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THE STABLE HOMOTOPY OF THE CLASSICAL GROUPS

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1. Introduction

Throughout this paper M shall denote a compact connected Riemann manifold of class C^∞. Let $\nu = (P, Q; h)$ be the triple consisting of two points P and Q on M together with a homotopy class h of curves joining P to Q. We will refer to such triples as *base points* on M.

Corresponding to $\nu = (P, Q; h)$ we define M^ν to be the set of all geodesics of *minimal* length which join P to Q and are contained in h.

There is an obvious map of the suspension of M^ν into M: one merely assigns to the pair (s, t), $s \in M^\nu$; $t \in [0, 1]$, the point on s which divides s in the ratio t to $1 - t$. (For fixed small $t > 0$, this map is 1 to 1 on M^ν and serves to define a topology on M^ν.) The induced homomorphism of $\pi_k(M^\nu)$ into $\pi_{k+1}(M)$ will be denoted by ν_*.

Let s be an arbitrary geodesic on M from P to Q. The index of s, denoted by $\lambda(s)$, is the properly counted sum of the conjugate points of P in the interior of s. We write $|\nu|$ for the first *positive* integer which occurs as the index of some geodesic from P to Q in the class h. In terms of these notions our principal result is the following theorem.

THEOREM I. *Let M be a symmetric space. Then for any base point ν on M, M^ν is again a symmetric space. Further, ν_* is onto in positive dimensions less than $|\nu|$ and is one to one in positive dimensions less than $|\nu| - 1$. Thus*:

$$\pi_k(M) = \pi_{k+1}(M) \qquad 0 < k < |\nu| - 1 . \tag{1.1}$$

As an example, let M be the n-sphere, $n \geqq 2$, and let $\nu = (P, Q)$ consist of two antipodes. (Because S^n is simply connected the class h is unique.) Then M^ν is the $(n-1)$-sphere, and $\nu_*: \pi_k(S^{n-1}) \to \pi_{k+1}(S^n)$ coincides with the usual suspension homomorphism. The integers which occur as indexes of geodesics joining P to Q, are seen to form the set $0, 2(n-1), 4(n-1)$, etc. Hence $|\nu| = 2(n-1)$, and (1.1) yields the Freudenthal suspension theorem. If $\nu = (P, Q)$ with Q not the antipode of P, then M^ν is a single point, while $|\nu|$ is seen to be $(n-1)$. In that case (1.1) merely implies that $\pi_k(S^n) = 0$ for $0 < k \leqq n - 2$.

At first glance the evaluation of $|\nu|$ may seem a formidable task.

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However on a symmetric space (see section 5) every pair of points (P, Q) is contained in a maximal flat geodesic torus T, and every index $\lambda(s)$ already occurs as the index of a geodesic joining P to Q on T. Further, for such a geodesic, $\lambda(s)$ is equal to the number of times s crosses the "singular" subtori of T. The disposition of these singular tori is well known. The computation of $|\nu|$ is therefore a routine matter.

Theorem I yields new results in the following manner: In view of the fact that with M the space M^ν is again symmetric, one may repeat the procedure of passing from M to M^ν. To facilitate the use of this iteration we will agree to call a sequence of symmetric spaces $\cdots M_1 \to M_2 \to M_3 \cdots$ a ν-sequence if at each step $M_i = M^\nu_{i+1}$ for some appropriate base point ν in M_{i+1}. For example, the sequence $\cdots S^n \to S^{n+1} \to S^{n+2} \cdots$ is a ν-sequence,

THEOREM II. *The following are three ν-sequences with the value of $|\nu|$ indicated at each step.*

$$\mathrm{U}(2n)/\mathrm{U}(n) \times \mathrm{U}(n) \xrightarrow{2n+2} \mathrm{U}(2n) \tag{1.2}$$

$$\begin{aligned} \mathrm{O}(2n)/\mathrm{O}(n) \times \mathrm{O}(n) &\xrightarrow{n+1} \mathrm{U}(2n)/\mathrm{O}(2n) \\ &\xrightarrow{2n+1} \mathrm{Sp}(2n)/\mathrm{U}(2n) \xrightarrow{4n+2} \mathrm{Sp}(2n) \end{aligned} \tag{1.3}$$

$$\begin{aligned} \mathrm{Sp}(2n)/\mathrm{Sp}(n) \times \mathrm{Sp}(n) &\xrightarrow{4n+1} \mathrm{U}(4n)/\mathrm{Sp}(2n) \\ &\xrightarrow{8n-2} \mathrm{SO}(8n)/\mathrm{U}(4n) \xrightarrow{8n-2} \mathrm{SO}(8n) \end{aligned} \tag{1.4}$$

Here we have used the standard notations and inclusions.

Notice that $|\nu|$ tends to ∞ with n at each step of these sequences. On the other hand it is well known that for each of the symmetric spaces involved, π_k becomes independent of $n \gg k$. (We will indicate these stable values of π_k by dropping the subscript n and using bold face type. For example, $\pi_k(\mathbf{U}/\mathbf{O}) = \pi_k\{\mathrm{U}(n)/\mathrm{O}(n)\}$ for $n \gg k$.) Finally, recall that in this notation $\pi_k(\mathbf{U}) = \pi_{k+1}(\mathbf{U}/\mathbf{U}\times\mathbf{U})$, $\pi_k(\mathbf{O}) = \pi_{+1}(\mathbf{O}/\mathbf{O}\times\mathbf{O})$ and $\pi_k(\mathbf{Sp}) = \pi_{k+1}(\mathbf{Sp}/\mathbf{Sp}\times\mathbf{Sp})$ $(k = 0, 1, \cdots)$, because in each instance the space on the right hand side represents the universal base space of the group in question. Combining these three observations with Theorem I, we obtain the following corollary to Theorem II.

COROLLARY. *The stable homotopy of the classical groups is periodic:*

$$\begin{aligned} \pi_k(\mathbf{U}) &= \pi_{k+2}(\mathbf{U}) \\ \pi_k(\mathbf{O}) &= \pi_{k+4}(\mathbf{Sp}) \\ \pi_k(\mathbf{Sp}) &= \pi_{k+4}(\mathbf{O}) \qquad k = 0, 1, \cdots \end{aligned} \tag{1.5}$$

The groups $\pi_k(\mathbf{U})$ are $0, Z$ for $k = 0, 1$. Hence $0, Z$ is the period of $\pi_*(\mathbf{U})$. In the case of **Sp**, one has the groups $0, 0, 0, Z$, for $k = 0, 1, 2, 3$ respectively. For **O** these first four groups are $Z_2, Z_2\, 0, Z$. Hence the period of $\pi_*(\mathbf{O})$ is $Z_2, Z_2, 0, Z, 0, 0, 0, Z$. Applying (1.3) and (1.4) one also obtains the stable homotopy of the other symmetric spaces. Thus:

$$\pi_k(\mathbf{Sp}/\mathbf{U}) = \pi_{k+1}(\mathbf{Sp}) \qquad k = 0, 1, 2\cdots \tag{1.6}$$
$$\pi_k(\mathbf{U}/\mathbf{O}) = \pi_{k+2}(\mathbf{Sp}) \qquad k = 0, 1, 2\cdots$$

while

$$\pi_k(\mathbf{O}/\mathbf{U}) = \pi_{k+1}(\mathbf{O}) \qquad k = 0, 1, 2\cdots \tag{1.7}$$
$$\pi_k(\mathbf{U}/\mathbf{Sp}) = \pi_{k+2}(\mathbf{O}) \qquad k = 0, 1, 2\cdots$$

(In the third formula we have replaced **SO**/**U** by **O**/**U** to obtain the correct value of π_0.)

The formulas (1.5) to (1.7) were already announced in [4]. The unitary groups were discussed by a different method in [5], where the unstable group $\pi_{2n}\{\mathrm{U}(n)\}$ was also evaluated as $Z/n!Z$.

The proof of Theorem I is summarized in this fashion: Let $\nu = (P, Q; h)$ be a base point, and let $\Omega_\nu M$ be the space of path from P to Q on M in the class h. We then construct a CW-model for $\Omega_\nu M$ which is of the form $K = M^\nu \cup e_1 \cup e_2$ etc., where the e_i are cells of dimension greater than or equal to $|\mu|$.

The existence of such a K follows readily from the Morse theory. For instance the deformations given in Seifert-Threlfall [10, pp. 34, 35] and can be interpreted us follows: Suppose that a smooth function f defined on a compact manifold N has a single nondegenerate critical point p, of index k in the range $a \leqq f \leqq b$, $a < f(p) < b$. Let N^a respectively N^b be the sets $f \leqq a$ and $f \leqq b$ on N. The assertion is, that then N^b is obtained from N^a by attaching a k-cell, e_k, to N^a. In symbols, $N^b = N^a \cup e_k$. (This point of view is also emphasized in notes by Pitcher [9], and R. Thom [12].)

To prove our theorem this interpretation of the Morse theory is first extended in two ways:

(A) The loopspace problem is reduced to the manifold problem.

(B) The notion of nondegeneracy is extended.

Thereafter it is shown that on a symmetric space the critical sets in the loopspace are nondegenerate for every choice of a base point.

The step (A) is already essentially contained in Morse [8]; while the notion of a nondegenerate critical manifold (step B) was introduced in [2][1].

The final step follows easily from the results of [6].

It is clear from this rough plan of the proof that considerable reviewing of more or less known material will be necessary to make the account intelligible. Because the theory of a nondegenerate function on a smooth manifold is by now well known, while some mystery still seems to hang over Morse's extension of this theory to loop spaces, we will review step (A) in greater detail than the other two steps.

2. Review of the Morse theory. A reduction theorem

Let $\mu = (P, Q)$ be any two points of M. The space of paths from P to Q on M is denoted by $\Omega_\mu M$ and is defined as follows:

DEFINITION 2.1. *The points of $\Omega_\mu M$ are the piecewise differentiable maps c: $[0,1] \to M$ which are parametrized proportionally to arc length, take 0 into P, and map 1 onto Q. The distance between two points c and c' in $\Omega_\mu M$ is given by:*

$$\rho_1(c, c') = \max_{t \in [0,1]} \rho\{c(t),\ c'(t)\} + |J(c) - J(c')|$$

where ρ is the metric on M, and J denotes the length function on $\Omega_\mu M$.

The advantage of this definition of $\Omega_\mu M$ is that $J(c)$, the length of c, is a continuous function of $\Omega_\mu M$. On the other hand $\Omega_\mu M$ is not complete.

If a is a real number, the subset of $\Omega_\mu M$ on which $J \leqq a$, is denoted by $\Omega_\mu^a M$, and is referred to as a half space of $\Omega_\mu M$. Such a half space is called regular if $\Omega_\mu^a M$ contains no geodesic of length a.

Let F be a continuous real valued function on a compact manifold N. The set $\{x \in N;\ F(x) \leqq a\}$ will be denoted by $F^a N$, or just N^a if the function is understood, and is also called a half-space for F on N. The half-space is called regular if F is of class C^∞ in some neighborhood of $F^a N$, and if F has no critical points at the level a. (In other words $dF(x) \neq 0$ if $F(x) = a$.)

The aim of this section is to show that every regular half space of $\Omega_\mu M$, is of the same homotopy type as a regular half-space of a manifold.

It turns out that if one steers a middle course between Morse and Seifert and Threlfall such a "model" for $\Omega_\mu^a M$ is easily constructed. We have just defined $\Omega_\mu M$ according to Seifert and Threlfall; for the rest

[1] The applications given in [2] are false, as was pointed out to me by A. S. Schwartz [11]. A distressingly simple example shows that the assertion [2, p. 253] to the effect that $\tilde{V}_{t,k}$ is a manifold is wrong. This mistake invalidates the computations for the circular connectivities of the n-sphere.

we follow, in spirit at least, Morse's account of thirty years ago.

Let φ_n: $M^n \to R$, be the function from the n^{th} cartesian product of M with itself, which assigns to $(x) = (x_1, \cdots, x_n)$ the number:

$$\varphi_n(x) = \rho^2(P, x_1) + \rho^2(x_1, x_2) + \cdots \rho^2(x_n, Q) .$$

were $\rho(x, y)$ denotes the distance between x and y on M, as before.

REDUCTION THEOREM I. *Let a be a positive number. Then there exists an integer n such that $\Omega_\mu^a M$ is of the same homotopy type as the half space $\varphi_n^b M^n$ of φ_n on M^n, where $b = a^2/n + 1$. Thus,*

$$\Omega_\mu^a M \approx \varphi_\mu^b M^n . \tag{2.1}$$

The statement (2.1) is new, although quite implicit in Morse's account. He, of course, did not have a definition of $\Omega_\mu M$ on which the length function was continuous. A slightly surprising technical phenomenon is that the function φ_n alone suffices to define a model for $\Omega_\mu^a M$. In Morse's original account, he essentially shows that $\Omega_\mu^a M$ is of the same homotopy type as the subset of M^n characterized by $\rho(x_i, x_{i+1}) < \bar{\rho}$; $\sum_{i=0}^{i=n} \rho(x , x_{i+1}) \leqq a$. (Here $x_0 = P$; $x_{n+1} = Q$).

PROOF OF (2.1). There exists a number $\bar{\rho} > 0$ such that two points of M with distance less than $\bar{\rho}$ have a unique shortest geodesic joining them. This shortest geodesic then varies smoothly with the end points, in particular $\rho^2(x, y)$ is a C^∞ function of x and y as long as $\rho(x, y) < \bar{\rho}$. Suppose now that n is chosen so large that:

$$a/\sqrt{n+1} < \bar{\rho} . \tag{2.4}$$

Under this condition on n we define maps α: $\Omega_\mu^a M \to \varphi_n^b M$ and β: $\varphi_n^b M^n \to \Omega_\mu^a M$ which constitute a homotopy equivalence. (For convenience we write φ for φ_n and denote $\varphi_n^b M^n$ by M_*^b in the sequel.)

DEFINITION OF α. Let $c \in \Omega_\mu^a M$. Then $\alpha(c) \in M^n$ is to be the point:

$$\alpha(c) = \{c(t_1), c(t_2), \cdots, c(t_n)\} ; \qquad t_i = i/n + 1 ;$$

Clearly α is a continuous function from $\Omega_\mu^a M$ to M^n. Next, $\varphi(\alpha c) = \sum_{i=0}^{i=n} \rho^2 \{c(t_i), c(t_{i+1})\}$. Each term of this sum is $\leqq (a/n+1)^2$ because c is parametrized proportionately to arc-length. Hence $\varphi(\alpha c) \leqq (a^2/n+1) = b$. The map α therefore take values in M_*^b.

DEFINITION OF β. If $x = (x_1, \cdots, x_n)$ is a point of $M_*^b = \varphi^b M^n$, then each of the numbers, $\{\rho(P, x_1), \rho(x_1, x_2) \cdots \rho(x_{n+1}, Q)\}$ is less than $a/\sqrt{n+1}$, hence less than $\bar{\rho}$. The unique geodesics joining consecutive points of the array $P, x_1, \cdots, x_n, Q$ are therefore well defined and combine to yield a curve, c, in $\Omega_\mu M$. By the Cauchy inequality the length of c does not exceed a. The correspondence $x \to c$ defines the map β.

LEMMA 2.1. *There exists a homotopy D_t, $0 \leqq t \leqq 1$ of $\Omega_\mu^a M$ on itself such that D_0 is the identity, and $D_1 = \beta \circ a$.*

The needed deformation is given explicitly in [10, p. 51]. One deforms the segment of c between t_i and t_{i+1}, into the geodesic chord joining $c(t_i)$ to $c(t_{i+1})$. The intermediate curves are geodesic segments from $c(t_i)$. $c(t_i + \varepsilon)$ followed by the original curve from $t_i + \varepsilon$ to t_{i+1}.

LEMMA 2.2. *There exists a homotopy Δ_t, $0 \leqq t \leqq 1$, of M_*^b on itself, such that Δ_0 is the identity, and, $\Delta_1 = \alpha \circ \beta$.*

This homotopy is to be found in Morse [8, p. 217]. If $x \in M_*^b$, $\beta(x)$ is a polygonal curve joining P to Q. Let c: $[0,1] \to M$ the parametrization of $\beta(x)$ which is proportional to arc length. Let $0 \leqq a_1, < \cdots, a_n \leqq 1$, be the pre-images under c of the points $x = \{x_1, \cdots, x_n\}$ on $\beta(x)$. The $\{a_i\}$ then correspond to the parameter values of the original vertices on $\beta(x)$. The composition $a \circ \beta$ takes x into $\{c(t_1), c(t_2), \cdots, c(t_n)\}$ where $t_i = i/n+1$. Hence if $a_i = t_i$, then the $a \circ \beta(x) = x$, and what is needed is a "universal" homotopy which takes the points a_i into the points t_i. The natural way of constructing this homotopy is to dispatch a_i on its way to t_i at a linear speed proportional to the distance to be traversed. In formulas, let

$$\begin{aligned} a_0^\tau &= 0 \\ a_0^\tau &= a_i(1-\tau) + \tau\, t_i\,, \qquad 0 \leqq \tau \leqq 1;\ i = 1, \cdots, n \\ a_{n+1}^\tau &= 1 \end{aligned}$$

The homotpy Δ_τ assigns to x the point $\{c(a_i^\tau)\}$ where $c = \beta(x)$. Clearly the a_i vary continuously with x for $x \in M_*^b$, so that Δ_τ is a proper homotopy. It remains to be checked that Δ_τ keeps M_*^b invariant. For this purpose it is sufficient to prove that $\varphi(\Delta_\tau x) \leqq \varphi(x)$; $0 \leqq \tau \leqq 1$.

Let $J(x)$ be the length of $\beta(x)$, and set $\delta_i = J(x)(a_i - a_{i-1})$. Thus $\sum_{i=1}^{i=n+1} \delta_i = J(x)$, while $\sum_{i=1}^{i=n+1} \delta_i^2 = \varphi(x)$. We also write $\{x_i^\tau\}$ for the coordinates $\Delta_\tau x$. Then:

$$\rho(x_i^\tau, x_{i+1}^\tau) \leqq \delta_{i+1}(1-\tau) + \tau(J(x)/n+1)\ ,$$

because $\beta(x)$ is parametrized proportionally to arc length. Hence:

$$\varphi(\Delta_\tau x) \leqq \sum_{i=}^{i=n+1} [\delta_i(1-\tau) + \tau(J(x)/n+1)]^2\ .$$

After expanding, the right hand side is seen to equal

$$\varphi(x) - 2\tau(\varphi(x) - \{J^2(x)/n + 1\}) + \tau^2(\varphi(x) - \{J^2(x)/n + 1\})$$

By the Cauchy inequality $\varphi(x) - J^2(x)/n + 1 \geqq 0$. Hence in the range $0 \leqq \tau \leqq 1$, $\varphi(\Delta_\tau x) \leqq \varphi(x)$. This completes the proof of the lemma, and hence of (2.1).

The statement (2.1) has a refinement which will be formulated next. Its purpose is to relate certain geometric properties of the geodesics in $\Omega_\mu^a M$ with the critical points of φ on M_*^b. Recall first the notion of the *index of a critical point*. If p is a critical point of the smooth function φ on the manifold N, the Hessian of φ, denoted by $H_p\varphi$, is the bilinear symmetric function on the tangent space N_p of N at p, which in terms of local coordinates is defined by $H_p\varphi(\partial/\partial x_\alpha, \partial/\partial x_\beta) = \partial^2\varphi/\partial x_\alpha \partial x_\beta$. The index of p as a critical point of φ is by definition the dimension of a maximal subspace of N_p on which the Hessian is negative definite. This integer is denoted by $\lambda_\varphi(p)$. Finally we briefly review the notion of a conjugate point on a geodesic. For details the reader is referred to [8] and [6].

If $s(\alpha, t)$ is a smooth family of geodesics, depending on a parameter α, then the vector field $\partial s(\alpha, t)/\partial\alpha|_{\alpha=0}$ along $s(0, t)$ is called a J-field along $s = s(0, t)$. The totality of such vector fields along s, forms a vector space J_s over the real numbers. If the length of s is less than ρ, every V in J_s is uniquely determined by its values at the end-points of s. In general, if P and Q are two points of s, Q is called a conjugate point of P (along s) of multiplicity k if the subspace of J_s, consisting of the fields which vanish at both P and Q, is of dimension precisely k.

REDUCTION THEOREM II. *The homotopy equivalence* $\alpha\colon \Omega_\mu^a M \to M_*^b$ *constructed in the proof of* (2.1) *has the following properties*:

(2.2) *Under* α *the geodesics of* $\Omega_\mu^a M$ *are mapped one to one onto the critical points of* φ *on* M_*^b.

(2.3) *If* s *is a geodesics of* $\Omega_\mu^a M$ *and* p *is its image under* α, *then*:

The dimension of the nullspace of $H_p\varphi$ *equals the multiplicity of* Q *as a conjugate point of* P *along* s.

The index $\lambda_\varphi(p)$ *is equal to the number* (*counted with multiplicities*) *of conjugate points of* P *in the interior of* s.

Except for a minor technicality, (2.2) and (2.3) are the content of Morse's index theorem. See [8, p. 91]. The technicality in question is the following one. Let ψ be the function $\rho(P, x_1) + \rho(x_1, x_2) + \cdots + \rho(x_n, Q)$. This function is smooth provided that no two consecutive coordinates coincide. Thus, except in a trivial case, the function ψ is smooth near the point p of (2.3), and, as will be shown in a moment, p is also a critical point of ψ. If in (2.3) we replace $\lambda_\varphi(p)$ by $\lambda_\psi(p)$ we obtain the statement of Morse. Note however that (2.2) with φ replaced by ψ is not true. Indeed, the critical sets of ψ are cells obtained by sliding the vertices along a given geodesic.

To prove our theorem it is therefore sufficient to establish (2.2) and the equality of $\lambda_\varphi(p)$ with $\lambda_\psi(p)$.

PROOF OF (2.2). If s is a geodesic segment of $\Omega_\mu^a M$ then $\beta \circ \alpha(s) = s$. Hence α imbeds this set of curves in M_*^b, and it remains to identify the critical points of φ on this set. Let $x \in M^n$, let X be a tangent vector to M^n at x, and consider the derivative $X\varphi$ of φ in the direction X. The point x is critical if and only if $X\varphi = 0$ for all X in the tangent space at x. Suppose that x has the coordinates $(x_1, \cdots, x_n)$ and that X has the corresponding components $(X_1, \cdots, X_n)$ in the natural product structure of the tangent space to M^n at x. Let s_i denote the geodesic segment from x_i to x_{i+1}, where we now set $x_0 = P$, $x_{n+1} = Q$, and let $\dot{s}_i^1$, respectively $\dot{s}_i^0$, be the unit tangent vector of s_i at x_{i+1} and x_i. By the well known first variation formula:

$$X \cdot \rho^2(x_i, x_{i+1}) = 2 \,|\, s_i \,|\, \{\langle \dot{s}_i^1, X_{i+1}\rangle - \langle \dot{s}_i^0 X_i\rangle\} ,$$

where $<, >$ denotes the inner product of the Riemannian structure, and $|\, s_i \,|$ denotes the length of s_i one obtains the expression:

$$X\varphi = 2 \sum_{i=0}^{i=n-1} \langle \,|\, s_i \,|\, \dot{s}_i^1 - |\, s_{i+1} \,|\, \dot{s}_{i+1}^0 X_{i+1} \rangle .$$

The components X_i of X are independent. Hence $X\varphi = 0$ for all X if and only if $\dot{s}_i^1 = \dot{s}_{i+1}^0$; $|\, s_i \,| = |\, s_{i+1} \,|$; $i = 1, \cdots, n-1$. In other words x is a critical point if and only if $\beta(x)$ is a geodesic, and $\alpha \circ \beta(x) = x$. This completes the proof of (2.2).

PROOF OF (2.3). Let A be the tangent space M_p^n. By varying the vertices of p along s, we single out a subspace $A^\natural$ of A on which $H_p\varphi$ is clearly positive definite. It therefore suffices to study the restriction of $H_p\varphi$ to a suitable complement of $A^\natural$ in A. Such a complement is furnished by the elements $X = \{X_i\}$ in A with each X_i perpendicular to s. Let this complement be denoted by $A^\#$, and suppose $X, Y \in A^\#$. For each segment s_i choose J-fields U_i and V_i. so that at the end points s_i, U_i coincides with X_{i-1} and X_i, while V_i coincides with Y_{i-1} and Y_i. We write this condition in the form $U_i^+ = X_{i+1}$; $U_i^- = X_i$, etc. Because $|\, s_i \,| < \rho$, the U_i, V_i are uniquely determined by X and Y. Now by the second variation formula,

$$H_p\varphi(X, Y) = k \sum \langle \Delta U_i^+ - \Delta U_{i+1}^-, V_i^+ \rangle$$

where ΔU_i denotes the covariant derivative of U_i along s, and k is equal to $(2/n+1) \times$ length of $\beta(x)$. For the function ψ we obtain similiarly the expression

$$H_p\psi(X, Y) = \sum \langle \Delta U_i^+ - \Delta U_{i+1}^-, V_i^+ \rangle$$

Thus on $A^\#$ these two Hessians differ only by a positive factor. On the complementary subspace $H_p\psi$ vanishes. Hence $\lambda_\varphi(p) = \lambda_\psi(p)$ as was to be shown.

REMARK. These formulas immediately prove the first part of (2.3). Indeed, a vector X is in the null space of $H_p\varphi$ if and only if the J-fields U_i along s_i fit together to form a global J-field along s which vanishes at both P and Q. In this manner Morse obtains the formula for the null space of $H_p\varphi$. Concerning the index formula, let me just remark that Morse obtains it by deforming Q along s into P, and observing that the index form $H_p\psi$ does not change during this deformation except when Q passes through conjugate points of P. At such points the index is shown to decrease by precisely the multiplicity of the conjugate point.

The two reduction theorems complete our original program of assigning to every regular half space of $\Omega_\mu^a M$ a regular half space of a compact manifold which is of the same homotopy type. (The fact that regularity is preserved under α follows from (2.2)). We will call the set M_*^b constructed in this section a model for $\Omega_\mu^a M$. If $\nu = (P, Q\colon h)$ is a base point, $\Omega_\nu M$ denotes the component of h in $\Omega_\mu M$ and the image of $\Omega_\nu^a M$ under α will be called a model for $\Omega_\nu^a M$. It is clear that the reduction theorem holds equally well in this new setting.

3. Review of the Morse Theory. The nondegenerate case

The classification of critical points according to index and nullity has topological implications which are usually expressed by the Morse inequalities. Actually however this "homology formulation" is proved by homotopy arguments. It is better therefore to state these implications in the language of CW-complexes [13]. In this manner homology consequences are easily accessible while the homotopy implications are not lost. (See [9] and [12].)

DEFINITION 3.1. (See [2].) *Let V be a smooth connected submanifold of the regular half space $N^a = f^a N$. Such a manifold is called a nondegenerate critical manifold of f on N^a if:*

(3.1) *Each point of V is a critical point of f.*

(3.2) *For any $p \in V$, the nullspace of $H_p f$ is the tangent space of V at p.*

An immediate consequence of (3.2) is that $\lambda_f(p)$ is a constant on V.

This integer is the index of V, and is written $\lambda_f(V)$. If V reduces to a point, H_pf is non-singular by the condition (3.2). The present notion therefore generalizes the classical definition of a nondegenerate critical point.

Let V be a nondegenerate critical manifold of f on N^a. We define the *negative bundle*, ξ_V, over V in the following manner.

Let a Riemannian structure be defined on N. At each point $p \in V$ the form H_pf then uniquely determines a linear self-adjoint transformation T_p on the tangent space of N at p, by the formula,

$$(3.3) \qquad \langle T_pX, Y\rangle = H_pf(X, Y) \qquad X, Y \in N_p .$$

These transformations combine to define a linear endomorphism, T, of the tangent space to N along V. By condition (3.2) the kernel of T is precisely the tangent space to V. Thus T is an automorphism of the normal bundle of V in N.

Now let ξ_V be the subbundle of this normal bundle which is spanned by the negative eigendirections of T. Thus the fiber of ξ_V at $p \in V$ is spanned by the normal vectors to V at p, for which $T_p \cdot Y = \lambda Y$, $\lambda < 0$. The fiber of ξ_V therefore has dimension $\lambda_f(V)$. If $\lambda_f(V) = 0$, we set ξ_V equal to V. The bundle ξ_V is independent of the Riemannian structure used.

Finally, recall the notion of attaching a vector bundle ξ, to a space Y to form the space $Y \cup \xi$.

In general if $\alpha: A \to Y$ is a map of a subset $A \subset X$ one forms the space $Y \cup_\alpha X$ by identfying $a \in A \subset X$ with $\alpha(a) \varepsilon Y$ in the disjoint union Y with X.

This attaching construction has the following elementary properties:

(3.4) The homotopy type of $Y \cup_\alpha X$ depends only on the homotopy type of α.

(3.5) If (X_1, A_1) is a deformation retract of (X, A) and if $\alpha_1 = \alpha \mid A_1$, then $Y \cup_{\alpha_1} X_1$ is of the same homotopy type as $Y \cup_\alpha X$.

When X is an n-cell e_n, and A is the bounding sphere of e_n, $Y \cup_\alpha e_n$ is referred to as Y with the cell e_n attached. If ξ is an orthogonal n-plane bundle, we form the space $Y \cup \xi$, by taking, in the above procedure, X equal to the set D_ξ of vectors of length $\leqq 1$ and setting A equal to $S_\xi = \partial D_\xi$. In this case we speak of Y with ξ attached, and if α is not explicitly in evidence just use the notation $Y \cup \xi$. If ξ is a 0-dimensional vector-bundle $Y \cup \xi$ stands for the disjoint union of Y with the base-space of ξ.

With this notation and terminology understood, the principal result of the nondegenerate Morse theory can be stated as follows:

THEOREM III. *Suppose that $N^a \subset N^b$ are two regular half-spaces of the function f on the compact manifold N.*

(3.6) *If f has no critical point in the range $a \leqq f \leqq b$ then N^a is a deformation retract of N^b.*

(3.7) *If f has a single nondegenerate critical manifold V in the range $a \leqq f \leqq b$, then N^b is of the same homotopy type as N^a with the negative bundle of f along V attached:*

$$N^b = N^a \cup \xi_V$$

where ξ_V is the negative bundle of f along V.

Immediate consequences in homotopy, [13], are:

COROLLARY 1. *Under the assumptions of* (3.7):

$$N^b = N^a \cup e_1 \cup \cdots \cup e_s \tag{3.8}$$

where the cells e_i, $i = 1, \cdots, s$, have dimension $\geqq \lambda_f(V)$. In particular.

$$\pi_r(N^b, N^a) = 0 \qquad \text{for } 0 \leqq r < \lambda(V)\,. \tag{3.9}$$

Using excision and Poincaré duality (3.2) implies:

COROLLARY 2. *Under the assumptions of* (3.7)

$$H^r(N^b, N^a; G) \approx H^r_c(\xi_V; G) = H^{r-\lambda}(V; G') \qquad \lambda = \lambda_f(V)\,. \tag{3.10}$$

Here the subscript c denotes compact cohomology, and by G' we mean the tensor of the coefficients G by the orientation sheaf of ξ_V.

REMARKS. In [2] we derived (3.10) with G specialized to Z_2. In this paper we will need only (3.9) but it seemed to me that (3.7) summarizes the situation better than any of the other versions. Remark that (3.10) implies (3.9) *if N^a is assumed to be simply connected*. On the other hand (3.8) yields (3.9) without this troublesome hypothesis.

The restriction that V be the only critical set of f in the range from a to b is not essential. If all the critical sets are nondegenerate, they are necessarily finite in number, so that if we denote them by V^i: $i = 1 \cdots s$; then Theorem III is easily modified to yield the formula

$$N^b = N^a \cup \xi_{V_1} \cup \cdots \cup \xi_{V_s}\,.$$

If N^a is triangulated, the attaching map of cell e_k can be deformed into the ($\dim e_k - 1$)-skeleton of N^a. In this way N^b becomes a CW-complex.

The case when V is a point, p, is completely treated in [10]. The present extension is best summarized by saying that what is done for a neighborhood of p in [10] can equally well be done in a normal neighborhood of V in the present case. On each fiber of such a neighborhood one encounters the nondegenerate critical point problem.

PROOF OF 3.6. Let N^b be endowed with a Riemann structure and denote the gradient of f corresponding to this structure by ∇f. If $p \in \overline{N^b - N^a}$, L_p shall denote the integral curve of $-\nabla f$ through p in its natural parameter. Because $df \neq 0$ on this set L_p is well defined. Further because $\overline{N^b - N^a}$ is compact, $|\nabla f| > \varepsilon_0 > 0$ on this set. Hence each L_p intersects $f^{-1}(a)$ at some point, say $h(p)$, and the function $p \to h(p)$ defines $f^{-1}(a)$ as a retract of $\overline{N^b - N^a}$. By assigning to p the point $h_t(p)$ on L_p which divides the segment from p to $h(p)$ in the ratio $1 : 1-t, f^{-1}(a)$ is seen to be a deformation retract $\overline{N^b - N^a}$. Hence (3.6) is true.

NOTE. The critical values of f form a closed set. Hence $N^{a-\varepsilon}$ is again a regular half-space of f when $\varepsilon > 0$ is small enough. Using this additional space it is easily seen that under the conditions of (3.6) N^b and N^a are in fact homeomorphic.

PROOF OF 3.7. We may assume that $f(V)=0$, and that f has no critical points in the range $[(-\varepsilon_0, 0) ; (0, \varepsilon_0)]$. It is also sufficient to prove that under these conditions $N^\varepsilon = N^{-\varepsilon} \cup \xi_V$ for some $0 < \varepsilon < \varepsilon_0$.

We have already defined $\xi = \xi_V$ as the negative bundle of f along V. Let ξ^+ be the negative bundle of function $-f$ along V. Then, clearly, the normal bundle η of V in N is the direct sum ξ^+ with ξ.

(3.11) $$\eta = \xi^+ \oplus \xi \; .$$

We let $\pi : \eta \to \xi$ be the natural projection. The length of a vector $X \in \eta$ is denoted by $|X|$ and the function $X \to |X|^2$ is denoted by φ.

Let $\rho : \eta \to N$ be the exponential map. This map is a homeomorphism in the vicinity of V included in η as the zero cross-section. Thus ρ induced a Riemann structure $(\, , \,)$ on this vicinity. The function $f \circ \rho$ will be donoted by f_*.

The condition that V is a nondegnerate critical manifold of f clearly implies that the function f_* restricted to any fiber of η has a nondegenerate critical point. More precisely the following is true:

(3.12) The function f_*, restricted to any fiber of ξ^+, $[\xi]$, has a nondegenerate minimum [maximum] at 0.

An easy computation now yields the following consequence:

(3.13) The function $(df_*, d\varphi)$, restricted to any fiber of $\xi^+[\xi]$ has a non-degenerate minimum [maximum] at 0.

The geometric interpretation of this remark is in turn :

(3.14) If $\varepsilon > 0$ is small enough the set $f_* \leqq \varepsilon$ on a fiber of ξ^+ is starshaped with respect to 0, and therefore linearly contractible.

(3.15) If $\mu > 0$ is small enough, the gradient of $-f_*$ points out of the set $\varphi(X) \leqq \mu$, at points with $\varphi(X) = \mu$, on any fiber of ξ^-.

Now, let X_μ^ε be the subset defined by:

$$X_\mu^\varepsilon = \{X \in \eta \mid f_*(X) \leqq \varepsilon;\ \varphi \circ \pi(X) \leqq \mu\} . \tag{3.16}$$

Then we can as a consequence of (3.14) and (3.15), find positive numbers ε and μ with the following properties:

(a) We have $\varepsilon < \varepsilon_0$.

(b) The map ρ is a homeomorphism on X_μ^ε.

(c) If $A_\mu^\varepsilon \subset X_\mu^\varepsilon$ is the subset of X_μ^ε on which $\varphi \circ \pi(X) = \mu$, then the pair $(X_\mu^\varepsilon \cap \xi^-, A_\mu^\varepsilon \cap \xi)$ is a deformation retract of $(X_\mu^\varepsilon, A_\mu^\varepsilon)$.

(d) The gradient of $-f$ points out of the set $\rho(X_\mu^\varepsilon)$ at the points of $\rho(A_\mu^\varepsilon)$.

Assume in the sequel that ε, μ have been chosen in the above manner. Also let $Y_\mu^\varepsilon = \overline{N - \rho(X_\mu^\varepsilon)}$. From (b) we conclude that $N^\varepsilon = Y_\mu^\varepsilon \cup_\alpha X_\mu^\varepsilon$ with attaching map $\alpha = \rho \mid A_\mu^\varepsilon$. From (c) it follows that $N^\varepsilon = Y_\mu^\varepsilon \cup \xi$. (Clearly the pair (D_ξ, S_ξ) is equivalent to the pair $(X_\mu^\xi \cap \xi, A_\mu^\varepsilon \cap \xi)$.) Finally, from (d) we conclude that at the boundary points of Y_μ^ε the gradient $-\nabla f$ points inward. Further there are no points with $\nabla f = 0$ on this set in the range $-\varepsilon \leqq f$, in view of (a). Hence $N^{-\varepsilon}$ is a deformation retract of Y_μ^ε by the argument used in the proof of (3.6). Thus N^ε is of the same homotopy type as $N^{-\varepsilon} \cup \xi$ as was to be shown.

REMARKS ON (3.8). This result follows from (3.7). One triangulates V and uses the preimages of these cells under the map $D_\xi \to V$ as the cells e_i.

The following is a different argument which proves (3.8) under the weaker hypothesis that (3.7) holds if V is a point. Let g be a function on V which has only nondegenerate critical points on V. Extend g to a function $\hat{g}$ on a normal neighborhood, B, of V in N by making $\hat{g}$ constant along the fibers, F, of B. Finally smooth $\hat{g}$ out to 0 inside a slightly bigger normal neighborhood. There results a C^∞ function $\hat{g}$ on M. Now consider the function $\hat{f} = f + \varepsilon\hat{g}$, with $\varepsilon > 0$. For ε sufficiently small $\hat{f}$ will have only nondegenerate critical points in the range $a \leqq f \leqq b$, and these will be precisely the critical points of g on V. Note that this part of the argument holds without the nondegeneracy hypothesis. All that is needed is that V be an isolated critical manifold. However, under such a general condition nothing can be said *a priori* about the indexes of the critical points of f. Under the nondegeneracy condition, $H_p f$ and $H_p g$ have complementary nullspaces at all critical points of f. Hence the indexes add, and are therefore $\geqq \lambda_f(V)$.

We close this section with the following easy corollary of Theorem I, corresponding to the case $\lambda_f(V) = 0$, i.e.. when $\xi_V = V$.

COROLLARY 3. *Let f be a smooth function on the compact manifold M. Assume that the critical set of f consists entirely of nondegenerate critical manifolds. Let M_* be the set on which f takes on its absolute minimum, and let $|f|$ denote the smallest index of the critical points of f on $M - M_*$. Then M is obtained from M_* by successively attaching cells of dimension no less then $|f|$. Thus: $M = M_* \cup e_1 \cup \cdots \cup e_i$; $\dim e_i \geqq |f|$.*

4. The suspension theorem

Le ν be a base point on M. The space $\Omega_\nu M$ is called nondegenerate if the set of geodesics in $\Omega_\nu M$ is the union of nondegenerate critical manifolds. Precisely, this condition should be formulated as follows: $\Omega_\nu M$ is nondegenerate if, given any regular half-space $\Omega_\nu^a M$, with model M_*^b, then the critical set of φ on M_*^b is the (necessarily) disjoint union of nondegenerate critical manifolds.

Combining the reduction Theorem III the following proposition becomes evident:

SUSPENSION THEOREM. *Let $\Omega_\nu M$ be nondegenerate. Let $\mathcal{V} = \mathcal{V}_\nu(M)$ be the collection of critical manifolds in $\Omega_\nu M$.*

Let $\mathcal{V}$ be well ordered, $\mathcal{V} = \{V_1, V_2, \cdots\}$, compatibly with the partial order defined on V by the length of the geodesics, and let $\xi_{V_i} = \xi_i$ be the negative bundle of V_i. Then $\Omega_\nu M$ has the same homotopy groups as the CW-*complex*:

$$K = \xi_1 \cup \xi_2 \cup \xi_3 \cup \cdots \tag{4.1}$$

We call this the suspension theorem because (1.1) follows from it trivially. Indeed, if $|\nu| > 1$, then only one of the critical manifolds V_i can have index 0, because $\Omega_\nu M$ is connected, (whence K is connected) and attaching a vector bundle of fiber dimension >1 does not change the number of components. Hence in this case V_1 has index 0 while all other V_i have index $\geqq |\nu|$. It follows that $M^\nu = V_1$. Thus going over to the corollary of Theorem III, K is of the form:

$$K = M^\nu \cup e_1 \cup e_2 \cup \cdots \qquad \dim e_i \geqq |\nu|. \tag{4.2}$$

Let $i : M^\nu \to \Omega_\nu M$ be the inclusion and let σ_* denote the suspension (in homotopy) from $\Omega_\nu M$ to M. Then $\sigma_* \circ i_*$; $\pi_k(M^\nu) \to \pi_{k+1}(M)$ agrees with the definition of ν_* given in the introduction. Hence by (4.2) we obtain the corollary:

COROLLARY (4.1). *Under the hypothesis of the suspension theorem,*

$$(4.3) \qquad \nu_*\colon \pi_r(M^\nu) \to \pi_{r+1}(M) \qquad 0 < r < |\nu| - 1$$

is an isomorphism onto.

For completeness, we state an immediate cohomology consequence of (4.1):

COROLLARY (4.2). *Under the hypothesis of the suspension theorem, $H^*(\Omega_\nu M; G)$ admits a spectral sequence E_r which converges to a graded group of $H^*(\Omega_\nu M; G)$ and whose E_1 term is given by:*

$$(4.4) \qquad E_1 = \sum H_c^*(\xi_i; G)$$

where ξ_i ranges over the negative bundles ξ_V; $V \subset \mathcal{V}$. (The subscript c denotes cohomology with compact supports.)

By Poincaré duality one has further that (in the notation of (3.10)):

$$(4.5) \qquad H_c^r(\xi_V; G) = H^{r-\lambda}(V; G'), \qquad \lambda = \lambda(V)\ .$$

REMARKS. Recall that nondegerate $\Omega_\nu M$ exist for every manifold M of the type we are considering. In fact nearly every base point, ν gives rise to an $\Omega_\nu M$ in which the geodesics are nondegenerate *critical points*. In that case (4.3) is quite uninteresting, however (4.4) is still useful; in particular, E_1 will then be free if G is taken as the integers. For instance, if M is a compact group, $E_1 = E_\infty$ is was shown in [3], while for compact symmetric spaces, in general, $E_1 = E_\infty$ at least mod 2. [6].

5. The proof of Theorem I

Theorem I follows from the suspension theorem of the last section once it is proved that:

(5.1) *If M is a symmetric space then $\Omega_\mu M$ is nondegenerate for every base point ν on M.*

(5.2) *With M, M^ν is again a symmetric space for every base point ν on M.*

Recall that the manifold M is called symmetric if the following condition is satisfied:

(5.3) *For every $P \in M$, there exists an isometry L_P, of M which keeps P fixed and reverses the geodesics through P.*

From the second condition it follows that $I_P^2 =$identity for every $P \in M$.

Another equivalent definition can be given in terms of the group of isometries of M. This group, which is known to be a compact Lie-group, will be denoted by G in the sequel. Using the fact that any two points of M can be joined by a geodesic one easily derives the following consequences of (5.3).

(5.4) The e-component G' of G acts transitively on M.

(5.5) If $K_P \in G$ is the stability of $P \in M$, then K_P is pointwise fixed under the automorphism A_P: $k \to I_P k I_P^{-1}$ of G.

(5.6) The e-component K'_P coincides with the e-component of the fixed point set of A_P in G.

The converse of (5.6) yields the alternate definition of symmetric spaces:

(5.7) If G is a compact group, and A is an involution of G, then in an invariant Riemannian structure, the coset space G/K is called a symmetric space if K' coincides with the e-component of the fixed point group of A.

In the sequel we assume M is a symmetric space with K_P the stability group of $P \in M$. The e-components of groups will be denoted by a dash, e.g., K'_P.

The action of K_P on M was discussed in [6], and was shown to be variationally complete.

As a consequence the following is true: (see [6, chapter II].)

PROPOSITION 5.1. *Let s be a nontrivial geodesic on M starting at P. Let Q be any point of s, and set K_{PQ} respectively K_s, equal to the subgroup of K'_P which keeps Q, respectively s, pointwise fixed. Then the multiplicity of Q as a conjugate point of P is equal to* $\dim K_{PQ}/K_s$.

The statement (5.1) is an immediate corollary of this proposition. Indeed, let $\nu = (P, Q; h)$ and let the set of geodesics in $\Omega_\nu M$ be denoted by $S_\nu M$. Clearly K'_{PQ} acts on $S_\nu M$, the orbit of $s \in S_\nu M$, being homeomorphic to K'_{PQ}/K'_s. In any model, M^b_*, for $\Omega^a_\nu M$ these orbits are certainly imbedded as smooth submanifolds. Now we see by Proposition 5.1 and (2.3) that the nullity of any point on such an orbit is equal to the dimension of the orbit. This is precisely the second condition for nondegeneracy. (see (3.2)).

There remains the statement (5.2). To prove it, we show that each orbit of K'_{PQ} on M^ν is a symmetric space. Let then V be the orbit of $s \in M^\nu$. We may assume that s does not degenerate, for then M^ν reduces to a point. Thus $V = K'_{PQ}/K_s$ and we have to produce an involution A of K_{PQ} whose fixed point set contains K'_s as e-component. Because s is a minimal geodesic in the $\Omega_\nu M$, no conjugate point of P occurs in the interior of s. In particular, the midpoint R of s is not conjugate to P along s. Hence $K'_s = K'_{PR}$ by Proposition 5.1.

Now $I_R P = Q$, and $I_R Q = P$. Hence if $k \in K_{PQ}$, then $I_R k I_R^{-1} \subset K_{PQ}$. Thus $A\colon K_{PQ} \to K_{PQ}$ defined by $A(k) = I_R k I_R^{-1}$ is an involution of K_{PQ}. On the other hand, the e-component of the fixed point set of A is precisely K'_{PR}. This proves (5.2) and completes the proof of Theorem I.

For future reference we close this section with the following theorem, which is a straightforward generalization of Theorem I of [6].

THEOREM IV. *Let ν be any base point on the symmetric space M. Then the spectral sequence*, (4.2), *attached to $\Omega_\nu M$ by the decomposition* (4.1), *is trivial over the integers* mod 2. *Thus*:

$$H^*(\Omega_\nu M;\, Z_2) = \sum H_c^*(\xi_V;\, Z_2) \qquad V \in \mathcal{V}_\nu(M)\,. \tag{5.8}$$

In the group case (5.8) *holds with integer coefficients.*

NOTE ON THE PROOF. The spectral sequence (4.2) is derived from the filtering of $K = \xi_0 \cup \xi_1 \cup \cdots$, by the subcomplexes $K_i = \xi_0 \cup \cdots \cup \xi_i$. Let $\alpha\colon S_{\xi_i} \to K_{i-1}$ be the attaching map of ξ_i. The problem is to show that α induces a trivial homomorphism in homology. Let $s \in V_i$ and consider the K cycle Γ_s as defined in [6]. This is a manifold fibered over V with a section $\sigma\colon V \to \Gamma$. One has a map of $\Gamma \to K_i$, which transforms ξ_i into the normal bundle of $\sigma(V)$ in Γ. Thus $\Gamma = \Gamma' \cup \xi_i$ corresponds to $K_i = K_{i-1} \cup \xi_i$ and in Γ' the attaching map α_* is always homologically trivial mod 2 (because ξ_i is the normal bundle of a section). If the fiber of Γ over V is orientable α_* will also be trivial over the integers.

The simplest application of Theorem IV is obtained by considering (5.8) in dimension 0. Because $\Omega_\nu M$ is always connected for any base point ν on M, (5.8) implies that M^ν is connected. This fact will also be apparent in the explicit computations of sections 7 and 8 which evaluate the integers $|\nu|$ of Theorem II.

Before proceeding to the proof of this theorem we have to review the basic conjugacy theorems for symmetric spaces which make the explicit computations possible. This is done in the next section.

6. The roots of a symmetric space

In this section G is to be a compact connected Lie group, in a left and right invariant metric, which an involution A. The full fixed point set under A is denoted by K, while the e-component of K is written K'. (Note that K thus plays the role of K_P in section 5.)

Let $\mathfrak{g}$ be the Lie algebra of G, and let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$$

be the decomposition of $\mathfrak{g}$ into the fixed point set of A, (this is $\mathfrak{k}$, the Lie algebra of K) and its orthogonal complement. Let $\mathfrak{h}_{\mathfrak{m}}$ be a maximal abelian subalgebra of $\mathfrak{m}$, and let $\mathfrak{h} \supset \mathfrak{h}_{\mathfrak{m}}$ be a Cartan subalgebra of $\mathfrak{g}$.

Let $\eta: G \to G$ be defined by: $\eta(g) = g \cdot A(g^{-1})$. Then $\eta(gk) = \eta(g)$ so that η is constant along the left cosets of K and in this manner defines a map $\eta_*: G/K \to G$. We also let M be the image of $\mathfrak{m}$ under the exponential map. Thus $M = e^{\mathfrak{m}}$. Then it is known [1], [7], that η_* is a homeomorphism of G/K onto M. Further the natural action of K on G/K now translates into the adjoint action of K on G restricted to M. In the sequel we will therefore always think of the symmetric space G/K as the subset $M \subset G$.

Let $T_{\mathfrak{m}}$ be the image of $\mathfrak{h}_{\mathfrak{m}}$ under the exponential map. This is a torus in M which is geodesically imbedded. Any torus of this form is called a maximal torus of M, and its dimension is the *rank of* M.

We write $W(G, K)$ or $W(M)$ for the group of automorphisms of $T_{\mathfrak{m}}$ which are induced by inner automorphisms of K'. The following are basic properties of maximal tori: (see [1], [6], [7])

(6.1) If T and T' are two maximal tori of M, then there exists a $k \in K'$ so that $T = kT'k^{-1}$.

(6.2) If X is a subset of $T_{\mathfrak{m}}$ and $k \in K$ has the property $kXk^{-1} \subset T_{\mathfrak{m}}$, then there exists an element σ of $W(G, K)$ so that $\sigma(x) = kxk^{-1}$, for all $x \in X$.

(6.3) Every point of M lies on a maximal torus of M.

We also have:

(6.4) The geodesics of M through e coincide with the one-parameter groups of G which lie in M.

(6.5) If $x \in m$, then the index of the geodesic segment:

$$\bar{x}(t) = e^{tx} \qquad 0 \leqq t \leqq 1.$$

in M is computed as follws:

Let $\Sigma(G) = \{\theta_i\}$, $i = 1, \cdots, m$, be a system of positive roots of G on $\mathfrak{h}$. Also if a is any real number, let $\| a \|$ denote the number 0 if $a = 0$, otherwise let $\| a \|$ be the greatest integer $< | a |$. With this understood, the index in question is given by:

$$\lambda(\bar{x}) = \sum_1^m \| \theta_i(x) \| \tag{6.6}$$

REMARKS.

(1) The formula (6.6) is to be found in [6], except for a factor 2 in the definition of the exponential map. This discrepancy is explained by the

fact that the inverse of $\gamma_*: G/K \to M$, is not given by the projection $M \to G/K$ induced by the natural map $\pi: G \to G/K$. Rather, one has $\gamma_*^{-1}(p) = \pi(\sqrt{p})$ where for $p \in M$, $\sqrt{p}$ is any point of M with $(\sqrt{p})^2 = p$. That this factor 2 could be done away with by considering M rather than G/K was pointed out to me by A. Borel.

(2) We can find distinct non-trivial forms $\{\varphi_i\}$, $i = 1, \cdots m'$, on $\mathfrak{h}_{\mathfrak{m}}$ such that each $\theta \in \Sigma(G)$ restricts to some $\pm \varphi_i$ on $\mathfrak{h}_{\mathfrak{m}}$. Such a system of forms is called a root system for M, and is denoted by $\Sigma(M)$. For each $\varphi \in \Sigma(M)$ let n_φ be the number of forms in $\Sigma(G)$ which restrict to $\pm \varphi$ on $\mathfrak{h}_{\mathfrak{m}}$. These integers are the multiplicities of the root forms of M. In terms of them, (6.6) is expressed by:

$$\lambda(\bar{x}) = \sum n_\varphi \| \varphi(x) \| \qquad \varphi \in \Sigma(M). \tag{6.7}$$

This formula has the following geometric interpretation: Consider the set of planes on which one of the root-forms $\varphi \in \Sigma(G/K)$ has an integral value. Then $\lambda(\bar{x})$ counts how many of these planes the line-segment tx, $0 \leqq t \leqq 1$, *crosses*, each crossing being counted by the appropriate multiplicity.

Finally, we recall the following facts:

(6.8) Let Λ_* be the lattice of those $x \in \mathfrak{h}_{\mathfrak{m}}$, for which the segment $\bar{x}(t) = e^{tx}$, $0 \leqq t \leqq 1$, represents a closed curve which is homotopic to zero in M. Then Λ_* is generated by elements $\mathfrak{h}_\varphi$, $\varphi \in \Sigma(M)$, characterized by:

$$\mathfrak{h}_\varphi \text{ is perpendicular to the plane } \varphi = 0, \text{ and } \varphi(\mathfrak{h}_\varphi) = 2\ .$$

(6.9) The representation of $W(M)$ on $\mathfrak{h}_{\mathfrak{m}}$ is generated by the reflections in the planes $\varphi = 0$ for $\varphi \in \Sigma(M)$.

These propositions enable us to survey the possible indexes of elements in $S_\nu M$ entirely in terms of the roots of G on $\mathfrak{h}$. Indeed, by (6.3) no generality is lost if we assume that the base-point $\nu = (P, Q; h)$ is of the form $P = e$; $Q \in T_{\mathfrak{m}}$. According to (5.1) the set $S_\nu M$ will consist of the cllection $\mathcal{CV}_\nu M$ of nondegenerate critical manifolds. If s is a geodesic of $V \in \mathcal{CV}_\nu M$, then V consists precisely of the set of geodesics ksk^{-1} where k is in the subgroup of K' keeping Q fixed. Hence, by (6.1), (6.2) and (6.4), each V contains geodesics which lie on $T_{\mathfrak{m}}$, and join e to Q. Further two such geodesics lie in the same V precisely if they are conjugate under $W(G, K)$.

We will adhere to the convention that if $x \in \mathfrak{h}_{\mathfrak{m}}$, then $\bar{x}$ represents the geodesic e^{tx}, $0 \leqq t \leqq 1$, in M. Because the geodesics on $T_{\mathfrak{m}}$ can be lifted into $\mathfrak{h}_{\mathfrak{m}}$ in the obvious fashion, our earlier conclusions can be summarized as follows:

PROPOSITION 6.1. *Let $x_\nu \in \mathfrak{h}_\mathfrak{m}$ be any point with $\bar{x}_\nu \in \Omega_\nu M$. Then if $x \in x_\nu + \Lambda_*$ there is a unique critical manifold $V_x \subset S_\nu M$ which contains $\bar{x}$. This manifold is homeomorphic to K'/K_x, where K_x is the centralizer of x in K'.*

The function $x \to V_x$ maps $x_\nu + \Lambda_*$ onto the set $\mathcal{CV}_\nu M$, and if $V_x = V_y$, $x, y, \in x_\nu + \Lambda_*$, then x and y are conjugate under the action of $W(G, K)$ on $\mathfrak{h}_\mathfrak{m}$.

COROLLARY. *The set of indexes $\lambda(s)$, $s \in S_\nu M$, consists of the integers $\lambda(\bar{x})$, computed according to* (6.7) *as x ranges over the points of $x_\nu + \Lambda_*$.*

In the next sections this proposition is applied to compute the values of $|\nu|$ given in Theorem II, case by case.

7. Computations when M is a group

If the compact connected group G is to be considered as a symmetric space, M, we must, to follow our general procedure, consider M as the subset (g, g^{-1}), $g \in G$, in $G \times G$. Then $M = G$, while $\mathfrak{h}_\mathfrak{m}$ corresponds to the anti-diagonal in $h \times h$. Thus in this case $\sum(M)$ is a positive root system for G each root being counted with multiplicity 2. The group K then corresponds to G acting on M by the adjoint action.

In each case to be considered, we will choose orthogonal coordinates in $\mathfrak{h}_\mathfrak{m}$, and so identify $\mathfrak{h}_\mathfrak{m}$ with R^l, the space of l-tuples of real numbers with the usual inner product $((x, y) = \sum x_i \cdot y_i$, where x_i, y_i are the coordinates of x and y respectively). The form which assigns to $x \in R^l$ its α^{th} coordinate will always be denoted by ω_α. The exponential map then gives rise to a map $R^l \to M$, which will be denoted by ρ. We will define this map in each case, and then give the root-system of M as it is expressed by the forms ω_α.

(7.1) *The unitary groups, $M = \mathrm{U}(2n)$.* Let d_α be the diagonal $2n \times 2n$ matrix with α^{th} entry $2\pi\sqrt{-1}$, and all other entries 0. Then $\rho: R^{2n} \to \mathrm{U}(2n)$ is given by:

$$\rho(x) = \exp\{\textstyle\sum \omega_\alpha(x)\, d_\alpha\} \qquad x \in R^{2n}.$$

and the root-forms of $M = \mathrm{U}(2n)$ are:

$$\Sigma(M)\colon \omega_\beta - \omega_\alpha \qquad 1 \leqq \alpha < \beta \leqq 2n.$$

It follows that $W(M)$ is permutation group of the coordinates in R^{2n}, and that Λ_* is generated by $\{1, -1, 0, 0, \cdots, 0\}$ and its transforms under $W(M)$.

Let $x_\nu \in R^{2n}$ be the element:

$$x_\nu = \{0, 0, \cdots, 0; 1, 1, \cdots, 1\} \quad (n \text{ entries } 0, n \text{ entries } 1)$$

and let $\nu = (P, Q; h)$ be the unique base point containing the curve $\bar{x}_\nu$ (Note that then $P = Q =$ identity). Thus K_{PQ} (in the sense of section 5) is equal to $\mathrm{U}(2n)$ and $K_{x_\nu} = \mathrm{U}(n) \times \mathrm{U}(n)$, whence $V_{x_\nu} = \mathrm{U}(2n)/\mathrm{U}(n) \times \mathrm{U}(n)$.

The points of $x_\nu + \Lambda_*$ are of the form: $x = \{a_1, \cdots, a_{2n}\}$ with $a_\alpha \in Z$; $\sum a_\alpha = n$. Let $b_1 < b_2 \cdots < b_k$ be the different integers which occur among the $\{a_i\}$, and assume that b_k occurs n_k times. Then according to (6.7):

$$\lambda(\bar{x}) = 2 \sum\nolimits_{\beta > \alpha} n_\alpha n_\beta (a_\beta - b_\alpha - 1)\,.$$

We conclude:

(1) If $x \in x_\nu + \Lambda_*$, with $\lambda(\bar{x}) = 0$ then x is conjugate to x_ν under $W(M)$.

(2) The next lowest value of λ on $x_\nu + \Lambda_*$ is $2(n+1)$. Up to conjugation by elements of $W(M)$ this value is taken on only at the points:

$$\{0, \cdots, 0; 0, 1, 1, \cdots, 1, 2\} \text{ and } \{-1, 0, 0, \cdots, 0, 1; 1, 1, \cdots, 1\}\,.$$

Hence:

(7.2) *In this case,* $M^\nu = V_{x_\nu} = \mathrm{U}(2n)/\mathrm{U}(n) \times \mathrm{U}(n)$, *while* $|\nu| = 2(n+1)$.

COROLLARY. *The sequence* (1.2) *is a ν-sequence.*

(7.3) *The orthogonal groups,* $M = \mathrm{SO}(2n)$. Let O_k be the $2n \times 2n$ matrix with only entry the diagonal box $2\pi\sqrt{-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ at the k^{th} level. Now ρ: $R^n \to \mathrm{SO}(2n)$ is given by: $\rho(x) = \exp\{\sum \omega_\alpha(x) O_\alpha\}$, and we have:

$$\Sigma(M)\colon \omega_\beta \pm \omega_\alpha\,; \qquad 1 \leqq \alpha < \beta \leqq n.$$

Further $W(M)$ is generated by the permutations $\omega_\alpha \to \omega_\beta$, and $\omega_\alpha \to -\omega_\beta$, $\alpha < \beta$; and Λ_* is generated by the element $\{1, -1, 0, \cdots, 0\}$ as a $W(M)$ module.

Let $x_\nu = \{1/2, 1/2, \cdots, 1/2\}$, and let ν be the base point determined by $\bar{x}_\nu$. Then $V_{x_\nu} = \mathrm{SO}(2n)/\mathrm{U}(n)$. By, (6.7) we see that $\lambda(\bar{x}) = 0$, x in $x_\nu + \Lambda_*$ implies x conjugate to x_ν under $W(M)$, while $|\nu|$ is given by $2(n-1)$. In fact the index of $\{\pm 1/2, 1/2, 1/2, \cdots, 3/2\}$ is precisely $2(n-1)$. Thus,

(7.4) *In this case* $M^\nu = \mathrm{SO}(2n)/\mathrm{U}(n)$, *while* $|\nu| = 2(n-1)$.

(7.5) *The symplectic groups,* $M = \mathrm{Sp}(n)$. Let $\mathrm{U}(n) \subset \mathrm{Sp}(n)$ be a standard inclusion, and let ρ: $R^n \to \mathrm{Sp}(n)$ be defined by the map $R^n \to \mathrm{U}(n)$ as in (7.1), (with n replaced by $2n$) followed by the inclusion. Then:

$$\Sigma(M)\colon \omega_\beta \pm \omega_\alpha;\ 2\omega_\alpha\,, \qquad 1 \leqq \alpha < \beta \leqq n.$$

$W(M)$: All signed permutaions.

Λ_*: Generated by $\{1, -1, 0, \cdots, 0\}$ as a $W(M)$-module.

Again, we choose $x_\nu = \{1/2, \cdots, 1/2\}$. Then $V_{x_\nu} = \mathrm{Sp}(n)/\mathrm{U}(n)$ as is easily seen. As before $V_{x_\nu} = M^\nu$. However now $\lambda(\{1/2, 1/2, \cdots, 3/2\}) = 2(n+1)$, and this is the value of $|\nu|$. Thus:

(7.6) *In this case,* $M^\nu = \mathrm{Sp}(n)/\mathrm{U}(n)$ *with* $|\nu| = 2(n+1)$.

8. The remaining computations. Proof of Theorem II

(8.1) *The space* $M = \mathrm{SO}(4n)/\mathrm{U}(2n)$. Let Q be the field of quaternions $x_0\cdot 1 + x_1\cdot i + x_2\cdot j + x_3\cdot k$; $x_s \in R^1$, where the $1, i, j, k$ are the usual quaternion units. We define the following endomorphisms of R^{4n}: E_0 the identity; E_1 is to take the α^{th} coordinate into minus the $(\alpha+2n)^{\text{th}}$ coordinate, while it takes the $(\alpha+2n)^{\text{th}}$ coordinate into the α^{th} one $(1 \leqq \alpha \leqq 2n)$. The endomorphism E_2 is to be represented by the matrix

$$\{O_1 + \cdots + O_n - O_{n+1} - \cdots - O_{2n}\}$$

where O_α is as defined in (7.3). The assignment $1 \to E_0$, $i \to E_1$, $j \to E_2$, defines a representation of Q on R^{4n}. Because $1, i$ generate a field isomorphic to the complex numbers, we see that the elements of $\mathrm{SO}(4n)$ which commute with E_1 form a subgroup $\mathrm{U}(2n) \subset \mathrm{SO}(4n)$. The elements of this subgroup which commute with E_2 in turn define $\mathrm{Sp}(n) \subset \mathrm{U}(2n)$. Hence if we set $G = \mathrm{SO}(4n)$, and let A be the inner automorphism by E_1, then A^2 is the identity and the fixed point set, K, of A is $\mathrm{U}(n)$. Thus $G/K = M$ is a symmetric space.

Let $R^{2n} \to \mathrm{SO}(4n)$ be defined as in (7.3) with n replaced by $2n$. Then R^{2n} corresponds to the Cartan algebra, $\mathfrak{h}$, of section (6), and we have to determine the inclusion $\mathfrak{h}_{\mathfrak{m}} \subset \mathfrak{h}$. It is not hard to see that this inclusion corresponds to a map $R^n \to R^{2n}$ given by

$$(x_1, \cdots, x_n) \to (x_1, \cdots, x_n; -x_1, \cdots, -x_n) .$$

Restricting the forms of (7.3) to this subspace, we obtain the following set of forms for $\Sigma(M)$: $\omega_\beta \pm \omega_\alpha$; $(1 \leqq \alpha < \beta \leqq n)$; $2\omega_\alpha$ $(1 \leqq \alpha \leqq n)$ Further the multiplicity of $\omega_\beta \pm \omega_\alpha$; $(\alpha \neq \beta)$ is 4, while that of $2\omega_\alpha$ is 1. Schematically we denote this set of forms by:

$$\Sigma(M): \quad \begin{array}{cc} \omega_\beta \mp \omega_\alpha & 2\omega_\alpha \\ 4 & 1 \end{array} \qquad 1 \leqq \alpha < \beta < n$$

(Thus the integer below the form denotes its multiplicity. This notation will be used throughout the sequel.) $W(M)$ and $\Lambda_*(M)$ are therefore the same as in (7.3)

Choose $x_\nu = \{1/2, \cdots, 1/2\}$, and let ν be the determined by $\bar{x}_\nu$. Note that $\bar{x}_\nu(t) = \exp(\pi\sqrt{-1}\, t\, E_2)$. It follows that in this case $K_{PQ} = \mathrm{U}(2n)$, while $K_{x_\nu} = \mathrm{Sp}(n)$. Thus $V_{x_\nu} = \mathrm{U}(2n)/\mathrm{Sp}(n)$. Just as previously, V_{x_ν} is actually M^ν, while $|\nu|$ is the index of $\{1/2, \cdots, 1/2, 3/2\}$, and thus given by $4n-2$. We conclude:

(8.2) *In this case* $M^\nu = \mathrm{U}(2n)/\mathrm{Sp}(n)$ *with* $|\nu| = 4n-2$.

(8.3) *The space* $M = \mathrm{U}(4n)/\mathrm{Sp}(2n)$. Let E_1 be the matrix described in the last section. Then it is well known that the subgroup of $\mathrm{U}(4n)$ whose elements satisfy the identity $U^t E_1 U = E_1$, form the linear symplectic group $\mathrm{Sp}(2n) \subset \mathrm{U}(4n)$. Let A be the automorphism of $\mathrm{U}(2n)$ which takes U into $E_1 \bar{U} E_1^{-1}$. (Here the bar denotes complex conjugation.) Then A^2 is the identity, and because $\bar{U}^t = U^{-1}$, the subgroup of $\mathrm{U}(2n)$ fixed under A is precisely $\mathrm{Sp}(n)$. Let $R^{2n} \to R^{4n}$ be the map:

$$(8.4) \qquad (x_1, \cdots, x_{2n}) \to (x_1, \cdots, x_{2n}, x_1, \cdots, x_{2n}) .$$

Then this map followed by the map $R^{4n} \to \mathrm{U}(4n)$ described in (7.1) describes ρ in this case. Restricting the forms of $\mathrm{U}(4n)$ according to (8.4) we obtain the following array for $\Sigma(M)$:

$$\Sigma(M): \quad \underset{4}{\omega_\beta - \omega_\alpha} \qquad 1 \leqq \alpha < \beta \leqq 2n.$$

Hence $W(M)$ and Λ_* are as described in (7.1). Accordingly choose $x_\nu = \{0, \cdots, 0, 1, \cdots, 1\}$, just as in (7.1), and let ν be determined by $\bar{x}_\nu$. This is then a closed curve in M. Thus K_{PQ} is represented by $\mathrm{Sp}(2n)$. The centralizer of $\bar{x}_\nu$ in $\mathrm{U}(4n)$ is clearly $\mathrm{U}(2n) \times \mathrm{U}(2n)$. Hence the centralizer in $\mathrm{Sp}(2n)$ is precisely $\mathrm{Sp}(n) \times \mathrm{Sp}(n)$. Thus V_{x_ν} is homeomorphic to $\mathrm{Sp}(2n)/\mathrm{Sp}(n) \times \mathrm{Sp}(n)$. Just as in (7.1) we see that $M^\nu = V_{x_\nu}$. However $|\nu|$ is now given by $4(n+1)$, because each root has weight 4 instead of 2. To summarize:

(8.5) *In this case* $M^\nu = \mathrm{Sp}(2n)/\mathrm{Sp}(n) \times \mathrm{Sp}(n)$ *while* $|\nu| = 4(n+1)$.

If we combine (7.4) with (8.2) and (8.5) we obtain the

COROLLARY. *The sequence* (1.4) *is a* ν*-sequence.*

(8.6) *The space* $M = \mathrm{Sp}(n)/\mathrm{U}(n)$. We will now interpret $\mathrm{Sp}(2n)$ as the group of $n \times n$ nonsingular matrixes with entries from Q which keep the symplectic product invariant. We also write $i[j]$ for the diagonal matrix $i \times$ Identity $[j \times$ Identity]. Consider the subgroup of $\mathrm{Sp}(n)$ which commutes with j. Because the elements of Q which commute with $j \in Q$ form a field isomorphic to C, this subgroup will be isomorphic to $\mathrm{U}(2n)$. Hence if A denotes the inner automorphism with j, then the fixed-point set of A is $\mathrm{U}(n)$. By a similar argument, the subgroup commuting with both i and j is the group $\mathrm{O}(n) \subset \mathrm{U}(n)$.

Let ρ: $R^n \to \mathrm{Sp}(n)$ be defined as in (7.1), except that $\sqrt{-1}$ is to be replaced by $i \in Q$, and $2n$ is to be replaced by n. Then $A\rho(x) = \rho(-x)$. Further the image of ρ is a maximal torus of $\mathrm{Sp}(n)$ as is seen from (7.5). This is therefore a case when $\mathfrak{h}_{\mathfrak{m}} = \mathfrak{h}$. If follows that the root system, $\Sigma(M)$, identical with $\Sigma(\mathrm{Sp}(n))$, except that each root has multiplity 1. Thus

$$\Sigma(M): \quad \underset{1}{\omega_\beta \pm \omega_\alpha} \quad \underset{1}{2\omega_\alpha} \qquad\qquad 1 \leqq \alpha < \beta \leqq n.$$

We chose x_ν as in (7.5), and ν correspondingly. If follows that the endpoint of $\bar{x}_\nu$ is minus the identity, whence $K_{PQ} = \mathrm{U}(n)$. The centralizer of x_ν must commute with j. Hence $K_{x_\nu} = \mathrm{O}(n)$. Thus $V_{x_\nu} = U(n)/\mathrm{O}(n)$. Using the results of (7.5) it follows that:

(8.7) *In this case* $M^\nu = \mathrm{U}(n)/\mathrm{O}(n)$ with $|\nu| = (n+1)$.

(8.8) *The space* $M = \mathrm{U}(2n)/\mathrm{O}(2n)$. It is clear that here the automorphism in question is the complex conjugation. We let $\rho: R^{2n} \to \mathrm{U}(2n)$ be defined precisely as in (7.1). We then see that this is again where $\mathfrak{h}_{\mathfrak{m}} = \mathfrak{h}$. Thus

$$\Sigma(M): \quad \underset{1}{\omega_\beta - \omega_\alpha} \qquad\qquad 1 \leqq \alpha < \beta \leqq 2n.$$

We choose x_ν just as in (7.1), whence $V_{x_\nu} = \mathrm{O}(2n)/\mathrm{O}(n) \times \mathrm{O}(n)$. By dividing the answer in (7.1) by 2, we finally obtain for $|\nu|$ the integer $(n+1)$. Thus:

(8.9) *In this case* $M^\nu = \mathrm{O}(2n)/\mathrm{O}(n) \times \mathrm{O}(n)$, *and* $|\nu| = (n+1)$.

Now combining (7.6) with (8.7) and (8.9) we obtain the

COROLLARY. *The sequence* (1.3) *is a* ν-*sequence.*

This then completes the proof of Theorem II. It might be useful for later reference, to summarize the computations of the last two sections in terms of the suspension theorem of section 4. In this summary, the symbol $X = Y \cup e_k \cdots$ will be interpreted to mean that X is obtained from Y by attaching cells of dimension $\geqq k$. With this understood we have shown that:

$$\begin{aligned} \Omega_\nu \mathrm{U}(2n) &\cong \mathrm{U}(2n)/\mathrm{U}(n) \times \mathrm{U}(n) \cup e_{2n+2} \cdots . \\ \Omega_\nu \mathrm{SO}(2n) &\cong \mathrm{SO}(2n)/\mathrm{U}(n) \cup e_{2n-2} \cdots . \\ \Omega_\nu \mathrm{Sp}(n) &\cong \mathrm{Sp}(n)/\mathrm{U}(n) \cup e_{2n+2} \cdots . \end{aligned} \tag{8.10}$$

Further,

$$\begin{aligned} \Omega_\nu \mathrm{Sp}(n)/\mathrm{U}(n) &\cong \mathrm{U}(n)/\mathrm{O}(n) \cup e_{n+1} \cdots \\ \Omega_\nu \mathrm{U}(2n)/\mathrm{O}(2n) &\cong \mathrm{O}(2n)/\mathrm{O}(n) \times \mathrm{O}(n) \cup e_{n+1} \cdots \end{aligned} \tag{8.11}$$

and

$$\begin{aligned} (8.12) \qquad \Omega_\nu SO(4n)/U(2n) &\cong U(2n)/Sp(n) \cup e_{4n-2} \cdots \\ \Omega_\nu U(4n)/Sp(2n) &\cong Sp(2n)/Sp(n) \times Sp(n) \cup e_{4n+4} \cdots \end{aligned}$$

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