Assembly and Morita invariance in the algebraic K-theory of Lawvere theories

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The algebraic K-theory of Lawvere theories provides a context for the systematic study of the stable homology of the automorphism groups of algebraic structures, such as the symmetric groups, the general linear groups, the automorphism groups of free groups, and many, many more. We develop this theory and present a wealth of old and new examples to compare our non-linear setting to the theories of modules over rings via assembly maps. For instance, a new computation included here is that of the algebraic K-theory of the Lawvere theory of Boolean algebras and all theories Morita equivalent to it, in terms of the stable homotopy groups of spheres. We give a comprehensive discussion of Morita invariance: The higher algebraic K-theory of Lawvere theories is invariant under passage to matrix theories, but, in general, not under idempotent modifications. We also prove that algebraic K-theory is a monoidal functor on the category of Lawvere theories with the Kronecker product as its monoidal product. This result enables us to embed the classical assembly maps in algebraic K-theory into our framework and discuss many other examples and extensions.

Algebraic K-theory and Lawvere theories. Quillen originally devised his higher algebraic K-theory for rings in terms of the general linear groups arising as the automorphism groups of the free modules. Shortly after, in particular through the work of Segal and Thomason, it was realized that infinite loop spaces can be constructed from general symmetric monoidal categories, not just categories of modules with respect to direct sum. The price we usually pay for gained generality is the increased difficulty, if not impossibility, of proving interesting results. In this paper, we propose Lawvere theories as a happy medium between the linear case of modules over rings, on the one hand, and, on the other hand, symmetric monoidal categories in general. These algebraic theories lie at the base of a categorical approach to universal algebra, and they encompass the standard algebraic theories of groups, rings, and modules, but also more exotic algebraic structures that are of interest because of their symmetries. One of several equivalent constructions of the algebraic K-theory space $\Omega^\infty\mathrm{K}(T)$ of a Lawvere theory T, the Quillen plus construction, makes it immediately clear that the homology of this space determines the stable homology of the diagram of automorphism groups of the free T–models T_r on r generators. The following summarizes Theorems 2.6 and 2.7 from the main text.

Theorem A. *For every Lawvere theory T, there is an equivalence*

$$\Omega^{\infty}\mathrm{K}(T) \simeq \mathrm{K}_0(T) \times \mathrm{B}\mathrm{colim}_r\,\mathrm{Aut}(T_r)^+$$

of spaces, and there is an isomorphism

$$\mathrm{colim}_r\,\mathrm{H}_*(\mathrm{Aut}(T_r)) \cong \mathrm{H}_*(\Omega^{\infty}_0\mathrm{K}(T))$$

between the stable homology of the automorphism groups of finitely generated free models T_r of the theory T and the homology of the zeroth component $\Omega^{\infty}_0\mathrm{K}(T)$ of the algebraic K-theory space $\Omega^{\infty}\mathrm{K}(T)$.

Stable homology computations. Several recent stable homology computations, such as the ones for the automorphism groups of free groups [Gal11] and for the Higman–Thompson groups [SW19], can be cast into our extended context: the algebraic K-theory of the Lawvere theory of groups is the sphere spectrum (Example 3.4), and the algebraic K-theory of the Lawvere theory of Cantor algebras is a Moore spectrum, depending on the arity (Example 3.7). We will use the following new computations to illustrate various aspects of the general theory.

Theorem B. *For the algebraic K-theory of the Lawvere theory* Boole *of Boolean algebras, we have*

$$\mathrm{K}_*(\mathrm{Boole}) \cong \pi_*(\mathbb{S})/2\text{–power torsion},$$

where the $\pi_*(\mathbb{S})$ are the stable homotopy groups of spheres.

Note that there is a canonical homomorphism $\mathrm{K}_*(\mathrm{Sets}) \to \mathrm{K}_*(\mathrm{Boole})$. In view of the isomorphism $\mathrm{K}_*(\mathrm{Sets}) \cong \pi_*(\mathbb{S})$ one might be tempted to expect this homomorphism to be surjective, but this is *not* the case (see Proposition 4.3). We obtain Theorem B as Corollary 4.2 to a more general result. Our Theorem 4.1 shows that there is a family of Lawvere theories Post_v, for integers $v \geqslant 2$, which specializes to the theory of Boolean algebras for $v = 2$ and for which Theorem B holds with 2 replaced by v.

Theorem C. *For every integer $v \geqslant 2$ we have* $\mathrm{K}_*(\mathrm{Post}_v) \cong \pi_*(\mathbb{S})/v$–power torsion.

Morita equivalence. Two rings are called Morita equivalent if they have equivalent categories of modules. Morita equivalent rings must have isomorphic higher algebraic K-groups. More generally, two Lawvere theories are called Morita equivalent if their categories of models are equivalent. This is the case if and only one of them is an idempotent modification of a matrix theory of the other. We prove:

Theorem D. *The higher algebraic K-theory of Lawvere theories is invariant under passage to matrix theories.*

Because we define the algebraic K-theory of Lawvere theories in terms of free models, there is no hope of extending this result to K_0: there are even Morita equivalent rings, such as the Leavitt algebras of Theorem 7.5 and their matrix algebras, that have non-isomorphic K_0's when those K-groups are defined using free modules only. The reason is, of course, the presence of projectives that are not free. Arguably, the ability to detect those non-free projectives is one desirable feature of lower K-theory. For rings, we could have built that feature into our theory by completing idempotents. In an additive category, all retracts have complements, and this completion would not change the higher K-theory, only K_0. However, for general Lawvere theories, this fix for K_0 is not possible without changing the higher K-theory: completing at idempotents can change higher K-groups. Since the Lawvere theories Post_ν are all Morita equivalent, our computations in Theorem C show that the higher algebraic K-theory of Lawvere theories is *not* Morita invariant.

Multiplicative properties. Later constructions of algebraic K-theory, such as Quillen's categorification of the Grothendieck construction, allow us to manipulate the resulting spectra more easily. In particular, Elmendorf and Mandell [EM06, EM09] (see also [BO20]), have extended the functoriality to an extent that enables us to prove that algebraic K-theory has good multiplicative properties.

Theorem E. *For any pair of Lawvere theories S and T, there is a natural morphism*

$$\mathrm{K}(S) \wedge \mathrm{K}(T) \longrightarrow \mathrm{K}(S \otimes T)$$

of K-theory spectra, where $S \otimes T$ is the Kronecker product of theories. These morphisms give K-theory the structure of a lax symmetric monoidal functor.

Assembly. The models for the Kronecker product $S \otimes T$ are the S–models in the category of T–models. For instance, if $S = \mathbb{Z}$ is the theory of abelian groups, this means that the models of $\mathbb{Z} \otimes T$ are the abelian group objects in the category of T–models. These models of $\mathbb{Z} \otimes T$ can always be described as the modules over a ring, which we also denote by $\mathbb{Z} \otimes T$. The arrow $\mathrm{K}(\mathbb{Z}) \wedge \mathrm{K}(T) \to \mathrm{K}(\mathbb{Z} \otimes T)$ from Theorem E is not an equivalence, in general; for instance, this fails for $T = \mathbb{Z}$ (see Theorem 7.8 for a more general statement). Regardless, the left hand side is often the best approximation we have to the right hand side. In fact, Theorem E allows us to give a new description of the assembly maps

$$\mathrm{K}(\mathbb{Z}) \wedge \Sigma^\infty_+(\mathrm{B}G) \longrightarrow \mathrm{K}(\mathbb{Z}G)$$

for the classical algebraic K-theory of group rings, as first defined by Loday [Lod76] in his thesis, and since then developed by many others [Wal78, Qui82, FJ93, WW95, DL98]; we refer to [HP04, Spe11] for a comparison, and to [Lüc19] for a recent survey. Our present paper produces the three constituent spectra $\mathrm{K}(\mathbb{Z})$, $\Sigma^\infty_+(\mathrm{B}G)$, and $\mathrm{K}(\mathbb{Z}G)$ from the same framework: as algebraic K-theory spectra $\mathrm{K}(T)$ of suitable Lawvere theories T. The suspension spectrum $\Sigma^\infty_+(\mathrm{B}G)$ of the classifying space $\mathrm{B}G$ of a discrete group G arises as the algebraic K-theory of the Lawvere theory of G–sets (see Example 3.3). We use our more general framework to give examples and non-examples of the assembly map being an equivalence, or at least rationally being an injection. One can think of this as the algebraic K-theory Novikov conjecture for Lawvere theories [BHM89, BHM93]. In the case of the theories of nilpotent groups of a given class c, as studied previously in [Szy14, Szy19], this leads us to various new interpolation schemes for the assembly map as $c \to \infty$.

Homological stability. The theory developed in this text is rooted in stable homology computations. The related issue of homological stability is not our concern here. We refer to the paper [R-WW17] by Randal-Williams and Wahl, which discusses the homological stability problem in a more general framework than ours. Still, one might at least wonder if the setting of Lawvere theories could lead to new results in that direction, too.

Outline. In Section 1, we start with a review of the relevant universal algebra of Lawvere theories. We include, in particular, Kronecker products, which we need for our discussion of assembly, and Morita invariance, which is generated by passage to matrix theories and impotent modifications. In Section 2, we define and compare models for the algebraic K-theory of Lawvere theories and show that the higher algebraic K-theory is invariant under passage to matrix theories. We pause the development of the theory in Section 3, where we collect examples of algebraic K-theory computations that were known, more often implicitly than explicitly, before this work. In Section 4, we compute the algebraic K-theory of the Lawvere theory of Boolean algebras and of the Morita equivalent theories of Post algebras, leading to a counterexample for Morita invariance in general. We then, in Section 5, turn to multiplicative matters and show that algebraic K-theory is a monoidal functor. Those results allow us to quickly give some first applications in Section 6, and then, in the final Section 7, to embed the classical Loday assembly map into our framework and to discuss many other new examples.

1 Lawvere theories

We need to review the basic notions and set up our notation for Lawvere theories [Law63]. In particular, we briefly discuss Kronecker products and Morita theory. Some textbook references are [Par69, Sch70, Bor94, ARV11].

Choose a skeleton $\mathbf{E}$ of the category of finite sets and (all) maps between them. For each integer $r \geqslant 0$ such a category has a unique object with precisely r elements, and there are no other objects. For the sake of explicitness, let us choose the model $\underline{r} = \{a \in \mathbb{Z} \,|\, 1 \leqslant a \leqslant r\}$ for such a set. A set with $r+s$ elements is the (categorical) sum (or co-product) of a set with r elements and a set with s elements.

Definition 1.1. A *Lawvere theory* $T = (\mathbf{F}_T, \mathrm{F}_T)$ is a pair consisting of a small category $\mathbf{F}_T$ together with a functor

$$\mathrm{F}_T \colon \mathbf{E} \longrightarrow \mathbf{F}_T$$

that is bijective on sets of objects and that preserves sums. This means that the canonical map $\mathrm{F}_T(\underline{r}) + \mathrm{F}_T(\underline{s}) \to \mathrm{F}_T(\underline{r} + \underline{s})$ induced by the canonical injections is an isomorphism for all sets $\underline{r}$ and $\underline{s}$ in $\mathbf{E}$.

The image of the set $\underline{r}$ with r elements under the functor $\mathrm{F}_T \colon \mathbf{E} \to \mathbf{F}_T$ will be written T_r, so that the object T_r is the sum in the category $\mathbf{F}_T$ of r copies of the object T_1.

1.1 First examples

We recall two of the most important classes of examples of Lawvere theories.

Example 1.2. Let A be a ring. Let $\mathbf{F}_A$ be the full subcategory of the category Mod_A of A–modules spanned by the modules $A^{\oplus r}$ for $r \geqslant 0$. This category is a skeleton of the category of finitely generated, free A–modules. Then the functor

$$\mathrm{F}_A \colon \mathbf{E} \to \mathbf{F}_A,$$

that sends the set with r elements to the free module $A^{\oplus r}$ with r generators is a Lawvere theory, called the theory of A–modules. Note that $A^{\oplus 0} = 0$ is the 0 module. In particular, for the initial ring $A = \mathbb{Z}$, we have the Lawvere theory of abelian groups.

Rings can be very complicated, and this is even more true for Lawvere theories, which are significantly more general.

Example 1.3. Let G be a group. Let $\mathbf{F}_G$ be (a skeleton of) the full subcategory of the category of G–sets on the free G–sets with finitely many orbits: those of the form $\coprod_r G$. Then the functor

$$\mathrm{F}_G \colon \mathbf{E} \to \mathbf{F}_G$$

sending r to $\coprod_r G$ is a Lawvere theory, called the theory of G–sets. In particular, for the trivial group $G = \{e\}$, we have the Lawvere theory E of sets.

Remark 1.4. Some authors prefer to work with the opposite category $\mathbf{F}_T^{\mathrm{op}}$, so that the object T_r is the *product* (rather than the co-product) of r copies of the object T_1. For example, this was Lawvere's convention when he introduced this notion in [Law63]. Our convention reflects the point of view that the object T_r should be thought of as the free T–model (or algebra) on r generators, covariantly in r (or rather in $\mathbf{E}$). To make this precise, recall the definition of a model (or algebra) for a theory T.

1.2 Models

Definition 1.5. Given a Lawvere theory T, a *T–model* (or *T–algebra*) is a presheaf X (of sets) on the category $\mathbf{F}_T$ that sends (categorical) sums in $\mathbf{F}_T$ to (categorical, i.e. Cartesian) products of sets. (This means that the canonical map $X(T_r + T_s) \to X(T_r) \times X(T_s)$ induced by the injections is a bijection for all sets $\underline{r}$ and $\underline{s}$ in $\mathbf{E}$.) We write $\mathbf{M}_T$ for the category of T–models, and we write $\mathbf{M}_T(X,Y)$ to denote the set of morphisms $X \to Y$ between T–algebras. Such a morphism is defined to be a map of presheaves, i.e., a natural transformation, so that $\mathbf{M}_T$ is a full subcategory of the category of presheaves on $\mathbf{F}_T$.

The values of a T–model are determined up to isomorphism by the value at T_1, and we often use the same notation for a model and its value at T_1.

Example 1.6. The categories of models for the Lawvere theories of Examples 1.2 and 1.3 are the categories of A–modules and G–sets, respectively. For example, the action of G on itself from the right gives for each $g \in G$ a G–map $g\colon \coprod_1 G \to \coprod_1 G$ in the category $\mathbf{F}_G$ of Example 1.3. Given a model $X\colon \mathbf{F}_G^{\mathrm{op}} \to \mathrm{Sets}$, the set maps $X(g)\colon X(\coprod_1 G) \to X(\coprod_1 G)$ combine to produce the action of the group G on the set $X(\coprod_1 G)$.

Example 1.7. The co-variant Yoneda embedding $\mathbf{F}_T \to \mathrm{Pre}(\mathbf{F}_T)$ sends the object T_r of $\mathbf{F}_T$ to the presheaf $T_s \mapsto \mathbf{F}_T(T_s, T_r)$ represented by it. Such a presheaf is readily checked to be a T–model. We refer to a T–model of this form as *free*. The definitions unravel to give natural bijections

$$\mathbf{M}_T(T_r, X) \cong X^r$$

for T–models X, so that T_r is indeed a free T–model on r generators.

We can summarize the situation as follows. The Yoneda embedding of $\mathbf{F}_T$ into presheaves on $\mathbf{F}_T$ factors through the category $\mathbf{M}_T$ of T–models:

$$\begin{tikzcd}
\mathbf{F}_T \arrow[r] \arrow[dr] & \mathbf{M}_T \arrow[d] \\
& \mathrm{Pre}(\mathbf{F}_T)
\end{tikzcd}$$

Both functors are fully faithful, and the free T–models are those in the (essential) image of the top functor.

Definition 1.8. A *morphism* $S \to T$ between Lawvere theories is a functor $L\colon \mathbf{F}_S \to \mathbf{F}_T$ that preserves sums and free models. This is equivalent to the condition that $\mathrm{F}_T \cong L \circ \mathrm{F}_S$, i.e., that L is a map under $\mathbf{E}$.

Often, a morphism $S \to T$ between Lawvere theories is described by giving a functor $R\colon \mathbf{M}_T \to \mathbf{M}_S$ that is compatible with the forgetful functors to the category $\mathbf{M}_E$ of sets. Then L is induced by the left adjoint to R, which exists for abstract reasons, namely by Freyd's adjoint functor theorem.

For any Lawvere theory T, the category $\mathbf{M}_T$ of T–models is complete and cocomplete. Limits are constructed levelwise, and the existence of colimits follows from the adjoint functor theorem. The category $\mathbf{M}_T$ becomes symmetric monoidal with respect to the (categorical) sum, and the unit object T_0 for this structure is also an initial object in the category $\mathbf{M}_T$.

In a slight generalization of Definition 1.5, we can define T–models in categories with (categorical) products. In particular, we may then consider T–models in other categories of models; this is what we are going to do now.

1.3 Kronecker products

Given Lawvere theories S and T, their Kronecker product $S \otimes T$ is a Lawvere theory that represents T–models in the category of S–models or, equivalently, S–models in the category of T–

models. These theories are described by Freyd [Fre66], and in Lawvere's thesis [Law68]. It follows from this description that there are morphisms

$$S \longrightarrow S \otimes T \longleftarrow T$$

of Lawvere theories.

Example 1.9. If S and T are the theories of modules over rings A and B, respectively, as in Example 1.2, then $S \otimes T$ is the theory of $(A \otimes B)$–modules [Bor94, 3.11.7b].

Example 1.10. If S and T are the theories of G–sets and H–sets for groups G and H, respectively, as in Example 1.3, then $S \otimes T$ is the theory of $(G \times H)$–sets: sets with commuting actions by G and H.

We can pair $S = \mathbb{Z}$, the Lawvere theory of abelian groups, with any Lawvere theory T to obtain a new Lawvere theory $\mathbb{Z} \otimes T$ whose models are the abelian group objects in the category of T–models. This theory $\mathbb{Z} \otimes T$ comes with a morphism

$$T \longrightarrow \mathbb{Z} \otimes T, \tag{1.1}$$

the *linearization*. Via the discussion following Definition 1.8, we can view the linearization morphism as induced by the left adjoint to the forgetful functor that takes an abelian group object in T–models to its underlying T–model. The models over $\mathbb{Z} \otimes T$ are essentially the modules over the endomorphism ring of the linearization of T_1.

The description of the Kronecker product $S \otimes T$ in terms of its models is not the most convenient for our purpose. We shall give another description of it following Hyland and Power [HP07]. Since the category of natural numbers (i.e. our skeleton $\mathbf{E}$ of the category of finite sets) has finite products as well as sums, for $\underline{r}, \underline{s} \in \mathbf{E}$, we have the product $\underline{r} \times \underline{s}$. Since $\mathbf{E}$ is skeletal, this product is the set $\underline{rs}$; we will consider $\underline{rs}$ as the r–fold sum of $\underline{s}$ with itself. Under this identification a morphism of sets $f \colon \underline{s} \to \underline{s}'$ in $\mathbf{E}$ induces a morphism $\underline{r} \times f \colon \underline{r} \times \underline{s} \to \underline{r} \times \underline{s}'$.

For any Lawvere theory S, we can extend this construction to the category $\mathbf{F}_S$. Given $r \in \mathbb{Z}$ and $S_s \in \mathbf{F}_S$, we define

$$\underline{r} \times S_s = \underbrace{S_s + \cdots + S_s}_{r}$$

to be the r–fold sum in $\mathbf{F}_S$ of S_s with itself. The skeletalness of the category $\mathbf{F}_S$ means this sum must be the object $S_{r \times s}$, but this identification provides a corresponding construction on morphisms. If $f \colon S_s \to S_{s'}$ is a morphism in $\mathbf{F}_S$, the functoriality of sums produces a morphism

$$\underline{r} \times f \colon \underline{r} \times S_s \to \underline{r} \times S_{s'}.$$

Conjugating by the symmetry $\underline{r} \times \underline{s} \to \underline{s} \times \underline{r}$ we similarly can construct $S_s \times \underline{r}$ and a morphism $f \times \underline{r} \colon S_s \times \underline{r} \to S_{s'} \times \underline{r}$. As an object in $\mathbf{F}_S$, we have $S_s \times \underline{r} = S_{r \times s} = \underline{r} \times S_s$.

Remark 1.11. For fixed r, the construction $S_s \mapsto \underline{r} \times S_s$ yields a functor $\mathbf{F}_S \to \mathbf{F}_S$ that is strong monoidal, as does the construction $S_s \mapsto S_s \times \underline{r}$.

Definition 1.12. Given two Lawvere theories S and T, the *Kronecker* (or *tensor*) *product* Lawvere theory $S\otimes T$ is defined by the universal property of admitting maps of Lawvere theories $S\to S\otimes T$ and $T\to S\otimes T$ so that the operations of S commute with the operations of T in the sense that for all $f\colon S_r\to S_s$ in $\mathbf{F}_S$ and $f'\colon T_{r'}\to T_{s'}$ in $\mathbf{F}_T$, the diagram

$$\begin{tikzcd}
(S\otimes T)_{r\times r'} \arrow[r, "\underline{r}\times f'"] \arrow[d, "f\times\underline{r}'"'] & (S\otimes T)_{r\times s'} \arrow[d, "f\times\underline{s}'"] \\
(S\otimes T)_{s\times r'} \arrow[r, "\underline{s}\times f'"'] & (S\otimes T)_{s\times s'}
\end{tikzcd}$$

commutes (in the category $\mathbf{F}_{S\otimes T}$). The vertical maps here should be interpreted as the image of the maps $f\times r'\colon S_{r\times r'}\to S_{s\times r'}$ in $\mathbf{F}_S$ under the map of Lawvere theories $S\to S\otimes T$, and similarly for the horizontal maps, *mutatis mutandis*.

Remark 1.13. Hyland and Power [HP07, Proposition 3.3] remark that the Kronecker product extends to a symmetric monoidal structure on the category of Lawvere theories with the theory E of sets as the unit theory. The construction of $S\otimes T$ can be done by hand, or it can be viewed as a special case of their work on pseudo-commutativity and, in particular, on the pseudo-closed structure of the 2–category of symmetric monoidal categories [HP02]. Much becomes easier in the present case because by definition, a map of Lawvere theories is a map under $\mathbf{E}$, i.e., a strictly commuting diagram of the following form:

$$\begin{tikzcd}
 & \mathbf{F}_S \arrow[dd, "L"] \\
\mathbf{E} \arrow[ur] \arrow[dr] & \\
 & \mathbf{F}_T
\end{tikzcd}$$

Thus, a natural transformation between such L must restrict to the identity natural transformation on $\mathbf{E}$. Since all objects in $\mathbf{F}_S$ are in the image of $\mathbf{E}$, this forces all natural transformations to be the identity. In other words, Lawvere theories naturally form a 1–category rather than a 2–category, so there is no room for the psubtlety of pseudoness.

1.4 Matrix theories

Given a Lawvere theory T and an integer $n\geqslant 1$, the *matrix theory* $\mathrm{M}_n(T)$ is the Lawvere theory such that the free $\mathrm{M}_n(T)$–model on a set X is the free T–model on the set $\underline{n}\times X$ (see [Wra71, Sec. 4]). In other words, the category $\mathbf{F}_{\mathrm{M}_n(T)}$ is the full subcategory of the category $\mathbf{F}_T$ consisting of the objects T_{nr} for $r\geqslant 0$.

More diagrammatically, if $\underline{n}\times -$ is the strong monoidal functor of Remark 1.11, then the underlying category of the Lawvere theory $\mathrm{M}_n(T)$ is the image of $\underline{n}\times -$ and the structure functor that defines $M_n(T)$ as a Lawvere theory is the composite

$$\mathbf{E}\longrightarrow\mathbf{F}_T\xrightarrow{\underline{n}\times -}\mathbf{F}_T.$$

It is easy to describe all $\mathrm{M}_n(T)$–models up to isomorphism: given a T–model X, we can construct an $\mathrm{M}_n(T)$–model on the n–th cartesian power X^n of X; the r–ary $\mathrm{M}_n(T)$–operations $(X^n)^r \to X^n$ are the maps such that all components $(X^n)^r \to X$ are nr–ary T–operations on X. In particular, we get a unary operation $X^n \to X^n$ for each self-map of the set $\underline{n}$, and so the monoid $\mathrm{End}(\underline{n})$ acts on the model X^n. Every model arises this way, up to isomorphism. Every $\mathrm{M}_n(T)$–model of the form X^n has an underlying T–model consisting of the operations that are themselves n–th powers, which gives a forgetful functor $\mathbf{M}_{\mathrm{M}_n(T)} \to \mathbf{M}_T$. Equivalently, there is a morphism

$$T \longrightarrow \mathrm{M}_n(T) \tag{1.2}$$

of Lawvere theories. From the diagrammatic perspective, this morphism is simply the above functor $\underline{n} \times -\colon \mathbf{F}_T \to \mathbf{F}_{\mathrm{M}_n(T)} \subset \mathbf{F}_T$, which by construction is a functor under $\mathbf{E}$ and hence a map of Lawvere theories. We readily observe that $\mathrm{M}_1(T) \cong T$ and $\mathrm{M}_m(\mathrm{M}_n(T)) \cong \mathrm{M}_{mn}(T)$.

Using the initial Lawvere theory E of sets, we can write $\mathrm{M}_n(T) \cong T \otimes \mathrm{M}_n(E)$, since an $\mathrm{M}_n(T)$–model is visibly a T–model in $\mathrm{M}_n(E)$–models (i.e. sets that are n–fold products). More generally, we have an isomorphism $\mathrm{M}_n(S \otimes T) \cong S \otimes \mathrm{M}_n(T)$ of theories.

Example 1.14. If T is the theory of modules over a ring A as in Example 1.2, then $\mathrm{M}_n(T)$ is the theory of modules over the matrix ring $\mathrm{M}_n(A)$.

Example 1.15. The Lawvere theory $\mathrm{M}_n(E)$ is the theory of $\mathrm{End}(\underline{n})$–sets.

1.5 Morita equivalence

We need to explain one more construction before we can come to Morita equivalence: idempotent modifications.

Let T be a Lawvere theory with an idempotent endomorphism $u\colon T_1 \to T_1$ of the free T–model T_1 on one generator. We write $u_n\colon T_n \to T_n$ for the n–fold sum, so that $u_1 = u$.

Lemma 1.16. *Consider the following properties for a morphism $f\colon T_r \to T_s$ in $\mathbf{F}_T$ with respect to a fixed idempotent u.*
(1) $f = u_s g u_r$ for some $g\colon T_r \to T_s$
(2) $u_s f = f = f u_r$
(3) $u_s f = f u_r$
Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3). *We have* (2) $\Leftarrow$ (3) *if and only if* $u = \mathrm{id}$.

We define $\mathbf{F}_T^u \leqslant \mathbf{F}_T$ to be the subcategory (!) consisting of the morphisms that satisfy condition (3) in Lemma 1.16 above. Note that (1) and (2) do not define a subcategory in general, because the identities satisfy (3), but not necessarily (1) or (2). However, we can define a new category structure on the subsets of $\mathbf{F}_T(T_r, T_s)$ of morphisms satisfying conditions (1) and (2): these subsets are closed under composition, and the u_r's act as new identities. This gives another category $\mathbf{F}_{uTu}$ and another Lawvere theory, the *idempotent modification* uTu of T with respect to the idempotent u. There is a functor $\mathbf{F}_T^u \to \mathbf{F}_{uTu}$ defined by $f \mapsto uf = ufu = fu$, and we can, in principle, compare the new Lawvere theory uTu to T using the zigzag

$$\mathbf{F}_{uTu} \longleftarrow \mathbf{F}_T^u \longrightarrow \mathbf{F}_T$$

of functors defined above, all of which are the identities on objects.

Definition 1.17. Two Lawvere theories S and T are called *Morita equivalent* if their categories $\mathbf{M}_S$ and $\mathbf{M}_T$ of models are equivalent.

An idempotent u is *pseudo-invertible* if there are morphisms $T_1 \to T_k$ and $T_k \to T_1$ such that their composition around $u_k \colon T_k \to T_k$ is the identity on T_1.

Proposition 1.18 ([Duk88], [McK96]). *A Lawvere theory is Morita equivalent to a given Lawvere theory T if and only if it is an idempotent modification of a matrix theory of T for some pseudo-invertible idempotent of the matrix theory.*

We refer to the textbook treatment in [ARV11, Ch. 6, 8, 15] for proofs of this and the following fact.

Proposition 1.19. *Two Lawvere theories are Morita equivalent if and only if their categories $\mathbf{F}_S$ and $\mathbf{F}_T$ of free models have equivalent idempotent completions.*

2 Algebraic K-theory

In this section, we define the algebraic K-theory spectrum $\mathrm{K}(T)$ of a Lawvere theory T, show how it encodes the stable homology of the automorphism groups of free T–models, and prove our positive results on Morita invariance.

2.1 The algebraic K-theory of Lawvere theories

We first specify the constructions of K-theory we use in this paper. Our primary approach is to view Lawvere theories as a special case of symmetric monoidal categories and apply the classic constructions of K-theory for the latter. There are several ways of approaching these constructions; we begin with a brief overview.

Let $\mathbf{S}$ denote a symmetric monoidal groupoid. For the following to make sense, it needs to satisfy an additional assumption, but we show in Proposition 2.4 that this is always the case for the categories we are interested in. We can then pass to Quillen's categorification $\mathbf{S}^{-1}\mathbf{S}$ of the Grothendieck construction. The canonical morphism $\mathrm{B}\mathbf{S} \to \mathrm{B}\mathbf{S}^{-1}\mathbf{S}$ between the classifying spaces is a group completion, and the target is an infinite loop space. We refer to [Gra76] and Thomason's particularly brief and enlightening discussion [Tho80] for detail. To build a K-theory spectrum $\mathrm{K}(\mathbf{S})$ with underlying infinite loop space $\Omega^\infty\mathrm{K}(\mathbf{S}) \simeq \mathrm{B}\mathbf{S}^{-1}\mathbf{S}$, we can use Segal's definition of the algebraic K-theory of a symmetric monoidal category in terms of Γ–spaces. The equivalence comes from [Seg74, §4], where he shows that $\Omega^\infty\mathrm{K}(\mathbf{S})$ is also a group completion of $\mathrm{B}\mathbf{S}$. Finally, for the multiplicativity properties we need later in Section 5, we in fact wish to use a variant of Segal's construction given by Elmendorf–Mandell [EM06] (see also [BO20]) which takes values in the category of symmetric spectra and builds the spaces of the K-theory spectrum "all at once" instead of iteratively.

Definition 2.1. Let T be a Lawvere theory. The *algebraic K-theory* of T is the spectrum

$$\mathrm{K}(T) = \mathrm{K}(\mathbf{F}_T^\times), \tag{2.1}$$

that is, the spectrum corresponding to the symmetric monoidal groupoid $\mathbf{F}_T^\times$ of isomorphisms in the symmetric monoidal category $\mathbf{F}_T$ of finitely generated free T–models, where the monoidal structure is given by the categorical sum.

Since the category $\mathbf{F}_T$ can be identified with the symmetric monoidal category of finitely generated free T–models, Definition 2.1 concerns the algebraic K-theory of finitely generated free T–models. In particular, the group $\mathrm{K}_0(T) = \pi_0\mathrm{K}(T)$ is the Grothendieck group of isomorphism classes of finitely generated free T–models. This group is always cyclic, generated by the isomorphism class $[T_1]$ of the free T–model on one generator. However, the group $\mathrm{K}_0(T)$ does not have to be infinite cyclic, as the Examples 2.10 and 3.7 show.

Remark 2.2. A morphism $S \to T$ of Lawvere theories (as in Definition 1.8) induces, via the left-adjoint functor $\mathbf{F}_S \to \mathbf{F}_T$, a morphism $\mathrm{K}(S) \to \mathrm{K}(T)$ of algebraic K-theory spectra. The left adjoint $\mathbf{F}_S \to \mathbf{F}_T$ sends the free S–model S_1 on one generator to the free T–model T_1 on one generator. It follows that the induced homomorphism $\mathrm{K}_0(S) \to \mathrm{K}_0(T)$ between cyclic groups is surjective, being the identity on representatives.

2.2 Stable homology and the plus construction

One reason for interest in the algebraic K-theory of Lawvere theories is the relation to the stable homology of the sequence of automorphism groups attached to a Lawvere theory. We now make this relation made precise.

Let T be a Lawvere theory. The automorphism groups of the free algebras T_r often turn out to be very interesting (see the Examples in Section 3 below). We use the notation $\mathrm{Aut}(T_r)$ for these groups.

Given integers $r, s \geqslant 0$, there is a *stabilization* homomorphism

$$\mathrm{Aut}(T_r) \longrightarrow \mathrm{Aut}(T_{r+s}) \tag{2.2}$$

that 'adds' the identity of the object T_s in the sense of the categorical sum $+$, and we use additive notation for this operation. More precisely, stabilization sends an automorphism u of T_r to the automorphism of T_{r+s} that makes the diagram

$$\begin{tikzcd} T_{r+s} \arrow[r, dashed] & T_{r+s} \\ T_r + T_s \arrow[u, "\cong"] \arrow[r, "u+T_s", "\cong"'] & T_r + T_s \arrow[u, "\cong"'] \end{tikzcd}$$

commute. By abuse of notation, this automorphism of the object T_{r+s} will sometimes also be denoted by $u + T_s$.

Remark 2.3. The alert reader will have noticed that we have not specified our choice of isomorphism $T_r + T_s \cong T_{r+s}$ in the preceding diagram, and we do not need to: all such choices obviously differ by some conjugation, so that they induce the same map in homology, which is all that matters for the purposes of this section.

Proposition 2.4. *For every Lawvere theory T, the stabilization maps*

$$\mathrm{Aut}(T_r) \longrightarrow \mathrm{Aut}(T_{r+1})$$

are injective.

Proof. It is enough to show that the kernels are trivial. This is clear for $r = 0$, since T_0 is initial, so that $\mathrm{Aut}(T_0)$ is the trivial group. For positive r we can choose a retraction ρ of the canonical embedding $\sigma \colon T_r \to T_{r+1}$. If u is in the kernel of the stabilization map, then we have the following commutative diagram.

$$\begin{tikzcd} T_r \arrow[r, "u"] \arrow[d, "\sigma"'] & T_r \arrow[d, "\sigma"'] \arrow[dr, equal] & \\ T_{r+1} \arrow[r, "\mathrm{id}"'] & T_{r+1} \arrow[r, "\rho"'] & T_r \end{tikzcd}$$

It implies $u = \mathrm{id}$. $\square$

Stabilization leads to a diagram

$$\mathrm{Aut}(T_0) \longrightarrow \mathrm{Aut}(T_1) \longrightarrow \mathrm{Aut}(T_2) \longrightarrow \mathrm{Aut}(T_3) \longrightarrow \cdots \tag{2.3}$$

of groups for every Lawvere theory T. We write $\mathrm{colim}_r \mathrm{Aut}(T_r)$ for the colimit of the diagram (2.3) with respect to the stabilization maps. This is the *stable automorphism group* for the Lawvere theory T.

Let us record the following group theoretical property of the stable automorphism groups. This is presumably well-known already in more or less generality. We nevertheless include an argument here for completeness' sake.

Proposition 2.5. *For every Lawvere theory T, the commutator subgroup of the stable automorphism group* $\operatorname{colim}_r \operatorname{Aut}(T_r)$ *is perfect.*

Proof. Given a commutator in the group $\operatorname{colim}_r \operatorname{Aut}(T_r)$, we can represent it as $[u,v]$ for a pair u,v of automorphisms in the group $\operatorname{Aut}(T_r)$ for some r. Allowing us thrice the space, in the group $\operatorname{Aut}(T_{3r})$ we have the identity

$$[u,v]+\operatorname{id}(T_{2r}) = [u+u^{-1}+\operatorname{id}(T_r), v+\operatorname{id}(T_r)+v^{-1}].$$

It therefore suffices to prove that each element of the form $w+w^{-1}$ is a commutator. This is a version of Whitehead's lemma that holds in every symmetric monoidal category: whenever there are automorphisms $w_1,\dots,w_n$ of an object such that their composition $w_1\cdots w_n$ is the identity, then $w_1+\cdots+w_n$ is a commutator. We apply this to the category $\mathbf{F}_T$ with respect to the monoidal product given by categorical sum $+$. □

After these preliminaries, we now move on to give another model for the algebraic K-theory space of a Lawvere theory T, one that uses the Quillen plus construction. This construction led to Quillen's historically first definition of the algebraic K-theory of a ring [Qui71] (see also [Wag72] and [Lod76]).

The plus construction can be applied to connected spaces X for which the fundamental groups have perfect commutator subgroups. It produces a map $X \to X^+$ into another connected space X^+ with the same integral homology, and such that the induced maps on fundamental groups are the abelianization. In fact, these two properties characterize the plus construction. By Proposition 2.5, the commutator subgroup of $\operatorname{colim}_r \operatorname{Aut}(T_r)$ is perfect. Therefore, the plus construction can be applied the classifying space $\operatorname{Bcolim}_r \operatorname{Aut}(T_r)$ in order to produce another space $\operatorname{Bcolim}_r \operatorname{Aut}(T_r)^+$.

Theorem 2.6. *For every Lawvere theory T, there is an equivalence*

$$\Omega^\infty \mathrm{K}(T) \simeq \mathrm{K}_0(T) \times \operatorname{Bcolim}_r \operatorname{Aut}(T_r)^+ \tag{2.4}$$

of spaces.

Proof. Quillen, in the his proof that the plus construction of K-theory agrees with the one obtained from the Q-construction, takes an intermediate step (see [Gra76, p. 224]): he shows that the plus construction, together with K_0, gives a space that is equivalent to the classifying space of his categorification $\mathbf{S}^{-1}\mathbf{S}$ of the Grothendieck construction of a suitable symmetric monoidal category $\mathbf{S}$. This part of his argument applies here to show that there is an equivalence

$$\mathrm{K}_0(T) \times \operatorname{Bcolim}_r \operatorname{Aut}(T_r)^+ \simeq \mathrm{B}((\mathbf{F}_T^\times)^{-1}\mathbf{F}_T^\times)$$

of spaces for every Lawvere theory T. The claim follows because we already know that the right hand side has the homotopy type of $\Omega^\infty \mathrm{K}(T)$. □

In general, there seems to be no reason to believe that such an artificial product as in (2.4) would form a meaningful whole. The present case is special because $\mathrm{K}_0(T)$ is generated by the isomorphism class of the free T–algebra T_1 of rank 1. Other constructions of the same homotopy type do not separate the group $\mathrm{K}_0(T)$ of components from the rest of the space. One way or another, note that all components of the algebraic K-theory space $\mathrm{K}(T)$ are equivalent; the group $\mathrm{K}_0(T)$ of components acts transitively on the infinite loop space $\Omega^\infty\mathrm{K}(T)$ up to homotopy.

Since the plus construction does not change homology, the definition of the algebraic K-theory space immediately gives the following result.

Theorem 2.7. *For every Lawvere theory T, there is an isomorphism*

$$\operatorname{colim}_r \mathrm{H}_*(\mathrm{Aut}(T_r)) \cong \mathrm{H}_*(\Omega_0^\infty\mathrm{K}(T))$$

between the stable homology of the automorphism groups of finitely generated free objects of the theory T and the homology of the zero component $\Omega_0^\infty\mathrm{K}(T)$ of the algebraic K-theory space $\Omega^\infty\mathrm{K}(T)$.

Ideally, the algebraic K-theory spectrum $\mathrm{K}(T)$ is more accessible and easier to understand and describe than the stable automorphism group $\operatorname{colim}_r \mathrm{Aut}(T_r)$. This is not at all plausible from the definition; only the now-classical methods of algebraic K-theory that have been developed over half a century that allow us to take this stance. From this perspective, Theorem 2.7 should be thought of as a computation of the group homology, once the spectrum $\mathrm{K}(T)$ is identified. The examples in Sections 3 and 4 give a taste of the flavor of some non-trivial (and non-linear) cases.

2.3 Morita invariance

We close this section by showing that the higher K-theory of a Lawvere theory T is invariant under passage to matrix theories $\mathrm{M}_n(T)$ discussed in Section 1.4. On the other hand, we shall see later that K-theory is *not* invariant under passage to idempotent completions. Therefore, the K-theory $\mathrm{K}(T)$ cannot be computed from the category of T–models alone.

Theorem 2.8. *For every Lawvere theory T, there is an equivalence*

$$\Omega_0^\infty\mathrm{K}(\mathrm{M}_n(T)) \simeq \Omega_0^\infty\mathrm{K}(T)$$

of infinite loop spaces.

Proof. We may use that the existence of isomorphisms $\mathrm{M}_n(T)_r \cong T_{n\times r}$ of models implies that we have isomorphisms

$$\mathrm{Aut}(\mathrm{M}_n(T)_r) \cong \mathrm{Aut}(T_{n\times r})$$

between the automorphism groups. Therefore, when we compare the diagrams (2.3), the one with the groups $\mathrm{Aut}(\mathrm{M}_n(T)_r)$ for $\mathrm{M}_n(T)$ naturally embeds as a cofinal subdiagram of the digram with the groups $\mathrm{Aut}(T_r)$ for T. We only see every n–th term, but the colimits can be identified, of course, and this proves the statement. □

Remark 2.9. The equivalence in Theorem 2.8 is not induced by the morphism $\mathrm{K}(T) \to \mathrm{K}(\mathrm{M}_n T)$ of spectra that comes from the canonical morphism (1.2) of theories. It *does* come from a morphism $\mathrm{K}(\mathrm{M}_n T) \to \mathrm{K}(T)$ of spectra in the other direction, but this morphism is not induced from a morphism of theories; it comes from the symmetric monoidal functor $\mathbf{F}_{\mathrm{M}_n(T)} \to \mathbf{F}_T$ given by the inclusion of $\mathbf{F}_{\mathrm{M}_n(T)}$ into $\mathbf{F}_T$ as the image of the functor $\underline{n} \times -$. This functor is defined by $\mathrm{M}_n(T)_r \cong T_{n\times r} \mapsto T_{n\times r}$ and so while it is essentially the identity on morphisms, it is not necessarily surjective on objects. In particular, it need not be surjective on the level of components, as is required for a map of Lawvere theories according to Remark 2.2.

In fact, as tempting as it might be to hope for an equivalence $\mathrm{K}(\mathrm{M}_n T) \simeq \mathrm{K}(T)$ of K-theory spectra, we *cannot* have that, in general, because of the difference in the groups K_0 of components:

Example 2.10. As explained in [SW19, Rem. 5.3] and Example 3.7 of the following section, the Cantor theories Cantor_a of arity $a \geqslant 2$ have $\mathrm{K}_0(\mathrm{Cantor}_a) = \mathbb{Z}/(a-1)$ finite. But by construction, the matrix theory $\mathrm{M}_n(\mathrm{Cantor}_a)$ only involves the elements represented by multiples of n in the group $\mathbb{Z}/(a-1)$. Therefore, if n is not coprime to $a-1$, then $\mathrm{K}_0(\mathrm{M}_n\mathrm{Cantor}_a)$ will be strictly smaller than $\mathrm{K}_0(\mathrm{Cantor}_a)$. In particular, the morphisms between $\mathrm{K}(\mathrm{Cantor}_a)$ and $\mathrm{K}(\mathrm{M}_n\mathrm{Cantor}_a)$ described in Remarks 2.2 and 2.9 are *not* equivalences in this case.

In Theorem 4.1, we provide examples of Lawvere theories that are Morita equivalent but have different higher K-theory. Specifically, there is a family of Lawvere theories that are all Morita equivalent to the theory of Boolean algebras. When we compute their algebraic K-theories, we find that they are all different.

3 Non-linear examples

The goal of this section is to demonstrate the interest in the algebraic K-theory $\mathrm{K}(T)$ of Lawvere theories T beyond what are arguably the most fundamental examples, the theories of modules over rings:

Example 3.1. Consider the theory of modules over a ring A, as in Example 1.2. The automorphism group of the free A–module A^r of rank r is the general linear group $\mathrm{Aut}(A^r) = \mathrm{GL}_r(A)$. The algebraic K-theory spectrum $\mathrm{K}(A)$ is Quillen's algebraic K-theory (actually, the 'free' version). In particular $\mathrm{K}(\mathbb{Z})$ is the K-theory spectrum of the Lawvere theory of abelian groups (in the guise of $\mathbb{Z}$–modules).

We can now move on to discuss non-linear examples: theories that are not given as modules over a ring.

3.1 Sets and variants

Example 3.2. Consider the initial theory E of sets. The automorphisms are just the permutations, and the automorphism group $\mathrm{Aut}\{1,\dots,r\} = \Sigma(r)$ is the symmetric group on r symbols. The algebraic K-theory is the sphere spectrum: $\mathrm{K}(E) \simeq \mathbb{S}$. This is one version of the Barratt–Priddy theorem [Pri71, BP72]. We provide details so that we can use the same notation later as well: Let $\mathrm{Q} \simeq \Omega^\infty\mathbb{S}$ denote the infinite loop space of stable self-maps of the spheres. The path components of the space Q are indexed by the degree of the stable maps, as a reflection of $\pi_0(\mathbb{S}) = \mathbb{Z}$, and we will write $\mathrm{Q}(r)$ for the component of maps of degree r. There are maps $\mathrm{B}\Sigma(r) \to \mathrm{Q}(r)$ which are homology isomorphisms in a range that increases with r by Nakaoka stability [Nak60]. These maps fit together to induce a homology isomorphism

$$\mathrm{B}\Sigma(\infty) \to \mathrm{Q}(\infty) \tag{3.1}$$

between the colimits. The stabilization $\mathrm{Q}(r) \to \mathrm{Q}(r+1)$ is always an equivalence, so that all the maps $\mathrm{Q}(r) \to \mathrm{Q}(\infty)$ to the colimit are equivalences as well. Passing to group completions, we get from the map (3.1) an equivalence

$$\Omega^\infty_0\mathrm{K}(E) \simeq \Omega^\infty_0\mathbb{S}$$

of infinite loop spaces, so that $\mathrm{K}(E) \simeq \mathbb{S}$ as spectra. We refer to Morava's notes [Mor] for more background and relations to the algebraic K-theory of the finite fields $\mathbb{F}_q$ when the number q of elements goes to 1.

Example 3.3. More generally, for any discrete group G, we can consider the Lawvere theory of G–sets. The algebraic K-theory of the theory of free G–sets is

$$\mathrm{K}(G\text{–Sets}) \simeq \Sigma^\infty_+(\mathrm{B}G),$$

the suspension spectrum of the classifying space $\mathrm{B}G$ (with a disjoint base point $+$). This observation is attributed to Segal. In particular, for the Lawvere theory $\mathbb{Z}$–sets, this gives

$$\mathrm{K}(\mathbb{Z}\text{–Sets}) \simeq \Sigma^\infty_+(\mathrm{B}\mathbb{Z}) \simeq \Sigma^\infty_+(\mathrm{S}^1) \simeq \mathbb{S} \vee \Sigma\mathbb{S}.$$

The theory $\mathbb{Z}$–sets is the theory of permutations: a model is a set together with a permutation of that set.

3.2 Groups and variants

Example 3.4. Consider the theory Groups of (all) groups. In this case, the automorphism groups $\mathrm{Aut}(\mathrm{F}_r)$ are the automorphism groups of the free groups F_r on r generators. The algebraic K-theory space has been shown to be the infinite loop space underlying the sphere spectrum by Galatius [Gal11]: the unit $\mathbb{S} \to \mathrm{K}(\mathrm{Groups})$ is an equivalence.

The theory of abelian groups has been dealt with in Example 3.1.

Example 3.5. There is an interpolation between the theory of all groups and the theory of all abelian groups by the theories Nil_c of nilpotent groups of a certain class c, with $1 \leqslant c \leqslant \infty$. There is a corresponding diagram

$$
\begin{tikzcd}
 & & \vdots \arrow[d] & \\
 & & \mathrm{K}(\mathrm{Nil}_3) \arrow[d] & \\
 & & \mathrm{K}(\mathrm{Nil}_2) \arrow[d] & \\
\mathbb{S} \arrow[r, equal] & \mathrm{K}(\mathrm{Groups}) \arrow[r] \arrow[ur] \arrow[uur] & \mathrm{K}(\mathrm{Abel}) \arrow[r, equal] & \mathrm{K}(\mathbb{Z})
\end{tikzcd}
$$

of algebraic K-theory spectra. This tower has been studied from the point of view of homological stability and stable homology in [Szy14] and [Szy19], respectively.

Example 3.6. In contrast to groups, the algebraic K-theory of the Lawvere theory Monoids of (associative) monoids (with unit) is easy to compute: the free monoid on a set X is modeled on the set of words with letters from that set, and it has a unique basis: the subset of words of length one, which can be identified with X. This implies that the automorphism group of the free monoid on r generators is isomorphic to the symmetric group $\Sigma(r)$, so that the map $\mathrm{K}(E) \to \mathrm{K}(\mathrm{Monoids})$ from the algebraic K-theory of the initial theory E of sets is an equivalence. By Example 3.2, we get an equivalence

$$\mathrm{K}(\mathrm{Monoids}) \simeq \mathbb{S}$$

of spectra. It follows, again from Galatius's theorem (see Example 3.4), that the canonical morphism $\mathrm{K}(\mathrm{Monoids}) \to \mathrm{K}(\mathrm{Groups})$ is an equivalence. It would be interesting to see a proof of this fact that does not depend on his result.

3.3 More exotic theories

Example 3.7. Let $a \geqslant 2$ be an integer. A *Cantor algebra* of arity a is a set X together with a bijection $X^a \to X$. The Cantor algebras of arity a are the models for a Lawvere theory Cantor_a, and its algebraic K-theory has been computed in [SW19]:

$$\mathrm{K}(\mathrm{Cantor}_a) \simeq \mathbb{S}/(a-1), \tag{3.2}$$

the Moore spectrum mod $a-1$. In particular, the spectrum $\mathrm{K}(\mathrm{Cantor}_2)$ is contractible. Note that the definition makes sense for $a = 1$ as well. In that case, we have an isomorphism between Cantor_1 and the Lawvere theory $\mathbb{Z}$–Sets of permutations, and the equivalence (3.2) is still true by Example 3.3.

Example 3.8. Lawvere theories can be presented by generators and relations. The 'generators' of a theory are specified in terms of a graded set $P = (P_a \,|\, a \geqslant 0)$, where P_a is a set of operations of arity a. There is a free Lawvere theory functor $P \mapsto T_P$ that is left adjoint to the functor that assigns to a theory the graded set of operations. For instance, let $[a]$ be the graded set that only has one element, and where the degree of that element is a. Then $T_{[a]}$ is the free theory generated by one operation of arity a. For instance, the Lawvere theory $T_{[0]}$ is the theory of pointed sets. The Lawvere theory $T_{[1]}$ is the theory of self-maps (or $\mathbb{N}$–sets): sets together with a self-map, and $T_{[2]}$ is the theory of magmas: sets equipped with a multiplication that does not have to satisfy any axioms. The free $T_{[a]}$–model on a set X is given by the set of all trees of arity a with leaves colored in X. This model has a unique basis: the trees of height 1, and we can argue as in Example 3.6 that $\mathrm{K}(T_{[a]}) \simeq \mathbb{S}$.

3.4 Inconsistent theories

Finally, we mention the two trivial (or *inconsistent*, in Lawvere's terminology) examples of theories where the free model functor is not faithful (see Lawvere's thesis [Law04, II.1, Prop. 3]).

Example 3.9. There is a theory such that all models are either empty or singletons. It has no operations in addition to the projections $X^n \to X$, and the relations are that all these projections are equal, so that $x_1 = x_2$ for all elements x_j in a set X that is a model.

Example 3.10. There is a theory such that all models are singletons. It has a 0–ary operation (constant) e, and the relation $x = e$ has to be satisfied for all x in a model X. Another way of describing the same Lawvere theory: this is the theory of modules over the trivial ring, where $0 = 1$. From this perspective, the theory is not so exotic after all!

For both of these examples, the algebraic K-theory spectra are obviously contractible.

4 Boolean algebras and Morita equivalent theories

In this section, we present new computations: we determine the algebraic K-theory of the Lawvere theory of Boolean algebras. Our methods allow us to deal more generally with the Lawvere theories of v–valued Post algebras. Boolean algebras form the case $v = 2$. The Lawvere theories of v–valued Post algebras are all Morita equivalent to each other. In fact, these form the set of all the Lawvere theories that are equivalent to the theory of Boolean algebras. As a consequence of our computations, we show that algebraic K-theory is not Morita invariant in general.

Boolean algebras and their relationship to set theory and logic are fundamental for mathematics and well-known. Post algebras were introduced by Rosenbloom [Ros42]. They are named after Post's work [Pos21] on non-classical logics with v truth values. Later references are Wade [Wad45], Epstein [Eps60], as well as the surveys by Serfati [Ser73] and Dwinger [Dwi77], to which we refer for defining equations and explicit models of the free algebras. In the following, we will only recall their definition as a Lawvere theory and what is necessary for our purposes.

We write $\mathrm{Map}(R,S)$ for the set of all maps from a set R to a set S. As before, we build on the specific finite sets $\underline{r} = \{a \in \mathbb{Z} \,|\, 1 \leqslant a \leqslant r\}$. For a fixed integer $v \geqslant 2$, we now consider the category of finite sets of the form $\mathrm{Map}(\underline{r},\underline{v})$, where r ranges over all integers $r \geqslant 0$, and all maps between these sets. By construction, this category has finite products, and every object $\mathrm{Map}(\underline{r},\underline{v})$ is the r–th power of the object $\mathrm{Map}(\underline{1},\underline{v}) = \underline{v}$. Therefore, the opposite category has finite co-products, and every object is a multiple of one object, the one corresponding to the set $\mathrm{Map}(\underline{1},\underline{v})$. This opposite category defines the Lawvere theory Post_v of *v–valued Post algebras*.

For $v = 2$, Post's v–valued logic specializes to the 2–valued Boolean logic, and we have

$$\mathrm{Post}_2 = \mathrm{Boole}, \tag{4.1}$$

the Lawvere theory of Boolean algebras. From our description above, this is a well-known consequence of Stone duality: the set of subsets of $\mathrm{Map}(\underline{r},\underline{2})$ is a free Boolean algebra on r generators, with 2^{2^r} elements in total.

Dukarm [Duk88, Sec. 3] notes that the Lawvere theories Post_v are all Morita equivalent to each other. After all, for any given integer $v \geqslant 2$, any finite set is a retract of a set of the form $\mathrm{Map}(\underline{r},\underline{v})$ for $r \geqslant 0$ large enough. There is no need for us to choose such a retraction. (The situation is comparable to the abstract existence of isomorphisms $\overline{\mathbb{Q}}_p \cong \mathbb{C}$ of fields between the algebraic closure $\overline{\mathbb{Q}}_p$ of the field $\mathbb{Q}_p$ of p–adic numbers and the field $\mathbb{C}$ of complex numbers, showing that the isomorphism type of $\overline{\mathbb{Q}}_p$ is independent of p.) In any event, it follows from the existence of such retractions that the idempotent completions of the categories of free v–valued Post algebras are equivalent to the category of non-empty finite sets, regardless of v. Since these idempotent completions are independent of the integer v, so is the Morita equivalence class of Post_v, by the results recalled in Section 2.3. The following theorem shows that, in contrast, higher algebraic K-theory detects the number v of truth values, and K-theory is therefore not fully Morita invariant.

Theorem 4.1. *For every integer $v \geqslant 2$ we have*

$$\mathrm{K}_*(\mathrm{Post}_v) \cong \pi_*(\mathbb{S})/v\text{–power torsion},$$

where the $\pi_*(\mathbb{S})$ are the stable homotopy groups of spheres.

We single out the case $v = 2$ for emphasis:

Corollary 4.2. *We have*

$$\mathrm{K}_*(\mathrm{Boole}) \cong \pi_*(\mathbb{S})/2\text{–power torsion}$$

for the algebraic K-theory of the Lawvere theory of Boolean algebras.

While Boolean algebras form a comparatively well-known algebraic structure, the v–valued Post algebras are certainly non-standard, and it might come as a surprise that we can prove such results without even revealing their defining operations, let alone the axioms that these operations are required to satisfy. However, as we hope the following proof makes clear, the ability to do so is precisely one of the benefits of our categorical methods.

Proof of Theorem 4.1. By definition, the category of free v–valued Post algebras is equivalent to the opposite of the full subcategory of the category of sets spanned by those sets of the form $\mathrm{Map}(\underline{r},\underline{v})$. Since these have different cardinalities for different values of r, the isomorphism type of the free v–valued Post algebra of rank r determines the rank r. Passing to group completion, we get $\mathrm{K}_0(\mathrm{Post}_v) \cong \mathbb{Z} \cong \pi_0(\mathbb{S})$, as claimed.

For the higher K-theory, we turn toward the automorphism groups. If X is an object in a category $\mathbf{C}$, we have

$$\mathrm{Aut}_{\mathbf{C}^{\mathrm{op}}}(X) \cong \mathrm{Aut}_{\mathbf{C}}(X)^{\mathrm{op}} \cong \mathrm{Aut}_{\mathbf{C}}(X).$$

Applied to our situation, this shows that the automorphism group of the free v–valued Post algebra of rank r is isomorphic to the group of permutations of the set $\mathrm{Map}(\underline{r},\underline{v})$ of cardinality v^r, and therefore to the symmetric group $\Sigma(v^r)$ acting on a set of v^r elements.

Stabilization leads us to the colimit of the diagram

$$\Sigma(1) \longrightarrow \Sigma(v) \longrightarrow \Sigma(v^2) \longrightarrow \cdots \longrightarrow \Sigma(v^r) \longrightarrow \cdots,$$

where the morphisms are given by multiplication with v: a permutation σ of $\underline{v}^r$ is sent to the permutation $\sigma \times \mathrm{id}_v$ of $\underline{v}^r \times \underline{v} = \underline{v}^{r+1}$, which looks just like v copies of the permutation σ acting on v disjoint copies of $\underline{v}^r$. In other words, $\sigma \times \mathrm{id}_v$ is a block sum of v copies of σ.

Picking up our notation from Example 3.2, we have maps $\mathrm{B}\Sigma(d) \to \mathrm{Q}(d)$ that fit together to form a commutative diagram as follows.

$$\begin{tikzcd}
\mathrm{B}\Sigma(1) \arrow[r, "\times v"] \arrow[d] & \mathrm{B}\Sigma(v) \arrow[r, "\times v"] \arrow[d] & \mathrm{B}\Sigma(v^2) \arrow[r, "\times v"] \arrow[d] & \cdots \\
\mathrm{Q}(1) \arrow[r, "\times v"'] & \mathrm{Q}(v) \arrow[r, "\times v"'] & \mathrm{Q}(v^2) \arrow[r, "\times v"'] & \cdots
\end{tikzcd}$$

This diagram can be used to compute the group completion of the upper colimit, which is the infinite loop space $\Omega^\infty_0\mathrm{K}(\mathrm{Post}_v)$ by Theorem 2.6. This time, in contrast to Example 3.2, the maps in the lower row are not equivalences, but multiplication by v in the infinite loop space structure on the $\mathrm{Q}(v^r) \simeq \mathrm{Q}(\infty) \simeq \mathrm{Q}(0)$. In other words, there is a homology isomorphism from the colimit $\mathrm{B}\Sigma(v^\infty)$ to the localization $\mathrm{Q}(0)[1/v]$ away from v. This homology isomorphism gives, after group completion, an equivalence

$$\Omega^\infty_0\mathrm{K}(\mathrm{Post}_v) \simeq \Omega^\infty_0\mathbb{S}[1/v]$$

of infinite loop spaces. Noting that the higher stable homotopy groups of the sphere are finite, and $A[1/v] = A/(v\text{–power torsion})$ for finite abelian groups A, we obtain the result of the theorem. □

We end this section with an observation which indicates that the relationship between the K-theories of the Lawvere theory E of sets and of Boolean algebras, or more generally v–valued Post algebras, is not as simple as Theorem 4.1 might suggest.

Proposition 4.3. *For each prime p, the homomorphism*

$$\pi_n(\mathbb{S}) \cong \mathrm{K}_n(E) \longrightarrow \mathrm{K}_n(\mathrm{Post}_p) \cong \pi_n(\mathbb{S})/p\text{–}power\ torsion,$$

induced by the universal arrow $E \to \mathrm{Post}_v$ of Lawvere theories, is not surjective. In particular, it is not the canonical surjection.

Proof. Every Boolean algebra has a natural structure of an $\mathbb{F}_2$–vector space. The addition is given by the symmetric difference

$$x+y = (x \vee y) \wedge \neg(x \wedge y) = (x \wedge \neg y) \vee (\neg x \wedge y).$$

In fact, the category of Boolean algebras is isomorphic to the category of Boolean rings, which are commutative rings where *every* element is idempotent. If 2 is idempotent, we have $4 = 2^2 = 2$, so that $2 = 0$, and the underlying abelian group is 2–torsion.

More generally, if p is a prime number, every p–Post algebra admits a natural structure of an $\mathbb{F}_p$–algebra in which every element x satisfies $x^p = x$ (see [Wad45] or [Ser73]).

It follows that the canonical morphism $\mathbb{S} \simeq \mathrm{K}(E) \to \mathrm{K}(\mathrm{Post}_p)$ of algebraic K-theory spectra factors through the algebraic K-theory $\mathrm{K}(\mathbb{F}_p)$ of the field $\mathbb{F}_p$.

$$\mathbb{S} \simeq \mathrm{K}(E) \longrightarrow \mathrm{K}(\mathbb{F}_p) \longrightarrow \mathrm{K}(\mathrm{Post}_p)$$

On the level of automorphism groups, these morphisms correspond to embeddings

$$\Sigma(r) \longrightarrow \mathrm{GL}_r(\mathbb{F}_p) \longrightarrow \Sigma(p^r)$$

of groups with images given by the subgroups of $\mathbb{F}_p$–linear bijections and the subgroup of that given by the permutation matrices.

Quillen [Qui72, Thm. 8(i)] has shown that $\mathrm{K}_{2j-1}(\mathbb{F}_q) \cong \mathbb{Z}/(q^j - 1)$ for all $j \geqslant 1$ and for all prime powers q. It follows that the p–torsion of the higher K-groups $\mathrm{K}_n(\mathbb{F}_p)$ of $\mathbb{F}_p$ is trivial. On the other hand, his computations [Qui76] showed that most of the stable homotopy of the spheres is contained in the kernel of the canonical morphisms $\mathbb{S} \to \mathrm{K}(\mathbb{Z}) \to \mathrm{K}(\mathbb{F}_p)$ of spectra: what is detected in the algebraic K-theory of finite fields is essentially the image of Whitehead's J-homomorphism. In particular, the kernel contains much more than just the p–power torsion. □

Remark 4.4. Morava, in his 2008 Vanderbilt talk [Mor], highlighted "the apparent fact that the spectrum $\mathbb{S}^\times$ defined by the symmetric monoidal category of finite pointed sets under Cartesian product has not been systematically studied." The spectrum which he denoted by $\mathbb{S}^\times$ can be modeled as the algebraic K-theory of a *many-sorted* Lawvere theory, where the sorts correspond to the prime numbers. It is not worth the effort to develop our theory in more generality just to cover that one example. Instead, we have contented ourselves with demonstrating how the theory we have developed so far suffices for us to deal with the local factors corresponding to each prime.

5 Multiplicative structure

Several modern constructions of algebraic K-theory machines have good multiplicative properties. In particular, we make use of Elmendorf and Mandell's version [EM06] of Segal's device, which builds a multiplicative form of K-theory from strict symmetric monoidal categories. The precise statement of their result is a little more involved than one might like because of the pseudoness discussed by Hyland and Power [HP02]. That is, since symmetric monoidal categories only form a "pseudo-monoidal category," we cannot formulate multiplicativity by saying something like "K-theory is a lax monoidal functor from symmetric monoidal categories to spectra," at least not without moving to a higher-categorical world. One of the points of working with Lawvere theories is that they are simultaneously comprehensive, flexible enough, and strictly lower-categorical. When restricted to Lawvere theories, K-theory *is* a lax monoidal functor in the usual, strict sense: this is what we show in Corollary 5.11.

Since one of our main goals is to use multiplicativity to understand assembly maps, we first isolate the part of Elmendorf–Mandell's general multiplicativity statement that we use to produce assembly-type maps, in Section 5.1. The presentation elides a number of the category-theoretical considerations but tells us precisely what kind of functors we'll need to produce assembly maps. In Section 5.2, we give a more categorically sophisticated and higher level discussion of multiplicativity, which in particular shows that K-theory is lax monoidal on Lawvere theories, as in Corollary 5.11. Because the proofs in these sections are somewhat technical, we have largely postponed them to Section 5.3.

Remark 5.1. We use the language of *multicategories* to describe the constructions in this section. All the multicategories we use are implicitly symmetric. Multicategories may be more familiar to some readers under the term (colored) *operad*, implied to have several objects. Our choice of terminology reflects that of the primary references [EM06, EM09] for this work. The terminological distinction is partly philosophical. In this work, the multicategories appear as generalizations of categories instead of as parameter spaces of operations. Of course, these roles are intimately linked, and we invite the reader to use their preferred term.

In this section, boldface uppercase letters $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$ will be symmetric monoidal categories. Our default notation for the monoidal product is $\oplus$ and 0 typically denotes the monoidal unit, with indices as in $\oplus = \oplus_{\mathbf{A}}$ and $0 = 0_{\mathbf{A}}$ if needed. By convention, we use "symmetric monoidal category" in this section for symmetric monoidal categories with strict unit, as our primarily references are written for this case. All symmetric monoidal categories can be strictified, so this does not represent a loss of generality.

5.1 A first phrasing of multiplicativity

One way to formulate multiplicativity is in terms of "bilinear functors." This formulation is analogous to thinking about bilinear maps between vector spaces, rather than the tensor product of vector spaces.

Definition 5.2. A *bilinear functor* of symmetric monoidal categories is a functor $P\colon \mathbf{A}\times\mathbf{B}\to\mathbf{C}$ together with natural distributivity isomorphisms

$$\delta_l\colon P(a,b)\oplus P(a',b)\to P(a\oplus a',b) \quad \text{and} \quad \delta_r\colon P(a,b)\oplus P(a,b')\to P(a,b\oplus b')$$

satisfying some unitality and compatibility conditions which are spelled out in [BO20, Def. 7.1].

Observe that the distributivity transformations mean in particular that P is strong monoidal "in each variable separately" in the sense that if we fix $a\in\mathbf{A}$, the functor $P(a,-)$ is strong monoidal and if we fix $b\in\mathbf{B}$, the functor $P(-,b)$ is strong monoidal.

Example 5.3. For any symmetric monoidal category $\mathbf{C}$, there is a "left unit" bilinear functor

$$u\colon \mathbf{E}^\times\times\mathbf{C}\to\mathbf{C}$$

given on objects by $u(\underline{n},c)=c^{\oplus n}$. The components of the distributivity natural transformation δ_l are the identity maps

$$c^{\oplus n}\oplus c^{\oplus n'}=c^{\oplus n+n'}$$

and the components of the distributivity natural transformation δ_r are the reordering isomorphisms

$$c^{\oplus n}\oplus c'^{\oplus n}\to(c\oplus c')^{\oplus n}.$$

One can similarly define a "right unit" bilinear functor $\mathbf{C}\times\mathbf{E}^\times\to\mathbf{C}$; here the left distributivity is given by reordering.

Remark 5.4. For a Lawvere theory S, the strong monoidal functor $\underline{r}\times-\colon \mathbf{F}_S\to\mathbf{F}_S$ of Remark 1.11, taking $S_s\mapsto\underline{r}\times S_s$, is $u(\underline{r},-)$.

Example 5.5. A *ring category* structure on a (strict) symmetric monoidal category $\mathbf{A}=(\mathbf{A},\oplus,0)$ consists of a bilinear functor $\otimes\colon\mathbf{A}\times\mathbf{A}\to\mathbf{A}$ and an object $1\in\mathbf{A}$ such that $1\otimes a=a=a\otimes 1$, and satisfying appropriate conditions (see [EM06, Def. 3.3]). This is also a *rig category* as defined by Baas, Dundas, Richter, and Rognes [BDRR13, §2.2].

Theorem 5.6 ([BO20, Thm. 7.4], cf. [EM06, Thm. 6.1]). *Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be symmetric monoidal categories (with strict units). A bilinear functor $\mathbf{A}\times\mathbf{B}\to\mathbf{C}$ of symmetric monoidal categories induces a morphism*

$$\mathrm{K}(\mathbf{A})\wedge\mathrm{K}(\mathbf{B})\to\mathrm{K}(\mathbf{C})$$

of spectra. This structure is associative and unital.
In the case where the bilinear functor $\mathbf{A}\times\mathbf{A}\to\mathbf{A}$ is the multiplication of a ring category as in Example 5.5, the induced map

$$\mathrm{K}(\mathbf{A})\wedge\mathrm{K}(\mathbf{A})\to\mathrm{K}(\mathbf{A})$$

is the multiplication of a ring structure on $\mathrm{K}(\mathbf{A})$.
In the case where the bilinear functor $\mathbf{E}^{\times} \times \mathbf{C} \to \mathbf{C}$ *is the left unit bilinear functor of Example 5.3, the induced map*

$$\mathbb{S} \wedge \mathrm{K}(\mathbf{C}) \simeq \mathrm{K}(\mathbf{E}^{\times}) \wedge \mathrm{K}(\mathbf{C}) \to \mathrm{K}(\mathbf{C})$$

is the left unit map for the smash product $\wedge$ *of spectra.*

This theorem is really just one consequence of the fact that Elmendorf–Mandell's K-theory is an enriched multifunctor from permutative categories to spectra. The slight extension to symmetric monoidal categories is in [BO20]. Since our Lawvere theories $\mathbf{F}_T^{\times}$ form permutative categories, this extension is not necessary for our work.

Remark 5.7. One way to think about Theorem 5.6 is that it tells us that K-theory is "morally lax monoidal," in the following sense. Symmetric monoidal categories do not form a symmetric monoidal category because there is, in general, no representing "tensor product" symmetric monoidal category "$\mathbf{A} \otimes \mathbf{B}$" for bilinear functors [HP02]. If such a tensor product exists, then there is a universal bilinear functor $\mathbf{A} \times \mathbf{B} \to \mathbf{A} \otimes \mathbf{B}$ and Theorem 5.6 provides the type of map of spectra $\mathrm{K}(\mathbf{A}) \wedge \mathrm{K}(\mathbf{B}) \to \mathrm{K}(\mathbf{A} \otimes \mathbf{B})$ needed to make the functor K lax monoidal. Since the tensor product doesn't always exist, Elmendorf and Mandell's approach is to work with the multicategory of permutative categories, in which n–ary maps are given by n–multilinear functors. They show that K-theory is a multifunctor from this multicategory to the category of spectra. Theorem 5.6 is an explicit statement of the fact that a map of multicategories takes binary maps to binary maps.

From Theorem 5.6, applied to the case of Lawvere theories, we can produce the following result. We give a detailed proof of it in Section 5.3.

Theorem 5.8. *For each pair of Lawvere theories S and T, there is a morphism*

$$\mathrm{K}(S) \wedge \mathrm{K}(T) \longrightarrow \mathrm{K}(S \otimes T) \tag{5.1}$$

of spectra that is natural in S and T and that induces multiplication at the level of π_0.

As a consequence of Theorem 5.8, a Lawvere theory T that has a multiplication $T \otimes T \to T$ produces a multiplication $\mathrm{K}(T) \wedge \mathrm{K}(T) \to \mathrm{K}(T)$ in spectra. We discuss this rather restrictive situation further in Section 6.1. Similarly, the left unit map $E \otimes T \to T$ of a Lawvere theory T yields the left unit map $\mathrm{K}(E) \wedge \mathrm{K}(T) \to \mathrm{K}(T)$ in spectra.

The category of Lawvere theories does have a symmetric monoidal structure, with tensor product given by the Kronecker product, and Theorem 5.8 is singling out the natural transformation that makes K-theory into a lax monoidal functor from the symmetric monoidal category of Lawvere theories to the category of spectra. We make this interpretation precise in the next subsection.

5.2 A second phrasing of multiplicativity

In the previous part of this section, we highlighted the key structure in a multiplicative formulation of K-theory that we will ultimately use to produce our assembly maps. In this section, we prove that this piece of structure is part of a larger framework. For this, we use a second phrasing of multiplicativity, which is also due to Elmendorf and Mandell [EM09]. In what follows, $\mathbf{Mult}_*$ denotes the category of based multicategories, which are implicitly symmetric and small.

Theorem 5.9 ([EM09]). *Based multicategories form a symmetric monoidal category* $\mathbf{Mult}_*$ *and the K-theory construction of* [EM06] *factors as the "underlying multicategory" functor U and a lax monoidal functor from* $\mathbf{Mult}_*$ *to spectra:*

$$\begin{tikzcd} \mathbf{PermCat} \arrow[d, "U"'] \arrow[dr, "\mathrm{K}"] & \\ \mathbf{Mult}_* \arrow[r, "\mathrm{K}"'] & \mathbf{Spectra} \end{tikzcd}$$

Note that the underlying multicategory $U\mathbf{C}$ of a permutative category $\mathbf{C}$ has a natural basepoint given by the unit object for the monoidal product.

Lawvere theories are, in particular, permutative categories and our definition of the K-theory of a Lawvere theory views the Lawvere theory as such. It thus suffices to prove that the composite of the embedding of Lawvere theories into permutative categories and the underlying multicategory functor is a lax symmetric monoidal functor from Lawvere theories to multicategories.

Theorem 5.10. *Let* $\iota\colon \mathbf{Lawvere} \to \mathbf{PermCat}$ *denote the embedding of Lawvere theories into permutative categories via* $T \mapsto \mathbf{F}_T$*. Let* $\iota^\times$ *denote embedding* $\mathbf{Lawvere} \to \mathbf{PermCat}$ *via* $T \mapsto \mathbf{F}_T^\times$*; we can view* $\iota^\times$ *as the composite of* ι *and the functor taking a permutative category to its subcategory of isomorphisms. Then the composite functors* $U\iota$ *and* $U\iota^\times$ *in the diagrams*

$$\begin{tikzcd} \mathbf{Lawvere} \arrow[r, "\iota"] \arrow[dr, "U\iota"'] & \mathbf{PermCat} \arrow[d, "U"] \\ & \mathbf{Mult}_* \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Lawvere} \arrow[r, "\iota^\times"] \arrow[dr, "U\iota^\times"'] & \mathbf{PermCat} \arrow[d, "U"] \\ & \mathbf{Mult}_* \end{tikzcd}$$

are both lax symmetric monoidal.

We give a detailed proof of Theorem 5.10 in Section 5.3.

Corollary 5.11. *The Elmendorf–Mandell construction of K-theory gives a lax monoidal functor*

$$\mathbf{Lawvere} \to \mathbf{Spectra}.$$

Proof. The previous two theorems demonstrate that Elmendorf and Mandell's K-theory construction factors as the composite of the lax monoidal functor $\mathrm{K}\colon \mathbf{Mult}_* \to \mathbf{Spectra}$ preceded by $U\iota^\times\colon \mathbf{Lawvere} \to \mathbf{Mult}_*$. □

5.3 Proofs

We end this section with the proofs of Theorems 5.8 and 5.10.

Proof of 5.8. In light of Theorem 5.6, it is sufficient to show that there is a bilinear functor $\mathbf{F}_S^\times \times \mathbf{F}_T^\times \to \mathbf{F}_{S\otimes T}^\times$ of symmetric monoidal categories. Essentially, this is the universal map that comes from the definition of the Kronecker product. However, given that general symmetric monoidal categories don't have such a monoidal product, it seems worthwhile to be fairly explicit about this.

Let S and T be Lawvere theories. We show that there is a bilinear functor

$$P\colon \mathbf{F}_S \times \mathbf{F}_T \to \mathbf{F}_{S\otimes T}$$

of symmetric monoidal categories with strict unit. Observe that a bilinear functor $\mathbf{A}\times\mathbf{B}\to\mathbf{C}$ restricts to a bilinear functor $\mathbf{A}^\times\times\mathbf{B}^\times\to\mathbf{C}^\times$ on the subcategories of isomorphisms in $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ because functors preserve isomorphisms and the natural distributivity maps are isomorphisms by definition. Hence it suffices to produce the bilinear functor P.

The functor P is defined on objects by $P(S_m,T_n)=(S\otimes T)_{m\times n}$. On morphisms, the arrow $P(f,g)$ is defined as either of the composites in the commuting diagram in $\mathbf{F}_{S\otimes T}$ that we obtain from Definition 1.12:

$$\begin{tikzcd}
(S\otimes T)_{m\times n} \arrow[r, "\underline{m}\times g"] \arrow[d, "f\times\underline{n}"'] & (S\otimes T)_{m\times n'} \arrow[d, "f\times\underline{n}'"] \\
(S\otimes T)_{m'\times n} \arrow[r, "\underline{m}'\times g"] & (S\otimes T)_{m'\times n'}
\end{tikzcd}$$

The fact that these composites agree implies that this assignment is functorial.

The distributivity natural transformations are in fact given by the identity morphisms:

$$\delta_l\colon P(S_m,T_n)\oplus P(S_{m'},T_n)=(S\otimes T)_{(m'\times n)+(m\times n)}=(S\otimes T)_{(m+m')\times n}$$

and similarly for δ_r. It is thus straightforward to check that the required unitality and compatibility conditions hold.

By construction, the map on the monoid of connected components of objects in $\mathbf{F}_S$, $\mathbf{F}_T$ and $\mathbf{F}_{S\otimes T}$ is induced by the multiplication $\mathbb{N}\times\mathbb{N}\to\mathbb{N}$, and hence the map

$$\pi_0\mathrm{K}(S)\wedge\pi_0\mathrm{K}(T)\to\pi_0\mathrm{K}(S\otimes T)$$

is also given by multiplication. $\square$

Proof of Theorem 5.10. A functor $F\colon\mathbf{C}\to\mathbf{D}$ is lax monoidal if we have a map $0_{\mathbf{D}}\to F(0_{\mathbf{C}})$ and natural maps $F(c_1)\oplus_{\mathbf{D}}F(c_2)\to F(c_1\oplus_{\mathbf{C}}c_2)$ that satisfy some standard axioms. We show that $U\iota$ satisfies this definition.

Recall the definition of the functor $U\colon \mathbf{PermCat} \to \mathbf{Mult}_*$. For a permutative category $\mathbf{C}$, the underlying multicategory $U\mathbf{C}$ has the same objects as the category $\mathbf{C}$ and for any $c_1,\ldots,c_n,d$, the set of n–ary morphisms $U\mathbf{C}(c_1,\ldots,c_n;d)$ is defined to be the morphism set $\mathbf{C}(c_1\oplus\cdots\oplus c_n,d)$. Composition is defined in the evident way.

First, we construct the map of unit objects. The unit object in $\mathbf{Mult}_*$ is the multicategory $\mathbf{1}_*$ with two objects 0 and 1. Here 0 is the basepoint and there is exactly one k–morphism $(0,\ldots,0)\to 0$ for each k. The only morphism of any arity involving 1 is the identity. The unit object in **Lawvere** is the Lawvere theory E of sets. We thus require a functor (of small based multicategories)

$$\mathbf{1}_* \to U\iota(E).$$

Since this functor is required to be based, it must send $0\in\mathbf{1}_*$ to the unit object $\underline{0}\in\mathbf{E}$, which is the basepoint in $U(\mathbf{E})$. Thus the data of this functor is equivalent to picking out a single object (together with its identity morphism) of $U\iota(E)$; the obvious choice is $1\mapsto\underline{1}$.

Next, we need the maps $U\iota(S)\otimes U\iota(T)\to U\iota(S\otimes T)$, where we are overloading the symbol $\otimes$ to represent both tensor product of multicategories and tensor product of Lawvere theories. These maps arise from the universal property of the tensor product of based multicategories. To be more precise, Elmendorf and Mandell define the tensor product of based multicategories so that a morphism of based multicategories $M_1\otimes M_2\to N$ is precisely the data of a based bilinear map $(M_1,M_2)\to N$. These bilinear maps are a multicategorical generalization of the bilinear functors in Definition 5.2; see [EM09, Definition 2.3] for the precise definition.

Since $U\colon \mathbf{PermCat}\to\mathbf{Mult}_*$ is a multifunctor and the binary morphisms in the category $\mathbf{Mult}_*$ are defined to be the based bilinear maps, it suffices to show that there is a binary morphism $\varphi\colon(\iota S,\iota T)\to\iota(S\otimes T)$ in the multicategory **PermCat**. In this case, $U\varphi$ must be a bilinear based map of multicategories $(U\iota S,U\iota T)\to U\iota(S\otimes T)$, and thus $U\varphi$ induces a morphism of based multicategories $(U\iota S)\otimes(U\iota T)\to U\iota(S\otimes T)$, as required.

For Lawvere theories S and T, we've thus reduced our problem to finding a binary morphism of permutative categories $\mathbf{F}_S\times\mathbf{F}_T\to\mathbf{F}_{S\otimes T}$. By definition, this is just a bilinear functor with strict unit and is precisely what we constructed in the Proof of Theorem 5.8, above.

We've thus constructed all the data making $U\iota\colon \mathbf{Lawvere}\to\mathbf{Mult}_*$ a lax monoidal functor. To complete the proof, one must show that the coherence maps are associative and unital. These are fairly straightforward to check using the universal property of the tensor product of multicategories: since the tensor product represents based bilinear maps of multicategories, it suffices to check that the required morphisms agree on that level.

For example, to check the commutativity of the left unit diagram

$$\begin{tikzcd}
\mathbf{1}_*\otimes U\iota(S) \arrow[r] \arrow[d] & U\iota(E)\otimes U\iota(S) \arrow[d] \\
U\iota(S) & U\iota(E\otimes S) \arrow[l]
\end{tikzcd}$$

we simply check that the two bilinear maps $\mathbf{1}_* \times U\imath(S) \to U\imath(S)$ agree. On objects, both are given by sending $(1, S_n) \mapsto S_n$. Any non-basepoint-involving morphism in $\mathbf{1}_* \times U\imath(S)$ is of the form $(\mathrm{id}_1, f\colon S_{n_1} \oplus \cdots \oplus S_{n_j} \to S_n)$, and both bilinear maps send this to f. In the map along the right hand side, this follows because $\mathrm{id}_1 \times f \in E \otimes S$ is sent to f under the unit map $E \otimes S \to S$ of Lawvere theories. The right unit diagram is similar.

For the associativity diagram, we can use the fact that the underlying multicategory functor U is full and faithful, and so it suffices to check that the two trilinear maps represented by the two composites

$$\begin{tikzcd}
(U\imath(S)\otimes U\imath(T))\otimes U\imath(V) \arrow[r] \arrow[d] & U\imath(S)\otimes(U\imath(T)\otimes U\imath(V)) \arrow[d] \\
U\imath(S\otimes T)\otimes U\imath(V) \arrow[d] & U\imath(S)\otimes U\imath(T\otimes V) \arrow[d] \\
U\imath((S\otimes T)\otimes V) \arrow[r] & U\imath(S\otimes(T\otimes V))
\end{tikzcd}$$

are given by the same functor $\mathbf{F}_S \times \mathbf{F}_T \times \mathbf{F}_V \to \mathbf{F}_{S\otimes T\otimes V}$. Inspecting the definitions shows that both functors send an object (S_l, T_m, V_n) to $(S \otimes (T \otimes V))_{lmn}$ and both send a morphism (f, g, h) to the composite $(f \times (\mathrm{id}\times \mathrm{id})) \circ (\mathrm{id}\times(g\times \mathrm{id})) \circ (\mathrm{id}\times(\mathrm{id}\times h))$, which by the definition of the Kronecker product agrees with the composite of these three maps in any other order.

To see that $U\imath^\times$ is also lax monoidal, it suffices to observe that the functor $\mathbf{1}_* \to U\imath(E)$ factors through $U\imath^\times(E)$ and, as observed in the above proof of Theorem 5.8, the binary morphism of permutative categories $\mathbf{F}_S \times \mathbf{F}_T \to \mathbf{F}_{S\otimes T}$ restricts to a binary morphism $\mathbf{F}_S^\times \times \mathbf{F}_T^\times \to \mathbf{F}_{S\otimes T}^\times$. $\square$

Remark 5.12. We could alternatively show that both of the two functors $U\imath\colon \mathbf{Lawvere} \to \mathbf{Mult}_*$ and $U\imath^\times\colon \mathbf{Lawvere} \to \mathbf{Mult}_*$ are lax monoidal functors as follows. As discussed in [EM09, Sec. 3], a lax monoidal functor between symmetric monoidal categories is simply a map between their underlying multicategories. Hence, it suffices to show that $U\imath$ and $U\imath^\times$ are multifunctors between the underlying multicategories of **Lawvere** and $\mathbf{Mult}_*$. In the diagram

$$\begin{tikzcd}
& \mathbf{Lawvere} \arrow[dl, "\imath"'] \arrow[dr, "\imath^\times"] & \\
\mathbf{PermCat} \arrow[rr, "(-)^\times"'] \arrow[d, "U"'] & & \mathbf{PermCat} \arrow[d, "U"] \\
\mathbf{Mult}_* & & \mathbf{Mult}_*
\end{tikzcd}$$

the maps U are multifunctors by Theorem 1.1 of [EM09] and we have already observed that $(-)^\times$ is a multifunctor. Hence both composites $\mathbf{Lawvere} \to \mathbf{Mult}_*$ are multifunctors if $\imath$ is. Elmendorf–Mandell's work ([EM06, Theorem 1.1] or the results of [EM09] stated as Theorem 5.9 above) then implies that K-theory of Lawvere theories, which is given by the composite

$$\mathbf{Lawvere} \xrightarrow{\imath^\times} \mathbf{PermCat} \xrightarrow{U} \mathbf{Mult}_* \xrightarrow{\mathrm{K}} \mathbf{Spectra},$$

is multiplicative.

Multilinearity of ι requires that for any map of Lawvere theories $S_1 \otimes \cdots \otimes S_k \to T$, we must have a k–linear functor of permutative categories $\mathbf{F}_{S_1} \times \cdots \times \mathbf{F}_{S_k} \to \mathbf{F}_T$. The universality of Kronecker products means we can reduce the construction of any such map to constructing a multilinear functor $\mathbf{F}_{S_1} \times \cdots \times \mathbf{F}_{S_k} \to \mathbf{F}_{S_1 \otimes \cdots \otimes S_k}$, extending our construction of the bilinear functor $\mathbf{F}_S \times \mathbf{F}_T \to \mathbf{F}_{S \otimes T}$ from Theorem 5.10.

Remark 5.13. Phrased ∞–categorically, this final description of multiplicativity of K-theory simply comes down to showing that ι: **Lawvere** $\to$ **PermCat** is a map of ∞–operads, and hence the composites $U\iota$ and $U\iota^{\times}$ are as well. Both the domain and codomain of these composites are ∞–operads coming from actual symmetric monoidal categories, and so one can describe maps of ∞–operads as straightforward lax monoidal functors. Note, however, that in comparing Lawvere theories and multicategories, we naturally pass through **PermCat**, which simply isn't a symmetric monoidal category. Hyland and Power [HP02] show that it only has a "weak" or "pseudo" monoidal structure, and the context of ∞–operads or multicategories is one way of providing elbow room for this weak structure. In fact, many of the subtle issues at the heart of multiplicative K-theory can be attributed to the need to consider a weak monoidal structure when thinking about permutative categories.

These remarks bring us to a peak of abstraction in our thinking about the multiplicativity of K-theory of Lawvere theories. In the next section, we return to the down-to-earth realm of applications of the concrete maps that multiplicativity produces.

6 First applications

In this section, we discuss some applications of the results of the preceding Section 5 that are not yet related to assembly.

6.1 Monoids in the category of Lawvere theories

Lawvere theories form a symmetric monoidal category with respect to $\otimes$; the theory of sets is the monoidal unit.

Definition 6.1. A *monoidal* Lawvere theory T is a monoid object in the symmetric monoidal category of Lawvere theories.

Which Lawvere theories T support such monoidal structures, and how many?

Proposition 6.2. *Any Lawvere theory T supports at most one monoidal structure.*

Proof. Since the monoidal unit E, the Lawvere theory of sets, is the initial object in the category, every Lawvere theory T has a canonical map $E \to T$ from the monoidal unit. Therefore, the question is: when does there exist a map $T \otimes T \to T$ that turns T into a monoid object in Lawvere theories?

Suppose we have a map $T \otimes T \to T$. Then every T–model determines a $T \otimes T$–model. What are these two potentially new T–structures on a given T–model? The unit axiom implies that both agree with the old structure. Thus, such a multiplication is automatically unique: the morphism $T \otimes T \to T$ is the inverse of the isomorphism $T \to T \otimes T$ given by the (left or right) unit. □

We see that to be a monoidal Lawvere theory is a *property*, not a *structure*, and theories that have this property are also called *commutative*. Such structures have been considered by Kock [Koc70, Koc71, Koc72] and, much more recently, in Durov's thesis [Dur].

Proposition 6.3. *If T is a commutative Lawvere theory, then its algebraic K-theory spectrum* $\mathrm{K}(T)$ *is a commutative ring spectrum.*

Proof. This follows immediately from the multiplicative properties of the K-theory functor for Lawvere theories (see Theorem 5.8 again). □

Example 6.4. The theory of modules over a given commutative ring is commutative.

For this reason, commutative Lawvere theories can be seen as generalizations of commutative rings.

Example 6.5. The theory of sets with an action of a fixed abelian group A is commutative.

6.2 Group actions

There is one important situation where the map (5.1) from Theorem 5.8 is an equivalence:

Proposition 6.6. *For any groups G and H, the map*

$$\mathrm{K}(G\text{–Sets}) \wedge \mathrm{K}(H\text{–Sets}) \longrightarrow \mathrm{K}(G\text{–Sets} \otimes H\text{–Sets})$$

from Theorem 5.8 is an equivalence.

Proof. Recall the equivalence

$$\mathrm{K}(G\text{–Sets} \otimes H\text{–Sets}) \simeq \mathrm{K}((G \times H)\text{–Sets}),$$

from Example 1.10: the two Lawvere theories G–Sets $\otimes$ H–Sets and $(G \times H)$–Sets are the same. Now we can use the equivalence $\mathrm{K}(G\text{–Sets}) \simeq \Sigma^\infty_+ \mathrm{B}G$ of spectra from Example 3.3 and the equivalence $\mathrm{B}(G \times H) \simeq \mathrm{B}(G) \times \mathrm{B}(H)$ of classifying spaces which induces the equivalence $\Sigma^\infty_+ \mathrm{B}(G \times H) \simeq \Sigma^\infty_+ \mathrm{B}G \wedge \Sigma^\infty_+ \mathrm{B}H$ of suspension spectra. □

6.3 Higher-dimensional Higman–Thompson groups

We continue the discussion of the Cantor theories from Example 3.7.

Example 6.7. Brin has introduced the *higher-dimensional Higman–Thompson groups* in [Bri04]. These are the automorphism groups of the free models for the Lawvere theories

$$\mathrm{Cantor}_{a(1)} \otimes \mathrm{Cantor}_{a(2)} \otimes \cdots \otimes \mathrm{Cantor}_{a(n)},$$

for a finite sequence of integers $a(n) \geqslant 2$ (see [M-PN13, DM-P14, M-PMN18, FN18]). The stable homology of these groups, by Theorem 2.7, is described by the algebraic K-theory spectrum $\mathrm{K}(\mathrm{Cantor}_{a(1)} \otimes \cdots \otimes \mathrm{Cantor}_{a(n)})$ of these Lawvere theories. From Theorem 5.8, we get a map

$$\mathrm{K}(\mathrm{Cantor}_{a(1)}) \wedge \mathrm{K}(\mathrm{Cantor}_{a(2)}) \wedge \cdots \wedge \mathrm{K}(\mathrm{Cantor}_{a(n)}) \longrightarrow \mathrm{K}(\mathrm{Cantor}_{a(1)} \otimes \cdots \otimes \mathrm{Cantor}_{a(n)}) \quad (6.1)$$

from the smash product of the algebraic K-theory spectra into it. The homotopy type of this smash product can be worked out, because the algebraic K-theory spectra are Moore spectra by [SW19], but it is not known whether the map (6.1) is an equivalence or not.

7 Assembly

In this section, we apply our results from Section 5 to assembly maps in the context of Lawvere theories.

Theorem 7.1. *For each Lawvere theory T there exists a unique* $\mathrm{K}(\mathbb{Z})$*–linear morphism*

$$\mathrm{K}(\mathbb{Z}) \wedge \mathrm{K}(T) \longrightarrow \mathrm{K}(\mathbb{Z} \otimes T) \quad (7.1)$$

between $\mathrm{K}(\mathbb{Z})$*–module spectra that extends the morphism* $\mathrm{K}(T) \to \mathrm{K}(\mathbb{Z} \otimes T)$ *induced by the linearization $T \to \mathbb{Z} \otimes T$ of the Lawvere theory T as defined in* (1.1).

Proof. This follows from Theorem 5.8, applied for the Lawvere theory $S = \mathbb{Z}$ of abelian groups. □

Definition 7.2. The map (7.1) is the *assembly map* for the Lawvere theory T.

7.1 The theories of group actions

We can now see how to recover one of the classical assembly maps in algebraic K-theory as a special case of our general assembly map (7.1).

Corollary 7.3. *For each group G there exists a unique* $\mathrm{K}(\mathbb{Z})$*–linear morphism*

$$\mathrm{K}(\mathbb{Z}) \wedge \Sigma^\infty_+(\mathrm{B}G) \longrightarrow \mathrm{K}(\mathbb{Z}G) \tag{7.2}$$

between $\mathrm{K}(\mathbb{Z})$*–module spectra, naturally in the group G.*

This is Loday's version of the assembly map in algebraic K-theory [Lod76]. There are obvious extensions to other coefficient rings than $\mathbb{Z}$.

Proof. If T is the theory of G–sets for a group G, then $\mathbb{Z} \otimes T$ is the theory of abelian groups with a linear G–action. These are precisely the modules over the group ring $\mathbb{Z}G$. Since we have a natural equivalence $\mathrm{K}(G\text{–Sets}) \simeq \Sigma^\infty_+(\mathrm{B}G)$ of spectra from Example 3.3, we get a map of spectra as indicated. $\square$

Example 7.4. The assembly map (7.2) is obviously an equivalence for the trivial group $G = e$. Less obviously, it is also an equivalence when $G \cong \mathrm{C}_\infty$ is infinite cyclic: we have $\mathbb{Z}\mathrm{C}_\infty \cong \mathbb{Z}[q^{\pm 1}]$ and Example 3.3, which together with Quillen's work on the algebraic K-theory of Laurent polynomial rings gives an equivalence

$$\mathrm{K}(\mathbb{Z}[q^{\pm 1}]) \simeq \mathrm{K}(\mathbb{Z}) \vee \Sigma\mathrm{K}(\mathbb{Z}) \simeq \mathrm{K}(\mathbb{Z}) \wedge (\mathrm{S}^0 \vee \mathrm{S}^1) \simeq \mathrm{K}(\mathbb{Z}) \wedge \Sigma^\infty_+(\mathrm{B}\mathrm{C}_\infty)$$

of spectra (see Grayson's paper [Gra76]).

The assembly map (7.2) fails to be an equivalence in general. For instance, the failure of surjectivity on π_1 is measured by the Whitehead group of G, and the Whitehead group is often nontrivial (take G of prime order $p \geqslant 5$). It is known that the map (7.2) is rationally injective for groups whose integral homology is of finite type by work of Bökstedt–Hsiang–Madsen [BHM89, BHM93] on the algebraic K-theoretic analogue of Novikov's conjecture. We refer to the surveys cited in the introduction for more recent results in this vein.

7.2 Other examples

We now present examples that show the interest of our assembly maps beyond the classical case of theories of group actions. We first continue the discussion of the Examples 3.7 and 6.7.

Theorem 7.5. *The assembly map* (7.1) *for the Lawvere theory* Cantor_a *of Cantor algebras of arity a is an equivalence.*

Proof. We start from [SW19], where the algebraic K-theory of Cantor_a is identified with the Moore spectrum $\mathbb{S}/(a-1)$. That Moore spectrum is the the cofiber of multiplication with $a-1$ on the sphere spectrum, so that $\mathrm{K}(\mathbb{Z}) \wedge \mathrm{K}(\mathrm{Cantor}_a)$ is the cofiber of multiplication with $a-1$ on $\mathrm{K}(\mathbb{Z})$.

Then we have the observation that $\mathbb{Z}\otimes\mathrm{Cantor}_a$ is the theory of modules over the Leavitt algebra L_a, the quotient of the free associative ring with unit on $2a$ generators, given as two vectors $R=(R_1,\dots,R_a)$ and $C=(C_1,\dots,C_a)$, modulo the ideal defined by the a^2+1 relations that ensure that the two square matrices $R^{\mathrm{t}}C$ and RC^{t} are the identity matrices. In other words, the modules over the ring L_a are the abelian groups M together with linear bijections $M^a\to M$, and these are precisely the models for $\mathbb{Z}\otimes\mathrm{Cantor}_a$.

Finally, the algebraic K-theory $\mathrm{K}(\mathrm{L}_a)$ has been computed in [ABC09], and the result shows that it is also the cofiber of multiplication with $a-1$ on $\mathrm{K}(\mathbb{Z})$. □

On the other extreme, there are Lawvere theories for which the assembly map is trivial because the target is contractible. For instance, in any abelian T–model M, all constants of T need to be equal, because there is a unique homomorphism $0=M^0\to M$ of abelian groups. This happens for rings with unit ($1=0$ implies $a=1\cdot a=0\cdot a=0$ for all a), but also for Boolean algebras:

Proposition 7.6. *Any abelian group object in the category of Boolean algebras is trivial.*

Proof. We first note that $0\wedge x=0$ and $1\wedge x=x$ hold in every Boolean algebra. If, in addition, we have $0=1$, then this implies $x=1\wedge x=0\wedge x=0$ for all x, and we are done. □

Corollary 7.7. *The assembly map for the Lawvere theory* Boole *is zero and, in particular, not rationally injective.*

Proof. It follows from Proposition 7.6 that $\mathbb{Z}\otimes\mathrm{Boole}$ is the theory of modules over the trivial ring, and the algebraic K-theory spectrum $\mathrm{K}(\mathbb{Z}\otimes\mathrm{Boole})\simeq\star$ is contractible.

On the other hand, the source $\mathrm{K}(\mathbb{Z})\wedge\mathrm{K}(\mathrm{Boole})\not\simeq\star$ of the assembly map is not contractible because of $\pi_0(\mathrm{K}(\mathbb{Z})\wedge\mathrm{K}(\mathrm{Boole}))\cong\mathrm{K}_0(\mathbb{Z})\otimes\mathrm{K}_0(\mathrm{Boole})\cong\mathbb{Z}\otimes\mathbb{Z}\cong\mathbb{Z}$. □

It is easy to generalize the preceding results from Boolean algebras to v–valued Post algebras; we omit the details.

7.3 More groups

The assembly map (7.1) for the theory of abelian groups is *not* an equivalence. In fact,

Theorem 7.8. *The assembly map*

$$\mathrm{K}(\mathbb{Z})\wedge\mathrm{K}(\mathrm{Nil}_c)\longrightarrow\mathrm{K}(\mathbb{Z}\otimes\mathrm{Nil}_c)=\mathrm{K}(\mathbb{Z})\tag{7.3}$$

for the theory Nil_c *of nilpotent groups of any given class* $c\geqslant 1$ *is not rationally injective.*

Proof. To see that, assume that it is rationally injective, and base change the $\mathrm{K}(\mathbb{Z})$–linear assembly map along the composition $\mathrm{K}(\mathbb{Z}) \to \mathrm{H}\mathbb{Z} \to \mathrm{H}\mathbb{Q}$ to get a $\mathrm{H}\mathbb{Q}$–linear map

$$\mathrm{H}\mathbb{Q} \wedge \mathrm{K}(\mathrm{Nil}_c) \longrightarrow \mathrm{H}\mathbb{Q},$$

which were then also rationally injective, contradicting the results in [Szy19]: the rational homology of $\mathrm{K}(\mathrm{Nil}_c)$ is non-trivial for all positive integers c. □

One might, therefore, say that a generalization of the Novikov conjecture to algebraic theories is impossible. On the other hand:

Example 7.9. The assembly map (7.1) for the theory of groups is an equivalence. This follows from Galatius's theorem (see Example 3.4) and the fact that (abelian) group objects in the category of groups are just abelian groups.

7.4 Non-linear assembly and a nilpotent interpolation

Example 7.9 might suggest using the theory of *all* groups (instead of the theory $\mathbb{Z}$ of abelian groups) and consider assembly-like maps

$$\mathrm{K}(\mathrm{Groups}) \wedge \mathrm{K}(T) \longrightarrow \mathrm{K}(\mathrm{Groups} \otimes T). \tag{7.4}$$

By Galatius's theorem, this is equivalent to the map

$$\mathrm{K}(T) \longrightarrow \mathrm{K}(\mathrm{Groups} \otimes T) \tag{7.5}$$

of spectra induced by the canonical morphism $T \to \mathrm{Groups} \otimes T$ in the sense that the obvious triangle commutes.

Inspired by Example 3.5, we find that there is an entire interpolation of assembly-style maps between Loday's assembly map (7.1) and the map (7.4): there is a tower

$$\mathrm{K}(\mathrm{Nil}_c) \wedge \mathrm{K}(T) \longrightarrow \mathrm{K}(\mathrm{Nil}_c \otimes T) \tag{7.6}$$

of maps of spectra, indexed by the intenger $c \geqslant 1$. Note how this differs from (7.3) in the way the theory Nil_c enters.

At the time of writing, it is not known to us whether the tower of spectra $\mathrm{K}(\mathrm{Nil}_c)$ converges (as $c \to \infty$) to the spectrum $\mathrm{K}(\mathrm{Groups}) \simeq \mathbb{S}$ or not. More generally, one may wonder whether or not the tower $\mathrm{K}(\mathrm{Nil}_c) \wedge \mathrm{K}(T)$ converges to $\mathrm{K}(\mathrm{Groups}) \wedge \mathrm{K}(T)$, or whether or not the tower $\mathrm{K}(\mathrm{Nil}_c \otimes T)$ converges to $\mathrm{K}(\mathrm{Groups} \otimes T)$. It would be interesting to pursue the question for which Lawvere theories T one or both of these is the case.

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References

[ARV11] J. Adámek, J. Rosický, E.M. Vitale. Algebraic theories. A categorical introduction to general algebra. Cambridge Tracts in Mathematics 184. Cambridge University Press, Cambridge, 2011. ◁ 4, 10

[ABC09] P. Ara, M. Brustenga, G. Cortiñas. K-theory of Leavitt path algebras. Münster J. Math. 2 (2009) 5–33. ◁ 33

[BDRR13] N.A. Baas, B.I. Dundas, B. Richter, J. Rognes. Ring completion of rig categories. J. Reine Angew. Math. 674 (2013) 43–80. ◁ 23

[BP72] M. Barratt, S. Priddy. On the homology of non-connected monoids and their associated groups. Comment. Math. Helv. 47 (1972) 1–14. ◁ 16

[BO20] A.M. Bohmann, A. Osorno. A multiplicative comparison of Waldhausen and Segal K-theory. Math. Z. 295 (2020) 1205–1243. ◁ 3, 11, 23, 24

[BHM89] M. Bökstedt, W.C. Hsiang, I. Madsen. The cyclotomic trace and the K-theoretic analogue of Novikov's conjecture. Proc. Nat. Acad. Sci. U.S.A. 86 (1989) 8607–8609. ◁ 3, 32

[BHM93] M. Bökstedt, W.C. Hsiang, I. Madsen. The cyclotomic trace and algebraic K-theory of spaces. Invent. Math. 111 (1993) 465–539. ◁ 3, 32

[Bor94] F. Borceux. Handbook of categorical algebra. 2. Categories and structures. Encyclopedia of Mathematics and its Applications 51. Cambridge University Press, Cambridge, 1994. ◁ 4, 7

[Bri04] M.G. Brin. Higher dimensional Thompson groups. Geom. Dedicata 108 (2004) 163–192. ◁ 31

[DL98] J.F. Davis, W. Lück. Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory. K-Theory 15 (1998) 201–252. ◁ 3

[DM-P14] W. Dicks, C. Martínez-Pérez. Isomorphisms of Brin–Higman–Thompson groups. Israel J. Math. 199 (2014) 189–218. ◁ 31

[Duk88] J.J. Dukarm. Morita equivalence of algebraic theories. Colloq. Math. 55 (1988) 11–17. ◁ 10, 19

[Dur] N. Durov. New Approach to Arakelov Geometry. arXiv:0704.2030 ◁ 30

[Dwi77] P. Dwinger. A survey of the theory of Post algebras and their generalizations. Modern uses of multiple-valued logic (Fifth Internat. Sympos., Indiana Univ., Bloomington, Ind., 1975) 51–75. Reidel, Dordrecht, 1977. ◁ 18

[EM06] A.D. Elmendorf, M.A. Mandell. Rings, modules, and algebras in infinite loop space theory. Adv. Math. 205 (2006) 163–228. ◁ 3, 11, 22, 23, 25, 28

[EM09] A.D. Elmendorf, M.A. Mandell. Permutative categories, multicategories and algebraic K-theory. Alg. Geom. Top. 9 (2009) 2391–2441. ◁ 3, 22, 25, 27, 28

[Eps60] G. Epstein. The lattice theory of Post algebras. Trans. Amer. Math. Soc. 95 (1960) 300–317. ◁ 18

[FJ93] F.T. Farrell, L.E. Jones. Isomorphism conjectures in algebraic K-theory. J. Amer. Math. Soc. 6 (1993) 249–297. ◁ 3

[Fre66] P. Freyd. Algebra valued functors in general and tensor products in particular. Colloq. Math. 14 (1966) 89–106. ◁ 7

[FN18] L. Funar, Y. Neretin. Diffeomorphism groups of tame Cantor sets and Thompson-like groups. Compos. Math. 154 (2018) 1066–1110. ◁ 31

[Gal11] S. Galatius. Stable homology of automorphism groups of free groups. Ann. of Math. 173 (2011) 705–768. ◁ 2, 16

[Gra76] D. Grayson. Higher algebraic K-theory II (after Daniel Quillen). Algebraic K-theory (Proc. Conf., Northwestern Univ., Evanston, Ill., 1976) 217–240. Lecture Notes in Math. 551. Springer, Berlin, 1976. ◁ 11, 13, 32

[HP04] I. Hambleton, E.K. Pedersen. Identifying assembly maps in K- and L-theory. Math. Ann. 328 (2004) 27–57. ◁ 3

[HP02] M. Hyland, J. Power. Pseudo-commutative monads and pseudo-closed 2–categories. J. Pure Appl. Algebra 175 (2002) 141–185. ◁ 8, 22, 24, 29

[HP07] M. Hyland, J. Power. The category theoretic understanding of universal algebra: Lawvere theories and monads. Computation, meaning, and logic: articles dedicated to Gordon Plotkin, 437–458. Electron. Notes Theor. Comput. Sci., 172. Elsevier Sci. B. V., Amsterdam, 2007. ◁ 7, 8

[Koc70] A. Kock. Monads on symmetric monoidal closed categories. Arch. Math. (Basel) 21 (1970) 1–10. ◁ 30

[Koc71] A. Kock. Closed categories generated by commutative monads. J. Austral. Math. Soc. 12 (1971) 405–424. ◁ 30

[Koc72] A. Kock. Strong functors and monoidal monads. Arch. Math. (Basel) 23 (1972) 113–120. ◁ 30

[Law63] F.W. Lawvere. Functorial semantics of algebraic theories. Proc. Nat. Acad. Sci. U.S.A. 50 (1963) 869–872. ◁ 4, 5

[Law68] F.W. Lawvere. Some algebraic problems in the context of functorial semantics of algebraic theories. 1968 Reports of the Midwest Category Seminar. II, 41–61. Springer, Berlin, 1968. ◁ 7

[Law04] F.W. Lawvere. Functorial semantics of algebraic theories and some algebraic problems in the context of functorial semantics of algebraic theories. Repr. Theory Appl. Categ. 5 (2004) 1–121. ◁ 18

[Lod76] J.-L. Loday. K-théorie algébrique et représentations de groupes. Ann. Sci. École Norm. Sup. 9 (1976) 309–377. ◁ 3, 13, 32

[Lüc19] W. Lück. Assembly maps. Handbook of Homotopy Theory, 851–890. H. Miller (editor). CRC Press, 2019. ◁ 3

[M-PMN18] C. Martínez-Pérez, F. Matucci, B. Nucinkis. Presentations of generalisations of Thompson's group V. Pacific J. Math. 296 (2018) 371–403. ◁ 31

[M-PN13] C. Martínez-Pérez, B. Nucinkis. Bredon cohomological finiteness conditions for generalisations of Thompson groups. Groups Geom. Dyn. 7 (2013) 931–959. ◁ 31

[McK96] R. McKenzie. An algebraic version of categorical equivalence for varieties and more general algebraic categories. Logic and algebra (Pontignano, 1994) 211–243. Lecture Notes in Pure and Appl. Math. 180. Dekker, New York, 1996. ◁ 10

[Mor] J. Morava. Some background for Manin's theorem $\mathrm{K}(\mathbb{F}_1) \simeq \mathbb{S}$. Talk at the Vanderbilt conference on Noncommutative Geometry and Geometry over the Field with One Element, 2008. www.alainconnes.org/docs/Morava.pdf ◁ 16, 21

[Nak60] M. Nakaoka. Decomposition theorem for homology groups of symmetric groups. Ann. of Math. 71 (1960) 16–42. ◁ 16

[Par69] B. Pareigis. Kategorien und Funktoren. Mathematische Leitfäden. B.G. Teubner, Stuttgart, 1969. ◁ 4

[Pos21] E.L. Post. Introduction to a general theory of propositions. Amer. J. Math. 41 (1921) 165–185. ◁ 18

[Pri71] S.B. Priddy. On $\Omega^\infty S^\infty$ and the infinite symmetric group. Algebraic topology (Proc. Sympos. Pure Math., Vol. XXII, Univ. Wisconsin, Madison, Wis., 1970) 217–220. Amer. Math. Soc., Providence, R.I., 1971. ◁ 16

[Qui71] D. Quillen. Cohomology of groups. Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, 47–51. Gauthier-Villars, Paris, 1971. ◁ 13

[Qui72] D. Quillen. On the cohomology and K-theory of the general linear groups over a finite field. Ann. Math. 96 (1972) 552–586. ◁ 21

[Qui76] D. Quillen. Letter from Quillen to Milnor on $\mathrm{Im}(\pi_i\mathrm{O} \to \pi_i^{\mathrm{s}} \to \mathrm{K}_i\mathbb{Z})$. Algebraic K-theory (Proc. Conf., Northwestern Univ., Evanston, Ill., 1976) 182–188. Lecture Notes in Math. 551. Springer, Berlin, 1976. ◁ 21

[Qui82] F. Quinn. Ends of maps. II. Invent. Math. 68 (1982) 353–424. ◁ 3

[R-WW17] O. Randal-Williams, N. Wahl. Homological stability for automorphism groups. Adv. Math. 318 (2017) 534–626. ◁ 4

[Ros42] P.C. Rosenbloom. Post algebras. I. Postulates and general theory. Amer. J. Math. 64 (1942) 167–188. ◁ 18

[Sch70] H. Schubert. Kategorien. II. Heidelberger Taschenbücher 66. Springer-Verlag, 1970. ◁ 4

[Seg74] G. Segal. Categories and cohomology theories. Topology 13 (1974) 293–312. ◁ 11

[Ser73] M. Serfati. Introduction aux algèbres de Post et à leurs applications. Cahiers du Bureau Universitaire de Recherche Opérationnelle. Série Recherche 21 (1973) 3–100. ◁ 18, 21

[Spe11] R. Sperber. Comparing assembly maps in algebraic K-theory. J. K-Theory 7 (2011) 145–168. ◁ 3

[Szy14] M. Szymik. Twisted homological stability for extensions and automorphism groups of free nilpotent groups. J. K-Theory 14 (2014) 185–201. ◁ 3, 17

[Szy19] M. Szymik. The rational stable homology of mapping class groups of universal nil-manifolds. Ann. Inst. Fourier 69 (2019) 783–803. ◁ 3, 17, 34

[SW19] M. Szymik, N. Wahl. The homology of the Higman–Thompson groups. Invent. Math. 216 (2019) 445–518. ◁ 2, 15, 17, 31, 32

[Tho80] R.W. Thomason. Beware the phony multiplication on Quillen's $\mathcal{A}^{-1}\mathcal{A}$. Proc. Amer. Math. Soc. 80 (1980) 569–573. ◁ 11

[Wad45] L.I. Wade. Post algebras and rings. Duke Math. J. 12 (1945) 389–395. ◁ 18, 21

[Wag72] J.B. Wagoner. Delooping classifying spaces in algebraic K-theory. Topology 11 (1972) 349–370. ◁ 13

[Wal78] F. Waldhausen. Algebraic K-theory of generalized free products. III, IV. Ann. of Math. 108 (1978) 205–256. ◁ 3

[WW95] M. Weiss, B. Williams. Assembly. Novikov conjectures, index theorems and rigidity, Vol. 2 (Oberwolfach, 1993) 332–352. London Math. Soc. Lecture Note Ser. 227. Cambridge Univ. Press, Cambridge, 1995. ◁ 3

[Wra71] G.C. Wraith. Algebras over theories. Colloq. Math. 23 (1971) 181–190. ◁ 8

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