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The Periodicity Theorem For The Classical Groups And Some Of Its Applications*

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In these lectures I hope to trace the development of a subject which has grown up during the past ten years or so and which is now generally known as K-theory. As my starting point I have chosen the periodicity phenomenon in the homotopy of the classical groups which it was my good fortune to discover in 1957. This starting point is partly justified because these lectures traditionally deal with some aspect of the lecturer's work, and thereafter the subject was essentially taken out of my hands: first, by Milnor and Kervaire, [27], [52] who independently used my results to settle an old question on division algebras over the reals, and then quite methodically by Atiyah–Hirzebruch [14] who, using a very general point of view which goes back on the one hand to Eilenberg–Steenrod and on the other hand to Grothendieck, transformed my naive computations into a powerful tool in algebraic topology.

Amongst the achievements of this topological K-theory we may now also count (1.) the solution of the vector-field problem on spheres, due to Adams [2]; (2.) an amazingly short solution of the Hopf conjecture, to the effect that amongst the spheres the 1, 3, and 7 spheres are the only groups in the sense of homotopy theory, due to Adams and Atiyah [6]; and finally the index theorem of Atiyah–Singer which not only generalizes the whole Riemann–Roch question to elliptic differential equations but also gives a completely new slant on the whole periodicity phenomenon [17], [18], and [19].

However, the main justification for starting with my essentially naive computations of twelve years ago, is that by tracing the subject from this point we will have to touch on many of the central concepts of modern algebraic topology in their historic order.

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1. The naive periodicity theorem

The homotopy group $\pi_k(X, x)$ of a space X at the point x has as its underlying set the homotopy classes $[S^n, X]_*$, of maps of an n-sphere with base point, $(S^n, *)$, into $(X, *)$.[1] Alternatively we may think of this set as the homotopy classes of maps of the pair $(I^n, \dot{I}^n)$ to $(X, *)$ where I^n is the unit cube in $\mathbf{R}^n$ and $\dot{I}^n$ is its boundary, and in this version the group structure is described by joining two cubes along standard faces and then reidentifying the resulting parallelepiped with the cube. This model for $\pi_k(X)$ (we will suppress base points whenever possible) also has the virtue that it leads directly to yet a third definition of $\pi_k(X)$. Namely, let ΩX denote the space of maps: $(I, \dot{I}) \to (X, *)$; $I = [0,1]$; in the compact open topology. By singling out one coordinate in I^n, one then obtains an obvious correspondence between $\pi_n(X)$ and $\pi_{n-1}(\Omega X)$, $n \geqslant 2$, which is easily seen to be an isomorphism. Thus one can—as was already pointed out by Hurewitz when he first defined these groups—define $\pi_k(X)$ purely in terms of the fundamental group π_1, by the formula

$$\pi_k(X) = \pi_1(\Omega \cdot \Omega \cdots \Omega X), \qquad k \geqslant 1; \quad \Omega \text{ taken } k-1 \text{ times}. \tag{1.1}$$

For $k = 0$, $\pi_k(X, *)$ denotes the set of arc-components of X with the component of $*$ singled out.[2]

In short then, homotopy groups are very easy to define, and quite obviously give rise to a covariant functor from topological spaces to groups: Thus every map $f : (X, *) \to (Y, *)$ induces a homomorphism $f_* : \pi_k(X) \to \pi_k(Y)$, which is the identity when f is the identity, and for the composition $g \circ f$ we have

$$(g \circ f)_* = g_* \circ f_* . \tag{1.2}$$

[1] Two maps $f, g : X \to Y$ are called homotopic if there is a 1-parameter family of maps F_t, $t \in [0, 1]$ such that $F_0 = f$ and $F_1 = g$. The set of homotpy classes of maps from X to Y is then often denoted by $[X, Y]$ and if X and Y are equipped with base points then $[X, Y]_*$ denotes the homotopy classes of basepoint preserving maps. Recall also that two spaces X, Y are of the same homotopy type if there exist maps $f : X \to Y$, $g : Y \to X$ so that both $f \circ g$ and $g \circ f$ are homotopic to the identity.

[2] ΩX is naturally a pointed space, the point path $\mu : I \to *$ playing the role of base point. By going one step further one can equally well identify $\pi_k(X)$ with the set of arc components: $\pi_0(\Omega^n X)$.

The simplest first properties of these functors are the following three:

$$\pi_k(X \times Y) = \pi_k(X) \times \pi_k(Y) \tag{1.3}$$

$$\pi_k(X) \text{ is abelian for } k > 1 \tag{1.4}$$

$$\pi_k(S^n) = 0 \qquad \text{for } k < n. \tag{1.5}$$

The first of these is very easy, the second uses the fact that really only one coordinate is used in the group law of $\pi_k(X)$ and then exploits the freedom in at least one other coordinate whenever $k \geqslant 2$. The third is based on the fact that any map $S^k \to S^n$ can be smoothed in its homotopy class, and smooth maps of $S^k \to S^n$, whether we interpret smooth as polyhedral or as differentiable, cannot be onto for $k < n$. Thus any $f : S^k \to S^n$ with $k < n$ will be homotopic to a map which avoids one point P of S^n and so can be shrunk to the antipode of P along the great circles through P.

A very important, but really no harder to prove, extension of (1.5) is now the following one: Suppose Y is a space and $\alpha : \dot{I}^n \to Y$ is a map. We then form a space

$$X = Y \underset{\alpha}{\cup} I^n \tag{1.6}$$

called "Y with an n-cell attached" by first taking the disjoint union

$$\tilde{X} = Y \cup I^n \tag{1.7}$$

and there identifying the points $p \in \dot{I}^n$ with the points $\alpha(p) \in Y$. The topology on X is the usual identification topology. It turns out that most of the reasonable spaces one meets in geometry can—up to homotopy—be thought of as being formed by successive adjunctions of such cells: For instance,

$$S^n = p \underset{\alpha}{\cup} I^n, \qquad \text{with} \quad \alpha : \dot{I}_n \to p \tag{1.8}$$

the unique map.

$$S^n \times S^m = p \underset{\alpha}{\cup} I^n \underset{\beta}{\cup} I^m \underset{\gamma}{\cup} I^{n+m}, \tag{1.9}$$

and more generally every polyhedron—i.e., a subset of $\mathbf{R}^n$ which the finite disjoint union of affine simplexes, which with a simplex contains all its faces—is in a very definite and obvious manner obtained by successively attaching the various simplexes.

Now the extension of (1.5) alluded to earlier is simply this:

PROPOSITION (1.10). *The inclusion $Y \to Y \cup_\alpha I^n$ induces isomorphisms in π_k for $k \leqslant n-2$ and is onto for $k = n-1$.*

There is also a fundamental extension of the property (1.3) to "twisted products" of various types. The twisted products which occur most often in geometry are of the following sort. We are given a space E together with a map $\pi : E \to X$ of E onto another space X, such that $\pi^{-1}(x)$ is homeomorphic to a fixed space F in the following uniform sense: There exists a covering $\{U_\alpha\}$ of X and homeomorphisms

$$\varphi_\alpha : U_\alpha \times F \to \pi^{-1}(U_\alpha)$$

such that the diagrams:

$$\begin{array}{ccc} U_\alpha \times F & \xrightarrow{\varphi_\alpha} & \pi^{-1}(U_\alpha) \\ \downarrow & & \downarrow \pi \\ U_\alpha & \xrightarrow{1} & U_\alpha \end{array},$$

where the vertical map on the left is the projection on the first factor, commute. In such a situation we refer to E as a twisted product of X and F. Examples of these abound; for instance if G is a Lie group and $H \subset G$ is a closed subgroup then the coset projection $G \to G/H$ gives rise to a twisted product structure on G, that is, G becomes the twisted product of G/H with H. More generally if $K \subset H$ are closed subgroups of G, then the natural map $G/K \xrightarrow{\pi} G/H$ defines G/K as a twisted product of G/H with H/K.

Now then, from the homotopy point of view the fundamental property for a twisted product projection $\pi : E \to X$ is the following covering homotopy theorem which goes back to Whitney, Eckmann, Steenrod [68]:

PROPOSITION (1.11). *(Covering homotopy property.) Given $\pi : E \to X$ as above, and a map $f : P \to E$, together with a homotopy $\bar{f}_t$ of the map $\bar{f} = \pi \circ f : P \to X$. Then there exists a homotopy f_t of f such that*

$$\pi \circ f_t = \bar{f}_t .$$

Note that for a genuine product this proposition is trivial—one simply leaves the fiber coordinate alone. In the general case, however, one has to proceed with care and ingenuity to make compatible choices in overlapping product structures.

A rather immediate consequence of this covering homotopy property, is now the extension of (1.3) which is called the *exact homotopy sequence of a twisted product*:

PROPOSITION (1.12). *Let $\pi : E \to X$ have the covering homotopy property, let $*$ be a base point in E, and let $F = \pi^{-1}(\pi *)$ be the fiber of π through $*$. Then there is an exact sequence of homotopy groups*:

$$\xrightarrow{\partial} \pi_k(F; *) \xrightarrow{i*} \pi_k(E, *) \xrightarrow{\pi*} \pi_k(X, \pi*) \xrightarrow{\partial} \pi_{k-1}(F, *) \longrightarrow$$

where $i*$ and $\pi*$ are induced by the natural maps $i : F \to E$ and π, while the definition of ∂ uses the covering homotopy property of π.

To obtain a clue as to the construction of ∂ consider the case ∂: $\pi_1(X) \to \pi_0(F)$. If $\alpha \in \pi_1(X)$ is represented by $\mu : (I, \dot{I}) \to (X, \pi*)$ we may think of μ as a homotopy of the map of a point into $\pi*$, and $* \in E$ as a lifting of this map. Hence by the covering homotopy property there exists a cover $\tilde{\mu} : [0, 1] \to E$ of this homotopy. Thus the value of $\tilde{\mu}$ at $1 \in [0, 1]$ will be in the fiber F through $*$. Now then, ∂ simply assigns to α the component of $\tilde{\mu}(1)$ in $\pi_0(F)$.

Several other comments are in order here. First of all, recall that a sequence of groups and homomorphisms is called exact if the image of every incoming homomorphism is the kernel of the outgoing homomorphism. For pointed sets, e.g., when $k = 0$ in (1.12), the definition is extended in the obvious way: the image set must map into the singled out element under the outgoing map. Secondly, a comment on terminology: Maps $E \to X$ with the covering homotopy properly play such a fundamental role in the body of algebraic topology that they are called *fiber maps* or "fiberings in the sense of Serre" or, simply, fiberings. Fiberings turn out to be the appropriate generalization of twisted products in homotopy theory, and this insight due to Serre, has revolutionized the subject in the last nineteen years. I will have more to say on this score later on.

With these fundamental first results of homotopy theory reviewed we may move on to the periodicity theorem for the classical groups. The classical groups are associated to the three basic complete topological fields over the real numbers **R**, which we denote by **R**, **C**, and **H** for the quaternions of Hamilton. If F is any one of these, $GL(n, F)$ denotes the group of automorphisms of F^n considered as a right module over F, and is referred to as the full linear group of dimension n. All of these topological groups are easily seen to be locally compact Lie groups and topologically they are homeomorphic to a product of their maximal compact subgroups (denoted by O_n, U_n, and Sp_n) and a Euclidean space. From the point of view of homotopy we may therefore consider these compact groups rather than the linear groups themselves.

A first consequence of Propositions (1.10), (1.12) is then the following stability theorem:

PROPOSITION (1.13). *The homotopy group: $\pi_k\{GL(n, F)\}$ is independent of n if n is large enough. More precisely, the inclusion $GL(n, F) \to GL(n+1, F)$ induces isomorphisms in π_k for $k \ll n$.*

To prove this theorem in the real case, say, one only has to observe that the homogeneous space O_{n+1}/O_n is homeomorphic to an n-sphere S^n. Hence, using the exact sequence (1.12) for this twisted product, one obtains the exact sequence:

$$\pi_{k+1}(S^n) \to \pi_k(O_n) \to \pi_k(O_{n+1}) \to \pi_k(S^n).$$

By (1.5) the outside terms disappear for $k + 1 < n$ so that the result follows.

We will denote the stable values of these groups by $\pi_k(O)$, $\pi_k(U)$ and $\pi_k(SP)$ respectively, and with this understood the periodicity theorem states the following:

PERIODICITY THEOREM: *The homotopy groups of O, U, and SP satisfy the recursion relation*:

$$\left.\begin{aligned} \pi_k(U) &= \pi_{k+2}(U) \\ \pi_k(O) &= \pi_{k+4}(SP) \\ \pi_k(SP) &= \pi_{k+4}(O). \end{aligned}\right\} \tag{1.14}$$

Thus $\pi_*(U)$ has period 2, while $\pi_*(O)$ and $\pi_*(SP)$ have period 8. The first few homotopy groups of these spaces can be computed without much trouble and were in any case known in 1957. It then follows from these computations that the period of π_* is given by:

$$\left.\begin{array}{rl} 0, \mathbf{Z} & \text{for } \pi_*(U) \\ 0, 0, 0, \mathbf{Z}\ \mathbf{Z}_2\ \mathbf{Z}_2\ 0\ \mathbf{Z} & \text{for } \pi_*(SP) \\ \mathbf{Z}_2\ \mathbf{Z}_2\ 0\ \mathbf{Z}\ 0\ 0\ 0\ \mathbf{Z} & \text{for } \pi_*(O). \end{array}\right\} \quad \text{and} \tag{1.15}$$

I think it is fair to say that the regularity of these homotopy groups came as a great surprise to all of us. The reason for this was that by 1957 we had already accepted the fact that the homotopy groups of spaces which one encounters in geometry are very difficult to compute, and quite irregular in structure. By some divine justice the π_k's of geometric finite objects such as polyhedra or manifolds seem to be as difficult to compute as they are easy to define. To gain some insight into this phenomenon let me digress here for a moment, in order to discuss the general problem of computing the homotopy groups of a polyhedron.

In the last twenty years tremendous advances have been made on this question, for instance it is known that if the fundamental group of a polyhedron is abelian, then all the homotopy groups of P are computable in the technical sense of recursive functions from the combinatorial information describing P. On the other hand this existence theorem—which was proved in its full generality by E. Brown, in 1957 [30] and depends on all the basic advances made by Serre, Cartan, Postnikov, Moore, and many others, is much too elaborate to be followed in practice. One can, however, approximate this procedure in important special cases. The main principles involved are then roughly the following ones: First of all one has the following basic theorem going back to Hurewicz:

THEOREM 1.16. *If $\pi_0, \ldots, \pi_{k-1}$ of a space Y all vanish; $k \geqslant 2$; then $\pi_k(Y) \simeq H_k(Y)$, where H_k denotes the k-th singular homotopy group of Y with integer coefficients.*

Concerning homology let me say here only that it is a covariant, homotopy invariant functor from spaces to abelian groups, which is harder to define than homotopy, but explicitly computable on polyhedra. In particular $H_k(P) = 0$ when $k > \dim P$.

Our next theorem deals with the simplest nontrivial spaces from the point of view of homotopy [38]:

THEOREM 1.17. (*Eilenberg—MacLane*). *Suppose Y is a space with vanishing homotopy save in dimension n, and that*:

$$\pi_n(Y) = \pi.$$

Then the homotopy type of Y, and in particular the homology of Y, is completely determined by the pair (π, n).

Conversely, given (π, n) *where* π *is an abelian group and* n *an integer* $\geqslant 1$, *there exists a space* $K(\pi, n)$*—which can in fact be constructed by the successive attaching of cells—which has the property that*

$$\pi_j\{K(\pi, n)\} = \begin{cases} 0 & j \neq n \\ \pi & j = n. \end{cases}$$

Note. Properly speaking the uniqueness part of this theorem is only true when Y is itself a space obtained by the successive attaching of cells—a so-called CW-complex. Alternatively one can define two spaces Y and Z to be of the same homotopy type if the functors $P \to [P, Y]$, and $P \to [P, Z]$ which they define on the category of polyhedra, are naturally isomorphic.

This theorem immediately raises the question of determining the homology of $K(\pi, n)$ purely in terms of (π, n) and this problem, which was initiated and partially solved by Eilenberg and MacLane [38], [42], was brought to a full solution by Serre, Moore and Cartan, using the methods initiated by Leray and Serre, Borel, Cartan, and the constructions of Steenrod [33]. The answer is elaborate and can not really be surveyed here. However, even a cursory insight shows that the $K(\pi, n)$ are only in very special instances finite polyhedra! In particular, the homology of $K(\mathbf{Z}, n)$, $n > 2$ has torsion of every type in arbitrarily high dimensions! To summarize, the $K(\pi, n)$—which should be thought of as the atomic spaces of homotopy theory—have a very complicated homology structure and are usually not encountered directly in geometry.

In view of (1.3) and (1.17) one can build spaces with arbitrarily assigned abelian homotopy groups $\{\pi_k\}$ simply by taking the product, $\prod_k K(\pi_k, k)$, of the corresponding Eilenberg–MacLane spaces. Unfortunately, however, not every space is of this homotopy type. On the other hand one does have a structure theorem of this sort if one allows the products to be "twisted". More precisely this is expressed by the following [62], [71]:

THEOREM 1.18. (*Postnikov, Moore, Whitehead*). *Let* Y *be an arc-connected space with homotopy groups* $\{\pi_k\}$, $k = 1, \ldots$. *Then there exists a sequence of fiberings*

$$Y_0 \xleftarrow{\pi_0} Y_1 \xleftarrow{\pi_1} Y_2 \longleftarrow$$

such that the fiber of π_k *is a* $K(\pi_k, k)$, *and such that* Y_n *is an* $(n - 1)$ *approximation to* Y *in the sense that the functors* $P \to [P, Y_n]$ *and* $P \to [P, Y]$ *are naturally isomorphic for all* P *of* dim $< n - 1$.

In the weak form in which I have just stated the Postnikov factorization, this conceptually fundamental fact is quite easy to prove. The construction is as follows: Let Y_n be the space obtained from Y by successively adding cells of dim $\geqslant n + 2$ and such that $\pi_k(Y_n) = 0$ for $k \geqslant n + 1$. Using (1.10) it is not hard to see that such a construction is possible.

By construction and the fact that $\pi_k(S^k) = \mathbf{Z}$, as follows from the Hurewitz theorem, the inclusion $Y \to Y_n$ then induces isomorphism in π_k for $k \leqslant n$. Having constructed Y_n, let Y_{n-1} be obtained by similarily "killing" the homotopy of Y_n in dimension n and above. Thus the inclusion $Y_n \xrightarrow{i} Y_{n-1}$ induces isomorphisms in π_k for $k \leqslant n - 1$.

Now there is a trick for converting any inclusion $A \to X$ (in fact any map) into a fibering! Simply define $L(A, X)$ as the space of paths

$$\mu : [0, 1] \to X$$

with $\mu(0) \in A$, and let

$$\sigma : L(A, X) \to X \tag{1.19}$$

be defined by

$$\sigma\mu = \mu(1).$$

The map σ is then quite trivially seen to have the covering homotopy property. On the other hand the inclusion $A \subset L(A, X)$ sending $a \in A$ into the "point path" a, is—also quite trivially—a homotopy equivalence. (One simply shrinks μ to its initial value along itself.)

Applying this construction to the inclusion

$$Y_n \to Y_{n-1},$$

we obtain the fibering

$$\tilde{Y}_n \xrightarrow{\sigma} Y_{n-1}, \qquad Y_n = L(Y_n, Y_{n-1})$$

with $\tilde{Y}_n$ of the homotopy type of Y_n. Now if we apply the homotopy sequence (1.12) it becomes apparent that the homotopy groups of the fiber F in σ are zero save in the one dimension n and then $\pi_n(F) = \pi_n(Y)$. Thus the fiber is a $K(\pi_n, n)$. Q.E.D.

The point we have reached now is that, as far as homotopy is concerned, every space is obtained from the basic Eilenberg–MacLane spaces by successive fiberings. Actually this philosophy, though reassuring, would be of little computational help were it not for the so-called "Spectral sequence of a fibering." In the 1940's Leray discovered that if E is a twisted product of X and a fiber F, then a definite but very complicated algebraic relation existed between the homology of E, X, and F. In a sense his theorem may be thought of as a fundamental extension of the Künneth theorem describing $H(X \times F)$ in terms of $H(X)$ and $H(F)$. For a general twisted product one of course can not expect an explicit description of $H(E)$ in terms of $H(X)$ and $H(F)$ because the twisting, which is hard to measure, must enter into the equation. However, Leray found that the effect of the twisting nevertheless obeys certain quite general laws [54]. In particular, these laws, which we refer to as the "Spectral sequence of the twisted product", are in certain cases sufficient to enable one to compute one of the variables $H(E)$, $H(X)$, and $H(F)$ if the other two are known. In 1951 Serre [65] established the fact that these same laws also hold for his more general fiberings (as defined above earlier) and thereby made this powerful tool applicable to homotopy theory. The results were spectacular. For instance: observe that the loop space ΩX may be thought of as the fiber of a fibering $E \xrightarrow{\sigma} X$ with E a contractible space. Indeed, simply set E equal to the space of maps

$$\mu : [0, 1] \to X$$

with $\mu(0) = * \in X$, and define σ by

$$\sigma(\mu) = \mu(1).$$

This is trivially a fibering, indeed it is our earlier construction applied to the inclusion of a point in X, so that E is of the homotopy type of $*$. We may therefore apply the spectral sequence with $H_k(E) = 0$, $k > 0$. When X is a sphere S^n, it then turns out that these data and the well-known fact that

$$H_k(S^n) = \begin{cases} \mathbf{Z} & k = 0, n \\ 0 & \text{otherwise} \end{cases}$$

lead to a *unique solution* for $H(\Omega S^n)$, namely:

$$H_k(\Omega S^n) = \begin{cases} \mathbf{Z} & k \equiv 0(n-1) \\ 0 & \text{otherwise.} \end{cases} \tag{1.20}$$

Computing further in the same vein Serre obtained some hold on the homology of the higher loopspaces of S^n and so, by the Hurewitz theorem, on the higher homotopy of S^n. For instance, he was able to show that the groups $\pi_k(S_n)$ were all finite for $k \geqslant 2n$. This same—really quite simple-minded fibering—also plays a crucial role in the computation of the homology of the $K(\pi, n)$'s. Simply observe that if X is a $K(\pi, n)$, then ΩX is a $K(\pi, n-1)$! Now $H(K(\pi, 1))$ is relatively easy to compute—and is in fact what we usually call the Eilenberg–MacLane homology of the group π—so that the spectral sequence yields an inductive procedure for passing from $H(K(\pi, 1))$ to $H(K(\pi, n))$.

Equipped with the spectral sequence the Postnikov factorization now becomes a powerful tool for computation of homotopy. The computation proceeds roughly like a crossword puzzle. Starting from a thorough understanding of the homology of a space X one tries to fit $K(\pi, n)$'s together so as to produce this homology. For instance, consider S^n, $n \geqslant 1$. From the fact that $H_n(S^n) = \mathbf{Z}$ we conclude that S^n "starts" with a $K(\mathbf{Z}, n)$. Now if S^n actually were a $K(\mathbf{Z}, n)$ the homology of $K(\mathbf{Z}, n)$ would have to vanish in dim $> n$. This happens for $n = 1$ and S^1 is actually a $K(\mathbf{Z}, 1)$. However, as soon as $n > 2$ torsion appears in $H\{K(\mathbf{Z}, n)\}$. Hence, looking at the first homology occuring in $H\{K(\mathbf{Z}, n)\}$ above dimension n one can determine the next constituent in the Postnikov system, etc. Of course, this procedure progressively leads to more and more ambiguities, and one can proceed further usually only by some new clue in the crossword puzzle. A clue which, by and large, must be found in some ad hoc manner! To refine this procedure to a foolproof computation one eventually has to descend from the purely homology level to a chain level. There, unfortunately, the computations become unmanageable—even for machines.

So much for this short excursion into the art of computation in homotopy. Using roughly the procedure described above, the stable homotopy groups of the classical groups had been computed in 1957 up to dim 10 for U, and up to dim 7 or so for O. As the homology of U_n is very regular and in particular has no torsion, there was really no expectation that U was made up entirely of $K(\mathbf{Z}, n)$'s. In fact, the early computations of the homotopy theorists yielded $\pi_{10}(U) = \mathbf{Z}_3$. This result was then called in question by computation of Borel and Hirzebruch who, inspired by quite different phenomena, predicted the value zero for this group, and it was this controversy which initially aroused my curiosity.

The original proof of Theorem 1, is based on the Morse theory and avoids $K(\pi, n)$'s or spectral sequences entirely [26]. Instead, one uses the

Morse theory to construct a CW-model for a component, Ω_*, of the loopspace on U_{2n}, which takes the following form:

$$\Omega_* = U_{2n}/U_n \times U_n \cup e_1 \cup e_2 \cup \cdots \tag{1.21}$$

with the dimension of the cells $\{e_i\}$ *to be attached, bounded from below by* $2n + 1$.

Now such a model for Ω_* implies the periodicity of $\pi_*(U)$ from principles we have already encountered: First of all $\pi_k(\Omega_*) = \pi_{k+1}\{U_{2n}\}$, $k > 0$, per definitionem. Secondly, by (1.10), this model implies that

$$\pi_k(\Omega_*) = \pi_k(U_{2n}/U_n \times U_n) \qquad k \leqslant 2n. \tag{1.22}$$

Finally applying the exact homotopy sequence to the fibering

$$U_{2n}/U_n \to U_{2n}/U_n \times U_n \tag{1.23}$$

with fiber $(U_n \times U_n)/U_n = U_n$ and keeping the stability theorem in mind one finds that

$$\pi_k(U_n) = \pi_{k+1}(U_{2n}/U_n \times U_n), \qquad k \leqslant n. \tag{1.24}$$

Combining all these equations, therefore yields the desired formula

$$\pi_{k-1}(U) = \pi_{k+1}(U) \qquad k > 0.$$

Let me now indicate how the Morse theory leads to the desired model for Ω_*.

Recall first of all the main assertion of the elementary Morse theory. Here one is dealing with a (finite dimensional) C^∞-manifold M, on which a C^∞ function φ is defined, and asks the question of how the "half-spaces of φ":

$$M^a = \{p \in M \mid \varphi(p) \leqslant a\},$$

evolve from one another. The nondegenerate Morse theory answers this question under the following assumptions on φ:

$$\varphi \textit{ is proper}, \text{ (i.e., } \varphi^{-1} \textit{ (compact) is compact}). \tag{1.25}$$

The critical points $\{p\}$ *of* φ *are nondegenerate in the sense that for every* $p \in \{p\}$ *the Hessian of* f *at* p *is nondegenerate.*[3] (1.26)

[3] Recall that p is critical for φ if and only if all partials of φ relative to local coordinates near p vanish at p; also that the Hessian $H_p\varphi$ is a quadratic form on the tangent-space to M at a critical point p, whose value of $X \in T_pM$ is the second derivative of φ in the direction X.

Under these assumptions the Morse theory asserts that:

THEOREM (1.27) (Morse). Part.1. *If φ has no critical point in the set $\varphi^{-1}[a, b]$, then M^a is diffeomorphic to M^b:*

$$M^a = M^b. \tag{1.28}$$

Part 2. *If there are no critical points of φ on $\varphi^{-1}(a)$ and $\varphi^{-1}(b)$ then up to homotopy M^b is obtained from M^a by attaching cells $\{e_p\}$ to M^a*

$$M^b = M^a \cup e_{p_1} \cup \cdots \cup e_{p_k} \tag{1.29}$$

where the $\{p_i\}$ are the critical points of φ in $\varphi^{-1}[a, b]$, and the dimension of e_{p_i} is equal to the index of inertia[4] *of the Hessian $H_{p_i}\varphi$:*

$$\dim e_p = \operatorname{index} H_{p_i}\varphi. \tag{1.30}$$

This theorem, though enunciated here in a more modern form, follows directly from Morse's argument of the 1920's, which carefully analyzes the deformations of M induced by the trajectories of a gradient-field of φ relative to a Riemannian structure on M [59].

We here only summarized the homotopy consequences of this analysis. A more refined conclusion is possible and leads one to the "handlebody theory" of Smale [66] which was the main tool in his fundamental work on the Poincaré conjecture in higher dimensions and his general study of the diffeomorphism types of manifolds.

Our concern here is not with this refinement, but rather with Morse's own extension of this theory to the loopspace on a manifold. The underlying philosophy of this development is that a properly posed variational problem—such as minimizing the length of path on a complete Riemannian manifold for instance—should define a function φ on a properly defined loopspace $\hat{\Omega}M$ to M, in such a manner that:

The space $\hat{\Omega} = \Omega M$ has the same homotopy type as the usual loopspace ΩM. (1.31)

The critical points of φ should correspond to the extremals of the variational problem. (1.32)

Theorem (1.27) *should hold.* (1.33)

[4] This index counts the number of negative squares occurring in a diagonal presentation of the quadratic form.

Thus the halfspaces $\hat{\Omega}^a = \{p \in \hat{\Omega} \mid \varphi(p) < a\}$ are to be of the same homotopy type for $a \in [a_0, a_1]$ if no critical point of φ exists in $\varphi^{-1}[a_0, a_1]$, and under a suitable nondegeneracy hypothesis on the extremals of the problem one should have a homotopy equivalence:

$$\hat{\Omega}^b \sim \hat{\Omega}^a \cup e_1 \cup \cdots e_k \tag{1.34}$$

whenever φ has no critical points on $\varphi^{-1}(a)$ and $\varphi^{-1}(b)$. Finally these cells should be in one-to-one correspondence with the critical points of φ on $\varphi^{-1}[a, b]$ and *their dimension should be computable from local data near the critical points.*

This program was carried out by Morse and a slight technical variant of his method was later developed by Seifert and Threllfall [69]. Both these approaches use polygonal approximations to the points of $\hat{\Omega}M$, and after a series of deformations, reduce the theorem to the finite dimensional case. More recently the theory of C^∞-locally Hilbertian manifolds has been greatly developed by Eells–Sampson, Palais, Smale, and others (see [37] for instance) and the above program can then be carried out directly in this framework. In my own work [26] I used a variant of the Seifert–Threllfall approach to establish a homotopy equivalence of the halfspaces $\hat{\Omega}^a M$ with the halfspace of the space $M_n = M \times \cdots \times M$, ($n$ copies) n large, on which the function

$$\varphi(x_1, \ldots, x_n) = \sum d^2(x_i, x_{i+1}) \qquad (d = \text{distance on } M)$$

is less than $b = (a^2/n)$. *Furthermore, under this homotopy equivalence the geodesics*—(I only treated the Riemannian case)—*of $\hat{\Omega}M$ went over precisely into the points $(x_1, \ldots, x_n)$ of M_n which cosrresponded to equally spaced points on a geodesic segment joining the base point of $\hat{\Omega}M$ to itself.*

This construction was carried out in the Seifert–Threllfall model for $\hat{\Omega}M$. That is, *the points of $\hat{\Omega}M$ are piecewise smooth maps from* $[0, 1]$ *to M, subject to the boundary conditions*: $\mu(0) = \mu(1) = P$, or *more generally*, boundary conditions of the form: $\nu = (P, Q, h)$ which are defined by:

$$\mu(0) = P, \qquad \mu(1) = Q, \qquad \mu \textit{ in a given homotopy class } h \textit{ of}$$
$$\textit{maps subject to the first conditions.}$$

This set is then made into a metric space $\Omega_\nu = \hat{\Omega}_\nu$, via the metric:

$$\sigma(\mu, v) = \max_t d(u(t), v(t)) + |L(u) - L(v)|.$$

Here d denotes the distance in the Riemannian metric on M, and $L(u)$ the length of the curve u in this metric.

The homotopy equivalence of $\Omega_v{}^a M$ with the φ-halfspace $M_n{}^b$ now enables one to carry out the Morse program directly by applying Theorem (1.10) to φ on $M_n{}^b$.

In the Riemannian case the index of a nondegenerate critical point of φ has a beautiful geometric interpretation, also essentially due to Morse. The assertion is the following one:

THEOREM (1.34) (Morse). *If on the geodesic segment $s \in \Omega_v M$ the endpoints are not conjugate, then the corresponding critical point of φ is nondegenerate; furthermore, the index of such a critical point is then the number (counted with multiplicity) of conjugate points of one endpoint in the interior of s.*

The implications of this extended Morse theory are maybe best illustrated by a concrete example. Consider S^n, $n > 2$ in its usual Riemannian structure and let (P, Q) be a pair of points on S^n which are not antipodal. The set $\mathfrak{C}(P, Q)$ of geodesics joining P to Q is then easily surveyed: Ordered by length they are in one-to-one correspondence with the integers, say s_0, s_1,...,etc. The endpoints are never conjugate, furthermore, the index of s_k is seen to be $(n - 1)$ {the number of times s_k contains the antipode of P in its interior}. Thus the index of s_k is precisely $k(n - 1)$.

Applying the Morse theory one obtains a model for ΩS^n which is of the form

$$\Omega S^n \simeq e_0 \cup e_{n-1} \cup e_{2(n-1)} \cup \cdots \tag{1.35}$$

where we have indexed the calls by their dimensions.

Using (1.35) and elementary properties of the homology functor it follows that the homology of ΩS^n is given by: $H_q(\Omega S^n) = \mathbf{Z}$ when $q \equiv 0$, mod$(n - 1)$ and vanishes otherwise. Thus one confirms the answer mentioned earlier, as computed by the spectral sequence, and Morse had essentially arrived at this result thirty years ago by this technique. Having computed $H_*(\Omega S^n)$ one may now turn the theory around, so to speak, and prove that *for every Riemannian metric on S^n and every pair of points (P, Q) on S^n there are an infinite number of geodesics joining P to Q.*

If all geodesics from P to Q are nondegenerate this is clear as otherwise ΩS^n would be constructable out of a finite number of cells which contradicts the existence of homology in arbitrary high dimensions. When certain geodesics joining P to Q are degenerate—i.e., P is conjugate along Q on them, one can still push this argument through, as even a degenerate critical point can only contribute a finite number of cells to the space.

As my next illustration, consider S^n, $n > 2$ again in its natural metric, but now let $\Omega_v S^n$ be based on an antipodal pair of points $(P, \bar{P})$. Now the set of geodesics $\mathfrak{C}(P, \bar{P})$ consists of a countable sequence of $(n-1)$-spheres. Let $S^{n-1} \subset \Omega_v S^n$ be the sphere consisting of geodesics of *minimal* length π, so that the other spheres consist of geodesics of length 3π, 5π, etc., and consider the half-space $\Omega_v{}^a M$ with $\pi < a < 3\pi$.

Now it is not difficult to show that the $\Omega_v{}^a M$ is of the same homotopy type as $S_\pi^{n-1} \subset \Omega_v{}^a M$. Further, note that on the geodesics of the sphere $S^{n-1}_{(2k+1)\pi}$, the number of conjugate points of P in the interior of the geodesic segment is precisely $(n-1)k$. Using this fact one can, by means of a function with just two critical points on S^{n-1}, perturb the function φ in the vicinity of the higher "critical manifolds" so as to have only non-degenerate critical points, the sphere $S_{(2k+1)\pi}$ contributing two critical points of index $(n-1)k$ and $(n-1)(k+1)$. It follows that ΩS^n admits a model of the form

$$\Omega S^n = S^{n-1} \cup e_{2(n-1)} \cup e_{2n-1} \cup \cdots . \tag{1.36}$$

Now the inclusion $S^{n-1} \xrightarrow{i} \Omega S^n$ induces a homomorphism i_* in homotopy which by (1.10) must be an isomorphism for dimensions $< 2n-3$. Combined with the isomorphism $\pi_k(\Omega X) \simeq \pi_{k+1}(X)$ we conclude that i_* induces an isomorphism:

$$\pi_k(S^{n-1}) \simeq \pi_{k+1}(S^n) \qquad k < 2n-3. \tag{1.37}$$

The homomorphism induced by i in this manner is called the "Suspension" and the isomorphism we have just established is precisely the weak "*Freudenthal Suspension Theorem*".

Classically derived in quite a different manner this result was the most general homotopy theorem concerning spheres known before the advent of pectral sequences. I note in passing that armed with this result it is then quite easy to give an inductive proof of Hopf's fundamental theorem to the effect that $\pi_n(S^n) \simeq \mathbf{Z}$, $n > 0$.

This last illustration is really the prototype of the proof of all the periodicity theorems. In the case of U_{2n} the argument runs as follows.

We pick (P, Q, h) on $U(2n)$ as $P =$ identity, $Q = -$identity, h the homotopy class of the path

$$\mu : [0, 1] \to U_{2n}$$

which sends t into the diagonal matrix with entries

$$(e^{i\pi t}, \cdots e^{-i\pi t}, e^{i\pi t}, \cdots , e^{-i\pi t}).$$

The set of minimal geodesics in the corresponding loopspace, Ω_ν, is then seen to consist of all the *conjugate* paths to μ, i.e., all paths of the form $t \to xu(t)\, x^{-1}$, $x \in U_{2n}$; this set is therefore homeomorphic to $U_{2n}/U_n \times U_n$.

By perturbing the function φ near the higher critical manifolds one then again estimates the dimension of the cells they contribute to Ω_ν and it then turns out that

$$\Omega_\nu U_{2n} = U_{2n}/U_n \times U_n \cup e_{2n+2} \cup \cdots$$

where the dots indicate cells of dimension $> 2n + 2$. This formula, as we have seen, then leads directly to the periodicity theorem. Q.E.D.

Clearly this procedure only works when one has great control over the geometry of the space. In particular one needs an oversight over the set of all geodesics joining two points and the number of conjugate points on any such geodesic. By and large the only spaces where such complete information is available are the *symmetric spaces*, (see [24], [28]) and for our purposes only the compact ones are of interest. Recall that such a space is the coset space of a compact Lie group G by a closed subgroup K which has the same identity component as the fixed point set of an involution. These spaces include the groups—simply write $G = G \times G/\Delta$ where Δ is the diagonal—the spheres, and for instance, the following spaces: SO_{2n}/U_n, Sp_n/U_n, U_n/O_n, U_{2n}/Sp_n, $Sp_{2n}/Sp_n \times Sp_n$, etc.

Now quite generally, one can show that if $\nu = (P, Q, h)$ is a "base point" on a symmetric space M and $\Omega_\nu M$ is the corresponding loopspace then the set of geodesics $\mathfrak{C}_\nu M$ in $\Omega_\nu M$ consist of a disjoint union of manifolds of geodesics all of which are naturally homogeneous spaces. Furthermore, the *manifold* of *minimal geodesics in* $\mathfrak{C}_\nu M$ *is itself again a symmetric space*. This phenomenon enables one to iterate the construction of $\Omega_\nu M$, and it is this iteration which leads one naturally to the other periodicity theorems. For instance, the sequence of models:

$$\begin{aligned}
\Omega_\nu SO_{2n} &\cong \{SO_{2n}/U_n\} \cup e_{2n-2} \cup \cdots \\
\Omega_\nu SO_{4n}/U_{2n} &\cong \{U_{2n}/Sp_n\} \cup e_{4n-2} \cup \cdots \qquad (1.38) \\
\Omega_\nu U_{4n}/Sp_{2n} &\cong \{Sp_{2n}/Sp_n \times Sp_n\} \cup e_{4n+4} \cup \cdots
\end{aligned}$$

leads to the formula

$$\pi_k(Sp) \simeq \pi_{k+4}(O) \qquad k \geqslant 0\,, \tag{1.39}$$

while the models

$$\begin{aligned} \Omega_v Sp_n &= \{Sp_n/U_n\} \cup e_{2n+2} \cup \cdots \\ \Omega_v Sp_n/U_n &= \{U_n/O_n\} \cup e_{n+1} \cup \cdots \\ \Omega_v U_n/O_n &= \{O_{2n}/O_n \times O_n\} \cup e_{n+1} \cup \cdots \end{aligned} \tag{1.40}$$

lead to the equation

$$\pi_k(O) = \pi_{k+4}(Sp) \qquad k \geqslant 0. \tag{1.41}$$

Of course, the homotopy of all the intermediary spaces such as Sp_n/U_n, etc., is also stable when $n \to \infty$ as can be shown quite easily, so that our procedure clearly also computes these stable homotopy groups.

Note that the last step in the derivation of (1.40) involves the isomorphism

$$\pi_k(Sp_{2n}/Sp_n \times Sp_n) \simeq \pi_{k+1}(Sp_n) \qquad k \ll n$$

which can be derived from an exact sequence argument—similar to the one we used for U, and which was in any case well-known. Alternatively, one can use the same method to obtain a model for $\Omega_v\{Sp_{2n}/Sp_n \times Sp_n\}$ which starts with Sp_n.

The same remarks apply to the isomorphism

$$\pi_k(O_{2n}/O_n \times O_n) \simeq \pi_{k+1}(O_n) \qquad k \ll n.$$

The formulas (1.39) and (1.41) give estimates of the dimensions in which our various models are applicable. By intelligently passing to $n = \infty$ we can, however, express their main import by saying that an 8-fold iteration of the loop construction leads back to the same space. For reasons which will become apparent in the next section, the best place to start this iteration is with the space BO, which is defined as the direct limit of the sequence of spaces

$$\to O_{2n}/O_n \times O_n \to O_{2(n+1)}/O_{n+1} \times O_{n+1} \to. \tag{1.42}$$

Keeping track of components, the maps implicit in (1.38) and (1.39) then can be interpreted as a map

$$\lambda_{\mathbf{R}} : BO \times \mathbf{Z} \to \Omega^8(BO \times \mathbf{Z}) \tag{1.43}$$

which induces an isomorphism in homotopy, and hence, as follows by an important but quite elementary theorem of J. H. C. Whitehead, a natural equivalence of the functors $P \to [P, \mathbf{Z} \times BO]$, and $P \to [P, \Omega^8\mathbf{Z} \times BO]$.

A similar construction is of course possible for the unitary group. Here one is led to the direct limit, BU, of the spaces $U_{2n}/U_n \times U_n$, and to a map

$$\lambda_{\mathrm{C}} : BU \times \mathbf{Z} \to \Omega^2 BU \times \mathbf{Z} \tag{1.44}$$

which is also a homotopy equivalence in the above sense.

The implications of the periodicity theorem in this guise will be discussed in the next section.

Let me just conclude here though, with a very cursory survey of the alternative proofs that the maps λ_{R} and λ_{C} induce homotopy equivalences which have been given since 1958. Toda was the first to construct λ_{C} and check its properties, by purely homotopy theoretical methods, i.e., without the Morse theory [70]. Thereafter, Cartan and Moore used the spectral sequence of a fibering and cohomology to yield the periodicity theorem [34]. Dyer and Lashoff also took a similar point of view [36]. In the early 1960's, Atiyah and I produced an elementary proof of the periodicity for BU by a construction inspired by the linearization procedure in ordinary differential equations of higher order [15]. This point of view was then extended to BO by Wood [73]. Also in the early 1960's, A. Shapiro and I noticed a periodicity phenomenon in certain representations of the Spinor group and were thus able to actually identify generators for $\pi_k(O)$ in terms of the Spinor groups. This point of view was developed greatly in the past few years by Karoubi and led to his theory of Banach categories and to a different proof of periodicity [50]. Finally, also in the past few years, Atiyah used the connection with the index theorem and the basic properties of Fredholm operators, to produce a very short proof of the periodicity. He also developed a theory of "complex bundles with involution" [12], from which the much more mysterious BO periodicity could be deduced from our elementary proof for BU.

2. The periodicity theorem in functorial form

Before describing the periodicity theorem in its functorial form a few remarks on homology or rather cohomology are in order. Historically homology and cohomology were first defined for polyhedra by an explicit algebraic construction depending on the combinatorial decomposition of the polyhedron. From this point of view the homotopy inveriance of these objects was then a difficult question which was only solved by Veblen and Alexander in the 1920's [8].

The modern approach is by and large in the opposite direction. One first defines a functor from spaces to abelian groups which is obviously homotopy-invariant and then checks that on the family of polyhedra, (or CW-complexes) $\mathbf{P}$, the functor is computable by a combinatorial formula. Now the simplest manner of constructing a homotopy invariant functor on this category is to choose a fixed space Y with base point, and in terms of it define a contravariant functor F from $\mathbf{P}$ to the category of pointed sets by assignment

$$F(X) = [X, Y]. \tag{2.1}$$

This F is contravariant because a map $g : X \to X'$ induces—by composition—a map, $F(g) : F(X') \to F(X)$, and it is a functor because one easily checks the characteristic identities

$$F(\text{identity}) = \text{identity}, \tag{2.2}$$

$$F(g \circ h) = F(h) \circ F(g). \tag{2.3}$$

Further F is obviously homotopy invariant, that is,

$$f \sim g \Rightarrow F(f) = F(g).$$

Functors of this simple type are called *representable*; also one calls Y the classifying *space* for F, or says that F is *represented by* Y.

Now suppose that Y is a topological group. The multiplication

$$Y \times Y \to Y$$

then naturally induces a group structure on $F(X)$ and if Y is commutative, then $F(X)$ will be commutative.

Actually, as stands to reason, Y need be only a "*group-up-to-homotopy*" for the corresponding functor to have values in groups, and such "group objects in homotopy theory", usually called H-spaces, are very easy to construct. Indeed the composition of loops, i.e., distributing the parameter first along the first and then along the second loop, defines a map

$$\mu : \Omega X \times \Omega X \to \Omega X$$

on any loopspace, which induces an H-structure on ΩX, and it is this

H-structure which ultimately accounts for the group structure in $\pi_1(X) = \pi_0(\Omega X)$. Further if $Y = \Omega\Omega X$ is a second loopspace, then this group structure is (homotopy) commutative. Thus any such Y furnishes one with a homotopy invariant functor from $\mathbf{P}$ to abelian groups.

Consider now the case when Y is one of our atomic objects, i.e., a $K(\pi, n)$, π abelian. From the fact that $\Omega K(\pi, n+1) \cong K(\pi, n)$, it follows that the functor

$$F : X \to [X, K(\pi, n)]$$

naturally has values in abelian groups, and—by virtue of the simplicity of a $K(\pi, n)$—there is some hope that this F should be combinatorially computable on a polyhedron. This turns out to be the case—*in fact one recaptures in this manner precisely the classical algorithm defining the* nth-*cohomology group* $H^n(P; \pi)$ *of* P *with coefficients* π.

For those familiar with this algorithm let me sketch an argument which yields this identification. Let the vertexes of P be ordered, and let $P^{(k)} \subset P$ be the k-skeleton of P, i.e., the sub-polyhedron consisting of simplexes of dim $\leqslant k$.

Suppose now that $f : P \to K(\pi, n)$ is a map. Using the property that $\pi_i(K(\pi, n)) = 0$ for $i < n$ we can inductively deform f to a new map, f_1, which sends $P^{(n-1)}$ into the base point of $K(\pi, n)$. Consider now the restriction of f_1 to the closure $\bar{\sigma}$, of the n-simplex σ with vertexes:

$$\sigma = \langle x_0, \dots, x_n \rangle, \qquad x_i \text{ in their natural order.}$$

Because all the faces of σ are mapped into the base point of $K(\pi, n)$ under f_1 this map determines an element $c_{f_1}(\sigma)$ in $\pi_n\{K(\pi, n)\} = \pi$.

In short c_{f_1} is a function from the n-simplexes of P, to π, i.e., an n-cochain on P with values in π.

Now the geometric fact that f_1 extends to $P^{(n+1)}$ is expressed by the following algebraic identity:

Whenever $\langle x_0, \dots, x_{n+1} \rangle$ *is an* $(n+1)$-*simplex of* P *then*

$$\sum (-1)^i c_f \langle x_0, \dots, \hat{x}_i, \dots, x_{n+1} \rangle = 0.$$

Here the hat over x_i indicates that the vertex x_i is to be deleted. The above identity is precisely the condition that c_{f_1} be a "cocycle" in the usual terminology. At this point we have, starting with f, constructed an "n-cocycle" on P. Next, one has to determine the ambiguity of the correspondence $f \to c_{f_1}$. The construction depended on the manner in which f was shrunk to f_1 on $P^{(n-1)}$, and it turns out that if c_{f_1} and c_{f_2} are the cocycles corresponding to two different shrinkings of f then $c_{f_1} - c_{f_2}$

is a cocycle of special type—called a "*coboundary*". Thus one finally obtains a well defined function:

$$F(P) \to \text{cocycles/coboundaries}.$$

The group on the right is the usual combinatorial nth-cohomology group of P with values in π—denoted by $H^n(P; \pi)$—and one can complete the above argument to show that our construction yields an isomorphism (Eilenberg–MacLane) [38]:

$$[P, K(\pi, n)] \simeq H^n(P; \pi). \tag{2.4}$$

The onto part of the construction uses the fact that $\pi_i\{K(\pi, n)\} = 0$ for $i > n$. This enables one to convert an n-cocycle, c, into a map, $P \to K(\pi, n)$ by sending $P^{(n-1)}$ into the base point of $K(\pi, n)$, mapping the n-simplexes by maps in the homotopy classes assigned to them by c, and then extending the map inductively over the skeletons of P.

Thus (2.4) asserts that the classical nth-cohomology construction $P \to H^n(P; \pi)$ is represented by $K(\pi, n)$ and so in particular proves its homotopy invariance.

Note furthermore that the family of spaces $\{K(\pi, n),\ n = 0, 1, 2,\ldots\}$ satisfy the condition:

$$\Omega K(\pi, n + 1) \simeq K(\pi, n). \tag{2.5}$$

An important consequence of (2.5) is now the following,

EXACTNESS PROPERTY OF COHOMOLOGY. *Let π be a fixed Abelian group, and let H^n denote the functor represented by $K(\pi, n), n \in \mathbf{Z}$.*[5] *Also let (P, Q) be a pair of polyhedra and define the relative groups $H^i(P, Q)$ by the formula*

$$H^i(P, Q) = \tilde{H}^i(P/Q) = \ker\{H^i(P/Q) \to H^i(*)\}$$

where P/Q is the space obtained from P by identifying Q to a point $*$. Then there is an infinite exact sequence:

$$\begin{tikzcd}
\arrow[r] & H^{n+1}(P,Q) \arrow[r] & H^{n+1}(P) \arrow[r] & H^{n+1}(Q) \arrow[r, "\delta"] & {} \\
\arrow[r] & H^n(P,Q) \arrow[r] & H^n(P) \arrow[r] & H^n(Q) \arrow[ull] &
\end{tikzcd} \tag{2.6}$$

which behaves naturally with respect to maps of pairs.

[5] For $n = 0$, $K(\pi, 0) = \pi$ so that $H^0(P; \pi) =$ locally constant functions from P to π. For $n < 0$ $K(\pi, n)$ is simply a base point.

The horizontal maps of this sequence are of course induced by the natural map $P \to P/Q$ and the inclusion $Q \subset P$ respectively, while δ is ultimately induced by the equivalence (2.5).

There are many ways of proving this property and they all depend in one way or another on the regularity properties of pairs of polyhedra (or finite CW-complexes) and on the elementary geometry of homotopy theory, rather than the special properties of the $K(\pi, n)$'s *other than the recursive formula* (2.5).

For instance the exactness of our sequence at $H^n(P)$ follows directly from the following elementary

HOMOTOPY EXTENSION THEOREM FOR POLYHEDRA: *Let $i : Q \to P$ be the inclusion of a sub-polyhedron into P. Also let*

$$f : P \to Y$$

be a map into a space Y and let $\hat{f}$ be the restriction of f to $Q : \hat{f} = f \mid A$. Then for any homotopy $\hat{f}_t$ of $\hat{f}$ there exists a homotopy f_t such that

$$f_t \mid A = \hat{f}_t \, .$$

This property is clearly dual to the covering homotopy property (1.11) and maps $i : Q \to P$ satisfying this condition are therefore often called "co-fiberings". The construction of δ in this sequence also plays a corresponding dual role to the construction of f in the sequence (1.12).

Let us recapitulate this development in the following manner. A sequence $F^* = \{F^n\}$, $n \in \mathbf{Z}$, of contravariant functors from $\mathbf{P}$ to abelian groups will be called a *cohomology theory* if the F^n are homotopy invariant and satisfy the exactness property (2.6) where $F^n(P, Q)$ is defined by $\tilde{F}^n(P/Q) = \ker\{F^n(P/Q) \to F^n(*)\}$.

With this understood, we see that the $K(\pi, n)$'s furnish us with a cohomology theory $H^* = \{H^n\}$,[6] which has the following two properties:

$$H^n(\text{point}) = \begin{cases} \pi & n = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

$$H^n(P) \simeq \text{classical } H^n(P; \pi). \tag{2.8}$$

Now historically the search for a good invariance proof of the classical groups $H^n(P; \pi)$ led to the definition of many cohomology theories

[6] Here we set $H^n(X) = 0$ for $n < 0$.

subject to (2.7). Thus one has the singular and Čech theories—and many others. Each of these starts from a different geometric point of view and produces a theory with different properties on different categories of spaces. However they all trivially satisfy (2.7) and were all eventually—quite laboriously in some instance—shown to satisfy (2.8).

This situation was finally cleared up by the following: [43]:

UNIQUENESS THEOREM (Eilenberg–Steenrod). *Any two cohomology theories subject to* (2.7) *agree with the classical theory of polyhedra.*[7]

In retrospect it seems quite natural that this theorem of the 1950's should have initiated research in more general cohomology theories—i.e., theories not satisfying (2.7). However this development had to wait until the 1960's (see for instance the work of Lima [55], and especially of G. W. Whitehead [72]) after Atiyah and Hirzebruch took this point of view with a particular theory [14], the so-called K-theory constructed out of vector-bundles over a space.

From our present vantage point the description of this cohomology theory is actually quite easy. First recall that the ultimate reason why the $K(\pi, n)$'s generate a cohomology theory in the recursion

$$\Omega \cdot K(\pi, n) = K(\pi, n-1). \tag{2.9}$$

This suggests that if F is a functor represented by $(Y, *)$ then the sequence of functors F^{-n}, n, $0, 1, \ldots, F^0 = F$; defined by

$$F^{-n}(P) = [P, \Omega^n Y] \tag{2.10}$$

should generate a "partial" cohomology theory, i.e., one satisfying the exactness axiom in dim < 0. The functor $F = F^0$ of course only has values in pointed sets, but its "derived functors" F^{-n} have natural group structure induced by the H structure of a loopspace.

Observe finally that the partial theory generated by F in this manner, has for its values on a point precisely the homotopy groups of Y:

$$F^{-n}(\text{point}) = \pi_0(\Omega^n Y) = \pi_n(Y; *). \tag{2.11}$$

Let us now apply this quite general principle to the functor KO, represented by the space $\mathbf{Z} \times BO$, with base point $0 \times p$, $p \in BO$:

$$KO(P) = [P, \mathbf{Z} \times BO]. \tag{2.12}$$

[7] Note that the usual "excision axiom" of Eilenberg–Steenrod holds trivially in our context by the assumed isomorphism $F^n(P, Q) \simeq \tilde{F}^n(P/Q)$.

The homotopy equivalence

$$\lambda_{\mathbf{R}} : \mathbf{Z} \times BO \to \Omega^8 Z \times BO$$

then *induces* an 8 *fold periodicity* in KO^{-n}:

$$\lambda_{\mathbf{R}}^* : KO^{-n-8}(P) \simeq KO^{-n},$$

and using this *periodicity we may now extend the definition* of KO^{-n} *to all* $n \in Z$. In short $\lambda_{\mathbf{R}}$ enables us to complete the partial theory generated by KO into a genuine cohomology theory $KO^* = \{KO^n\}$; $n \in Z$, which is periodic of period 8 and satisfies the point axiom:

$$KO^n(\text{point}) = \{\mathbf{Z}, \mathbf{Z}_2, \mathbf{Z}_2, 0, \mathbf{Z}, 0, 0, 0\} \tag{2.13}$$

for $n \equiv 0, -1, \ldots, -8 \bmod 8$.

Similarly $\lambda_{\mathbf{C}}$ induces a cohomology theory KU^*, $\{KU^n\}$ with period 2 and having the "point" values

$$KU^n(\text{point}) = \{\mathbf{Z}, 0\} \qquad \text{for } n \equiv 0, 1, \bmod 2. \tag{2.14}$$

The philosophy that the periodicity phenomenon can be used to construct periodic cohomology theories now has many consequences. First of all one can embark on a program of extending the classical constructions to these new theories, and in particular to compute the values of this theory on spaces other than the spheres. For instance consider the complex projective space $\mathbf{C}P_n$. Applying the exactness axiom to the inclusion $\mathbf{C}P_{n-1} \subset \mathbf{C}P_n$ and recalling that $\mathbf{C}P_n/\mathbf{C}P_{n-1} = S^{2n}$ is immediately seen to compute KU of CP_n:

$$\begin{aligned} &KU^0(\mathbf{C}P_n) \simeq \mathbf{Z} \oplus \cdots \oplus \mathbf{Z}(n+1) \text{ copies} \\ &KU^1(\mathbf{C}P_n) = 0. \end{aligned} \tag{2.15}$$

In fact a similar argument shows in general that any P which is obtained by successively attaching even dimensional cells, has a KU^0 isomorphic to the direct sum of $\mathbf{Z}$'s one for each cell, while $KU^1 = 0$.

A more systematic approach to the computability of these theories leads one to try to extend the uniqueness theorem in this context. This then naturally leads to a, by that time not unexpected, spectral sequence which was first written down for these K-theories by Atiyah and Hirzebruch [14], but which is clearly valid for all cohomology theories. Let me state it here, primarily for those acquainted with these concepts:

THEOREM 2.16. *Let F^* be a cohomology theory. Then there is a spectral sequence with E_2-term:*

$$E_2^{p,q} = H^p(P; F^q(\text{point})) \tag{2.17}$$

which converges to a graded group associated to $F^*(P)$.

Note that here H^p denotes the *classical cohomology*, and the qualitative content of the theorem is that from the computable terms $H^p(P;F^q(\text{point}))$ one can, by a series of homology operations *satisfying certain dimensional restrictions*, arrive at the groups $F^*(P)$-modulo certain extensions.

This vague piece of information is nevertheless a powerful tool for computations in certain fortuitous cases. For instance when applied to a theory, F^*, satisfying the Eilenberg–Steenrod axiom (2.7), the assertion of (2.17) immediately yields the uniqueness theorem. Thus (2.17) really is the proper generalization of that result. Applied to KU it also immediately yields the computation of $KU(\mathbf{C}P_n)$. Other quite routine consequences are:

PROPOSITION 2.18. *Let $\mathbf{Q}$ denote the rationals. Then*

$$KU^n(P) \otimes \mathbf{Q} \simeq \sum H^m(P; \mathbf{Q}) \qquad m \equiv n \bmod 2$$

$$KO^n(P) \otimes \mathbf{Q} \simeq \sum H^m(P; \mathbf{Q}) \qquad m \equiv n \bmod 4.$$

Thus over the rationals nothing new can be expected from these theories, *furthermore the same result would hold for any periodic theories with the prescribed values on points.*

The real power of these theories therefore ultimately rests in the geometric roots of the functors KO and KU, and we will turn to this geometric interpretation next. For this purpose a short review of the basic facts concerning vector bundles, as found in Steenrod's book [68] of the 1950's for instance, will be essential.

Recall first of all that an n-dimensional vector bundle E over X is a twisted product E over X with fiber $\mathbf{R}^n$, for which the *twisting preserves the vector-space structure of* $\mathbf{R}^n$. Precisely, there must exist a covering $\{U_\alpha\}$ of X and local product representations

$$\varphi_\alpha : U_\alpha \times \mathbf{R}^n \to \pi^{-1}(U_\alpha)$$

such that on the overlaps $U_\alpha \cap U_\beta$, the map

$$\varphi_\beta^{-1} \circ \varphi_\alpha : U_\alpha \cap U_\beta \times \mathbf{R}^n \to U_\alpha \cap U_\beta \times \mathbf{R}^n \tag{2.19}$$

restricts to a linear isomorphism of $\mathbf{R}^n$ on each fiber.

Intuitively a vector bundle E is therefore best thought of as a locally trivial family of vector space $E_x = \pi^{-1}(x)$, parametrized by the points of X.

Two vector bundles E and E' over X are called isomorphic if there exists a map

$$\varphi : E \to E'$$

which preserves fibers and induces a *linear isomorphism* $\varphi_x : E_x \to E_x'$, on the fibers over $x \in X$.

The simplest bundle is the product bundle $X \times R^n \to X$ and a bundle isomorphic to it is called a *trivial* bundle.

We let $\mathrm{Vect}_n(X)$ denote the isomorphism classes of n-dimensional vector bundles with the trivial bundle as base point, and extend the function $\mathrm{Vect}_n(X)$ to a contravariant functor from **P** to pointed sets by the "fibered product" construction:

Given E over X and $f : X' \to X$, let $f^{-1}E$ be the bundle over X' consisting of the pairs $(x', e) \in X' \times E$ for which $f(x') = \pi(e)$.

Intuitively $f^{-1}E$ simply changes the parameter space—thus $f^{-1}E$ consist of the family $\{E_{f(x')}\}$, $x' \in X'$.

The functor $X \to \mathrm{Vect}_n(X)$ is clearly rooted in geometry. Indeed the family of tangent planes $T_m(M)$, $m \in M$ for a differentiable manifold M are easily seen to fit together into a vector bundle, $T(M)$, the "tangent bundle of M"—and the "position" of $T(M)$ in $\mathrm{Vect}_n(M)$, $n = \dim M$ is of great interest in many geometric problems.

Similarly if $V \subset M$ is a smooth submanifold of M, the normal bundle of $N(V)$ is well defined and has, as fiber over v, the vector-space

$$N_v(V) = T_v(M)/T_v(V).$$

In this example $N(V)$ can also be thought of as an abstract model for a *tubular neighborhood of V in M.*

Quite generally one is led to vector bundles in geometry whenever one attempts to linearize nonlinear problems.

Let us now consider the functors Vect_n from the point of view of a cohomology theory. A first disappointing observation is that these functors seem to have no natural group structure. On the other hand one notes that the Vect_n's are related by a large number of natural transformations.

This comes about because, as is easily seen, any *natural construction on vector spaces naturally extends to vector bundles by performing the con-*

struction on each fiber. Thus the construction of the dual of a vector space leads to the notion of the dual E^* to a bundle E. Similarly one has the notion of the direct sum $E \oplus F$, and the tensor product $E \otimes F$ of two bundles over a fixed space. Note that one therefore also has a natural definition of the exterior powers $\lambda^n E$, and the symmetric powers $\mathrm{sym}^n(E)$, of a vector bundle. In fact any representation

$$\rho : GL(n, R) \to GL(m, R)$$

extends to an operation $\rho(X)$ from $\mathrm{Vect}_n(X)$ to $\mathrm{Vect}_m(X)$ which is natural in the sense that if $X \xrightarrow{f} Y$ is a map, then

$$\rho(Y) \circ f^{-1} = f^{-1} \circ \rho(X).$$

These quite immediate facts are nevertheless of great importance as will be seen later on.

We turn next to the question of representability, and so, in particular, the homotopy invariance of Vect_n on polyhedra. Here the fundamental result going back to Steenrod, is the following:

THEOREM (2.20). *The functors* Vect_n *from* $\mathbf{P}$ *to pointed sets are represented by the infinite Grassmanian* G_n . *Thus*

$$\mathrm{Vect}_n(P) = [P, G_n],$$

where G_n *is the direct limit of the spaces*

$$O_{n+k}/O_n \times O_k \subset O_{n+k+1}/O_n \times O_{k+1} \qquad k = 1, 2, \ldots .$$

In particular on polyhedra of $\dim < N$, Vect_n *is represented by the finite Grassmannians*

$$G_{n,m} = O_{n+m}/O_n \times O_m , \qquad m \gg N.$$

This theorem follows from a much more general theorem in fiber bundles, but can also be given a more direct and elementary proof. To sketch this in briefly, consider first of all the homotopy invariance question. This property may be derived from the homotopy covering property for twisted products with the aid of the following observation.

If E and F are bundles of dimension n over X, we can construct a space $\mathrm{Iso}(E, F)$ as the set of pairs (x, φ_x), with $x \in X$, and $\varphi_x : E_x \to F_x$ an isomorphism. This set is topologized by its obvious inclusion in the bundle $\mathrm{Hom}(E, F) = E^* \otimes F$, and so also inherits a projection

$\tau : (x, \varphi_x) \to x$ onto X. Furthermore, Iso(E, F) *is now seen to be the twisted product of* X *and* $GL(n, R)$ *under* τ, and in terms of it an isomorphism $\varphi : E \simeq F$ precisely amounts to a section

$$s : X \to \mathrm{Iso}(E, F),$$

of τ, that is, a map s with the property $\tau \circ s = 1$.

To return to the homotopy invariance question, consider the space $X = P \times I$, let $i_0 : P \to P \times I$ be the inclusion $p \to (p, 0)$, and $\pi : X \to P$ the natural projection. Also let E be an arbitrary bundle over X, and set F equal to:

$$F = \pi^{-1} \circ i_0^{-1} \circ E.$$

Finally let $h_t : P \times I \to P \times I$ be the homotopy retraction of $P \times I$ to $P \times 0$ given by

$$h_t(p, \alpha) = (p, ta), \qquad 0 \leqslant t \leqslant 1.$$

Then in the diagram:

$$\begin{array}{ccc} & & \mathrm{Iso}(E, F) \\ & & \downarrow \\ X & \xrightarrow[h_t]{} & X \end{array}$$

h_0 has a lifting (given by the identity map) and so by the covering homotopy theorem h_1 has one also.

This implies the isomorphism:

$$E \simeq \pi^{-1} \circ s_0^{-1} E,$$

and so leads to the homotopy invariance for Vect_n.

To explain why the Grassmanian classifies Vect_n is also not difficult. Geometrically $G_{n,m}$ may be thought of as the *set of m-dimensional subspaces* in a Euclidean $(n + m)$ space V.

Now consider the trivial bundle:

$$\tilde{V} = V \times G_{n,m}$$

over $G_{n,m}$, and let S be the subset of $\tilde{V}$ consisting of all pairs $v \in V$, $A \in G_{n,m}$ with $v \in A$:

$$S = \{(v, A), v \in A\}.$$

The projection of (v, A) on A, then defines S as a vector bundle over $G_{n,m}$ which we call the *universal sub-bundle of* $\tilde{V}$. Similarly the subset

$$Q = \{(v, A); v \in \text{orthocomplement of } A\}$$

defines the universal *quotient bundle over* $G_{n,m}$ and we clearly have the bundle formula:

$$S \oplus Q = \tilde{V}. \tag{2.21}$$

Now let E be an n-dimensional bundle over P. We claim that for m large enough (with respect to the dimension of P) there will be a map $f{:}P \to G_{n,m}$ such that

$$f^{-1}Q \simeq E. \tag{2.22}$$

Once this general fact is established it is not hard to see that the assignment $[P, G_{n,m}] \to \mathrm{Vect}_n(P)$ which sends $f : X \to G_{n,m}$ into $f^{-1}Q$ is one-to-one-and onto when m is large enough.

To construct f, we use the triviality of a vector bundle over any simplex of P (this follows from the homotopy invariance) together with a partition of one argument, to construct a finite number of sections $s_0 ,..., s_{n+m}$ of E with the property that the $\{s_i(x)\} \in E_x$ span E_x for every $x \in P$. Now let V be the space generated by these sections over $\mathbf{R}$ and define a map $f : P \to G_{n,m}$ by assigning to $x \in P$, the subspace $f(x)$ consisting of all $v \in V$ which vanish at x. Evaluating the section in the orthocomplement of $f(x)$ at x, then establishes an isomorphism of $Q_{f(x)}$ with E_x and so yields the desired formula (2.22).

So much for a discussion of Theorem (2.20). Apart from establishing the basic cohomological properties of Vect_n, note that Theorem (2.20) also connects this functor with our earlier homotopy groups.

Indeed, using the exact sequence of the fibering

$$O_{n+m}/O_m \to O_{n+m}/O_n \times O_m$$

with fiber $O_n \times O_m/O_n = O_n$, we see that, just as in (1.24),

$$\pi_k(O_n) \simeq \pi_{k+1}(G_{n,m}) \qquad m \gg n.$$

Now by Theorem 1 we know that as a pointed set

$$\mathrm{Vect}_n(S^{k+1}) = \pi_{k+1}(G_{n,m}), \qquad m \gg k. \tag{2.23}$$

In short

$$\mathrm{Vect}_n(S^{k+1}) \simeq \pi_k(O_n), \qquad n \gg k \tag{2.24}$$

so that we may interpret the periodicity theorem as evaluating the functor $\mathrm{Vect}_n(S^{k+1})$ for n large compared to k.

At this point we are ready to geometrically construct the basic functor $K_S(P)$, of "stable bundles over P". For this purpose define two bundles E, F over P to be stably equivalent:

$$E \approx_S F, \tag{2.25}$$

if and only if they become isomorphic after suitable number of trivial bundles is added to each of them, i.e., if and only if:

$$E \oplus 1_k \simeq F + 1_l, \qquad 1_k \text{ the trivial } k\text{-dimensional bundle } \mathbf{R}^k \times P. \tag{2.26}$$

Now define $K_S(P)$ simply as the *stable equivalence classes of bundles over* **P**. This functor is already very close to the functor KO introduced earlier. Indeed by fitting the representation spaces G_n together intelligently it is not hard to see that:

THEOREM (2.27). *The functor K_S is represented by the space BO introduced in Section* 1:

$$K_S(P) = [P; BO].$$

A first virtue of this "geometric" construction of K_S is that $K_S(P)$, $P \in \mathbf{P}$, is easily seen to carry a natural abelian group-structure. Indeed the direct sum operation on bundles clearly induces an additive structure on $K_S(P)$, with the class of a trivial bundle playing the role of zero. On the other hand Equation (2.19) together with representability of $\mathrm{Vect}_n(P)$ by a finite Grassmanian immediately implies that for every bundle E over P there exists a bundle E^1 such that

$$E + E^1 = \text{trivial bundle}. \tag{2.28}$$

Thus in $K_S(P)$ every element has an inverse relative to addition, and so $K_S(P)$ is naturally an abelian group under the direct sum operation.

There is a variant of this procedure, given by Grothendieck's K-construction, [23] which he applied very successfully in several quite different algebraic contexts and which as we will see, naturally leads to our earlier functor KO over polyhedra.

A sequence of bundles

$$0 \longrightarrow E' \xrightarrow{\varphi} E \xrightarrow{\psi} E'' \longrightarrow 0$$

and maps φ, ψ over X is called *exact* if each map preserves fibers, is linear on the fibers, and the induced sequence of maps at each $x \in X$:

$$0 \longrightarrow E_x' \xrightarrow{\varphi_x} E_x \xrightarrow{\psi_x} E_x' \longrightarrow 0 \tag{2.29}$$

is exact.

Now let $\mathscr{E}(X)$ denote the free group generated by the vector bundles over X,[8] and write $[E]$ for the generator in $\mathscr{E}(X)$ determined by the bundle E. Next let $R(X)$ be subgroup of $\mathscr{E}(X)$ generated by elements of the forms $[E] - [E'] - [E'']$, the triples (E', E, E'') ranging over the exact sequences $0 \to E' \to E \to E'' \to 0$. Finally define $KO(X)$ as the quotient group

$$KO(X) \simeq \mathscr{E}(X)/R(X). \tag{2.30}$$

The advantage of this definition is first of all that it yields a group for every space X, whereas the stabilizing construction K_S only yields a group if every bundle over X has an inverse in the sense of (2.28). On **P** *the two are trivially related though.* To see this, let $\gamma(E) \in KO(P)$ be the class of $[E]$, and let $\gamma_s(E) \in K_S(P)$ denote the stable class of E in $K_S(P)$.

Also let

$$\dim : KO(P) \to H^0(P; \mathbf{Z})$$

be induced by the function which assigns to E its dimension over each component of P.

Now then, the relation is question is simply that

$$K_S(P) \simeq \ker\{\dim : KO(P) \to H^0(P, \mathbf{Z})\}, \tag{2.31}$$

and in fact

$$KO(P) = K_S(P) \oplus H^0(P, \mathbf{Z}). \tag{2.32}$$

Over a connected P the isomorphism (2.32) is induced by the map which sends $\gamma(E)$ to $\gamma_s(E) + \dim E$.

[8] Bundles may have different dimensions over different components of x.

In view of (2.27) this isomorphism now naturally leads to the desired identification

$$KO(P) \simeq [P, \mathbf{Z} \times BO] \simeq KO(P). \tag{2.33}$$

Note. Actually the introduction of exact sequences is quite superfluous in topology, because it is easily seen that over polyhedra every such exact sequence splits in the sense that the middle term becomes isomorphic to a direct sum of the two end terms. However, in algebraic geometry and many other contexts this is not the case.

A first advantage of this geometric interpretation of the KO theory via the K-construction is now that the tensor product on bundles quite naturally defines a ring structures on $KO(P)$. Using the obvious fact that $\dim : KO \to H^0$ is a ring homomorphism, and the identification (2.32), $K_S(P)$ of course also inherits a ring structure, however this multiplication is not so obviously apparent.

A more striking example of the advantage of our new interpretation for KO is furnished by Grothendieck's extension of the exterior power operations λ^k from bundles to $KO(P)$. He first observes that if a function $E \to \varphi(E)$ from vector bundles over P to an abelian group A is additive in the sense that for every exact sequence

$$0 \to E' \to E \to E'' \to 0,$$

$\varphi(E') + \varphi(E'') = \varphi(E)$, then φ determines a unique homomorphism

$$\hat{\varphi} : KO(P) \to A$$

which induces φ in the sense that

$$\varphi(E) = \hat{\varphi}\{\gamma(E)\}.$$

(This is for instance the principle by which the dimension function induced the homomorphism dim.) Once this is granted, we may use the well known identity

$$\lambda^m(E + F) = \sum_{p+q=m} \lambda^p(E) \otimes \lambda^q(F) \tag{2.34}$$

to define exterior powers in $KO(P)$. Indeed let $A(P) \subset KO(P)[[t]]$ be the *multiplicative* group of formal power series in t with coefficients in $KO(P)$, starting with (the trivial bundle) 1. Now given a vector bundle E let $\lambda_t(E) \in A(P)$ be defined by the formula

$$\lambda_t(E) = \sum_{i=0}^{\infty} t^i \gamma(\lambda^i E). \tag{2.35}$$

The relation (2.34) immediately shows that $E \to \lambda_t(E)$ is additive and so induces an (additive!) homomorphism—also denoted by λ_t,

$$\lambda_t : KO(P) \to A(P) \tag{2.36}$$

whose components, λ^i yield natural operations

$$\lambda^i : KO(P) \to KO(P). \tag{2.37}$$

The existence of these operations, which play a crucial role in the theory, are not at all easy to see from the identification $KO(P) = [P, \mathbf{Z} \times BO]$.

In short, from various points of view the "geometric" $KO(P)$ is a good functor with values in rings and carrying various operations deduced from linear algebra, whereas the functor represented by $\mathbf{Z} \times BO$ naturally leads to an 8-fold periodic cohomology theory whose values on the spheres is given by the periodicity theorem.

Let us now try and reconcile these two quite different personalities of the functor KO more directly than via the representability theorems for Vect_n.

To do this we need the notion of suspending a polyhedron. First of all, we define the cone CP on P, as the polyhedron obtained from $P \times I$ by identifying $P \times [1]$ with a point—the base point of the cone. Thus the cone on S^n is the n-disc. Now define the suspension SP, of P as the space obtained from two copies of CP by gluing them together along $P \times 0 \subset CP$:

$$SP = CP \underset{P}{\cup} CP.$$

Thus the suspension of S^n is S^{n+1}:

$$SS^n = S^{n+1}.$$

The suspension SP will be taken to have as base point one of the base points of CP and is thus considered as a *pointed space*. With this understood the suspension is now quite trivially seen to play an *adjoint role* to the *loop functor* Ω, in the sense that if Y is a space with base point and P^+ denotes the disjoint union of P and a base point $*$, then

$$[P, \Omega Y] = [P^+, \Omega Y]_* \simeq [SP^+, Y]_* . \tag{2.38}$$

This relation leads to a basic duality which prevades homotopy theory. For instance just as ΩY is a group object, SP is always seen to form a "cogroup object," and one can put this concept at the base of the group structures in homotopy theory. In particular, the partial cohomology theory derived from a functor F, represented by Y with base point $*$, can—by virtue of [2.38]—be reinterpreted by the formula:

$$F^{-n}(P) = [S^n P^+, Y]_* .$$

Thus, for instance, from this point of view the periodicity theorem asserts that

$$KO(S^{n+8}P) \approx KO(S^n P). \tag{2.39}$$

Finally, one observes that the space $S^n \times P$ is very closely related to $S^n P$. Indeed if P is connected and $p \times q$ is a point of $S^n \times P$, then elementary geometry shows that

$$S^n \times P/S^n \times q \cup p \times P \simeq S^n P.$$

From this fact, and the fact that each of the subsets $p \times P$, $S^n \times q$ are retracts of $S^n \times p$ it finally follows that for any cohomology theory $\{F^n\}$:

$$F^m(S^n \times P) = \tilde{F}^m(S^n \vee P) \oplus F^m(S^n P), \quad S^n \vee P = S^n \times q \cup p \times P; \tag{2.40}$$

where $\tilde{F}^m(S^n \vee P)$ is the kernel of the map $F^m(S^n \vee P) \to F^m(p \times q)$ induced by the map: $p \times q \to S^n \vee P$.

Consideration of this type immediately lead one to the conclusion that a *quite equivalent formulation of the real periodicity theorem is the formula*

$$KO(P \times S^8) \simeq KO(P) \otimes KO(S^8). \tag{2.41}$$

Now for our geometric KO there is a natural map

$$KO(P) \otimes KO(Q) \to KO(P \times Q) \tag{2.42}$$

induced by the tensor product of bundles: Given E over P and F over Q

simply send $\gamma(E) \otimes \gamma(F)$ into $\gamma(\tilde{E} \otimes \tilde{F})$ where $\tilde{E}$ and $\tilde{F}$ are the pull-backs of E and F under the natural maps on the factors.

These remarks therefore naturally lead us to the following *functorial form* of the periodicity:

THE REAL PERIODICITY THEOREM: *The tensor product of bundles induces an isomorphism*

$$KO(P) \otimes KO(S^8) \simeq KO(P \times S^8). \tag{2.43}$$

One can, of course, derive a completely parallel geometric interpretation for KU—using complex vector bundles throughout—and then the corresponding periodicity takes the form:

COMPLEX PERIODICITY THEOREM: *The tensor product of complex bundles induces an isomorphism*:

$$KU(P) \otimes KU(S^2) \simeq KU(P \times S^2). \tag{2.44}$$

Originally it took quite a bit of work to translate the maps $\lambda_{\mathbf{R}}$ and $\lambda_{\mathbf{C}}$ into the formulas (2.43) and (2.44). However, the more modern approach is to try and prove these relations directly, and I would like to close this section with an outline of two such direct assaults on (2.44).

Let me start with the elementary proof which Atiyah and I developed in 1964 [15].

The basic philosophy of this proof is that $P \times S^2$ is thought of as a family of 2-spheres S^2_p, $p \in P$, parametrized by P, and one attempts to extend an elementary computation of $KU(S^2)$ to describe $KU(P \times S^2)$ as a *module over* $KU(P)$.

First note that every bundle on a polyhedron Q which is the union of two subpolyhedra:

$$Q = A \cup B, \qquad A \cap B = C \tag{2.45}$$

can be constructed by the following clutching procedure: Given E on A, F on B, and an isomorphism

$$\varphi : E \mid C \to F \mid C \tag{2.46}$$

one forms the bundle (E, φ, F) over Q by simply using φ_x to identify E_x with F_x over C. Note also that it then follows from the homotopy invariance of Vect_n that the *isomorphism class of* (E, φ, F) *depends only* on the *isomorphism class of the clutching function* φ.

We now apply this procedure to construct bundles over $P \times S^2$, from bundles over P. For this purpose think of S^2 as $\mathbf{C} \cup \infty$, and single out the subsets

$$D^0 = \{z \mid |z| \leqslant 1\}, \qquad D^\infty = \{z \mid |z| \geqslant 1\}, \qquad z \in S^2.$$

Also let π denote the projection $P \times S^2 \to P$, and let $\pi_0 = \pi \mid P \times D^0$, $\pi_\infty = \pi \mid P \times D^\infty$.

Our clutching data now consist of a bundle E on P, together with an isomorphism

$$\varphi : \pi_0^{-1}E \to \pi_\infty^{-1}E$$

defined on $P \times D^0 \cap P \times D^\infty = P \times S^1$, and the bundle over $P \times S^2$ resulting from such data will, for simplicity, be denoted by (E, φ, E).

Note here, that φ amounts to a function which to each $p \in P$, $z \in S^1 = D^0 \cap D^\infty$ assigns an *automorphism*

$$\varphi_p(z) : E_p \to E_p\,. \tag{2.47}$$

In particular, then the clutching function:

$$\varphi_p(x) = \text{multiplication by } z^m \tag{2.48}$$

is universally defined on all E over P and yields bundles (E, z^m, E) over $P \times S^2$.

The bundle $(1, z^{-1}, 1)$ (where 1 denotes the trivial line bundle over P) is traditionally denoted by H, and with this understood the following identities are immediate:

$$\begin{aligned}(E, 1, E) &\cong \pi^{-1}E\\ (E, z^m, E) &\cong \pi^{-1}E \otimes H^{-m}.\end{aligned} \tag{2.49}$$

Now when P reduces to a point it was well-known that $KU(S^2)$ is freely generated by [1] and $[H]$, or—as will be more convenient—by [1], and $[H] - [1]$, Hence our task will be to show, in general, that as a module over $KU(P)$, these elements freely generate $KU(P \times S^2)$. Thus to every bundle F over $P \times S^2$ we have to find elements $\alpha_0(F)$ and $\alpha_1(F)$ in $KU(P)$ such that in $KU(P \times S^2)$ the decomposition

$$[F] = \alpha_0(F) + \alpha_1(F) \otimes \{[H] - [1]\}, \tag{2.50}$$

is valid.

First of all the constant term $\alpha_0(F)$ is easily disposed of. Indeed if $s_q : P \to P \times S^2$, is the section $p \to p \times q$, $q \in S^1$, then $s_q^{-1} H \simeq 1$. Applying s_q^{-1} to (2.50) therefore evaluates $\alpha_0(F)$ as:

$$\alpha_0(F) = s_q^{-1} F.$$

The difficulty is therefore in the construction of the element $\alpha_1(F)$ which we will, for simplicity sake, from now on denote by $\alpha(F)$.

Our solution of this problem proceeds in the following five steps:

Step 1. One first remarks that for every F on $P \times S^2$ there exists a clutching function φ, so that

$$F = (s_q^{-1} F, \varphi, s_q^{-1} F). \tag{2.51}$$

This result follows from the homotopy invariance of Vect_n because $P \times D^0$, $P \times D^\infty$ both have $s_q(P)$ as a deformation retract.

Step 2. One shows next that every (E, φ, E) is isomorphic to an (E, φ', E) with φ' of the following "Laurent form":

$$\varphi_p'(z) = \sum_{|k|<N} a_k(p)\, z^k, \qquad a_k(p) \in \mathrm{Aut}(E_p). \tag{2.52}$$

This assertion is proved by expanding $\varphi_p(z)$ into a Fourier series and then approximating $\varphi_p(z)$ by a sufficiently high finite Ceasaro sum $\varphi_p{}^N(z)$. The resulting $\varphi_p{}^N(z)$ will then be homotopic to $\varphi_p(z)$ and has the form (2.52). Due to the compactness of P there is no trouble in extending the approximation theorem from a fixed $p \in P$ to all of P.

Step 3. By tensoring with a suitable power of H, it is first established that the crux of the matter lies in the construction of α for bundles $F = (E, \varphi, E)$, with φ in the *polynomial form*:

$$\varphi_p(z) = \sum_{0 \leqslant k \leqslant n-1} a_k(p)\, z^k; \qquad a_k(p) \in \mathrm{Aut}(E_p), \tag{2.53}$$

and for these, one now shows that

$$F \oplus (n-1)\, \pi^{-1} E \simeq (nE, az + b, nE) \tag{2.54}$$

by a procedure gleaned from the theory of ordinary differential equations.

Indeed in matrix form the linear clutching function $az + b$, acting on $E \oplus E \oplus \cdots \oplus E$ (n times), is given by the expression

$$az + b = \begin{pmatrix} 0, & a_0, & \cdots & a_n \\ -z, & 1 & & \\ & & 1 & \\ & & -z, & 1 \end{pmatrix},$$

and the isomorphism in question follows from the identity

$$az + b = \begin{pmatrix} 1 & p_1 & p_n \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} p & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ -z & 1 & & \\ & & -z & 1 \end{pmatrix}$$

where the p_i are inductively defined as $p_0 = p$, $zp_{k+1}(z) = p_k(z) - p_k(0)$ —after the nilpotent parts of the two matrices on the left and right are deformed to zero.

At this stage we come to the most interesting point. By combining the above reductions we have arrived at a *linear* clutching function

$$\varphi_p(z) = a(p)\, z + b(p),$$

whereas our final answer only involves the clutching functions z and 1. Hence it suggests itself that the bundle E on which φ acts, should decompose into two parts

$$E \cong E_+ \oplus E_- .$$

so that φ is deformable into z on E_+ and into 1 on E_-. This actually can be arranged by the following construction which is familiar in spectral theory.

Step 5. Let $F = (E, \varphi, E)$ be given by a linear clutching function $\varphi = az + b$. For each $p \in P$ let

$$Q_p : E_p \to E_p$$

be the endomorphism defined by the integral

$$Q_p = \frac{1}{2\pi i} \int_{|z|=1} \varphi_p(z)^{-1}\, d\varphi_p . \qquad (2.55)$$

Then Q_p is seen to be a projection operator $Q_p{}^2 = Q_p$, which decomposes E into the direct sum of bundles

$$E = E_+ \oplus E_-, \qquad E_{+p} = Q_p E_p \tag{2.56}$$

corresponding to the subspaces where $\varphi_p(z)$ becomes singular with $|z| < 1$ and with $|z| > 1$, respectively. By shrinking the constant form b to zero on E_+, and the linear term to zero on E_- one then establishes an isomorphism:

$$F \cong \pi^{-1}E_+ \otimes [H^{-1}] \oplus \pi^{-1}E_-$$

from which it follows that

$$\alpha(F) = -[E_+]. \tag{2.57}$$

These steps carried out carefully now lead to the periodicity. Roughly speaking one has here replaced the deformations of the Morse theory by two types of elementary deformations. First the deformation furnished by the Fourier expansions to get into the realm of algebra, and then the deformations furnished by the finite dimensional spectral theorem.

This proof has several advantages over my earlier one. First of all one notices, that the procedure just described extends, practically word for word, to two quite different generalizations of the periodicity theorem. The first of these extends the formula (2.44) to the case when $Q \xrightarrow{\pi} P$ is a *twisted product of P with S^2* of the following type. Let L be a line-bundle, i.e., a 1-dimensional complex vector bundle over P. Now from the direct sum $L \oplus 1$ over P, and finally let $Q = \mathbf{P}(L \oplus 1)$ be the projectivisation of $L \oplus 1$. Thus the fiber of Q at P consists of the one dimensional subspaces of $L_p \oplus 1_p$. With this understood one has the following:

THOM THEOREM IN KU-THEORY: *As a module over $KU(P)$, the ring $KU\{P(L+1)\}$ is given by*:

$$KU\{P(L+1)\} \simeq KU(P)[t]/(t-1)([L]\,t-1). \tag{2.58}$$

Another extension to which the same procedure is applicable but which I will not discuss here is when a fixed compact groups acts on all the spaces in sight [18].

Let me now finally indicate the framework of ideas in which Atiyah has recently [12] constructed the most penetrating proof of this theorem. He was led to this point of view by his work with Singer on the general index problem for elliptic operators. In their final solution of this question K-theory and the periodicity played an essential role, so that he was led to try for a proof of the periodicity in this framework. The argument runs as follows. First of all recall that if $\mathscr{H}$ is an ∞-dimensional separable Hilbert space over $\mathbf{C}$, then a bounded linear transformation

$$T : \mathscr{H} \to \mathscr{H}$$

is called a *Fredholm operator* if its kernel and cokernel are finite dimensional.

Such an operator therefore has a natural index attached to it:

$$\operatorname{index}(T) = \dim \ker T - \dim \operatorname{coker} T, \tag{2.59}$$

which is classically well known to be invariant *under perturbations* of T. This led Atiyah to interpret the index of T as the element

$$[\ker T] - [\operatorname{coker} T] \in K(\text{point})$$

and then to extend the classical invariance property of the index to construct a natural map *from a family of Fredholm operators* $T = \{T_p\,, p \in P\}$ to an element index $(T) \in KU(P)$. Put differently, let $\mathscr{F}$ denote the subspace of Fredholm operators in the Banach algebra A of bounded linear transformations on $\mathscr{H}$. Then this extended index furnishes one with a map

$$\operatorname{index} : [P, \mathscr{F}] \to KU(P). \tag{2.60}$$

Actually one can push on now and using Kuiper's theorem [53] that the unitary group of $\mathscr{H}$ is contractible *prove that* $\mathscr{F}$ *actually classifies* KU as was done by Palais [60], and Janich [49] so that this index *finally furnishes one with an isomorphism*

$$[P, \mathscr{F}] \simeq KU(P). \tag{2.61}$$

In this framework, Atiyah constructs the operation α of (2.50) in the following manner. We again start with a bundle $F = (E, \varphi, E)$ over $P \times S^2$. Next let $\mathscr{H}$ be the Hilbert space of square integrable functions on the circle $| z | = 1$; let $\mathscr{H}_0 \subset \mathscr{H}$ be the closed subspace generated by the function z^m, $m \geqslant 0$, and write

$$M : \mathscr{H} \to \mathscr{H}_0$$

for the orthogonal projection of $\mathscr{H}$ on $\mathscr{H}_0$.

Now for each $p \in P$, let

$$T_p : E_p \otimes \mathscr{H}_0 \to E_p \otimes \mathscr{H}_0 \tag{2.62}$$

be defined as the composition

$$T_p = M \circ \text{multiplication by } \varphi_p(z).$$

The nonsingularity of $\varphi_p(z)$ for $|z| = 1$ now leads to the result that T_p is a Fredholm operator, so that the T_p constitute a family of Fredholm operators $T(\varphi) = \{T_p\}$ on the family of Hilbert spaces $E_p \otimes \mathscr{H}_0$, $p \in P$. Now the earlier index function is easily seen to extend to this situation also, to yield an element index $(T) \in KU(P)$. With this understood, simply set

$$\alpha(E, \varphi, E) = \text{index } T. \tag{2.63}$$

Thereafter the proof follows quite easily from general properties of the index and KU.

In short, within this framework one not only obtains the rather attractive model, $\mathscr{F}$, for BU; one also finds a more natural definition of the crucial operator α from $KU(P \times S^2)$ to $KU(P)$.

There remains of course the question of how Atiyah was led to precisely this construction of α. To gain some insight into this, consider the case $P =$ point, and let us compute $\alpha(1, z, 1)$. In the basis for $\mathscr{H}_0$ furnished by the function $\{z^i\}$ $i = 0, 1, \ldots$, the operator T simply takes the form: $Tz_i = z_{i+1}$. Thus index $(T) = -1$. Similarly one finds that the index of the operators T corresponding to $(1, z^m, 1)$ is m, and so by the invariance under homotopy,

$$\alpha(1, \varphi(z), 1) = -\text{winding number of } \varphi(z), \qquad |z| = 1. \tag{2.64}$$

Thus the general α may be considered as the natural generalization of the winding number of a curve in the plane, and this was in a sense also the guiding principle of our earlier elementary proof.

3. Some applications

The state of the art in topology has traditionally been tested on certain problems which arise in geometry and I would now like to report briefly, but in some detail, on the performance of K-theory in this arena.

Let me start then with parallelizability questions, for which K-theory is clearly pertinent because it is fashioned out of the functors Vect_n. The general problem here is to determine the position of the tangent bundle $T(M)$ in $\mathrm{Vect}_m(M)$ when M is a connected compact differentiable manifold of dimension m. For instance, the vector field problem deals with the question of how many trivial bundles can be split off from $T(M)$. Thus here one seeks to find the largest integer k so that

$$T(M) \simeq k \cdot 1 \oplus E, \tag{3.1}$$

with $k \cdot 1$ denoting the k-fold direct sum of the trivial bundle 1. More generally one can ask for the existence of decompositions

$$T(M) \simeq E_1 \oplus E_2 \oplus \cdots \oplus E_n, \tag{3.2}$$

with the E_j restricted in some specific manner.

When $T(M)$ trivializes completely we call M parallelizable so that a natural starting point is to decide which simple compact manifolds are parallelizable. Now if $T = T(M)$, is to be trivial, then

$$t = [T] - m[1] \in \tilde{K}O(M)$$

must be the zero element. Hence if both $\tilde{K}O(M)$ and t can be computed, then this condition immediately furnishes nonparallelizability theorems. Of course, note that t might well be zero, without T being isomorphic to $m \cdot 1$. Indeed $t = 0$ means simply that

$$T \oplus l \cdot 1 \simeq (m + l) \cdot 1 \qquad \text{for some integer } l.$$

For instance, in the case of the n-sphere, S^n, the imbedding $S^n \in \mathbf{R}^{n+1}$ with obviously trivial normal bundle immediately shows that

$$T(S^n) \oplus 1 \simeq (n + 1) \cdot 1$$

so that this method fails completely for the spheres.

The situation is vastly different on the real projective space $\mathbf{R}P_n$. Indeed, let S denote the sub-bundle of $\mathbf{R}P_n$, which we already defined for all Grassmannians. In this case S is clearly a line bundle. Furthermore, if

$$0 \to S \to \tilde{\mathbf{R}}^{n+1} \to Q \to 0 \tag{3.3}$$

is the exact sequence defining S, then it is not hard to check that

$$T(\mathbf{R}P_n) \simeq S \otimes Q.$$

Tensoring with S in (3.3) therefore yields the exact sequence

$$0 \to S^2 \to S \otimes \tilde{\mathbf{R}}^{n+1} \to T(\mathbf{R}P_n) \to 0. \tag{3.4}$$

On the other hand, $S^2 \simeq 1$ for every real line bundle, as is not hard to see—by choosing a Riemannian structure on the fibers, for instance. Hence the sequence (3.4) implies that in $\widetilde{KO}(\mathbf{R}P_n)$ the tangent class is given by:

$$[T] = (n+1)[S] - [1],$$

and therefore

$$t = (n+1)\{[S] - [1]\}. \tag{3.5}$$

Now $\widetilde{KO}(\mathbf{R}P_n)$ can be computed explicitly, and the answer is as follows:

THEOREM 3.6. *Let a_n denote the dimension of the real* Spin *representation of* Spin(n). *Thus the first* 8 *values of a_n are*

$$\{1, 2, 4, 4, 8, 8, 8, 8\}$$

and $a_{n+8} = 16a_n$. Then

$$\widetilde{KO}(\mathbf{R}P_{n-1}) = \mathbf{Z}/a_n\mathbf{Z} \tag{3.7}$$

with generator $\zeta = [S] - [1]$.

The proof of this theorem is not easy. The result was first noted by A. Shapiro and myself, in connection with our description of the generators of $\pi_k(O)$ in the Spin-group [16], but a complete proof was only published by Adams [2] in his work on the vector field problem on the spheres. Essentially he threw the book at it, using the spectral sequence and comparison exact sequences linking KU and KO.

Now granting (3.7) one sees immediately that $t = 0$ if and only if a_n divides $(n + 1)$. This occurs only for $n = 1, 2, 4, 8$. Hence one has the corollary:

COROLLARY (3.8). $\mathbf{R}P_k$ *is parallelizable only for* $n = 1, 3, 7$.

Note that this same computation also solves the question of the possible dimensions of division algebras over $\mathbf{R}$.

Indeed, if $\mathbf{R}^k$ is a division algebra over $\mathbf{R}$ the multiplication

$$\mu : \mathbf{R}^k \times \mathbf{R}^k \to \mathbf{R}^k,$$

when restricted to $S^{k-1} \times \mathbf{R}^k$ yields a map

$$\mu' : S^{k-1} \times \mathbf{R}^k \to \mathbf{R}^k$$

which clearly has the property that

$$\mu'(-x, y) = -\mu'(x, y)$$

and is furthermore linear in y. This, after a little reflection, is seen to imply that

$$n \cdot S \simeq n \cdot 1, \qquad \text{over } \mathbf{R}P_{n-1}\,. \tag{3.9}$$

Thus, again a_n must divide n and so one arrives at the

COROLLARY 3.10. *Division algebras over* $\mathbf{R}$ *occur only in dimensions* 1, 2, 4, 8.

The original proofs, both of Kervaire and of Milnor, derived these theorems by a much more conceptually complicated route from the periodicity theorem for U.

Let me now very briefly indicate other techniques which have led to results in these questions when the above most direct approach fails.

We first ask quite generally, when an element $x \in K(X)$ (here K denotes either KO or KU) is representable by an n-dimensional bundle: i.e., *when does there exist a bundle E, with fiber-dimension n such that*

$$[E] = x \quad \text{in } K(X)\,?$$

Now clearly $\lambda^i E = 0$ for $i > n$ for such "honest" n-bundles. Hence an immediate necessary condition on x is that the $\lambda^i(x) = 0$ for $i > n$.

Put differently, if *k trivial bundles can be split off* $E \in \mathrm{Vect}_n(X)$, *then in* $K(X)$:

$$\lambda^i\{[E] - k[1]\} = 0 \qquad \text{for } i > n - k. \tag{3.11}$$

Now by the definition of the λ^i:

$$\lambda_t k[1] = \{\lambda_t[1]\}^k = (1 + t)^k \tag{3.12}$$

and therefore

$$\lambda_t\{[T] - k[1]\} = \lambda_t[T]/(1 + t)^k,$$

so that the vanishing of the λ^i in high dimensions *expresses a nontrivial condition on* $[T]$ *in* $K(X)$.

In this way the λ^i can be used to generate obstructions to splitting off trivial bundles from a bundle E whose K-class is known. More generally one can treat the following question.

Let $\lambda : H \to GL(n, \mathbf{R})$ be a homomorphism. One can then ask what the obstruction to reducing the structure group[9] of a bundle E in $\mathrm{Vect}_n(X)$ to H (relative to λ) is. Now given any representation, say Δ, of H:

$$\Delta : H \to GL(m, \mathbf{R})$$

such a reduction $\tilde{E}$ of E furnishes one with a new bundle, $\Delta(\tilde{E})$, in $\mathrm{Vect}_m(X)$, and one can—by purely *representation theoretic means* compute identities which link certain functions of $\Delta(\tilde{E})$ with $\lambda^i[E]$. Thus knowing only $[E]$ and the $\lambda^i[E]$ one may, in certain instances, conclude that there is no element at all in $K(X)$ which satisfies these identities. In that case then, $\Delta(\tilde{E})$ can not exist, that is, $[E]$ cannot be represented by a bundle E which admits a reduction to H.

A notable example of this procedure is, for instance, the following recent nonimmersion theorem of Feder [44]:

THEOREM 3.13. *Let* $\alpha(n)$ *denote the number of* 1's *in the* diadic *expansion of n. Then* $\mathbf{C}P_n$, $n > 3$, *n odd, cannot be immersed in* $\mathbf{R}^{4n-2\alpha(n)}$.

Similar results were obtained earlier by Atiyah–Hirzebruch; however, there again the original approach was not purely K-theoretic but also involved the comparison of $K(X)$ with $H^*(X)$. On the whole, a systematic exploitation of these representation theoretic methods still seems to be missing.

Before proceeding to our main application, a word is in order as to why these procedures apply to immersion problems. First recall that an immersion $f : M \to \mathbf{R}^n$ is a smooth map whose differential

$$df : T(M) \to f^{-1}T(\mathbf{R}^n)$$

[9] We refer the reader to Steenrod's text of fiber bundles for a definition of this notion.

is injective at each point. Now because $T(\mathbf{R}^n) = n \cdot 1$, it follows that an immersion produces an exact sequence

$$0 \to T(M) \to n \cdot 1 \to Q \to 0$$

over M with $\dim Q = n - m$. *Thus if an immersion in* $\mathbf{R}^n$ *is possible, then the class* $[T(M)] - n[1]$ *in* $K(X)$ *must have a representative of dimension* $(n - m)$, and so one is back to the same sort of "desuspension question".

In conclusion I would now like to describe in greater detail how the KU functor leads to a solution of the Hopf conjecture. I have chosen this application, because the vector field problem is solved by similar though more elaborate techniques and the index problem is such a long and interesting story that I would hardly do it justice here. Rather, a whole set of these lectures should some day be devoted to it.

The question is simply this: *Which of the spheres,* S^m, $m > 0$ *are group objects in homotopy theory?* Put differently, which S^m admit a law of multiplication

$$f : S^m \times S^m \to S^m, \tag{3.14}$$

which obeys the group axioms up to homotopy? Note that one may also consider this question to be the natural extension of the $\mathbf{R}$-division algebra problem to topology, because the law of multiplication of such an algebra in $\mathbf{R}^n$ easily induces a homotopy group-law on S^{n-1}.

Now an immediate consequence of the existence of a homotopy unit $* \in S^m$, is that if

$$S^m \underset{i_2}{\overset{i_1}{\rightrightarrows}} S^m \times S^m$$

are the inclusions $p \to (p, *)$ and $p \to (*, p)$, respectively, then

$$p \to (p, *) \cdot p \to (*, p)$$

$$i_1 \circ f \sim i_2 \circ f \sim 1. \tag{3.15}$$

Conversely it is not hard to show that if an f subject to (3.15) can be found, then it induces an H structure on S^m.

A first stab at determining the values of n for which such f cannot be found is therefore simply to apply KU^* to both (3.14) and (3.15) and see whether a contradiction arises for certain values of m.

Now recall that we have already found $KU(S^{2n})$ to be $\mathbf{Z} \oplus \mathbf{Z}$, and we may choose as generators for this group the trivial bundle 1 and an "interesting" virtual bundle η_n of dimension zero, generating $\widetilde{KU}(S^{2n})$. Thus

$$KU(S^{2n}) = \mathbf{Z} \cdot 1 \oplus \mathbf{Z}\eta_n . \tag{3.16}$$

Next consider $KU(S^{2n} \times S^{2m})$. We of course have our tensor product map:

$$KU(S^{2n}) \otimes KU(S^{2m}) \to KU(S^{2n} \times S^{2m}) \tag{3.17}$$

and I claim that by using the already alluded to fact that:

$$S^{2n} \times S^{2m}/S^{2n} \vee S^{2m} = S^{2(n+m)}, \tag{3.18}$$

it follows directly from the periodicity theorem and the exact sequence of KU-theory, that (3.17) is an isomorphism. Thus

$$KU(S^{2n} \times S^{2m}) = \mathbf{Z}1 + \mathbf{Z}1 \otimes \eta_m + \mathbf{Z}\eta_n \otimes 1 + \mathbf{Z}\eta_n \otimes \eta_m , \tag{3.19}$$

and the sign of the η_n, $n = 1, 2, \ldots$, can be chosen so that under the map

$$S^{2n} \times S^{2m} \xrightarrow{\tau} S^{2(n+m)}$$

induced by (3.18),

$$\tau^* \eta_{n+m} = \eta_n \otimes \eta_m . \tag{3.20}$$

So much for the additive structure of the groups involved. To compute the ring structure of $KU(S^2)$ recall our generator $H = (1, z^{-1}, 1)$ of the previous section.

Following out explicitly our *linearization procedure* for the bundle $H^{-2} = (1, z^2, 1)$ then easily yields the identity:

$$([H] - 1)^2 = 0. \tag{3.21}$$

Thus if we choose $\eta_1 = (H - 1)$, we obtain $\eta_1^2 = 0$. Hence by (3.20) and induction it follows that quite generally:

$$\eta_n^2 = 0, \qquad \eta_n \in \widetilde{KU}(S^{2n}). \tag{3.22}$$

Returning to our problem, assume first that m is even, $m = 2n$. Applying f^* to (4.1) therefore must result in a formula of the type:

$$f^* \eta_n = a\eta_n \otimes 1 + b_1 \otimes \eta_n + e\eta_n \otimes \eta_n , \tag{3.23}$$

with a, b, c integers. On the other hand (3.15) and (3.20) clearly imply that $a = b = 1$.

Let us next compute the square of $f^*\eta_n$. From the commutativity of the tensor product one immediately sees that the answer is $ab\eta_n \otimes \eta_n = \eta_n \otimes \eta_n \neq 0$. On the other hand $\eta_n^2 = 0$ and therefore $f^*\eta_n^2 = 0$. Hence (3.23) is impossible and n cannot be even.

Assume therefore m odd, $m = 2n - 1$. If one tries a similar argument in this case no contradiction is found and so at first sight all odd spheres seem to admit an H-structure, as far as the ring structure of KU theory is concerned. By the way the ring structure of the classical theory, H^*, also only serves to eliminate the even case.

To proceed further, in either case we have to make a geometrical construction which in some sense crystalizes the implication of (3.15) on f. This construction, which goes back to Hopf [47], converts f into a homotopy element $h(f) \in \pi_{4n-1}(S^{2n})$ and was used by Hopf to describe the first known nontrivial elements in the higher homotopy of the spheres. The construction is best indicated by the diagram:

$$\begin{array}{ccc} CS^m \times S^m & \xrightarrow{h_+} & CS^m \\ \cup & & \cup \\ S^m \times S^m & \xrightarrow{f} & S^m \\ \cap & & \cap \\ S^m \times CS^m & \xrightarrow{h_-} & CS^m \end{array} \tag{3.24}$$

where C denotes the cone as before. What is described here, is that the map f extends to a map $h_- : S^n \times CS^n \to CS^n$ by simply sending $(p \times (t, q))$ to $(t, f'p \times q)$, with t the parameter in $[0, 1]$ describing a point of CS^n. Similarly f extends to h_+ above, and these two combine to give a map

$$h(f) : S^{2m+1} \to S^{m+1} \tag{3.25}$$

because $CS^m \times S^m \cup S^m \times CS^m$ clearly yields S^{2m+1}.

Now Hopf's brilliant conclusion concerning $h(f)$ was, that the condition (3.15) on f implies that inverse images of two generic points in S^{n+1} under $h(f)$ "link" with "linking-number" 1. Note that when $n = 1$, then

$$h(f) : S^3 \to S^2$$

and generic inverse images will be circles so that "linking" can in that case be understood in its most prosaic sense. Thereafter he showed that the linking number $l(g)$ of any map

$$g : S^{4n-1} \to S^{2n}$$

is a homotopy invariant of g which is zero on the trivial class. In fact $g \to l(g)$ induces a homomorphism

$$\pi_{4n-1}(S^{2n}) \to \mathbf{Z}$$

called the Hopf invariant.

In particular then the element $h(f)$ has Hopf-invariant one, for any H-space structure f on S^n so that f generates an infinite cyclic subgroup of $\pi_{4n-1}(S^{2n})$. For example, when $n = 1$, the group structure on S_1 thus shows that $\pi_3(S^2)$ has an infinite cyclic subgroup, and similarly the group S^3 of unit quaternions produces an element with Hopf invariant 1 in $\pi_7(S^4)$.

Hopf's linking argument was later translated into cohomology by Steenrod [68]. His construction is as follows:

Given a map $g : S^{4n-1} \to S^{2n}$, let

$$X_g = S^{2n} \underset{g}{\cup} e^{4n}$$

be the space obtained from S^{2n} by attaching a cell of dimension $4n$, via the attaching map g. We then clearly have $S^{2n} \subset X_g$ and $X_g/S^{2n} = S^{4n}$. From this it follows via the exact sequence that

$$H^{2n}(X_g) = \mathbf{Z}, \qquad H^{4n}(X_g) = \mathbf{Z}$$

while all other cohomology in dimension > 0 vanishes. Now let x, y be generators of these groups, respectively. Steenrod showed that the ring structure of X_g recaptures Hopf's linking number, in the sense that

$$x^2 = \pm l(g) \cdot y. \tag{3.26}$$

At this stage then, one has the implication: *If S^{n-1} is a group object, then there exists a 2-cell complex*

$$X = S^{2n} \underset{g}{\cup} e_{4n} \tag{3.27}$$

with cohomology generators $x \in H^{2n}(X)$, and $y \in H^{4n}(X)$, subject to the relation $x^2 = y$. Thus the Hopf conjecture can be settled if one shows that complexes X with the above cohomology ring simply do not exist unless $n = 1, 2$, or 4.

A first big step in this direction was taken by Adem in the 1950's, who showed that such complexes, X, could exist only if n is a power of 2. He was able to deduce this result from relations between the Steenrod operations $\{Sq^i\}$. Roughly the story is this: In the 1940's Steenrod made the fundamental discovery [67] that the squaring operation $Sq^i : H^i(P; \mathbf{Z}_2) \to H^{2i}(P; \mathbf{Z}_2)$ sending x to x^2, could be extended to a *natural* operation

$$Sq^i : H^n(P; \mathbf{Z}_2) \to H^{n+i}(P; \mathbf{Z}_2)$$

raising dimensions by i on all of $H^*(X; \mathbf{Z}_2)$.

Thereafter Adem [7] showed that certain relations existed between these and *in particular that the Sq^{2^k} generate all the Sq^i under composition.*

It follows immediately that no X of our type can exist unless n is a power of 2.

Adams finally settled the Hopf conjecture [6] by showing that even the Sq^{2^k}, with $k > 3$, are decomposable, but only by so called secondary operations. These are operations which are defined only on classes on which certain primary operations vanish, and have values in cosets of H^* which can again be described by primary operations. His argument is very deep and difficult.

Let us now tackle this same problem with the functor KU, and see whether the existence of a X of our type is compatible with the relations which exist between the operations λ^i in KU.

Now in principle it is easily seen that there exist universal relations expressing $\lambda^i \circ \lambda^j$ as a polynomial in the λ^k,s.

Indeed this is just a translation of the well known fact that the exterior powers of the standard representation of $U(n)$ are the basic irreducible representations of $U(n)$. However, the computation becomes rather involved. We will therefore switch to the slightly weaker but much more tractable operations ψ_k, first defined by Adams.

To define the ψ_k's, in terms of the λ^i,s, let

$$\psi_t = \sum_0^\infty t^i \psi^i, \qquad \psi^0(x) = \dim x$$

be defined in terms of our element $\lambda_t \in KU[[t]]$ by the formula:

$$\psi_t(x) = \psi^0(x) - t \frac{d}{dt} \log\{\lambda_{-t}(x)\} \qquad x \in KU(P). \tag{3.28}$$

Since all coefficients in this power series are integers the ψ's are well defined by (3.28). Furthermore the *logarithmic derivative* changes the basic formula

$$\lambda_t(x + y) = \lambda_t(x) \cdot \lambda_t(y),$$

into

$$\psi_t(x + y) = \psi_t(x) + \psi_t(y).$$

Thus the ψ^i are additive!

Let us next compute $\psi_t(x)$ when $x = [L]$ with L a line bundle. Then clearly

$$\lambda_t(x) = 1 + tx,$$

so that

$$\psi_t(x) = 1 + \frac{tx}{1 - tx} = 1 + tx + t^2x^2 + \cdots .$$

In short:

$$\psi^k(x) = x^k, \qquad \text{when } x = [L]. \tag{3.29}$$

Concerning the ψ^k operations one now has the following basic result:

PROPOSITION (3.30). *For $x, y \in KU(P)$*

$$\begin{aligned} \psi^k(x + y) &= \psi^k(x) + \psi^k(y) \\ \psi^k(x \otimes y) &= \psi^k(x) \otimes \psi^k(y) \\ \psi^k\{\psi^l(x)\} &= \psi^{kl}(x). \end{aligned} \tag{3.31}$$

If p is a prime then, $\psi^p(x) \equiv x^p \bmod p.$ (3.32)

If $\eta \in KU(S^{2n})$ has $\dim \eta = 0$ *then,*
$$\psi^k(\eta) = k^n\eta. \tag{3.33}$$

All but the last of these properties are immediately apparent from the following fundamental *splitting principle*, which states that:

Any two natural operations which agree on a direct sum of line bundles agree on all elements of $KU(P)$. (3.34)

Indeed, the additivity of the $\{\psi^k\}$ together with (4.16) immediately yield the next two formulas of Proposition (4.18) *on a direct sum of line bundles*. Hence the relations are true generally.

Concerning the proof of this splitting principle let me only say that it follows from an extension of our "Thom" Theorem (2.58) to twisted products, with projective space as fiber. This allows one to construct for every bundle E over P, a space $Y(E)$ over P, so that

(1) $KU(P)$ injects into $KU\{Y(E)\}$ and
(2) E pulled back to $Y(E)$ splits into line bundles.

This construction is again copied directly from a similar argument in the theory of characteristic classes, which was used with great success by Borel and Hirzebruch [20]–[22].

Finally note that when $n = 1$ the last formula follows directly from the formulas (3.21) and (3.29), that is from the relation:

$$\eta_1 = [H] - 1, \qquad \eta_1^2 = 0.$$

Indeed:

$$\begin{aligned}\psi^k\eta_1 &= [H]^k - [1]^k\\ &= \{[H] - 1\}(1 + [H] + \cdots + [H]^{k-1})\\ &= k([H] - 1).\end{aligned}$$

Finally, using induction on k and (3.20) yields the desired result.

We are now nearly in a position to tackle the problem with KU-theory. First, however, the Steenrod translation of the linking number has to be translated further into KU-theory. This leads to the following result:

Let X be of the "Steenrod type":

$$X = S^{2n} \cup_g e_{4n},$$

with linking number $l(g) = 1$. Let $X^j \to S^{4n}$ be the collapsing map $X \to X/S^{2n} = S^{4n}$, and let $y \in KU(X)$ be the element

$$y = j^*\eta_{2n}, \qquad \eta_{2n} \in KU(S^{4n})$$

with η_{2n} our earlier generator.

Next, let $i = S^{2n} \to X$ be the inclusion. From the exact sequence it then follows that there exists an element $x \in KU(X)$ such that

$$i^*x = \eta_n, \qquad \eta_n \in KU(S^{2n}).$$

The exact sequence further teaches one that $1, x, y$ now freely generate $KU(X)$ and that $i^*y = 0$. *Thus x is well defined modulo a multiple of y.*

Next consider the element x^2. We have $(i^*x)^2 = 0$, whence,

$$x^2 = Hy$$

for some integer H. *Now the crucial point in the translation of the Steenrod argument to KU-theory, is that linking number* 1 *forces H to be an odd number.*

The stage is now set for the "Postcard proof" of Adams–Atiyah [6].

Consider the elements $\psi^2(x)$ and $\psi^3(x)$. By Proposition (4.17) we must have:

$$\psi^2(x) = 2^n x + ay, \qquad \psi^3(x) = 3^n x + by.$$

Further since $\psi^2(x) \equiv x^2 \bmod 2$, a must be *odd*. Now from $\psi^k(y) = j^*\psi^k(\eta_n) = k^{2n}y$ we conclude that

$$\psi^6(x) = \psi^3\{\Psi^2(x)\} = 6^n x + (2^n b + 3^{2n} a)\, y$$

$$\psi^6(x) = \psi^2\{\Psi^3(x)\} = 6^n x + (2^{2n} b + 3^n a)\, y.$$

Hence $2^n b + 3^{2n} a = 2^{2n} b + 3^n a$ or $2^n(2^n - 1)b = 3^n(3^n - 1)a$. Finally, because a is odd, 2^n must divide $3^n - 1$ which by elementary number theory can happen only if $n = 1, 2$, or 4. Q.E.D.

4. Concluding remarks

The penalty for starting at the beginning is that one rarely gets to the end. In the present instance this means that I will not be able to do justice to the "work in progress" aspect of the subject.

Partly the difficulty is that by now K-theory is "standard equipment" for a topologist or a geometer and even of some enterprising analysts, so that interesting connections or applications crop up everywhere. The equivariant K-theory of Atiyah and Segal is a notable instance of this, with many connections of cobordism theory and its applications to transformation groups as studied so exhaustively by Conner and Floyd [35], Bredon [29], and others.

One of course also has a growing list of spaces whose K-theory is completely determined—maybe the most notable instance being the simply connected Lie groups whose KU^* ring is always an exterior algebra. This beautiful result is due to Hodgkins [45]. Anderson and Hodgkin have also studied the K-theory of the Eilenberg–MacLane spaces [10]. Here the results are rather negative, essentially there are no interesting or unexpected bundles on these spaces.

Finally, K-theory also plays a role in some of the central questions of present-day topology—that is, questions concerning the various types of geometric structures which can be imposed on a manifold. (See for instance [9].) Thus in Sullivan's "triangularability theorems at odd primes", K-theory plays a vital role. For example, he asserts that in this world of odd primes, a complex P satisfying Poincaré duality, admits a piecewise linear structure if and only if $K^*(P)$ also satisfies a duality condition quite analogous to Poincaré duality.

One may view these modern developments as concerning themselves with the structure groups of S^{n-1}, other than O_n. Thus one has the group PL_n, of piecewise linear automorphisms of S^{n-1}, the group Top_n of topological automorphisms and finally the monoid G_n of homotopy-equivalences of S^{n-1}. Now in the same manner in which the sequence $\{O_n\}$ gave rise to KO-theory the other families just defined give rise to representable functors KPL, $KTOP$, and KG respectively, which are the basic objects in the obstruction theory of geometric structures. These theories are obviously linked by natural homomorphisms

$$KO \to KPL \to KTOP \to KG$$

and the composition of these arrows

$$J : KO \to KG,$$

is referred to as the J-homomorphism. Further, the image of $KO(P)$ under J is denoted by $J(P)$, and this object is of great interest in many ways. The group $J(P)$ is always finite and one of the most exciting recent endeavors has been the *attempt to compute* $J(P)$ *in terms of* $KO(P)$ *as a* ψ^k *module.* This work initiated and highly developed by Adams [3], [1] led him to the following:

CONJECTURE (Adams). *If* $x \in KO(P)$, *then*

$$k^m J\{(x) - \psi^k(x)\} = 0$$

for m large enough.

At the time of this writing the status of the Adams conjecture is a cloudy one. Certainly no proof accessible to an honest topologist has been published. What has happened is that in 1968 D. Quillen [63] published a most stimulating "plan of proof" which has its starting point the Frobenius automorphism in algebraic geometry, and leads to the Adams conjecture via voluminuous procedures which involve the Etalé Cohomology of Grothendieck on the one hand, and the localization theory of Artin–Mazur on the other.

Quillen's idea has since then been seriously developed by Sullivan, who steering his own course—influenced by Lubkin rather than Grothendieck—and using a more concrete form of localization than that of Artin and Mazur, seems now on the verge of writing down a proof of the Adams conjecture which at least some of us topologists might understand.

In any case it seems clear that this framework of ideas will have a far reaching effect on topology in the years to come.

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