Green functors

Recall that the letter R denotes a commutative and associative ring with unit, and G denotes a finite group.

1. Two equivalent definitions

There are (at least) two equivalent definitions of Green functors for G over R.

1.1. Using the poset of subgroups of G. This definition of Green functors uses the poset of subgroups of G. It goes back to Green ([7])

Definition 1.1.1 : *A Green functor A for G over R is a Mackey functor for G over R, together with an R-algebra structure on $A(H)$, for each subgroup H of G. The Mackey structure and the algebras structures have to be compatible in the following sense :*

- *If $H \subseteq K$ are subgroups of G, and if $x \in G$, the maps r_H^K and $c_{x,H}$ are maps of R-algebras.*
- *(Frobenius relations) If $H \subseteq K$ are subgroups of G, if $a \in A(H)$ and $b \in A(K)$, then*

$$b(t_H^K a) = t_H^K\Big((r_H^K b)a\Big) \qquad (t_H^K a)b = t_H^K\Big(a(r_H^K b)\Big) \quad .$$

A morphism of Green functors $f : A \to B$ is a morphism of Mackey functors such that for each subgroup H of G, the map $f_H : A(H) \to B(H)$ is a map of R-algebras.

Remark 1.1.2 : The R-algebras considered here are always supposed unital, and the maps of R-algebras must preserve identity elements.

1.2. Using the category of G-sets. The following definition of Green functors ([3] Section 2.2) is analogous to the Dress definition of Mackey functors.

Definition 1.2.1 : *A Green functor A for G over R is a Mackey functor for G over R endowed for any G-sets X and Y with bilinear maps*

$$A(X) \times A(Y) \to A(X \times Y)$$

denoted by $(a, b) \mapsto a \times b$ which are bifunctorial, associative, and unitary, in the following sense:

- *(Bifunctoriality) If $f : X \to X'$ and $g : Y \to Y'$ are morphisms of G-sets, then the squares*

$$\begin{array}{ccc} A(X) \times A(Y) & \xrightarrow{\times} & A(X \times Y) \\ {\scriptstyle A_*(f) \times A_*(g)} \downarrow & & \downarrow {\scriptstyle A_*(f \times g)} \\ A(X') \times A(Y') & \xrightarrow[\times]{} & A(X' \times Y') \end{array}$$

$$\begin{array}{ccc} A(X) \times A(Y) & \xrightarrow{\times} & A(X \times Y) \\ {\scriptstyle A^*(f) \times A^*(g)} \uparrow & & \uparrow {\scriptstyle A^*(f \times g)} \\ A(X') \times A(Y') & \xrightarrow[\times]{} & A(X' \times Y') \end{array}$$

  are commutative.

- *(Associativity) If X, Y and Z are G-sets, then the square*

$$\begin{array}{ccc} A(X) \times A(Y) \times A(Z) & \xrightarrow{Id_{A(X)} \times (\times)} & A(X) \times A(Y \times Z) \\ {\scriptstyle (\times) \times Id_{A(Z)}} \downarrow & & \downarrow {\scriptstyle \times} \\ A(X \times Y) \times A(Z) & \xrightarrow[\times]{} & A(X \times Y \times Z) \end{array}$$

  is commutative, up to identifications $(X \times Y) \times Z \simeq X \times Y \times Z \simeq X \times (Y \times Z)$.

- *(Unitarity) If $\bullet$ denotes the G-set with one element, there exists an element $\varepsilon_A \in A(\bullet)$ such that for any G-set X and for any $a \in A(X)$*

$$A_*(p_X)(a \times \varepsilon_A) = a = A_*(q_X)(\varepsilon_A \times a)$$

  denoting by p_X (resp. q_X) the (bijective) projection from $X \times \bullet$ (resp. from $\bullet \times X$) to X.

If A and B are Green functors for the group G, a morphism of Green functors $f : A \to B$ is a morphism of Mackey functors such that for any G-sets X and Y, the square

$$\begin{array}{ccc} A(X)\times A(Y) & \stackrel{\times}{\longrightarrow} & A(X\times Y) \\ {\scriptstyle f_X\times f_Y}\downarrow & & \downarrow{\scriptstyle f_{X\times Y}} \\ B(X)\times B(Y) & \underset{\times}{\longrightarrow} & B(X\times Y) \end{array}$$

is commutative. The composition of morphisms of Green functors is the composition of morphisms of Mackey functors. The category of Green functors for G over R is denoted by $\mathsf{Green}_R(G)$.

Remark 1.2.2 : The most concise way to express this definition (Street [9]) uses the monoidal structures on G-set and R-Mod, given respectively by direct product of G-sets and tensor product of R-modules. More precisely, let $C : G\text{-}\mathsf{set} \times G\text{-}\mathsf{set} \to G\text{-}\mathsf{set}$ denote the direct product functor, and $T : R\text{-}\mathsf{Mod} \times R\text{-}\mathsf{Mod} \to R\text{-}\mathsf{Mod}$ denote the tensor product functor. A Green functor for G over R is just a Mackey functor for G over R, viewed as a bivariant functor from G-set to RG-Mod, which is monoidal, i.e. endowed with a natural transformation $T \circ A \to A \circ C$ of bivariant functors, which has to be compatible with the various isomorphisms corresponding to the associativity and unit objects of the monoidal structures.

1.3. Equivalence of the definitions.

• [1 $\to$ 2] If A is a Green functor for the first definition, recall that the corresponding Mackey functor in the sense of Dress is defined for a finite G-set X by

$$A(X) = \Big(\bigoplus_{x\in X} A(G_x)\Big)^G \quad .$$

If Y is another finite G-set, define a product map $A(X)\times A(Y) \to A(X\times Y)$ in the following way : if $u = (u_x)_{x\in X} \in A(X)$ and $v = (v_y)_{y\in Y} \in A(Y)$, then the product $u \times v$ is defined by

$$(u\times v)_{x,y} = r^{G_x}_{G_{x,y}}(u_x)\, r^{G_y}_{G_{x,y}}(v_y) \quad ,$$

for $x \in X$ and $y \in Y$, where $G_{x,y} = G_x \cap G_y$. The identity element ε of A is the identity element of the algebra $A(G) = A(\bullet)$.

• [2 → 1] If A is a Green functor in the second sense, then recall that $A(H)$ is defined by $A(H) = A(G/H)$ for a subgroup H of G. Then the product $A(H) \times A(H) \to A(H)$ is defined by

$$a.b = A^*(\delta_{G/H})(a \times b) \quad ,$$

where $\delta_{G/H}$ is the diagonal inclusion $G/H \to (G/H) \times (G/H)$. The identity element ε_H of $A(H)$ is equal to $r_H^G \varepsilon$, where ε is the identity element of A.

2. Examples

2.1. Representations rings. The representation groups associated to subgroups of G have generally a natural ring structure, for which they can be viewed as Green functors :

• If k is a field, then the tensor product of RH-modules (over k) induces a ring structure on $R_k(H)$, for $H \subseteq G$. The identity element is the image of the trivial module k. One can check that R_k is a Green functor for G over $\mathbb{Z}$.

• If H is a subgroup of G, then the direct product of H-sets gives a ring structure on the Burnside group $B(H)$. Its identity element is the image of the trivial H-set $\bullet$. This endows the Burnside functor B with a Green functor structure.

Recall that if X is a finite G-set, then $B(X)$ is the Grothendieck group of the category of finite G-sets over X. If Y is another finite G-set, then the product $B(X) \times B(Y) \to B(X \times Y)$ is induced by the obvious product sending a G-set U over X and a G-set V over Y to their direct product $U \times V$ over $X \times Y$.

More generally, the functor RB is a Green functor for G over R. It is an initial object in the category $\mathsf{Green}_R(G)$.

2.2. Cohomology rings. Let K be a subgroup of G. The cup product in cohomology defines a ring structure on

$$H^{\oplus}(K, R) = \bigoplus_{l=0}^{\infty} H^l(K, R) \quad ,$$

for which $H^{\oplus}(-, R)$ becomes a Green functor. The subfunctor $H^0(-, R)$ is also a Green functor, usually denoted by FP_R. More generally, if A is a G-algebra over R, the functor FP_A is a Green functor for G over R.

3. Modules over a Green functor

Green functors can be viewed as generalized R-algebras : a Green functor for the trivial group over R is nothing but an R-algebra. Similarly, there are (at least) two equivalent definitions of the notion of module over a Green functor :

3.1. Using the poset of subgroups of G.

Definition 3.1.1 : *Let A be a Green functor for G over R. A module M over A (or an A-module) is a Mackey functor for G over R, together with a structure of $A(H)$-module on $M(H)$, for each subgroup H of G. The Mackey structure and the module structures have to be compatible in the following sense :*

- *If $H \subseteq K$ are subgroups of G, and if $x \in G$, then*

$$\forall a \in A(K), \forall m \in M(K),\ r_H^K(am) = r_H^K(a) r_H^K(m) \quad ,$$

$$\forall a \in A(H), \forall a \in M(H),\ c_{x,H}(am) = c_{x,H}(a) c_{x,H}(m) \quad .$$

- *(Frobenius relations) If $H \subseteq K$ are subgroups of G, then*

$$\forall a_i n A(K), \forall m \in M(H),\ a(t_H^K m) = t_H^K\Big((r_H^K a) m\Big) \quad ,$$

$$\forall a \in A(H), \forall m \in M(K),\ (t_H^K a) m = t_H^K\Big(a (r_H^K m)\Big) \quad .$$

A morphism of A-modules Green functors $f : M \to N$ is a morphism of Mackey functors such that for each subgroup H of G, the map $f_H : M(H) \to N(H)$ is a map of $A(H)$-modules.

3.2. Using the category of G-sets.

Definition 3.2.1 : *A module M over the Green functor A for G over R is a Mackey functor for G over R, endowed for any G-sets X and Y with bilinear maps*

$$A(X) \times M(Y) \to M(X \times Y)$$

denoted by $(a, m) \mapsto a \times m$ which are bifunctorial, associative, and unitary, in the following sense:

- *(Bifunctoriality) If* $f : X \to X'$ *and* $g : Y \to Y'$ *are morphisms of* G*-sets, then the squares*

$$\begin{array}{ccc} A(X) \times M(Y) & \xrightarrow{\times} & M(X \times Y) \\ {\scriptstyle A_*(f) \times M_*(g)} \downarrow & & \downarrow {\scriptstyle M_*(f \times g)} \\ A(X') \times M(Y') & \xrightarrow[\times]{} & M(X' \times Y') \end{array}$$

$$\begin{array}{ccc} A(X) \times M(Y) & \xrightarrow{\times} & M(X \times Y) \\ {\scriptstyle A^*(f) \times M^*(g)} \uparrow & & \uparrow {\scriptstyle M^*(f \times g)} \\ A(X') \times M(Y') & \xrightarrow[\times]{} & M(X' \times Y') \end{array}$$

are commutative.

- *(Associativity) If* X*,* Y *and* Z *are* G*-sets, then the square*

$$\begin{array}{ccc} A(X) \times A(Y) \times M(Z) & \xrightarrow{Id_{A(X)} \times (\times)} & A(X) \times M(Y \times Z) \\ {\scriptstyle (\times) \times Id_{A(Z)}} \downarrow & & \downarrow {\scriptstyle \times} \\ A(X \times Y) \times M(Z) & \xrightarrow[\times]{} & M(X \times Y \times Z) \end{array}$$

is commutative, up to identifications $(X \times Y) \times Z \simeq X \times Y \times Z \simeq X \times (Y \times Z)$*.*

- *(Unitarity) For any* G*-set* X *and for any* $m \in M(X)$

$$A_*(q_X)(\varepsilon_A \times m) = m$$

denoting by q_X *the (bijective) projection from* $\bullet \times X$ *to* X*.*

A morphism of A*-modules* $f : M \to N$ *is a morphism of Mackey functors such that for any* G*-sets* X *and* Y*, the square*

$$\begin{array}{ccc} A(X) \times M(Y) & \xrightarrow{\times} & M(X \times Y) \\ {\scriptstyle Id \times f_Y} \downarrow & & \downarrow {\scriptstyle f_{X \times Y}} \\ A(X) \times N(Y) & \xrightarrow[\times]{} & N(X \times Y) \end{array}$$

is commutative. The composition of morphisms of A*-modules is the composition of morphisms of Mackey functors. The category of* A*-modules is denoted by* A-Mod*.*

3.3. Examples.

• Let V be an RG-module. Then the functor FP_V is a module over the Green functor FP_R. More generally, if $l \in \mathbb{N}$, the cohomology functor $H^l(-,V)$ and the homology functor $H_l(-,V)$ are FP_R-modules.

Conversely, it has been observed by Puig that the FP_R-modules can be characterized among the Mackey functor over R as the *cohomological* Mackey functors (recall that it means that $t_H^K r_H^K$ is equal to the multiplication by the index $|K:H|$, for any subgroups $H \subseteq K$ of G).

• Let M and N be Mackey functors for G over R. If X is a finite G-set, define

$$\mathcal{H}(M,N)(X) = \mathrm{Hom}_{\mathsf{Mack}_R(G)}(M, N_X) \quad .$$

This definition can be extended to give a natural Mackey functor structure on $\mathcal{H}(M,N)$. In the case $M=N$, there is similarly a natural Green functor structure on $\mathcal{H}(M,M)$. If A is any Green functor for G over R, it is equivalent to give M a structure of A-module, or to give a morphism of Green functors $A \to \mathcal{H}(M,M)$ ([3] 2.1.2).

Since the Green functor RB is an initial object in the category $\mathsf{Green}_R(G)$, it follows that the category of Mackey functors for G over R is equivalent to the category of RB-modules.

• The previous example shows that the category $\mathsf{Mack}_R(G)$ admits an *internal* Hom *construction*. There is also an internal tensor product in $\mathsf{Mack}_R(G)$: if M and N are Mackey functors for G over R, this construction gives another Mackey functor $M\hat{\otimes}N$ for G over R. The value of $M\hat{\otimes}N$ at the finite G-set X can be defined as

$$(M\hat{\otimes}N)(X) = \hat{M} \otimes_{\mu_R(G)} \widehat{N_X} \quad ,$$

where $\hat{M}$ is the $\mu_R(G)$-module associated to M, viewed as a right $\mu_R(G)$-module using the anti-automorphism of the Mackey algebra.

This construction is functorial in M and N, and there is an adjunction

$$\mathrm{Hom}_{\mathsf{Mack}_R(G)}(M\hat{\otimes}N, P) \cong \mathrm{Hom}_{\mathsf{Mack}_R(G)}\Big(N, \mathcal{H}(M,P)\Big) \quad .$$

• If A is a Green functor for G over R, the opposite Green functor A^{op} is the Mackey functor A, equipped with the opposite product $\times^{op}$, defined for finite G-sets X and Y by

$$\forall a \in A(X), \forall b \in A(Y),\ a \times^{op} b = A_*\begin{pmatrix} {\scriptstyle y,x} \\ \downarrow \\ {\scriptstyle x,y} \end{pmatrix} (b \times a) \quad ,$$

where $\binom{y,x}{\downarrow \atop x,y}$ is the map from $Y \times X$ to $X \times Y$ sending (y,x) to (x,y). The identity element of A^{op} is the identity element of A.

There is an obvious notion of right module over a Green functor, and the category of right modules over A is equivalent to the category of left modules over A^{op}.

If A and B are Green functors for G over R, there is also an obvious notion of (A,B)-bimodule, or A-module-B. Moreover the tensor product $A\hat{\otimes}B^{op}$ has a natural structure of Green functor, with canonical Green functor homomorphisms $A \to A\hat{\otimes}B^{op}$ and $B^{op} \to A\hat{\otimes}B^{op}$, and the category of A-modules-B is equivalent to the category of $A\hat{\otimes}B^{op}$-modules.

• Let A be a Green functor. Then A is an A-module-A. If M is an A-module, and if X is a finite G-set, there is a natural structure of A-module on the Dress construction M_X. This gives an endofunctor of the category A-Mod, which is self adjoint. Moreover, if M is an A-module, then

$$\mathrm{Hom}_{A\text{-}\mathsf{Mod}}(A_X, M) \cong M(X) \quad .$$

In particular the modules A_X are projective A-modules, and the module A_Ω is a progenerator of the category A-Mod.

4. Associated categories and algebras

4.1. The category associated to a Green functor. Let A be a Green functor for G over R. Let $\mathcal{C}_A$ denote the following category :

- The objects of $\mathcal{C}_A$ are the finite G-sets.
- If X and Y are finite G-sets, then

$$\mathrm{Hom}_{\mathcal{C}_A}(X,Y) = A(Y \times X) \quad .$$

- If X, Y, and Z are finite G-sets, then the composition of the morphisms $f \in A(Y \times X)$ and $g \in A(Z \times Y)$ in $\mathcal{C}_A$ is the element $g \circ f$ of $A(Z \times X)$ defined by

$$g \circ f = A_*\binom{z,y,x}{\downarrow \atop z,x} A^*\binom{z,y,x}{\downarrow \atop z,y,y,x} (g \times f) \quad ,$$

where $\binom{z,y,x}{\downarrow \atop z,x}$ is the map from $Z \times Y \times X$ to $Z \times X$ sending (z,y,x) to (z,x), and $\binom{z,y,x}{\downarrow \atop z,y,y,x}$ is the map from $Z \times Y \times X$ to $Z \times Y \times Y \times X$ sending (z,y,x) to (z,y,y,x).

- The identity morphism of the finite G-set X is the element

$$A_*\begin{pmatrix} x \\ \downarrow \\ x,x \end{pmatrix} A^*\begin{pmatrix} x \\ \downarrow \\ \bullet \end{pmatrix}(\varepsilon)$$

of $A(X\times X)$, where $\begin{pmatrix} x \\ \downarrow \\ x,x \end{pmatrix}$ is the diagonal inclusion from X to $X\times X$, and $\begin{pmatrix} x \\ \downarrow \\ \bullet \end{pmatrix}$ is the unique map from X to the trivial G-set $\bullet$.

Proposition 4.1.1 : [[3] 3.3.5] *The category of A-modules is equivalent to the category of R-linear functors from $\mathcal{C}_A$ to R-Mod.*

4.2. The algebra associated to a Green functor. Since A_Ω is a pro-generator of the category A-Mod, it follows that A-Mod is equivalent to the category of modules over the algebra

$$\mu(A) = \mathrm{End}_{A\text{-}\mathsf{Mod}}(A_\Omega) \quad ,$$

which is also isomorphic to the algebra $\mathrm{End}_{\mathcal{C}_A}(\Omega) = A(\Omega\times\Omega)$. The algebra $\mu(A)$ can also be defined by generators and relations using the following result :

Proposition 4.2.1 : *Let A be a Green functor for the group G. Then $\mu(A)$ is isomorphic to the R-algebra defined by the following generators and relations :*

- *The generators are:*
  - *The elements t_K^H and r_K^H, for $K\subseteq H\subseteq G$.*
  - *The elements $c_{x,H}$ for $x\in G$ and $H\subseteq G$.*
  - *The elements $\lambda_{K,a}$ for $K\subseteq G$ and $a\in A(K)$.*
- *The relations are :*
  - *The relations of the Mackey algebra for r_K^H, t_K^H, and $c_{x,H}$, i.e.*

$$t_K^H t_L^K = t_L^H, \quad r_L^K r_K^H = r_L^H \quad \forall L\subseteq K\subseteq H$$

$$c_{y,{}^xH}c_{x,H} = c_{yx,H} \quad \forall x,y,H$$

$$t_H^H = r_H^H = c_{h,H} \quad \forall h\in H$$

$$c_{x,H}t_K^H = t_{{}^xK}^{{}^xH}c_{x,K}, \quad c_{x,K}r_K^H = r_{{}^xK}^{{}^xH}c_{x,H} \quad \forall x,K,H$$

$$\sum_H t_H^H = \sum_H r_H^H = 1$$

$$r_K^H t_L^H = \sum_{x\in K\backslash H/L} t_{K\cap {}^xL}^K c_{x,K^x\cap L} r_{K^x\cap L}^L \quad \forall K\subseteq H\supseteq L$$

the other products of r_H^K, t_H^K and $c_{g,H}$ being zero.

– *The additional following relations :*

$$\lambda_{K,a}+\lambda_{K,a'} = \lambda_{K,a+a'}, \quad \lambda_{K,a}\lambda_{K,a'} = \lambda_{K,aa'} \quad \forall a,a' \in A(K),\ \forall K \subseteq G$$

$$\lambda_{K,z\varepsilon_K} = zt_K^K \quad \forall K \subseteq G, \quad z \in R$$

$$r_K^H \lambda_{H,a} = \lambda_{K,r_K^H(a)} r_K^H \quad \forall a \in A(H),\ \forall K \subseteq H \subseteq G$$

$$\lambda_{H,a} t_K^H = t_K^H \lambda_{K,r_K^H(a)} \quad \forall a \in A(H),\ \forall K \subseteq H \subseteq G$$

$$t_K^H \lambda_{K,a} r_K^H = \lambda_{H,t_K^H(a)} \quad \forall a \in A(K),\ \forall K \subseteq H \subseteq G$$

$$\lambda_{{}^xH,c_{x,H}(a)} c_{x,H} = c_{x,H} \lambda_{H,a} \quad \forall x \in G,\ \forall a \in A(H),\ \forall H \subseteq G$$

5. Simple modules and simple Green functors

5.1. Simple modules. The classification and description of simple Mackey functors can be generalized to an arbitrary Green functor, in the following form ([3] Chapter 11) :

Notation 5.1.1 : *Let A be a Green functor for G over R. If H is a subgroup of G, denote by $\overline{A}(H)$ the Brauer quotient of A in H, defined by*

$$\overline{A}(H) = A(H)/\sum_{K \subset H} t_K^H A(K) \quad ,$$

and denote by $a \mapsto \bar{a}$ the projection map from $A(H)$ to $\overline{A}(H)$. This map induces an R-algebra structure on $\overline{A}(H)$, together with a natural action of the group $\overline{N}_G(H)$. Denote by $\hat{A}(H)$ the semi-direct product $\overline{A}(H) \otimes R\overline{N}_G(H)$.

Let V be a simple $\hat{A}(H)$-module. If X is a finite G-set, then $R(X^H)$ is a $\overline{N}_G(H)$-module. Set

$$S_{H,V}(X) \simeq Tr_1^{\overline{N}_G(H)}\Big(\mathrm{Hom}_R(R(X^H), V)\Big)$$

If $f : X \to Y$ is a morphism of finite G-sets, then for $\alpha \in S_{H,V}(X)$ and $y \in Y^H$, set

$$S_{H,V*}(f)(\alpha)(y) = \sum_{\substack{x \in X^H \\ f(x)=y}} \alpha(x)$$

If $g : Y \to X$ is a morphism of finite G-sets, then for $\alpha \in S_{H,V}(X)$ and $y \in Y^H$, set

$$S_{H,V}^*(g)(\alpha)(y) = \alpha g(y)$$

Finally if $a \in A(X)$ *and* $f \in S_{H,V}(Y)$*, then define a morphism* $a \times f$ *from* $R\Big((X \times Y)^H\Big) = R(X^H \times Y^H)$ *to* V *by*

$$(a \times f)(x,y) = \Big(\overline{A^*(m_x)(a)} \otimes 1\Big).f(y)$$

where m_x *is the morphism of* G*-sets from* G/H *to* X *defined by* $m_x(uH) = ux$*.*

One can show that these definitions give an A-module structure on $S_{H,V}$. Moreover :

Proposition 5.1.2 : *Let* A *be a Green functor for the group* G*.*

1. *If* S *is a simple* A*-module, and* H *is a minimal subgroup for* S*, then* $V = S(H)$ *is a simple* $\hat{A}(H)$*-module, and* S *is isomorphic to* $S_{H,V}$*.*
2. *Conversely, if* H *is a subgroup of* G*, and* V *is a simple* $\hat{A}(H)$*-module, then* $S_{H,V}$ *is a simple* A*-module, the group* H *is minimal for* $S_{H,V}$*, and moreover* $S_{H,V}(H) \simeq V$*.*
3. *Let* H *and* K *be subgroups of* G*. If* V *is a simple* $\hat{A}(H)$*-module, and if* W *is a simple* $\hat{A}(K)$*-module, then the modules* $S_{H,V}$ *and* $S_{K,W}$ *are isomorphic if and only if the pairs* (H,V) *and* (K,W) *are conjugate under* G*.*

5.2. Simple Green functors. A simple Green functor is a Green functor A for G over R, such that A is simple as A-module-A (or $A\hat{\otimes}A^{op}$-module). This is equivalent to requiring that A has no non-trivial *functorial two-sided ideal*, in the sense of Thévenaz ([10]). The simple Green functors can be described using the following result (for details, see [3] 11.5) :

Proposition 5.2.1 : [Thévenaz [10] Theorem 12.11]

1. *Let* A *be a simple Green functor for* G*. Then there exists a subgroup* M *of* G*, a normal subgroup* H *of* M*, and a simple algebra* S *on which* M/H *acts projectively, such that*

   $$A \simeq \mathrm{Ind}_M^G \mathrm{Inf}_{M/H}^M FP_S \quad .$$

   The triple (M,H,S) *is unique up to conjugation by* G *(and up to isomorphism of* M/H*-algebras for* S*).*
2. *Conversely, if* $H \trianglelefteq M$ *are subgroups of* G*, if* S *is a simple algebra on which* M/H *acts projectively, then* $\mathrm{Ind}_M^G \mathrm{Inf}_{M/H}^M FP_S$ *is a simple Green functor.*

6. An example of related construction

6.1. Crossed G-monoids.

Definition 6.1.1 : *A G-monoid is a monoid endowed with a left G-action by monoid automorphisms. A G-monoid which is a group is called a G-group.*

A morphism of G-monoids is a G-equivariant monoid homomorphism.

Example 6.1.2 : Denote by G^c the set G on which the group G acts by conjugation. Then the multiplication map $G^c \times G^c \to G^c$ endows G^c with a structure of G-group. More generally, if N is a normal subgroup of G, then G acts on N by conjugation, and N is a G-group for this action.

Definition 6.1.3 : *A crossed G-monoid is a pair (Γ, φ), where Γ is a G-monoid, and $\varphi : \Gamma \to G^c$ is a morphism of G-monoids. A morphism $\theta : (\Gamma, \varphi) \to (\Gamma', \varphi')$ of crossed G-monoids is a morphism of G-monoids $\theta : \Gamma \to \Gamma'$ such that $\varphi' \circ \theta = \varphi$.*

6.2. Associated Green functors.

Proposition 6.2.1 : [[4],[1]]*Let (Γ, φ) be a crossed G-monoid. If A is a Green functor for G over R, let A_Γ denote the Mackey functor obtained by the Dress construction from the G-set Γ. If X and Y are finite G-set, define a product map $\times_\Gamma : A_\Gamma(X) \otimes_R A_\Gamma(Y) \to A_\Gamma(X \times Y)$ by*

$$\forall a \in A_\Gamma(X),\ \forall b \in A_\Gamma(Y),\ a \otimes b \mapsto a \times_\Gamma b = A_* \begin{pmatrix} {\scriptstyle x,\gamma,y,\gamma'} \\ \downarrow \\ {\scriptstyle x,\varphi(\gamma)y,\gamma\gamma'} \end{pmatrix} (a \times b) \quad ,$$

where $\begin{pmatrix} {\scriptstyle x,\gamma,y,\gamma'} \\ \downarrow \\ {\scriptstyle x,\varphi(\gamma)y,\gamma\gamma'} \end{pmatrix}$ is the map from $X \times \Gamma \times Y \times \Gamma$ to $X \times Y \times \Gamma$ sending (x, γ, y, γ') to $(x, \varphi(\gamma)y, \gamma\gamma')$. Let moreover ε_{A_Γ} denote the element $A_* \begin{pmatrix} \bullet \\ \downarrow \\ 1_\Gamma \end{pmatrix} (\varepsilon_A)$ of $A(\Gamma) \cong A_\Gamma(\bullet)$, where $\begin{pmatrix} \bullet \\ \downarrow \\ 1_\Gamma \end{pmatrix}$ is the map sending the unique element of $\bullet$ to the identity element of Γ.

Then A_Γ is a Green functor for G over R, and the correspondence $A \mapsto A_\Gamma$ is an endo-functor of the category $\mathsf{Green}_R(G)$.

In particular, the evaluation $A_\Gamma(\bullet) \cong A(\Gamma)$ of A_Γ at the trivial G-set is an R-algebra. One can express the product formula for this algebra in the decomposition $A(\Gamma) \cong \Big(\oplus_{\gamma \in \Gamma} A(G_\gamma) \Big)^G$:

Proposition 6.2.2 : *Let a and b be elements of $A(\Gamma)$. Then for $\gamma \in \Gamma$, the γ component of $a \times_\Gamma b$ is given by*

$$(a \times_\Gamma b)_\gamma = \sum_{\substack{(\alpha,\beta)\in G_\gamma\backslash\Gamma\times\Gamma \\ \alpha\beta=\gamma}} t_{G_{\alpha,\beta}}^{G_\gamma}(r_{G_{\alpha,\beta}}^{G_\alpha} a_\alpha . r_{G_{\alpha,\beta}}^{G_\beta} b_\beta) \quad .$$

6.3. Examples.

• Let B denote the Burnside Green functor, and let $\Gamma = G^c$. Then the ring $B(\Gamma) = B(G^c)$ is called the *crossed Burnside ring*. It is the Grothendieck ring of the monoidal category of *crossed G-sets*, i.e. G-sets over G^c. This ring has been studied by Yoshida (see also [2])

• Let A denote the cohomology Green functor $H^\oplus(-, R)$, and let $\Gamma = G^c$. Then one can show that the algebra $A(\Gamma)$ is isomorphic to the Hochschild cohomology algebra of the group algebra RG. In this case, the above product formula has been conjectured by Cibils ([5]) and Cibils and Solotar ([6]), and proved by Siegel and Witherspoon ([8]).

• (Cibils) Let A denote the Grothendieck ring of the category of finitely generated RG-modules, for relations given by direct sum decompositions. If $\Gamma = G^c$, then the ring $A(\Gamma)$ is isomorphic to the Grothendieck ring of Hopf bimodules for the Hopf algebra RG.

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