Bisets and associated functors

Recall that the letter R denotes a commutative and associative ring with unit, and G denotes a finite group.

1. Functors between categories of G-sets

In view of Dress's definition of Mackey functors, it is natural to consider the following problem : if G and H are finite groups, find all functors from H-set to G-set which preserve disjoint unions and cartesian squares. If F is such a functor and M is a Mackey functor for G over R, then the composition $M \circ F$ will be a Mackey functor for H over R.

1.1. Bisets and related products.

Definition 1.1.1 : *If G and H are groups, a (G,H)-biset, or G-set-H, is a set U with a left G action and a right H action which commute, i.e. are such that*

$$\forall g \in G, \forall u \in U, \forall h \in H,\ (gu)h = g(uh) \quad .$$

If G, H, and K are groups, if U is a G-set-H, and V is an H-set-K, the product $U \times_H V$ is the quotient of the direct product $U \times V$ by the right action of H given by

$$\forall (u,v) \in U \times V, \forall h \in H,\ (u,v)h = (uh, h^{-1}v) \quad .$$

It is a G-set-K for the action induced by

$$\forall g \in G, \forall k \in K, \forall (u,v) \in U \times V,\ g(u,v)k = (gu, vk) \quad .$$

The product $U \circ V$ is the set of pairs (u,v) in $\times V$ such that whenever $uh = u$ for some $h \in H$, there exists $k \in K$ with $hv = vk$. The set $U \circ V$ is invariant under the right action of H on $U \times V$, and the set of H-orbits on $U \circ V$ is denoted by $U \circ_H V$. It is a sub-G-set-K of $U \times_H V$.

Proposition 1.1.2 : *Let G, H, K and L be groups. Let U be a (G,H)-biset, let V be an (H,K)-biset, and let W be a (K,L)-biset. Then the canonical bijection*

$$(U \times V) \times W \to U \times (V \times W)$$

induces isomorphisms of (G, L)*-bisets*

$$(U \times_H V) \times_K W \cong U \times_H (V \times_K W)$$

$$(U \circ_H V) \circ_K W \cong U \circ_H (V \circ_K W) \quad .$$

Remark 1.1.3 : Let G and H be finite groups, and let U be a finite G-set-H. If X is a finite H-set, consider X as an H-set-$\{1\}$. Then $U \circ_H X$ is a G-set-$\{1\}$, i.e. a G-set. Note that in this case

$$U \circ_H X = \{(u, x) \in U \times X \mid \forall h \in H,\ uh = u \Rightarrow hx = x\}/H \subseteq U \times_H X \quad ,$$

If $f : X \to Y$ is a morphism of G-sets, denote by $U \circ_H f$ the map from $U \circ_H X$ to $U \circ_H Y$ defined by $(U \circ_H f)(u, x) = (u, f(x))$. This shows that any finite (G, H)-biset U gives rise to a functor $U \circ_H -$ from H-set to G-set.

1.2. Classification of functors.

Proposition 1.2.1 : [[1] Théorème 1] *Let G and H be finite groups.*

1. *If $F : H\text{-}\mathsf{set} \to G\text{-}\mathsf{set}$ is a functor which commutes with disjoint unions, and preserves cartesian squares, then there exists a finite (G, H)-biset U such that F is isomorphic to the functor $U \circ_H -$. Such a biset U is unique up to isomorphism.*
2. *Conversely, if U is a finite (G, H)-biset, then the functor $U \circ_H -$ commutes with disjoint unions and preserves cartesian squares.*
3. *If K is another finite group, and if V is a finite (H, K)-biset, then the composition of the functors $V \circ_K -$ and $U \circ_H -$ is isomorphic to the functor $(U \circ_H V) \circ_K -$.*

1.3. Examples.

• Let H be a subgroup of G, and let U be the set G, viewed as an (H, G)-biset by left and right multiplication. Then the functor $U \circ_G -$ is isomorphic to the *restriction* functor $\mathrm{Res}_H^G : G\text{-}\mathsf{set} \to H\text{-}\mathsf{set}$.

• Let H be a subgroup of G, and let U be the set G, viewed as a (G, H)-biset by left and right multiplication. Then the functor $U \circ_H -$ is isomorphic to the *induction* functor $\mathrm{Ind}_H^G : H\text{-}\mathsf{set} \to G\text{-}\mathsf{set}$.

• Let N be a normal subgroup of G, and set $H = G/N$. Let U be the set H, viewed as a (G, H)-biset, by right multiplication, and by projection $G \to H$ followed by left multiplication. Then the functor $U \circ_H -$ is isomorphic to the *inflation* functor $\mathrm{Inf}_H^G : H\text{-}\mathsf{set} \to G\text{-}\mathsf{set}$.

• Let N be a normal subgroup of G, and set $H = G/N$. Let U be the set H, viewed as a (H, G)-biset, by left multiplication, and by projection $G \to H$ followed by right multiplication. Then the functor $U \circ_G -$ is isomorphic to the *fixed points by* N functor $\mathrm{Fix}(N) : G\text{-}\mathsf{set} \to H\text{-}\mathsf{set}$.

• Let $f : G \to H$ be a group isomorphism, and let U be the set H, viewed as an (H, G)-biset by left multiplication, and by f followed by right multiplication. Then the functor $U \circ_G -$ is the *transport by* f functor $\mathrm{Iso}_G^H(f) : G\text{-}\mathsf{set} \to H\text{-}\mathsf{set}$.

• One can show that any functor $F : G\text{-}\mathsf{set} \to H\text{-}\mathsf{set}$ which commutes with disjoint unions and preserves pull-back squares is isomorphic to a disjoint union of a finite number of functors which are composed of the five previous cases, i.e. of the form

$$\mathrm{Ind}_K^G \circ \mathrm{Inf}_{K/M}^K \circ \mathrm{Iso}_{L/N}^{K/M}(f) \circ \mathrm{Fix}(N) \circ \mathrm{Res}_L^H \quad ,$$

for some subgroups $M \trianglelefteq K \subseteq G$ and $N \trianglelefteq L \subseteq H$, and some group isomorphism $f : L/N \to K/M$.

2. Composition with bisets

2.1. Functors between categories of Mackey functors.

Proposition 2.1.1 : *Let G and H be finite groups, and let U be a finite (G, H)-biset. If M is a Mackey functor for G over R, viewed as a bivariant functor $G\text{-}\mathsf{set} \to R\text{-}\mathsf{Mod}$, denote by $M \circ U-$ the bivariant functor $H\text{-}\mathsf{set} \to R\text{-}\mathsf{Mod}$ obtained by precomposition of M with the functor $U \circ_H : H\text{-}\mathsf{set} \to G\text{-}\mathsf{set}$.*

Then $M \circ U$ is a Mackey functor for H over R, and the correspondence $M \mapsto M \circ U$ is an R-linear exact functor from $\mathsf{Mack}_R(G)$ to $\mathsf{Mack}_R(H)$.

This composition of Mackey functors with bisets can be extended to Green functors :

Proposition 2.1.2 : *Let G and H be finite groups, and let U be a finite (G,H)-biset. If A is a Green functor for G over R, and if X and Y are finite H-sets, define a product $\times^U : (A\circ U)(X)\times(A\circ U)(Y)\to(A\circ U)(X\times Y)$ by*

$$\forall a\in(A\circ U)(X),\forall b\in(A\circ U)(Y),\ a\times^U b = A^*\begin{pmatrix}{\scriptstyle u,x,y}\\ \downarrow\\ {\scriptstyle u,x,u,y}\end{pmatrix}(a\times b) \quad ,$$

where $\begin{pmatrix}{\scriptstyle u,x,y}\\ \downarrow\\ {\scriptstyle u,x,u,y}\end{pmatrix}$ is the map from $U\circ_H(X\times Y)$ to $(U\circ_H X)\times(U\circ_H Y)$ sending $\big(u,(x,y)\big)$ to $\big((u,x),(u,y\big)$. Let moreover $\varepsilon_{A\circ U}$ denote the element $A^*(p_{U/H})(\varepsilon_A)$ of $A(U/H)=(A\circ U)(\bullet)$, where $p_{U/H}$ is the unique map from U/H to $\bullet$.

Then $A\circ U$ is a Green functor for H over R for the product $\times^U$ and the identity element $\varepsilon_{A\circ U}$.

Similarly, if M is an module over the Green functor A, then $M\circ U$ is a module over the Green functor $A\circ U$:

Proposition 2.1.3 : *Let G and H be finite groups, and let U be a finite (G,H)-b iset. Let A be a Green functor for G over R, and let M be an A-module. If X and Y are finite H-sets, define a product $\times^U : (A\circ U)(X)\times(M\circ U)(Y)\to(M\circ U)(X\times Y)$ by*

$$\forall a\in(A\circ U)(X),\forall m\in(M\circ U)(Y),\ a\times^U b = M^*\begin{pmatrix}{\scriptstyle u,x,y}\\ \downarrow\\ {\scriptstyle u,x,u,y}\end{pmatrix}(a\times m) \quad .$$

*Then $M\circ U$ is a module over the Green functor $A\circ U$, for the product $\times^U$, and the correspondence $M\mapsto M\circ U$ is an R-linear exact functor from A-*Mod *to $A\circ U$-*Mod.

2.2. Examples.

• Let H be a subgroup of G, and let U be the set G, viewed as an (H,G)-biset by left and right multiplication. Then the functor $M\mapsto M\circ U$ is the *induction functor* for Mackey functors. If M is a Mackey functor for H over R, this induced Mackey functor $\mathrm{Ind}_H^G M$ can be computed by

$$(\mathrm{Ind}_H^G M)(X) = M(\mathrm{Res}_H^G X)$$

for a finite G-set X.

• Let H be a subgroup of G, and let U be the set G, viewed as a (G,H)-biset by left and right multiplication. Then the functor $M\mapsto M\circ U$ is the *restriction functor* for Mackey functors. If M is a Mackey functor for G over R, this restricted Mackey functor $\mathrm{Res}_H^G M$ can be computed by

$$(\mathrm{Res}_H^G M)(X) = M(\mathrm{Ind}_H^G X)$$

for a finite H-set X.

• Let N be a normal subgroup of G, and set $H = G/N$. Let U be the set H, viewed as a (G,H)-biset, by right multiplication, and by projection $G \to H$ followed by left multiplication. Then the functor $M \mapsto M \circ U$ is the *deflation functor* for Mackey functors. If M is a Mackey functor for G over R, this deflated Mackey functor $\mathrm{Def}_H^G M$ can be computed by

$$(\mathrm{Def}_H^G M)(X) = M(\mathrm{Inf}_H^G X)$$

for a finite H-set X.

• Let N be a normal subgroup of G, and set $H = G/N$. Let U be the set H, viewed as a (H,G)-biset, by left multiplication, and by projection $G \to H$ followed by right multiplication. Then the functor $M \mapsto M \circ U$ is the *inflation functor* for Mackey functors. If M is a Mackey functor for H over R, this inflated Mackey functor $\mathrm{Inf}_H^G M$ can be computed by

$$(\mathrm{Inf}_H^G M)(X) = M(X^N)$$

for a finite G-set X.

3. Adjoint constructions.

3.1. Left and right adjoints.

Proposition 3.1.1 : [[2] Chapters 9 and 10]

1. *Let G and H be finite groups, and let U be a finite (G,H)-biset. The functor*
$$M \mapsto M \circ U : \mathsf{Mack}_R(G) \to \mathsf{Mack}_R(H)$$
admits a left adjoint $\mathcal{L}_U$ and a right adjoint $\mathcal{R}_U$.
2. *If G, H, and K are finite groups, if U is a finite (G,H)-biset and V is a finite (H,K)-biset, there are isomorphisms of functors*
$$\mathcal{L}_U \circ \mathcal{L}_V \cong \mathcal{L}_{U\circ_H V} \qquad \mathcal{R}_U \circ \mathcal{R}_V \cong \mathcal{R}_{U\circ_H V} \quad .$$

3. *If A is a Green functor for H over R, then $\mathcal{L}_U(A)$ has a natural structure of Green functor for G over R. The correspondence $A \mapsto \mathcal{L}_U(A)$ is a functor from $\mathsf{Green}_R(H)$ to $\mathsf{Green}_R(G)$.*

4. *If M is an A-module, then $\mathcal{L}_U(M)$ has a natural structure of $\mathcal{L}_U(A)$-module, and the correspondence $M \mapsto \mathcal{L}_U(M)$ is a functor from A-Mod to $\mathcal{L}_U(A)$-Mod.*

Remark 3.1.2 : The explicit description of the functors $\mathcal{L}_U(M)$ is rather complicated in the general case (see [2] Chapter 9). However, a partial description can be obtained by the following argument, which also gives a proof for the existence of the functors $\mathcal{L}_U$ and $\mathcal{R}_U$: recall that the Mackey algebra $\mu_R(G)$ can be identified with $RB(\Omega_G \times \Omega_G)$, where $\Omega_G = \bigsqcup_{K \subseteq G} G/K$. If M is a Mackey functor for G over R, then $M(\Omega_G)$ is a $\mu_R(G)$-module. Set similarly $\Omega_H = \bigsqcup_{L \subseteq H} H/L$, and identify $\mu_R(H)$ with $RB(\Omega_H \times \Omega_H)$.

Now consider

$$R\mathcal{B}_U = RB\Big(\Omega_G \times (U \circ_H \Omega_H)\Big) \quad .$$

This is a $\Big(\mu_R(G), \mu_R(H)\Big)$-bimodule, for the actions extending linearly the following products : suppose that $\Big(X, (a, b)\Big)$ is a G-set over $\Omega_G \times \Omega_G$, that $\Big(Y, (c, d)\Big)$ is an H-set over $\Omega_H \times \Omega_H$, and that $\Big(Z, (e, f)\Big)$ is a G-set over $\Omega_G \times (U \circ_H \Omega_H)$. Build the following diagram

$$\begin{tikzcd}[column sep=small]
 & & & E \arrow[dl, "k"'] \arrow[dr, "l"] & & & \\
 & & C \arrow[dl, "g"'] \arrow[dr, "h"] & & D \arrow[dl, "i"'] \arrow[dr, "j"] & & \\
 & X \arrow[dl, "a"'] \arrow[dr, "b"] & & Z \arrow[dl, "e"'] \arrow[dr, "f"] & & U \circ_H Y \arrow[dl, "U \circ_H c"'] \arrow[dr, "U \circ_H d"] & \\
\Omega_G & & \Omega_G & & U \circ_H \Omega_H & & U \circ_H \Omega_H
\end{tikzcd}$$

where all the squares are pull-back squares. Then the left and right actions on $R\mathcal{B}_U$ are defined by

$$\Big(X, (a, b)\Big).\Big(Z, (e, f)\Big).\Big(Y, (c, d)\Big) = \Big(E, (agk, (U \circ_H d)jl)\Big) \quad .$$

It is easy to this from this definition that there is an isomorphism of left $\mu_R(G)$-modules

$$R\mathcal{B}_U \cong RB_{U \circ_H \Omega_H}(\Omega_G) \quad .$$

In particular $R\mathcal{B}_U$ is projective and finitely generated as $\mu_R(G)$-module. Moreover, one can show that if M is a Mackey functor for G over R, then the natural isomorphism of R-modules

$$(M\circ U)(\Omega_H)=M(U\circ_H\Omega_H)\cong \mathrm{Hom}_{\mu_R(G)}\Big(R\mathcal{B}_U,M(\Omega_G)\Big)$$

is an isomorphism or $\mu_R(H)$-modules. Thus if N is a Mackey functor for H over R, this gives by standard arguments

$$\mathcal{L}_U(N)(\Omega_G)\cong R\mathcal{B}_U\otimes_{\mu_R(H)}N(\Omega_H)\quad .$$

But since $R\mathcal{B}_U$ is projective and finitely generated as a left $\mu_R(G)$-module, one has also that

$$(M\circ U)(\Omega_H)\cong R\mathcal{B}_U^\sharp\otimes_{\mu_R(G)}M(\Omega_G)\quad ,$$

where

$$\begin{aligned} R\mathcal{B}_U^\sharp &= \mathrm{Hom}_{\mu_R(G)}\Big(R\mathcal{B}_U,\mu_R(G)\Big)\\ &= \mathrm{Hom}_{\mu_R(G)}\Big(R\mathcal{B}_U,RB_{\Omega_G}(\Omega_G)\Big)\\ &= (RB_{\Omega_G}\circ U)(\Omega_H)\cong RB\Big((U\circ_H\Omega_H)\times\Omega_G\Big)\\ &\cong R\mathcal{B}_U\quad . \end{aligned}$$

In other words $R\mathcal{B}_U^\sharp$ can be identified to $R\mathcal{B}_U$, viewed as a $\Big(\mu_R(H),\mu_R(G)\Big)$-bimodule for the action defined by

$$\forall a\in\mu_R(H),\forall b\in\mu_R(G),\forall c\in R\mathcal{B}_U,\ a.c.b=\sigma_G(b)c\sigma_H(a)\quad ,$$

where σ_G (resp. σ_H) is the anti-automorphism of $\mu_R(G)$ (resp.$\mu_R(H)$).

Now it follows that if N is a Mackey functor for H over R,

$$\mathcal{R}_U(N)(\Omega_G)\cong\mathrm{Hom}_{\mu_R(H)}\Big(R\mathcal{B}_U^\sharp,N(\Omega_H)\Big)\quad .$$

3.2. Examples.

- Let H be a subgroup of G. Then the functors

$$\mathsf{Mack}_R(G)\ni M\mapsto \mathrm{Res}_H^G M\in\mathsf{Mack}_R(H)$$

$$\mathsf{Mack}_R(H) \ni N \mapsto \mathrm{Ind}_H^G N \in \mathsf{Mack}_R(G)$$

are mutual left and right adjoint functors ([4] Prop. 4.2).

• Let $N \trianglelefteq G$, and $H = G/N$. Then then inflation functor $\mathrm{Inf}_H^G : \mathsf{Mack}_R(H) \to \mathsf{Mack}_R(G)$ has a left adjoint $M \mapsto M^N$, and a right adjoint $M \mapsto M_N$, which can be computed as follows : if K/N is a subgroup of $H = G/N$, then

$$M^N(K/N) = M(K)/\sum_{N \not\subseteq L \subseteq K} t_L^K M(L) \qquad M_N(K/N) = \bigcap_{N \not\subseteq L \subseteq K} \mathrm{Ker}\ r_L^K \quad .$$

In the same situation, the left and right adjoints to the deflation functor Def_H^G are described in Section 9.9.3 of [2].

4. Biset functors

4.1. Bisets as morphisms. The previous sections suggest the following idea : if G and H are finite group, then a finite (G, H)-biset is a kind of generalized morphism from H to G. At least it gives a way in many situations to transport to the group G known structures associated to H. This idea can be formalized as follows :

Definition 4.1.1 : *Let $\mathcal{C}_R$ denote the following category :*

- *The objects of $\mathcal{C}_R$ are finite groups.*
- *If H and G are finite groups, then*

$$\mathrm{Hom}_{\mathcal{C}_R}(H, G) = RB(G \times H^{op})$$

  is the Burnside group of (G, H)-bisets, with coefficients in R.
- *Composition of morphisms is obtained by linearity from the product $(U, V) \mapsto U \times_H V$.*
- *The identity morphism of the group G is the biset G itself, for left and right multiplication.*

Let $\mathcal{F}_R$ denote the category of R-linear functors from $\mathcal{C}_R$ to R-Mod.

Remark 4.1.2 : It is often interesting to consider proper subcategories of $\mathcal{C}_R$ (consisting for example only of finite p-groups, for some prime p), which need not be full subcategories (it may be appropriate to consider only those bisets which are free on the right, or on the left, or both, in building the Hom-sets).

In particular if one allows only left and right free bisets as morphisms, the objects of $\mathcal{F}_R$ are called *global Mackey functors* : this is because left and right free bisets can be viewed as disjoint unions of bisets composed of restrictions, group isomorphisms, and inductions.

It is also interesting to replace the composition product $\times_H$ by the composition product $\circ_H$. This gives examples of categories with the same objects, the same morphisms, but with different composition product. Such categories may have completely different properties.

4.2. Examples.

• Let $R = \mathbb{Z}$. Let $R_{\mathbb{Q}}(G)$ denote the Grothendieck group of the category of (finite dimensional) $\mathbb{Q}G$-modules. If H is another finite group, if L is a finite dimensional $\mathbb{Q}H$-module, and if U is a finite (G,H)-biset, then $\mathbb{Q}U \otimes_{\mathbb{Q}H} L$ is a finite dimensional $\mathbb{Q}G$-module.

This construction gives a linear map $I_U : R_{\mathbb{Q}}(H) \to R_{\mathbb{Q}}(G)$. If U' is another finite (G,H)-biset, then clearly $I_{U \sqcup U'} = I_U + I_{U'}$. Moreover, if K is another finite group, and if V is a finite (H,K)-biset, it is clear that $I_U \circ I_V = I_{U \times_H V}$. Hence the correspondence $G \mapsto R_{\mathbb{Q}}(G)$ can be viewed as an object of $\mathcal{F}_{\mathbb{Z}}$.

• Let $R = \mathbb{Z}$, and let k be a field of characteristic $p > 0$. Let $R_k(G)$ be the Grothendieck group of the category of the category of (finite dimensional) kG-modules, for relations given by short exact sequences. The previous example cannot be extended to this situation, because the functor $L \mapsto kU \otimes_{kH} L$ does not preserve short exact sequences in general. It does indeed if the module kU is flat as a right kH-module, which happens for example if U is free on the right. This shows the interest of restricting the set of morphisms in the category $\mathcal{C}_R$.

• Let p be a prime number, and let $k = \mathbb{F}_p$ be the field with p elements. If G is a finite p-group, let $D_k(G)$ denote the Dade group of endo-permutation kG-modules. One can show (see [3]) that the correspondence $G \mapsto D_k(G)$ is a functor on the full subcategory of $\mathcal{C}_{\mathbb{Z}}$ consisting of finite p-groups. This shows the interest of restricting the class of objects in the category $\mathcal{C}_R$.

• Let $F(G) = K_0\Big(\mu_R(G)\Big)$ be the Grothendieck group of finitely generated $\mu_R(G)$-modules. If H is a finite group and U is a finite (G,H)-biset, then the functor $\mathcal{L}_U$ is left adjoint to an exact functor. Hence it maps projective $\mu_R(H)$-modules to projective $\mu_R(G)$-modules. This defines a linear map

$F(U) : F(H) \to F(G)$. If K is another finite group and if V is a finite (H,K)-biset, it is clear that $F(U) \circ F(V) = F(U \circ_H V)$. In other words, we get a functor on the category $\mathcal{C}_{\mathbb{Z}}$, equipped with the product $\circ_H$ instead of $\times_H$.

4.3. Simple biset functors. The category $\mathcal{F}_R$ is an abelian category. One can try to classify and describe its simple objects.

Lemma 4.3.1 : *There is a direct sum decomposition*

$$\mathrm{End}_{\mathcal{C}_R}(G) = \mathcal{A}_G \oplus \mathcal{I}_G$$

where $\mathcal{I}_G$ is a two sided ideal of $\mathrm{End}_{\mathcal{C}_R}(G)$, and $\mathcal{A}_G$ is a subalgebra isomorphic to the algebra over R of the group $\mathrm{Out}(G)$ of outer automorphisms of G.

In particular, any $R\mathrm{Out}(G)$-module can be viewed as a module for the algebra $\mathrm{End}_{\mathcal{C}_R}(G)$.

Notation 4.3.2 : *Let H be an object of $\mathcal{C}_R$, and let V be an $R\mathrm{Out}(H)$-module. If G is an object of $\mathcal{C}_R$, set*

$$L_{H,V}(G) = \mathrm{Hom}_{\mathcal{C}_R}(H,G) \otimes_{\mathrm{End}_{\mathcal{C}_R(H)}} V \quad ,$$

where the right $\mathrm{End}_{\mathcal{C}_R}(H)$-module structure on $\mathrm{Hom}_{\mathcal{C}_R}(H,G)$ is given by composition of morphisms in $\mathcal{C}_R$.

If $f : G \to G'$ is a morphism in $\mathcal{C}_R$, define a map $L_{H,V}(f) : L_{H,V}(G) \to L_{H,V}(G')$ by composition with f on the left.

Clearly $L_{H,V}$ is an R-linear functor from $\mathcal{C}_R$ to R-Mod, i.e. an object of $\mathcal{F}_R$.

Proposition 4.3.3 :

1. *Let H be an object of $\mathcal{C}_R$, and let V be a simple $R\mathrm{Out}(H)$-module. Then $L_{H,V}$ admits a unique maximal proper subfunctor $J_{H,V}$. The quotient $S_{H,V} = L_{H,V}/J_{H,V}$ is a simple object of $\mathcal{F}_R$, and $S_{H,V}(K) = 0$ for any object K of $\mathcal{C}_R$ with $|K| < |H|$.*
2. *If S is a simple object of $\mathcal{F}_R$, let H be an object of $\mathcal{C}_R$ of minimal order such that $S(H) \neq 0$. Then $S(H)$ is a simple $R\mathrm{Out}(H)$-module, and $S \cong S_{H,V}$.*
3. *If H (resp. K) is an object of $\mathcal{C}_R$, and if V (resp. W) is a simple $R\mathrm{Out}(H)$-module (resp. a simple $R\mathrm{Out}(K)$-module), then the functors $S_{H,V}$ and $S_{K,W}$ are isomorphic if and only if there is a group isomorphism $\varphi : H \to K$ such that $\varphi(V) = W$.*

Remark 4.3.4 : This proposition gives a nice classification of the simple objects in $\mathcal{F}_R$. Unfortunately, this does not give any easy way to compute the evaluations $S_{H,V}(G)$ of a simple functor at the finite group G. For example, suppose that $R = k$ is a field, that $\mathcal{C}_R$ consists of all finite groups, that all finite bisets are allowed as morphisms, and that the composition product is $\times_H$.

Let H be a finite group, and let $V = k$ be the trivial $k\mathrm{Out}(H)$-module. If G is a finite group, let $B_H(G)$ denote the k-vector space with basis the set of conjugacy classes of pairs (K, L) of subgroups of G, with $L \trianglelefteq K$ and $K/L \cong H$. Define a bilinear form on $B_H(G)$, with values in k, by

$$\langle (K, L) \mid (K', L') \rangle = |\{x \in K\backslash G/K' \mid K.{}^xL' = L.{}^xK' \quad K \cap {}^xL' = L \cap {}^xK'\}| \quad .$$

Then one can show that $S_{H,k}(G) \simeq B_H(G)/\mathrm{Rad}\langle\ |\ \rangle$. It is far from obvious however even to compute in general the k-dimension of this space.

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