. By a result of Brown and Peterson, the homomorphism $\Omega^{\text{framed}}_k \to \Omega^{SU}_k$ is zero if $k \neq 8l + 1, 8l + 2$ [5]. In particular, if $\Sigma \in \Gamma_{4n-1}$, then Σ is the boundary of an SU-manifold M. Since this is both a spin manifold and a U-manifold, one can use the orientations for KO-theory and K-theory above to show that the decomposable Chern numbers of M satisfy both the weakly complex and spin Riemann-Roch integrality conditions. Hence the argument of Theorem 3.8 shows that Σ is the boundary of an SU-manifold with decomposable numbers zero. (See Remark 2.5.)

4. An invariant and computations

In this section we first define the invariant $f: \Gamma_{4n-1} \to \mathbf{Z}_{\theta_n}$. Theorem 1.3 of the introduction then follows fairly easily. The remainder of the section is devoted to proving Theorem 1.4, and describing the maps in diagram 1.1.

The main step is Lemma 4.5 which uses Theorems 1.5 and 1.6 to compute the Pontrjagin class of the generator of $\pi_{4n}(BPL)/(\text{torsion}) = \mathbf{Z}$.

Let $\Sigma \in \Gamma_{4n-1}$. Following Theorem 3.8, Let $\Sigma = \partial M' = \partial M''$ where M' is a spin manifold, M'' is a U manifold and the decomposable numbers of both vanish. Define homomorphisms $f'\colon \Gamma_{4n-1} \to \mathbf{Z}_{8\theta_n}$ for all n and $f''\colon \Gamma_{4n-1} \to \mathbf{Z}_{8\theta_n}$ for even n, by

$$f'(\Sigma) = \text{index}\,(M') \in \mathbf{Z} \bmod 8\theta_n\mathbf{Z}$$

and

$$f''(\Sigma) = \text{index}\,(M'') \in \mathbf{Z} \bmod 8\theta_n\mathbf{Z}\ .$$

Here θ_n is the integer defined in 1.2.

Note that f' and f'' are well-defined by Corollary 2.4. That is, if $\Sigma = \partial M_1 = \partial M_2$, with M_1, M_2 spin manifolds, then $M = M_1 \cup_\Sigma (-M_2)$ is a closed, spin manifold with decomposable numbers zero. According to Corollary 2.4, $8\theta_n$ divides index $(M) = \text{index}\,(M_1) - \text{index}\,(M_2)$. If M_1, M_2 are U-manifolds, Lemma 3.5 guarantees that M is also a U-manifold. Again Corollary 2.4 applies.

Clearly, f' and f'' are homomorphisms. For instance, if $\Sigma_1 = \partial M_1$ and $\Sigma_2 = \partial M_2$, one can form the connected sum at the boundary $\Sigma_1 \# \Sigma_2 = \partial(M_1 \# M_2)$. Then

$$f'(\Sigma_1 \# \Sigma_2) = \text{index}\,(M_1 \# M_2) = \text{index}\,(M_1) + \text{index}\,(M_2)\ .$$

Remark 4.1. Actually $f' = f''$ for even n, for as noted in Remark 3.9, our methods could be used to show that $\Sigma = \partial M$ where M is an SU-manifold with decomposable numbers zero. Thus $f'(\Sigma) = f''(\Sigma) = \text{index}\,(M)$. We will not need the (weaker) homomorphism $f''\colon \Gamma_{4n-1} \to \mathbf{Z}_{4\theta_n}$ defined for odd n by using U-manifolds and Corollary 2.4.

The following theorem allows us to improve the invariants f', f'' by a factor of 8. The proof will be given in § 5.

THEOREM 4.2. (i) *Let M' be a (topological) $8n + 4$ manifold with $w_1(M') = w_2(M') = 0$. Then 8 divides $I(M') = \text{index}\,(M')$.*

(ii) *Let M'' be an almost smooth $4n$ U-manifold with decomposable Chern numbers zero. Then 8 divides $I(M'') = \text{index}\,(M'')$.*

It follows from Theorem 4.2 and Remark 4.1 that, if M^{4n} is an almost smooth spin manifold with decomposable numbers zero, then 8 divides $I(M^{4n})$. Thus define $f\colon \Gamma_{4n-1} \to \mathbf{Z}_{\theta_n}$, for all n, by $f(\Sigma) = (1/8)I(M^{4n}) \in \mathbf{Z} \bmod \theta_n\mathbf{Z}$, where $\Sigma = \partial M^{4n}$, and M is a spin manifold with decomposable numbers zero.

THEOREM 4.3. *bP_{4n} is a direct summand of Γ_{4n-1}. That is,*

$$\Gamma_{4n-1} \cong bP_{4n} \oplus \pi^S_{4n-1}/\text{im}(J).$$

PROOF. The Milnor generator Σ_0 of bP_{4n} bounds a framed manifold of index 8 [18]. Thus $f(\Sigma_0) = 1$. For odd n, $bP_{4n} \xrightarrow{\cong} \mathbf{Z}_{\theta_n}$, hence f is a splitting homomorphism for the exact sequence

$$\text{(B)} \qquad 0 \longrightarrow bP_{4n} \longrightarrow \Gamma_{4n-1} \longrightarrow \pi^S_{4n-1}/\text{im}\,(J) \longrightarrow 0\ .$$

For even n, either $bP_{4n} \cong \mathbf{Z}_{\theta_n}$ or $bP_{4n} \cong \mathbf{Z}_{2\theta_n}$. In the first case (all known cases), we are done as before. In the second case, from 1.2 we have $2\theta_n = 2^{2n-1}\cdot\theta'_n$ where θ'_n is odd. Then an elementary abelian group argument, which we leave to the reader, using

(1) the exact sequence (B),

(2) the homomorphism $f\colon \Gamma_{4n-1} \to \mathbf{Z}_{\theta_n}$ with $f(\Sigma_0) = 1$,

(3) the result of Adams that, if $\alpha \in \pi^S_{4n-1}$ is a 2-torsion element, $n > 2$, then $2^{2n-2}\alpha = 0$,

implies that Γ_{4n-1} contains no 2-torsion summand of order greater than 2^{2n-1}. Thus bP_{4n} must be a direct summand. The fact (3) is a consequence of the deep result that there are no elements of filtration greater than $2n$ nor less than 3 in the Adams spectral sequence in the stem π^S_{4n-1}, $n > 2$ [3].

We now seek further information on the groups and maps in the diagram discussed in the introduction.

4.4

$$\begin{tikzcd}
 & \pi_{4n}(F/PL) \arrow[r, equal] \arrow[d] & \pi_{4n}(F/PL) \arrow[d] \\
\pi_{4n}(BO) \arrow[r] \arrow[d, equal] & \pi_{4n}(BPL) \arrow[r] \arrow[d] & \pi_{4n-1}(PL/O) \arrow[d] \\
\pi_{4n}(BO) \arrow[r] & \pi_{4n}(BF) \arrow[r] & \pi_{4n-1}(F/O)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbf{Z} \arrow[r, equal] \arrow[d, "\gamma"] & \mathbf{Z} \arrow[d, "\Theta"'] & \mathbf{Z}_{\theta_n} \\
\mathbf{Z} \arrow[r, "\alpha"] \arrow[d, equal] & \mathbf{Z} + T \arrow[r, "\beta"] \arrow[d, "J_{PL}"] & \Gamma_{4n-1} \arrow[d] \arrow[ur, "f"'] & \\
\mathbf{Z} \arrow[r, "J"] & \pi^S_{4n-1} \arrow[r] \arrow[dl, "e"] & \pi^S_{4n-1}/\text{im}(J) & \\
\mathbf{Z}_{j_n} & & &
\end{tikzcd}$$

Here T denotes the torsion subgroup of $\pi_{4n}(BPL)$. Note that the Pontrjagin class of a bundle over S^{4n} defines a non-trivial homomorphism $p_n\colon \pi_{4n}(BPL) \to Q$. Thus $T = \ker\,(p_n)$.

It is convenient to introduce the odd integers

j'_n = largest odd factor of j_n

θ'_n = largest odd factor of θ_n.

Choose a generator η of the infinite summand of $\pi_{4n}(BPL)$.

LEMMA 4.5. *For $n > 2$, we have*

$$\gamma(1) = j'_n\cdot(\eta) + t_1$$
$$\alpha(1) = 2^{d_n}\cdot\theta'_n\cdot(\eta) + t_2\ ,$$

where γ and α are the maps indicated in diagram 4.4, $t_1, t_2 \in T$, and $2^{d_n} = \theta_n\cdot j'_n/\theta'_n\cdot j_n$. (For $n > 2$, one has $d_n \geqq 1$ [1].)

PROOF. Milnor has shown that $2\theta_n\cdot\gamma(1) = 2j_n\cdot\alpha(1)$, [22]. (In fact,

$\theta_n \cdot \gamma(1) = j_n \cdot \alpha(1)$ whenever $|\operatorname{im}(J)| = j_n$.) Since j'_n and θ'_n are relatively prime [22] it follows immediately that

$$\gamma(1) = j'_n \cdot b(\eta) + t_1$$
$$\alpha(1) = 2^{d_n}\theta'_n \cdot b(\eta) + t_2$$

for some integer b, and $t_1, t_2 \in T$.

Since $\gamma(1) \in \pi_{4n}(BPL)$ is the generator of the subgroup of fibre homotopically trivial bundles over S^{4n}, we know that its Pontrjagin class is the same as the Pontrjagin class of the almost framed Milnor manifold of index 8. Thus from Lemma 2.1 (v), $p_n(\gamma(1)) = a_n(2n-1)!j_n/\theta_n$, and hence $p_n(\eta) = a_n(2n-1)!j_n/\theta_n j'_n \cdot b$.

Let $\beta(\eta) = \Sigma = \partial M^{4n}$ where M^{4n} is a spin manifold with decomposable numbers zero. According to Lemma 3.1, $\nu = \xi - d^*(\eta)$ where ν is the normal bundle of $\hat{M} = M \cup_\Sigma C\Sigma$, ξ is a spin vector bundle over $\hat{M}$, and $d\colon \hat{M} \to S^{4n}$ is a map of degree one.

Let $U \in \widetilde{KO}(T(\nu))$ be the Thom class of § 3. The top dimensional component of $\mathrm{ph}(\xi)\mathrm{ph}(U)$ is in $a_n\mathbf{Z}$. Since the decomposable numbers of ξ vanish, this implies that $p_n(\xi)/(2n-1)! \in a_n\mathbf{Z}$. Thus, $-p_n(\xi) = a_n(2n-1)!c$ for some $c \in \mathbf{Z}$. Now

$$I(M) = \frac{-8\theta_n p_n(\nu)}{a_n(2n-1)!j_n} = \frac{8\theta_n c}{j_n} + \frac{8}{j'_n b}$$

since $\nu = \xi - d^*(\eta)$. Since $d_n \geqq 1$, we see that

$$\frac{8\theta_n c}{j_n} + \frac{8}{j'_n b} = \frac{8(\theta'_n c' b + 1)}{j'_n \cdot b}$$

for some even integer c'. By Theorem 4.2, $I(M)$ is an integer divisible by 8, and it is immediate that $b = 1$. This proves the lemma.

For a finite abelian group G, denote its 2-primary and odd-primary summands by ${}_2G$ and ${}_oG$, respectively. The next two results describe ${}_2T$ and ${}_oT$, where $\pi_{4n}(BPL) \cong \mathbf{Z} + T$.

THEOREM 4.6. *The PL J-homomorphism induces an isomorphism* $J_{PL}\colon {}_2T \xrightarrow{\cong} {}_2\pi^S_{4n-1}$ *for* $n > 2$.

PROOF. From the exact sequence

$$0 \longrightarrow \mathbf{Z} \xrightarrow{\gamma} \mathbf{Z} + T \xrightarrow{J_{PL}} \pi^S_{4n-1} \longrightarrow 0\ ,$$

we see that J_{PL} injects T in π^S_{4n-1}. Dividing by T yields a new exact sequence

$$0 \longrightarrow \mathbf{Z} \xrightarrow{\gamma'} \mathbf{Z} \longrightarrow \pi^S_{4n-1}/J_{PL}(T) \longrightarrow 0\ .$$

Lemma 4.5 implies that $\gamma'(1) = j'_n$, hence $\pi^S_{4n-1}/J_{PL}(T) \cong \mathbf{Z}_{j'_n}$ is an odd torsion group. The proposition follows immediately.

THEOREM 4.7. *The maps J_{PL} and β induce isomorphisms*

$$J_{PL}\colon {}_oT \xrightarrow{\cong} {}_o(\ker e) \subset \pi^S_{4n-1}$$

$$\beta\colon {}_oT \xrightarrow{\cong} {}_o(\ker f) \subset \Gamma_{4n-1}\ .$$

In particular, ${}_oT \cong {}_o\big(\pi^S_{4n-1}/\operatorname{im}(J)\big)$.

PROOF. Let $\sigma \in T$, and let $\beta(\sigma) = \Sigma = \partial M^{4n}$ where M^{4n} is a spin manifold with decomposable numbers zero. Again, we have $\nu = \xi - d^*\sigma$. Since $\sigma \in T$, $p_n(\sigma) = 0$, hence

$$I(M) = \frac{-8\theta_n p_n(\xi)}{a_n(2n-1)!j_n}\ .$$

As in the proof of Lemma 4.5, we have $-p_n(\xi) = a_n(2n-1)!c$ for some $c \in \mathbf{Z}$. Thus $f(\Sigma) = (1/8)I(M) = \theta_n c/j_n \in \mathbf{Z} \bmod \theta_n\mathbf{Z}$. Since j'_n and θ'_n are relatively prime [22], it follows that j'_n divides c. Thus, $f(\Sigma)$ has order a power of 2 in $\mathbf{Z} \bmod \theta_n\mathbf{Z}$. In particular, if $\sigma \in {}_oT$ is an odd torsion element, then $f(\Sigma) = 0$. Hence β maps ${}_oT$ into ${}_o(\ker f) \subset \Gamma_{4n-1}$. Similarly,

$$\hat{A}^{-1}(\xi)_{4n} = \frac{-\operatorname{num}(B_n/4n)p_n(\xi)}{(2n-1)!j_n} = \frac{a_n\cdot\operatorname{num}(B_n/4n)\cdot c}{j_n}$$

has order a power of 2 in $Q/\mathbf{Z}$. It then follows from Lemma 3.5 that $e\big(J_{PL}(\sigma)\big) = 0$ if $\sigma \in {}_oT$. Hence J_{PL} maps ${}_oT$ into ${}_o(\ker e) \subset \pi^S_{4n-1}$. The isomorphisms

$$\pi^S_{4n-1}/J_{PL}(T) \xrightarrow{\cong} \mathbf{Z}_{j'_n} \xrightarrow{\cong} {}_o(\pi^S_{4n-1}/\ker e)$$

then guarantee by the 5-lemma that $J_{PL}\colon {}_oT \xrightarrow{\cong} {}_o(\ker e)$ is an isomorphism. Since ${}_o(\ker f) \xrightarrow{\cong} {}_o(\ker e) \xrightarrow{\cong} {}_o\big(\pi^S_{4n-1}/\operatorname{im}(J)\big)$, we also see that $\beta\colon {}_oT \xrightarrow{\cong} {}_o(\ker f)$ is an isomorphism.

Theorems 4.6 and 4.7 imply Theorem 1.4 of the introduction.

Finally, we relate the invariants e and f on the 2-primary components of the groups involved. Note that ${}_2\mathbf{Z}_{j_n}$ is generated by the residue class of $\operatorname{num}(B_n/4n)\cdot j'_n$.

THEOREM 4.8. *Let $n > 2$. Define $h\colon {}_2\mathbf{Z}_{j_n} \to \mathbf{Z}_{\theta_n}$ by $h\big(\overline{\operatorname{num}(B_n/4n)\cdot j'_n}\big) = 2^{d_n}\cdot\theta'_n(\bar{1})$. Then the following diagram commutes.*

$$\begin{tikzcd}
{}_2T \arrow[rr, "\beta"] \arrow[dd, "J_{PL}"', "\cong"] & & {}_2\Gamma_{4n-1} \arrow[d, "f"] \\
 & & \mathbf{Z}_{\theta_n} \\
{}_2\pi^S_{4n-1} \arrow[r, "e"'] & {}_2\mathbf{Z}_{j_n} \arrow[ur, "h"'] &
\end{tikzcd}$$

PROOF. Let $\sigma \in {}_2T$, $\beta(\sigma) = \Sigma = \partial M^{4n}$, and $\nu = \xi - d^*\sigma$ as above. We showed in the proof of Theorem 4.7 that $f\beta(\sigma) = \theta_n c/j_n$ where $a_n(2n-1)!c = -p_n(\xi)$, and that $eJ_{PL}(\sigma) = (1/a_n)\hat{A}^{-1}(\xi)_{4n} = \operatorname{num}(B_n/4n)\cdot c/j_n$. Since $\operatorname{num}(B_n/4n)$ is odd, we see that $f\beta(\sigma)$ and $eJ_{PL}(\sigma)$ have the same order in the 2-primary cyclic groups ${}_2\mathbf{Z}_{\theta_n}$ and ${}_2\mathbf{Z}_{j_n}$ respectively, namely, the order of c/j_n in $Q/\mathbf{Z}$. The theorem follows.

If we combine the results 4.3 and 4.5–4.8, we see that, for all $n > 2$ such that $\operatorname{im}(J) \cong \mathbf{Z}_{j_n}$, the groups and maps in diagram 1.1 can be described as follows:

$$\begin{tikzcd}
 & \pi_{4n}(F/PL) \arrow[r, phantom, "="] \arrow[d] & \pi_{4n}(F/PL) \arrow[d] & \\
\pi_{4n}(BO) \arrow[r] \arrow[d, equal] & \pi_{4n}(BPL) \arrow[r] \arrow[d] & \pi_{4n-1}(PL/O) \arrow[d] & = \\
\pi_{4n}(BO) \arrow[r] & \pi_{4n}(BF) \arrow[r] & \pi_{4n-1}(F/O) &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathbf{Z} \arrow[r, phantom, "="] \arrow[d, "\gamma"] & \mathbf{Z} \arrow[d, "\Theta"'] \\
\mathbf{Z} \arrow[r, "\alpha"] \arrow[d, equal] & \mathbf{Z} \oplus {}_2(\mathbf{Z}_{j_n}) \oplus \pi^S_{4n-1}/\operatorname{im}(J) \arrow[r, "\beta"] \arrow[d, "J_{PL}"] & \mathbf{Z}_{\theta_n} \oplus \pi^S_{4n-1}/\operatorname{im}(J) \arrow[d] \\
\mathbf{Z} \arrow[r, "J"] & \mathbf{Z}_{j_n} \oplus \pi^S_{4n-1}/\operatorname{im}(J) \arrow[r] & \pi^S_{4n-1}/\operatorname{im}(J) .
\end{tikzcd}$$

The maps J and Θ are the natural projections onto the factors $\mathbf{Z}_{j_n}$ and $\mathbf{Z}_{\theta_n}$, respectively. The splittings of π^S_{4n-1} and Γ_{4n-1} correspond to the invariants $e\colon \pi^S_{4n-1} \to \mathbf{Z}_{j_n}$ and $f\colon \Gamma_{4n-1} \to \mathbf{Z}_{\theta_n}$. J_{PL} and β map the summand $\pi^S_{4n-1}/\operatorname{im}(J)$ of $\pi_{4n}(BPL)$ isomorphically onto $\ker(e)$ and $\ker(f)$, respectively. On the summand ${}_2(\mathbf{Z}_{j_n})$, J_{PL} and β are inclusions into $\mathbf{Z}_{j_n}$ and $\mathbf{Z}_{\theta_n}$. If η generates the summand $\mathbf{Z} \subset \pi_{4n}(BPL)$, then $\gamma(1) = j'_n\cdot(\eta)$, and $\alpha(1) = 2^{d_n}\theta'_n(\eta) + (\sigma)$ where σ generates ${}_2(\mathbf{Z}_{j_n}) \subset \pi_{4n}(BPL)$.

Remark 4.9. We have been ignoring the cases $n = 1$ or 2. The results here are well-known [27]:

$$\pi_4(BPL) = \mathbf{Z} \qquad \Gamma_3 = 0$$
$$\pi_8(BPL) = \mathbf{Z} + \mathbf{Z}_4 \qquad \Gamma_7 = bP_8 = \mathbf{Z}_{28} .$$

These are exceptional because $2^{d_n} = \theta_n j'_n/\theta'_n j_n < 1$ only for $n = 1, 2$, hence Lemma 4.5 does not hold in these dimensions.

5. Proof of Theorem 4.2

Let B be a symmetric bilinear form over the integers (that is, a symmetric matrix with integral coefficients) with determinant ± 1. If the diagonal entries of B are even, then the signature of the form is divisible by 8 [23]. In a $4n$-manifold this condition can be checked on the cup product pairing

$$H^{2n}(M^{4n}, \mathbf{Z}) \otimes H^{2n}(M^{4n}, \mathbf{Z}) \longrightarrow H^{4n}(M^{4n}, \mathbf{Z}) = \mathbf{Z}$$

rather easily by reducing cohomology classes mod 2. For if $x \in H^{2n}(M^{4n}, \mathbf{Z})$, then x^2 is even if $\bar{x}^2 = \mathrm{Sq}^{2n}\bar{x} = V_{2n} \cdot \bar{x} = 0$, where $\bar{x} \in H^{2n}(M^{4n}, \mathbf{Z}_2)$ is the reduction of the integral class x, and $V_{2n} \in H^{2n}(M^{4n}, \mathbf{Z}_2)$ is the Wu class. V_{2n} is, of course, given by a polynomial in the Stiefel-Whitney classes of the tangent bundle of M^{4n} [19].

PROOF OF 4.2 (i). Let M^{8n+4} be a manifold with $w_1(M) = w_2(M) = 0$. By the Adem relations $\mathrm{Sq}^{4n+2} = \mathrm{Sq}^2\mathrm{Sq}^{4n} + \mathrm{Sq}^1\mathrm{Sq}^{4n}\mathrm{Sq}^1$. Hence for any $x \in H^{4n+2}(M, \mathbf{Z}_2)$ we have

$$\begin{aligned} x^2 &= \mathrm{Sq}^{4n+2}x = \mathrm{Sq}^2(\mathrm{Sq}^{4n}x) + \mathrm{Sq}^1(\mathrm{Sq}^{4n}\mathrm{Sq}^1x) \\ &= (w_2 + w_1^2)\cdot(\mathrm{Sq}^{4n}x) + w_1\cdot(\mathrm{Sq}^{4n}\mathrm{Sq}^1x) = 0 \,. \end{aligned}$$

Thus the cup product form is even, and 8 divides index(M^{8n+4}).

PROOF OF 4.2 (ii). Consider an almost smooth manifold M^{4n} with a U-structure on the normal bundle of $M^{4n} - (\mathrm{pt})$. As in § 3 we can write $\nu = \xi + d^*\sigma$ where ν is the normal microbundle of M, ξ is a complex vector bundle over M, σ is a PL bundle over S^{4n}, and $d\colon M^{4n} \longrightarrow S^{4n}$ is a map of degree one. Assume the decomposable Chern numbers of ξ vanish. We will construct a cobordism between M^{4n} and a manifold N^{4n} in which the cup product form is even. Then 8 divides index(N^{4n}) = index(M^{4n}).

By performing surgeries on embedded circles in M, we may assume that M is simply connected [23]. Since $\pi_1(BU) = 0$, the U-structure on $M^{4n} - (\mathrm{pt})$ can be preserved. In this and subsequent surgery we stay in the smooth part of M, that is, away from the "bad" point.

Atiyah and Hirzebruch have shown that, for a complex bundle ξ, the total Wu class V is given by [9]

$$V = \sum_{j=0}^{\infty} V_j(\xi) \equiv \sum_{j=0}^{\infty} 2^j T_j(c_1, \cdots, c_j) \qquad \mathrm{mod}\ 2$$

where $c_i = c_i(\xi)$ and T_j is the Todd polynomial. This formula makes sense because the polynomial $2^j T_j$ has an odd denominator.

For the manifold M above, the Wu class V_{2n} of M coincides with $V_{2n}(-\xi)$ because of the relation $\nu = \xi + d^*\sigma$. In particular, V_{2n} is the reduction of an integral class, say $c_M \in H^{2n}(M, \mathbf{Z})$, which is a polynomial in the Chern classes $c_j(\xi)$. Suppose that c_M is a torsion class in $H^{2n}(M, \mathbf{Z})$. Then the cup product form on M is even because, for $x \in H^{2n}(M, \mathbf{Z})$, we have $0 = x\cdot c_M \equiv \bar{x}\cdot V_{2n} = \bar{x}^2 \bmod 2$. In general, we will construct a cobordism between M^{4n} and a U-

manifold N^{4n} for which c_N is a torsion class.

First, note that $c_M^2 = 0$ because c_M^2 is a decomposable Chern number of ξ. The argument below was used by Lashof in a more general situation [Lashof, *Poincaré duality and cobordism*, Trans. Amer. Math. Soc. 109 (1963), 257–277]. For completeness, we include some of the details of our special case.

LEMMA 5.1. *There is an integer* $b \neq 0$ *and a map* $f\colon M^{4n} \to S^{2n}$ *with* $f^*(x_{2n}) = b \cdot c_M$ *where* $x_{2n} \in H^{2n}(S^{2n}, \mathbf{Z})$ *is a generator.*

PROOF. This is a special case of a general result on integral cohomology classes whose square is zero [Berstein, Comment. Math. Helv. 35 (1961), 9–15]. A more elementary proof could be given for Lemma 5.1 since, except for the final obstruction in $H^{4n}(M^{4n}, \pi_{4n-1}(S^{2n}))$, we are in the stable range. Finiteness of the last obstruction follows from $c_M^2 = 0$.

By transverse regularity, f can be factored up to homotopy, as below [19].

5.2

$$\begin{tikzcd}
M^{4n} \arrow[r, "g"] & T(e_{2n}) \arrow[r] & S^{2n} \\
& L^{2n} \arrow[ul, "i"] \arrow[u] \arrow[r] & \mathrm{pt} \arrow[u]
\end{tikzcd} .$$

Here $L^{2n} = f^{-1}(\mathrm{pt})$ has a trivial normal bundle, e_{2n}, in M^{4n} and g collapses $M - L \times D^{2n}$ to the point at infinity in $T(e_{2n})$. If $\Phi\colon H^*(L^{2n}) \xrightarrow{\cong} H^*(T(e_{2n}))$ is the Thom isomorphism, then $g^*\Phi(1) = f^*(x_{2n}) = b \cdot c_M$. In particular, since $g\,|_{M-L\times D^{2n}} = 0$, we see that $b \cdot c_M\,|_{M-L\times D^{2n}} = 0$.

The framing e_{2n} of L in M and the complex normal bundle ξ of $M - (\mathrm{pt})$, give a complex structure, ξ_L, on the stable normal bundle of L.

LEMMA 5.3. *The Chern numbers of* (L^{2n}, ξ_L) *vanish, hence* (L^{2n}, ξ_L) *bounds a U-manifold.*

PROOF. A standard property of the Thom-Gysin map $g^*\Phi\colon H^*(L) \to H^*(M)$ is the following:

5.4 If $y \in H^*(M)$ and $a \in H^*(L)$, then

$$y \smile g^*\Phi(a) = g^*\Phi(i^*y \smile a) .$$

In particular, let c be a $2n$-dimensional monomial in Chern classes. Since $i^*c(\xi) = c(\xi_L)$, we have by 5.4

$$g^*\Phi(c(\xi_L)) = g^*\Phi(i^*c(\xi)) = c(\xi) \smile g^*\Phi(1) = bc(\xi)c_M = 0 ,$$

since this is a decomposable Chern number of ξ. Since $g^*\Phi$ is an isomorphism in the top dimension, we conclude that $c(\xi_L) = 0$, as desired.

Thus let $(L^{2n}, \xi_L) = \partial(W^{2n+1}, \xi_W)$ where, of course, ξ_W is a complex structure on the stable normal bundle of W, extending ξ_L. Now the manifold

$$Q = M \times I \cup_{L \times D^{2n} \times 1} W \times D^{2n}$$

naturally inherits a U-structure and gives a cobordism between M^{4n} and N^{4n} where

$$N^{4n} = M^{4n} - L^{2n} \times \mathring{D}^{2n} \cup_{L^{2n} \times S^{2n-1}} W^{2n+1} \times S^{2n-1} . \tag{5.5}$$

We want to prove that $c_N \in H^{2n}(N, \mathbf{Z})$ is a torsion class. First, we may assume that $\pi_0(L) = \pi_1(L) = 0$, for instance by constructing a suitable framed cobordism, if necessary, using 0 and 1 dimensional framed surgeries on L in M. Also, we may assume that $\pi_0(W) = \pi_1(W) = 0$. Thus, $H_1(L) = H_1(W) = H^1(W) = H_1(W, L) = 0$ and, by Poincaré duality, $H^{2n-1}(L) = H^{2n}(W) = 0$.

Consider the following portion of the Mayer-Vietoris sequence for the decomposition of N in 5.5.

$$\cdots H^{2n}(M^{4n} - L^{2n} \times \mathring{D}^{2n}) \oplus H^{2n}(W^{2n+1} \times S^{2n-1}) \xleftarrow{i_1^* + i_2^*} H^{2n}(N^{4n})$$

$$\xleftarrow{\delta} H^{2n-1}(L^{2n} \times S^{2n-1}) \xleftarrow{j^*} H^{2n-1}(M^{4n} - L^{2n} \times \mathring{D}^{2n}) \oplus H^{2n-1}(W^{2n+1} \times S^{2n-1}) \cdots .$$

Since $H^{2n-1}(L^{2n}) = 0$ and $H^{2n-1}(W^{2n+1} \times S^{2n-1}) \to H^{2n-1}(S^{2n-1})$ is onto, we see that j^* is onto, hence $\delta = 0$. It thus suffices to show that $(i_1^* + i_2^*)(c_N)$ is a torsion element. First, $H^{2n}(W) = H^1(W) = 0$, hence $H^{2n}(W^{2n+1} \times S^{2n-1}) = 0$ and $i_2^* c_N = 0$. Finally, c_N is a Chern class of $\xi_N = \xi_Q|_N$. Since also $\xi_M = \xi_Q|_M$, we have by naturality $i_1^*(c_N) = c_M|_{M - L \times D^{2n}}$. But we saw above that $b \cdot c_M|_{M - L \times D^{2n}} = 0$, hence $i_1^*(c_N)$ is a torsion class. This completes the proof.

Added in proof. Theorems 1.3 and 1.4 have been proved independently by D. Frank.

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