Topological resolutions in $K(2)$-local homotopy theory at the prime 2

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Abstract

We provide a topological duality resolution for the spectrum $E_2^{h\mathbb{S}_2^1}$, which itself can be used to build the $K(2)$-local sphere. The resolution is built from spectra of the form E_2^{hF} where E_2 is the Morava spectrum for the formal group of a supersingular curve at the prime 2 and F is a finite subgroup of the automorphisms of that formal group. The results are in complete analogy with the resolutions of Goerss, Henn, Mahowald and Rezk (*Ann. of Math.* (2) 162 (2005) 777–822) at the prime 3, but the methods are of necessity very different. As in the prime 3 case, the main difficulty is in identifying the top fiber; to do this, we make calculations using Henn's centralizer resolution.

Contents

Chromatic stable homotopy theory uses the algebraic geometry of smooth one-parameter formal groups to organize calculations and the search for large scale phenomena. In particular, the chromatic filtration on the category of the p-local finite spectra corresponds to the height filtration for formal groups. The layers of the chromatic filtration are given by localization with respect to the Morava K-theories $K(n)$, with $n \geqslant 0$. Thus, to understand the homotopy type of a finite spectrum X we begin by addressing $L_{K(n)}X$ for all prime numbers p and all $0 \leqslant n < \infty$. A useful and inspirational guide to this point of view can be found in the table in [**22**, Section 2].

If $n = 0$, $K(0) = H\mathbb{Q}$ and L_0X is the rational homotopy type of X. For $n \geqslant 1$, the basic computational tool in $K(n)$-local homotopy theory is the $K(n)$-local Adams–Novikov spectral sequence

$$H^s(\mathbb{G}_n, (E_n)_tX) \Longrightarrow \pi_{t-s}L_{K(n)}X.$$

Here $\mathbb{G}_n$ is the automorphism group of a pair $(\mathbb{F}_q, \Gamma_n)$ where $\mathbb{F}_q$ is a finite field of characteristic p and Γ_n is a chosen formal group of height n over $\mathbb{F}_q$. Then E_n is the Morava (or Lubin–Tate) E-theory defined by $(\mathbb{F}_q, \Gamma_n)$. We will give more details and make precise choices in Section 1.

First suppose that p is large with respect to n. (To be precise, we may take $2p - 2 > \max\{n^2, 2n + 2\}$ or $p > 3$ if $n = 2$.) Then the Adams–Novikov spectral sequence for $X = S^0$

collapses and will have no extensions, so the problem becomes algebraic, although by no means easy. See for example, [**6**, **27**, **35**], for the case $n = 2$ and $p > 3$. However, if p is small with respect to n, the group $\mathbb{G}_n$ has finite subgroups of p-power order and the spectral sequence will usually have differentials and extensions, so the problem is no longer purely algebraic. At this point, topological resolutions become a useful way to organize the contributions of the finite subgroups. The key to unlocking this idea is the Hopkins–Miller theorem, which implies that $\mathbb{G}_n$, and hence all of its finite subgroups, act on E_n and that $L_{K(n)}S^0 \simeq E_n^{h\mathbb{G}_n}$. See [**11**] for this and more.

The prototypical example is at $n = 1$ and $p = 2$. Adams and Baird [**9**], and Ravenel [**31**] showed that here we have a fiber sequence

$$L_{K(1)}S^0 \to KO \xrightarrow{\psi^3 - 1} KO$$

where KO is 2-complete real K-theory. For a suitable choice of a height one formal group Γ_1 we can take $E_1 = K$, where K is 2-complete complex K-theory. Then $\mathbb{G}_1 = Aut(\Gamma_1, \mathbb{F}_2) \cong \mathbb{Z}_2^\times$ is the units in the 2-adic integers, and $C_2 = \{\pm 1\} \subseteq \mathbb{Z}_2^\times$ acts through complex conjugation. We can then rewrite this fiber sequence as

$$L_{K(1)}S^0 = E_1^{h\mathbb{G}_1} \longrightarrow E_1^{hC_2} \xrightarrow{\psi^3 - 1} E_1^{hC_2},$$

and ψ^3 is a topological generator of $\mathbb{Z}_2^\times/\{\pm 1\} \simeq \mathbb{Z}_2$.

For higher heights the topological resolutions will not be simple fiber sequences, but finite towers of fibrations with the successive fibers built from E^{hF_i} where F_i runs over various finite subgroups of $\mathbb{G}_n$.

In [**16**] the authors generalized the fiber sequence of the $K(1)$-local case to the case $n = 2$ and $p = 3$. One way to say what they proved is the following. Let us write $E = E_2$ to simplify notation. First we have a split short exact sequence of groups

$$\{1\} \to \mathbb{G}_2^1 \longrightarrow \mathbb{G}_2 \xrightarrow{N} \mathbb{Z}_3 \to \{1\}$$

where N is obtained from a determinant (see (1.7)) and hence a fiber sequence

$$L_{K(2)}S^0 \longrightarrow E^{h\mathbb{G}_2^1} \xrightarrow{\psi - 1} E^{h\mathbb{G}_2^1},$$

where ψ is any element of $\mathbb{G}_2$ which maps by N to a topological generator of $\mathbb{Z}_3$. Then, second, there exists a resolution of $E^{h\mathbb{G}_2^1}$

$$E^{h\mathbb{G}_2^1} \to E^{hG_{24}} \to \Sigma^8 E^{hSD_{16}} \to \Sigma^{40} E^{hSD_{16}} \to \Sigma^{48} E^{hG_{24}}.$$

Here *resolution* means each successive composition is null-homotopic and all possible Toda brackets are zero modulo indeterminacy; thus, the sequence refines to a tower of fibrations

The maximal finite subgroup of $\mathbb{G}_2$ of 3-power order is a cyclic group C_3 of order 3; it is unique up to conjugation in $\mathbb{G}_2^1$. The group G_{24} is the maximal finite subgroup of $\mathbb{G}_2$ containing C_3. The subgroup SD_{16} is the semidihedral group of order 16. Because of the symmetry of this resolution, and because $\mathbb{G}_2^1$ is a virtual Poincaré duality group of dimension 3, this is called a *duality resolution*. This resolution and related resolutions were instrumental in exploring the $K(2)$-local category at primes $p > 2$. See [**5**, **13**–**15**, **17**, **21**, **27**]. The latter paper makes a thorough exploration of what happens at $p > 3$.

The main theorem of this paper provides an analog of the duality resolution at $n = 2$ and $p = 2$. The prime 2 is much harder for a number of reasons. First, the maximal finite 2-subgroup

of $\mathbb{G}_2$ is not cyclic, but isomorphic to Q_8, the quaternion group of order 8. Second, every finite subgroup of $\mathbb{G}_2$ that we consider contains the central $C_2 = \{\pm 1\}$ of $\mathbb{G}_2$; therefore, the homotopy groups of the relevant fixed point spectra E^{hF} are much more complicated. This means that the strategy of proof of [16] will not work and we need to find another way.

We now state our main result. As our chosen formal group Γ_2 we will use the formal group of a supersingular elliptic curve over $\mathbb{F}_4$. The curve will be defined over $\mathbb{F}_2$, so that $\mathbb{G}_2 \cong \mathbb{S}_2 \rtimes \text{Gal}(\mathbb{F}_4/\mathbb{F}_2)$ where $\mathbb{S}_2 = \text{Aut}(\Gamma_2/\mathbb{F}_4)$ is the group of automorphisms of Γ_2 over $\mathbb{F}_4$. Once again we have fiber sequence

$$L_{K(2)}S^0 \longrightarrow E^{h\mathbb{G}_2^1} \xrightarrow{\psi - 1} E^{h\mathbb{G}_2^1}.$$

We have $E^{h\mathbb{G}_2^1} = (E^{h\mathbb{S}_2^1})^{h\text{Gal}}$ where $\text{Gal} = \text{Gal}(\mathbb{F}_4/\mathbb{F}_2)$ and $\mathbb{S}_2^1 = \mathbb{S}_2 \cap \mathbb{G}_2^1$. For many computational applications, the difference between $E^{h\mathbb{G}_2^1}$ and $E^{h\mathbb{S}_2^1}$ is innocuous. See Lemma 1.37. Then our main result is this:

THEOREM 1. *There exists a resolution of $E^{h\mathbb{S}_2^1}$ in the $K(2)$-local category at the prime 2*

$$E^{h\mathbb{S}_2^1} \to E^{hG_{24}} \to E^{hC_6} \to \Sigma^{48}E^{hC_6} \to \Sigma^{48}E^{hG_{24}} .$$

The spectrum E^{hC_6} is 48-periodic, so $E^{hC_6} \simeq \Sigma^{48}E^{hC_6}$; this suspension is there only to emphasize the symmetry in the resolution.

Once again resolution means each successive composition is null-homotopic and all possible Toda brackets are zero modulo indeterminacy; thus, the sequence again refines to a tower of fibrations. This result has already had applications: it is an ingredient in Agnès Beaudry's analysis of the chromatic splitting conjecture at $p = n = 2$. See [4].

The apparent similarity of Theorem 1 with the prime 3 analog is tantalizing, especially the suspension factor on the last term, but we as yet have no conceptual explanation. The proof at the prime 2 can be adapted to the prime 3 but in both cases it comes down to a very specific, prime dependent calculation.

A very satisfying feature of chromatic height 2 is the connection with the theory of elliptic curves. The subgroup $G_{24} \subseteq \mathbb{S}_2$ is the automorphism group of our chosen supersingular curve inside the automorphisms of its formal group. Appealing to Strickland [37] we can use this to get formulas for the action of G_{24} on E_*, a necessary beginning to group cohomology calculations. Furthermore, we know that

$$E^{hG_{48}} \simeq (E^{hG_{24}})^{h\text{Gal}} \simeq L_{K(2)}Tmf$$

where Tmf is the global sections of the sheaf of E_∞-ring spectra on the compactified stack of generalized elliptic curves provided by Hopkins and Miller. See [12]. Similarly, E^{hC_6} is the localization of global sections of the similar sheaf for elliptic curves with a level 3 structure. See [28]. We will not need anything like the full power of the Hopkins–Miller theory here, although we do use some of the calculations that arise from this point of view. See Section 2.

We will prove Theorem 1 in three steps.

First we prove, in Section 3, that there exists a resolution

$$E^{h\mathbb{S}_2^1} \to E^{hG_{24}} \to E^{hC_6} \to E^{hC_6} \to X$$

where

$$E_*X \cong E_*E^{hG_{24}}$$

as twisted $\mathbb{G}_2$-modules. This resolution and the necessary algebraic preliminaries were announced in [20], and grew out of the work surrounding [16]. The algebraic preliminaries are discussed in detail in [3].

Second, in Section 4 we examine the Adams–Novikov Spectral Sequence

$$H^*(G_{24}, E_*) \Longrightarrow \pi_* X.$$

Using a comparison of our resolution with a second resolution, due to Henn, we show, roughly, that certain classes $\Delta^{8k+2} \in H^0(G_{24}, E_{192k+48})$ are permanent cycles — which would certainly be necessary if our main result is true. The exact result is in Corollary 4.7. These calculations were among the main results in the first author's thesis [8] and the key ideas for the entire project can be found there. This ratifies a comment of Mark Mahowald that there is a class in $\pi_{45} L_{K(2)} S^0$ which supports non-zero multiplications by η and $\overline{\kappa}$ and the only way this could happen is if Δ^2 is a permanent cycle. This insight was, we think, the result of long hours of contemplation of the results of Shimomura and Wang [34] and, indeed, one reason for this entire project is to find some way to catch up with those amazing calculations. We will have more to say about this element of Mahowald's in Remark 4.8.

Third and finally, in Section 5 we use a variation of this same comparison argument to produce the equivalence $\Sigma^{48} E^{hG_{24}} \simeq X$. See Theorem 5.8. At the very end we add a remark about the possibility or impossibility of resolutions for $L_{K(2)} S^0$ itself. See Remark 5.10.

There is a number of sections of preliminaries. Section 1 provides the usual background on the $K(n)$-local category as well as some more specific information on mapping spaces. Section 2 pulls together what we need about the homotopy groups of various fixed point spectra. This draws from many sources, and we try to be complete there.

An essential ingredient in our argument is the existence of the other topological resolution for $E^{h\mathbb{S}_2^1}$, Henn's centralizer resolution from [20, §3.4]. We have less control over the maps in this resolution, but, as has happened before ([14]; see also [13]), this resolution provides essential information needed to solve ambiguities in the duality resolution. It is also much closer to being an Adams–Novikov-style resolution as it is based on relative homological algebra. We cover some of this material at the end of Section 3.

1. Recollections on the $K(n)$-local category

We begin with the standard material on the $K(n)$-local category, the Morava stabilizer group, and Morava E-theory, also known as Lubin–Tate theory. We then get specific at $n = 2$ and $p = 2$, discussing the role of formal groups arising from supersingular elliptic curves. We review some material on the homotopy type of the spectrum of maps between various fixed point spectra derived from Morava E-theory and then spell out some details of the $K(n)$-local Adams–Novikov Spectral Sequence. Finally, we discuss the role of the Galois group.

1.1. The $K(n)$-local category

Fix a prime p and let $n \geqslant 1$. Let Γ_n be a formal group of height n over the finite field $\mathbb{F}_p$ of p elements. Then for any finite extension $i : \mathbb{F}_p \subseteq \mathbb{F}_q$ of $\mathbb{F}_p$, we form the group $\mathrm{Aut}(\Gamma_n/\mathbb{F}_q)$ of the automorphisms of $i^*\Gamma_n$ over $\mathbb{F}_q$. We fix a choice of Γ_n with the property that any extension $\mathbb{F}_{p^n} \subseteq \mathbb{F}_q$ gives an isomorphism

$$\mathrm{Aut}(\Gamma_n/\mathbb{F}_{p^n}) \stackrel{\cong}{\longrightarrow} \mathrm{Aut}(\Gamma_n/\mathbb{F}_q). \tag{1.1}$$

The usual Honda formal group satisfies these criteria: this has a formal group law which is p-typical and with p-series $[p](x) = x^{p^n}$. However, if $n = 2$, then the formal group of a supersingular elliptic curve defined over $\mathbb{F}_p$ will also do, and this will be our preferred choice at $p = 2$. Define

$$\mathbb{S}_n = \mathrm{Aut}(\Gamma_n/\mathbb{F}_{p^n}). \tag{1.2}$$

If we choose a coordinate for Γ_n, then any element of $\mathbb{S}_n$ defines a power series $\phi(x) \in x\mathbb{F}_{p^n}[[x]]$, invertible under composition, and the assignment $\phi(x) \mapsto \phi'(0)$ defines a map

$$\mathbb{S}_n \longrightarrow \mathbb{F}_{p^n}^\times.$$

This is a split surjection and we define S_n to be the kernel of this map; this is the p-Sylow subgroup of the profinite group $\mathbb{S}_n$. We then get an isomorphism $S_n \rtimes \mathbb{F}_{p^n}^\times \cong \mathbb{S}_n$.

Define the (big) *Morava stabilizer group* $\mathbb{G}_n$ as the automorphism group of the pair $(\mathbb{F}_{p^n}, \Gamma_n)$. Since Γ_n is defined over $\mathbb{F}_p$, there is an isomorphism

$$\mathbb{G}_n \cong \mathrm{Aut}(\Gamma/\mathbb{F}_{p^n}) \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) = \mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p). \tag{1.3}$$

We will often write $\mathrm{Gal} = \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ when the field extension is understood.

We next must define Morava K-theory. There are many variants, all of which have the same Bousfield class and define the same localization; we will choose a variant which works well with Morava E-theory. Let $K(n) = K(\mathbb{F}_{p^n}, \Gamma_n)$ be the 2-periodic ring spectrum with homotopy groups

$$K(n)_* = \mathbb{F}_{p^n}[u^{\pm 1}]$$

and with associated formal group Γ_n. Here the class u is in degree -2. The group $F_0 = \mathbb{F}_{p^n}^\times \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ acts on $K(n)$ and

$$\mathbb{F}_p[v_n^{\pm 1}] \cong (K(n)^{hF_0})_* = K(n)_*^{F_0}$$

where $v_n = u^{-(p^n-1)}$. The spectrum $K(n)^{hF_0}$ is thus a more classical version of Morava K-theory.

We will spend a great deal of time working in the $K(n)$-local category and, when doing so, all our spectra will implicitly be localized. In particular, we emphasize that we often write $X \wedge Y$ for $L_{K(n)}(X \wedge Y)$, as this is the smash product internal to the $K(n)$-local category.

We now define the Morava spectrum $E = E_n = E(\mathbb{F}_{p^n}, \Gamma_n)$. (We will suppress the n on E_n whenever possible to help ease the notation.) This is the complex oriented, Landweber exact, 2-periodic, E_∞-ring spectrum with

$$E_* = (E_n)_* \cong \mathbb{W}[[u_1, \ldots, u_{n-1}]][u^{\pm 1}] \tag{1.4}$$

with u_i in degree 0 and u in degree -2. Here $\mathbb{W} = W(\mathbb{F}_{p^n})$ is the ring of Witt vectors on $\mathbb{F}_{p^n}$. Note that E_0 is a complete local ring with residue field $\mathbb{F}_{p^n}$; the formal group over E_0 is a choice of universal deformation of the formal group Γ_n over $\mathbb{F}_{p^n}$. (We will be specific about this choice at $n = p = 2$ below in Subsection 1.2.) The group $\mathbb{G}_n = \mathrm{Aut}(\Gamma_n) \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ acts on $E = E_n$, by the Hopkins–Miller theorem [**18**] and we have (see Subsection 1.5) a spectral sequence for any closed subgroup $F \subseteq \mathbb{G}_n$,

$$E_2^{s,t} = H^s(F, \pi_t E) \Longrightarrow \pi_{t-s} E^{hF}. \tag{1.5}$$

We will collectively call these by the name Adams–Novikov Spectral Sequence. See also Lemma 1.29. If $F = \mathbb{G}_n$ itself, then $E^{h\mathbb{G}_n} \simeq L_{K(n)}S^0$ and we are computing the homotopy groups of the $K(n)$-local sphere.

Various subgroups of $\mathbb{G}_n$ will play a role in this paper, especially at $n = 2$ and $p = 2$. The right action of $\mathrm{Aut}(\Gamma_n)$ on $\mathrm{End}(\Gamma_n)$ defines a determinant map $\det\colon \mathbb{S}_n = \mathrm{Aut}(\Gamma_n/\mathbb{F}_{p^n}) \to \mathbb{Z}_p^\times$ which extends to a determinant map

$$\mathbb{G}_n \cong \mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \xrightarrow{\det \times 1} \mathbb{Z}_p^\times \times \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \xrightarrow{p_1} \mathbb{Z}_p^\times. \tag{1.6}$$

Define the *reduced determinant* (or *reduced norm*) N to be the composition

$$\mathbb{G}_n \xrightarrow{\det} \mathbb{Z}_p^\times \longrightarrow \mathbb{Z}_p^\times/C \cong \mathbb{Z}_p. \tag{1.7}$$

$$\underset{N}{\underbrace{\phantom{\mathbb{G}_n \xrightarrow{\det} \mathbb{Z}_p^\times \longrightarrow}}}$$

where $C \subseteq \mathbb{Z}_p^\times$ is the maximal finite subgroup. For example, $C = \{\pm 1\}$ if $p = 2$. There are isomorphisms $\mathbb{Z}_p^\times/C \cong \mathbb{Z}_p$ and we choose one. Write $\mathbb{G}_n^1$ for the kernel of N, $\mathbb{S}_n^1 = \mathbb{S}_n \cap \mathbb{G}_n^1$, and $S_n^1 = S_n \cap \mathbb{G}_n^1$. The map $N : S_n \to \mathbb{Z}_p$ is split surjective and we have semi-direct product decompositions for each of the groups $\mathbb{G}_n$, $\mathbb{S}_n$, and S_n; for example, there is an isomorphism

$$\mathbb{S}_n^1 \rtimes \mathbb{Z}_p \cong \mathbb{S}_n.$$

If n is prime to p, we can choose a central splitting and the semi-direct product is actually a product, but that is not the case of interest here.

1.2. Deformations from elliptic curves

Here we spell out what we need from the theory of elliptic curves at $p = 2$; this will give us a preferred formal group and a preferred universal deformation. Choose Γ_2 to be the formal group obtained from the elliptic curve C_0 over $\mathbb{F}_2$ defined by the Weierstrass equation

$$y^2 + y = x^3. \tag{1.8}$$

This is a standard representative for the unique isomorphism class of supersingular curves over $\overline{\mathbb{F}}_2$; see [33, Appendix A]. Because C_0 is supersingular, Γ_2 has height 2, as the notation indicates. Following [37] let C be the elliptic curve over $\mathbb{W}(\mathbb{F}_4)[[u_1]]$ defined by the Weierstrass equation

$$y^2 + 3u_1 xy + (u_1^3 - 1)y = x^3. \tag{1.9}$$

This reduces to C_0 modulo the maximal ideal $\mathfrak{m} = (2, u_1)$; the formal group G of C is a choice of the universal deformation of Γ_2.

Again turning to [33, Appendix A], we have

$$G_{24} \overset{\text{def}}{=} \operatorname{Aut}(C_0/\mathbb{F}_4) \cong Q_8 \rtimes \mathbb{F}_4^\times \tag{1.10}$$

where $\mathbb{F}_4^\times \cong C_3$ acts on Q_8 as the 3-Sylow subgroup of $\operatorname{Aut}(Q_8)$. Define

$$G_{48} \overset{\text{def}}{=} \operatorname{Aut}(\mathbb{F}_4, C_0) \cong \operatorname{Aut}(C_0/\mathbb{F}_4) \rtimes \operatorname{Gal}(\mathbb{F}_4/\mathbb{F}_2). \tag{1.11}$$

Since any automorphism of the pair $(\mathbb{F}_4, C_0)$ induces an automorphism of the pair $(\mathbb{F}_4, \Gamma_2)$ we get a map $G_{48} \to \mathbb{G}_2$. This map is an injection and we identify G_{48} with its image.

REMARK 1.12. Let $C_0[3]$ be the subgroup scheme of C_0 consisting of points of order 3; over $\mathbb{F}_4$, this becomes abstractly isomorphic to $\mathbb{Z}/3 \times \mathbb{Z}/3$. The group G_{48} acts linearly on $C_0[3]$ and choosing a basis for the $\mathbb{F}_4$-points of $C_0[3]$ determines an isomorphism $G_{48} \cong GL_2(\mathbb{Z}/3)$. This restricts to an isomorphism $G_{24} \cong SL_2(\mathbb{Z}/3)$.

REMARK 1.13. The following subgroups will play an important role in this paper.

(1) $C_2 = \{\pm 1\} \subseteq Q_8$.
(2) $C_6 = C_2 \times \mathbb{F}_4^\times$.
(3) C_4, any of the subgroups of order 4 in Q_8.
(4) G_{24} and G_{48} themselves.

The subgroup C_4 is not unique, but it is unique up to conjugation in G_{24} and in $\mathbb{G}_2$. In particular, the homotopy type of E^{hC_4} is well defined.

REMARK 1.14. We have been discussing G_{24} as a subgroup of $\mathbb{S}_2$, but it can also be thought of as a quotient. Inside of $\mathbb{S}_2$ there is a normal torsion-free pro-2-subgroup K which has the property that the composition

$$G_{24} \longrightarrow \mathbb{S}_2 \longrightarrow \mathbb{S}_2/K$$

is an isomorphism. Thus we have a decomposition $K \rtimes G_{24} \cong \mathbb{S}_2$. See [3] for details. The group K is a Poincaré duality group of dimension 4.

1.3. The functor E_*X

We define

$$E_*X = \pi_* L_{K(n)}(E \wedge X).$$

Despite the notation, $E_*(-)$ is not a homology theory, as it does not take arbitrary wedges to sums, but it is our most sensitive algebraic invariant on the $K(n)$-local category.

Here are some properties of $E_*(-)$. See [24, §8, Appendix A] for more details. Let $\mathfrak{m} \subseteq E_0$ be the maximal ideal and $L = L_0$ be the 0th derived functor of completion at $\mathfrak{m}$. Recall that an E_0-module is L-complete if the natural map $M \to L(M)$ is an isomorphism. If $M = L(N)$, then M is L-complete; that is, $L(N) \to L^2(N)$ is an isomorphism for all N. Thus the full subcategory of L-complete modules is a reflexive sub-category of all continuous E_0-modules. By [24, Proposition 8.4], E_*X is L-complete for all X.

If N is an E_0-module, there is a short exact sequence

$$0 \to \lim_k^1 \mathrm{Tor}_1^{E_0}(E_0/m^k, N) \to L(N) \to \lim_k N/m^k N \to 0. \tag{1.15}$$

Hence if M is L-complete, then M is $\mathfrak{m}$-complete, but it will be complete and separated only if the right map is an isomorphism. See [16, §2] for some precise assumptions which guarantee that E_*X is complete and separated. All of the spectra in this paper will meet these assumptions.

Since $\mathbb{G}_n$ acts on E, it acts on E_*X in the category of L-complete modules. This action is twisted because it is compatible with the action of $\mathbb{G}_n$ on the coefficient ring E_*. We will call L-complete E_0-modules with such a $\mathbb{G}_n$-action either *twisted $\mathbb{G}_n$-modules*, or *Morava modules*.

For example, let $F \subseteq \mathbb{G}_n$ be a closed subgroup. Then there is an isomorphism of twisted $\mathbb{G}_n$-modules

$$E_* E^{hF} \cong \mathrm{map}(\mathbb{G}_n/F, E_*) \tag{1.16}$$

where $\mathrm{map}(-,-)$ denotes the set of continuous maps. On the right-hand side of this equation, E_* acts on the target and the $\mathbb{G}_n$-action is diagonal. This needs a bit of care, as the proof of this result given in [11] is given for the Honda formal group and it may not be true in general. However, in analyzing the proof of the crucial Theorem 2 of [11] we see that the isomorphism of (1.16) follows from the more basic isomorphism

$$E_0 E \cong \mathrm{map}(\mathbb{G}_n, E_0).$$

The standard proof of this isomorphism for the Honda formal group (see [36, Theorem 12], for example) requires only that our formal group be defined over $\mathbb{F}_p$ and satisfy the stabilization requirement of (1.1).

REMARK 1.17. We add here that [24, Theorem 8.9] implies that the functor

$$X \mapsto E_*X = \pi_* L_{K(n)}(E_n \wedge X)$$

detects weak equivalences in the $K(n)$-local category. Given a map $f : X \to Y$, the class of spectra Z so that $L_{K(n)}(Z \wedge f)$ is a weak equivalence is closed under cofibrations and retracts; hence if E is in this class, then $L_{K(n)}S^0$ is in this class.

1.4. Mapping spectra

We collect here some basics about the mapping spectra $F(E^{hF_1}, E^{hF_2})$ for various subgroups F_1 and F_2 of $\mathbb{G}_n$.

REMARK 1.18. Let $F \subseteq \mathbb{G}_n$ be a closed subgroup. We begin with the equivalence

$$E \wedge E^{hF} \simeq \mathrm{map}(\mathbb{G}_n/F, E) \tag{1.19}$$

from the local smash product to the localized spectrum of continuous maps. It is helpful to visualize this map as sending $x \wedge y$ to the function $gF \mapsto x(gy)$. We will continue this mnemonic below: using point-wise defined functions to indicate maps of spectra which cannot be defined that way. We hope the readers can fill in the details themselves; if not, complete details can be found in [**16**, § 2].

The action of $\mathbb{G}_n$ on E in (1.19) defines the Morava module structure of $E_* E^{hF}$; under the isomorphism of (1.19) this maps to the diagonal action on the functions

$$(h\phi)(g) = h\phi(h^{-1}g).$$

Note that

$$(E_* E^{hF})^{\mathbb{G}_n} = \mathrm{map}_{\mathbb{G}_n}(\mathbb{G}_n/F, E_*) \cong (E_*)^F.$$

By [**11**, Theorem 2], this extends to an isomorphism

$$H^*(\mathbb{G}_n, E_* E^{hF}) \cong H^*(F, E_*).$$

See also Lemma 1.29.

REMARK 1.20. We now recall some results from [**16**]. If $X = \lim X_i$ is a profinite set, let $E[[X]] = \lim E \wedge X_i^+$ where the $+$ indicates a disjoint basepoint. Then if F_1 is a closed subgroup of $\mathbb{G}_n$, we have an equivalence

$$E[[\mathbb{G}_n/F_1]] \simeq F(E^{hF_1}, E) \tag{1.21}$$

defined as follows. Let F_E denote the function spectrum in E-modules. Then

$$E[[\mathbb{G}_n/F_1]] \simeq F_E(\mathrm{map}(\mathbb{G}_n/F_1, E), E)$$

$$\simeq F_E(E \wedge E^{hF_1}, E)$$

$$\simeq F(E^{hF_1}, E).$$

Next note that the equivalence of (1.21) is $\mathbb{G}_n$-equivariant with the following actions: in $F(E^{hF_1}, E)$ we act on the target and on $E[[\mathbb{G}_n/F_1]]$ we act as follows:

$$h\left(\sum a_g gF_1\right) = \sum h(a_g) hgF_1.$$

We can now make the following deductions. First suppose $F_1 = U$ is open (and hence closed), so that $\mathbb{G}_n/F_1 = \mathbb{G}_n/U$ is finite. Let F_2 be finite. Then we have equivalences

$$\prod_{F_2 \backslash \mathbb{G}_n/U} E^{hF_x} \simeq E[[\mathbb{G}_n/U]]^{hF_2}$$

$$\simeq F(E^{hU}, E)^{hF_2}$$

$$\simeq F(E^{hU}, E^{hF_2}). \tag{1.22}$$

The product in the source is over the double coset space, and for a double coset $F_2 x U$,

$$F_x = F_2 \cap xUx^{-1} \subseteq F_2.$$

Since it depends on a choice of x, the group F_x is defined only up to conjugation, but the fixed point spectrum E^{hF_x} is well defined up to weak equivalence.

The first map of (1.22) sends $a \in E^{hF_x}$ to the sum

$$\sum_{gF_x \in F_2/F_x} (ga)\ gxU.$$

We say a word about the naturality of the equivalence of (1.22). Suppose $U \subseteq V \subseteq \mathbb{G}_n$ is a nested pair of open subgroups. Then for each double coset F_2xU we get a double coset F_2xV, a nested pair of subgroups

$$F_x = F_2 \cap xUx^{-1} \subseteq F_2 \cap xVx^{-1} = G_x,$$

and a transfer map

$$\mathrm{tr}_x : E^{hF_x} \longrightarrow E^{hG_x}$$

associated to this inclusion. Then we have a commutative diagram

$$\begin{array}{ccc}
\prod_{F_2 \backslash \mathbb{G}_n/U} E^{hF_x} & \xrightarrow{\simeq} & E[[\mathbb{G}_n/U]]^{hF_2} \\
{\scriptstyle \mathrm{tr}} \downarrow & & \downarrow {\scriptstyle g} \\
\prod_{F_2 \backslash \mathbb{G}_n/V} E^{hG_x} & \xrightarrow{\simeq} & E[[\mathbb{G}_n/V]]^{hF_2}
\end{array} \qquad (1.23)$$

where the map tr is the sum of the transfer maps and the map g induced by the quotient $\mathbb{G}_n/U \to \mathbb{G}_n/V$.

For a more general closed subgroup F_1 write $F_1 = \cap_i U_i$ where $U_i \subseteq \mathbb{G}_n$ is open. Then, for F_2 finite we get a weak equivalence

$$\mathrm{holim}_i \prod_{F_2 \backslash \mathbb{G}_n/U_i} E^{hF_x} \simeq F(E^{hF_1}, E^{hF_2}) \qquad (1.24)$$

where the product is as before and

$$F_x = F_2 \cap xU_ix^{-1} \subseteq F_2$$

depends on i and, as in (1.23), there may be transfer maps in the transition maps for the homotopy limit. As F_2 is finite and the product in (1.24) is finite, this implies that the image of

$$\prod_{F_2 \backslash \mathbb{G}_n/U_j} \pi_* E^{hF_x} \longrightarrow \prod_{F_2 \backslash \mathbb{G}_n/U_i} \pi_* E^{hF_x}$$

is independent of j for large j. It follows that there will be no $\lim^1$ term for the homotopy groups of the inverse limit and hence there is an isomorphism

$$\lim_i \prod_{F_2 \backslash \mathbb{G}_n/U_i} \pi_* E^{hF_x} \cong \pi_* F(E^{hF_1}, E^{hF_2}) \qquad (1.25)$$

1.5. The $K(n)$-local Adams–Novikov Spectral Sequence

This is the main technical tool of this paper, and we give a few details of the construction and some of its properties. We begin with some algebra from [**24**, Appendix A].

Let $\mathfrak{m} \subset E_0 \cong \mathbb{W}[[u_1, \ldots, u_{n-1}]]$ be the maximal ideal. An L-complete E_0-module M is *pro-free* if any one of the following equivalent conditions is satisfied:

(1) $M \cong L(N) \cong N_{\mathfrak{m}}^{\wedge}$ for some free E_0-module N;
(2) the sequence $(p, u_1, \ldots, u_{n-1})$ is regular on M;
(3) $\mathrm{Tor}_1^{E_0}(\mathbb{F}_{p^n}, M) = 0$;
(4) M is projective in the category of L-complete modules.

Proposition A.13 of [24] implies that a continuous homomorphism $M \to N$ of pro-free modules is an isomorphism if and only if $M/\mathfrak{m}M \to N/\mathfrak{m}N$ is an isomorphism.

If $F \subseteq \mathbb{G}_n$ is a closed subgroup, then $E_* E^{hF}$ is pro-free, by (1.16).

By [24, Proposition 8.4], if X is a spectrum with $K(n)_* X$ concentrated in even degrees, then $E_* X$ is pro-free, concentrated in even degrees. Furthermore, if $E_* X$ is pro-free and concentrated in even degrees, then

$$K(n)_* X \cong \mathbb{F}_{p^n} \otimes_{E_0} E_* X \cong (E_* X)/\mathfrak{m}.$$

REMARK 1.26.   We now come to the Adams–Novikov Spectral Sequence in the $K(n)$-local category. As in [36], Proposition 15, this spectral sequence is obtained by the standard cosimplicial cobar complex for $E = E_n$ in the $K(n)$-local category. Thus we have

$$E_2^{s,t} \cong \pi^s \pi_t(E^\bullet \wedge X) \Longrightarrow \pi_{t-s} L_{K(n)} X.$$

The smash products are in the $K(n)$-local category. It is a consequence of the proof of [36, Proposition 15] that this spectral sequence converges strongly and has a horizontal vanishing line at E_∞.

We would now like to rewrite the E_2-term as group cohomology, at least under some hypotheses. For any group G, let EG be the standard contractible simplicial G-set with $(EG)_s = G^{s+1}$. This is a bar construction. If G is a topological group, then EG is a simplicial space.

Let M be an L-complete Morava module. We define

$$H^s(\mathbb{G}_n, M) = \pi^s \mathrm{map}_{\mathbb{G}_n}(E\mathbb{G}_n, M)$$

where $\mathrm{map}_{\mathbb{G}_n}(-,-)$ denotes the group of continuous $\mathbb{G}_n$-maps. Since there is an isomorphism $\mathrm{map}_{\mathbb{G}_n}(\mathbb{G}_n^{s+1}, M) \cong \mathrm{map}(\mathbb{G}_n^s, M)$, we have that $H^*(\mathbb{G}_n, M)$ is the sth cohomology group of a cochain complex

$$M \longrightarrow \mathrm{map}(\mathbb{G}_n, M) \longrightarrow \mathrm{map}(\mathbb{G}_n^2, M) \longrightarrow \cdots$$

PROPOSITION 1.27.   Suppose $X = Y \wedge Z$ where $K(n)_* Y$ is concentrated in even degrees and Z is a finite complex. Then we have an isomorphism

$$\pi^s \pi_t(E^\bullet \wedge X) \cong H^s(\mathbb{G}_n, E_t X)$$

and the $K(n)$-local Adams–Novikov Spectral Sequence reads

$$H^s(\mathbb{G}_n, E_t X) \Longrightarrow \pi_{t-s} L_{K(n)} X.$$

Proof.   If $Z = S^0$, then $E_* X$ is pro-free in even degrees. The result can be found in [1, Theorem 4.3]. The authors there work with the version of Morava E-theory obtained from the Honda formal group, but they need only the isomorphism $E_0 E \cong \mathrm{map}(\mathbb{G}_n, E_0)$, which holds in our case. See the remarks after (1.16). The key idea is that this last isomorphism can be extended to an isomorphism

$$E_*(E^{\wedge s} \wedge X) \longrightarrow \mathrm{map}(\mathbb{G}_n^s, E_* X)$$

for any spectrum X with $E_* X$ pro-free and in even degrees. For more general Z, the isomorphism on E_2-terms follows from the five lemma. $\square$

In some crucial cases, it is possible to reduce the E_2-term to group cohomology over a finite group. Let $F \subseteq \mathbb{G}_n$ be a finite subgroup and N an L-complete twisted F-module. Define an

L-complete module $N\uparrow_F^{\mathbb{G}_n}$ as the set of continuous maps $\phi : \mathbb{G}_n \to N$ such that $\phi(hx) = h\phi(x)$ for $h \in F$. This becomes a Morava module with

$$(g\phi)(x) = \phi(xg)$$

with $g \in \mathbb{G}_n$ and there is an isomorphism

$$\mathrm{map}_{\mathbb{G}_n}(\mathbb{G}_n^{s+1}, N\uparrow_F^{\mathbb{G}_n}) \cong \mathrm{map}_F(\mathbb{G}_n^{s+1}, N)$$

of groups of continuous maps. Let $H^s(F, N) = \pi^s \mathrm{map}_F(EF, N)$.

LEMMA 1.28 (Shapiro Lemma). *Let* $F \subseteq \mathbb{G}_n$ *be a finite subgroup and suppose* N *is an* L-complete twisted F-module with the property that $N/\mathfrak{m}N$ is finite dimensional over $\mathbb{F}_{p^n}$. *Then there is an isomorphism*

$$H^*(\mathbb{G}_n, N\uparrow_F^{\mathbb{G}_n}) \cong H^*(F, N)$$

and, under these hypotheses on N, *there is an isomorphism*

$$H^*(F, N) \cong \lim_k H^*(F, N/m^k N).$$

Proof. Since N is L-complete and $N/\mathfrak{m}N$ is finite, the short exact sequence of (1.15) implies $N \cong N_{\mathfrak{m}}^\wedge$.

Choose a nested sequence $U_{i+1} \subseteq U_i \subseteq \mathbb{G}_n$ of finite index subgroups of $\mathbb{G}_n$ with the property that $\cap U_i = \{e\}$. Then for all $s \geqslant 0$ we have

$$\mathrm{map}_{\mathbb{G}_n}(\mathbb{G}_n^{s+1}, N\uparrow_F^{\mathbb{G}_n}) \cong \mathrm{map}_F(\mathbb{G}_n^{s+1}, N)$$

$$\cong \lim_k \mathrm{map}_F(\mathbb{G}_n^{s+1}, N/\mathfrak{m}^k N)$$

$$\cong \lim_k \mathrm{colim}_i \mathrm{map}_F((\mathbb{G}_n/U_i)^{s+1}, N/\mathfrak{m}^k N).$$

The last isomorphism follows from the fact that $N/m^k N$ is discrete and finite. Since F is finite, we have $F \cap U_i = \{e\}$ for all i greater than some i_0. Then for $i > i_0$ it follows that $(\mathbb{G}_n/U_i)^{\bullet+1}$ is a contractible simplicial free F-set and, thus, we have an isomorphism

$$\pi^s \mathrm{map}_F((\mathbb{G}_n/U_i)^{\bullet+1}, N/\mathfrak{m}^k N) \cong H^s(F, N/\mathfrak{m}^k N).$$

Again since F is finite, $H^s(F, N/\mathfrak{m}^k N)$ is a finite abelian group. Both isomorphisms of the lemma now follow. $\qquad\square$

LEMMA 1.29. *Let* Y *be a spectrum equipped with an isomorphism of Morava modules*

$$E_* Y \cong \mathrm{map}(\mathbb{G}_n/F, E_*) \cong E_* E^{hF}$$

where $F \subseteq \mathbb{G}_n$ *is a finite subgroup. Then for all finite spectra* Z, *there is an isomorphism*

$$H^*(\mathbb{G}_n, E_*(Y \wedge Z)) \cong H^*(F, E_* Z).$$

Proof. We have an isomorphism of Morava modules

$$E_*(Y \wedge Z) \xrightarrow{\;\cong\;} \mathrm{map}(\mathbb{G}_n/F, E_* Z)$$

where the action of $\mathbb{G}_n$ on the target is by conjugation: $(g\phi)(x) = g\phi(g^{-1}x)$. There is an isomorphism of Morava modules

$$\mathrm{map}(\mathbb{G}_n/F, E_* Z) \cong (E_* Z)\uparrow_F^{\mathbb{G}_n}$$

adjoint to evaluation at eF. The result follows from the Shapiro Lemma 1.28. $\qquad\square$

REMARK 1.30. Suppose $Y = E^{hF}$. Then Lemma 1.29 follows from the variant of [**11**, Theorem 2] appropriate for our formal group; indeed, we need only require that F be closed. However, the proof relies on the construction of E^{hF} given in that paper and does not *a priori* apply if we only know $E_*Y \cong E_*E^{hF}$ — as will be the case in our application.

REMARK 1.31. There is a map of Adams–Novikov Spectral Sequences

$$
\begin{array}{ccc}
\mathrm{Ext}^s_{BP_*BP}(\Sigma^t BP_*, BP_*) & \Longrightarrow & \mathbb{Z}_{(2)} \otimes \pi_{t-s}S^0 \\
\downarrow & & \downarrow \\
H^s(\mathbb{G}_2, E_t) & \Longrightarrow & \pi_{t-s}L_{K(2)}S^0
\end{array}
$$

but it takes a little care to define. Let $G(x,y) \in E^0[[x,y]]$ be the formal group law of the supersingular elliptic curve of (1.9); since G is the formal group of a Weierstrass curve, it has a preferred coordinate. Let $G_*(x,y) = uG(u^{-1}x, u^{-1}y) \in E^*[[x,y]]$. Then G_* is a formal group over E^* with coordinate in cohomological degree 2 and, therefore, is classified by a map $\mathbb{Z}_{(2)} \otimes MU_* \to E_*$. Since G_* is not evidently 2-typical, it need not be classified by a map $BP_* \to E_*$. However, over a $\mathbb{Z}_{(2)}$-algebra, the Cartier idempotent gives an equivalence between the groupoid of all formal group laws and the groupoid of 2-typical formal group laws; hence we have a diagram of spectral sequences as needed:

$$
\begin{array}{ccc}
\mathrm{Ext}^s_{BP_*BP}(\Sigma^t BP_*, BP_*) & \Longrightarrow & \mathbb{Z}_{(2)} \otimes \pi_{t-s}S^0 \\
\cong \uparrow & & \uparrow = \\
\mathbb{Z}_{(2)} \otimes \mathrm{Ext}^s_{MU_*MU}(\Sigma^t MU_*, MU_*) & \Longrightarrow & \mathbb{Z}_{(2)} \otimes \pi_{t-s}S^0 \\
\downarrow & & \downarrow \\
H^s(\mathbb{G}_2, E_t) & \Longrightarrow & \pi_{t-s}L_{K(2)}S^0 \,.
\end{array}
$$

We will use this below in the section on the cohomology of G_{48}.

1.6. *The action of the Galois group*

We now turn to analyzing $E^{h\mathbb{S}_2}$ as an equivariant spectrum over the Galois group. As above, we will write $\mathrm{Gal} = \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$, so that $\mathbb{G}_n \cong \mathbb{S}_n \rtimes \mathrm{Gal}$.

We begin with the following elementary fact: the map $\mathbb{Z}_p \to \mathbb{W}$ is Galois with Galois group Gal; thus, it is faithfully flat, étale, and the shearing map

$$
\mathbb{W} \otimes_{\mathbb{Z}_p} \mathbb{W} \to \mathrm{map}(\mathrm{Gal}, \mathbb{W})
$$

sending $a \otimes b$ to the function $g \mapsto ag(b)$ is an isomorphism. In fact, this shearing map is certainly an isomorphism modulo p; then the statement for $\mathbb{W}$ follows from Nakayama's lemma. Faithfully flat descent now implies that the category of $\mathbb{Z}_p$-modules is equivalent to the category of twisted $\mathbb{W}[\mathrm{Gal}]$-modules under the functor $M \mapsto \mathbb{W} \otimes_{\mathbb{Z}_p} M$; the inverse to this functor sends N to N^{Gal}.

This extends to the following result.

LEMMA 1.32. *Let $K \subseteq \mathbb{G}_n$ be a closed subgroup and let $K_0 = K \cap \mathbb{S}_n$. Suppose the canonical map*

$$
K/K_0 \longrightarrow \mathbb{G}_n/\mathbb{S}_n \cong \mathrm{Gal}
$$

is an isomorphism. Then for any twisted $\mathbb{G}_n$-module M we have isomorphisms

$$H^*(K, M) \cong H^*(K_0, M)^{\mathrm{Gal}}$$

$$H^*(K_0, M) \cong \mathbb{W} \otimes_{\mathbb{Z}_p} H^*(K, M).$$

Proof. The subgroup $\mathbb{S}_n$ acts on E_0 through $\mathbb{W}$-algebra homomorphisms; hence it acts on M through $\mathbb{W}$-module homomorphisms. It follows that we can write the functor $(-)^K$ of invariants as a composite functor

$$\text{twisted } \mathbb{G}_n\text{-modules} \xrightarrow{(-)^{K_0}} \text{twisted } \mathbb{W}[\mathrm{Gal}]\text{-modules} \xrightarrow{(-)^{\mathrm{Gal}}} \mathbb{Z}_p\text{-modules.}$$

As we just remarked, the second of these two functors is an equivalence of categories and in particular it is an exact functor. The first equation follows. The second equation follows from the first and the fact that the inverse to $(-)^{\mathrm{Gal}}$ is the functor $M \mapsto \mathbb{W} \otimes_{\mathbb{Z}_p} M$. $\qquad\square$

We now give a fact seemingly known to everyone, but hard to find in print. Drew Heard was the first to point out an error in our original argument; others followed quickly. We learned the following replacement from Mike Hopkins, Agnès Beaudry, and the referee. We extend our thanks to everyone.

LEMMA 1.33. *For all p and all $n \geqslant 1$ we have isomorphisms*

$$H^0(\mathbb{G}_n, E_0) \cong \mathbb{Z}_p$$

$$H^0(\mathbb{S}_n, E_0) \cong \mathbb{W} = W(\mathbb{F}_{p^n}).$$

Furthermore, $H^0(\mathbb{G}_n, E_t) = H^0(\mathbb{S}_n, E_t) = 0$ if $t \neq 0$.

Proof. By Lemma 1.32 we need only do the case of $\mathbb{S}_n$.

It is also sufficient to prove this when Γ_n is the Honda formal group over $\mathbb{F}_{p^n}$. Any other height n formal group becomes isomorphic to the Honda formal group over the algebraic closure of $\mathbb{F}_p$, and the general result could then be deduced from Galois descent.

Write ϕ for lift of Frobenius to Witt vectors $\mathbb{W}$. For the Honda formal group, $\mathbb{S}_n$ is the group of units in the endomorphism ring

$$\mathrm{End}(\Gamma_n) \cong \mathbb{W}\langle S \rangle / (S^n - p)$$

where $\mathbb{W}\langle S \rangle$ is non-commutative polynomial ring over $\mathbb{W}$ on a variable S with $Sa = \phi(a)S$ when $a \in \mathbb{W}$. Thus we have an inclusion $\mathbb{W}^\times \subset \mathbb{S}_n$. Every $k \in \mathbb{Z}_p^\times \subseteq \mathbb{S}_n$ corresponds to an automorphism $\Gamma_n \to [k]_{\Gamma_n}$ of Γ_n; in particular $\mathbb{Z}_p^\times$ acts trivially on E_0 and if $u \in E_{-2}$ is the invertible generator, we have $k_* u = ku$. If follows that $H^0(\mathbb{Z}_p^\times, E_t) = 0$ if $t \neq 0$.

This leaves the case $t = 0$. Write $\mathbb{K} = \mathbb{W}[p^{-1}]$. Since all elements of $\mathbb{S}_n$ fix the constants $\mathbb{W} \subseteq E_0$ there is an inclusion

$$\mathbb{W} \cong H^0(\mathbb{W}^\times, \mathbb{W}) \to H^0(\mathbb{W}^\times, E_0).$$

Furthermore, since E_0 is torsion-free, it is sufficient to show $H^0(\mathbb{W}^\times, E_0[p^{-1}]) \cong \mathbb{K}$. By [10], Lemma 4.3, there are power series $w_i = w_i(u_1, u_2, \ldots, u_{n-1}) \in \mathbb{K}[[u_1, \ldots, u_{n-1}]]$ and an inclusion

$$\mathbb{W}[[w_1, \ldots, w_{n-1}]] \longrightarrow \mathbb{K}[[u_1, \ldots, u_{n-1}]]$$

onto a $\mathbb{G}_n$-equivariant sub-algebra which becomes an isomorphism after inverting p in the source. Thus it is sufficient to show that

$$H^0(\mathbb{W}^\times, J) = 0$$

where $J = \mathbb{W}[[w_1, \ldots, w_{n-1}]]/\mathbb{W}$.

By [**10**], Proposition 3.3, the action of $\mathbb{W}^\times$ on $\mathbb{W}[[w_1, \ldots, w_{n-1}]]$ is diagonal: if $a \in \mathbb{W}^\times$, then

$$a_* w_i = \phi^i(a) a^{-1} w_i.$$

Let $\omega \in \mathbb{W}$ be a primitive $(p^n - 1)$st root of unity and set $a = 1 + p\omega$. Then $\phi(1 + p\omega) = 1 + p\omega^p$. If some $i_j \neq 0$ it follows that

$$(1 + p\omega)_* w_1^{i_1} \ldots w_{n-1}^{i_{n-1}} \neq w_1^{i_1} \ldots w_{n-1}^{i_{n-1}}$$

as needed. $\square$

REMARK 1.34. Note that the proof of Lemma 1.33 actually shows we need only relatively small subgroups of $\mathbb{G}_n$ to get the full invariants. Specifically, we have

$$\mathbb{Z}_p \cong H^0(\mathbb{W}^\times \rtimes \mathrm{Gal}, E_0)$$

and if $t \neq 0$, then $H^0(\mathbb{Z}_p^\times, E_t) = 0$. We will not need this stronger result.

REMARK 1.35. In the proof of Lemma 1.36 below we will use the following observation. Suppose we have a spectral sequence $\{E_r^{s,t}\}$ that is multiplicative; that is, $E_r^{*,*}$ is a bigraded ring which is commutative up to sign and d_r satisfies the Leibniz rule, again up to sign. Further suppose $R \subseteq E_r^{0,0}$ is a commutative subring of d_r-cycles and $R \subseteq S$ is an étale extension in $E_r^{0,0}$. Then every element of S is a d_r-cycle. To see this, note that d_r restricted to S is a derivation over R and any such derivation must vanish; indeed, depending on your foundations, the vanishing of such derivations may even be part of your definition of étale.

LEMMA 1.36. *For all p and all $n \geqslant 1$ there is a* Gal-*equivariant equivalence*

$$\mathrm{Gal}^+ \wedge L_{K(n)} S^0 \to E^{h\mathbb{S}_n}.$$

Proof. We first prove we have an injection $\mathbb{W} \to \pi_0 E^{h\mathbb{S}_n}$. We begin with the isomorphism $\mathbb{W} \cong H^0(\mathbb{S}_n, E_0)$ of Lemma 1.33. Since $\mathbb{Z}_p \subseteq \mathbb{W}$ is an étale extension and the Adams–Novikov Spectral Sequence for $E^{h\mathbb{S}_n}$ is a spectral sequence of rings, all of $\mathbb{W}$ survives to E_∞ and the edge homomorphism provides a surjection $\pi_0 E^{h\mathbb{S}_n} \to \mathbb{W}$ of rings. The kernel of this map nilpotent as an ideal because the spectral sequence has a horizontal vanishing line at E_∞. See Remark 1.26. We then have a diagram

$$
\begin{array}{ccc}
\mathbb{Z}_p & \longrightarrow & \pi_0 E^{h\mathbb{S}_n} \\
{\scriptstyle \subseteq} \downarrow & \nearrow & \downarrow \\
\mathbb{W} & \underset{=}{\longrightarrow} & \mathbb{W}
\end{array}
$$

Again since $\mathbb{Z}_p \subseteq \mathbb{W}$ is étale, the dashed arrow can be completed uniquely. This yields the injection we need.

Now define $\omega : S^0 \to E^{h\mathbb{S}_n}$ to be a representative of the homotopy class defined by a primitive $(p^n - 1)$st root of unity in $\mathbb{W}$. We can extend ω to a Gal-equivariant map $f : \mathrm{Gal}^+ \wedge S^0 \to E^{h\mathbb{S}_2}$ inducing the splitting $\mathbb{W} \to \pi_0 E^{h\mathbb{S}_2}$; here we use that the map $\mathbb{Z}_p[\mathrm{Gal}] \to \mathbb{W}$

$$\sum a_g g \longmapsto \sum a_g g(\omega)$$

is an isomorphism. The map f extends to an isomorphism of twisted $\mathbb{G}_n$-modules

$$E_* f : E_*(\mathrm{Gal}^+ \wedge S^0) \cong \mathrm{map}(\mathrm{Gal}, E_*) \cong \mathrm{map}(\mathbb{G}_n/\mathbb{S}_n, E_*) \cong E_* E^{h\mathbb{S}_n}$$

thus completing the argument. $\square$

The following result is a topological analog of Lemma 1.32.

LEMMA 1.37. *Let $K \subseteq \mathbb{G}_n$ be a closed subgroup and let $K_0 = K \cap \mathbb{S}_n$. Suppose the canonical map*

$$K/K_0 \longrightarrow \mathbb{G}_n/\mathbb{S}_n \cong \mathrm{Gal}$$

is an isomorphism. Then there is a Gal-equivariant equivalence

$$\mathrm{Gal}^+ \wedge E^{hK} \to E^{hK_0}.$$

Proof. This follows from Lemma 1.36. Define a map $E^{h\mathbb{S}_n} \wedge E^{hK} \to E^{hK_0}$ by the composition

$$E^{h\mathbb{S}_n} \wedge E^{hK} \to E^{hK_0} \wedge E^{hK_0} \to E^{hK_0} \tag{1.38}$$

where the first map is given by the inclusion and the last map is multiplication. We have

$$E_*(E^{h\mathbb{S}_n} \wedge E^{hK}) \cong \mathrm{map}(\mathbb{G}_n/\mathbb{S}_n, E_*) \otimes_{E_*} \mathrm{map}(\mathbb{G}_n/K, E_*)$$

$$\cong \mathrm{map}(\mathbb{G}_n/\mathbb{S}_n \times \mathbb{G}_n/K, E_*).$$

In E_*-homology, the map of (1.38) then becomes the map

$$\mathrm{map}(\mathbb{G}_n/\mathbb{S}_n, E_*) \otimes_{E_*} \mathrm{map}(\mathbb{G}_n/K, E_*) \cong \mathrm{map}(\mathbb{G}_n/\mathbb{S}_n \times \mathbb{G}_n/K, E_*)$$

$$\longrightarrow \mathrm{map}(\mathbb{G}_n/K_0, E_*)$$

induced by the maps on cosets

$$\mathbb{G}_n/K_0 \to \mathbb{G}_n/K_0 \times \mathbb{G}_n/K_0 \to \mathbb{G}_n/\mathbb{S}_n \times \mathbb{G}_n/K$$

where the first map is the diagonal and the second map is projection. Since $K/K_0 \cong \mathbb{G}_n/\mathbb{S}_n$, this map on cosets is an isomorphism; therefore, the map of (1.38) is an E_*-isomorphism. By Remark 1.17 this map is a weak equivalence. $\qquad \square$

REMARK 1.39. Combining Lemmas 1.32 and 1.37 yields an isomorphism of spectral sequences

$$
\begin{array}{ccc}
\mathbb{W} \otimes H^*(K, E_*) & \Longrightarrow & \mathbb{W} \otimes \pi_* E^{hK} \\
\cong \downarrow & & \downarrow \cong \\
H^*(K_0, E_*) & \Longrightarrow & \pi_* E^{hK_0}
\end{array}
$$

where the differentials on the top line are the $\mathbb{W}$-linear differentials extended from the spectral sequence for K.

REMARK 1.40. At $n = 2$ and $p = 2$, Lemma 1.37 applies to the case of $K = G_{48}$; then $K_0 = G_{24}$. This implies that any of the spectra

$$X(i,j) \overset{\mathrm{def}}{=} \Sigma^{24i} E^{hG_{48}} \vee \Sigma^{24j} E^{hG_{48}}$$

has the property that $E_* X(i,j) \cong E_* E^{hG_{24}}$. But $X(i,j) = \Sigma^{24i} E^{hG_{24}}$ if and only if $i \equiv j \bmod 8$. This means that in some of the arguments we give to prove our main result we will have to produce two homotopy classes rather than one. See Theorem 5.8.

2. The homotopy groups of homotopy fixed point spectra

Here we collect what we will need about the homotopy groups of E^{hF}, where F runs through the finite subgroups of $\mathbb{G}_2^1$ of Remark 1.13. We will be working entirely at $n = p = 2$ and using the formal group from the supersingular curve of (1.8). Much of what is needed is in the literature and we will do our best to give references. However, much of what is written is for

calculations over Hopf algebroids, which is not quite what we are doing, and the results need translation. In addition, many of the results as written include some variant of the phrase 'we neglect the *bo*-patterns'. We make this thought precise with the following *ad hoc* definition.

DEFINITION 2.1. Let $F \subseteq \mathbb{G}_2$ be any finite subgroup containing $C_2 = \{\pm 1\}$. Then we define the *bo-patterns* $L_1(\pi_* E^{hF})$ of $\pi_* E^{hF}$ to be the image of the map in homotopy

$$\pi_* E^{hF} \longrightarrow \pi_* L_{K(1)} E^{hF}.$$

We also define the *pure $K(2)$-classes* $M_2(\pi_* E^{hF})$ to be the kernel of the same map.

Thus we have a short exact sequence

$$0 \to M_2(\pi_* E^{hF}) \to \pi_* E^{hF} \to L_1(\pi_* E^{hF}) \to 0.$$

Note that the *bo*-patterns are defined as a quotient. In most cases, this sequence is not split as sequence of modules over the homotopy groups of spheres.

REMARK 2.2. The name *bo*-patterns is something of a misnomer, as KO-patterns would be more accurate. Here KO is 8-periodic 2-complete real K-theory. In all our examples we will have an isomorphism

$$R(F) \otimes_{\mathbb{Z}_2} KO_* \cong \pi_* L_{K(1)} E^{hF}$$

for some $\mathbb{Z}_2$-algebra $R(F)$ in degree zero. While $R(F) \otimes_{\mathbb{Z}_2} KO_*$ is 8-periodic, $L_1(\pi_* E^{hF})$ will typically have $8k$-periodicity for some $k > 1$. As a warning, we mention that this isomorphism is simply as rings; we are not claiming $L_{K(1)} E^{hF}$ is a KO-algebra.

REMARK 2.3. Here is more detail, to explain our thinking.
Let us write $S/2^n$ for the $\mathbb{Z}/2^n$-Moore spectrum. Then there is a weak equivalence

$$L_{K(1)} X \simeq \operatorname{holim} v_1^{-1}(X \wedge S/2^n)$$

and, if X is $K(2)$-local, a corresponding localized Adams–Novikov Spectral Sequence

$$\lim v_1^{-1} H^*(\mathbb{G}_2, (E_* X)/2^n) \Longrightarrow \pi_* L_{K(1)} X. \tag{2.4}$$

This spectral sequence does not obviously converge.
Now suppose $C_2 = \{\pm 1\} \subseteq F \subset G_{48}$. Then the spectral sequence (2.4) for $X = E^{hF}$ becomes

$$\lim v_1^{-1} H^*(F, (E_*)/2^n) \Longrightarrow \pi_* L_{K(1)} E^{hF}. \tag{2.5}$$

Using Strickland's formulas [**37**] it is possible to show that

$$\lim v_1^{-1} H^*(F, (E_*)/2^n) \cong \lim v_1^{-1} H^*(C_2, (E_*)/2^n)^{F/C_2}$$

and that

$$v_1^{-1} H^*(C_2, (E_*)/2^n) \cong \mathbb{W}((u_1))[u^{\pm 2}, \eta]$$

where $\mathbb{W}((u_1)) = \lim(\mathbb{W}/2^n)[[u_1^{\pm 1}]]$ and $\eta \in H^1(C_2, E_2)$ detects the class of the same name in $\pi_1 S^0$. (See (2.6).) From this it follows that the spectral sequence (2.5) is completely determined by the standard differential $d_3(v_1^2) = \epsilon \eta^3$, where $v_1^2 = u_1^2 u^{-2}$ and $\epsilon \in \mathbb{F}_4((u_1))^\times$ is a unit. We can conclude that the spectral sequence converges and

$$\pi_* L_{K(1)} E^{hF} \cong \mathbb{W}((u_1))^F \otimes_{\mathbb{Z}_2} KO_*$$

and in particular, that $\pi_* L_{K(1)} E^{hF}$ and $L_1(\pi_* E^{hF})$ are both concentrated in degrees congruent to 0, 1, 2, and 4 modulo 8. It then remains to analyze the pure $K(2)$-local classes.

Now, not much of what we just wrote is explicitly in print, and it would take quite a few pages to prove in detail. But we will put together what we can from the existing literature to cover the main points case-by-case below. See Propositions 2.7, 2.11, and 2.17.

2.1. The homotopy groups of E^{hC_2} and E^{hC_6}

The standard source here is Mahowald–Rezk [**28**], which uses the Hopf algebroid approach, but also uses the same elliptic curve we have chosen. So the translation is straightforward. Here is a summary.

The central $C_2 \subseteq \mathbb{G}_2$ acts trivially on E_0 and by multiplication by -1 on u; hence

$$H^*(C_2, E_*) \cong \mathbb{W}[[u_1]][u^{\pm 2}, \alpha]/(2\alpha) \tag{2.6}$$

where $\alpha \in H^1(C_2, E_2)$ is the image of the generator of $H^1(C_2, \mathbb{Z}\langle \mathrm{sgn} \rangle)$ under the map which sends the generator of the sign representation to u^{-1}. Since

$$v_1 = u_1 u^{-1} \in H^0(C_2, E_2/2)$$

and the class $u_1 \alpha \in H^1(C_2, E_2)$ is the image of v_1 under the integral Bockstein, the class $\eta \in \pi_1 S^0$ is detected by $u_1 \alpha$. We will also write $\eta = u_1 \alpha$.

PROPOSITION 2.7. *The class $b \overset{\mathrm{def}}{=} u_1^2 u^{-2}$ reduces to v_1^2 in $v_1^{-1} H^*(C_2, E_*/2)$. There is an isomorphism*

$$\mathbb{W}((u_1))[b^{\pm 1}, \eta]/(2\eta) \cong \lim v_1^{-1} H^*(C_2, E_*/2^n).$$

The standard differential $d_3(v_1^2) = \eta^3$ (see Lemma 2.21) forces a differential

$$d_3(u^{-2}) = \epsilon u_1 \alpha^3$$

where $\epsilon \in \mathbb{F}_2[[u_1]]^\times$. Using the Mahowald–Rezk transfer argument [**28**, Proposition 3.5] we have $\nu \in \pi_3 S^0$ is non-zero in $\pi_3 E^{hC_2}$ and detected by α^3; this in turn forces a differential

$$d_7(u^{-4}) = \alpha^7 = \alpha \nu^2.$$

The spectral sequence collapses at E_7 and we have the following result.

PROPOSITION 2.8. *The homotopy ring $\pi_* E^{hC_2}$ is periodic of period 16 with periodicity generator e_{16} detected by u^{-8}. The bo-patterns $L_1(E^{hC_2})$ are concentrated in degrees congruent to 0, 1, 2, and 4 modulo 8 and the group of pure $K(2)$-local classes $M_2(E^{hC_2})$ is generated by the classes*

$$\alpha^i e_{16}^k, \quad k \in \mathbb{Z}, \ i = 3, 4, 5, 6.$$

To get the homotopy of E^{hC_6}, with $C_6 = C_2 \times \mathbb{F}_4^\times$, we need to know the action of $\mathbb{F}_4^\times$. We can use Strickland's calculations [**37**] or interpret the Mahowald–Rezk results. Let $\omega \in \mathbb{F}_4^\times$ be a primitive cube root of unity. Then, $\omega_* u = \omega u$ and $\omega_* u_1 = \omega u_1$; it follows that $\omega_* \alpha = \omega^{-1} \alpha$. The next result can be deduced from these formulas and the fact that

$$\pi_* E^{hC_6} \cong (\pi_* E^{hC_2})^{\mathbb{F}_4^\times}.$$

PROPOSITION 2.9. *The homotopy ring $\pi_* E^{hC_6}$ is periodic of period 48 with periodicity generator e_{16}^3 detected by u^{-24}. The bo-patterns $L_1(E^{hC_6})$ are concentrated in degrees congruent to 0, 1, 2, and 4 modulo 8 and the group of pure $K(2)$-local classes $M_2(E^{hC_6})$ is generated by the classes*

$$e_{16}^{3k} \alpha^3 \quad e_{16}^{3k} \alpha^6 \quad e_{16}^{3k+1} \alpha^4 \quad e_{16}^{3k+2} \alpha^5$$

of degrees $48k + 3$, $48k + 6$, $48k + 20$, and $48k + 37$ respectively. Furthermore, the homotopy class $\nu \in \pi_3 S^0$ is detected by the class α^3 and the class $\overline{\kappa} \in \pi_{20} S^0$ is detected by $e_{16}\alpha^4$.

2.2. The homotopy groups of E^{hC_4}

The standard reference for this calculation is Behrens–Ormsby [**7**, §2.1], especially Theorem 2.1.3 and Perspective 2, after Remark 2.1.9. See also their Figure 2. Again they use Hopf algebroids. Here we hit another small problem: the supersingular elliptic curve they use is different from the one we have chosen, and thus they have a different action of C_4 on a different version of Morava E-theory. There are two possible solutions. One is to do the calculations over again, using Strickland's formulas. The other is to note that the two supersingular curves become isomorphic over the algebraically closed field $\overline{\mathbb{F}}_2$ and to use descent to make the calculations. Using either method we obtain the following result. Let $i \in C_4$ be a generator and let

$$z = u_1 + i_* u_1 \in H^0(C_4, E_0).$$

There are further cohomology classes

$$b_2 \in H^0(C_4, E_4) \quad \delta \in H^0(C_4, E_8)$$

and

$$\gamma \in H^1(C_4, E_6) \quad \xi \in H^2(C_4, E_8).$$

Let $\eta \in H^1(C_4, E_2)$ and $\nu \in H^1(C_4, E_4)$ be the images of the like-named classes from the BP-based Adams–Novikov Spectral Sequence (see Remark 1.31).

PROPOSITION 2.10.   *There is an isomorphism*

$$H^*(C_4, E_*) \cong \mathbb{W}[[z]][b_2, \delta^{\pm 1}, \eta, \nu, \gamma, \xi]/R$$

where R is the ideal of relations given by

$$2\eta = 2\gamma = 4\xi = 0$$

and

$$b_2^2 \equiv z^2 \delta \mod 2$$

and

$$\delta\eta^2 = b_2\xi = \gamma^2 \quad b_2\gamma = z\delta\eta$$

and

$$b_2\eta = z\gamma \quad \gamma\eta = z\xi$$

and the final relations involving ν:

$$\nu^2 = 2\xi \quad 2\nu = z\nu = \eta\nu = b_2\nu = \gamma\nu = 0$$

Proof. This can be obtained from Perspective 2 (after Remark 2.1.9) of [**7**] by a three-step process. First set $\tilde{\gamma} = \gamma$, $\tilde{j} = z - 2$, and $\beta = \delta^{-1}\xi$. Second, invert δ. Finally, complete at the maximal ideal of $H^0(C_4, E_0)$. ☐

This result is displayed in Figure 1, presented in the standard Adams format: the x-axis is $t - s$; the y-axis is s. In this chart, the square box $\square$ represents a copy of $\mathbb{W}[[z]]$, the circle $\circ$
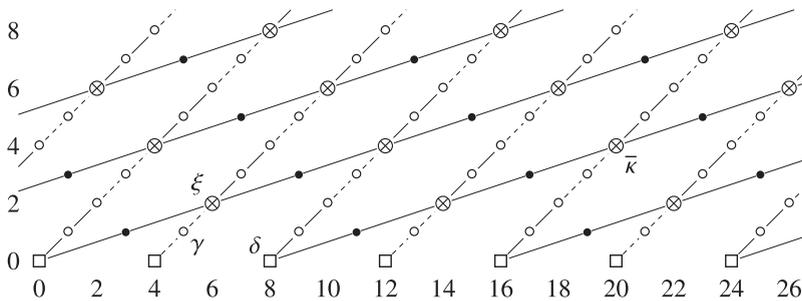
FIGURE 1. *The cohomology of C_4.*

a copy of $\mathbb{F}_4[[z]]$, and the crossed circle $\otimes$ a copy of $\mathbb{W}[[z]]/(4, 2z)$ generated by a class of the form $\xi^j \delta^i$. A solid dot is a copy of $\mathbb{F}_4$ annihilated by z; it is generated by a class of the form $\xi^j \nu$. The solid lines are multiplication by η or ν, as needed, and a dashed line indicates that $x\eta = zy$, where x and y are generators in the appropriate bidegree.

Note that $H^*(C_4, E_*)$ is 8-periodic with the periodicity class δ and that

$$\times \xi : H^s(C_4, E_*) \to H^{s+2}(C_4, E_{*+8})$$

is onto for $s \geqslant 0$ and an isomorphism for $s > 0$. In fact $\delta^{-1}\xi \in H^2(C_4, E_0)$ is, up to multiplication by a unit, the image of the periodicity class for the group cohomology of C_4 under the inclusion of trivial coefficients:

$$\mathbb{Z}/4 \cong H^4(C_4, \mathbb{Z}_2) \cong H^2(C_4, \mathbb{W}) \to H^s(C_4, E_0).$$

PROPOSITION 2.11. *Modulo 2 we have an equivalence $b_2 \equiv v_1^2$ and then an isomorphism*

$$\mathbb{W}((z))[b_2^{\pm 1}, \eta]/(2\eta) \cong \lim v_1^{-1} H^*(C_4, E_*/2^n).$$

The differentials and extensions in this spectral sequence go exactly as in Behrens and Ormsby [**7**, Theorem 2.3.12]. We end with the following result.

PROPOSITION 2.12. *The homotopy ring $\pi_* E^{hC_4}$ is periodic of period 32 with periodicity generator e_{32} detected by δ^4. The bo-patterns $L_1(E^{hC_4})$ are concentrated in degrees congruent to 0, 1, 2, and 4 modulo 8 and the group of pure $K(2)$-local classes $M_2(E^{hC_4})$ is generated by the classes $e_{32}^k x$ where x is from the following table*

Class	Degree	Order	E_2-name
a	1	2	$\delta^{-1}\xi\nu$
$a\eta$	2	2	$\delta^{-2}\xi^2\nu^2$
ν	3	4	ν
$a\nu$	4	2	$\delta^{-1}\xi\nu^2$
ν^2	6	2	ν^2
ϵ	8	2	$\delta^{-2}\xi^4$
$\nu^3 = \eta\epsilon$	9	2	$\overline{\kappa}^2\delta^{-5}\xi\nu$
κ	14	2	$\delta\nu^2$
b	19	4	$\delta^2\nu$
$\overline{\kappa}$	20	4	$\delta\xi^2$
$\eta\overline{\kappa}$	21	2	$\eta\xi^3$
$b\nu$	22	4	$\delta^2\nu^2$
c	27	2	$\delta^2\gamma\xi$
$c\eta$	28	2	$\delta^{-1}\xi^6$
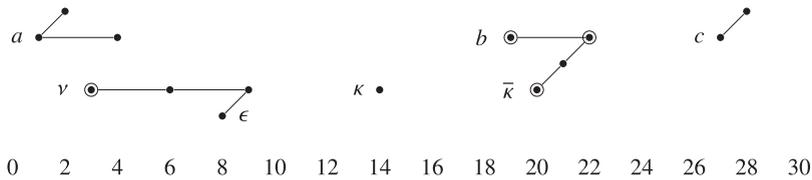
FIGURE 2. *The pure $K(2)$ classes in $\pi_* E^{hC_4}$.*

The pure $K(2)$ classes in $\pi_* E^{hC_4}$ are presented in Figure 2: the horizontal bar is multiplication by ν, the diagonal bar is multiplication by η. Note that $\overline{\kappa}\eta^2 = 2b\nu$. The horizontal scale is the degree of the element, but the vertical scale has no meaning. Many of the additive and multiplicative relations are given by exotic extensions in this spectral sequence and the meaning of the original Adams–Novikov filtration becomes attenuated as a result; see Behrens–Ormsby [**7**], especially figure 9, for details.

2.3. *The homotopy groups of $E^{hG_{24}}$ and $E^{hG_{48}}$*

Remarks 1.39 and 1.40 yield an isomorphism of spectral sequences

$$
\begin{array}{ccc}
\mathbb{W} \otimes_{\mathbb{Z}_2} H^*(G_{48}, E_*) & \Longrightarrow & \mathbb{W} \otimes \pi_* E^{hG_{48}} \\
\cong \downarrow & & \downarrow \cong \\
H^*(G_{24}, E_*) & \Longrightarrow & \pi_* E^{hG_{24}}.
\end{array}
$$

Therefore, we focus on the case of G_{48}. Here the standard sources are [**2**, **12**, **23**] although it requires some translation in each case to get the results we want.

The ring $H^0(G_{48}, E_*)$ is isomorphic to the ring of modular forms for supersingular elliptic curves at the prime 2. Then there are elements

$$
c_4 \in H^0(G_{48}, E_8) \quad c_6 \in H^0(G_{48}, E_{12}) \quad \Delta \in H^0(G_{48}, E_{24})
$$

obtained from the modular forms of the same name for our supersingular curve. Since this curve is smooth, Δ is invertible and the j-invariant of our curves $j = c_4^3/\Delta \in H^0(G_{48}, E_0)$ is defined. Then we get an isomorphism

$$
\mathbb{Z}_2[[j]][c_4, c_6, \Delta^{\pm 1}]/\left(c_4^3 - c_6^2 = (12)^3 \Delta, \Delta j = c_4^3\right) \cong H^0(G_{48}, E_*).
$$

Modulo 2 we get a slightly simpler answer:

$$
\mathbb{F}_2[[j]][v_1, \Delta^{\pm 1}]/\left(j\Delta = v_1^{12}\right) \cong H^0(G_{48}, E_*/2).
$$

Modulo 2 we have congruences

$$
c_4 \equiv v_1^4 \quad \text{and} \quad c_6 \equiv v_1^6. \tag{2.13}
$$

To describe the higher cohomology, we make a table of multiplicative generators. For each x, the bidegree of x is (s, t) if $x \in H^s(G_{48}, E_t)$. All but μ detect the elements of the same name in $\pi_* S^0$. Furthermore, all elements but $\overline{\kappa}$ are in the image of the map (see Remark 1.31)

$$
\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*) \to H^*(\mathbb{G}_2, E_*) \to H^*(G_{48}, E_*).
$$

Hence we also give the name (the 'MRW' is for Miller–Ravenel–Wilson) of a preimage. The Greek letter notation is that of [**30**].
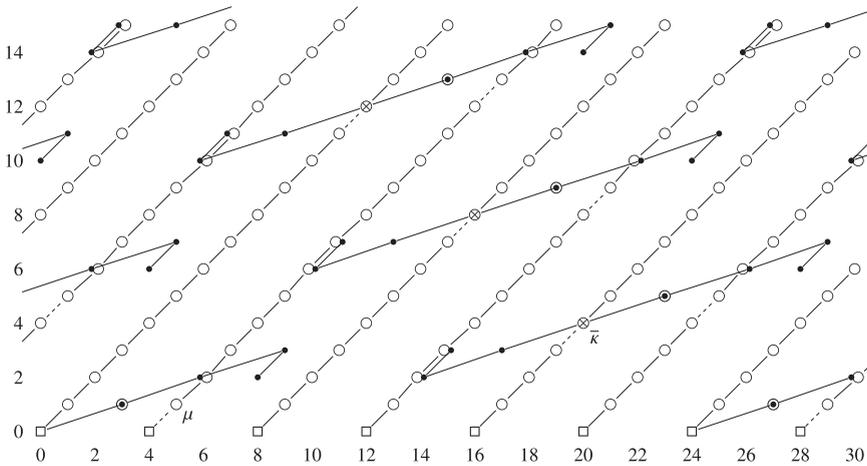
FIGURE 3. *The cohomology of G_{48}.*

Class	Bidegree	Order	MRW
η	(1,2)	2	α_1
ν	(1,4)	4	$\alpha_{2/2}$
μ	(1,6)	2	α_3
ϵ	(2,10)	2	β_2
κ	(2,16)	2	β_3
$\overline{\kappa}$	(4,24)	8	$-$

The class $\overline{\kappa} \in \pi_{20}S^0$ is detected by the image of β_4 in $H^2(\mathbb{G}_2, E_*)$. The class μ has a special role which we discuss in Lemma 2.21, but we would like to note right away that

$$v_1^2\eta \equiv \mu \quad \text{modulo } 2. \tag{2.14}$$

The following result is actually much easier to visualize than to write down. See the Figure 3.

THEOREM 2.15. *There is an isomorphism*

$$H^0(G_{48}, E_*)[\eta, \nu, \mu, \epsilon, \kappa, \overline{\kappa}]/R \cong H^*(G_{48}, E_*)$$

where R is the ideal defined by

(1) *the order of the elements of positive cohomological degree:*

$$2\eta = 4\nu = 2\mu = 2\epsilon = 2\kappa = 8\overline{\kappa} = 0;$$

(2) *the relations for ν:*

$$\eta\nu = 2\nu^2 = \nu^4 = \mu\nu = 0;$$

(3) *the relations for ϵ:*

$$\eta\epsilon = \nu^3, \quad \nu\epsilon = \epsilon^2 = \mu\epsilon = 0;$$

(4) *the relations for κ:*

$$\nu^2\kappa = 4\overline{\kappa}, \quad \eta^2\kappa = \epsilon\kappa = \kappa^2 = \mu\kappa = 0;$$

(5) *the elements annihilated by modular forms*

$$c_4\nu = c_6\nu = c_4\epsilon = c_6\epsilon = c_4\kappa = c_6\kappa = 0;$$
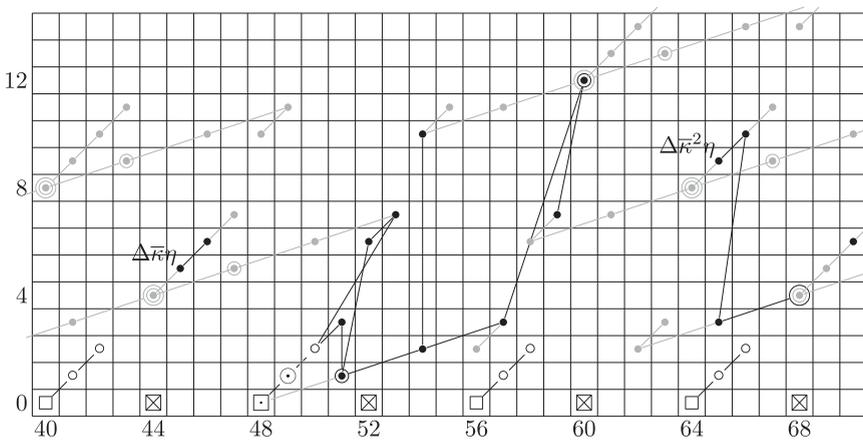
FIGURE 4. *The homotopy groups* $\pi_i E^{hG_{48}}$ *for* $40 \leqslant i \leqslant 70$.

(6) *the relations between* $\overline{\kappa}$ *and modular forms;*

$$c_4 \overline{\kappa} = \Delta \eta^4, \quad c_6 \overline{\kappa} = \Delta \eta^3 \mu;$$

(7) *and the relations indicated by the congruences of* (2.13) *and* (2.14):

$$\mu^2 = c_4 \eta^2 \quad c_4 \mu = c_6 \eta \quad c_6 \mu = c_4^2 \eta.$$

This result is presented graphically in Figure 3. We present it as the E_2 page of the Adams–Novikov Spectral Sequence. The cohomology is 24-periodic on Δ, and the spectral sequence fills the entire upper-half plane. In Figure 3, the square box $\square$ represents a copy of $\mathbb{Z}_2[[j]]$, the circle $\circ$ a copy of $\mathbb{F}_2[[j]]$, and the crossed circle $\otimes$ a copy of $\mathbb{Z}_2[[j]]/(8, 2j)$ generated by a class of the form $\Delta^i \overline{\kappa}^j$. The solid bullet represents a class of order 2 annihilated by j and the doubled bullet a class of order 4 annihilated by j; these last classes are always of the form $\Delta^i \overline{\kappa}^j \nu$. The solid lines are multiplication by η or ν, as needed, and a dashed line indicates that $x\eta = jy$, where x and y are generators in the appropriate bidegree.

REMARK 2.16. (1) Many of the later relations can be rephrased as relations for multiplication by $j = c_4^3/\Delta$. For example Theorem 2.15 (4) implies

$$j\nu = j\epsilon = j\kappa = 0$$

and (5) implies

$$j\overline{\kappa} = c_4^2 \eta^4$$

and (6) implies

$$j\mu = c_4^2 c_6 \Delta^{-1} \eta.$$

These last two equations explain the dashed lines in Figure 4.

(2) Multiplication by $\overline{\kappa} : H^s(G_{48}, E_t) \to H^{s+4}(G_{48}, E_{t+24})$ is surjective and an isomorphism if $s > 0$. In fact, up to a unit, $\Delta^{-1}\overline{\kappa} \in H^4(G_{48}, E_0)$ is the image of the periodicity class in group cohomology for Q_8 under the inclusion of trivial coefficients:

$$\mathbb{Z}/8 \cong H^4(Q_8, \mathbb{W})^{G_{48}/Q_8} \cong H^4(G_{48}, \mathbb{W}) \to H^4(G_{48}, E_0).$$

The congruence (2.13) and the relations of Theorem 2.15 now give the following result. Note that the class c_4 becomes invertible in $\lim v_1^{-1} H^*(G_{48}, E_*/2^n)$ and we may define $b_2 = c_6/c_4$.

(*Warning*: This class b_2 is related to, but not quite the same, as the class b_2 of Proposition 2.10. Both uses of b_2 appear in the literature.)

PROPOSITION 2.17. *The class b_2 reduces to v_1^2 in $v_1^{-1}H^*(G_{48}, E_*/2)$. There are isomorphisms*

$$\mathbb{Z}_2((j))[b_2^{\pm 1}, \eta]/(2\eta) \cong \lim v_1^{-1}H^*(G_{48}, E_*/2^n).$$

and

$$\mathbb{F}_2((j))[v_1^{\pm 1}, \eta] \cong v_1^{-1}H^*(G_{48}, E_*/2).$$

Under the reduction map $H^*(G_{48}, E_*) \to H^*(G_{48}, E_*/2)$ we have

$$c_4 \mapsto v_1^4$$

$$c_6 \mapsto v_1^6$$

$$c_4\overline{\kappa} \mapsto v_1^4\overline{\kappa} = \Delta\eta^4$$

$$\mu \mapsto v_1^2\eta.$$

Under the localization map $H^*(G_{48}, E_*) \to v_1^{-1}H^*(G_{48}, E_*/2)$ we have

$$\Delta \mapsto v_1^{12}/j$$

$$\overline{\kappa} \mapsto v_1^8\eta^4/j$$

and that ν, ϵ, and κ map to zero.

We have the following; see [**2**, **12**, **23**].

PROPOSITION 2.18. *The homotopy ring $\pi_* E^{hG_{48}}$ is periodic of period 192 with periodicity generator detected by Δ^8. The bo-patterns $L_1(E^{hG_{48}})$ are concentrated in degrees congruent to 0, 1, 2, and 4 modulo 8.*

REMARK 2.19. We will not try to enumerate the pure $K(2)$-classes of $M_2(E^{hG_{48}})$; this information is known (by the same references as for Proposition 2.18), but we will not need that information in its entirety and it is rather complicated to write down. What we will need can be read off of Figure 4, which is adapted from the charts created by Tilman Bauer [**2**, Section 8].

This chart shows a section of the E_∞-page of the Adams–Novikov Spectral Sequence

$$E_2^{s,t} = H^s(G_{48}, \pi_t E) \Longrightarrow \pi_{t-s}E^{hG_{48}}.$$

It is in the standard Adams bigrading $(t - s, s)$.

Some additive and multiplicative extensions are displayed as well. The non-zero permanent cycles are in black; some other elements, mostly built from patterns around elements of the form $\Delta^j\overline{\kappa}^i$, have been left in gray for orientation, even though they do not last to the E_∞-page. Bullets with circles are elements of order 4; bullets with two circles are elements of order 8. Vertical lines are extensions by multiplication by 2, lines raising homotopy degree by 1 are η-extensions, lines raising homotopy degree by 3 are ν-extensions.

The lines $0 \leqslant s \leqslant 2$ display the *bo*-patterns; the adorned boxes and circles all represent ideals of either $\mathbb{Z}_2[[j]]$ or $\mathbb{F}_2[[j]]$:

$$\square \cong \mathbb{Z}_2[[j]]$$

$$\boxtimes \cong (2) \subseteq \mathbb{Z}_2[[j]]$$

$$\square \cong (4, j) \subseteq \mathbb{Z}_2[[j]]$$

$$\circ \cong \mathbb{F}_2[[j]]$$

$$\odot \cong (j) \subseteq \mathbb{F}_2[[j]].$$

Elements not falling into one of these patterns are annihilated by j. The η-extension from $(t - s, s) = (65, 3)$ entry is ambiguous. We mark it as non-zero because we may choose, as Bauer does, the two generators of the group of pure $K(2)$-classes in $\pi_{65}E^{hG_{48}}$ to be

$$e[45, 5]\overline{\kappa} \quad \text{and} \quad e[51, 1]\kappa$$

where $e[45, 5] \in \pi_{45}E^{hG_{48}}$ and $e[51, 1] \in \pi_{51}E^{hG_{48}}$ are generators detected by $\Delta\overline{\kappa}\eta$ and $\Delta^2\nu$, respectively. The class $e[51, 1]\kappa$ is detected on the $s = 3$ line by $\Delta^2\kappa\nu$ and $e[51, 1]\kappa\eta \neq 0$.

We now record, from Figure 4, some data about our crucial homotopy classes.

LEMMA 2.20. *There is an isomorphism*

$$\mathbb{Z}/2 \cong \pi_{45}E^{hG_{48}}.$$

The generator is detected by the class

$$\Delta\overline{\kappa}\eta \in H^5(G_{48}, E_{50}).$$

The class $\Delta\overline{\kappa}^2\eta^2 \in H^{10}(G_{48}, E_{76})$ *is a non-zero permanent cycle detecting a generator of the subgroup* $\pi_{66}E^{hG_{48}}$ *of the pure $K(2)$-classes of that degree.*

We close with some remarks on the role of μ in the d_3 differentials.

LEMMA 2.21. *Let $\mu \in H^1(\mathbb{G}_2, E_6)$ be the image of the class*

$$\alpha_3 \in \mathrm{Ext}^1_{BP_*BP}(\Sigma^6 BP_*, BP_*).$$

Then in any of the Adams–Novikov Spectral Sequences

$$H^s(F, E_t) \Longrightarrow \pi_{t-s}E^{hF}$$

and for any $x \in H^*(F, E_*)$ we have

$$d_3(x\mu) = d_3(x)\mu + x\eta^4.$$

In the spectral sequences

$$H^s(F, E_t/2) \Longrightarrow \pi_{t-s}(E^{hF} \wedge S/2) \tag{2.22}$$

we have

$$d_3(v_1^2 x) = d_3(x)v_1^2 + x\eta^3 + y$$

where $y\eta = 0$. Finally, in the spectral sequence (2.22) *we have*

$$d_3(v_1^4 x) = v_1^4 d_3(x).$$

Proof. In the Adams–Novikov Spectral Sequence

$$\mathrm{Ext}^s_{BP_*BP}(\Sigma^t BP_*, BP_*) \Longrightarrow \mathbb{Z}_{(2)} \otimes \pi_{t-s}S^0.$$

we have $d_3(\alpha_3) = \eta^4$. (In fact, by [**30**, Corollary 4.23], $\eta^4 \neq 0$ at E_2 and $E_2^{1,6} \cong \mathbb{Z}/2$ generated by α_3. The differential is then forced.) Since the fixed point spectral sequence is a module over

this standard Adams–Novikov Spectral Sequence, the first formula follows. The second formula follows because $v_1^2 \eta = \alpha_3$ in

$$\operatorname{Ext}^s_{BP_*BP}(\Sigma^t BP_*, BP_*/2).$$

The third formula follows from the fact that $S/2$ has a v_1^4-self map. □

3. Algebraic and topological resolutions

In this section we review the centralizer resolution constructed by Hans-Werner Henn [20, §3.4] and then begin the construction of the topological duality resolution. The details of the algebraic duality resolution can be found in [3]. The two resolutions have complementary features. While we will not try to make this thought completely precise, the duality resolution reflects, in an essential way, the fact that the group $\mathbb{S}_2^1$ is a virtual Poincaré duality group of dimension 3. The centralizer resolution on the other hand, is much closer to being an Adams–Novikov tower as there is an underlying relative homological algebra in the spirit of Miller [29]. See Remark 3.24.

3.1. The centralizer resolution

Henn's centralizer resolutions grew out of his paper [19] which used the centralizers of elementary abelian subgroups of S_n to detect elements in the cohomology of S_n. At the prime 2, this approach needs a slight modification, as the maximal finite 2-group in S_2 is Q_8, which is not elementary abelian.

REMARK 3.1. In (1.10) we defined $G_{24} \subseteq \mathbb{S}_2^1$ as the image of a group of automorphisms of a supersingular elliptic curve. The group $\mathbb{S}_2^1$ fits into a short exact sequence

$$1 \longrightarrow \mathbb{S}_2^1 \longrightarrow \mathbb{S}_2 \overset{N}{\longrightarrow} \mathbb{Z}_2 \longrightarrow 1$$

where N is the reduced determinant map of (1.7). Let

$$\pi = 1 + 2\omega$$

be an element of $\mathbb{S}_2$, where $\omega \in \mathbb{W}^\times$ is a cube root of unity. Note that π is not an element of $\mathbb{S}_2^1$ because $N(\pi) = 3$. Then we define $G_{24}' := \pi G_{24} \pi^{-1} \subseteq \mathbb{S}_2^1$. This is a subgroup isomorphic to G_{24}, but not conjugate to G_{24} in $\mathbb{S}_2^1$.

Note that multiplication by π defines an equivalence $E^{hG_{24}} \simeq E^{hG_{24}'}$. For complete details on this and more, see [3].

We now have the following result from [20, §3.4]. This is the *algebraic centralizer resolution*.

THEOREM 3.2. *There is an exact sequence of continuous $\mathbb{S}_2^1$-modules*

$$0 \to \mathbb{Z}_2[[\mathbb{S}_2^1/C_6]] \to \mathbb{Z}_2[[\mathbb{S}_2^1/C_2]] \to \mathbb{Z}_2[[\mathbb{S}_2^1/C_6]] \oplus \mathbb{Z}_2[[\mathbb{S}_2^1/C_4]]$$

$$\to \mathbb{Z}_2[[\mathbb{S}_2^1/G_{24}]] \oplus \mathbb{Z}_2[[\mathbb{S}_2^1/G_{24}']] \overset{\epsilon}{\longrightarrow} \mathbb{Z}_2 \to 0. \tag{3.3}$$

The map ϵ is the sum of the augmentation maps.

We will call this a resolution, even though the terms are not projective as $\mathbb{Z}_2[[\mathbb{S}_2^1]]$- modules. It is an $\mathcal{F}$-projective resolution, an idea we explore in Remark 3.24.

REMARK 3.4. Suppose we write

$$P_0 = \mathbb{Z}_2[[\mathbb{S}_2^1/G_{24}]] \times \mathbb{Z}_2[[\mathbb{S}_2^1/G_{24}']]$$

$$P_1 = \mathbb{Z}_2[[\mathbb{S}_2^1/C_6]] \times \mathbb{Z}_2[[\mathbb{S}_2^1/C_4]]$$

$$P_2 = \mathbb{Z}_2[[\mathbb{S}_2^1/C_2]]$$

$$P_3 = \mathbb{Z}_2[[\mathbb{S}_2^1/C_6]].$$

Then for any profinite $\mathbb{S}_2^1$-module M, we get a spectral sequence

$$E_1^{p,q} \cong \mathrm{Ext}_{\mathbb{Z}_2[[\mathbb{S}_2^1]]}^q(P_p, M) \Longrightarrow H^{p+q}(\mathbb{S}_2^1, M).$$

The E_1-terms can all be written as group cohomology groups; for example

$$E_1^{0,q} \cong H^q(G_{24}, M) \times H^q(G_{24}', M).$$

We will call this the *algebraic centralizer resolution spectral sequence*. In many applications, the distinction between the groups G_{24} and G_{24}' disappears. For example, if M is a $\mathbb{G}_2$-module (such as $E_n X$ for some spectrum X) then multiplication by π induces an isomorphism $H^q(G_{24}, M) \cong H^q(G_{24}', M)$.

REMARK 3.5. We can induce the resolution (3.3) of $\mathbb{S}_2^1$-modules up to a resolution of $\mathbb{G}_2$-modules and obtain an exact sequence

$$0 \to \mathbb{Z}_2[[\mathbb{G}_2/C_6]] \to \mathbb{Z}_2[[\mathbb{G}_2/C_2]] \to \mathbb{Z}_2[[\mathbb{G}_2/C_6]] \oplus \mathbb{Z}_2[[\mathbb{G}_2/C_4]]$$

$$\to \mathbb{Z}_2[[\mathbb{G}_2/G_{24}]] \oplus \mathbb{Z}_2[[\mathbb{G}_2/G_{24}]] \to \mathbb{Z}_2[[\mathbb{G}_2/\mathbb{S}_2^1]] \to 0.$$

Since G_{24}' is conjugate to G_{24} in $\mathbb{G}_2$, we have $\mathbb{Z}_2[[\mathbb{G}_2/G_{24}]] \cong \mathbb{Z}_2[[\mathbb{G}_2/G_{24}']]$ as $\mathbb{G}_2$-modules and we have made that substitution. If F is any closed subgroup of $\mathbb{G}_2$, then the equivalence of (1.19) gives us an isomorphism of twisted $\mathbb{G}_2$-modules

$$\mathrm{Hom}_{\mathbb{Z}_2[[\mathbb{G}_2]]}(\mathbb{Z}_2[[\mathbb{G}_2/F]], E_*) \cong E_* E^{hF}.$$

Combining these observations, we get an exact sequence of twisted $\mathbb{G}_2$-modules

$$0 \to E_* E^{h\mathbb{S}_2^1} \to \begin{matrix} E_* E^{hG_{24}} \\ \times \\ E_* E^{hG_{24}} \end{matrix} \to \begin{matrix} E_* E^{hC_6} \\ \times \\ E_* E^{hC_4} \end{matrix} \to E_* E^{hC_2} \to E_* E^{hC_6} \to 0. \qquad (3.6)$$

We then have the following result; this is the *topological centralizer resolution* of [20, Theorem 12].

THEOREM 3.7. *The algebraic resolution of* 3.6 *can be realized by a sequence of spectra*

$$E^{h\mathbb{S}_2^1} \xrightarrow{p} \begin{matrix} E^{hG_{24}} \\ \times \\ E^{hG_{24}} \end{matrix} \to \begin{matrix} E^{hC_6} \\ \times \\ E^{hC_4} \end{matrix} \to E^{hC_2} \to E^{hC_6}$$

All compositions and all Toda brackets are zero modulo indeterminacy.

REMARK 3.8. The vanishing of the Toda brackets in this result has several implications. To explain these and for future reference we write, echoing the notation of Remark 3.4:

$$F_0 = E^{hG_{24}} \times E^{hG_{24}}$$

$$F_1 = E^{hC_6} \times E^{hC_4}$$

$$F_2 = E^{hC_2}$$

$$F_3 = E^{hC_6}. \qquad (3.9)$$

Then the resolution of Theorem 3.7 can be refined to a tower of fibrations under $E^{h\mathbb{S}_2^1}$:

$$
\begin{array}{ccccccc}
E^{h\mathbb{S}_2^1} & \longrightarrow & Y_2 & \longrightarrow & Y_1 & \longrightarrow & E^{hG_{24}} \times E^{hG_{24}} = F_0 \\
\uparrow & & \uparrow & & \uparrow & & \\
\Sigma^{-3} F_3 & & \Sigma^{-2} F_2 & & \Sigma^{-1} F_1 & &
\end{array}
\tag{3.10}
$$

Alternatively we could refine the resolution into a tower over $E^{h\mathbb{S}_2^1}$. Let us write

$$
\begin{array}{ccc}
X & \dashleftarrow & Z \\
& \searrow \quad \nearrow & \\
& Y &
\end{array}
$$

for a cofiber sequence (that is, a triangle) $X \to Y \to Z \to \Sigma X$. Then we have a diagram of cofiber sequences

$$
\begin{array}{ccccccc}
E^{h\mathbb{S}_2^1} & \overset{}{\dashleftarrow} & C_1 & \dashleftarrow & C_2 & \dashleftarrow & C_3 \\
& {}_{p}\searrow \quad \nearrow & & \searrow \quad \nearrow & & \searrow \quad {}^{\cong}\nearrow & \\
& F_0 & & F_1 & & F_2 & & F_3
\end{array}
\tag{3.11}
$$

where each of the compositions $F_{i-1} \to C_i \to F_i$ is the map $F_{i-1} \to F_i$ in the resolution.

The towers (3.10) and (3.11) determine each other. This is because there is a diagram with rows and columns cofibration sequences

$$
\begin{array}{ccccc}
\Sigma^{-(s+1)} C_{s+1} & \longrightarrow & E^{h\mathbb{S}_2^1} & \longrightarrow & Y_s \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma^{-s} C_s & \longrightarrow & E^{h\mathbb{S}_2^1} & \longrightarrow & Y_{s-1} \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma^{-s} F_s & \longrightarrow & * & \longrightarrow & \Sigma^{-s+1} F_s
\end{array}
\tag{3.12}
$$

Using the tower over $E^{h\mathbb{S}_2^1}$ of (3.11) we get a number of spectral sequences; for example, if Y is any spectrum, we get a spectral sequence for the function spectrum $F(Y, E^{h\mathbb{S}_2^1})$

$$
E_1^{s,t} = \pi_t F(Y, F_s) \Longrightarrow \pi_{t-s} F(Y, E^{h\mathbb{S}_2^1}).
\tag{3.13}
$$

Up to isomorphism, this spectral sequence can be obtained from the tower of (3.10); this follows from (3.12).

REMARK 3.14. It is direct to calculate $E_* C_s$ and $E_* Y_s$ for the layers of the two towers. If we define $K_s \subseteq E_* F_s$ to be the image of $E_* F_{s-1} \to E_* F_s$, then $E_* C_s \cong K_s$ and, more, if we apply E_* to (3.11) we get a collection of short exact sequences:

$$
\begin{array}{ccccccc}
E_* E^{h\mathbb{S}_2^1} & \overset{0}{\dashleftarrow} & E_* C_1 & \overset{0}{\dashleftarrow} & E_* C_2 & \overset{0}{\dashleftarrow} & E_* C_3 \\
& \searrow \quad \nearrow & & \searrow \quad \nearrow & & \searrow \quad {}^{\cong}\nearrow & \\
& E_* F_0 & & E_* F_1 & & E_* F_2 & & E_* F_3.
\end{array}
$$

Note that this implies that each of the dotted arrows of (3.11) has Adams–Novikov filtration one. Finally, the cofibration sequence $E^{h\mathbb{S}_2^1} \to Y_s \to \Sigma^{-s} C_{s+1}$ induces a short exact sequence

$$
0 \to E_* E^{h\mathbb{S}_2^1} \to E_* Y_s \to \Sigma^{-s} K_{s+1} \to 0.
$$

3.2. *The duality resolution, first steps*

We have the *algebraic duality resolution* from [**3**]. The groups G_{24} and G_{24}' are defined in Remark 3.1.

THEOREM 3.15.   *There is an exact sequence of continuous $\mathbb{S}_2^1$-modules*

$$0 \to \mathbb{Z}_2[[\mathbb{S}_2^1/G_{24}']] \to \mathbb{Z}_2[[\mathbb{S}_2^1/C_6]] \to \mathbb{Z}_2[[\mathbb{S}_2^1/C_6]] \to \mathbb{Z}_2[[\mathbb{S}_2^1/G_{24}]] \xrightarrow{\epsilon} \mathbb{Z}_2 \to 0 \qquad (3.16)$$

where ϵ is the augmentation.

REMARK 3.17.   As in Remark 3.4 we get a spectral sequence. Suppose we write

$$Q_0 = \mathbb{Z}_2[[\mathbb{S}_2^1/G_{24}]]$$

$$Q_1 = Q_2 = \mathbb{Z}_2[[\mathbb{S}_2^1/C_6]]$$

$$Q_3 = \mathbb{Z}_2[[\mathbb{S}_2^1/G_{24}']]$$

Then for any profinite $\mathbb{S}_2^1$-module M, such as $E_*X = (E_2)_*X$ for some finite spectrum X, we get a spectral sequence

$$E_1^{p,q} \cong \mathrm{Ext}^q_{\mathbb{Z}_2[[\mathbb{S}_2^1]]}(Q_p, M) \Longrightarrow H^{p+q}(\mathbb{S}_2^1, M),$$

which we will call the *algebraic duality resolution spectral sequence*.

As in Remark 3.5 and (3.6) we immediately have the following consequence.

COROLLARY 3.18.   *There is an exact sequence of twisted $\mathbb{G}_2$-modules*

$$0 \to E_*E^{h\mathbb{S}_2^1} \to E_*E^{hG_{24}} \to E_*E^{hC_6} \to E_*E^{hC_6} \to E_*E^{hG_{24}} \to 0.$$

The first of these maps is induced by the map on homotopy fixed point spectra $E^{h\mathbb{S}_2^1} \to E^{hG_{24}}$ induced by the subgroup inclusion $G_{24} \subseteq \mathbb{S}_2^1$.

We would now like to prove the following result, paralleling Theorem 3.7; it also appears in [**20**]. The main work of the next two sections and, indeed, the main theorem of this paper is to identify X.

PROPOSITION 3.19.   *The algebraic resolution of 3.18 can be realized by a sequence of spectra*

$$E^{h\mathbb{S}_2^1} \xrightarrow{q} E^{hG_{24}} \to E^{hC_6} \to E^{hC_6} \to X$$

with $E_*X \cong E_*E^{hG_{24}}$ as a twisted $\mathbb{G}_2$-module. All compositions and all Toda brackets are zero modulo indeterminacy.

REMARK 3.20.   A consequence of the last sentence of this result is that this resolution can be refined to a tower of fibrations under $E^{h\mathbb{S}_2^1}$

$$(3.21)$$

or to a tower over $E^{h\mathbb{S}_2^1}$

$$(3.22)$$

As in Remark 3.14 the dotted arrows have Adams–Novikov filtration 1. Examining this last diagram, we see that X can be *defined* as the cofiber of $D_2 \to E^{hC_6}$ and it will follow, as in Remark 3.14, that $E_*X \cong E_*E^{hG_{24}}$. Thus Proposition 3.19 is equivalent to the following result. See also [**8**, **20**].

LEMMA 3.23. *The truncated resolution*

$$0 \to E_*E^{h\mathbb{S}_2^1} \to E_*E^{hG_{24}} \to E_*E^{hC_6} \to E_*E^{hC_6}$$

can be realized by a sequence of spectra

$$E^{h\mathbb{S}_2^1} \longrightarrow E^{hG_{24}} \to E^{hC_6} \to E^{hC_6}$$

such that all compositions are zero and the one Toda bracket is zero modulo indeterminacy.

Proof. The map $E^{h\mathbb{S}_2^1} \xrightarrow{q} E^{hG_{24}}$ is the inclusion map on homotopy fixed point spectra induced by the subgroup inclusion $G_{24} \subseteq \mathbb{S}_2^1$. To realize the other maps and to show that the compositions are zero, we prove that the Hurewicz map

$$\pi_0 F(E^{hF}, E^{hC_6}) \longrightarrow \mathrm{Hom}_{Mor}(E_0 E^{hF}, E_0 E^{hC_6})$$

to the category of Morava modules is an isomorphism for $F = \mathbb{S}_2^1$ or $F = G_{24}$. To see this, first note that there is an isomorphism

$$\pi_0 F(E^{hF}, E^{hC_6}) = \pi_0 E[[\mathbb{G}_2/F]]^{hC_6} \cong H^0(C_6, E_0[[\mathbb{G}_2/F]]).$$

This follows from (1.25) and the fact that $\pi_0 E^{hK} = H^0(K, E_0)$ whenever $K \subseteq C_6$; see §2.1. Then we can finish the argument by using (1.16) and (1.21) to show

$$H^0(C_6, E^0[[\mathbb{G}_2/F]]) \cong \mathrm{Hom}_{E^0[[\mathbb{G}_2]]}(E^0[[\mathbb{G}_2/C_6]], E^0[[\mathbb{G}_2/F]])$$

$$\cong \mathrm{Hom}_{Mor}(E_0 E^{hF}, E_0 E^{hC_6}),$$

where $\mathrm{Hom}_{E^0[[\mathbb{G}_2]]}(-,-)$ is the group of continuous homomorphisms of profinite $E^0[[\mathbb{G}_2]]$-modules and $\mathrm{Hom}_{Mor}(-,-)$ is the group of homomorphisms of Morava modules.

This leaves the Toda bracket. To see that it is zero modulo indeterminacy we show that the indeterminacy is the entire group. To be specific, we show that the inclusion $E^{h\mathbb{S}_2^1} \to E^{hG_{24}}$ induces a surjection

$$\pi_* F(E^{hG_{24}}, E^{hC_6}) \longrightarrow \pi_* F(E^{h\mathbb{S}_2^1}, E^{hC_6}).$$

Using (1.21), we can rewrite this map as

$$\pi_* E[[\mathbb{G}_2/G_{24}]]^{hC_6} \longrightarrow \pi_* E[[\mathbb{G}_2/\mathbb{S}_2^1]]^{hC_6}.$$

Again using (1.21), the inclusion $E^{h\mathbb{S}_2^1} \to E^{hG_{24}}$ induces a map of C_6-spectra

$$E[[\mathbb{G}_2/G_{24}]] \simeq F(E^{hG_{24}}, E) \to F(E^{h\mathbb{S}_2^1}, E) \simeq E[[\mathbb{G}_2/\mathbb{S}_2^1]].$$

It is thus sufficient to show that the quotient map on cosets

$$\mathbb{G}_2/G_{24} \longrightarrow \mathbb{G}_2/\mathbb{S}_2^1$$

has a C_6-splitting. Since $\mathbb{G}_2 \cong \mathbb{S}_2 \rtimes \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$, every coset in $\mathbb{G}_2/\mathbb{S}_2^1$ has a representative of form $\pi^i \phi^\epsilon \mathbb{S}_2^1$ where π is as in Remark 3.1, $\phi \in \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$ is the Frobenius, $i \in \mathbb{Z}_2$, and $\epsilon = 0$ or 1. The splitting is then given by

$$\pi^i \phi^\epsilon \mathbb{S}_2^1 \longmapsto \pi^i \phi^\epsilon G_{24}. \qquad\qquad \square$$

3.3. *Comparing the two resolutions*

There is a map from the centralizer tower to the duality tower; we will not prove that here. In the end we will only need a small part of the data given by such a map, and what we need is in Remark 3.27.

REMARK 3.24.   The underlying algebra for the centralizer resolutions fits well the relative homological algebra usually deployed in building an Adams–Novikov tower; this goes back to Miller in [**29**], among other sources.

Here is more detail. Let $\mathcal{F}$ be the set of conjugacy classes of finite subgroups of $\mathbb{S}_2^1$. A continuous $\mathbb{S}_2^1$-module P is $\mathcal{F}$-*projective* if the natural map

$$\bigoplus_{F \in \mathcal{F}} \mathbb{Z}_2[[\mathbb{S}_2^1]] \otimes_{\mathbb{Z}_2[[F]]} P \longrightarrow P$$

is split surjective, where F runs over representatives for the classes in $\mathcal{F}$. The class of $\mathcal{F}$-projectives is the smallest class of continuous $\mathbb{S}_2^1$-modules closed under finite sums, retracts, and containing all induced modules $\mathbb{Z}_2[[\mathbb{S}_2^1]] \otimes_{\mathbb{Z}_2[[F]]} M$, where M is a continuous F-module.

The class of $\mathcal{F}$-projectives defines a class of $\mathcal{F}$-exact morphisms, there are enough $\mathcal{F}$-projectives, there are $\mathcal{F}$-projective resolutions, and so on. All of this and more is discussed in [**20**, § 3.5].

Comparing the two towers now begins with the following result, see Remark (d) after [**20**, Proposition 17].

PROPOSITION 3.25.   *The centralizer resolution* (3.3) *is an $\mathcal{F}$-projective resolution of the trivial $\mathbb{S}_2^1$-module $\mathbb{Z}_2$.*

Thus if we write $P_\bullet \to \mathbb{Z}_2$ for the centralizer resolution (3.3) and $Q_\bullet \to \mathbb{Z}_2$ for the duality resolution (3.16), then standard homological algebra gives us a map of resolutions, unique up to chain homotopy

$$
\begin{array}{ccc}
Q_\bullet & \longrightarrow & \mathbb{Z}_2 \\
{\scriptstyle g_\bullet}\downarrow & & \downarrow{\scriptstyle =} \\
P_\bullet & \longrightarrow & \mathbb{Z}_2.
\end{array}
$$

The map $g_0 : Q_0 \to P_0$ can be chosen to be the inclusion onto the first factor

$$Q_0 = \mathbb{Z}_2[[\mathbb{S}_2^1/G_{24}]] \xrightarrow{i_1} \mathbb{Z}_2[[\mathbb{S}_2^1/G_{24}]] \oplus \mathbb{Z}_2[[\mathbb{S}_2^1/G'_{24}]] = P_0. \qquad (3.26)$$

REMARK 3.27.   This immediately gives a map from the centralizer resolution spectral sequence of Remark 3.4 to the duality resolution spectral sequence of Remark 3.17. This map is independent of the choice of $g_\bullet$ at the E_2-page. This can be lifted to a map from the centralizer tower to the duality tower, although we do not need that here and will not prove it. We note

that (3.26) implies there is a commutative diagram where the horizontal maps are the edge homomorphisms of the two spectral sequences

$$
\begin{array}{ccc}
H^*(\mathbb{S}_2^1, E_*) & \xrightarrow{\ p^*\ } & H^*(G_{24}, E_*) \times H^*(G_{24}, E_*) \\
{\scriptstyle =}\big\downarrow & & \big\downarrow {\scriptstyle (g_0)^*} \\
H^*(\mathbb{S}_2^1, E_*) & \xrightarrow[\ q^*\ ]{} & H^*(G_{24}, E_*)
\end{array}
$$

and $(g_0)^*$ is projection onto the first factor. This can be realized by a diagram of spectra, where the map g_0 is again projection onto the first factor

$$
\begin{array}{ccc}
E^{h\mathbb{S}_2^1} & \xrightarrow{\ p\ } & E^{hG_{24}} \times E^{hG_{24}} \\
{\scriptstyle =}\big\downarrow & & \big\downarrow {\scriptstyle g_0} \\
E^{h\mathbb{S}_2^1} & \xrightarrow[\ q\ ]{} & E^{hG_{24}}.
\end{array}
$$

4. Constructing elements in $\pi_{192k+48}X$

We now turn to the analysis of the homotopy groups of X, where $\Sigma^{-3}X$ is the top fiber in the duality tower; see Proposition 3.19. We have an isomorphism of Morava modules $E_*X \cong E_*E^{hG_{24}}$ and hence, by Proposition 1.27 and Lemma 1.29, a spectral sequence

$$
E_2^{s,t} = H^*(G_{24}, E_*) \Longrightarrow \pi_*X.
$$

See also Remark 1.30. The cohomology if G_{24} is discussed in Theorem 2.15. In this section we show, roughly, that $\Delta^{8k+2} \in H^0(G_{24}, E_{192k+48})$ is a permanent cycle — which would certainly be necessary if our main result is true. The exact result is below in Corollary 4.7. In the next section, we will use this and a mapping space argument to finish the identification of the homotopy type X.

The statements and the arguments in this section have a rather fussy nature because the spectrum X has no *a priori* ring or module structure and, in particular, the $\mathbb{W}$-algebra structure on $H^*(G_{24}, E_*)$ does not immediately extend to a $\mathbb{W}$-module structure on π_*X.

The results of this section were among the main results in the first author's PhD thesis [**8**] and the key ideas for the entire project can be found there.

We begin by combining Remark 1.39 and Lemma 2.20 to obtain the following result. Note that $45 \equiv 5$ modulo 8, so there is no contributions from the *bo*-patterns in that degree. In all degrees the *bo*-patterns lie in Adams–Novikov filtration at most 2. The following is an immediate consequence of Lemmas 1.37 and 2.20.

LEMMA 4.1. *There is an isomorphism*

$$
\mathbb{F}_4 \cong \pi_{45}E^{hG_{24}}.
$$

We can chose an $\mathbb{F}_4$ generator detected by the class

$$
\Delta\overline{\kappa}\eta \in H^5(G_{24}, E_{50}).
$$

The class $\Delta\overline{\kappa}^2\eta^2 \in H^{10}(G_{24}, E_{76})$ is a non-zero permanent cycle detecting an $\mathbb{F}_4$ generator of the subgroup of $\pi_{66}E^{hG_{24}}$ consisting of the elements of Adams–Novikov filtration greater than 2.

Let $p : E^{h\mathbb{S}_2^1} \to E^{hG_{24}} \times E^{hG_{24}}$ be the augmentation in the topological centralizer resolution of Theorem 3.7. This is the same map as from the top to the bottom of the centralizer resolution tower (3.10).

LEMMA 4.2. *Let* $k \in \mathbb{Z}$. *The map*

$$p_* : \pi_{192k+45}E^{h\mathbb{S}_2^1} \longrightarrow \pi_{192k+45}(E^{hG_{24}} \times E^{hG_{24}})$$

is surjective. If $x \in \pi_{192k+45}E^{h\mathbb{S}_2^1}$ *has the property that* $p_*(x) \neq 0$, *then* x *has Adams–Novikov filtration at most 5,* $x\overline{\kappa}\eta \neq 0$, *and* $x\overline{\kappa}\eta$ *is detected by a class of Adams–Novikov filtration at most 10.*

Proof. For the first statement we examine the homotopy spectral sequence of the centralizer tower (3.13). In this case this spectral sequence reads

$$E_1^{s,t} = \pi_t F_s \implies \pi_{t-s}E^{h\mathbb{S}_2^1}.$$

The fibers F_s are spelled out in (3.9). We are asking that the edge homomorphism

$$p_* : \pi_{192k+45}E^{h\mathbb{S}_2^1} \longrightarrow \pi_{192k+45}F_0$$

be surjective. The crucial input is that

$$\pi_k E^{hC_6} \subseteq \pi_k E^{hC_2} = 0$$

for $k = 45$, 46, and 47 and that $\pi_{45}E^{hC_4} = 0$. See Proposition 2.9 and Figure 2.

The final statement follows from Lemma 4.1. □

Now let $q : E^{h\mathbb{S}_2^1} \to E^{hG_{24}}$ be the augmentation in the topological duality resolution of Proposition 3.19. This is also the projection from the top to the bottom of the duality resolution tower (3.21). Let $i : \Sigma^{-3}X \to E^{h\mathbb{S}_2^1}$ be the map from the top fiber of the duality tower. Consider the commutative diagram

$$
\begin{array}{ccccc}
\pi_{192k+45}\Sigma^{-3}X & \xrightarrow{\ i_*\ } & \pi_{192k+45}E^{h\mathbb{S}_2^1} & \xrightarrow{\ =\ } & \pi_{192k+45}E^{h\mathbb{S}_2^1} \\
{\scriptstyle r}\big\downarrow & & {\scriptstyle p_*}\big\downarrow & & {\scriptstyle q_*}\big\downarrow \\
\mathbb{F}_4 & \longrightarrow & \pi_{192k+45}(E^{hG_{24}} \times E^{hG_{24}}) & \xrightarrow[(g_0)_*]{} & \pi_{192k+45}E^{hG_{24}}.
\end{array}
\tag{4.3}
$$

The bottom row is short exact and induced by the map between the resolutions. See Remark 3.27. The map r is defined by this diagram and the fact that the composition

$$\pi_*\Sigma^{-3}X \longrightarrow \pi_* E^{h\mathbb{S}_2^1} \xrightarrow{\ q_*\ } \pi_* E^{hG_{24}}$$

is zero.

PROPOSITION 4.4. *The map*

$$r : \pi_{192k+45}\Sigma^{-3}X \to \mathbb{F}_4$$

is surjective. If $y \in \pi_{192k+45}\Sigma^{-3}X$ *is any class so that* $r(y) \neq 0$, *then* y *is detected by a class*

$$f \overset{\text{def}}{=} f(j)\Delta^{8k+2} \in H^0(G_{24}, E_{192k+48}) \cong W[[j]]\Delta^{8k+2}.$$

Furthermore

$$f\overline{\kappa}\eta \in H^5(G_{24}, E_{192k+74})$$

is a non-zero permanent cycle in the spectral sequence for π_*X.

Proof. In the diagram (4.3), the map p_* is onto. Let x be any element in $\pi_{192k+45}E^{h\mathbb{S}_2^1}$ such that $p_*(x) \neq 0$ and $q_*(x) = (g_0)_* p_*(x) = 0$. Then x must have filtration at least 1 in the homotopy spectral sequence of the duality tower. Since $\pi_k E^{hC_6} = 0$ for $k = 46$ and $k = 47$, by Proposition 2.12, any such element must be the image of a class $y \in \pi_* \Sigma^{-3}X$. This shows r is surjective.

The map $\Sigma^{-3}X \to E^{h\mathbb{S}_2^1}$ raises Adams–Novikov filtration by 3; see the diagram of (3.22) and the remarks thereafter. If $r(y) = p_* i_*(y) \neq 0$, then by Lemma 4.2, y must have Adams–Novikov filtration at most 2; however, by Theorem 2.15 and the chart of Figure 3 we have that

$$H^1(G_{24}, E_{192k+49}) = 0 = H^2(G_{24}, E_{192k+50}).$$

Thus y must have filtration 0. Similarly $0 \neq y\bar{\kappa}\eta$ must have filtration at least 5 and at most 7. Again we examine the chart of Figure 3 to find it must have filtration 5 and be detected in group cohomology, as claimed. $\qquad\square$

Recall that we are writing $S/2$ for the mod 2 Moore spectrum.

PROPOSITION 4.5. *Let* $y \in \pi_{192k+48}X$ *be detected by*

$$f = f(j)\Delta^{8k+2} \in H^0(G_{24}, E_*).$$

If $r(y) \neq 0$, *then* f *and* $f\bar{\kappa}\eta$ *are non-zero permanent cycles in the spectral sequence*

$$H^*(G_{24}, E_*/2) \Longrightarrow \pi_*(X \wedge S/2).$$

Proof. This follows from Proposition 4.4 and the fact that

$$H^5(G_{24}, E_{192k+74}) \to H^5(G_{24}, E_{192k+74}/2)$$

is injective. This last statement can be deduced from the long exact sequence in cohomology induced by the short exact sequence $0 \to E_* \to E_* \to E_*/2 \to 0$. See Figure 3. $\qquad\square$

The crucial theorem then becomes:

THEOREM 4.6. *Let* $y \in \pi_{192k+48}X$ *and let*

$$f = f(j)\Delta^{8k+2} \in H^0(G_{24}, E_*)$$

be the image of y *under the edge homomorphism*

$$\pi_* X \longrightarrow H^0(G_{24}, E_*).$$

If $f(j) \equiv 0$ *modulo* $(2, j)$, *then* $r(y) = 0$.

Proof. We will show that if $f(0) \equiv 0$ modulo 2, then $f\bar{\kappa} = 0$ in $E_4^{*,*}(X \wedge S/2)$. We can then apply Proposition 4.5 to obtain the integral statement.

Under the assumption $f(0) \equiv 0$ modulo 2 we have

$$f = jg(j)\Delta^{8k+2} \in H^0(G_{24}, E_*/2).$$

We will show that in the Adams–Novikov Spectral Sequence for $L_{K(2)}(X \wedge S/2)$ we have

$$d_3(v_1^8 g(j)\Delta^{8k+2}\mu) = f\bar{\kappa}.$$

The result will follow.

We appeal to Theorem 2.15, Remark 2.16, and the chart of Figure 3. We have

$$j\bar{\kappa} = c_4^2 \eta^4 = v_1^8 \eta^4$$

and, hence, that

$$f\overline{\kappa} = jg(j)\Delta^{8k+2}\overline{\kappa} = v_1^8 g(j)\Delta^{8k+2}\eta^4.$$

Since $f\overline{\kappa}$ is a d_3-cycle, d_3 is η-linear, and since

$$H^0(G_{24}, (E/2)_{192k+48}) \xrightarrow{\eta^4} H^4(G_{24}, (E/2)_{192k+56})$$

is injective we have that

$$d_3(v_1^8 g(j)\Delta^{8k+2}) = 0.$$

It now follows from Lemma 2.21 that

$$d_3(v_1^8 g(j)\Delta^{8k+2}\mu) = v_1^8 g(j)\Delta^{8k+2}\eta^4 = f\overline{\kappa}.$$

This is what we promised.   □

The next result has a slightly complicated statement because we do not know yet that $\pi_* X$ is a $\mathbb{W}$-module.

COROLLARY 4.7.   *There is a commutative diagram*

$$
\begin{array}{ccc}
\pi_{192k+48}X & \longrightarrow & H^0(G_{24}, E_{192k+48}) \\
{\scriptstyle r}\downarrow & & \downarrow{\scriptstyle \epsilon} \\
\mathbb{F}_4 & \xrightarrow{\ \cong\ } & \mathbb{F}_4
\end{array}
$$

where the bottom map is some possibly non-trivial isomorphism of groups and

$$\epsilon(f(j)\Delta^{8k+2}) = f(0) \mod (2, j).$$

There are homotopy classes $x_{k,i} \in \pi_{192k+48}X$, $i = 1, 2$ detected by classes

$$f_i(j)\Delta^{8k+2} \in H^0(G_{24}, E_{192k+48})$$

so that $f_1(0)$ and $f_2(0)$ span $\mathbb{F}_4$ as an $\mathbb{F}_2$ vector space.

   Proof.   This is an immediate consequence of Theorem 4.6.   □

REMARK 4.8.   Lemma 4.4 produces classes $f \in \pi_{45}\Sigma^{-3}X$ for which $f\overline{\kappa}\eta \neq 0$. The image of any such f in $\pi_{45}E^{h\mathbb{S}_2^1}$ is non-zero and has Adams–Novikov filtration at least 3. It is natural to ask what we know about these classes. In particular, does one of these classes come from the homotopy groups of sphere itself under the unit map $\pi_* S^0 \to \pi_* E^{h\mathbb{S}_2^1}$?

   There are classes $x \in \pi_{45}S^0$ detected in the Adams Spectral Sequence by $h_4^3 = h_3^2 h_5$. Note any such class has filtration 3. There is a choice for x that seem to support non-zero multiplications by many of the basic elements in $\pi_* S^0$, including η and $\overline{\kappa}$. It would be easy to guess that this class is mapped to the class we have constructed, but we've not yet settled this one way or another.

   For more about this class in $\pi_{45}S^0$, see the chart 'The E_∞-page of the classical Adams Spectral Sequence' in [26]. The fact that some class detected by $h_3^2 h_5$ can have a non-zero $\overline{\kappa}$ multiplication can be found in [25, Lemma 4.114]. See also table 33 of that paper. Note that Isaksen is careful to label this lemma as tentative, as this is in the range where the homotopy groups of spheres still need exhaustive study. In the [32, Table A.3.3], a related class is posited to be detected in the Adams–Novikov Spectral Sequence by the class γ_4, also of filtration 3, but note the question mark there.

5.   The mapping space argument

We would like to extend the results of the Section 4 in the following way. Let $\iota : S^0 \to E^{hG_{48}}$ be the unit and r the composition

$$\pi_{192k+48}X \cong \pi_{192k+45}(\Sigma^{-3}X) \to \pi_{192k+45}E^{hG_{24}} \cong \mathbb{F}_4$$

defined in (4.3).

THEOREM 5.1.   *The composite*

$$\pi_{192k+48}F(E^{hG_{48}}, X) \xrightarrow{\iota_*} \pi_{192k+48}X \xrightarrow{r} \mathbb{F}_4$$

is surjective.

We can use this result to build maps out of $E^{hG_{48}}$ as follows. Recall from (1.21) that if $F \subseteq \mathbb{G}_2$ is a closed subgroup, then there is an isomorphism $E^*[[\mathbb{G}_2/F]] \cong E^*E^{hF}$ of $E^*[[\mathbb{G}_2]]$-modules. Also, Proposition 3.19 and the Universal Coefficient theorem give an isomorphism $E^*X \cong E^*[[\mathbb{G}_2/G_{24}]]$.

Consider the following diagram. Note we are using that $E^{-48} = E_{48}$.

$$
\begin{array}{ccc}
\pi_{48}F(E^{hG_{48}}, X) & \xrightarrow{\quad\quad\quad \iota_* \quad\quad\quad} & \pi_{48}X \\
{\scriptstyle H}\downarrow & & \downarrow{\scriptstyle H} \\
\mathrm{Hom}_{E^0[[\mathbb{G}_2]]}(E^0[[\mathbb{G}_2/G_{24}]], E^{-48}[[\mathbb{G}_2/G_{48}]]) & \xrightarrow{\iota^*} & \mathrm{Hom}_{E^0[[\mathbb{G}_2]]}(E^0[[\mathbb{G}_2/G_{24}]], E^{-48}) \\
& & \downarrow{\scriptstyle \cong} \\
& & \mathbb{W}[[j]]\Delta^2 \cong (E_{48})^{G_{24}} \\
& & \downarrow{\scriptstyle \epsilon} \\
& & \mathbb{F}_4
\end{array}
$$

$$(5.2)$$

where the maps labeled H are the Hurewicz maps for $E^*(-)$ and the map ϵ reduces mod $(2, j)$. By Corollary 4.7, the vertical composition on the right is r up to some automorphism of $\mathbb{F}_4$. Proposition 4.4 and Corollary 4.7 then yield the following corollary to Theorem 5.1, using the case when $k = 0$.

COROLLARY 5.3.   *Let $f(j)\Delta^2 \in H^0(G_{24}, E_{48})$. Then there is a map*

$$\phi : \Sigma^{48}E^{hG_{48}} \to X$$

so that $\iota_*(\phi) \equiv f(0)$ modulo 2.

We will use this result to show that there is an equivalence $\Sigma^{48}E^{hG_{24}} \to X$. See Theorem 5.8.

We now begin the proof of Theorem 5.1. Let $p : E^{h\mathbb{S}_2^1} \to E^{hG_{24}} \times E^{hG_{24}}$ be the projection from the top to the bottom of the centralizer resolution tower.

LEMMA 5.4.   *Let $k \in \mathbb{Z}$. The map*

$$p_* : \pi_{192k+45}F(E^{hG_{48}}, E^{h\mathbb{S}_2^1}) \to \pi_{192k+45}\big(F(E^{hG_{48}}, E^{hG_{24}}) \times F(E^{hG_{48}}, E^{hG_{24}})\big)$$

is surjective.

Proof. We apply $F(E^{hG_{48}}, -)$ to the centralizer tower and examine the resulting spectral sequence in homotopy. See (3.13). The spectral sequence reads

$$E_1^{s,t} = \pi_t F(E^{hG_{48}}, F_s) \Longrightarrow \pi_{t-s} F(E^{hG_{48}}, E^{h\mathbb{S}_2^1})$$

and the fibers F_s are described in (3.9). Thus we need to know

$$0 = \pi_{192k+45} F(E^{hG_{48}}, E^{hC_6} \vee E^{hC_4})$$

$$= \pi_{192k+46} F(E^{hG_{48}}, E^{hC_2})$$

$$= \pi_{192k+47} F(E^{hG_{48}}, E^{hC_6}).$$

We can use (1.24). Note that C_2 is central, so all of the subgroups F_x contain C_2. Therefore, the crucial input is as before:

$$\pi_k E^{hC_6} \subseteq \pi_k E^{hC_2} = 0$$

for $k = 45$, 46, and 47 and that $\pi_{45} E^{hC_4} = 0$. See Propositions 2.8, 2.9, and 2.12. See also Figure 2. $\qquad\square$

Let $q : E^{h\mathbb{S}_2^1} \to E^{hG_{24}}$ be the projection from the top to the bottom of the duality resolution tower. Using Remark 3.27 we now can produce a commutative diagram, where we have abbreviated $F(E^{hG_{48}}, Y)$ as $F(Y)$ and we are writing $n = 192k + 45$.

$$
\begin{array}{ccccc}
\pi_n F(\Sigma^{-3} X) & \longrightarrow & \pi_n F(E^{h\mathbb{S}_2^1}) & \xrightarrow{\ =\ } & \pi_n F(E^{h\mathbb{S}_2^1}) \\
\downarrow{\scriptstyle r} & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle q_*} \\
\pi_n F(E^{hG_{24}}) & \longrightarrow & \pi_n\big(F(E^{hG_{24}}) \times F(E^{hG_{24}})\big) & \xrightarrow{(g_0)_*} & \pi_n F(E^{hG_{24}}) \\
\downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} \\
\pi_n E^{hG_{24}} & \longrightarrow & \pi_n\big(E^{hG_{24}} \times E^{hG_{24}}\big) & \xrightarrow[\ (g_0)_*\ ]{} & \pi_n E^{hG_{24}}.
\end{array}
$$

(5.5)

The maps labeled $(g_0)_*$ are induced from the map $g_0 : E^{hG_{24}} \times E^{hG_{24}} \to E^{hG_{24}}$ discussed in Remark 3.27. It is projection onto the first factor. The lower two rows are split short exact, the maps labeled ι_* are all onto, and the maps p_* and q_* are onto. The map r is then defined by the requirement that the upper right square commute. It is the analog of the map r of (4.3), and in fact we have a commutative diagram

$$
\begin{array}{ccc}
\pi_{192k+45} F(E^{hG_{48}}, \Sigma^{-3} X) & \xrightarrow{\ \iota_*\ } & \pi_{192k+45} \Sigma^{-3} X \\
\downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} \\
\pi_{192k+45} F(E^{hG_{48}}, E^{hG_{24}}) & \xrightarrow[\ \iota_*\ ]{} & \pi_{192k+45} E^{hG_{24}} \cong \mathbb{F}_4.
\end{array}
$$

Since $E^{hG_{24}}$ is an $E^{hG_{48}}$-module spectrum, the evaluation map

$$\iota : F(E^{hG_{48}}, E^{hG_{24}}) \to E^{hG_{24}}$$

is split. Hence, Theorem 5.1 follows from the next result.

LEMMA 5.6. *The map*

$$r : \pi_{192k+45} F(E^{hG_{48}}, \Sigma^{-3} X) \to \pi_{192k+45} F(E^{hG_{48}}, E^{hG_{24}})$$

is onto.

Proof. The proof is an exact copy of the first part of the argument for Proposition 4.4, generalized to mapping spaces; that is, we examine the homotopy spectral sequence built from the duality tower for $F(E^{hG_{48}}, E^{h\mathbb{S}_2^1})$. In the diagram (5.5), the map p_* is onto, and if

$$x \in \pi_{192k+45} F(E^{hG_{48}}, E^{h\mathbb{S}_2^1})$$

is any element so that $p_*(x) \neq 0$ and $q_*(x) = (g_0)_* p_*(x) = 0$, then x must have filtration at least 1 in the homotopy spectral sequence of the duality tower. Since $\pi_k F(E^{hG_{48}}, E^{hC_6}) = 0$ for $k = 46$ and $k = 47$, using (1.24) and Propositions 2.8 and 2.9, any such element must be the image of a class from $\pi_* F(E^{hG_{48}}, \Sigma^{-3} X)$. $\qquad\square$

REMARK 5.7. Note that all of these arguments would work with replacing $E^{hG_{48}}$ with $E^{hG_{24}}$.

The next result is our main theorem.

THEOREM 5.8. *There is an equivalence*

$$\Sigma^{48} E^{hG_{24}} \xrightarrow{\simeq} X$$

realizing the given isomorphism of Morava modules

$$E_* E^{hG_{24}} \xrightarrow{\cong} E_* X.$$

Proof. We actually use the given isomorphism of Morava modules to produce a (non-equivariant) equivalence

$$C_2^+ \wedge \Sigma^{48} E^{hG_{48}} \simeq \Sigma^{48}(E^{hG_{48}} \vee E^{hG_{48}}) \to X.$$

Here $C_2 = \mathrm{Gal} = \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$. Then we will apply Lemma 1.37.

We begin with some algebra. Recall from Remark 1.14 that the 2-Sylow subgroup $S_2 \subseteq \mathbb{S}_2$ can be decomposed as $K \rtimes Q_8$. Since $G_{24} = Q_8 \rtimes \mathbb{F}_4^\times$ and $G_{48} = G_{24} \rtimes \mathrm{Gal}$ we have that $E^*[[\mathbb{G}_2/G_{24}]]$ and $E^*[[\mathbb{G}_2/G_{48}]]$ are free $E^*[[K]]$ modules of rank 2 and rank 1, respectively. Since K is a finitely generated pro-2-group, the ring $E^0[[K]]$ is a complete local ring with maximal ideal $\mathfrak{m}_K$ given by the kernel of the reduced augmentation

$$E^0[[K]] \longrightarrow E^0 \longrightarrow \mathbb{F}_4.$$

We will produce a map

$$f : \Sigma^{48}(E^{hG_{48}} \vee E^{hG_{48}}) \to X$$

so that the map of $E^0[[K]]$-modules

$$E^* f : E^{48} X \longrightarrow E^{48}\big(\Sigma^{48}(E^{hG_{48}} \vee E^{hG_{48}})\big)$$

is an isomorphism modulo $\mathfrak{m}_K$. Then, by the appropriate variant of Nakayama's Lemma (see [**16**, Lemma 4.3]), it will be an isomorphism of $E^0[[K]]$-modules and, hence, of $E^0[[\mathbb{G}_2]]$-modules, as required.

Since $G_{24} \cong Q_8 \rtimes \mathbb{F}_4^\times$ and $\mathbb{G}_2 \cong ((K \rtimes Q_8) \rtimes \mathbb{F}_4^\times) \rtimes \mathrm{Gal}$, we have an isomorphism of K-sets $\mathbb{G}_2/G_{24} \cong K \sqcup K\phi$ where ϕ is the Frobenius in the Galois group. Then we have the following

commutative diagram. It is an expansion of the diagram (5.2). In order to simplify notation, here we write Hom_G for $\mathrm{Hom}_{E^0[[G]]}$.

$$\mathrm{Hom}_{\mathbb{G}_2}(E^0[[\mathbb{G}_2/G_{24}]], E^{-48}[[\mathbb{G}_2/G_{48}]]) \longrightarrow \mathrm{Hom}_K(E^0[[K]] \oplus E^0[[K\phi]], E^{-48}[[K]])$$

with vertical maps ι^* on both sides, to

$$\mathrm{Hom}_{\mathbb{G}_2}(E^0[[\mathbb{G}_2/G_{24}]], E^{-48}) \longrightarrow \mathrm{Hom}_K(E^0[[K]] \oplus E^0[[K\phi]], E^{-48})$$

with left vertical map $\cong$ and right vertical map $\subseteq$, to

$$(E_{48})^{G_{24}} \cong \mathbb{W}[[j]] \longrightarrow E_{48}^2 \cong (\mathbb{W}[[u_1]])^2$$

with vertical maps down to

$$\mathbb{F}_4 \longrightarrow \mathbb{F}_4^2 .$$

The top two horizontal maps are forgetful maps, remembering only the K action. The third horizontal map is the composition

$$\mathbb{W}[[j]] \xrightarrow{\ \subseteq\ } \mathbb{W}[[u_1]] \xrightarrow{\ 1 \times \phi\ } (\mathbb{W}[[u_1]])^2$$

and the bottom map is $x \mapsto (x, \phi(x))$. The top vertical maps are induced by the map $E^*(E^{hG_{48}}) \to E^* S^0$ given by the unit, the middle vertical map evaluates a homomorphism at $1 \in E^0[[\mathbb{G}_2/G_{24}]]$; note we are again using that $E^{-48} \cong E_{48}$. The final map is reduction modulo the maximal ideals in both cases.

By Corollary 5.3 we can produce two maps

$$f_i : \Sigma^{48} E^{hG_{48}} \to X, \quad i = 1, 2$$

so that $(f_i)_*(\iota) \equiv \omega^i \Delta^2$ modulo $(2, j)$, where $\omega \in \mathbb{F}_4$ is the primitive cube root of unity. We now examine the fate of

$$E^0(f_i) : E^0[[\mathbb{G}_2/G_{24}]] \longrightarrow E^{-48}[[\mathbb{G}_2/G_{48}]]$$

as we work from the upper left to the bottom right of this diagram. Using the formulas of the previous paragraph we have

$$E^0(f_1) \mapsto (\omega, \omega^2) \quad \text{and} \quad E^0(f_2) \mapsto (\omega^2, \omega).$$

Finally, let

$$f = f_1 \vee f_2 : \Sigma^{48}(E^{hG_{48}} \vee E^{hG_{48}}) \to X.$$

If we apply $E^*(-)$ to this map we get a map

$$E^* f : E^0[[\mathbb{G}_2/G_{24}]] \to E^{-48}[[\mathbb{G}_2/G_{48}]] \times E^{-48}[[\mathbb{G}_2/G_{48}]]$$

which yields a map of K-modules

$$E^* f : E^0[[K]]^2 \longrightarrow E^{-48}[[K]]^2$$

which, modulo the maximal ideal $\mathfrak{m}_K$, gives the map

$$\mathbb{F}_4^2 \longrightarrow \mathbb{F}_4^2$$

given by the matrix

$$\begin{pmatrix} \omega & \omega^2 \\ \omega^2 & \omega \end{pmatrix}$$

with determinant $\omega^2 + \omega = 1$. Thus $E^* f$ is an isomorphism, as needed. $\qquad \square$

We can now complete the proof of Theorem 1 and construct the topological duality resolution. Proposition 3.19 and Theorem 5.8 imply the following. Recall from Theorem 2.9 that E^{hC_6} is 48-periodic.

COROLLARY 5.9. *There exists a resolution of $E^{h\mathbb{S}_2^1}$ in the $K(2)$-local category at the prime* 2

$$E^{h\mathbb{S}_2^1} \to E^{hG_{24}} \to E^{hC_6} \to \Sigma^{48} E^{hC_6} \to \Sigma^{48} E^{hG_{24}}.$$

REMARK 5.10. In [**16**], working at the prime 3, the authors were able to produce a topological resolution for $L_{K(2)}S^0 = E^{h\mathbb{G}_2}$ itself by the same methods that produced the resolution for $E^{h\mathbb{G}_2^1}$. The resolution for the sphere was essentially a double of that for $E^{h\mathbb{G}_2^1}$. Not only do the methods of [**16**] not apply to the case $p = 2$, it is very unlikely that there is a topological resolution of the sphere, or for $E^{h\mathbb{S}_2}$, which could be obtained by doubling the resolution of $E^{h\mathbb{S}_2^1}$. There are any number of difficulties, but the first obstacle is that we have only a semi-direct decomposition $\mathbb{S}_2^1 \rtimes \mathbb{Z}_2 \cong \mathbb{S}_2$ at $p = 2$, rather than the product decomposition $\mathbb{G}_2^1 \times \mathbb{Z}_3 \cong \mathbb{G}_2$ at the prime 3. This makes the algebra much harder, and it only gets worse from there. Other short topological resolutions are possible, of course, and could be very instructive. This is the subject of current research by Agnès Beaudry and Hans-Werner Henn.

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