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Unstable localization and periodicity

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Introduction

In the 1980's, remarkable advances were made by Ravenel, Hopkins, Devinatz, and Smith toward a global understanding of stable homotopy theory, showing that some major features arise "chromatically" from an interplay of periodic phenomena arranged in a hierarchy (see [20], [21], [28]). We would like very much to achieve a similar understanding in unstable homotopy theory and shall describe some progress in that direction. In particular, we shall explain and extend some results of our papers [4], [11], and some closely related results of Dror Farjoun and Smith [17], [18], [19].

Periodic phenomena in stable homotopy theory are quite effectively exposed by localizations with respect to various periodic homology theories such as the Morava K-theories [6], [27]. This approach remains promising in unstable homotopy theory, but a different sort of localization, called the *W-nullification* or *W-periodization* for a chosen space W, now seems more fundamental and effective. It simply trivializes the $[W, -]_*$-homotopy of spaces in a universal way. In Section 1 of this article, we recall the general theory of nullifications, including some crucial properties which have only recently been discovered. In Section 2, we introduce a corresponding theory of nullifications for spectra which we apply to determine nullifications of Eilenberg-MacLane spaces and other infinite loop spaces.

In Section 3, we begin to classify spaces according to the nullification functors which they produce, and prove a classification theorem for finite suspension complexes similar to the Hopkins-Smith classification theorem for finite spectra. In Section 4, we study the arithmetic nullifications, which act very much like classical localizations and completions of spaces. We apply them to determine arbitrary nullifications of Postnikov spaces and to extend the classification results of Section 3 beyond finite suspension complexes.

In Section 5, we present an unstable chromatic tower providing successive approximations to a space, incorporating higher and higher types of periodicity. In Section 6, we introduce a sequence of monochromatic homotopy categories containing the successive fibres of chromatic towers. Using work of Kuhn [22] and others, we show that the nth stable monochromatic homotopy category embeds as a categorical retract of its unstable counterpart. Finally, in Section 7, we apply some of the preceding work to obtain general results on E_*-acyclicity and E_*-equivalences of spaces for various spectra E.

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For simplicity, we work primarily in the pointed homotopy category Ho_* of CW-complexes and use the natural free and pointed function complexes, $\mathrm{map}(X,Y)$ and $\mathrm{map}_*(X,Y)$, in Ho_*.

1 Nullifications of spaces

For spaces $W, Y \in Ho_*$, we say that Y is *W-null* or *W-periodic* if $W \to *$ induces an equivalence $Y \simeq \mathrm{map}(W,Y)$. When Y is connected, this just means that $\mathrm{map}_*(W,Y) \simeq *$ or equivalently that $[\Sigma^i W, Y] = *$ for each $i \geq 0$. A *W-nullification* or *W-periodization* of X consists of a map $\alpha : X \to X'$ such that X' is W-null and

$$\mathrm{map}(\alpha, Y) : \mathrm{map}(X',Y) \simeq \mathrm{map}(X,Y)$$

for each W-null space Y. By [4, Cor. 7.2], [11], or [17], we have

THEOREM 1.1. *For each $W, X \in Ho_*$, there exists a W-nullification of X.*

This is unique up to equivalence and will be denoted by $\alpha : X \to P_W X$. Roughly speaking, $P_W X$ may be constructed from X by repeatedly attaching mapping cones to trivialize maps coming in from W and its suspensions, continuing to an appropriate transfinite colimit. Among the best known examples are

EXAMPLE 1.2. If $W = S^{n+1}$, then $P_W X$ is the nth Postnikov section of X.

EXAMPLE 1.3. If $W = S^1 \cup_p e^2$, then $P_W X$ is the Anderson localization [2], [14] of X away from p. This is equivalent to the standard localization $X[1/p]$ when X is simply connected.

The W-nullification is actually a special case of the very general *f-localization* introduced in [4, Cor. 7.2] and [17] for a map f of spaces, and many results on W-nullifications can at least partially be generalized to f-localizations.

As seen from Example 1.2, the W-nullification need not preserve fibrations. However, by [11, 4.1] and [18], it mixes with the ΣW-nullification to give

THEOREM 1.4. *For a space $W \in Ho_*$ and a fibre sequence $F \to X \to B$ of pointed spaces with B connected, there is a natural fibre sequence $P_W F \to \overline{X} \to P_{\Sigma W} B$ together with a natural P_W-equivalence $X \to \overline{X}$ where $\overline{X}$ is ΣW-null.*

We may obtain $\overline{X}$ as the orbit space of $P_W F$ under the principal action by $P_W \Omega B$. The following case, first noted by Dror Farjoun, is particularly useful.

COROLLARY 1.5. *For $W \in Ho_*$, P_W preserves each fibre sequence $F \to X \to B$ of pointed spaces such that B is W-null and connected.*

In the natural Postnikov tower

$$P_W X \leftarrow P_{\Sigma W} X \leftarrow P_{\Sigma^2 W} X \leftarrow \dots,$$

we long suspected that the higher fibres might be Eilenberg-MacLane spaces for many choices of W beyond the classical spheres. We very much wanted to prove such a result because we knew that it would imply strong fibration theorems for nullification functors and allow us to bring some important parts of stable localization and periodicity theory into the unstable realm. In 1991, we finally succeeded by using a version of

KEY LEMMA 1.6. *For a connected space V and a connected ΣV-null H-space Y, the inclusion $V \subset SP^\infty V$ induces an equivalence* $\mathrm{map}_*(SP^\infty V, Y) \simeq \mathrm{map}_*(V, Y)$.
Proof. This follows by [11, Cor. 6.9] since $\mathrm{map}_*(V, Y)$ is homotopically discrete with $\pi_1 Y$ acting trivially on $[V, Y]$.

A space is called a *GEM* when it is equivalent to a product of Eilenberg-MacLane spaces $K(G_n, n)$ for a sequence of abelian groups $\{G_n\}_{n\geq 1}$. For the connected ΣV-null H-space Y, the Key Lemma shows that each map $V \to Y$ has a canonical factorization through the GEM

$$SP^\infty V \simeq \prod_{n=1}^{\infty} K(H_n V, n).$$

This easily implies

THEOREM 1.7. *For a space W and a connected H-space X, if $P_W X \simeq *$ then $P_{\Sigma W} X$ is a GEM.*
Proof. $P_{\Sigma W} X$ is a connected H-space since nullification functors preserve finite products. Moreover,

$$\mathrm{map}_*(\Sigma P_{\Sigma W} X, P_{\Sigma W} X) \simeq \mathrm{map}_*(P_{\Sigma W} X, \Omega P_{\Sigma W} X) \simeq *$$

since $P_W(P_{\Sigma W} X) \simeq P_W X \simeq *$ and $\Omega P_{\Sigma W} X$ is W-null. Hence, by the Key Lemma, $P_{\Sigma W} X$ is a retract of the GEM $SP^\infty P_{\Sigma W} X$.

This immediately generalizes to f-localizations, and a relative version is given by Dror Farjoun and Smith [19]. For p prime and $n \geq 1$, we say that a space $W \in Ho_*$ satisfies the *n-supported p-torsion condition* when $\widetilde{H}_* W$ is $(n-1)$-connected p-torsion with $H_n(W; Z/p) \neq 0$. We now recover the following result of [11, 7.2] and [19].

THEOREM 1.8. *For connected spaces W, $X \in Ho_*$ and $i \geq 1$, the fibre F of the Postnikov map $P_{\Sigma^{i+1} W} X \to P_{\Sigma^i W} X$ is a GEM. Moreover, when W satisfies the n-supported p-torsion condition, $F \simeq K(G, n+i)$ for some p-torsion abelian group G.*
Proof. The space F is 1-connected and $\Sigma^{i+1} W$-null with $P_{\Sigma^i W} F \simeq *$ by Corollary 1.5. Thus F is an H-space by [19, 2.1], and is a GEM by Theorem 1.7. The last statement follows as in [11, 7.6], where the obvious H-space ΩF is used instead of F.

The fibre of the lowest Postnikov map $P_{\Sigma W} X \to P_W X$ can be much more complicated: it equals X when X is acyclic and $W = X$, but it remains a GEM when X is a connected H-space. Theorem 1.8 combines with Theorem 1.4 to give the strong fibration theorem of [11, Thm. 8.1] and [19].

THEOREM 1.9. *For a connected space $W \in Ho_*$ and a fibre sequence $F \to X \to B$ of pointed spaces with B connected, the fibre E of the map*

$$P_{\Sigma W}F \to \mathrm{fib}(P_{\Sigma W}X \to P_{\Sigma W}B)$$

is a GEM. Moreover, when W satisfies the n-supported p-torsion condition, then $E \simeq K(G,n)$ for some p-torsion abelian group G.

Thus $P_{\Sigma W}$ preserves fibre sequences up to a "small abelian error term" E.

2 Nullifications of spectra

We now introduce nullifications of spectra and show that they have almost the same basic properties as nullifications of spaces, but with easier proofs. By virtue of Theorem 2.10 below, they determine the unstable nullifications of Eilenberg-MacLane spaces and other infinite loop spaces.

We work in the homotopy category Ho^s of CW-spectra [1] and call a spectrum E *connective* when $\pi_i E = 0$ for $i < 0$. For spectra W and Y, we let $F^c(W,Y)$ denote the connective cover of the function spectrum $F(W,Y)$, and say that Y is *W-null* or *W-periodic* when $F^c(W,Y) \simeq 0$. This means that $[W,Y]_i \cong 0$ for each $i \geq 0$. A *W-nullification* or *W-periodization* of a spectrum X consists of a map $\alpha : X \to X'$ of spectra such that X' is W-null and

$$F^c(\alpha, Y) : F^c(X', Y) \simeq F^c(X, Y)$$

for each W-null spectrum Y.

THEOREM 2.1. *For each W, $X \in Ho^s$, there exists a W-nullification of X.*
Proof. We may view Ho^s as the associated homotopy category of the closed simplicial model category of spectra in [12, 2.4] and apply [4, Cor. 7.2] to give W-nullifications in Ho^s.

The W-nullification in Ho^s is unique up to equivalence and will be denoted by $\alpha : X \to P_W X$. It is a special case of the f-localization which exists in Ho^s for each map f of spectra. The W-nullification mixes with the ΣW-nullification to give

THEOREM 2.2. *For $W \in Ho^s$ and a cofibre sequence $F \to X \to B$ of spectra, there is a natural cofibre sequence $P_W F \to \overline{X} \to P_{\Sigma W} B$ together with a natural P_W-equivalence $X \to \overline{X}$ where $\overline{X}$ is ΣW-null.*
Proof. Use the cofibre sequence of $P_W(\Sigma^{-1}B) \to P_W F$.

COROLLARY 2.3. *For $W \in Ho^s$, P_W preserves each cofibre sequence $F \to X \to B$ of spectra such that B is W-null.*

To obtain stronger (co)fibration results, we let H be the spectrum of integral homology and use

KEY LEMMA 2.4. *For spectra $V, Y \in Ho^s$, if $F^c(\Sigma V, Y) \simeq 0$, then the Hurewicz map $V \to H \wedge V$ induces an equivalence $F^c(H \wedge V, Y) \simeq F^c(V, Y)$.*
Proof. This follows since the cofibre of the unit map $S \to H$ is 1-connected.

A spectrum X is called a *stable GEM* if it is equivalent to a wedge (and thus a product) of Eilenberg-MacLane spectra $\{\Sigma^n HG_n\}_{n \in Z}$. This happens if and only if X admits a module structure over the ring spectrum H (i.e. a map $H \wedge X \to X$ in Ho^s satisfying the associativity and unit conditions). As in 1.7, the Key Lemma implies

THEOREM 2.5. *For spectra W, $X \in Ho^s$, if $P_W X \simeq 0$, then $P_{\Sigma W} X$ is a stable GEM with a canonical H-module structure.*

This immediately generalizes to f-localizations. For p prime and $n \in Z$, we say that a spectrum W satisfies the *n-supported p-torsion condition* when $H_* W$ is $(n-1)$-connected p-torsion with $H_n(W; Z/p) \neq 0$. As in 1.8, we deduce

THEOREM 2.6. *For spectra W, $X \in Ho^s$ and $i \in Z$, the fibre F of the Postnikov map $P_{\Sigma^{i+1} W} X \to P_{\Sigma^i W} X$ is a stable GEM with a canonical H-module structure. Moreover, when W satisfies the n-supported p-torsion condition, then $F \simeq \Sigma^{n+i} HG$ for some p-torsion abelian group G.*

This combines with Theorem 2.2 to give a strong fibration theorem

THEOREM 2.7. *For $W \in Ho^s$ and a cofibre sequence $F \to X \to B$ of spectra, the fibre E of the map*

$$P_W F \to \mathrm{fib}(P_W X \to P_W B)$$

is a stable GEM. Morever, when W satisfies the n-supported p-torsion condtion, then $E \simeq \Sigma^{n-1} HG$ for some p-torsion abelian group G.

Since smash products with connective spectra preserve P_W-equivalences in Ho^s, we have

PROPOSITION 2.8. *For $W \in Ho^s$, if A is a connective ring spectrum and M is an A-module spectrum, then $\alpha : M \to P_W M$ is a map of A-module spectra.*

COROLLARY 2.9. *For $W \in Ho^s$, if M is a stable GEM, then so is $P_W M$. Moreover, if $M = \Sigma^n HG$, then $\pi_i P_W M = 0$ unless $i = n$, $n+1$.*
Proof. This follows since $P_W M$ is an H-module spectrum by 2.8, and since each H-module map out of $\Sigma^n HG$ has a retract of the required form.

To relate the stable and unstable nullifications, we use the adjoint functors $\Sigma^\infty : Ho_* \to Ho^s$ and $\Omega^\infty : Ho^s \to Ho_*$.

THEOREM 2.10. *For a space $W \in Ho_*$ and a spectrum $E \in Ho^s$, the natural map*

$$P_W \Omega^\infty E \to \Omega^\infty P_{\Sigma^\infty W} E$$

is an equivalence.

Proof. Let $Ho^{sc} \subset Ho^s$ be the full subcategory of connective spectra. The proof in [8, Thm. 1.1] is easily adapted to show the existence of an idempotent functor $T : Ho^{cs} \to Ho^{cs}$ and $\eta : Id \to T$, such that for any $X \in Ho^{cs}$ the map $\Omega^\infty \eta : \Omega^\infty X \to \Omega^\infty TX$ is a W-nullification. Moreover, (T, η) must be equivalent to the idempotent functor $(P_{\Sigma^\infty W}, \alpha)$ on Ho^{cs} since a connective spectrum X is $\Sigma^\infty W$-null if and only if $\Omega^\infty X$ is W-null. Thus, $\Omega^\infty \alpha : \Omega^\infty E \to \Omega^\infty P_{\Sigma^\infty W} E$ is a W-nullification for all connective E, and hence for all E.

We may now destabilize the preceding corollary to give

COROLLARY 2.11. *For spaces W, $Y \in Ho_*$, if Y is a GEM, then so is $P_W Y$. Moreover, if $Y = K(G, n)$, then $\pi_i P_W Y = 0$ unless $i = n$, $n+1$.*

2.12. TRIVIALIZATIONS OF SPECTRA. In [5, 1.7] for $W \in Ho^s$, we introduced $[W, -]_*$-*trivializations* of spectra. These may be defined in the same way as W-nullifications, using $F(W, -)$ instead of $F^c(W, -)$. From the present standpoint, $[W, -]_*$-trivializations of spectra are just $\Sigma^* W$-nullifications, where $\Sigma^* W$ is the wedge of $\{\Sigma^n W\}_{n \in Z}$. They always preserve cofibre sequences of spectra since $\Sigma(\Sigma^* W) \simeq \Sigma^* W$.

3 Nullity classes

We can now begin to classify spaces and spectra according to the nullification functors which they produce. For spaces X, $Y \in Ho_*$, we say that X *kills* Y when the following equivalent conditions hold:

(i) each X-null space is Y-null;
(ii) $Y \to *$ is an X-nullification;
(iii) $P_X Y \simeq *$.

We say that X and Y have the *same nullity* when they kill each other and thus produce equivalent nullifications. The resulting *nullity classes* or *P-classes* $\langle X \rangle$ have a partial ordering, where $\langle X \rangle \geq \langle Y \rangle$ means that X kills Y, and have operations

$$\bigvee_\alpha \langle X_\alpha \rangle = \langle \bigvee_\alpha X_\alpha \rangle \qquad\qquad \langle X \rangle \wedge \langle Y \rangle = \langle X \wedge Y \rangle$$

with the expected properties as explained more fully in [11, Sect. 9] and [18]. However, we warn that the inequality "$\geq$" may be defined oppositely.

The above notions extend immediately to spectra, and we write $\langle E \rangle^s$ for the nullity class of $E \in Ho^s$. By Theorem 2.10 we have

PROPOSITION 3.1. *For a space $W \in Ho_*$ and connective spectrum $X \in Ho^s$, the condition $\langle W \rangle \geq \langle \Omega^\infty X \rangle$ is equivalent to $\langle \Sigma^\infty W \rangle^s \geq \langle X \rangle^s$. Thus $\langle W \rangle \geq \langle \Omega^\infty \Sigma^\infty W \rangle$ and $\langle \Sigma^\infty \Omega^\infty X \rangle^s \geq \langle X \rangle^s$.*

COROLLARY 3.2. *Let V, $W \in Ho_*$ be connected spaces.*

(i) *If $\langle V\rangle \geq \langle W\rangle$, then $\langle SP^\infty V\rangle \geq \langle SP^\infty W\rangle$.*
(ii) $\langle V\rangle \geq \langle SP^\infty V\rangle$.
(iii) *If V is a GEM, then $\langle V\rangle = \langle SP^\infty V\rangle$.*

Proof. This follows by 3.1 since $SP^\infty V \simeq \Omega^\infty(H \wedge \Sigma^\infty V)$ and since a GEM is a homotopy retract of its infinite symmetric product.

THEOREM 3.3. *For a connected space $W \in Ho_*$ and $k \geq 1$,*

$$\langle \Sigma W\rangle = \langle \Sigma^k W\rangle \vee \langle SP^\infty \Sigma W\rangle.$$

Proof. By Theorem 1.8 for $i \geq 1$, $P_{\Sigma^{i+1}W}(\Sigma^i W)$ is a GEM killed by $\Sigma^i W$. Hence, by 3.2, it is also killed by $SP^\infty \Sigma^i W$, and we have

$$\langle \Sigma^i W\rangle \leq \langle \Sigma^{i+1} W\rangle \vee \langle SP^\infty \Sigma^i W\rangle.$$

This inductively implies

$$\langle \Sigma W\rangle \leq \langle \Sigma^k W\rangle \vee \langle SP^\infty \Sigma W\rangle,$$

and the opposite inequality is evident.

This theorem enables us to partially destabilize the Hopkins-Smith classification of finite CW-spectra [20], [21], [28]. Over a finite prime p and for $n \geq 0$, let $K(n)$ denote the nth Morava K-spectrum, where $K(0) = HQ$. The *p-type* of a space X is the smallest integer n such that $\widetilde{K}(n)_* X \neq 0$, or is ∞ when $\widetilde{K}(n)_* X = 0$ for all n. It is denoted by $\mathrm{type}_p X$. We shall see in Corollary 7.2 that $\mathrm{type}_p X = \infty$

if and only if $\widetilde{H}_*(X; Z_{(p)}) = 0$. By Mitchell [26] or Hopkins-Smith [21], for each positive integer n, there exists a finite p-torsion complex of p-type n. We say that two spaces $X, Y \in Ho_*$ have the *same stabilized nullity* if $\langle X\rangle \geq \langle \Sigma^i Y\rangle$ and $\langle Y\rangle \geq \langle \Sigma^j X\rangle$ for some $i, j \geq 0$. The resulting *stabilized nullity classes* $\{X\}$ are partially ordered with finite wedge and smash operations. As noted by Dror Farjoun in the p-local case, the Hopkins-Smith classification shows

THEOREM 3.4. *For finite connected complexes V, $W \in Ho_*$, the condition $\{V\} = \{W\}$ holds if and only if $\mathrm{type}_p V = \mathrm{type}_p W$ for each prime p.*

Proof. Taking suspensions, we can assume that V and W are 1-connected. Given that $\mathrm{type}_p V = \mathrm{type}_p W$ for each prime p, we apply the "thick subcategory theorem" as in [11, 9.14] to deduce that $\{V_{(p)}\} = \{W_{(p)}\}$ for each p. When $\widetilde{H}_*(V; Q) = 0$ this implies that $\{V\} = \{W\}$ by wedge decomposition. When $\widetilde{H}_*(V; Q) \neq 0$, the p-types of V and W are all 0, and hence $\{V_{(p)}\} = \{S^1_{(p)}\} = \{W_{(p)}\}$ for each p. Thus $\{V\} \geq \{M\} \leq \{W\}$ for each finite complex M with $\widetilde{H}_*(M; Q) = 0$. Using a cofibre sequence $B \to \Sigma^i V \to M$ where B is a wedge of spheres and M is as above, we deduce that $\{V\} = \{S^1\} = \{W\}$.

The *Z/p-connectivity* of a space X is the largest integer n such that $\widetilde{H}_n(X; Z/p) = 0$, or is ∞ when $\widetilde{H}_*(X; Z/p) = 0$. It is denoted by $\mathrm{conn}_p X$ and is a nullity class invariant since it may be expressed as a cohomological connectivity. As shown p-locally in [11, 9.15], the Hopkins-Smith classification destabilizes to

THEOREM 3.5. *For finite connected complexes V, $W \in Ho_*$, the condition $\langle \Sigma V\rangle = \langle \Sigma W\rangle$ is equivalent to the joint conditions:*

(i) $\mathrm{type}_p \Sigma V = \mathrm{type}_p \Sigma W$ *for each prime* p;
(ii) $\mathrm{conn}_p \Sigma V = \mathrm{conn}_p \Sigma W$ *for each prime* p.

Proof. This follows from Theorems 3.3 and 3.4 since condition (ii) implies $\langle SP^\infty \Sigma V\rangle = \langle SP^\infty \Sigma W\rangle$.

The preceding results 3.2-3.5 have the expected versions for spectra, culminating in

THEOREM 3.6. *For finite CW spectra X, $Y \in Ho^s$, the condition $\langle X\rangle^s = \langle Y\rangle^s$ is equivalent to the joint conditions:*

(i) $\mathrm{type}_p X = \mathrm{type}_p Y$ *for each prime* p;
(ii) $\mathrm{conn}_p X = \mathrm{conn}_p Y$ *for each prime* p.

3.7. RELATED CLASSIFICATIONS OF SPECTRA. For a spectrum $X \in Ho^s$, we let $\langle X\rangle^t$ be the class of all spectra Y such that $[Y,-]_*$ and $[X,-]_*$ have the same trivial spectra, and thus give the same trivialization functors (2.12). As in [5], we let $\langle X\rangle$ be the class of all spectra Y such that the homology theories X_* and Y_* have the same acyclic spectra, and thus give the same localization functors. In general $\langle X\rangle^s \subset \langle X\rangle^t \subset \langle X\rangle$, and for a finite CW spectrum X the class $\langle X\rangle^t = \langle X\rangle$ is determined by the Hopkins-Smith invariants $\{\mathrm{type}_p X\}_p$.

4 The arithmetic nullifications

When W is a wedge of 1-connected Moore spaces, the W-nullification acts very much like a classical localization or completion functor, transforming homotopy groups in an elementary arithmetic way. We shall describe these arithmetic nullifications quite explicitly, and then apply them to determine arbitrary nullifications of Postnikov spaces and to extend our nullity classification results.

For a sequence $\{G_i\}_{i\geq 2}$ of abelian groups, let $M(G_i, i)$ be the Moore space with $H_i M(G_i, i) = G_i$ and take the wedge

$$MG(n) = M(G_2, 2) \vee \cdots \vee M(G_n, n).$$

Let J be the set of all primes p such that $p : G_i \cong G_i$ for $2 \leq i \leq n$. By [11, Sect. 5], we have

THEOREM 4.1. *For a space $Y \in Ho_*$ and $m > n$, there is a natural isomorphism*

$$\pi_m P_{MG(n)} Y \cong \pi_m Y \otimes Z_{(J)}$$

when $G_2, \ldots, G_n$ are all torsion, and there is a splittable natural short exact sequence

$$0 \to \prod_{p\in J} \mathrm{Ext}(Z_{p^\infty}, \pi_m Y) \to \pi_m P_{MG(n)} Y \to \prod_{p\in J} \mathrm{Hom}(Z_{p^\infty}, \pi_{m-1} Y) \to 0$$

when $G_2, \ldots, G_n$ are not all torsion.

The required Ext-p-completion is discussed in [13] and is given by

$$\mathrm{Ext}(Z_{p^\infty}, N) \cong \lim_n N/p^n N$$

when the p torsion elements of N are of bounded order. To extend Theorem 4.1, we need another algebraic notion. For abelian groups B and X, we call X *B-null* or *B-reduced* when $\mathrm{Hom}(B, X) = 0$. Each abelian group A has a maximal B-null quotient group $A//B$ as in [11, 5.1]. For instance, when B is p-torsion with $B/pB \neq 0$, then $A//B$ is the quotient of A by its p-torsion subgroup. We shall need

LEMMA 4.2. *If A is J-local for a set J of primes, then so is $A//B$ for all B.*
Proof. This follows when $Q \otimes B \neq 0$ since each Q-null quotient of A is J-local.

Now let

$$MG = M(G_2, 2) \vee M(G_3, 3) \vee \dots$$

be an infinite wedge of Moore spaces. Let $\overline{G}_{n+1} = G_{n+1}$ when G_2, $G_3 \dots, G_{n+1}$ are all torsion or when $G_{n+1} \otimes Q \neq 0$, and let $\overline{G}_{n+1} = G_{n+1} \oplus Q$ otherwise.

THEOREM 4.3. *For a space $Y \in Ho_*$, there is a natural isomorphism*

$$\pi_{n+1} P_{MG} Y \cong \begin{cases} \pi_{n+1} Y & \text{for } n < 1 \\ (\pi_{n+1} P_{MG(n)} Y)//\overline{G}_{n+1} & \text{for } n \geq 1. \end{cases}$$

Proof. This follows since

$$\pi_{n+1} P_{MG} Y \cong \pi_{n+1} P_{MG(n+1)} Y \cong \pi_{n+1} P_{MG(n) \vee N} Y \cong \pi_{n+1} P_N P_{MG(n)} Y$$

for $N = M(\overline{G}_{n+1}, n+1)$ by [11, Sect. 5] and Lemma 4.2.

In this theorem, we could replace $\overline{G}_{n+1}$ by $G_2 \oplus \cdots \oplus G_{n+1}$, but not by G_{n+1} as seen from

EXAMPLE 4.4. For $G_2 = Z[1/p]$, $G_3 = Z/p$, and $Y = K(\bigoplus_j Z/p^j, 3)$, we have $\pi_3 P_{MG} Y = 0$ while $(\pi_3 P_{MG(2)} Y)//G_3 \neq 0$.

Theorems 4.1 and 4.3 combine to express $\pi_* P_{MG} Y$ algebraically in terms of $\pi_* Y$, and hence

COROLLARY 4.5. *A space $Y \in Ho_*$ is killed by MG if and only if Y is 1-connected and $K(\pi_n Y, n)$ is killed by MG for each $n \geq 2$.*

For a 1-connected space W, we let MHW denote the associated wedge $M(\widetilde{H}_* W)$ of Moore spaces, and we note that MHW kills W since it successively kills the homology groups of W. Thus, there is a natural map $P_W Y \to P_{MHW} Y$ for $Y \in Ho_*$.

THEOREM 4.6. *For a 1-connected space W and connected Postnikov space $Y \in Ho_*$, the map $P_W Y \to P_{MHW} Y$ is an equivalence.*

Proof. The homotopy fibre F of this map is a 1-connected Postnikov space with $P_{MHW}F \simeq *$ by Corollary 1.5. Hence for $n \geq 2$, $\langle MHW \rangle \geq \langle K(\pi_n F, n) \rangle$ by Corollary 4.5, and $\langle W \rangle \geq \langle SP^\infty W \rangle = \langle SP^\infty MHW \rangle \geq \langle K(\pi_n F, n) \rangle$ by Corollary 3.2. Hence, W kills F and $Y \to P_{MHW}Y$ is a W-nullification.

By this theorem, the W-nullification always acts arithmetically on Postnikov spaces. Of course, it acts arithmetically on arbitrary spaces when $\langle W \rangle = \langle MHW \rangle$.

THEOREM 4.7. *If $W \in Ho_*$ is 1-connected with $\langle \Sigma^r W \rangle = \langle \Sigma^r MHW \rangle$ for some $r \geq 0$, then $\langle W \rangle = \langle MHW \rangle$.*

Proof. We may assume $\langle \Sigma W \rangle = \langle \Sigma MHW \rangle$ and must show $P_W Y \simeq P_{MHW}Y$ for each connected $Y \in Ho_*$. Since MHW is a suspension and since $P_W Y$ is ΣMHW-null, the fibre F of $P_W Y \to P_{MHW}Y$ is a GEM by Theorem 1.8. Since MHW kills the GEM F, it kills $K(\pi_n F, n)$ for $n \geq 2$. As in the proof of 4.6, this implies that W kills $K(\pi_n F, n)$ for $n \geq 2$. Thus W kills the GEM F, and $P_W Y \to P_{MHW}Y$ is a P_W-equivalence. Since $P_{MHW}Y$ is W-null, we have $P_W Y \simeq P_{MHW}Y$ as required.

We can now supplement Theorem 3.5 with a nullity classification theorem for some nonsuspension spaces.

THEOREM 4.8. *Let V, $W \in Ho_*$ be finite 1-connected complexes such that* $\text{type}_p V$ *and* $\text{type}_p W$ *belong to $\{0, 1, \infty\}$ for all p. Then the condition $\langle V \rangle = \langle W \rangle$ is equivalent to the joint conditions:*

(i) $\text{type}_p V = \text{type}_p W$ *for all p;*
(ii) $\text{conn}_p V = \text{conn}_p W$ *for all p.*

Proof. We have $\langle \Sigma V \rangle = \langle \Sigma MHV \rangle = \langle \Sigma MHW \rangle = \langle \Sigma W \rangle$ by Theorem 3.5, and thus have $\langle V \rangle = \langle W \rangle$ by Theorem 4.7.

The nullity classes covered by this theorem have canonical representatives of the form $\bigvee_{p \in J} M(Z/p, n_p)$ or $S^n \vee \bigvee_{p \in J} M(Z/p, n_p)$ where J is a finite set of primes and n_p, $n \geq 2$ are integers with $n_p < n$ for each $p \in J$. For instance, when W is a finite 1-connected complex with $H_*(W; Q) \neq 0$, its nullity class has a canonical representative of the second sort, where n is the smallest integer with $\widetilde{H}_n(W; Q) \neq 0$.

We do not know whether the assumption that the p-types belong to $\{0, 1, \infty\}$ is actually required in Theorem 4.8. However, the 1-connectivity is required. For instance, a homology n-sphere M^n and S^n satisfy the joint conditions, but $\langle M^n \rangle \neq \langle S^n \rangle$ when $P^{n-1}M^n$ is noncontractible.

5 Chromatic towers

In [11] we introduced an unstable chromatic tower $\{P_{v_n} X\}_{n \geq 0}$ providing successive approximations to a space X, incorporating higher and higher types of periodicity. We now present a simple version of this tower and compare it with a stable chromatic tower of Ravenel [29] and others. We work over a fixed prime p.

5.1 AN UNSTABLE CHROMATIC TOWER. For each $n \geq 0$, we choose a finite p-torsion complex $\overline{V}_n$ of p-type $n+1$ such that $\text{conn}_p \overline{V}_n$ is minimal. For instance, we may choose $\overline{V}_0 = S^1 \cup_p e^2$ and choose $\overline{V}_1$ for p odd to be the cofibre of the Adams map

$$A : \Sigma^{2p-2}(S^2 \cup_p e^3) \to S^2 \cup_p e^2$$

constructed in [15]. In general, $\text{conn}_p \overline{V}_n$ must be at least n by [11, 9.16], and actually equals n in the only known cases above. By Theorem 3.5 the nullity class $\langle \Sigma \overline{V}_n \rangle$ is well-defined and satisfies

$$\langle \Sigma \overline{V}_n \rangle = \langle \Sigma \overline{V}_n \vee \Sigma \overline{V}_{n+1} \rangle > \langle \Sigma \overline{V}_{n+1} \rangle.$$

The $\Sigma \overline{V}_n$-nullification of a space $X \in Ho_*$ is called the *v_n-periodization* and is denoted by $P_n X$. There is a natural *unstable chromatic tower*

$$P_0 X \leftarrow P_1 X \leftarrow P_2 X \leftarrow \cdots ,$$

which has the obvious convergence property, holim $P_n X \simeq X$, since $\pi_i P_n X \cong \pi_i X$ for $i < \text{conn}_p \overline{V}_n + 2$. The v_n-periodization $P_n X$ may perhaps "become active" at a higher dimension than the more sophisticated $P_{v_n} X$ of [11, 10.2], but there is essential agreement since $\pi_i P_n X \cong \pi_i P_{v_n} X$ for $i > \text{conn}_p \overline{V}_n + 2$. To explain the chromatic properties of our tower, we recall

5.2. THE v_n-PERIODIC HOMOTOPY GROUPS. For a finite p-torsion complex $W \in Ho_*$ of p-type n, a *v_n-map* is a map $\omega : \Sigma^d W \to W$ with $d > 0$ such that $\widetilde{K}(n)_* \omega$ is an isomorphism and $\widetilde{K}(m)_* \omega = 0$ for all $m \neq n$. For instance, the above Adams map is a v_1-map. The Hopkins-Smith "periodicity theorem" [21] ensures that each finite p-torsion complex of p-type n has a v_n-map after sufficient suspension, and that such a v_n-map is unique up to stable iteration. For each $n \geq 1$, we choose a finite p-torsion complex V_{n-1} of p-type n having a v_n-map ω. Then for a space $Y \in Ho_*$, we obtain the *v_n-periodic homotopy groups*

$$v_n^{-1} \pi_*(Y; V_{n-1}) = Z[\omega, \omega^{-1}] \otimes_{Z[\omega]} \pi_*(Y; V_{n-1})$$

by inverting the action of ω on $\pi_*(Y; V_{n-1})$. These depend on V_{n-1}, but not on the choice of ω. By [11, Thm. 11.5], we have

THEOREM 5.3. *For a space $X \in Ho_*$ and $n \geq 0$, the v_n-periodization $X \to P_n X$ induces*

$$v_m^{-1} \pi_*(P_n X; V_{m-1}) \cong \begin{cases} v_m^{-1} \pi_*(X; V_{m-1}) & \text{for } 1 \leq m \leq n \\ 0 & \text{for } m > n. \end{cases}$$

Thus we may regard the spaces $\{P_n X\}_{n \geq 0}$ as successive approximations to X capturing higher and higher types of periodicity at the prime p. To isolate the nth type of periodicity, we simply take the fibre $\widetilde{P}_n X$ of the tower map $P_n X \to P_{n-1} X$.

COROLLARY 5.4. *For a space $X \in Ho_*$ and $n \geq 1$, $\widetilde{P}_n X$ is an n-connected p-torsion space with*

$$v_m^{-1}\pi_*(\widetilde{P}_n X; V_{m-1}) \cong \begin{cases} v_n^{-1}\pi_*(X; V_{n-1}) & \text{for } m = n \\ 0 & \text{for } m \neq n. \end{cases}$$

Since the cofibre of $\omega : \Sigma^d V_{n-1} \to V_{n-1}$ has p-type $n+1$, we find

$$v_n^{-1}\pi_t(X; V_{n-1}) \cong \pi_t(P_n X; V_{n-1}) \cong \pi_t(\widetilde{P}_n X; V_{n-1})$$

for $n \geq 1$ and $t \geq 2$, so that the v_n-periodic homotopy groups of X are exposed as ordinary homotopy groups of $P_n X$ and $\widetilde{P}_n X$. Our unstable chromatic tower is closely related to

5.5. A STABLE CHROMATIC TOWER. For a spectrum E, we obtain the *stable chromatic tower* $\{L'_n E\}_{n\geq 0}$ of Ravenel [29] and others by letting $L'_n E$ be the $[W_n, -]_*$-trivialization (2.12) of E for a p-torsion finite CW-spectrum W_n of p-type $n+1$. This tower must be distinguished from Ravenel's original chromatic tower $\{L_n E\}_{n\geq 0}$ in view of his refutation of the telescope conjecture. The fibre of the tower map $L'_n E \to L'_{n-1} E$ is denoted by $M'_n E$, and the tower $\{L'_n E\}_{n\geq 0}$ sorts the v_n-periodic homotopy groups of a spectrum E in the same way as the tower $\{P_n X\}_{n\geq 0}$ sorts the v_n-periodic homotopy groups of a space X. The chromatic tower of a spectrum E and that of $\Omega^\infty E$ are related by a natural map

$$\{P_n \Omega^\infty E\}_{n\geq 0} \to \{\Omega^\infty L'_n E\}_{n\geq 0}.$$

THEOREM 5.6. *There are induced isomorphisms $\pi_i P_n \Omega^\infty E \cong \pi_i \Omega^\infty L'_n E$ and $\pi_i \widetilde{P}_n \Omega^\infty E \cong \pi_i \Omega^\infty M'_n E$ for $i > \mathrm{conn}_p \overline{V}_n + 2$.*

Proof. For $W_n = \Sigma^\infty(\Sigma \overline{V}_n)$, the $[W_n, -]_*$-trivialization of E is given by the homotopy colimit of the $\Sigma^{-k} W_n$-nullifications of E as $k \to \infty$. Hence by Theorems 2.6 and 2.10,

$$\pi_i P_n \Omega^\infty E \cong \pi_i P_{W_n} E \cong \pi_i L'_n E$$

for $i > \mathrm{conn}_p \overline{V}_n + 2$.

6 The monochromatic homotopy categories

Working over a fixed prime p for $n \geq 1$, we let $\widetilde{P}_n Ho_* \subset Ho_*$ and $M'_n Ho^s \subset Ho^s$ be the full subcategories whose objects are equivalent to the nth chromatic layers $\widetilde{P}_n X$ and $M'_n E$ of spaces $X \in Ho_*$ and spectra $E \in Ho^s$. We call $\widetilde{P}_n Ho_*$ and $M'_n Ho^s$ *monochromatic homotopy categories* and now show that $M'_n Ho^s$ embeds faithfully as a categorical retract of its unstable counterpart $\widetilde{P}_n Ho_*$.

THEOREM 6.1. *For $n \geq 1$, the functor $\widetilde{P}_n \Omega^\infty : M'_n Ho^s \to \widetilde{P}_n Ho_*$ has a left inverse Φ_n.*

The required functor Φ_n is given by the following theorem which extends results of Kuhn [22], Davis-Mahowald [16], and the author [10].

THEOREM 6.2. *For $n \geq 1$, there exists a functor $\Phi_n : Ho_* \to M'_n Ho^s$ such that:*

(i) *there is a natural equivalence* $\Phi_n \Omega^\infty E \simeq M'_n E$ for $E \in Ho^s$;
(ii) *the functor Φ_n preserves fibre sequences and homotopy direct limits of directed systems of pointed spaces;*
(iii) *if $V_{n-1} \in Ho_*$ is a finite p-torsion complex of p-type n with a v_n-map, then $v_n^{-1}\pi_*(X; V_{n-1}) \cong [V_{n-1}\Phi_n X]_*$ and $\Omega^\infty F(\Sigma^2 V_{n-1}, \Phi_n X) \simeq$* $\mathrm{map}_*(\Sigma^2 V_{n-1}, P_n X)$ *for $X \in Ho_*$;*
(iv) *if $f : X \to Y$ is a map in Ho_* with $f_* : v_n^{-1}\pi_*(X; V_{n-1}) \cong v_n^{-1}\pi_*(Y; V_{n-1})$, then $\Phi_n f : \Phi_n X \simeq \Phi_n Y$.*

This will be proved in 6.8. To avoid some technical difficulties, we shall construct Φ_n as the composite of functors $\widehat{\Phi}_n : Ho_* \to \widehat{M'_n}Ho^s$ and $\Gamma_n : \widehat{M'_n}Ho^s \simeq M'_n Ho^s$ where $\widehat{M'_n}Ho^s$ is a different form of the nth monochromatic stable homotopy category. We first explain

6.3. THE FUNCTORS Γ_n AND $\widehat{\Gamma}_n$. For $n \geq 0$ let W_n be a p-torsion finite CW-spectrum of p-type $n+1$. As in [5], we say that a spectrum C is $[W_n, -]_*$-*colocal* if each $[W_n, -]_*$-equivalence of spectra is a $[C, -]_*$-equivalence, and say that a map of spectra $X' \to X$ is a $[W_n, -]_*$-*colocalization* if it is a $[W_n, -]_*$-equivalence with X' $[W_n, -]_*$-colocal. Each spectrum X has a natural $[W_n, -]_*$-colocalization given by the fibre of the $[W_n, -]_*$-trivialization $X \to L'_n X$, and we let $\Gamma_n Ho^s \subset Ho^s$ denote the full subcategory of $[W_n, -]_*$-colocal spectra. In addition, each spectrum X has a natural W_{n*}-localization $X \to \widehat{\Gamma}_n X$ as in [6], and we let $\widehat{\Gamma}_n Ho^s \subset Ho^s$ denote the full subcategory of W_{n*}-local spectra. The functor $\widehat{\Gamma}_n : Ho^s \to Ho^s$ is right adjoint to $\Gamma_n : Ho^s \to Ho^s$ since there are natural equivalences $\widehat{\Gamma}_n X \simeq F(\Gamma_n S, X)$ and $\Gamma_n X \simeq X \wedge \Gamma_n S$ by [5, p. 375]. In unpublished work [7, 2.7], we noted

THEOREM 6.4. *For $n \geq 0$, there are adjoint equivalences of categories $\Gamma_n : \widehat{\Gamma}_n Ho^s \simeq \Gamma_n Ho^s : \widehat{\Gamma}_n$.*
Proof. For $X \in \widehat{\Gamma}_n Ho^s$, the map $\Gamma_n X \to X$ is a W_{n*}-localization since its cofibre $L'_n X$ is W_{n*}-acyclic. Hence the adjunction unit $X \to \Gamma_n \widehat{\Gamma}_n X$ is an equivalence, and the adjunction counit is likewise.

For $n = 0$, this theorem gives a correspondence between spectra with p-torsion homotopy groups and those with Ext-p-complete homotopy groups. To identify spectra in $\Gamma_n Ho^s$ and $\widehat{\Gamma}_n Ho^s$ for $n \geq 0$, we need

LEMMA 6.5. *A spectrum E belongs to $\Gamma_n Ho^s$ if and only if $\pi_* E$ is p-torsion and $v_i^{-1}\pi_*(E; W_{i-1}) = 0$ for each $i \leq n$.*
Proof. These conditions hold if and only if $L'_n E \simeq 0$.

LEMMA 6.6. *If p acts nilpotently on a spectrum E and if a v_i-map of W_{i-1} acts nilpotently on $F(W_{i-1}, E)$ for each $i \leq n$, then E belongs to both $\Gamma_n Ho^s$ and $\widehat{\Gamma}_n Ho^s$.*
Proof. For a W_{n*}-acyclic spectrum A, $F(A, E)$ is trivial since it is both $[W_n, -]_*$-trivial and $[W_n, -]_*$-colocal by Lemma 6.5. Hence E is W_{n*}-local.

Finally we need

6.7. The equivalent categories $M'_n Ho^s$ and $\widehat{M}'_n Ho^s$. For $n \geq 1$, the nth chromatic layer of a spectrum E is now given by $M'_n E = \Gamma_{n-1} L'_n E$, and the nth monochromatic homotopy category $M'_n Ho^s$ consists of the $[W_n, -]_*$-trivial $[W_{n-1}, -]_*$-colocal spectra. Similarly, we let $\widehat{M}'_n E = \widehat{\Gamma}_{n-1} L'_n E$ and the homotopy category $\widehat{M}'_n Ho^s$ consists of the $[W_n, -]_*$-trivial W_{n-1*}-local spectra. These would be the $K(n)_*$-local spectra if the telescope conjecture were valid. We may view $\widehat{M}'_n Ho^s$ and $M'_n Ho^s$ as alternative forms of the nth stable monochromatic homotopy category since there are adjoint equivalences

$$\Gamma_n : \widehat{M}'_n Ho^s \simeq M'_n Ho^s : \widehat{\Gamma}_n$$

by Theorem 6.4. In both $\widehat{M}'_n Ho^s$ and $M'_n Ho^s$, each $[W_{n-1}, -]_*$-equivalence is a homotopy equivalence. However, $\widehat{M}'_n Ho^s$ is closed under homotopy inverse limits, while $M'_n Ho^s$ is closed under homotopy direct limits.

6.8. Proof of Theorem 6.2. In [22], Kuhn constructed a functor $\phi_n : Ho_* \to Ho^s$ for $n \geq 1$ such that $\phi_n \Omega^\infty : Ho^s \to Ho^s$ is the $K(n)_*$-localization. His work may now be adapted to give a functor $\widehat{\Phi}_n : Ho_* \to Ho^s$ for $n \geq 1$ such that $\widehat{\Phi}_n \Omega^\infty : Ho^s \to Ho^s$ is $\widehat{M}'_n$, and the resulting functor $\Phi_n = \Gamma_n \widehat{\Phi}_n$ has the required properties. In more detail, choose a sequence $C_1 \to C_2 \to C_3 \to \cdots$ of finite p-torsion spectra of p-type $n-1$ with homotopy direct limit $\Gamma_{n-1} S$, by starting with $C_0 = 0$ and successively attaching finite sets of "$\Sigma^i W_{n-1}$-cells" to give a sequence of complexes C_k over S with

$$\operatorname{colim}_k [W_{n-1}, C_k]_* \cong [W_{n-1}, S]_*.$$

By the Hopkins-Smith "periodicity theorem", the complexes C_k for $k \geq 1$ can successively be equipped with v_n-maps $\omega_k : \Sigma^{d_k} C_k \to C_k$ such that each ω_{k+1} is compatible with a power of ω_k. As in [22], for $X \in Ho_*$, there are associated "function spectra" $\phi(C_k, P_n X)$ of C_k into $P_n X$. Each $\phi(C_k, P_n X)$ belongs to $\widehat{M}'_n Ho^s$ by Lemma 6.6, and there are natural equivalences

$$\phi(C_k, P_n \Omega^\infty E) \simeq \phi(C_k, \Omega^\infty L'_n E) \simeq F(C_k, L'_n E)$$

for $E \in Ho^s$. We construct the spectrum $\widehat{\Phi}_n X$ as the homotopy inverse limit of the tower $\{\phi(C_k, P_n X)\}_{k \geq 1}$, working in the underlying categories of spaces and spectra as in [10]. Each $\widehat{\Phi}_n X$ belongs to $\widehat{M}'_n Ho^s$, and there are natural equivalences

$$\widehat{\Phi}_n(\Omega^\infty E) \simeq F(\Gamma_{n-1} S, L'_n E) \simeq \widehat{M}'_n E$$

for $E \in Ho^s$. By [10] and 6.7, the functor $\Phi_n = \Gamma_n \widehat{\Phi}_n$ has the required properties.

7 E_*-acyclicity and E_*-equivalences of spaces

We shall apply some of the preceding work to obtain general results on E_*-acyclicity and E_*-equivalences of spaces for various spectra E. For p prime and $E \neq 0$, the E_*-acyclicity of $K(Z/p,n)$ implies that of $K(Z/p,n+1)$, and we define the *p-transition* $\mathrm{tran}_p E$ of E to be the largest integer n such that $\widetilde{E}_* K(Z/p,n) \neq 0$, or to be ∞ when $\widetilde{E}_* K(Z/p,n) \neq 0$ for all n. For instance, $\mathrm{tran}_p HZ/p = \infty$ and $\mathrm{tran}_p K(n) = n$ by [30]. In [9], we proved

THEOREM 7.1. *Each E_*-equivalence of spaces is an $H_i(-;Z/p)$-equivalence for $i \leq \mathrm{tran}_p E$. The condition* $\mathrm{tran}_p E = 0$ *holds if and only if* $E \simeq E[1/p]$.

COROLLARY 7.2. *If E is a p-local spectrum with* $\mathrm{tran}_p E = 0$ *or* ∞, *then the E_*-equivalences of spaces are the same as the $H_*(-;G)$-equivalences for* $G = Z/p$, $Z_{(p)}$, or Q

Thus, for an infinite wedge $E = \bigvee_{i=0}^{\infty} K(n_i)$ of Morava K-spectra with $n_i < n_j$ for $i < j$, the E_*-equivalences of spaces are the same as the $H_*(-;Z_{(p)})$-equivalences when $n_0 = 0$ and as the $H_*(-;Z/p)$-equivalences when $n_0 > 0$. In view of this corollary, we are primarily interested in p-local spectra E with extraordinary p-transitions $\mathrm{tran}_p E = n$ where $0 < n < \infty$. In general, if a loop space ΩX is E_*-acyclic, then so is the space X, but the converse will obviously fail when E has an extraordinary p-transition. We now show that such failures are quite limited.

THEOREM 7.3. *If a simply connected H-space X is E_*-acyclic for a spectrum E, then $(\Omega X)_E$ is an E_*-local GEM and $B(\Omega X)_E$ is an E_*-acyclic GEM, where $B(\Omega X)_E$ denotes the classifying space of the E_*-localized loop space.*

Proof. As in the proof of Theorem 1.7, we have

$$\mathrm{map}_*\big(\Sigma B(\Omega X)_E, B(\Omega X)_E\big) \simeq \mathrm{map}_*\big(B(\Omega X)_E, (\Omega X)_E\big) \simeq *$$

because $B(\Omega X)_E$ is E_*-acyclic and $(\Omega X)_E$ is E_*-local. Thus, by the Key Lemma 1.6, $B(\Omega X)_E$ is a retract of $SP^\infty B(\Omega X)_E$ and is therefore a GEM.

This may also be deduced from the f-generalization of Theorem 1.7 and is closely related to results of [19]. It implies

THEOREM 7.4. *Let E be a p-local spectrum with* $\mathrm{tran}_p E = n$ *where* $0 < n < \infty$. *If X is an E_*-acyclic $(n+1)$-connected H-space and $\pi_{n+2} X$ is torsion, then ΩX is also E_*-acyclic.*

Proof. By Theorem 7.3, $K\big(\pi_i(\Omega X)_E, i\big)$ is E_*-local for all i, and either $\pi_* E$ or $\pi_*(\Omega X)_E$ is torsion. Thus by Lemma 7.5 below, $\pi_{n+1}(\Omega X)_E$ is torsion-free and $\pi_i(\Omega X)_E = 0$ for $i \geq n+2$. Hence the map $\Omega X \to (\Omega X)_E$ is nullhomotopic, and ΩX is E_*-acyclic.

It is straightforward to show

LEMMA 7.5. *For E as above, an Eilenberg-MacLane space $K(G,i)$ is E_*-acyclic if G is torsion and $i \geq n+1$, or if $\pi_* E$ is torsion and $i \geq n+2$.*

We now investigate E_*-equivalences in the full subcategory $Ho_{n+2} \subset Ho_*$ of $(n+2)$-connected spaces, letting $X\langle k\rangle$ denote the k-connected cover of a space X.

PROPOSITION 7.6. *For E as above and for a map $g : X \to Y$ in the Ho_{n+2}, the following are equivalent:*

(i) *g is an E_*-equivalence;*
(ii) *g is an $(EZ/p)_*$-equivalence and an $(EQ)_*$-equivalence;*
(iii) *$g\langle k\rangle : X\langle k\rangle \to Y\langle k\rangle$ is an E_*-equivalence for all k.*

Proof. We have (i) $\Leftrightarrow$ (ii) since E is p-local, and obtain (ii) $\Leftrightarrow$ (iii) since the maps $X\langle k\rangle \to X\langle k-1\rangle$ are $(EZ/p)_*$-equivalences by Lemma 7.5.

We let $\widetilde{\Omega} : Ho_{n+2} \to Ho_{n+2}$ denote the $(n+2)$-connected loop functor $\widetilde{\Omega} X = (\Omega X)\langle n+2\rangle$, and we say that a map $g : X \to Y$ in Ho_{n+2} is a *durable E_*-equivalence* when $\widetilde{\Omega}^m g : \widetilde{\Omega}^m X \to \widetilde{\Omega}^m Y$ is an E_*-equivalence for all $m \geq 0$.

THEOREM 7.7. *Let E be a p-local spectrum with* $\mathrm{tran}_p E = n$ *where* $0 < n < \infty$. *A map $g : X \to Y$ in Ho_{n+2} is a durable E_*-equivalence under each of the following conditions:*

(i) *g, $\widetilde{\Omega} g$, and $\widetilde{\Omega}^2 g$ are E_*-equivalences;*
(ii) *g and $\widetilde{\Omega} g$ are E_*-equivalences, and the fibre of g is an H-space.*

Proof. Use Theorem 7.4 to show that $\widetilde{\Omega}^m(\mathrm{fib}\, g)$ is E_*-acyclic for all m.

When $E = K(1)$ or KZ/p, we have convenient homotopical criteria for durability.

THEOREM 7.8. *A map $g : X \to Y$ in Ho_3 is a durable $K(1)_*$-equivalance (or $K_*(-;Z/p)$-equivalence) if and only if it satisfies the following equivalent conditions:*

(i) *$g_* : v_1^{-1}\pi_*(X;Z/p) \cong v_1^{-1}\pi_*(Y;Z/p)$;*
(ii) *$\Phi_1 g : \Phi_1 X \simeq \Phi_1 Y$;*
(iii) *$\widetilde{P}_1 g_* : \pi_j \widetilde{P}_1 X \cong \pi_j \widetilde{P}_1 Y$ for $j >$* $\mathrm{conn}\, \overline{V}_1 + 2$.

Proof. This is proved for (i) in [11, Thm. 14.7] using work of Thompson to verify the "only if" part. The equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) follow by Corollary 5.4 and Theorem 6.2.

This is an unstable version of the result, proved in [6] using work of Mahowald and Miller, that a map of spectra is a $K_*(-;Z/p)$-equivalence if and only if it is a $v_1^{-1}\pi_*(-;Z/p)$-equivalence. Theorems 7.7 and 7.8 provide tools for studying the $K_*(-;Z/p)$-homology of iterated loop spaces in some previously inaccessible cases. For instance, in [11] we used the "if" part of 7.8 to deduce that the Snaith map

$$s : \Omega_0^{2n+1} S^{2n+1} \to QRP^{2n}$$

is a $K_*(-;Z/2)$-equivalence from Mahowald's result that it is a $v_1^{-1}\pi_*(-;Z/p)$-equivalence.

This confirmed an old conjecture of Miller-Snaith [25] and allowed us to determine $K_*(\Omega_0^{2n+1}S^{2n+1};Z/2)$ from their computation of $K_*(QRP^{2n};Z/2)$. Lisa Langsetmo has likewise determined $K_*(\Omega^j S^{2n+1};Z/p)$ for all $j < 2n$ using $v_1^{-1}\pi_*(-;Z/p)$-equivalences of Mahowald-Thompson [24], and we are currently obtaining similar results with S^{2n+1} replaced by an H-space X such that $K^*(X;Z_p^\wedge)$ is a p-adic exterior algebra.

Theorem 7.8 cannot easily be generalized to durable $K(n)_*$-equivalences, although the implications (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) clearly remain valid for $v_n^{-1}\pi_*(-;V_{n-1})$, Φ_n, $\widetilde{P}_n$, and $\overline{V}_n$. To see the difficulty for $n = 2$, note that $U\langle 4\rangle \in Ho_4$ has $v_2^{-1}\pi_*(U\langle 4\rangle;V_1) = 0$, but also has $\widetilde{K}(2)_*U\langle 4\rangle \neq 0$ by Theorem 7.4, because $\widetilde{K}(2)_*BSU \neq 0$ since the map $BU \to CP^\infty$ is not a $K(2)_*$-equivalence by [30, p. 709] and [31, p. 394]. One might reasonably try to generalize Theorem 7.8 to durable $K(n)_*$-equivalences by strengthening condition (i) to "$g_* : v_i^{-1}\pi_*(X;V_{i-1}) \cong v_i^{-1}\pi_*(Y;V_{i-1})$ for $1 \le i \le n$" and similarly strengthening conditions (ii) and (iii). These homotopy conditions do indeed imply that a map $g : X \to Y$ in Ho_{n+2} is a durable $K(n)_*$-equivalence by [11, Sect. 13], and the converse would also follow if we knew that the $K(n)_*$-equivalences of spaces were the same as the $(L'_n SZ/p)_*$-equivalences. Such a generalization of Theorem 7.8 would become quite plausible if each $K(n)_*$-equivalence of spaces were shown to be a $K(i)_*$-equivalence for $1 \le i \le n$, without the usual finiteness assumptions.

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