ANDREW BLUMBERG

THE BURNSIDE CATEGORY

ADRIAN CLOUGH · ARUN DEBRAY · ERNIE FONTES · DAN FREED · TOM GANNON · GILL GRINDSTAFF · NICKY REYES
YURI SULYMA · RICHARD WONG · YIXIAN WU · ZHU YAOGUANG · NEŽA ŽAGER KORENJACK · VAL ZAKHAREVICH

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0.1. Introduction

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These notes were taken in UT Austin's M392C (Topics in Algebraic Topology) class in Spring 2017, taught by Andrew Blumberg. I (Arun) live-TEXed them using `vim`, so there may be typos; please send questions, comments, complaints, and corrections to `a.debray@math.utexas.edu`. Alternatively, these notes are hosted on Github at `https://github.com/adebray/equivariant_homotopy_theory`, and you can submit a pull request. Thanks to Rustam Antia-Riedel, Omar Antolín Camarena, Gill Grindstaff, Charles Rezk, Sean Tilson, and Yixian Wu for catching a few errors; to Andrew Blumberg, Ernie Fontes, Tom Gannon, Tyler Lawson, Richard Wong, Valentin Zakharevich, an anonymous reader, and the users of the homotopy theory Mathoverflow chatroom for some clarifications and suggestions; to Adrian Clough, Richard Wong, and Yuri Sulyma for adding some examples, remarks, and references; and to Richard Wong for the cover page.

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These notes are an overview of equivariant stable homotopy theory. We're in the uncomfortable position where this is a big subject, a hard subject, and one that is poorly served by its textbooks. Algebraic topology is like this in general, but it's particularly acute here. Nonetheless, here are some references:

- Adams, "Prerequisites (on equivariant stable homotopy) for Carlsson's lecture." [**Ada84**]. This is old, and some parts of it don't reflect how we do things now.
- The Alaska notes [**May96**], edited by May, are newer, and are written by many authors. Some parts are a grab bag, and some parts (e.g. the rational equivariant bits) aren't entirely right. The notes are also not a textbook.
- Appendix A of Hill-Hopkins-Ravenel [**HHR16**]. This is a paper which resolved an old conjecture on manifolds using equivariant stable homotopy theory, but let this be a lesson on referee reports: the authors were asked to provide more background, and so wrote a 150-page appendix on this material. Their suffering is your gain: the appendix is a well-written introduction to equivariant stable homotopy theory, albeit again not a textbook.

There are arguably two very serious modern applications of equivariant stable homotopy theory:

- The first is trace methods in algebraic K-theory: Hochschild homology and its topological cousins are equipped with natural S^1-actions (the same S^1-action coming from field theory). This is how people other than Quillen compute algebraic K-theory.
- The other major application is Hill-Hopkins-Ravenel's settling of the Kervaire invariant 1 conjecture in [**HHR16**].

The nice thing is, however you feel about the applications, both applications require developing new theory in equivariant stable homotopy theory. Hill-Hopkins-Ravenel in particular required a clarification of the foundations of this subject which has been enlightening.

In these notes, we hope to cover the foundations of equivariant stable homotopy theory. On the one hand, this will be a modern take, insofar as we emphasize the norm and presheaves on orbit categories (these will be explained in due time), the modern emerging consensus on how to think of these things, different than what's written in textbooks. The former is old, but has gained more attention recently; the latter is new. Moreover, there's an increasing sense that a lot of the foundations here are best done in ∞-categories. We will not take this approach in order to avoid getting bogged down in ∞-categories; moreover, the notes are supposed to be rigorous. It will sometimes be clear to some people that ∞-categories lie in the background, but we won't talk very much about them.

We'll cover some old topics such as Smith theory and the Segal conjecture, and newer ones such as trace methods and Hill-Hopkins-Ravenel, depending on student interest. We will not have time to discuss many topics, including equivariant cobordism or equivariant surgery theory.

Prerequisites. If you don't know these prerequisites, that's okay; it means you're willing to read about them on your own.

- Foundations of unstable homotopy theory at the level of May's *A Concise Course in Algebraic Topology* [**May99**]. For example, we'll discuss equivariant CW complexes, so it will help to know what a CW complex is.
- A little bit of category theory, e.g. found in Mac Lane [**Mac78**] or Riehl [**Rie16**].

- These notes will not require much in the way of simplicial methods (simply because it's hard to reconcile simplicial methods with non-discrete Lie groups), but you will want to know the bar construction. An excellent source for this is [**Rie14**, Chapter 4].
- A bit of abstract homotopy theory, e.g. what a model structure is. Good sources for model categories are [**Rie14**, Part III] and [**Hov99**].

If you don't know these, feel free to ask the professor for references. His advisor suggested learning nonequivariant stable homotopy theory by reading Lewis-May-Steinberger's book [**LMS86**] on equivariant stable homotopy theory and letting $G = *$, but this may not appeal to everyone. In any case, perhaps this isn't necessarily a good reference for nontrivial groups.

Unstable equivariant questions are very natural, and somewhat reasonable. But stable questions are harder; they ultimately arise from reasonable questions, but their formulations and answers are hard: even discussing the equivariant analogue of $\pi_0 S^0$ (see (2.7.3)) requires some representation theory — and yet of course it should. Thus there's a lot of foundations behind hard calculations.

Categories of topological spaces. The category of topological spaces we consider is Top, the category of compactly generated, weak Hausdorff spaces (and continuous maps); we'll also consider Top_*, the category of based, compactly generated, weak Hausdorff spaces and continuous, based maps. This is an important and old trick which eliminates some pathological behavior in quotients. It's reasonable to imagine that point-set topology shouldn't be at the heart of foundational issues, but there are various ways to motivate this, e.g. to make Top more resemble a topos or the category of simplicial sets.

Definition 0.1.1. Let X be a topological space.

- A subset $A \subseteq X$ is **compactly closed** if for every compact Hausdorff space Y and $f : Y \to X$, $f^{-1}(A)$ is closed.
- X is **compactly generated** if every compactly closed subset of X is closed.
- X is **weak Hausdorff** if for every compact Hausdorff space Y and continuous map $f : Y \to X$, $f(Y)$ is a closed subset of X.[1]

The intuition behind compact generation is that the topology is determined by compact Hausdorff spaces. The weak Hausdorff condition is strictly stronger than T_1 (points are closed), but strictly weaker than being Hausdorff. Any space you can think of without trying to be pathological will meet these criteria.

There is a functor k from all spaces to compactly generated spaces which adds the necessary closed sets. This has the unfortunate name of **k-ification** or **kaonification**;[2] by putting the compactly generated topology on $X \times X$, we mean taking $k(X \times X)$. There's also a "weak Hausdorffification" functor w which makes a space weakly Hausdorff, which is some kind of quotient.[3]

When computing limits and colimits, it's often desirable to compute them in the category of spaces and then apply k and w to return to Top. This works correctly for limits, but for colimits, w is particularly badly behaved: you cannot compute the colimit in Top by computing it in Set and figuring out the topology. In general, you'll need to take a quotient.

Nonetheless, there are nice theorems which make things work out anyways.

Proposition 0.1.2. *Let $Z = \operatorname{colim}(X_0 \to X_1 \to X_2 \to \dots)$ be a sequential colimit (sometimes called a **telescope**); if each X_i is weak Hausdorff, then so is Z.*

Proposition 0.1.3. *Consider a diagram*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \\
C &
\end{tikzcd}$$

where f is a closed inclusion. If A, B, and C are weakly Hausdorff, then $B \amalg_A C$ is weakly Hausdorff.

---

[1] When X is compactly generated, this is equivalent to being **k-Hausdorff**, i.e. the diagonal map $\Delta : X \to X \times X$ is closed when $X \times X$ has the compactly generated topology. See [**Str09**, **Rez**] for more details.

[2] Kaonification is of course distinct from koanification, the process which makes statements more confusing.

[3] The k functor is right adjoint to the forgetful map, which tells you what it does to limits.

These are the two kinds of colimits people tend to compute, so this is reassuring.

One reason we require regularity on our topological spaces is the following, which is not true for topological spaces in general.

Lemma 0.1.4. *Let X, Y, and Z be in* Top*; then, the natural map*

$$\mathrm{Map}(X \times Y, Z) \longmapsto \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$

is a homeomorphism.

Enrichments. The categories Top and Top_* are enriched over themselves (as are categories of G-spaces, which we'll see soon). This merits a brief digression into enriched categories.

Definition 0.1.5. Let $(\mathsf{V}, \otimes, \mathbf{1})$ be a symmetric monoidal category.[4] Then, an **enrichment** of a category C over V means

- for every $x, y \in \mathsf{C}$, there is a hom-object $\underline{\mathsf{C}}(x, y)$, which is an object in V,
- for every $x \in \mathsf{C}$, there is a unit V-morphism $\mathbf{1} \to \underline{\mathsf{C}}(x, x)$,
- composition $\underline{\mathsf{C}}(x, y) \otimes \underline{\mathsf{C}}(y, z) \to \underline{\mathsf{C}}(x, z)$ is associative and unital, and
- the underlying category is recovered as $\mathsf{C}(x, y) = \mathrm{Map}(\mathbf{1}, \underline{\mathsf{C}}(x, y))$.

A great deal of category theory can be generalized to enriched categories, including V-enriched functors, V-enriched natural transformations, V-enriched limits and colimits, and more. The canonical reference is Kelley [**Kel84**], available free and legally online. It covers just about everything we need except for the Day convolution, which can be read from Day's thesis [**Day70**]. Another good source, with a view towards homotopy theory, is [**Rie14**, Chapter 3].

Definition 0.1.6. Let C and D be enriched over V. Then, an **enriched functor** $F : \mathsf{C} \to \mathsf{D}$ is an assignment of objects in C to objects in D and maps $\underline{\mathsf{C}}(x, y) \to \underline{\mathsf{D}}(Fx, Fy)$ that are V-morphisms, and commute with composition.

A category enriched over Top is called a **topological category**.

Exercise 0.1.7. Work out the definition of enriched natural transformations.

This brings us to the beginning.

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[4] Briefly, this means V has a tensor product $\otimes$ and a unit $\mathbf{1}$; there are certain axioms these must satisfy.

CHAPTER 1

Unstable equivariant homotopy theory

1.1. G-spaces

Let G be a group. We'll generally restrict to finite groups or compact Lie groups; this is not because these are the only interesting groups, but rather because they are the only ones we really understand. If you can come up with a good equivariant homotopy theory for discrete infinite groups, you will be famous. Throughout, keep in mind the examples C_p (the cyclic group of order p, sometimes also denoted $\mathbb{Z}/p$), C_{p^n}, the symmetric group Σ_n, and the circle group S^1.

There's a monad[1] M_G on Top which sends $X \mapsto G \times X$, and analogously a monad M_G^* on Top_* sending $X \to G_+ \wedge X$. One can define the category of **G-spaces** $G\mathsf{Top}$ (resp. **based G-spaces** $G\mathsf{Top}_*$) to be the category of algebras over M_G (resp. M_G^*). This is probably not the most explicit way to define G-spaces, but it makes it evident that $G\mathsf{Top}$ and $G\mathsf{Top}_*$ are complete and cocomplete.

More explicitly, $G\mathsf{Top}$ is the category of spaces $X \in \mathsf{Top}$ equipped with a continuous action $\mu\colon G \times X \to X$. That is, μ must be associative and unital. Associativity is encoded in the commutativity of the diagram

$$\begin{tikzcd}
G \times G \times X \arrow[r, "1\times\mu"] \arrow[d, "m"] & G \times X \arrow[d, "\mu"] \\
G \times X \arrow[r, "\mu"] & X.
\end{tikzcd}$$

The morphisms in $G\mathsf{Top}$ are the **G-equivariant** maps $f : X \to Y$, i.e. those commuting with μ:

$$\begin{tikzcd}
G \times X \arrow[r] \arrow[d, "\mu_X"] & G \times Y \arrow[d, "\mu_Y"] \\
X \arrow[r, "f"] & Y.
\end{tikzcd}$$

It's possible (but not the right idea) to let $\underline{G}$ denote[2] the category with an object $*$ such that $\underline{G}(*,*) = G$. Then, $G\mathsf{Top}$ is also the category of functors $\underline{G} \to \mathsf{Top}$, with morphisms as natural transformations. This realizes $G\mathsf{Top}$ as a **presheaf category**; it will eventually be useful to do something like this, but in a different way described by Elmendorf's theorem (Theorem 1.3.8).

When we write $\mathrm{Map}(X,Y)$ in $G\mathsf{Top}$ or $G\mathsf{Top}_*$, we could mean three things:

(1) The set of G-equivariant maps $X \to Y$.
(2) The space of G-equivariant maps $X \to Y$ in the subspace topology of all maps from $X \to Y$. As this suggests, $G\mathsf{Top}$ admits an enrichment over Top (resp. $G\mathsf{Top}_*$ admits an enrichment over Top_*).
(3) The G-space of all maps $X \to Y$, where G acts by conjugation: $f \mapsto g^{-1}f(g\cdot)$. This realizes $G\mathsf{Top}$ as enriched over itself, and similarly for $G\mathsf{Top}_*$.

Each of these is useful in its own way: for constructions it may be important to be self-enriched, or to only look at G-equivariant maps. We will let $\mathrm{Map}^G(X,Y)$ or $\mathrm{Map}(X,Y)$ denote (2) or its underlying set (1), and $G\,\mathrm{Map}(X,Y)$ denote (3).[3]

It turns out you can recover Map^G from $G\,\mathrm{Map}$: the equivariant maps are the fixed points under conjugation of all maps. This is written $(G\,\mathrm{Map}(X,Y))^G = \mathrm{Map}^G(X,Y)$.

---

[1] We're going to say more about monads in §2.4.

[2] There isn't really a standard notation for this category, but the closest is BG. This notation emphasizes the fact that groupoids are Quillen equivalent to 1-truncated spaces.

[3] Later, when we discuss G-spectra, we will use $F(X,Y)$ to denote function spectrum of X and Y as a G-spectrum, or $F_G(X,Y)$ when G needs to be explicit.

Throughout this class, “subgroup” will mean “closed subgroup” unless specified otherwise.

Definition 1.1.1. Let X be a G-set and $H \subseteq G$ be a subgroup. Then, the **H-fixed points** of X is the space $X^H := \{x \in X \mid hx = x \text{ for all } h \in H\}$. This is naturally a WH-space, where $WH = NH/H$ (here NH is the normalizer of H in G).[4]

Definition 1.1.2. The **isotropy group** of an $x \in X$ is $G_x := \{h \in G \mid hx = x\}$.

Isotropy groups are useful in the following two ways.

(1) Often, it will be helpful to reduce questions from $G\mathsf{Top}$ to Top using $(-)^H$.
(2) It’s also useful to induct over isotropy types.

Now, we’ll see some examples of G-spaces.

Example 1.1.3. Let H be a subgroup of G; then, the **orbit space** G/H is a useful example, because it corepresents the fixed points by H. That is, $X^H \cong \mathrm{Map}(G/H, X)$. These spaces will play the role of points when we build things such as equivariant CW complexes. $\triangleleft$

Example 1.1.4. Let $H \subset G$ as usual and $U \colon G\mathsf{Top} \to H\mathsf{Top}$ be the forgetful functor. Then, U has both left and right adjoints:

- The left adjoint sends X to the **balanced product** $G \times_H X := G \times X/\sim$, where $(gh, x) \sim (g, hx)$ for all $g \in G$, $h \in H$, and $x \in X$. Despite the notation, this is *not* a pullback! (In the based case, the balanced product is $G_+ \wedge_H X$.) G acts via the left action on G. This is called the **induced G-action** on $G \times_H X$.
- The right adjoint is $\mathrm{Map}^H(G, X)$ (or $\mathrm{Map}^H(G_+, X)$ in the based case), the space of H-equivariant maps $G \to X$, with G-action $(gf)(g') = f(g'g)$. This is called the **coinduced G-action** on $\mathrm{Map}^H(G, X)$.[5] Sometimes this is also denoted $F_H(G, X)$. $\triangleleft$

Remark. Here is a categorical perspective on “change of group.” Quite generally, a group homomorphism $G \xrightarrow{f} H$ induces adjunctions

$$\begin{tikzcd} G\mathsf{Top} \arrow[r, shift left=3, "f_!"] \arrow[r, phantom, shift left=1.5, "\perp"] \arrow[r, phantom, shift right=1.5, "\perp"] \arrow[r, shift right=3, "f_*"'] & H\mathsf{Top} \arrow[l, "f^*" description] \end{tikzcd}.$$

These are given by $f_!(X) := H \times_G X$ and $f_*(X) := \mathrm{Map}^G(H, X)$ for a G-space X, where H is given the structure of a G-space by f. When $H = *$, an H-space is just a space, and $f_!(X) = X_G$ is the space of orbits while $f_*(X) = X^G$ is the space of fixed points. Observe that similar statements hold for categories of modules, given a ring homomorphism $R \xrightarrow{f} S$.

In fact, these are both cases of very general abstract nonsense. Let BG denote the category with one object $*$ with $\mathrm{Hom}(*, *) = G$; as we have said above, we can (naïvely) write $G\mathsf{Top}$ as the functor category Top^{BG}. A group homomorphism $G \xrightarrow{f} H$ induces a functor $BG \xrightarrow{F} BH$ (it is not quite true that the two are equivalent—think about why this is). Now $f^* \colon H\mathsf{Top} \to G\mathsf{Top}$ is just restriction along F:

$$\begin{tikzcd} BG \arrow[r, dotted, "f^*(Y)"] \arrow[d, "F"'] & \mathsf{Top} \\ BH \arrow[ur, "Y"'] & \end{tikzcd}$$

According to abstract nonsense, restriction along F has left and right adjoints, called **left and right Kan extension along F**, respectively:

$$\begin{tikzcd} BG \arrow[r, "X"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow\eta"] & \mathsf{Top} \\ BH \arrow[ur, dotted, "f_!(X) = \mathrm{Lan}_F X"'] & \end{tikzcd} \qquad \begin{tikzcd} BG \arrow[r, "X"] \arrow[d, "F"'] \arrow[dr, phantom, "\Uparrow\epsilon"] & \mathsf{Top} \\ BH \arrow[ur, dotted, "f_*(X) = \mathrm{Ran}_F X"'] & \end{tikzcd}$$

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[4] If $H \trianglelefteq G$, then X^H is also a G/H-space.

[5] This actually *is* a group action, since if $a, b, g \in G$, then $(a(bf))(g) = (bf)(ga) = f(gab) = (ab(f))(g)$.

These diagrams do not commute, but there are natural transformations $X \overset{\eta}{\Longrightarrow} f^*f_!(X)$ and $f^*f_*(X) \overset{\epsilon}{\Longrightarrow} X$. When H is the trivial group, BH is the trivial category, and it is known that left/right Kan extensions of a functor X along a functor to the trivial category pick out the colimit/limit of X. That is, still viewing a G-space X as a functor $BG \to \mathsf{Top}$, we have $X_G = \operatorname*{colim}_{BG} X$ and $X^G = \lim_{BG} X$.

For an example-driven introduction to Kan extensions, we recommend [**Rie16**, Chapter 6]. Like much of category theory, this is ultimately all trivial, but it may be highly non-trivial to understand why it is trivial. ◀

Example 1.1.5. Let V be a finite-dimensional real representation of G, i.e. a real inner product space on which G acts in a way compatible with the inner product. (This is specified by a group homomorphism $G \to \mathrm{O}(V)$.) The one-point compactification of V, denoted S^V, is a based G-space; the unit disc $D(V)$ and unit sphere $S(V)$ are unbased spaces, but we have a quotient sequence

$$\begin{tikzcd} S(V)_+ \arrow[r] & D(V)_+ \arrow[r] & S^V. \end{tikzcd}$$

If $V = \mathbb{R}^n$ with the trivial G-action, S^V is S^n with the trivial G-action, so these generalize the usual spheres; thus, these S^V are called **representation spheres**. ◀

We will let S^n denote $S^{\mathbb{R}^n}$, our preferred model for the n-sphere with trivial G-action.

Exercise 1.1.6. Show that $S^V \wedge S^W \cong S^{V\oplus W}$.

Definition 1.1.7. A **G-homotopy** is a map $h\colon X \times I \to Y$ in $G\mathsf{Top}$, where G acts trivially on I. We generally think of it, as usual, as interpolating between $h(-,0)$ and $h(-,1)$. This is the same data as a path in $G\operatorname{Map}(X,Y)$. A **G-homotopy equivalence** between X and Y is a map $f : X \to Y$ such that there exists a $g : Y \to X$ and G-homotopies $gf \sim \mathrm{id}_X$ and $fg \sim \mathrm{id}_Y$.

1.2. G-CW complexes and Whitehead's theorem

"It's nice to write down, but oh so false."

After defining a G-homotopy, the (well, a) natural question that might arise: what are G-weak equivalences and G-CW complexes? This closely relates to obstruction theory: CW complexes are test objects.

To define G-CW complexes, we need cells. One choice is $G/H \times D^{n+1}$ and $G/H \times S^n$, where the actions on D^{n+1} and S^n are trivial. This is a plausible choice (and in fact, will be the right choice), but it's not clear why — why not $G \times_H D(V)$ or $G \times_H S(V)$ for some H-representation V? Ultimately, this comes from a (quite nontrivial) theorem that these can be triangulated in terms of the cells $G/H \times D^{n+1}$ and $G/H \times S^n$.[6] This is one of several triangulation results proven in the 1970s which are now assumed without comment, but if you like this kind of math then it's a very interesting story.

Definition 1.2.1. A **G-CW complex** is a sequential colimit of spaces X_n, where X_{n+1} is a pushout

$$\begin{tikzcd} \coprod G/H \times S^n \arrow[r] \arrow[d] & X_n \arrow[d] \\ \coprod G/H \times D^{n+1} \arrow[r] & X_{n+1}, \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

where H varies over all closed subgroups of G.

That is, it's formed by attaching cells just as usual, though now we have more cells.

This immediately tells you what the homotopy groups have to be: $[G/H \times S^n, X]$, which by an adjunction game is isomorphic to $\pi_n(X^H)$. We let $\pi_n^H(X) := \pi_n(X^H)$. Thus, we can define weak equivalences.

Definition 1.2.2. A map $f : X \to Y$ of G-spaces is a **weak equivalence** if for all subgroups $H \subset G$, $f_*\colon \pi_n^H(X) \to \pi_n^H(Y)$ is an isomorphism.

These homotopy groups have a more complicated algebraic structure: they're indexed by the lattice of subgroups of G and the integers. This is fine (you can do homological algebra), but some things get more complicated, including asking what the analogue of connectedness is! (We'll broach this in Definition 1.2.10.)

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[6] Illman's thesis [**Ill72**] is a reference, albeit not the most accessible one.

One quick question: do we need all subgroups H? What if we only want finite-index ones? The answer, in a very precise sense, is that if you're willing to use fewer subgroups, you get fewer cells $G/H \times S^n$, and that's fine, and you get a different kind of homotopy theory.

Finally, the Whitehead theorem (Corollary 1.2.14) is true for G-CW complexes. This follows for the same reason as in May's course: it follows word-for-word after proving the equivariant HELP (homotopy extension lifting property) lemma (Theorem 1.2.11), which is true by the same argument as in the nonequivariant case.

G-CW complexes are just like the CW complexes we know and love, but with new cells G/H indexed by the closed subgroups $H \subset G$. The idea is that you're building up a space by attaching different spaces with different isotropy groups (G/H has isotropy group H, just by construction).

Example 1.2.3 (Zero-dimensional complexes)**.** The zero-dimensional complexes are G/H or disjoint unions $\amalg_i G/H_i$. This is an instance of the slogan that "orbits are points." Keep in mind that if G is a compact Lie group, this might not be zero-dimensional in other, more familiar kinds of dimension. ◀

Example 1.2.4. Let S^1 act on $\mathbb{R}^2$ by rotation along the origin. This also induces a C_n-action, as $C_n \subseteq S^1$ as the n^{th} roots of unity. Let V denote this C_n-space.

Let $D(V)$ denote the unit disc in V, and S^V denote its one-point compactification, a representation sphere. Then, $D(V)$ looks like wedges of pie, as the origin is fixed. On S^V, the point at infinity is also fixed, so we obtain a beachball.

Now let's consider V as an S^1-space, and write down the CW structure on S^V. There are two fixed points, and each one is a 0-cell $S^1/S^1 \times *$, but there is one 1-cell $S^1 \times I$ attached to the endpoints (thought of as a meridian rotated around the sphere).

Now let's consider the beachball for C_2 on S^V, where there are two hemispheres and C_2 rotates by a half-turn. What's the G-CW structure on this?

- There are two 0-cells $C_2/C_2 \times *$, corresponding to the two fixed points, the north and south poles.
- There is a single free 1-cell $C_2 \times I$, corresponding to the boundary of the hemispheres.
- There is a single 2-cell $C_2 \times D^2$. ◀

We discussed other prospective cells $G \times_H S(V)$ and $G \times_H D(V)$, but can be decomposed in terms of the actual cells we use. One point to observe about these cells is that G does not act on them by permuting non-equivariant cells around, but rather in a more complicated way; there is virtue in the simplicity of the G-cells we have chosen to work with.

Exercise 1.2.5. C_2 also acts on S^2 by the antipodal map, which has no fixed points. Write a C_2-CW cell structure for this C_2-space.

Example 1.2.6. The torus $S^1 \times S^1$ has an S^1-action given by $z(z_1, z_2) = (zz_1, z_2)$. With this action, the torus can be viewed as an S^1-CW complex with one 0-cell $S^1/e \times *$ and one 1-cell $S^1 \times [0,1]$, with the attaching map sending 0 and 1 to $*$. Note that the largest cell we used here was a 1-cell, whereas in the nonequivariant construction of the torus, we are required to use a 2-cell. Check out Figure 1 for a picture. ◀

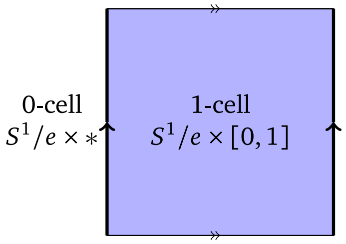

FIGURE 1. The S^1-CW structure on the torus in Example 1.2.6. There is one 0-cell and one 1-cell.

Example 1.2.7. Let T be a solid equilateral triangle in the plane, so $D_6 = \langle r, s \mid r^3 = s^2 = 1, srs = r^{-1} \rangle$ acts on it by rotations and reflections.

- There are three 0-cells: the center is a fixed point, so a $D_6/D_6 \times *$. The three vertices are an orbit, with the stabilizer of each vertex conjugate to $\langle s \rangle$ inside D_6, so they form a 0-cell of the form $D_6/\langle s \rangle \times *$. Similarly, the midpoints of each edge are an orbit with each stabilizer conjugate to $\langle s \rangle$, so they're also a 0-cell of the form $D_6/\langle s \rangle \times *$.

- There are three 1-cells: the three line segments from the center to a vertex are an orbit for $D_6/\langle s\rangle$, so form a $D_6/\langle s\rangle \times [0,1]$. Similarly, the three line segments from the center to the midpoint of an edge form a $D_6/\langle s\rangle \times [0,1]$. The six line segments from a vertex to the center of a midpoint are a free orbit of D_6, hence form a $D_6/e \times [0,1]$.
- The triangle minus these 0- and 1-cells is a free orbit, a $D_6/e \times D^2$.

See Figure 2 for a picture. ⊲

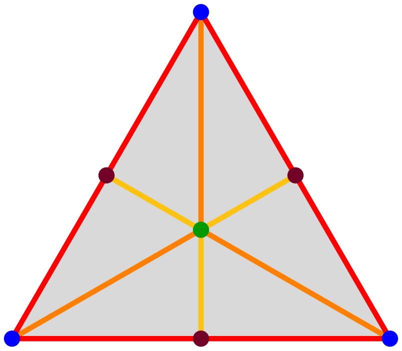

FIGURE 2. The D_6-equivariant structure on a solid triangle, as in Example 1.2.7. Each cell is depicted in a different color. The three 0-cells are purple ($D_6/\langle s\rangle \times *$), blue ($D_6/\langle s\rangle \times *$), and green ($D_6/D_6 \times *$); the three 1-cells are yellow ($D_6/\langle s\rangle \times [0,1]$), orange ($D_6/\langle s\rangle \times [0,1]$), and red ($D_6/e \times [0,1]$); the one 2-cell is gray ($D_6/e \times D^2$).

REMARK. At this point in class, the professor mentioned that these notes are hosted on Github at `https://github.com/adebray/equivariant_homotopy_theory`. Since there aren't very many sources for learning this material, and existing ones tend to have few examples, the hope is that these notes can be turned into a good source of lecture notes for learning this material. So as you're learning this material, feel free to add examples, insert comments (e.g. "this section is confusing/unmotivated"), and let me know if you want access to the repository. ⊲

REMARK.

(1) There is a technical issue of a G-CW structure on a product of G-CW complexes; namely, there are technical difficulties in cleanly putting a G-CW structure on $G/H_1 \times G/H_2$ involving triangulation. We won't digress into this: it's straightforward for finite groups, but a theorem for compact Lie groups, and required revisiting the foundations. Similarly, if $H \subset G$, we'd like the forgetful functor $G\mathsf{Top} \to H\mathsf{Top}$ to send G-CW complexes to H-CW complexes. This is again possible, yet involves technicalities.

(2) A nicer fact is that computing the fixed points of a G-CW complex is straightforward. Recall that $(-)^H$ is a right adjoint, which can be seen by realizing it as the limit of the diagram

$$\begin{tikzcd}\bullet \arrow[loop above, "H"] \arrow[r] & \mathsf{Top}.\end{tikzcd}$$

Thus, we don't expect it to commute with colimits in general. However, it does commute with many important ones, as in the following proposition. ⊲

Proposition 1.2.8. *The fixed point functor $(-)^H$ commutes with*

(1) *pushouts where one leg is a closed inclusion, and*
(2) *sequential colimits along closed inclusions.*

This is great, because it means we can commute $(-)^H$ through the construction of a G-CW complex! In particular, on each cell,

$$(G/K \times D^n)^H \cong (G/K)^H \times D^n,$$

so we need to understand $(G/K)^H \cong \mathrm{Map}^G(G/H, G/K)$. We will return to this important point.

Two approaches to the Whitehead theorem. We'll now discuss some homotopy theory of G-spaces and the Whitehead theorem. The first will be a hands-on proof using the HELP lemma. This is an elegant approach to unstable homotopy theory due to Peter May in which one lemma gives quick proofs of several theorems. In the equivariant case, it allows a quick reduction to the non-equivariant case; it will be useful to see a proof of this nature. Ultimately, we will take a different approach involving model categories, and this will be the second perspective.

Definition 1.2.9. Let $X, Y \in \mathsf{Top}$ and $f : X \to Y$ be continuous. Then, f is n**-connected** if $\pi_q(f)\colon \pi_q(X) \to \pi_q(Y)$ is an isomorphism when $q < n$ and surjective when $q = n$.

We wish to generalize this to the equivariant case.

Definition 1.2.10. Let $\theta\colon \{\text{conjugacy classes of subgroups of } G\} \to \{x \in \mathbb{Z} \mid x \geq -1\}$.

- A map $f : X \to Y$ of G-spaces is θ**-connected** if for all $H \subset G$, f^H is $\theta(H)$-connected.
- A G-CW complex is θ**-dimensional** if all cells of orbit type G/H have (nonequivariant) dimension at most $\theta(H)$.

Theorem 1.2.11 (Equivariant HELP lemma)**.** *Let A, X, Y, and Z be G-CW complexes such that $A \subseteq X$ is θ-dimensional and let $e\colon Y \to Z$ be a θ-connected G-map. Given $g : A \to Y$, $h\colon A \times I \to Z$, and $f : X \to Z$ such that $eg = hi_0$ and $fi = hi_1$, there exist maps $\tilde{g} : X \to Y$ and $\tilde{h}\colon X \times I \to Z$ that make the following diagram commute:*

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd] & & A \times I \arrow[dl, "h"'] \arrow[dd] & & A \arrow[ll, "i_1"'] \arrow[dl, "g"'] \arrow[dd] \\
& Z & & Y \arrow[ll, "e"'] & \\
X \arrow[rr, "i_0"'] \arrow[ur, "f"] & & X \times I \arrow[ul, dashed, "\tilde{h}"] & & X \arrow[ll, "i_1"] \arrow[ul, dashed, "\tilde{g}"]
\end{tikzcd}$$

This is a massive elaboration of the idea of a Hurewicz cofibration. The best way to understand this is to prove it (though it's not an easy proof).

In the non-equivariant case, one reduces to working one cell at a time, inductively extending over the cells of X not in A.[7] In this case, look at $S^{n-1} \subseteq D^n$. Now you just do it: at this point, there's no way to avoid writing down explicit homotopies.

Exercise 1.2.12. Think about this argument, and then read the proof in [**May99**].

The equivariant case is very similar: in the same way, one can reduce to inductively attaching a single cell in the case where X is a finite CW complex. This comes via a map $G/H \times S^{n-1} \to G/H \times D^n$, but the only interesting content is in the nonequivariant part, so we can reduce again to $S^{n-1} \to D^n$ with trivial G-action! This allows us to finish the proof in the same way. It also says that the homotopy theory of G-spaces is lifted from ordinary homotopy theory, in a sense that model categories will allow us to make precise.

The first consequence of Theorem 1.2.11 is:

Theorem 1.2.13. *Let $e\colon Y \to Z$ be a θ-connected map and $e_*\colon [X, Y] \to [X, Z]$ be the map induced by composition.*

- *If X has dimension less than θ, e_* is a bijection.*[8]
- *If X has dimension θ, e_* is a surjection.*

The proof is an exercise; filling in the details is a great way to get your hands on what the HELP lemma is actually doing. Hint: consider the pairs $\varnothing \to X$ and $X \times S^0 \to X \times I$, and apply the HELP lemma.

Corollary 1.2.14 (Equivariant Whitehead theorem)**.** *Let $e\colon Y \to Z$ be a weak equivalence of G-CW complexes. Then, e is a G-homotopy equivalence.*

---

[7] This requires reducing to the case where X is a finite CW complex, but taking a sequential colimit recovers the theorem for all CW complexes X.

[8] We say that X has dimension less than θ if for all closed subgroups $H \subset G$, all cells of orbit type G/H have (nonequivariant) dimension at most n for some $n \leq \theta(H)$.

PROOF. This is also a standard argument: using Theorem 1.2.13, e_* is a bijection, so we can pull back $\mathrm{id}_Z \in [Z, Z]$ to an inverse $(e_*)^{-1}(\mathrm{id}_Z) \in [Z, Y]$, which is a homotopy inverse to e. ⊠

One can continue and prove the cellular approximation theorem in this way, and so forth. We won't do this, because we'll approach it from a model-categorical perspective.

One thing that's useful, not so much for these notes as for enriching your life, is to learn how to approach this from the perspective of abstract homotopy theory, learning about disc complexes and so forth. You can prove theorems such as the HELP lemma and its consequences in a general setting, and then specialize them to the cases you need. This is a great way to "just do it" without needing model categories.

Anyways, we'll now define a model structure on $G\mathsf{Top}$ and $G\mathsf{Top}_*$. If you don't know what a model category is, now is a good time to review.

Proposition 1.2.15. *There is a model structure on* $G\mathsf{Top}$ *(and on* $G\mathsf{Top}_*$*) defined by the following data.*

Fibrations: *The maps* $f : X \to Y$ *such that for all* $H \subset G$, $f^H : X^H \to Y^H$ *is a fibration.*
Weak equivalences: *The maps* $f : X \to Y$ *such that for all* $H \subset G$, $f^H : X^H \to Y^H$ *is a weak equivalence.*

So we once again parametrize everything over subgroups of G and use fixed points. This is a cofibrantly generated model category; the cofibrations are specified by generators of acyclic cofibrations in a similar manner to Top. That is, in Top, one can choose generators $I = \{S^{n-1} \to D^n\}$ and $J = \{D^n \to D^n \times I\}$; in $G\mathsf{Top}$, we instead take $I_G = \{G/H \times I\}$ and $J_G = \{G/H \times J\}$.

These are cells that we used to define G-CW complexes, and this is no coincidence: it's a general fact about cofibrantly generated model categories that follows from the small object argument[9] that cofibrant objects are retracts of "cell complexes" built from the things in I, and cofibrations are retracts of cellular inclusions of cell complexes. In this sense, CW complexes are inevitable.

The Whitehead theorem (Corollary 1.2.14) now falls out of the general theory of model categories.

Theorem 1.2.16 (Whitehead theorem for model categories)**.** *Let* $f : X \to Y$ *be a weak equivalence of cofibrant-fibrant objects in a model category. Then,* f *is a homotopy equivalence.*

In Top and $G\mathsf{Top}$, all objects are fibrant, so this is particularly applicable.

1.3. Elmendorf's theorem

> *"What's bad about this proof?"*
> *"It appeals to machinery we didn't develop in this class?"*
> *"No, that's perfectly fine."*

In this section, we take up Elmendorf's theorem, which provides another model for $G\mathsf{Top}$ as Top-valued presheaves on a category called the orbit category. Its proof gives us an opportunity to discuss the bar construction, which is ubiquitous in homotopy theory.

Definition 1.3.1. The **orbit category** $\mathscr{O}_G$ is the full subcategory of $G\mathsf{Top}$ on the objects G/H.

That is, its objects are the spaces G/H, where $H \subset G$ is closed, and its morphisms are $\mathrm{Map}^G(G/H, G/K) \cong (G/K)^H$. These maps are the same thing as subconjugacy relations, i.e. those of the form

$$gHg^{-1} \subseteq K, \tag{1.3.2}$$

since for all $h \in H, h(gK) = gK$ if and only if $K = g^{-1}hgK$ if and only if $gHg^{-1} \subseteq K$. A G-map $f : G/H \to G/K$ is completely specified by what it does to the identity coset $f(eH) = gK$, and this g implies the subconjugacy relation (1.3.2), since, as above, $h(gK) = gK$ for all $h \in H$.

There's another description of the orbit category.

Proposition 1.3.3. *Let G be a finite group. Then, the orbit category* $\mathscr{O}_G$ *is equivalent to the category of finite transitive G-sets and G-maps.*

The observation that ignites the proof is that if $x \in X$ has isotropy group H, then its orbit space is isomorphic to G/H.

---

[9] The small object argument is a beautiful piece of basic mathematics that everybody should know. If you don't know it, your homework is to read enough about model categories to get to that point. In general, there may be large objects and transfinite induction, but for the case we care about large cardinals won't arise.

Definition 1.3.4. Given a G-space X, we obtain a presheaf on the orbit category, namely a functor $X^{(-)}\colon \mathscr{O}_G^{\mathsf{op}} \to \mathsf{Top}$, by sending $G/H \to X^H$. This assignment itself is a functor $\psi\colon G\mathsf{Top} \to \mathsf{Fun}(\mathscr{O}_G^{\mathsf{op}}, \mathsf{Top})$.

Proposition 1.3.5. $\mathsf{Fun}(\mathscr{O}_G^{\mathsf{op}}, \mathsf{Top})$ *has a projective model structure where the weak equivalences and fibrations are taken pointwise.*

The point is the following result, a revisionist interpretation of Elmendorf's theorem. Elmendorf's original proof [**Elm83**] showed these two categories have the same homotopy theory, but his proof was more explicit and did not use model categories.

Theorem 1.3.6 (Elmendorf [**Elm83**, **Ste16**])**.** *ψ is the right adjoint in a Quillen equivalence; the left adjoint θ is evaluation at G/e.*

That is, these two model categories have the same homotopy theory.

Exercise 1.3.7. Check that evaluation at G/e is a left adjoint to ψ.

It's also possible to state Elmendorf's theorem in a more general form.

Theorem 1.3.8 (Elmendorf)**.** *The functor $G\mathsf{Top} \to \mathsf{Fun}(\mathscr{O}_G^{\mathsf{op}}, \mathsf{Top})$ determined by $X \mapsto \big(G/H \mapsto X^H\big)$ induces an equivalence of $(\infty, 1)$-categories, where the weak equivalences on the left and right are specified by a family $\mathscr{F}$.*

Without delving into $(\infty, 1)$-categories, this means

- the homotopy categories are equivalent, and
- homotopy limits and colimits behave identically.

In other words, from the perspective of abstract homotopy theory, these are the same.

Let X be a finite G-set. Then, X is the coproduct (disjoint union) of a bunch of orbits:

$$X \cong \coprod_i G/H_i.$$

The way you see this is that for any $x \in X$, its orbit is isomorphic to G/G_x. This is yet another manifestation of the slogan that "orbits are points." But it also implies that, rather than just presheaves on $\mathscr{O}_G$, one could work with certain presheaves on the category of finite G-sets, and this perspective will turn out to be useful. By "certain" we mean a compatibility with orbits.

Definition 1.3.9. By a **family** of subgroups $\mathscr{F}$ of G, we mean a collection of subgroups of G closed under conjugation and taking subgroups.

Examples include the set of all subgroups, the set of just the identity, and the set of finite subgroups. The latter is useful for some S^1-equivariant spaces, where one tends to lose control of the S^1-fixed points, but the finite subgroups behave better.

Definition 1.3.10. Let $\mathscr{F}$ be a specified family of subgroups of G.

- In $G\mathsf{Top}$, the weak equivalences specified by $\mathscr{F}$ are the maps $f : X \to Y$ such that $f^H : X^H \to Y^H$ is a weak equivalence for all $H \in \mathscr{F}$.
- For $\mathsf{Fun}(\mathscr{O}_G^{\mathsf{op}}, \mathsf{Top})$, a weak equivalence specified by $\mathscr{F}$ is a pointwise weak equivalence at G/H for all $H \in \mathscr{F}$.

We'll give two proofs of Theorem 1.3.8. The first will be model-categorical.

Recall[10] if $F\colon \mathsf{C} \underset{\perp}{\rightleftarrows} \mathsf{D} : G$ is a Quillen adjunction, then the left and right derived functors $(\mathbf{L}F, \mathbf{R}G)$ is an adjunction on the homotopy categories $(\operatorname{Ho}\mathsf{C}, \operatorname{Ho}\mathsf{D})$. If K denotes fibrant replacement in D and Q denotes cofibrant replacement in C, then the derived functors are $\mathbf{L}F = FQ$ and $\mathbf{R}G = GK$.[11]

Definition 1.3.11. That (F, G) is a **Quillen equivalence** means that for any cofibrant $X \in \mathsf{C}$ and fibrant $Y \in \mathsf{D}$, then $FX \to Y$ is a weak equivalence iff its adjoint $X \to GY$ is.

---

[10] If this is not review to you, then exercise: learn this material!

[11] This does require cofibrant and fibrant replacement to be functorial, which is not true in every model category, but will be true for pretty much everything we study.

This is equivalent to asking that $(\mathbf{L}F, \mathbf{R}G)$ are equivalences of categories.

This is a kind of curious way to look at an equivalence of categories. One says that $G \colon \mathsf{D} \to \mathsf{C}$ **creates the weak equivalences** of D if for every morphism f of D, f is a weak equivalence iff Gf is.

Lemma 1.3.12. *If G creates the weak equivalences of D and for all cofibrant X the unit map $X \to GFX$ is a weak equivalence, then (F, G) is a Quillen equivalence.*

This is a useful tool for extending model categories along free-forgetful adjunctions; for example, if you have a model category and want to understand abelian group or ring objects in this category, often their weak equivalences are detected by the forgetful functor.

PROOF SKETCH OF THEOREM 1.3.8. We want to apply Lemma 1.3.12 to the adjunction

$$\begin{tikzcd} \mathsf{Fun}(\mathscr{O}_G^{\mathsf{op}}, \mathsf{Top}) \arrow[r, shift left, "\theta"] \arrow[r, phantom, "\perp"] & G\mathsf{Top}, \arrow[l, shift left, "\psi"] \end{tikzcd}$$

where $\theta : X \mapsto X(G/e)$ is evaluation at G/e and $\psi \colon Y \mapsto \{Y^H\}$. The first condition, that ψ detects the weak equivalences, is straightforward, so we need to check that $X \mapsto \{X(G/e)^H\}$ is a weak equivalence for all cofibrant X.

Cellular objects model the generating cofibrations, so cofibrant objects are retracts of cellular objects. Since weak equivalences are preserved under retracts, then we can check on cellular objects. Here it's easier, since $(-)^H$ commutes with the relevant colimits and is suitably cellular. ⊠

The missing steps in this proofs can be filled in by explicitly identifying the cofibrant objects in $\mathsf{Fun}(\mathscr{O}_G^{\mathsf{op}}, \mathsf{Top})$. These are free diagrams on the orbit category; not hard to write down, but messy enough to avoid on the chalkboard.

REMARK. Elmendorf's original proof of his theorem was in the 1980s and did not use model categories, even though Quillen had already introduced them at the time. Until the mid-1990s (30 years after [CITE ME: "Homotopical algebra"]), many homotopy theorists avoided them, thinking of them as formal gobbledygook. However, about the time EKMM introduced a symmetric monoidal category of spectra, people began realizing they were unavoidable. ◂

You might not like the given proof of Elmendorf's theorem because it's extremely inexplicit: cofibrant replacement is an infinite process, and many of the steps involved are quite abstract. The next proof will be more explicit, building a (homotopical) right adjoint to ψ.

This proof will go through the **bar construction**, a categorical tool that's extremely useful. References for it include May's "Geometry of iterated loop spaces" [May72], Riehl's monograph [Rie14], and Vogt's "Tensor products of functors."

SECOND PROOF OF THEOREM 1.3.8. Let $M \colon \mathscr{O}_G \to \mathsf{Top}$ realize orbits as spaces: G/H is sent to the topological space G/H, and an equivariant map f is forgotten to a continuous map f.

Given an $X \in \mathsf{Fun}(\mathscr{O}_G^{\mathsf{op}}, \mathsf{Top})$, let

$$\Phi(X) := |B_\bullet(X, \mathscr{O}_G, M)|$$

denote the geometric realization of the simplicial bar construction. Let's be a little more explicit about this. $B_\bullet(X, \mathscr{O}_G, M)$ is a simplicial space that sends

$$[n] \longmapsto \coprod_{G/H_{n-1} \to \cdots \to G/H_0} X(G/H_0) \times M(G/H_{n-1}).$$

As usual, the face maps are defined by composition, and the degeneracies by inserting the identity map. Since G acts on $M(-)$ simplicially (i.e., in a way compatible with the face and degeneracy maps), then $|B_\bullet(X, \mathscr{O}_G, M)|$ is a G-space (passing through the coend formula for the geometric realization).

If $H \subseteq G$, we want to understand $\Phi(X)^H$. Because the G-action passed through geometric realization,

$$\Phi(X)^H \cong |B_\bullet(X, \mathscr{O}_G, M^H)| \cong |B_\bullet(X, \mathscr{O}_G, \mathrm{Map}_{\mathscr{O}_G}(G/H, -))|.$$

Let $X(G/H)$ denote the constant simplicial space $[n] \mapsto X(G/H)$. Then, by general theory of the bar construction for any corepresented functor, there's a simplicial map

$$B_\bullet(X, \mathscr{O}_G, \mathrm{Map}_{\mathscr{O}_G}(G/H, -)) \longrightarrow X(G/H) \tag{1.3.13}$$

defined by composing and applying X, and this is a simplicial homotopy equivalence (you can write down a retraction).[12] Thus, $\Phi(X)^H \cong X(G/H)$. In other words, Φ is a homotopy inverse, since taking H-fixed points of $\Phi(X)$ gives back what you started with. ⊠

$\Phi(X)$ is still an infinite-dimensional object, but it's much more explicit, and you can work with it.

Applications of this perspective. We'll be able to use Elmendorf's theorem to make some constructions that would be hard to imagine without the orbit category.

Definition 1.3.14. Let $\mathscr{F}$ be a family of subgroups of G. Then, the **classifying space!for a family of subgroups** for $\mathscr{F}$ is specified by the universal property that if Z has $\mathscr{F}$-isotropy, then $[Z, E\mathscr{F}]$ has a unique element. An explicit construction is to let $\widetilde{E}\mathscr{F}$ denote the presheaf on the orbit category where

$$\widetilde{E}\mathscr{F}(G/H) := \begin{cases} *, & H \in \mathscr{F} \\ \varnothing, & H \notin \mathscr{F}, \end{cases}$$

and let $E\mathscr{F} := \Phi(\widetilde{E}\mathscr{F})$.

If you unwind the definition, this is the bar construction applied to G in the category of G-spaces with weak equivalences given by $\mathscr{F}$, meaning it deserves to be called a classifying space.

REMARK. One use for $E\mathscr{F}$ is for when you want to focus attention on a family of subgroups. One common example is S^1-spaces, in which there are many constructions that are fixed by the finite subgroups of S^1, so having that family $\mathscr{F}$ is helpful, and for this one can smash with $E\mathscr{F}$.

There are various other applications. One is called **isotropy separation**, which splits up $\mathscr{F}$ into pieces that can be detected with different kinds of isotropy subgroups, and one can induct on this in nice cases. ◀

Another useful notion is the G-connected components.

Definition 1.3.15. Let X be a G-space and $x \in X^G$. Let Y_x be the presheaf on the orbit category sending H to the connected component containing $x \in X^H$. Then, the **G-connected component** of x is $\Phi(Y_x)$.

The third useful application is defining Eilenberg-Mac Lane spaces. This will lead us to cohomology (and then to Smith theory and other things).

REMARK. Another application of Elmendorf's theorem, which we will not discuss in detail (unless we get to the slice filtration), is Postnikov towers. They're constructed in the same way, by either using the small object argument or killing homotopy groups. ◀

1.4. Bredon cohomology

"Smith's theorem was proven by Smith, hence the name."

Another consequence of Elmendorf's theorem (Theorem 1.3.6) is that presheaves on the orbit category that are valued in abelian groups are Eilenberg-Mac Lane G-spaces. Homotopy classes of maps into these G-spaces gives what's known as Bredon cohomology; we'll introduce this and compute several examples.

Definition 1.4.1. Let G be a finite group, A **coefficient system** is a presheaf $X \in \mathsf{Fun}(\mathscr{O}_G^{\mathsf{op}}, \mathsf{Ab})$

Elmendorf's theorem says that for any coefficient system, we have an Eilenberg-Mac Lane G-space. You could say here that (Bredon) cohomology is completely determined: cohomology is the things represented by Eilenberg-Mac Lane spaces. But it will be good to see it explicitly. Bredon cohomology is explicit, but there are serious drawbacks: it has poor formal properties, and you need a lot of geometric insight to compute things. We'll later see that this abelian category (meaning we can do homological algebra) is the wrong one; it produces a $\mathbb{Z}$-graded cohomology theory (or rather, one graded on subgroups of $\mathbb{Z}$); this will be the wrong answer, especially if you want Poincaré duality, and the right answer uses a grading by the representation ring. But we'll get there.

---

[12] This is called an **extra degeneracy argument** in the literature. There's an observation probably due to John Moore which approximately says that if you have a simplicial object with an extra degeneracy condition playing well with the preexisting ones, then it must be contractible; this argument is applied to the fiber of (1.3.13).

Remark. If G is a compact Lie group, the proper definition of a coefficient system is an Ab-valued presheaf on $h\mathscr{O}_G$, the homotopy category of the orbit category. (For finite groups these definitions coincide.) In other words, given G a compact Lie group and a family of subgroups $\mathscr{F}$ of G, the homotopy orbit category $h\mathscr{O}_G$ has objects G/H, where $H \in \mathscr{F}$, and has morphisms the homotopy classes of maps $G/H \to G/K$. ⋖

Example 1.4.2. Let us consider $h\mathscr{O}_{S^1}$, the homotopy orbit category for S^1 with $\mathscr{F}$ the family of finite subgroups of S^1.

We first consider the orbit category $\mathscr{O}_{S^1}$. Since S^1 is abelian, we have a morphism $G/H \to G/K$ if and only if $H \subseteq K$. In fact,

$$\mathrm{Hom}_{\mathscr{O}_{S^1}}(G/H, G/K) = \begin{cases} G/K, & \text{if } H \subseteq K \\ 0, & \text{otherwise.} \end{cases}$$

The homotopy orbit category $h\mathscr{O}_{S^1}$ is not too much more difficult: since S^1 is connected, if $H \subseteq K$, then all maps $G/H \to G/K$ are homotopic (they are maps of circles of the same degree). Hence, we have

$$\mathrm{Hom}_{h\mathscr{O}_{S^1}}(G/H, G/K) = \begin{cases} 1, & \text{if } H \subseteq K \\ 0, & \text{otherwise.} \end{cases}$$ ⋖

Coefficient systems have an important role in Bredon cohomology, which is calculated with coefficients in a coefficient system, and whose construction makes use of a chain complex of coefficient systems. In this way, coefficient systems play the part in Bredon cohomology that abelian groups do in CW cohomology. Indeed, coefficient systems form an abelian category; it may be helpful to think of it as the category of "right $\mathscr{O}_G$-modules," even if that isn't literally true.

Here are some examples of coefficient systems (which are often denoted with underlines).

Example 1.4.3.

(1) $\underline{\mathbb{Z}}$ will denote the **constant coefficient system** with coefficients in $\mathbb{Z}$, i.e. the functor which sends all objects to $\mathbb{Z}$ and all morphisms to $\mathrm{id}_{\mathbb{Z}}$. You can replace $\mathbb{Z}$ with your favorite abelian group.

(2) For a G-space X, the coefficient system $\underline{\pi}_n(X)$ ($n \geq 2$) sends $G/H \mapsto \{\pi_n X^H\}$. This is an example of a general formula: given a functor Top → Ab, we can compose to obtain a map $\mathsf{Fun}(\mathscr{O}_G^{\mathrm{op}}, \mathsf{Top}) \to \mathsf{Fun}(\mathscr{O}_G^{\mathrm{op}}, \mathsf{Ab})$.

(3) In the same way, $\underline{H}_n(X)$ sends $G/H \mapsto \{H_n(X^H; \mathbb{Z})\}$. ⋖

We will now define Bredon cohomology, due to Bredon [Bre67], which is the analogue of CW cohomology for G-CW complexes.

Definition 1.4.4. We first define a chain complex of coefficient systems $\underline{C}_\bullet(X)$, which is the analogue of the CW chain complex. It sends an orbit G/H to the CW chains of X^H.[13]

Let X be a G-CW complex and X_n denote its n-skeleton. Let

$$\underline{C}_n(X) := \underline{H}_n(X_n, X_{n-1}; \mathbb{Z}),$$

i.e. the coefficient system sending $G/H \mapsto H_n((X^H)_n, (X^H)_{n-1}; \mathbb{Z}) = C_n^{\mathrm{CW}}(X^H)$. The differential at G/H is the CW chain complex differential for X^H, i.e. the connecting morphism in the long exact sequence of the triple $((X^H)_n, (X^H)_{n-1}, (X^H)_{n-2})$. One should check that this commutes with the bonding maps for $\underline{C}_n(X)$, but it does, so this works.

Using this chain complex, we define Bredon homology and cohomology.

Definition 1.4.5.

- The **Bredon cohomology** with coefficients in a coefficient system M is

$$H_G^n(X; M) := H^n(\mathrm{Hom}_{\mathsf{Fun}(\mathscr{O}_G^{\mathrm{op}}, \mathsf{Ab})}(\underline{C}_*(X), M)).$$

  That is, we take the chains on X and compute the maps into M; since $\mathsf{Fun}(\mathscr{O}_G^{\mathrm{op}}, \mathsf{Ab})$ is an abelian category, this is a cochain complex of abelian groups, and we can take its homology to obtain a sequence of abelian groups.

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[13] This requires knowing how to obtain a CW structure on X^H given a G-CW structure on X. If G is finite, this is easy to see; for general compact Lie groups, though, this requires a triangulation argument. One wants the resulting coefficient system to be independent of the choice of triangulation, but as in the nonequivariant case, this is proven via an axiomatic characterization of cohomology.

- For homology to be a covariant functor, we need the coefficient system M to be a functor $\mathscr{O}_G \to \mathsf{Ab}$ rather than $\mathscr{O}_G^{\mathrm{op}} \to \mathsf{Ab}$ (e.g. $\underline{H}^*(X)$ for a G-space X, which sends $G/H \mapsto \{H^n(X^H)\}$).[14] With M such a coefficient system, the **Bredon homology** with coefficients in M is
$$H_n^G(X;M) := H_n(\underline{C}_n(X) \otimes_{\mathscr{O}_G} M).$$
By this tensor product, we mean a coend:
$$\begin{aligned}\underline{C}_n(X) \otimes_{\mathscr{O}_G} M &= \int^{G/H \in \mathscr{O}_G} \underline{C}_n(X)(G/H) \otimes M(G/H)\\ &= \bigoplus_{G/H} \underline{C}_n(X)(G/H) \otimes M(G/H)\,/\sim,\end{aligned}$$
where if $f \in \mathrm{Map}_{\mathscr{O}_G}(G/H, G/K)$, $(f^*y, z) \sim (y, f_*z)$.[15]

The whole philosophy of Bredon (co)homology is to understand equivariant cohomology and homology through the fixed-point sets and the lattice of subgroups of G.

Now we'll compute some simple examples of Bredon homology and cohomology over C_2. Its orbit category is simple:

(1.4.6)
$$\begin{tikzcd} C_2/e \arrow[loop above] \arrow[d] \\ C_2/C_2. \end{tikzcd}$$

The map $C_2/e \to C_2/C_2$ crushes C_2/e to a point, and the map $C_2/e \to C_2/e$ exchanges the two points.

Example 1.4.7. Let C_2 act on S^2 by rotation by π. We'll compute the Bredon cohomology of this C_2-space with coefficients in $\underline{\mathbb{Z}}$.

Exercise 1.4.8. If $\sigma\colon C_2 \to \mathrm{GL}_1(\mathbb{R})$ denotes the sign representation, verify this C_2-action on S^2 makes it into the representation sphere $S^{2\sigma}$.

To compute the Bredon cohomology of $S^{2\sigma}$, we first need a C_2-CW structure on it. We've already computed one in Example 1.2.4: the two fixed points are two 0-cells $C_2/C_2 \times D^0$, a great circle through them is the single 1-cell $C_2/e \times D^1$, and the two hemispheres are the single 2-cell $C_2/e \times D^2$. See Figure 3 for a picture.

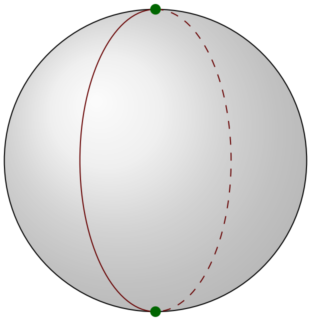

FIGURE 3. A C_2-CW structure on $S^{2\sigma}$, the 2-sphere with a C_2-action by rotation through 180°. The two green dots are the two 0-cells $C_2/C_2 \times D^0$; the red circle is the single 1-cell $C_2/e \times D^1$, and the gray hemispheres are the single 2-cell $C_2/e \times D^2$.

Next we compute $\underline{C}_*(S^{2\sigma})$. A coefficient system is determined by a map $M_1 \to M_2$ and an involution on M_2. In our case, $\underline{C}_k(X)(C_2/e) = \mathbb{Z} \oplus \mathbb{Z}$ for $k = 0, 1, 2$, and the involution flips the two factors. Therefore we obtain a

---

[14] Would it be appropriate to call such a functor an "efficient system?"

[15] Tensor products are particular instances of coends; instead of inducing an equivalence $mr \otimes n \sim m \otimes rn$, you flip a map across the two objects. One might write $y \cdot f$ for f^*y and $f \cdot z$ for f_*z to emphasize this point of view.

chain complex of coefficient systems which is

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "f_1"] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "f_2"] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

at C_2/e and

(1.4.9)
$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

at C_2/C_2. Next we need to determine the differentials. By definition, these are determined by the cellular boundary maps, i.e. the attaching maps. Let (x, y) denote the standard basis of $\mathbb{Z}\oplus\mathbb{Z}$.

- f_1 sends $(x, y) \mapsto (x + y, x + y)$.
- f_2 sends $(x, y) \mapsto (x - y, y - x)$,

Next, we compute the Hom from this coefficient system to the constant coefficient system $\underline{\mathbb{Z}}$. These groups are evidently determined by what happens at the C_2/e spot. For the first two terms, we are looking at maps $\mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}$ that are equivariant (where $g \in C_2$ acts by transposition on the left and trivially on the right). Such a map is determined by where it sends $(1, 0)$, and so these two terms are isomorphic to $\mathbb{Z}$. The last term is simply all homomorphisms $\mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z}$, which is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$. Therefore, we have the cochain complex

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & 0. \end{tikzcd}$$

Computing the differentials, we find that the first map is $(a, b) \mapsto a + b$ and the second map is 0.

Taking the cohomology of this complex then produces

$$H^n(S^{2\sigma}; \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z}, & n = 0, 2 \\ 0, & \text{otherwise.} \end{cases}$$

Now, we'll compute the Bredon homology of $S^{2\sigma}$, again with coefficients in the constant functor valued in $\mathbb{Z}$. First we need to compute the tensor product $\underline{C}_\bullet(X) \otimes_{\mathscr{O}_{C_2}} \underline{\mathbb{Z}}$:

$$\begin{tikzcd} \mathbb{Z}^2 & \mathbb{Z}^2/\langle[1,-1]\rangle \arrow[l, "\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)"'] & \mathbb{Z}^2/\langle[1,-1]\rangle. \arrow[l, "0"'] \end{tikzcd}$$

Hence the Bredon homology of $S^{2\sigma}$ is

$$H_k^{C_2}(S^{2\sigma}; \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z}^2, & k = 0 \\ \mathbb{Z}, & k = 2 \\ 0, & \text{otherwise.} \end{cases}$$

TODO: shouldn't it be $\mathbb{Z}$, 0, $\mathbb{Z}$? ◁

Remark. We're used to reading off properties of a space from its cohomology, and this is still true here, if harder. For example, H^0 tells us the number of connected components of S^2/C_2. ◁

Exercise 1.4.10. Generalize Example 1.4.7 to $S^{n\sigma}$, the n-sphere with a C_2-action given by a half-turn.

Example 1.4.11. Let C_2 act on S^2 by the antipodal action.

Exercise 1.4.12. Identify this action with that of the unit sphere $S(3\sigma)$ in 3σ (i.e. a direct sum of three copies of the sign representation).

We get a C_2-CW structure of TODO. We'll again compute the Bredon cohomology of S^2 with coefficients in $\underline{\mathbb{Z}}$. The chain complex $\underline{C}_\bullet(S(3\sigma))$ is

$$\begin{tikzcd} \mathbb{Z}^2 \arrow[loop above, "\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)"] & \mathbb{Z}^2 \arrow[l, "\left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)"'] \arrow[loop above, "\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)"] & \mathbb{Z}^2 \arrow[l, "\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)"'] \arrow[loop above, "\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)"] \\ 0 \arrow[u] & 0 \arrow[l] \arrow[u] & 0. \arrow[l] \arrow[u] \end{tikzcd}$$

Now we need to take $\mathrm{Hom}(-,\underline{\mathbb{Z}})$. In each degree we get a one-dimensional space of maps generated by $(1,1)$. The differential $\partial\colon C^0 \to C^1$ sends

$$(1,1) \longmapsto (1,1)\circ\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix},$$

and the differential $\partial\colon C^1 \to C^2$ sends

$$(1,1) \longmapsto (1,1)\circ\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} = \begin{pmatrix}2\\ 2\end{pmatrix}.$$

Hence the cochain complex is

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z},$$

so the Bredon cohomology is

$$H^k_{C_2}(S(3\sigma);\underline{\mathbb{Z}}) = \begin{cases}\mathbb{Z}, & k=0\\ \mathbb{Z}/2, & k=1\\ 0, & \text{otherwise,}\end{cases}$$

which is exactly the cohomology of $\mathbb{RP}^2 = S(3\sigma)/C_2$. $\triangleleft$

This is no coincidence: if G acts freely on X, then the Bredon cohomology and homology is that of the quotient.

Exercise 1.4.13. Let G act freely on a CW complex X, let $M\colon \mathscr{O}_G^{\mathrm{op}} \to \mathsf{Ab}$, and let $N\colon \mathscr{O}_G \to \mathsf{Ab}$. Show that

$$H^*_G(X;M) \cong H^*(X/G;M(G/e)) \qquad \text{and} \qquad H^G_*(X;N) \cong H_*(X/G;N(G/e)).$$

Corollary 1.4.14. *If G acts on X trivially, $H^*_G(X;M) \cong H^*(X;M(e))$, and similarly for homology.*

Example 1.4.15. Let's generalize to C_n. Let C_n act on S^2 by rotation through an angle $2\pi/n$. If V denotes the 2-dimensional C_n-representation generated by a rotation through $2\pi/n$, then this C_n-space is the representation sphere S^V.

The C_2-CW structure on $S^{2\sigma}$ from Example 1.4.7 generalizes to the "beachball" C_n-CW structure on S^V:

- There are two 0-cells $C_n/C_n \times D^0$, which are the fixed points.
- There is a single 1-cell $C_n/e \times D^1$, which is n equally spaced meridians.
- There is a single 2-cell $C_n/e \times D^2$, which fills in the rest of the sphere.

We'll first compute Bredon cohomology with respect to the trivial family $\mathscr{F}_0$ of subgroups of C_n, i.e. just C_n and e. The orbit category looks similar to the one for C_2 (1.4.6), but there are more automorphisms of C_n/e: $\mathrm{Map}^{C_n}(C_n/e, C_n/e) \cong (C_n/e)^{\{e\}}$, so there are n of them. Namely, for $0 \le i < n$, let φ_i send $x \mapsto x + i \bmod n$; these are equivariant, so we've found them all. Thus the orbit category for the trivial family of subgroups of C_n is

$$\begin{tikzcd} C_n/e \arrow[loop above, "{\varphi_0,\dots,\varphi_{n-1}}"] \arrow[d] \\ C_n/C_n. \end{tikzcd}$$

Since φ_1 generates $\mathrm{Aut}_{\mathscr{O}_{C_n}}(C_n/C_n)$, we'll keep track of it and leave the rest of the φ_i implicit.

First, we calculate $\underline{C}_\bullet(S^V)$. At C_n/e, this is just the CW chains for S^2, but with the nonequivariant CW structure induced by forgetting the C_n-action on S^V. That is, there are two 0-cells, n 1-cells, and n 2-cells:

$$\mathbb{Z}^2 \longleftarrow \mathbb{Z}^n \longleftarrow \mathbb{Z}^n.$$

At C_n/C_n, this is the CW chains for the fixed points, which are an S^0:

$$\mathbb{Z}^2 \longleftarrow 0 \longleftarrow 0$$

just as in (1.4.9).

Next we should figure out the bonding maps. Since the fixed points are the 0-skeleton, the maps in $\underline{C}_0(S^V)$ are all the identity. In degrees 1 and 2, $\varphi_1^*\colon \mathbb{Z}^n \to \mathbb{Z}^n$ is the shift matrix sending the standard basis vector $e_i \mapsto e_{i+1 \bmod n}$.

Next the differentials. Let e_1 denote the north pole, e_2 denote the south pole, $f_1, \dots, f_n$ denote the 1-cells and $g_1, \dots, g_n$ denote the 2-cells. Orient the 1-cells as pointing north and the 2-cells as counterclockwise if the north pole is at the top.

- For $d \colon \underline{C}_1(S^V) \to \underline{C}_0(S^V)$, $d(f_i) = e_1 - e_2$ and thus is the matrix
$$A := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ -1 & -1 & \cdots & -1 \end{pmatrix}.$$
- For $d \colon \underline{C}_2(S^V) \to \underline{C}_1(S^V)$, $d(g_i) = f_{i+1 \bmod n} - f_i$, so its matrix is
$$B := \begin{pmatrix} -1 & & & 1 \\ 1 & -1 & & \\ & 1 & \ddots & \\ & & \ddots & \\ & & & -1 \end{pmatrix}.$$

Thus $\underline{C}_\bullet(X)$ is

$$\begin{tikzcd}
\mathbb{Z}^2 \arrow[loop above, "\mathrm{id}"] & \mathbb{Z}^n \arrow[l, "A"'] \arrow[loop above, "\varphi_1^*"] & \mathbb{Z}^n \arrow[l, "B"'] \arrow[loop above, "\varphi_1^*"] \\
\mathbb{Z}^2 \arrow[u, "\mathrm{id}"'] & 0 \arrow[l] \arrow[u] & 0. \arrow[l] \arrow[u]
\end{tikzcd} \tag{1.4.16}$$

Next we apply $\mathrm{Hom}(-, \underline{\mathbb{Z}})$. The cochain complex is

$$\mathbb{Z}^2 \xrightarrow{d^0} \mathbb{Z} \xrightarrow{d^1} \mathbb{Z},$$

and the differentials are:

- $d^0 \colon \mathbb{Z}^2 \to \mathbb{Z}$ is the matrix $\begin{pmatrix} 1 & -1 \end{pmatrix}$.
- $d^1 \colon \mathbb{Z} \to \mathbb{Z}$ is the zero map.

Hence

$$H^k_{G;\mathscr{F}_0}(S^V) = \begin{cases} \mathbb{Z}, & k = 0, 2 \\ 0, & \text{otherwise.} \end{cases} \tag{1.4.17}$$

It would also be good to compute the Bredon cohomology with respect to the complete family, but depending on n, this could get complicated. If n is prime, the complete and trivial families coincide, so we're done. A slightly more interesting example is $n = p^2$, where p is prime. In this case the orbit category is

$$\begin{tikzcd}
C_{p^2}/e \arrow[loop right, "C_{p^2}"] \arrow[d] \\
C_{p^2}/C_p \arrow[loop right, "C_p"] \arrow[d] \\
C_{p^2}/C_{p^2}.
\end{tikzcd} \tag{1.4.18}$$

To compute $\underline{C}_\bullet(S^V)$, we need to know the H-CW structures on $(S^V)^H$, where H ranges over the subgroups of C_{p^2}.

- For $H = e$, this is the C_{p^2}-CW structure we started with: there are two fixed, 0-cells, one free 1-cell, and one free 2-cell.
- For $H = C_p$, $(S^V)^{C_p}$ is just the two poles, so there are two free 0-cells.
- For $H = C_{p^2}$, this is the ordinary CW structure on the poles: there are two 0-cells.

So the chain complex is not very different from (1.4.16):

$$\begin{tikzcd}
\mathbb{Z}^2 \arrow[loop above, "\mathrm{id}"] & \mathbb{Z}^{p^2} \arrow[l, "A"'] \arrow[loop above, "\varphi_1^*"] & \mathbb{Z}^{p^2} \arrow[l, "B"'] \arrow[loop above, "\varphi_1^*"] \\
\mathbb{Z}^2 \arrow[u, "\mathrm{id}"'] \arrow[loop left, "\mathrm{id}"] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] \\
\mathbb{Z}^2 \arrow[u, "\mathrm{id}"'] & 0 \arrow[l] \arrow[u] & 0. \arrow[l] \arrow[u]
\end{tikzcd}$$

Therefore the Bredon cohomology is the same as in (1.4.17). In fact, this generalizes to $n \neq p^2$, because for any $H \subset C_n$, $(S^V)^H$ is the two poles. The orbit category is fancier, but the Bredon cohomology is exactly the same. ◂

Example 1.4.19. Our next example will be a C_4-space with nontrivial C_2-fixed points. Namely, consider a star graph X with one central vertex, six peripheral vertices, and six edges as in Figure 4.

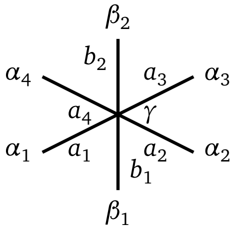

FIGURE 4. A star graph X which C_4 acts on by sending $\alpha_i \mapsto \alpha_{i+1 \bmod 4}$, and by $\beta_i \mapsto \beta_{i+1 \bmod 2}$. If we think of this as coordinate axes, C_4 acts by rotation on the xy-plane and reflection on the z-axis.

Let C_4 act by sending $\alpha_i \mapsto \alpha_{i+1 \bmod 4}$ and $\beta_i \mapsto \beta_{i+1 \bmod 2}$. That is, it reflects the βs and rotates the αs. Hence X has a C_4-CW structure as follows.

- A single C_4/C_4 0-cell at the center γ.
- A single C_4/C_2 0-cell $\{\beta_1, \beta_2\}$.
- A single C_4/e 0-cell $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.
- A single C_4/C_2 1-cell, which is the edges b_1 and b_2.
- A single C_4/e 1-cell, which is the edges $a_1, \dots, a_4$.

The orbit category is as in (1.4.18). Keep in mind that there are multiple maps $C_4/e \rightrightarrows C_4/C_2$, but they're all conjugates of each other under the C_4-action on C_4/e.

To compute $\underline{C}_\bullet(X)$, we look at the fixed points under $\{e\}$, C_2, and C_4.

- The fixed points under $\{e\}$ are of course the whole space, and so the CW chain complex is

$$\mathbb{Z} \cdot \{b_1, b_2, a_1, a_2, a_3, a_4\} \longrightarrow \mathbb{Z} \cdot \{\gamma, \beta_1, \beta_2, \alpha_1, \alpha_2, \alpha_3, \alpha_4\}$$

  (seven total 0-cells, and six total 1-cells).
- The C_2-fixed points is the line between β_1 and β_2, so its CW chain complex is

$$\mathbb{Z} \cdot \{b_1, b_2\} \longrightarrow \mathbb{Z} \cdot \{\gamma, \beta_1, \beta_2\}.$$

- The C_4-fixed points are just the center point, so the chain complex is $0 \to \mathbb{Z}$.

With bases for chain groups as above, the differentials are

$$d_{C_4} = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & & & & \\ 0 & 1 & & & & \\ & & 1 & 0 & 0 & 0 \\ & & 0 & 1 & 0 & 0 \\ & & 0 & 0 & 1 & 0 \\ & & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad d_{C_2} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

i.e. $\underline{C}_\bullet(X)$ is

$$\begin{tikzcd}
\mathbb{Z}^6 \arrow[loop left, "C_4"] & \mathbb{Z}^7 \arrow[l, "d_{C_4}"'] \arrow[loop right, "C_4"] \\
\mathbb{Z}^3 \arrow[u, hook] \arrow[loop left, "C_2"] & \mathbb{Z}^2 \arrow[l, "d_{C_2}"'] \arrow[u, hook] \arrow[loop right, "C_2"] \\
\mathbb{Z} \arrow[u, hook] & 0. \arrow[l] \arrow[u]
\end{tikzcd}$$

The C_4-actions in the top row are the regular C_4-representation on the subspace generated by the α_i or the edges they touch. Similarly, the C_2-actions in the middle row are the regular representations on the subspaces generated by the β_j and their edges. The inclusions in the diagram $\mathbb{Z}^k \hookrightarrow \mathbb{Z}^m$ are as the first k components.

Applying Hom(–, $\underline{\mathbb{Z}}$), we get a cochain complex

$$\mathbb{Z}^3 \xrightarrow{\left(\begin{smallmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{smallmatrix}\right)} \mathbb{Z}^2.$$

so the cohomology is that of a point, which is reassuring. ◁

Example 1.4.20. Let's compute Bredon cohomology with coefficients in a non-constant coefficient system. For C_2, this is the data of an abelian group A, an involution $\varphi : A \to A$, and a map $\mathbb{Z} \to A$ equivariant with respect to φ. For example, let's take $\mathbb{Z}[i]$ with φ acting by complex conjugation, and call this coefficient system M.

Let C_2 act on S^2 by reflection across a meridian m; this is the representation sphere $S^{1+\sigma}$. It has a C_2-CW structure as follows.

- One 0-cell e of the form $C_2/C_2 \times D^0$ at the north pole.
- One 1-cell f of the form $C_2/C_2 \times D^1$, which is the meridian m.
- One 2-cell g of the form $C_2/e \times D^2$, which is everything else.

For a picture, see Figure 5.

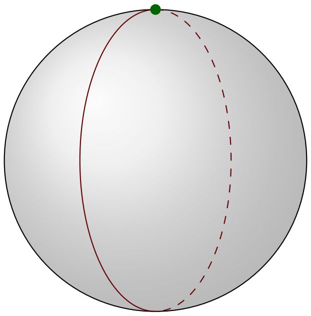

FIGURE 5. A C_2-CW structure for $S^{1+\sigma}$, the sphere where C_2 acts by reflection across a meridian. There's a single 0-cell $C_2/C_2 \times D^0$ (the green dot), a single 1-cell $C_2/C_2 \times D^1$ (the red circle), and a single 2-cell $C/2\varepsilon \times D^2$ (the gray hemispheres).

Let g_1 and g_2 denote the two nonequivariant 2-cells that make up g. Then, orient f and g such that $\partial f = e - e = 0$ and $\partial g_1 = \partial g_2 = f$.

Then, $\underline{C}_\bullet(S^{1+\sigma})$ is

$$\begin{tikzcd}
\mathbb{Z}\cdot e \arrow[loop above, "\mathrm{id}"] & \mathbb{Z}\cdot f \arrow[l, "0"'] \arrow[loop above, "\mathrm{id}"] & \mathbb{Z}\cdot\{g_1,g_2\} \arrow[l, "(1\ 1)"'] \arrow[loop above, "\mathrm{swap}"] \\
\mathbb{Z}\cdot e \arrow[u, "\mathrm{id}"'] & \mathbb{Z}\cdot f \arrow[l, "0"'] \arrow[u, "\mathrm{id}"'] & 0. \arrow[l] \arrow[u]
\end{tikzcd}$$

Now we map to M. We have two copies of $\mathrm{Hom}(\underline{\mathbb{Z}},M)=\mathbb{Z}$, and at degree 2 we get $\mathrm{Hom}_{\mathbb{Z}[C_2]}(\mathbb{Z}[C_2],\mathbb{Z}[i])$, which is free of rank 2. The cochain complex is

$$\mathbb{Z}\xrightarrow{0}\mathbb{Z}\xrightarrow{\text{rank }1}\mathbb{Z}^2,$$

so the Bredon cohomology is

$$H^k_{C_2}(S^{1+\sigma};M)=\begin{cases}\mathbb{Z}, & k=0,2\\ 0, & \text{otherwise.}\end{cases}$$ ⊲

We want cohomology to have nice formal properties analogous to the Eilenberg-Steenrod axioms. We'll think of H_G as a general G-equivariant cohomology theory of pairs $H^n_G(X,A;M)$, but in our case this will just be $H^n_G(X/A;M)$.

(1) H^*_G should be invariant under weak equivalences: a weak equivalence $X\to Y$ should induce an isomorphism $H^n_G(Y;M)\xrightarrow{\cong}H^n_G(X;M)$.

(2) Given a pair $A\subseteq X$, we get a long exact sequence

$$\cdots\longrightarrow H^n_G(X,A;M)\longrightarrow H^n_G(X;M)\longrightarrow H^n_G(A;M)\xrightarrow{\delta}H^{n+1}(X,A;M)\longrightarrow\cdots$$

(3) The excision axiom: if $X=A\cup B$, then

$$H^n_G(X/A;M)\cong H^n_G(B/(A\cap B);M).$$

(4) The Milnor axiom:

$$H^n_G\Big(\bigvee_i X_i;M\Big)=\prod_i H^n_G(X_i;M).$$

(5) Finally, for now we impose the dimension axiom: our points are orbits G/J, so we ask that $H^n(G/H;M)$ is concentrated in degree 0.

Some of these are easier than others: Bredon cohomology is manifestly homotopy-invariant in the same ways as ordinary cohomology, so invariance under weak equivalence and the Milnor axiom are immediate, and excision follows because if all spaces involved are CW complexes, $X/A\cong B/(A\cap B)$.

What takes more work is the dimension axiom and the long exact sequence. We'll show that $\underline{C}_n(X)$ is a projective object, and hitting projective objects with Hom produces a long exact sequence by homological algebra. (Recall that an object P in a category where you can do homological algebra is **projective** if $\mathrm{Hom}(P,-)$ is exact, which is equivalent to maps to P lifting across surjections $M\twoheadrightarrow P$.)

Proof of the dimension and long exact sequence axioms. Observe that $\underline{C}_n(X)$ splits as a direct sum of pieces $H_n(G/H_+\wedge S^n)\cong\widetilde{H}_0(G/H)$ indexed by the cells G/H of X. At G/K, $H_0(G/H)=\mathbb{Z}[\pi_0(G/H)^K]$. This is a free abelian group, and we'll directly use the lifting criterion to prove this is projective. That is, we'll write down an isomorphism

$$\varphi\colon \mathrm{Hom}_{\mathsf{Fun}(\mathscr{O}_G^{\mathrm{op}},\mathsf{Ab})}(\underline{H}_0(G/H),M)\xrightarrow{\cong}M(G/H).$$

This immediately proves $\underline{H}_0(G/H)$ is projective: evaluating a coefficient system on an exact sequence produces an exact sequence, and we've shown $\mathrm{Hom}(\underline{H}_0(G/H),-)$ is evaluation of a coefficient system.

The map φ takes a homomorphism θ and applies it to $\mathrm{id}_{G/H}$, which produces something in $M(G/H)$. Why is this an isomorphism? The Yoneda lemma is a fancy answer, but you can prove it in a more elementary manner.

Exercise 1.4.21. Calculate that any $\theta\in\mathrm{Hom}_{\mathsf{Fun}(\mathscr{O}_G^{\mathrm{op}},\mathsf{Ab})}(H_0(G/H),M)$ is determined by where the identity $\mathrm{id}\in\mathrm{Map}_{\mathscr{O}_G}(G/H,G/H)$ is sent, implying φ is an isomorphism.

Thus, we've effectively calculated the value at G/H, proving the dimension axiom as well. ⊠

The Eilenberg-Steenrod axioms hold for Bredon homology, and the proof is the same.

REMARK. Let's foreshadow a little bit. In ordinary homotopy theory, one can show that the Eilenberg-Steenrod axioms plus the value on points determine a cohomology theory, and this is still true in the equivariant case. But then one wonders about Brown representability and what happens when you remove the dimension axiom — and indeed there are lots of interesting examples of generalized equivariant cohomology theories. ◀

Exercise 1.4.22. We constructed a functor $\underline{H}\colon \mathsf{Fun}(\mathscr{O}_G^{\mathsf{op}}, \mathsf{Top}) \to \mathsf{Fun}(\mathscr{O}_G^{\mathsf{op}}, \mathsf{Ab})$. Show that $\underline{H}M$ represents $H_G^n(-;M)$.

This slick definition of $\underline{H}$ is one of the advantages of working with presheaves on the orbit category.

REMARK. You might also want to have a universal coefficient sequence, but it's more complicated. The short exact sequence in ordinary homotopy theory depends on the existence of short projective resolutions. Here, we have enough projectives and injectives, but resolutions are longer. Thus, taking an injective resolution of M and filtering the resulting double complex, one obtains a spectral sequence

$$\mathrm{Ext}^{p,q}(\underline{C}_*(X), M) \Longrightarrow H_G^{p+q}(X;M).$$

Warning: indexing might be slightly off.

There's a corresponding Tor spectral sequence. ◀

Since the category is more complicated, one expects to have to do more work. But sometimes there are nice results nonetheless; Smith theory, which we discuss in the next section, is an example.

1.5. Smith theory and the localization theorem

"It seems that the people registered for the class and the people showing up for class are disjoint."

Smith theory is an example application of Bredon cohomology. This theorem is very old, from the 1940s, so none of the cohomology in the statement is equivariant.

Theorem 1.5.1 (Smith)**.** *Let G be a finite p-group and X be a finite G-CW complex such that (the underlying topological space of) X is an $\mathbb{F}_p$-cohomology sphere.*[16] *Then, X^G is either empty or an $\mathbb{F}_p$-cohomology sphere of smaller dimension.*

There are sharper statements, but we can prove this one. It's the start of a long program to understand $H_*(X^G)$ using algebraic data calculated from X and the action of G on X. One useful tool in this is the **Borel construction** (well, *a* Borel construction) $EG \times_G X$ (where this is the usual balanced product, not a pullback). This is a "fattened up" version of X/G.

Definition 1.5.2. The **Borel cohomology** of X is $H_G^*(X) := H^*(EG \times_G X)$.

Warning! This notation is potentially confusing. Outside of the field of equivariant homotopy theory, "equivariant cohomology" generally means Borel cohomology, not Bredon cohomology. In equivariant homotopy theory, H_G^* can be used to denote both. We will always make the coefficient system for Bredon cohomology explicit, so that $H_G^*(X)$ unambiguously refers to Borel cohomology, as is standard in the literature.[17]

The finiteness in Theorem 1.5.1 is key: Elmendorf's theorem lets us build a C_p-complex with non-equivariant homotopy type S^n and any set of fixed points. Thus, in the infinite-dimensional case, we should be looking at a different thing than the fixed points, namely the homotopy fixed points.

Definition 1.5.3. The **homotopy fixed points** of a G-space X is $X^{hG} := \mathrm{Map}(EG, X)^G$.

This can also be defined as a homotopy limit. This is "easy" to compute (relative to the rest of equivariant homotopy theory, that is), as there are spectral sequences, nice models for EG, and other tools.

The terminal map $EG \to *$ induces a map $X^G \to X^{hG}$. The Sullivan conjecture is all about when this happens: when you p-complete, things become really nice. (There will be a precise statement next lecture.)

Returning to Theorem 1.5.1, we can use Bredon cohomology to give an easy, modern proof. The idea is to adroitly choose coefficient systems such that we recover $H^n(X)$, $H^n(X^G)$, and $H^n((X/G)/G)$ from Bredon cohomology. They'll fit into exact sequences, and using tools like Mayer-Vietoris, we'll get inequalities on the

---

[16] A space X is an $\mathbb{F}_p$**-cohomology sphere** if there is an isomorphism of graded abelian groups $H^*(X;\mathbb{F}_p) \cong H^*(S^n;\mathbb{F}_p)$ for some n.

[17] The nLab uses a different convention: Borel cohomology is $H_G^*(X;A)$, and Bredon cohomology is $\mathbf{H}^*(X_G)$.

ranks of groups. This will also use the fact that a short exact sequence of coefficient systems induces a long exact sequence on H_G^*. The proof will be beautiful and short, unlike Smith's original proof!

Theorem 1.5.1 is sufficiently classical that there are several different proofs. Ours illuminates Bredon cohomology at the expense of obscuring the overarching goal of Smith theory. We'll try to be consistent with the notation in [**May87**, **May96**]. We'll also discuss another proof by Dwyer and Wilkerson in "Smith theory revisited" [**DW88**], a short, beautiful paper which is highly recommended. It uses the unstable Steenrod algebra to prove Theorem 1.5.1 and more.

PROOF OF THEOREM 1.5.1. First, we can quickly reduce to the case where $G = \mathbb{Z}/p$: if $H \subset G$ is a normal subgroup, then $X^G \cong (X^H)^{G/H}$, so you can induct on the order of the group using the Sylow theorems. Thus, we will assume $G = \mathbb{Z}/p$, whose orbit category is simple:

$$\begin{tikzcd}
\bullet \arrow[loop above, "g"] \arrow[d] \\
\bullet
\end{tikzcd}$$

There is a cofiber sequence

$$\begin{tikzcd}
X^G_+ \arrow[r] & X_+ \arrow[r] & X/X^G,
\end{tikzcd}$$

which is not particularly deep. We're going to construct three special coefficient systems L, M, and N such that

$$\begin{aligned}
H_G^*(X;L) &\cong \widetilde{H}^*((X/X^G)/G;\mathbb{F}_p)\\
H_G^*(X;M) &\cong H^*(X;\mathbb{F}_p)\\
H_G^*(X;N) &\cong H^*(X^G;\mathbb{F}_p).
\end{aligned}$$

These will fit into exact sequences which will imply the inequalities we wanted.[18]

How do we construct custom coefficient systems? Since these constructions commute with colimits, it suffices to determine them via computation at $n = 0$ and $X = G/H$ for subgroups $H \subset G$, meaning just e and G.

For L, we want to recover $\widetilde{H}^0((X/X^G)/G;\mathbb{F}_p)$ for $X = G/e$ and $X = G/G$.

- For $X = G/e$, we want $\widetilde{H}^0((G/\varnothing)/G;\mathbb{F}_p) \cong \widetilde{H}^0(G_+/G) \cong \mathbb{F}_p$.
- For $X = G/G$, we get $\widetilde{H}^0((*/*)/G;\mathbb{F}_p) = 0$.

So we conclude $L(G/e) = \mathbb{F}_p$ and $L(G/G) = 0$.

For M, a similar calculation shows we need $H^0(G/e) \cong \mathbb{F}_p[G]$[19] and $H^0(G/G) = \mathbb{F}_p$, and for N, we need $N(G/e) = 0$ and $N(G/G) \cong \mathbb{F}_p$.

REMARK. Almost everything in this proof generalizes; we will only need X to be an $\mathbb{F}_p$-cohomology sphere in order to know dimensions of a few things. But this technique of customized coefficient systems can be used elsewhere. ◀

Let I denote the **augmentation ideal** of $\mathbb{F}_p[G]$, i.e. the kernel of the map $\mathbb{F}_p[G] \to \mathbb{F}_p$ sending all $g \mapsto 1$. We will let I^n refer to the coefficient system which assigns I^n to G/e and 0 to G/G.

As coefficient systems, $M/I \cong \mathbb{F}_p$, and therefore there is a short exact sequence of coefficient systems

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] & M \arrow[r] & N \oplus L \arrow[r] & 0.
\end{tikzcd}$$

This implies a long exact sequence in Bredon cohomology:

$$\begin{tikzcd}
\cdots \arrow[r] & H_G^q(X;I) \arrow[r] & H_G^q(X;M) \arrow[r] & H_G^q(X;N \oplus L) \arrow[r] & H_G^{q+1}(X;I) \arrow[r] & \cdots
\end{tikzcd}$$

Exactness at $H^q(X;N \oplus L)$ implies

$$\operatorname{rank} H_G^q(X;N) + \operatorname{rank} H_G^q(X;L) \le \operatorname{rank} H_G^q(X;M) + \operatorname{rank} H_G^{q+1}(X;I).$$

---

[18] TODO: M is H^0 applied to some spaces and map. What are they?

[19] Here, $\mathbb{F}_p[G]$ is the **group ring**, the $\mathbb{F}_p$-algebra of functions $G \to \mathbb{F}_p$ with addition taken pointwise and multiplication determined by requiring $\delta_g \cdot \delta_h = \delta_{gh}$, where δ_g is a delta function, equal to 1 at g and 0 elsewhere.

That is,

$$\operatorname{rank} H^q(X^G;\mathbb{F}_p) + \operatorname{rank} \widetilde{H}^q((X/X^G)/G;\mathbb{F}_p) \le \operatorname{rank} H^q(X;\mathbb{F}_p) + \operatorname{rank} H_G^{q+1}(X;I). \tag{1.5.4}$$

We'll use this to strongly constrain $\operatorname{rank} H^q(X^G;\mathbb{F}_p)$, but first we need another inequality coming from another exact sequence. Namely, the following sequence of coefficient systems is exact:

$$0 \longrightarrow L \longrightarrow M \longrightarrow I \oplus N \longrightarrow 0.$$

This is because $I^p = 0$ and for $0 \le n \le p-1$, $I^n/I^{n+1} \cong \mathbb{F}_p$. In particular, $I^{p-1} \cong \mathbb{F}_p \cong L$, so we can think of M/L as M/I^{p-1}. Now we play the same game: the induced long exact sequence is

$$\cdots \longrightarrow H_G^q(X;L) \longrightarrow H_G^q(X;M) \longrightarrow H_G^q(X;I\oplus N) \longrightarrow H_G^{q+1}(X;L) \longrightarrow \cdots$$

which implies

$$\operatorname{rank} H_G^q(X;N) + \operatorname{rank} H_G^q(X;I) \le \operatorname{rank} H_G^q(X;M) + \operatorname{rank} H_G^{q+1}(X;L),$$

i.e.

$$\operatorname{rank} H^q(X^G;\mathbb{F}_p) + \operatorname{rank} H_G^q(X;I) \le \operatorname{rank} H^q(X;\mathbb{F}_p) + \operatorname{rank} H^{q+1}((X/X^G)/G;\mathbb{F}_p). \tag{1.5.5}$$

Let's use this to prove

$$\operatorname{rank} \widetilde{H}^q((X/X^G)/G;\mathbb{F}_p) + \sum_{i=q}^{q+r} \operatorname{rank} H^i(X^G;\mathbb{F}_p) \le \operatorname{rank} \widetilde{H}^{q+r+1} + \sum_{i=q}^{q+r} \operatorname{rank} H^i(X;\mathbb{F}_p). \tag{1.5.6}$$

Let

$$\begin{aligned} a_q &:= \operatorname{rank} H^q(X^G;\mathbb{F}_p)\\ b_q &:= \operatorname{rank} H^q(X;\mathbb{F}_p)\\ c_q &:= \operatorname{rank} H^q((X/X^G)/G;\mathbb{F}_p)\\ d_q &:= \operatorname{rank} H_G^q(X;I). \end{aligned}$$

Then, (1.5.4) and (1.5.5) say

$$a_q + c_q \le b_q + d_{q+1} \qquad \text{and} \qquad a_q + d_q \le b_q + c_{q+1}.$$

Now, adding (1.5.4) for q even and (1.5.5) for q odd proves (1.5.6).

When $q = 0$ and r is large, the finite-dimensionality of X implies that

$$\sum_i \operatorname{rank} H^i(X^G;\mathbb{F}_p) \le \sum_i \operatorname{rank} H^i(X;\mathbb{F}_p). \tag{1.5.7}$$

This is already an interesting bound, especially relative to the amount of work we've put in.

Specializing to X an $\mathbb{F}_p$-cohomology sphere, (1.5.7) means

$$\sum_i \operatorname{rank} H^i(X^G;\mathbb{F}_p) \le 2.$$

We want to show this sum isn't 1 (so that we get the cohomology of a sphere) and that the top nonzero rank is at most n. We will do this with another short exact sequence of coefficient systems:

$$0 \longrightarrow I^{n+1} \longrightarrow I^n \longrightarrow L \longrightarrow 0.$$

From this, we get another long exact sequence. Applying the Euler characteristic, we obtain that

$$\chi(X) = \chi(X^G) + p\widetilde{\chi}((X/X^G)/G). \tag{1.5.8}$$

Here

$$\widetilde{\chi}(Y) := \sum_i (-1)^i \operatorname{rank} \widetilde{H}^i(Y)$$

is the **reduced Euler characteristic**.

Equation (1.5.8) already implies that $\chi(X) \equiv \chi(X^G) \bmod p$, so $\sum \operatorname{rank} H^*(X^G;\mathbb{F}_p) \ne 1$ in our case.

Exercise 1.5.9.

(1) Think about choices of q and r that allow you to deduce $m \le n$, finishing the proof.
(2) Small changes need to be made to this argument when $p = 2$; what are they? ⊠

This is an appealing proof: some fairly simple calculations and a dash of formal theory very effectively led to the result. We'll give another proof with different advantages and disadvantages.

Smith theory naturally leads to questions about how to recover $H^*(X^G)$ algebraically from some equivariant cohomology theory on X. For example, we could ask about Borel cohomology $H_G^*(X) := H^*(EG \times_G X)$, where EG is a free G-space that's nonequivariantly contractible (which is simple to construct with the bar construction or through Elmendorf's theorem).

In the following, all cohomology is understood to have coefficients in $\mathbb{F}_p$. Recall that the Bredon cohomology of X is defined to be $H_G^*(X) := H^*(EG \times_G X)$. For a subgroup $H \subset G$, let $S_H \subset H^*(BG)$ be the multiplicative set generated by the classes in $H^2(BG)$ that are images of the Bockstein homomorphism $H^1(BG) \to H^2(BG)$ of the elements that are nontrivial in $H^1(BH)$. This uses the fact that Borel cohomology is an $H^*(BG)$-module: $H^*(EG \times_G *) \cong H^*(EG/G) = H^*(BG)$, and using the terminal map $X \to *$ we get a map $H^*(BG) \to H^*(EG \times_G X)$.

Theorem 1.5.10. *Let G be a finite p-group and $H \subset G$ be a subgroup. Then, there is an isomorphism* $S_H^{-1}H_G^*(X) \xrightarrow{\cong} S_H^{-1}H_G^*(X^H)$.

There's a rich theory of unstable modules over the Steenrod algebra $\mathscr{A}_p$, which could fill a whole semester. There's a functor Un which produces unstable $\mathscr{A}_p$-modules, in a sense by only keeping the unstable part.

Theorem 1.5.11 (Dwyer-Wilkerson [DW88]).

$$H^*(X^G) \cong \mathbb{F}_p \otimes_{H^*(BG)} \mathrm{Un}(S_G^{-1}H^*(EG \times_G X)).$$

The proof uses arguments that were hard to think of, but easy to follow.

We'll use these theorems to prove Smith's theorem using the Serre spectral sequence for

$$X \longrightarrow EG \times_G X \longrightarrow BG.$$

This will be the nicest kind of spectral sequence argument: everything degenerates.

Theorem 1.5.10 is an example of a general class of **localization theorems** in equivariant cohomology. In these theorems, one considers the fiber sequence $X_+^G \to X_+ \to X/X^G$, and wants to show that for some functor E, $E(X_+^G) \cong E(X_+)$. This boils down to showing $E(X/X^G)$ vanishes, which will always follow from showing that E vanishes on G-spaces whose G-actions are free away from the basepoint. In general, this will reduce to considering cells, so one considers $E(G/H_+ \wedge \bigvee_\alpha S^{q_\alpha})$ for some wedge of spheres.

In our case, $E(G/H_+ \wedge \bigvee_\alpha S^{q_\alpha})$ is

$$S_H^{-1}H_G^*\left(G/H_+ \wedge \bigvee_\alpha S^{q_\alpha}\right) = S_H^{-1}H^*\left(EG \times_G \left(G/H_+ \wedge \bigvee_\alpha S^{q_\alpha}\right)\right)$$

Now, the term $EG \times_G G/H \cong EG/H \cong BH$, so we have a piece that looks like $H^*(BH)$, which is how BH inserts itself into the argument.

Exercise 1.5.12. Finish the Serre spectral sequence proof of Theorem 1.5.1. Hint: there's a simple geometric reason why the spectral sequence collapses, which is what makes this whole thing go.

Recall that the Borel cohomology $H^*(EG \times_G X)$ is an $H^*(BG)$-module, through the map $EG \times_G X \to EG \times_G * = BG$. Thus, we should compute $H^*(BG)$ as a ring. In the case we care about, $G = (\mathbb{Z}/p)^n$; let's start with $n = 1$.

There's a nice geometric model for $B\mathbb{Z}/p$ as an "infinite-dimensional lens space:" let S^∞ denote the unit sphere in an infinite-dimensional complex Hilbert space,[20] and give it a C_p-action by $z \mapsto e^{2\pi i/p}z$. Then, the infinite-dimensional lens space is S^∞/C_p.

The quotient $S^\infty \twoheadrightarrow S^\infty/C_p$ is a covering map, and S^∞ is contractible. The proof goes through some version of the Eilenberg swindle:

- There is a homotopy from id_{S^∞} to $s : (x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots)$. In fact, it's a straight-line homotopy.
- There is a straight-line homotopy from s to $(0, x_1, \dots) \mapsto (1, 0, 0, \dots)$.

Thus S^∞/C_p is the quotient of a contractible space by a free G-action, so it deserves to be called $B\mathbb{Z}/p$.

You can set up a cell structure on $B\mathbb{Z}/p$ as with finite-dimensional lens spaces, and therefore compute that

$$H^k(B\mathbb{Z}/p) \cong \begin{cases} \mathbb{Z}/p, & k \text{ even} \\ 0, & k \text{ odd}. \end{cases}$$

---

[20] Alternatively, you could choose S^∞ to be the colimit of S^n for all n, through the inclusion $S^n \hookrightarrow S^{n+1}$ at the equator. These two choices are not homeomorphic, but produce homotopy equivalent models of $B\mathbb{Z}/p$.

Using the universal coefficient theorem, you can then deduce that

$$H^k(B\mathbb{Z}/p;\mathbb{Z}/p) \cong \mathbb{Z}/p$$

for all k. Now we want to deduce the ring structure.

Recall that if

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] & L \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

is a short exact sequence, it induces a long exact sequence in cohomology:

$$\begin{tikzcd} \cdots \arrow[r] & H^k(-;M) \arrow[r] & H^k(-;L) \arrow[r] & H^k(-;N) \arrow[r, "\beta"] & H^{k+1}(-;M) \arrow[r] & \cdots \end{tikzcd}$$

Let's apply this to the short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot p"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}/p \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{Z}/p \arrow[r, "\cdot p"] & \mathbb{Z}/p^2 \arrow[r] & \mathbb{Z}/p \arrow[r] & 0. \end{tikzcd}$$

This is a map of short exact sequences, inducing a map of their long exact sequences.

(1.5.13)
$$\begin{tikzcd} \cdots \arrow[r] & H^q(X;\mathbb{Z}) \arrow[r] \arrow[d] & H^q(X;\mathbb{Z}) \arrow[r] \arrow[d] & H^q(X;\mathbb{Z}/p) \arrow[r] \arrow[d] & H^{q+1}(X;\mathbb{Z}) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & H^q(X;\mathbb{Z}/p) \arrow[r] & H^q(X;\mathbb{Z}/p^2) \arrow[r] & H^q*(X;\mathbb{Z}/p) \arrow[r, "\beta"] & H^{q+1}(X;\mathbb{Z}/p) \arrow[r] & \cdots \end{tikzcd}$$

The map $\beta : H^q(X;\mathbb{Z}/p) \to H^{q+1}(X;\mathbb{Z}/p)$ will be called the **Bockstein homomorphism**, and is a simple example of a cohomology operation.

We now assume p is odd.

Lemma 1.5.14. *If n is odd, the Bockstein for $B\mathbb{Z}/p$ is an isomorphism; if n is even, it's* 0.

PROOF. First, observe that the diagram

$$\begin{tikzcd} H^n(B\mathbb{Z}/p;\mathbb{Z}) \arrow[r, "f"] & H^n(B\mathbb{Z}/p;\mathbb{Z}/p) \arrow[r, "g"] \arrow[dr, "\beta"'] & H^{n+1}(B\mathbb{Z}/p;\mathbb{Z}) \arrow[r] \arrow[d, "\pi"] & H^{n+1}(B\mathbb{Z}/p;\mathbb{Z}) \\ & & H^{n+1}(B\mathbb{Z}/p;\mathbb{Z}/p) & \end{tikzcd}$$

commutes, and the top row is exact.

- If n is even, f is an isomorphism, so $g = 0$, so $\beta = 0$.
- if n is odd, g and π are surjections, so β is a surjection between two $\mathbb{F}_p$-vector spaces of the same rank, so β is an isomorphism. ⊠

Next, a nice way to compute the cup product. The map $B\mathbb{Z}/p \to \mathbb{CP}^\infty$ is cellular, and is a homeomorphism when restricted to even-dimensional cells. As the cell structure determines the cup product structure, the cup products on $H^{\text{even}}(B\mathbb{Z}/p;\mathbb{Z}/p)$ and $H^{\text{even}}(\mathbb{CP}^\infty;B\mathbb{Z}/p) \cong \mathbb{Z}/p[y_1]$ agree.[21] On the odd-dimensional cells, the cup product is graded commutative rather than strictly commutative, so we get an exterior algebra. Therefore we conclude that

$$H^*(B\mathbb{Z}/p;\mathbb{Z}/p) \cong \Lambda(x_1) \otimes \mathbb{Z}/p[y_1],$$

where $|x_1| = 1$, $|y_1| = 2$, and $\beta x_1 = y_1$.

By essentially the same argument, one can compute the cohomology ring for $B(\mathbb{Z}/p)^n$.

Proposition 1.5.15. *As rings, there is an isomorphism*

$$H^*(B(\mathbb{Z}/p)^n;\mathbb{Z}/p) \cong \Lambda(x_1,\dots,x_n) \otimes \mathbb{Z}/p[y_1,\dots,y_n],$$

where $\beta x_i = y_i$.

---

[21] By $\mathbb{CP}^\infty$, we mean the colimit of $\mathbb{CP}^n$ over all n, and similarly for $\mathbb{RP}^\infty$. However, there are models for them based on the unit sphere in an infinite-dimensional Hilbert space modulo an S^1 (resp. $\mathbb{Z}/2$) action. These are homotopic to, but not homeomorphic to, the colimit realizations of $\mathbb{CP}^\infty$ and $\mathbb{RP}^\infty$.

Exercise 1.5.16. Figure out the slight changes needed for $p = 2$.

Recall that the localization theorem asserted that if G is a finite p-group, $S_H^{-1}H^*(EG\times_G X)\cong S_H^{-1}H^*(EG\times_G X^G)$, where S is the multiplicative system generated by images of the Bockstein homomorphism in $H^*(BG;\mathbb{Z}/p)$. We'll prove this in the case when G is abelian.

We mentioned above that it's possible to inductively reduce to considering $H^*(BH)\otimes H^*\big(\bigvee S^{q_i}\big)$. It's now clear[22] that something in S restricts to 0 in $H^*(BH)$, completing the proof (in the abelian case).

The localization fails terribly for infinite nonabelian compact Lie groups G. For example, for any topological space K, there exists a G-CW complex X such that X is nonequivariantly contractible, X is finite-dimensional, and $X^G \simeq K$.

1.6. The Sullivan conjecture

> *"The number of children he had was a monotonically increasing function."*

The Sullivan conjecture is really Sullivan's attack on Adams' conjecture, and is a very important story. We won't prove the conjecture, because it's hard, but the context around it was a major motivation for a lot of the work in algebraic and geometric topology in the past 40 years. Sullivan wrote up some notes for a class of his at MIT, which have been published as [**Sul05**], and you should read them: they are enlightening and contain all of the jokes he told in class!

Theorem 1.6.1 (Sullivan conjecture)**.** *Let G be a finite, abelian p-group. Then, $X^{hG}\to X^G$ is an equivalence on p-completions.*

Recall that $X^{hG} := \mathrm{Map}(EG,X)^G$, so this asserts a weak equivalence (after p-completing) $\mathrm{Map}(EG,X)^G\to\mathrm{Map}(*,X)^G$.

By **p-completion**, we mean Bousfield localization at $\mathbb{F}_p$ cohomology. This produces the category of spaces where equivalences are detected by $H^*(-;\mathbb{F}_p)$. The most familiar example of completion is **rationalization**, a localization where equivalences are detected by rational homotopy groups, and one of Sullivan's biggest achievements was providing a completely algebraic description of the rational homotopy category in [**Sul77**]. More broadly, he had the insight that to study a problem in homotopy theory, one could localize at $\mathbb{Q}$ and at each $\mathbb{F}_p$ and study each piece, which has been a very fruitful approach.

p-completion falls into the collection of basic life skills for homotopy theorists, so if you haven't seen it before, you should read about it. The standard reference is [**BK72b**], but this is 500 pages and hard to read.

Sullivan's conjecture is an algebro-geometric attack on the Adams conjecture. This was within Sullivan's program to find algebraic models of manifolds. This is still being done today, and is what led Sullivan to think about string topology and related things.

Let X be a manifold. We first have the homotopical data of X, $C^*(X;\mathbb{Q})$ and $C^*(X;\mathbb{F}_p)$, which are "commutative rings."[23] That X is a manifold means we can see Poincaré duality, which doesn't appear in all $\mathbb{E}_\infty$ dg algebras. But we still need some way to encode additional geometric obstructions, e.g. a way to encode the tangent and normal bundles.

It's been an interesting, but as yet unsuccessful, attack to use a Frobenius algebra structure to try to obtain this geometric data. It's neat to think about what the $\mathbb{E}_\infty$ analogue of a Frobenius algebra is, and this is intimately related to Lurie's approach to the cobordism hypothesis [**Lur09b**], thinking about fully dualizable objects in a symmetric monoidal ∞-category.

This circle of ideas is also related to surgery theory; there's been lots of cool work by smart people in it, and L-theory was invented basically as an algebraic home for these geometric objects.

Sullivan was interested in the Adams conjecture because it says that one can identify the tangent bundle inside K-theory $K(X)$ with its Adams operations ψ^k, as fiberwise homotopy types.

Sullivan's idea, motivated by Quillen, was to use the theory of étale homotopy types. This translates some questions about scheme theory into homotopy theory. For example, if X is a variety (more generally a scheme), one can assign some profinite topological object, built out of something like a system of hypercovers.[24] So if you take the profinite completion of the complex points $X(\mathbb{C})^\wedge$, it has an action of the absolute Galois group

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[22] TODO: I didn't follow this proof in class.

[23] They're not literally commutative; instead, they're $\mathbb{E}_\infty$ dg algebras. They also have more structure as modules over the Steenrod algebra.

[24] If you don't know what this is, it's an example of an extremely interesting construction which you should look up sometime.

$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ — and another crazy interpretation of the Adams conjecture is that the profinite completion of $K(X)$ can be interpreted in this way, and has an action of $(\widehat{\mathbb{Z}})^*$ ($\widehat{\mathbb{Z}}$ is the Galois group of the maximal abelian extension over $\mathbb{Q}$). The conjecture is that this action is by the Adams operations.

There's a deep and inadequately understood story (which could be an opportunity for you) connecting p-adically completed complex K-theory $KU_p^\wedge$ to number theory, specifically the Iwasawa algebra. Adams noticed this, but it's too interesting to be a coincidence.

Anyways, stable fiberwise homotopy types are invariant under this $(\widehat{\mathbb{Z}})^*$-action, which led Sullivan to ask questions about $(X(\mathbb{C})^\wedge)^{hC_2}$ versus $X(\mathbb{R})^\wedge$. The references [**Sul05**, **Sul74**] are both excellent for this.

For reasons of scope, we can't go into too much more detail, but you should definitely look this stuff up. The takeaway is that equivariant homotopy theory has been motivated by seemingly unrelated questions about manifolds. There's been a lot of interesting interplay between algebraic and geometric topology in the last half century, and this is one of the sites of contact.

1.7. Question-and-answer session: 2/2/17

Question 1.7.1. So, why does anyone care about Spanier-Whitehead duality?

One answer is that Spanier-Whitehead duality formally implies Poincaré duality. Poincaré duality is a remarkable fact about the cohomology of manifolds, which is one good reason to care.

Another is that it's the real setting for the Euler characteristic — really. The Euler characteristic arises as a trace, and therefore some kind of categorical duality should appear.

Being more specific, let's consider a closed symmetric monoidal category C with unit $\mathbb{S}$ (so secretly we're thinking of the stable homotopy category). Spanier-Whitehead duality consists of two maps $\varepsilon\colon X \wedge Y \to \mathbb{S}$ and $M : \mathbb{S} \to X \wedge Y$, such that the composition

$$X \cong \mathbb{S} \wedge X \xrightarrow{M\wedge \mathrm{id}} X \wedge Y \wedge X \xrightarrow{\mathrm{id}\wedge(\varepsilon\circ\tau)} X \wedge \mathbb{S} \cong X$$

is the identity.

Using an adjunction, you can show $X \cong F(Y, \mathbb{S})$ and $Y \cong F(X, \mathbb{S})$, and so X is the dual of its dual. This has lots of formal consequences, including Poincaré duality, the construction of things such as the Euler characteristic, and more. The Spanier-Whitehead dual of X is often denoted DX.

Suppose that we're in an algebraic category, say $H\mathbb{Q}$-module spectra (the rational stable homotopy category). Then, $H\mathbb{Q}$ is the unit, so $DX = C^*(X;\mathbb{Q})$ (maps from X to $\mathbb{Q}$). That is, the Spanier-Whitehead dual embeds the unstable homotopy theory of X, as long as you remember the $\mathbb{E}_\infty$-ring structure on DX (which is bizarre, e.g. it's not connective). There's a quite nontrivial theorem that $\mathbb{E}_\infty$ maps between DX and DY correspond to maps of *spaces* between X and Y.

This sort of data is useful for algebraicizing manifolds, and that would be nice for classifying manifolds, a goal with fairly broad applications outside of homotopy theory. And Spanier-Whitehead duality has important consequences on this: Sullivan began it by showing [**Sul77**] that the $\mathbb{E}_\infty$-ring structure on $C^*(X;\mathbb{Q})$ controls the rationalization $X_\mathbb{Q}$, and Mandell [**Man01**] ended it by showing that the $\mathbb{E}_\infty$ structure on $C^*(X;\mathbb{F}_p)$ controls $X_p^\wedge$.

Question 1.7.2. What are some of the obstacles to extending equivariant homotopy theory to groups that aren't compact Lie groups?

One cornerstone of homotopy theory is the Pontrjagin-Thom construction, which depends on the weak Whitney embedding theorem: that a manifold can be embedded in some high-dimensional $\mathbb{R}^N$. This still works equivariantly *but only for compact Lie groups* — a G-manifold X can be embedded in some high-dimensional G-representation. This is again an important piece in the equivariant Pontrjagin-Thom construction, and its failure for noncompact groups is one major reason things don't work. There are other things that require compactness (e.g. descending chain arguments). People have tried and not gotten anywhere.

In stable homotopy theory, there's a modification of the orbit category into something called the Burnside category, and we'll see that it controls a lot of the stable homotopy theory of G-spaces. In fact, if you have something that strongly resembles the Burnside category, you have something that looks like equivariant stable homotopy theory. Clark Barwick and his collaborators have been working on studying presheaves on things that look like Burnside categories.

A lot of it boils down to the fact that compact Lie groups have a tractable representation theory.

Question 1.7.3. What does it mean that K-theory and the Adams operations determine the tangent bundle?

It's a somewhat implicit statement: any possible algebraic statement about the tangent bundle can be translated into an equivalent statement in $K(X)$ that uses the Adams operations.

Question 1.7.4. How many notions of G-spectra are there?

One way you talk about equivariant stable homotopy theory is a **universe**, a countably infinite-dimensional inner product space containing the irreducible representations you care about infinitely often. Then, you have Spanier-Whitehead duality for G/H iff G/H embeds in the universe, and you get different flavors of homotopy theory depending on which universes you use. There are lots of models here — depending on what you mean, spectra objects or diagrams on $\mathscr{O}_G$ might not be the right thing for naïve spectra; instead, you need transfers, so you have to consider sheaves on the Burnside category.

The choice of universe also affects which suspensions you invert: if V is in your universe, you can invert Σ^V.

Of course, there are probably many different point-set models for stable homotopy, but they'll give you the same answer.

Question 1.7.5. Can you go over how to design a coefficient system such that its cohomology is something obtained from X?

For example, let's try to determine M such that $H_G^*(X;M) \cong H^*(X)$. The reason it suffices to determine what $M(G/H)$ is on all closed subgroups $H \subset G$ is the dimension axiom!

So if $G = \mathbb{Z}/p$, you can assign $M(G/e) := H^0(G/e)$ and $M(G/G) := H^0(G/G)$, and the map between them is H^0 of the map $G/e \to G/G$.

Question 1.7.6. So far, we've mostly seen G-equivariant homotopy theory where G is a finite cyclic group. Is there anything interesting for nonabelian groups, etc?

Part of the problem is that computing examples is not easy, and it gets much harder when you have a complicated lattice of subgroups. [LMS86] is 500 pages, and there are scarcely any examples, because the computations for nontrivial examples are so hard!

Peter May invented this stuff and set a bunch of grad students to work on it, but there wasn't a lot of buzz until Carlsson proved the Segal conjecture, and then again more recently with Hill-Hopkins-Ravenel. So there haven't been a lot of computations, period. If you do make an interesting computation with, say, the monster group, by all means write it up!

So for the most part people have studied cyclic groups and S^1. There's been a little discussion of dihedral actions, and some stuff with the symmetric groups.

There's also some stuff done for profinite groups, e.g. [Fau08]. This is in some sense easy to set up formally, especially because you mostly care about finite-index subgroups. People who study Galois actions (e.g. Carlsson's program to lift the Quillen-Lichtenbaum conjecture into this context) care about this.

Question 1.7.7. So we've spent some time looking at EG, a contractible space with a free G-action, and its quotient BG. What things do people do with these objects?

BG is a classifying space for principal G-bundles (and therefore for $G = \mathrm{O}_n$ or U_n, also vector bundles of rank n). That is, homotopy classes of maps $X \to BG$ are identified with isomorphism classes of principal G-bundles on X. There's a book by May, "Classifying spaces and fibrations," which is excellent and goes into great detail on this stuff. Because $B\mathrm{U}_n$ classifies complex vector bundles of rank n, it's used to construct complex K-theory (and same for $B\mathrm{O}_n$ and real K-theory).

In our case, smashing with EG is often a way to localize.

A third reason to care about this is group cohomology and other purely algebraic stuff. The group cohomology of G is the cohomology of BG, and there are plenty of applications of group cohomology.

So though BG may be infinite-dimensional, it's very simple homotopically.

Question 1.7.8. What's going on with the construction to the right adjoint to the functor ψ in the proof of Elmendorf's theorem? What's a coend?

Adrian said some stuff here; I wasn't able to get it down. He motivated the bar construction as the thing whose homotopy colimits are ordinary colimits, I think?

Anyways, our setup is that we have a presheaf on the orbit category $X \in \mathsf{Fun}(\mathscr{O}_G^{\mathrm{op}}, \mathsf{Top})$ and want to produce a G-space. The right adjoint[25] to ψ is the geometric realization of the bar construction $B_\bullet(X, \mathscr{O}_G, M)$. The

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[25] Recall that the left adjoint was evaluation at G/e.

bar construction is a generalization of the bar resolution to compute the derived tensor product: the functor $M\colon \mathscr{O}_G \to \mathsf{Top}$ sending G/H to the space G/H is thought of as a "right $\mathscr{O}_G$-module;" instead of tensoring a bunch of elements together, we get a bunch of arrows, meaning we can replace with a coproduct. In some sense, it would be nice to take a tensor product with X, but we have to do so in a derived sense, hence the bar construction.

A coend is a functor that behaves like geometric realization: there's two functors with opposite variance, and you want to glue along their common edge, just like in geometric realization.

Question 1.7.9. How did Elmendorf formalize his proof, given that it was done before model categories were available?

He didn't: Elmendorf's paper is eminently readable, and simply provides an equivalence of homotopy categories. It was not lifted into model categories unti much later. This stuff was all put into use fairly recently: for example, not that long ago, it was known to experts but not written down that a left Quillen adjoint that's part of a Quillen equivalence preserves homotopy limits.

Here, someone asked about the Freudenthal suspension theorem, and this led to a digression.

REMARK. Modern cryptography depends on some hardness assumptions, that some functions, such as the discrete log in a finite field, are hard to compute (but easy to check answers to). There's a paper by Impagliazzo [**Imp95**] which asks what cryptography and security would look like if certain assumptions were false or true, with cute names for different worlds. Imagine doing that for the Freudenthal suspension theorem — what if the stable range were at about n instead of about $2n$? What if it were $n/2$? ◀

Question 1.7.10. In non-equivariant rational homotopy theory, there's a standard, completely algebraic description of the rational homotopy category. Does this also work for the rational equivariant homotopy category?

This is very hard — someone was working on this, but the work actually depended on an incorrect calculation, and some of it had to be redone. There are people working on an algebraic model for rational G-spectra, but the algebraic models are *very* complicated.

Question 1.7.11. We've seen that representations of G play a huge role in equivariant homotopy theory; these could be thought of as relating to K-theory of BG (e.g. the representation ring $\mathrm{RO}(G)$ is $KO^0(BG)$). More generally, do $KO(BG)$ or $MO(BG)$ play a special role "controlling" the G-equivariant stable category?

People certainly care about computing K-theory or bordism of BG. As to whether those control the G-equivariant stable category—that might be true, but I don't think anything like that has been said.

Question 1.7.12. This question comes from GitHub: suppose you're defining a coefficient system M over $G = C_p$. This is determined by $M(C_p/e)$, $M(C_p/C_p)$, and a map between them. How do you get the map?

The map is determined by functoriality: in the cases we used to prove Theorem 1.5.1, which is where this question arose, $M(C_p/e)$ and $M(C_p/C_p)$ are both defined to be H^0 of something, so the map between them is H^0 of the map $C_p/e \to C_p/C_p$.

Question 1.7.13. Do you want to talk about the unstable Steenrod functor Un?

You don't want to know.

Question 1.7.14. Does Un appear in the theorem as a technical necessity or for deep reasons?

It's very deep, and there's a whole theory of unstable modules over the Steenrod algebra. This involves some extra restrictions.

Question 1.7.15. So how many kids did Denis Sullivan have?

About 7 or 8 by the end of [**Sul05**]; one is now also a math professor. There are lots of other anecdotes in the notes: once he was frustrated with some calculations he was doing, so he hurled the book into the Atlantic ocean. While he was on a boat to Oxford.

Question 1.7.16. Do we have any previews of the Clark school, besides what few things they've posted on the arXiv?

They've been doing some really hard stuff: the idea is that G-spectra should be presheaves of spectra on Burnside categories, or statements like G-spectra being the initial stable G-symmetric monoidal category… you know what the theorems should be, but the proofs are technical. Riehl-Verity technology should make some of this easier, hopefully.

Question 1.7.17. What does it mean that "the Spanier-Whitehead dual of a point involves taking shifts?"

We've been a little careless about basepoints: this is really the pointed point, so S^0; thus, we really should have said the Spanier-Whitehead duals of spheres.

Question 1.7.18. The Fruedenthal suspension theorem requires inclusion of the basepoint to be a cofibration. If you add a disjoint basepoint, is that the case?

Yes. You can think of a cofibration as a slightly weaker version of a closed inclusion. This is no big technical obstacle: the **whisker construction** replaces the basepoint with a line segment, so the homotopy theory is unchanged and the other end of the line segment can be a basepoint that's cofibrantly included.

Question 1.7.19. What's a natural example of a genuine G-spectrum of interest?

One is the equivariant sphere spectrum. Thinking of cohomology theories, there's equivariant K-theory KU_G, different kinds of equivariant bordism, and KR (a genuine equivariant spectrum that's built from K-theory).

Question 1.7.20. Is there an equivariant approach to de Rham theory, e.g. starting with manifolds and equivariant differential forms?

Rationally, this can work, but integrally there are issues. Relatedly, defining a G-manifold is not too hard, and there are nice things like the equivariant tubular neighborhood theorem, but there are some subtleties.

Question 1.7.21. One of the interpretations of the homotopy category of a model category is localization at the weak equivalences. This suggests that one only needs to look at weak equivalences. So if you only specify weak equivalences, do you get a unique model category?

No, you might not get one at all, and if you do, it won't be unique. What it means is (modulo some scary set-theoretic issues) that the homotopy theory only depends on the weak equivalences. If you have a category with weak equivalences, you can form the **Dwyer-Kan localization**, an associated simplicial category, and then take the associated quasicategory or ∞-category. Under some hypotheses, you can then retreat to a model-categorical structure. One commonly heard analogy is that a model category is like choosing a basis for a vector space (where the basis-independent answer reflects the underlying ∞-category).

Another perspective is that model categories are like axiomatic obstruction theory, axiomatizing the cellular inclusions (cofibrations) or extension questions common when considering CW theory.

Question 1.7.22. One way to define the stable category is to invert the canonical map from a finite wedge of things to a product of those things. How can we make this precise?

The goal is to localize with respect to something. There's some set-theoretical issues that Bousfield was good at addressing, but if you have a set of morphisms, you can localize with respect to them, and that's what we're doing here. The Stacks project is a good resource for learning about localization.

The fact that finite wedges and finite products are the same is another way that spectra resemble abelian groups, for which finite products and finite coproducts agree.

Question 1.7.23. Where's the best place to learn non-equivariant stable homotopy theory?

Adams' book [**Ada74**], with good calculations, but its construction of the stable category isn't so pleasant. It might be better to consider [**MMSS01**]. It assumes you already know why you care, then takes the historical constructions and shows how they all relate together. It may seem scary, but looking at the constructions shows they're all not so bad.

Adams' infinite loop spaces book [**Ada78**] is good bedtime reading.

Question 1.7.24. We talked about the notion of an excisive functor, and there's a notion of an excisive pair in unstable homotopy theory. Are the two related?

Both deserve the name "excisive" because they satisfy some sort of Mayer-Vietoris principle: excisive functors take pushouts to pullbacks, for example.

CHAPTER 2

Building the equivariant stable category

2.1. Dualities: Alexander, Spanier-Whitehead, Atiyah, Poincaré

"The sun, it burns!"

Before constructing the equivariant stable category, we'll provide some motivation, much of which is also good motivation for the nonequivariant stable category. There's a choice here: you could just say "let's take the category of orthogonal G-spectra," but having some motivation for why we're doing what we're doing is important.

There are lots of ways to think about where stabilization comes from.

(1) The Freudenthal suspension theorem says that if X is nondegenerately based (meaning the based inclusion map $* \hookrightarrow X$ is a cofibration) and $n-1$-connected, then $\pi_q(X) \to \pi_{q+1}(\Sigma X)$ is an isomorphism for $q < 2n-1$ and a surjection when $q = 2n-1$.[1] It's easier to see that cohomology groups are stable under suspension, but this tells us that homotopy groups stabilize in a range that increases at about twice the rate that the connectivity of X does. Since $\Sigma^n X$ is at least n-connected, this suggests you could replace X by the sequence $X, \Sigma X, \dots, \Sigma^n X, \dots$, and keep track of that instead, regarding it as a repository for the **stable homotopy groups** $\pi_n^S(X) \coloneqq \operatorname{colim}_k \pi_{n+k}(\Sigma^k X)$. One way to think of this is as formally making homotopy theory into a homology theory (which it isn't *a prioiri*); you end up taking the same kind of colimit.

You could do this equivariantly: we have representation spheres. But it's not entirely clear what to do.

(2) Another perspective is that the stable category is the result of inverting the canonical map[2]

$$\bigvee_{i=1}^{k} X \longrightarrow \prod_{i=1}^{k} X. \tag{2.1.1}$$

Again, this is something you can think about making precise; the stable category is the initial triangulated category constructed from Top in which (2.1.1) is an isomorphism. In particular, this forces the homotopy category to be additive.

Again, we could do this equivariantly.

(3) Suppose we have a functor F from finite CW complexes to Top such that $F(*) = *$, and suppose F commutes with filtered colimits and preserves weak equivalences (e.g. if it's topologically or simplicially enriched, for formal reasons). By taking colimits, we can obtain a functor $\widehat{F}$ from CW complexes to Top. We say F is **excisive** if it takes pushouts to pullbacks; this is an old perspective, which was used to show the Dold-Thom theorem, that the infinite symmetric product SP^∞ is a cohomology theory. In any case, if F is excisive, $\{F(S^n)\}$ represents a cohomology theory. Namely, the homotopy pushout

$$\begin{tikzcd}
S^n \arrow[r] \arrow[d] & D^{n+1} \arrow[d] \\
D^{n+1} \arrow[r] & S^{n+1}
\end{tikzcd}$$

creates suspension, and via F, becomes a pullback creating Ω. Asking what the excisive functors are leads to the stable category, and is also something you could do equivariantly.

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[1] The map comes from the loop-suspension adjunction, which gives us a unit $X \to \Omega\Sigma X$, hence a map $\Omega^q X \to \Omega^{q+1}\Sigma X$, and the map on homotopy groups is π_0 of that map. This is the based version of the mapping space and Cartesian product adjunction: $\Sigma X \coloneqq S^1 \wedge X$ and $\Omega X \coloneqq \mathrm{Map}(S^1, X)$ are adjoint functors.

[2] The existence of this map follows from the universal property of the product.

(4) Another perspective: what is the Spanier-Whitehead dual of a point? Taking shifts, what's the Spanier-Whitehead dual of S^n? Equivariantly, one wants to know the Spanier-Whitehead duals of G/H_+. This is important for defining Poincaré duality, etc. Nonequivariantly, the best way to answer this is the Pontrjagin-Thom construction, which not only answers this, but provides a deep understanding for what the stable homotopy groups of the spheres are. In this chapter, we'll do this equivariantly, and it will tell us what the spheres are.

Pursuant to motivating the stable category, this section is about duality. This is a reinvention of the original construction of the stable category — Spanier's original construction of spectra was motivated by answering questions on duality, and we'll proceed similarly, if a bit ahistorically. Since we're in the equivariant setting, the answers will be slightly different.

Alexander duality is a tale as old as time, from what could be called the prehistory of algebraic topology.[3]

Theorem 2.1.2 (Alexander duality [**Ale15**]). *Let $K \subset S^n$ be compact, locally contractible, and nonempty.*[4] *Then, K and $S^n \setminus K$ are **Alexander dual** in that there is an isomorphism*

$$\widetilde{H}^{n-i-1}(K;\mathbb{Z}) \cong \widetilde{H}_i(S^n \setminus K;\mathbb{Z}).$$

The proof isn't too hard, e.g. Hatcher does it. This is closely related to considering embeddings in Euclidean spaces, after you take the one-point compactification.

The good part of this proof is that it doesn't depend on the embedding. But there are a few drawbacks:

(1) K does not determine the homotopy type of $S^n \setminus K$. Knot theory is full of examples, and they tell you that the issues arise for the fundamental group.
(2) n can vary, and if you embed $S^n \hookrightarrow S^{n+1}$ as the equator, you get different statements.

Motivated by the second issue, Spanier defined the S-category in the 1950s.[5]

Definition 2.1.3. The ***S*-category** S is the category whose objects are the objects in Top and whose morphisms are

$$\mathrm{Map}_{\mathsf{S}}(X,Y) := \operatorname*{colim}_n \mathrm{Map}_{\mathsf{Top}}(\Sigma^n X, \Sigma^n Y).$$

By the Freudenthal suspension theorem, the hom-sets stabilize at some finite n. Spanier then defined ***S*-duality**, which we might call **Spanier-Whitehead duality**, by specifying that X and Y are S-dual if $Y \cong S^n \setminus X$ in the S-category.

REMARK. The S-category has some issues: it's neither complete nor cocomplete. We like gluing stuff together, so this is unfortunate. ◀

Spanier proved that to every $X \to S^n$, you can assign a dual $D_n X$, that $\Sigma D_n = D_{n+1}$, and $D_{n+1}\Sigma = D_n$. That is, duality commutes with suspension, so in S, every $X \to S^n$ has a unique S-dual: the duals inside S^n and S^{n+1} are the same in the S-category for sufficiently large n.

Spanier and Whitehead then asked one of their graduate students, Elon Lima, to formalize this S-category, leading to the first notion of the category of spectra [**Lim58**].

An axiomatic setting for duality. The formal setting for duality is a **closed symmetric monoidal category** C. We're not going to spell out the whole definition, but here are some important parts.

- C is symmetric monoidal, meaning there's a tensor product $\wedge\colon \mathsf{C}\times\mathsf{C} \to \mathsf{C}$, which is (up to natural isomorphism) associative and commutative, and has a unit S. Commutativity is ensured by the **flip map** $\tau\colon X\wedge Y \xrightarrow{\cong} Y \wedge X$.
- There is an internal **mapping object** $F(X,Y) \in \mathsf{C}$ for any $X, Y \in \mathsf{C}$.
- The functors $- \wedge X$ and $F(X,-)$ are adjoint (just like the tensor-hom adjunction).

The unit and counit of the tensor-hom adjunction are used to define duality.

Definition 2.1.4. The **evaluation map** is the unit $X \wedge F(X,Y) \to Y$, and the **coevaluation map** is the counit $X \to F(Y, Y \wedge X)$. The **dual** of X is $DX = F(X,S)$.

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[3] If you read between the lines, you can find it in Poincaré's works, but it's from the 1940s or 50s stated explicitly.

[4] Classically, one works simplicially, picking a triangulation of S^n and letting K be a subpolyhedron.

[5] The name "S-category" is somewhat misleading: in those days, suspension was sometimes denoted S instead of Σ to make typeseting easier, and the S in S-category stood for suspension, not spheres.

You also get a natural map $\nu\colon F(X,Y)\wedge Z \to F(X,Y\wedge)$.

Exercise 2.1.5. Check that $X \cong F(S,X)$, which follows directly from the axioms.

The adjoint of ε is a map $X \to DDX$.

There are a few good references for this: Dold-Puppe [DP61] is one, and [LMS86] is another, though it presents a somewhat old way of doing things.

Definition 2.1.6. $X \in \mathsf{C}$ is **strongly dualizable** if there exists an $\eta\colon S \to X \wedge DX$ such that the following diagram commutes.

$$\begin{tikzcd} S \arrow[r, "\eta"] \arrow[d] & X \wedge DX \arrow[d, "\tau"] \\ F(X,X) & DX \wedge X. \arrow[l, "\nu"'] \end{tikzcd} \tag{2.1.7}$$

Here, the left-hand map comes from an adjunction to the identity $\mathrm{id}_X : X \cong X \wedge S \to X$. The lower map is more explicitly $\nu\colon F(X,S)\wedge X \to F(X,X)$.

Example 2.1.8. Let R be a commutative ring and $\mathsf{C} = \mathsf{Mod}_R$, and let X be a free R-module. If $\{v_i\}$ is a basis for X and $\{f_i\}$ is the dual basis, then the map $\eta\colon R \to X \otimes_R \mathrm{Hom}_R(X,R)$ is the map sending

$$1 \longmapsto \sum v_i \otimes f_i.$$

If you unravel what (2.1.7) is saying, it says that the map

$$x \longmapsto \sum_i f_i(x)v_i$$

must be the identity. Thus, X is strongly dualizable iff X is finitely generated and projective. That is, *X is strongly dualizable iff it's a retract of a finite-rank free module*, which is a perspective that will be useful later. ◂

Another way to think of this is that X is strongly dualizable with dual Y iff there exist maps $\varepsilon : X \wedge Y \to S$ and $\eta : S \to X \wedge Y$ such that the compositions

$$X \cong S \wedge X \xrightarrow{\eta\wedge\mathrm{id}_X} X \wedge Y \wedge X \xrightarrow{\mathrm{id}_X\wedge\varepsilon} X \wedge S \cong X$$

and

$$Y \cong Y \wedge S \xrightarrow{\mathrm{id}_Y\wedge\eta} Y \wedge X \wedge Y \xrightarrow{\varepsilon\wedge\mathrm{id}_Y} S \wedge Y \cong Y$$

are the identity.[6] From this and a diagram chase, you get some nice results.

Proposition 2.1.9. *If X and Y are dual, then there are isomorphisms $Y \overset{\cong}{\to} F(X,S)$ and $X \overset{\cong}{\to} DDX$.*

To paraphrase Lang, the best way to learn this is to prove all the statements without looking at the proofs, like all diagram chases.

If you like string calculus, you can think of these in terms of S- or Z-shaped diagrams. In this form, these results are sometimes known as the **Zorro lemmas**.

Another consequence of this formulation is that $-\wedge DX$ is right adjoint to $-\wedge X$, so by uniqueness of adjoints there's a natural isomorphism $-\wedge DX \cong F(X,-)$.

Atiyah duality. The Whitney theorem tells us that for any manifold M and sufficiently large n, there's an embedding $M \hookrightarrow \mathbb{R}^n$. This means we can compute the Spanier-Whitehead dual of a manifold, which is the setting of Atiyah duality. We'll assume M is compact.

By the tubular neighborhood theorem, there's an $\varepsilon > 0$ and a tubular neighborhood M_ε such that M_ε is the disc bundle of the normal bundle $\nu \to M$.[7]

Thom's thesis [Tho54] supplied an amazing connection between cobordism groups and the stable homotopy groups of the spheres by way of the **Pontrjagin-Thom map** $S^n \to \mathbb{R}^n/(\mathbb{R}^n \setminus M_\varepsilon)$ (heuristically, crushing everything outside M_ε), and $\mathbb{R}^n/(\mathbb{R}^n \setminus M_\varepsilon) \cong D(V)/S(V)$, the **Thom space** $T\nu$ of the normal bundle to M.

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[6] In some presentations, this is how duality in a symmetric monoidal category is defined; the two approaches are equivalent.

[7] Let E be a vector bundle, and choose a metric on E; then, the **disc bundle** $D(E)$ is the subset of vectors with $\|x\| \le 1$, and the **sphere bundle** $S(E)$ is the subset of vectors with $\|x\| = 1$. These are fiber bundles with fiber D^n and S^{n-1}, respectively.

Thus, we get a sequence of maps

$$S^n \longrightarrow T\nu \longrightarrow T\nu \wedge M_+,$$

whose composition is called the **Thom diagonal**. This should look remarkably like the duality map η. To construct ε, let $s\colon M \to \nu$ be the zero section; then, the composite

$$M \xrightarrow{\Delta} M \times M \xrightarrow{s\times\mathrm{id}} \nu \times M.$$

has trivial normal bundle, and the Pontrjagin-Thom construction yields a map

$$T\nu \wedge M_+ \longrightarrow \Sigma^n M_+,$$

and projecting M to a point, we get

$$\varepsilon\colon T\nu \wedge M_+ \longrightarrow S^n.$$

The following theorem is nice, but quite nontrivial, at least from this approach.

Theorem 2.1.10 (Atiyah duality). *η and ε exhibit $T\nu$ and M_+ as n-dual in the S-category.*

We'd like $T\nu$ and $\Sigma^{-n}M_+$ to be dual, but we don't know how to show that yet. More accurately, let $\Sigma^\infty\colon \mathsf{Top} \to \mathsf{S}$ be the functor that sends spaces and maps to themselves, so it makes a little more sense to say that $\Sigma^\infty M_+$ and $\Sigma^{-n}\Sigma^\infty T\nu$ are dual in the S-category.

One surprising consequence is that the tangent bundle and normal bundle define stable homotopy invariants through the Thom spectrum, which raises the question of what $T\nu$ looks like for particular examples.

Another is that we can immediately prove Poincaré duality, assuming the Thom isomorphism theorem. Namely, we can establish an isomorphism

$$[DX, H\mathbb{Z}] \cong [S, X_+ \wedge H\mathbb{Z}]. \tag{2.1.11}$$

Here, we're using the corepresentability of cohomology: we know $H^n(-,\mathbb{Z})$ is represented by $K(\mathbb{Z}, n)$, and stitch these together (somehow) into $H\mathbb{Z}$. So the left-hand side is $H^*(DX;\mathbb{Z})$, and the right-hand side is $H_*(X;\mathbb{Z})$.

Applying Theorem 2.1.10 to (2.1.11),

$$[\Sigma^{-n}T\nu, H\mathbb{Z}] \cong H^{m-*}(X;\mathbb{Z}).$$

If X is orientable, then the cohomology of the Thom spectrum is the same as the cohomology of X. The Thom isomorphism theorem establishes a degree shift that gets rid of the dependency on n, the dimension of ambient space.

The equivariant setting. We'd like to establish Atiyah duality for G/H.

Definition 2.1.12. A **G-manifold** is a manifold M with a G-action by smooth maps.

The theorems of differential topology that we need hold for G-manifolds. Namely, there is an equivariant tubular neighborhood theorem, etc. In the smooth case, this goes back to the work of Andrew Gleason in the 1930s, and in the PL case to Lashof in the 1950s. If you like manifold topology, these are really nice proofs to read, avoiding triangulation arguments.

We need one key fact.

Theorem 2.1.13 (Equivariant Whitney's theorem [Mos57, Pal57]). *Let M be a compact G-manifold. Then, there is a G-equivariant embedding $M \hookrightarrow V$, where V is some finite-dimensional real G-representation.*

Now we proceed as before: all of the arguments are exactly the same, including the equivariant Pontrjagin-Thom construction.

But this means that suspension has to be smashing with S^V, the representation sphere for V. So in order for manifolds to have duals (meaning, in order to establish Poincaré duality), we need an S-category whose morphisms are

$$\mathrm{Map}_S(X, Y) := \operatorname*{colim}_V \mathrm{Map}(\Sigma^V X, \Sigma^V Y),$$

where $\Sigma^V X := S^V \wedge X$. This is not sequential, but there is an equivariant Freudenthal suspension theorem that stabilizes it.

Poincaré duality is one of the oldest results in algebraic topology, and if you don't have it in your theory, what are you even doing? So to obtain Poincaré duality, *we are inexorably forced to smash with representation spheres that G-manifolds embed in*, namely finite-dimensional real representations. So many treatments of equivariant homotopy theory start with defining Σ^V and then they're off to the races, but this is why they're doing this. If you don't need Poincaré duality, then you could do something different.

As before, we want a diagram category akin to the orbit category, but we want Atiyah duality to manifest in it.

Question 2.1.14. In this setting, what replaces the orbit category?

The naïve choice is the functor category from $\mathscr{O}_G^{\mathrm{op}}$ to the equivariant S-category, but this doesn't see enough: it's like only choosing the trivial representation. This is nontrivial to see, though.

So we need extra structure in the orbit category, and this will come through extra structure in $\mathscr{O}_G$. We'll add extra maps between G/H and G/K called **transfer maps**.

Let M be a G-manifold embedded in some G-representation V, ν be its normal bundle, and τ be its tangent bundle. Then, $\tau \oplus \nu$ is trivial, so the Thom construction applied to it is just smashing with S^V. Thus, we obtain a sequence of maps

$$S^V \longrightarrow T\nu \longrightarrow T(\nu \oplus \tau) \longrightarrow S^V \wedge M_+. \tag{2.1.15}$$

Exercise 2.1.16. If you project $M \to *$, you get a map $S^V \to S^V$. Show that the degree of this map is the Euler characteristic $\chi(M)$.

If $K \subset H$ are subgroups of G, then (2.1.15) defines a map $S^V \to S^V \wedge (H/K)_+$, and we can induce this to get a map

$$(G/H)_+ \wedge S^V \longrightarrow (G/K)_+ \wedge S^V.$$

These are the additional maps we need. We'll go over them again in the next section, but these define the Burnside category, which suffices to define the equivariant stable category! What's really nice is how this derives from some of the oldest constructions and questions in homotopy theory: Thom's thesis can be considered the beginning of modern homotopy theory.

2.2. Transfers and the Burnside category

"The Burnside category is really nice, but the Sideburn category is a bit hairier."

The Burnside category is a souped-up version of the orbit category that accounts for transfers. There are at least five ways of defining it: we'll start with Definition 2.2.2 and later provide an alternative, Definition 3.1.5.

Recall that if M is a G-manifold, we can equivariantly embed it in a finite-dimensional real G-representation V, so that we obtain a sequence of maps $S^V \to T\nu \to T(\nu \oplus \tau) \cong S^V \wedge M_+ = \Sigma^V M_+$. (Here ν is the normal bundle of M and τ is the tangent bundle.) Thinking of this as a transfer $M \to *$, we'd like to do something similar for G/H. Namely, if $K \subset H$ are subgroups of G, we'd like to define a transfer map $G/K \to G/H$, a stable map (i.e. one that lives in $\operatorname{colim} \operatorname{Map}(\Sigma^V G/H, \Sigma^V G/K)$). This will be a "wrong-way" map.

Applying $G \times_H -$ to $H/K \to *$, we get a map $G/K \to G/H$. The transfer for $G/K \to G/H$ is via the transfer $H/K \to *$, induced up from a map $H/K \hookrightarrow V$. Here, V is a G-representation, and the embedding is H-equivariant, where H acts on V by restriction. Thus, we obtain a map $S^V \to S^V \wedge H/K$.

We want to understand how induction works on H-spaces whose structure has been defined by restriction from G-spaces.

Exercise 2.2.1. Show that if X is a G-space and $i_H^* X$ is the H-space defined by restriction, $G \wedge_H i_H^* X \cong G/H_+ \wedge X$.

So our map $G \wedge_H S^V \to G \wedge_H (S^V \wedge H/K_+)$ is identified with a map $G/H_+ \wedge S^V \to G/K_+ \wedge S^V$.

Consider the maps in $\mathrm{Hom}_{\mathscr{O}_G}(G/H_1, G/H_2)$ that are determined by subconjugacy, i.e. $G/H_1 \xrightarrow{\cong} G/g^{-1}H_1 g \to G/H_2$ (we assume $g^{-1}H_1 g \subset H_2$). There's a covariant functor from $\mathscr{O}_G$ to the (equivariant) S-category which sends a map f to its representative in the colimit, and we also have a contravariant functor: $G/H_1 \to G/H_2$ takes the transfer map, then the inverse of the conjugation isomorphism.

Definition 2.2.2. The **Burnside category** B_G is the full subcategory of the S-category spanned by the orbits G/H.

It's a theorem that every map in B_G is a composition of two maps in the images of the two functors $\mathscr{O}_G \to \mathsf{S}$ described above, which is pretty neat.

The Burnside category is enriched in Top, because the S-category is.

Definition 2.2.3. The **algebraic Burnside category** is $\pi_0(B_G)$, which is enriched in Ab. A **Mackey functor** is a functor $\pi_0 B_G \to \mathsf{Ab}$.

Mackey functors will be our replacement for coefficient systems. They arise in other contexts, so you could care about Mackey functors without caring about equivariant stable homotopy theory (well, you probably secretly do anyways). For that reason, there are definitions of Mackey functors that are less homotopical.

We'd like to define the category of G-spectra to be "spectral Mackey functors,"[8] i.e. functors from B_G to spectra. We need to develop tools for this first: we haven't developed a good model for spectra (in particular, a point-set model, not a model of the stable homotopy category), and we want these functors to be enriched, which will require some more work. We'll now begin to develop the tools to surmount these problems.

REMARK. Once we have a good category Sp^G of G-spectra, we can provide another, equivalent definition of the Burnside category, as the full subcategory of Sp^G spanned by $\Sigma^\infty_+ G/H$. ◀

Recollections about transfers. One concrete example of a transfer map comes from a finite cover $\widetilde{X} \to X$: summing over the fibers defines a transfer map $H^*(\widetilde{X};\mathbb{Q}) \to H^*(X;\mathbb{Q})$.

Transfers also arise in group cohomology $H^*(G;M)$, where M is a G-module. This is computed by resolving $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module by some resolution $P_\bullet$, then computing $H^*(\mathrm{Hom}_G(P_\bullet, M))$. Now suppose $H \subset G$; there are two adjoints to the restriction functor $\mathsf{Mod}_G \to \mathsf{Mod}_H$.

- The left adjoint sends an H-module $N \mapsto \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N$. This is called the **induced G-module** $\mathrm{Ind}_H^G N$.
- The right adjoint sends $N \mapsto \mathrm{Map}_H(\mathbb{Z}[G], N)$. This is called the **coinduced G-module**, written $\mathrm{CoInd}_H^G N$.

These definitions, and their names, should look familiar.

Now suppose $H \subset G$ is finite-index, e.g. when G is a finite group. Then

$$\mathrm{Ind}_H^G N \cong \bigoplus_{g \in G/H} gN \qquad \text{and } \mathrm{CoInd}_H^G N \cong \prod_{g\in G/H} gN.$$

In an additive category, finite sums and finite products are isomorphic, so $\mathrm{Ind}_H^G N \cong \mathrm{CoInd}_H^G N$. When G is finite, the analogous statement about TODO certain sums and products uniquely characterizes the equivariant stable category.[9]

So if $H \subset G$, we'd like to build a transfer map $H^*(H;M) \to H^*(G;M)$, where M is a G-module made an H-module by restricting. Induction is left adjoint to the restriction functor i_H^*, so the counit of the adjunction is a map $\eta\colon \mathrm{Ind}_H^G i_H^* M \to M$. In particular, $H^*(H;M) \cong H^*(G;\mathrm{Ind}_H^G M) \xrightarrow{\eta} H^*(G;M)$, and this is the desired transfer map.

As group cohomology $H^*(G;M)$ is the cohomology of BG in the local system defined by M, you could ask whether the transfer map is induced from a map $BG \to BH$ (in Top). This is true, and the map is a fibration.[10]

2.3. Diagram spectra

This section is motivated by [MMSS01], an excellent paper that constructs the point-set stable category using diagram spectra. You should absolutely read this paper; it's a masterwork of exposition and making things look simple and clear in retrospect.

The approach of diagram spectra is different from, but equivalent to, the approaches taken in [LMS86, May96]. Our goal is to define a complete, cocomplete, symmetric monoidal category Sp such that

- the S-category is a full subcategory of Sp, and
- there is a symmetric monoidal functor $\Sigma^\infty : \mathsf{Top} \to \mathsf{Sp}$, which is left adjoint to a right adjoint $\Omega^\infty : \mathsf{Sp} \to \mathsf{Top}$.

There's a sense in which Sp is the smallest category satisfying these hypotheses, or that you get it by adding limits and colimits to S. In particular, we are constructing a stable analogue of the category of topological spaces, *not* its homotopy category.

REMARK. Historically, [CITE ME: Boardman] constructed the stable homotopy category as a formal completion of the S-category. Then, people tried to find "point-set models," stable model categories whose homotopy categories

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[8] As devloped in the model categorical setting in [GM13], and in the ∞-categorical setting in [Bar17].

[9] This is an instance of some life advice from Mike Hill: in order to avoid being confused by constructions in equivariant stable homotopy theory, try computing the analogous construction in group cohomology first. Group cohomology is very concrete, and so it's definitely worth thinking this through in this case.

[10] If you don't restrict to finite subgroups, this is not quite true: if $H \subset G$ is a subgroup of infinite index, the map in question only exists on the spectrum level.

are isomorphic to Boardman's category. There are several options, but explicit proofs that their homotopy categories are equivalent to Boardman's are rare in the literature. ◂

Definition 2.3.1. By a **diagram** D we mean a small category, which we assume is enriched in Top and symmetric monoidal. The category of D**-spaces** is the category $\mathsf{Fun}(\mathsf{D},\mathsf{Top})$ of enriched functors.

The category of D-spaces is symmetric monoidal under **Day convolution**. The idea is to build a symmetric monoidal product via left Kan extension: if F and G are D-spaces, the functor $F \mathbin{\overline{\wedge}} G\colon (d_1,d_2)\mapsto F(d_1)\wedge G(d_2)$ is a $(\mathsf{D}\times\mathsf{D})$-space. To produce a D-space from this, let $\boxtimes\colon \mathsf{D}\to\mathsf{D}$ be the monoidal product on D, and consider the left Kan extension

(2.3.2)
$$\begin{tikzcd}
\mathsf{D}\times\mathsf{D} \arrow[r, "F\overline{\wedge}G"] \arrow[d, "\boxtimes"'] & \mathsf{Top} \\
\mathsf{D} \arrow[ur, dashed, "F\wedge G"'] &
\end{tikzcd}$$

This is our symmetric monoidal product $F\wedge G$. More explicitly,

$$\begin{aligned}(F\wedge G)(z) &= \operatorname*{colim}_{x\boxtimes y=z} F(x)\wedge G(y)\\ &:= \int^{x,y\in\mathsf{D}} F(x)\wedge G(y)\wedge \mathsf{D}(x\boxtimes y, z).\end{aligned}$$

This looks like the usual convolution, and the analogy with harmonic analysis can be taken further, e.g. in an unpublished paper of Isaksen-Behrens.

For any $d\in\mathsf{D}$, there's an **evaluation** functor $\mathrm{Ev}_d\colon \mathsf{Fun}(\mathsf{D},\mathsf{Top}_*)\to\mathsf{Top}_*$ sending $X\mapsto X(d)$. It's adjoint to $F_d\colon \mathsf{Top}_*\to\mathsf{Fun}(\mathsf{D},\mathsf{Top}_*)$ defined by

$$(F_dA)(e) := \mathrm{Map}_{\mathsf{D}}(d,e)\wedge A_+.$$

The unit for the symmetric monoidal structure on $\mathsf{Fun}(\mathsf{D},\mathsf{Top})$ is F_0S^0.

Let R be a **commutative monoid object** in $\mathsf{Fun}(\mathsf{D},\mathsf{Top})$, which approximately means there are maps $F_0S^0\to R$, $S^0\to R(0)$, and a unital, associative, commutative map $R(d)\wedge R(e)\to R(d\boxtimes e)$. For example, the unital condition is that the composition

$$R(d)\wedge S^0 \longrightarrow R(d)\wedge R(0) \longrightarrow R(d\boxtimes 0)\cong R(d)$$

must be the identity.

In this case, we can define the category Mod_R of R**-modules** in $\mathsf{Fun}(\mathsf{D},\mathsf{Top})$, those D-spaces M with an action map $\mu\colon R\wedge M\to M$ (satisfying the usual conditions). This is also a symmetric monoidal category (this requires R to be commutative), defined in the same way as the tensor product of modules over a ring: $M\wedge_R N$ is the coequalizer

$$M\wedge R\wedge N \rightrightarrows M\wedge N \longrightarrow M\wedge_R N.$$

Example 2.3.3 (Prespectra)**.** Let $\mathsf{D}=\mathbb{N}$, with only the identity maps. This is symmetric monoidal under addition: $[m]\boxtimes[n] := [m+n]$. The assignment $S_{\mathbb{N}}\colon [n]\mapsto S^n$ is a monoid in $\mathbb{N}$-spaces, and the category of $S_{\mathbb{N}}$-modules is classically called **prespectra**; the monoidal structure is the identification of $S^m\wedge S^n\cong S^{m+n}$.

Warning! $S_{\mathbb{N}}$ is *not* a commutative monoid! $S^n\wedge S^m \ncong S^m\wedge S^n$.

This was the cause of thirty years of pain and suffering in the community — they didn't know they were unhappy. People knew what the smash product should be on the homotopy category, and wanted a point-set model that's symmetric monoidal, unlike this example. ◂

Symmetric spectra are one answer, which we won't use in these notes. If all of this had been stated in terms of the Day convolution from the get-go, people probably would have figured out symmetric spectra as early as the 1960s, but hindsight is always clearer, and here we are. Symmetric spectra were introduced in [HSS00]; see [Sch12] for a detailed introduction.

Example 2.3.4 (Orthogonal spectra [May80])**.** Let $\mathscr{I}$ denote the category whose objects are finite-dimensional real inner product spaces V, and whose morphisms $\mathscr{I}(V,W)$ are the linear isometric isomorphisms $V\to W$.

In this category, $V\oplus W$ and $W\oplus V$ aren't equal, but are isomorphic, and the isomorphism between them is reflected in the flip between $S^n\wedge S^m$ and $S^m\wedge S^n$. In particular, the assignment $S_{\mathscr{I}}\colon V\mapsto S^V$ (the one-point compactification of V) is a *commutative* monoid, so the category of $S_{\mathscr{I}}$-modules is a symmetric monoidal category, called the category of **orthogonal spectra**. This is the model of the stable category that we will use. ◂

Example 2.3.5 ($\mathscr{W}$-spaces [And74])**.** Let $\mathscr{W}$ be the category of finite CW complexes (with either all maps or cellular maps; it doesn't really matter). $\mathscr{W}$-spaces are already like spectra, in a sense, in that they're modules over the identity functor $i\colon \mathscr{W} \hookrightarrow \mathsf{Top}$. There's a map $\varphi\colon A \to \mathrm{Map}(B, A \wedge B)$ sending $a \mapsto (b \mapsto a \wedge b)$, so if F is a $\mathscr{W}$-space, we have a sequence of maps

$$\begin{tikzcd} A \arrow[r, "\varphi"] & \mathrm{Map}(B, A \wedge B) \arrow[r] & \mathrm{Map}(F(B), F(A \wedge B)). \end{tikzcd}$$

Taking its adjoint defines a map

$$A \wedge F(B) \longrightarrow F(A \wedge B),$$

so F is a module over i. ◂

The assignment $n \mapsto \mathbb{R}^n$ defines a functor $\mathbb{N} \to \mathscr{I}$, and therefore a functor from prespectra to orthogonal spectra; with the right model structures, this induces an equivalence of their homotopy categories. Similarly, the assignment $V \mapsto S^V$ defines a functor $\mathscr{I} \to \mathscr{W}$, hence a functor from orthogonal spectra to $\mathscr{W}$-spaces, and this also will induce a homotopy equivalence.

Example 2.3.6 (Γ-spaces)**.** Let D be the category of finite based sets and based maps, e.g. $n_+ = \{0, 1, \dots, n\}$ with 0 as the basepoint. D-spaces are called **Γ-spaces**, and agree with Segal's notion of Γ-spaces [Seg74], which are defined differently. The multiplication comes from the map $\psi\colon 2_+ \to 1_+$ sending $1, 2 \mapsto 1$.

Let $d_i : n_+ \to 1_+$ send $j \mapsto \delta_{ij}$ (i.e. 1 if $i = j$, and 0 otherwise). A Γ-space is **special** if the induced map

$$X(n_+) \xrightarrow{\varphi_n} \prod_n X(1_+)$$

is a weak equivalence; it's **very special** if in addition the composition

$$\begin{tikzcd} X(1_+) \times X(1_+) & X(2_+) \arrow[l, "\varphi_2"', "\simeq"] \arrow[r, "\psi_*"] & X(1_+) \end{tikzcd}$$

induces a commutative monoid structure on $\pi_0 X(1_+)$.

Kan extension defines a functor from D to the category of finite CW complexes, and working with π_*-equivalences of these, one obtains a model structure on the category of Γ-spaces. This is Quillen equivalent to the category of **connective spectra**, i.e. those whose negative homotopy groups vanish. ◂

In the equivariant case, there's even more structure, and notions of "extra special" Γ-spaces, as we will see in §4.5.

Definition 2.3.7. A prespectrum is an **Ω-prespectrum** if for all n, $X_n \xrightarrow{\cong} \Omega^m X_{m+n}$.

Definition 2.3.8. If X is a prespectrum and $q \in \mathbb{Z}$, the q^{th} **homotopy group** of X is

$$\pi_q(X) := \operatorname*{colim}_n \pi_{n+q} X(n).$$

A **π_*-isomorphism** of prespectra is a map that induces an isomorphism on all homotopy groups.

Notice that negative homotopy groups exist, and may be nontrivial.

Remark. This is one approach to defining the stable category, and is not the only one. In [Ada74] (which is an excellent book), Adams uses a more naïve viewpoint of "cells first, maps later" which doesn't require such abstraction, but it would be a huge mess to prove that his model is complete or cocomplete. The diagram spectra approach rigidly separates point-set techniques (easy, but not as useful) from operations on the homotopy category (more useful, but harder), and this separation is often useful. The ∞-categorical perspective mashes it all together, which can be confusing, but is the only setting in which you can prove things such as the stable category being initial. ◂

We'll reintroduce G-actions soon, and this is pretty slick using orthogonal spectra: we can replace $\mathscr{I}$ with the category of finite-dimensional G-representations with invariant inner products. Orthogonal spectra also have a really nice homotopy theory relative to symmetric spectra (which have other advantages that don't apply as much to us).

Because diagram categories are presheaves on nice categories, they inherit some good properties from Top_*; in particular, they are complete and cocomplete, and limits and colimits may be taken pointwise. This is also true for categories of modules over monoids in D-spaces, though it requires more work to prove: computing colimits is a bit harder, just like how the free product of groups is more complicated than the direct product. Categories of rings are *not* bicomplete, though.

We'll use prespectra and orthogonal spectra to define Quillen equivalent models for the stable homotopy category. As such, familiar constructions from stable homotopy theory can be constructed as prespectra and orthogonal spectra.

Example 2.3.9 (Suspension spectra). Let $X \in \mathsf{Top}_*$. The **suspension spectrum** of X, denoted $\Sigma^\infty X$, is the stable homotopy type corresponding to the homotopy type of X.[11]

- In prespectra, the suspension spectrum of X is $\Sigma^\infty X\colon [n] \mapsto S^n \wedge X$. The $S_{\mathbb{N}}$-module structure is the data of the structure maps $S^m \wedge (S^n \wedge X) \to S^{m+n} \wedge X$.
- To define an orthogonal spectrum E, one must define for each finite-dimensional real inner product space V a pointed space $E(V)$ with an $\mathrm{O}(V)$-action and for each pair of such inner product spaces V and W, a structure map $S^V \wedge E(W) \to E(V \oplus W)$ that's $\mathrm{O}(V) \times \mathrm{O}(W)$-equivariant.

  Wanting to do this for every space forces our hand: we have to use the trivial action. The suspension spectrum of X sends $V \mapsto S^V \wedge X$, where the $\mathrm{O}(V)$-action is the usual action in the first component and the trivial action in the second component. The structure maps $S^V \wedge S^W \wedge X \to S^{V\oplus W} \wedge X$ are $\mathrm{O}(V) \times \mathrm{O}(W)$-equivariant, as desired.

The suspension spectrum of S^0 is the sphere spectrum $S_{\mathbb{N}}$ or $S_{\mathscr{I}}$. ◅

Example 2.3.10 (Eilenberg-Mac Lane spectra). If A is an abelian group, the **Eilenberg-Mac Lane spectrum** HA is the spectrum that represents cohomology with coefficients in A. If in addition A is a commutative ring, HA is a commutative monoid in spectra, which defines the ring structure on A-cohomology.

- There is an identification $K(A,n) \xrightarrow{\cong} \Omega K(A,n+1)$; let $i_n\colon \Sigma K(A,n) \to K(A,n+1)$ be its adjoint. As an $\mathbb{N}$-space, $HA\colon [n] \mapsto K(A,n)$; the map i_n makes it into a prespectrum. If A is also a commutative ring, one obtains maps $K(A,m) \wedge K(A,n) \to K(A,m+n)$, and these satisfy the axioms to ensure that HA is a commutative ring spectrum.
- For orthogonal spectra, the construction is more complicated, since we must choose a model for $K(A,n)$ with an O_n-action on it. In particular, we must assume A is countable. Given an inner product space V, the A**-linearization** $A[S^V]$ of S^V is, as a set, A tensored with the reduced free abelian group on K (so the basepoint maps to zero), topologized as the quotient

  $$\coprod_{k=0}^{\infty} A^k \times (S^V)^k \twoheadrightarrow A[S^V], \qquad (a_1,\dots,a_k,x_1,\dots,x_k) \longmapsto \sum_{j=1}^{k} a_j \cdot x_j.$$

  This is a model for $K(A, \dim V)$, but has an $\mathrm{O}(V)$-action induced from the $\mathrm{O}(V)$-action on S^V. The structure map $S^V \wedge A[S^W] \to A[S^{V\oplus W}]$ sends

  $$v \wedge \left(\sum_j a_j \cdot w_j\right) \longmapsto \sum_j a_j \cdot (v \wedge w_j), \tag{2.3.11}$$

  and this is $\mathrm{O}(V) \times \mathrm{O}(W)$-equivariant, so defines an orthogonal spectrum. If A is a commutative ring, the ring spectrum structure on HA is defined by the multiplication maps $\mu\colon A[S^V] \wedge A[S^W] \to A[S^{V\oplus W}]$ sending

  $$\mu\colon \left(\sum_i a_i x_i\right), \left(\sum_j b_j y_j\right) \longmapsto \sum_{i,j} (a_i b_j)(x_i \wedge y_j),$$

  and the unit maps $e\colon S^V \to A[S^V]$ send $x \mapsto 1 \cdot x$. This construction is discussed in more detail in [Sch12, Example I.1.14] and [Sch17, Example V.1.9]. ◅

Example 2.3.12 (Thom spectra). TODO ◅

Definition 2.3.13. Let $f : X \to Y$ be a map of D-spaces (or prespectra or orthogonal spectra).

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[11] For a space X without basepoint, X_+ denotes the based space $X \amalg *$, with the extra point as a basepoint. Then, one considers $\Sigma^\infty X_+$.

- f is a **level equivalence** if for all $d \in \mathsf{D}$, $f(d)\colon X(d) \xrightarrow{\simeq} Y(d)$ is a weak equivalence.
- f is a **level fibration** if for all $d \in \mathsf{D}$, $f(d)$ is a fibration.

That is, f is a natural transformation, and it acts through weak equivalences (resp. fibrations).

Theorem 2.3.14. *The category of* D*-spaces has a model structure, called the* ***level model structure****, in which the weak equivalences are the level equivalences and the fibrations are level fibrations. Moreover, this model category is cofibrantly generated.*

Exercise 2.3.15. Starting with the usual model structure on Top_*, construct the level model structure.

The “cofibrantly generated” part means that cofibrant objects behave like CW complexes, and in particular there is a theory of cellular objects. If $F_d\colon \mathsf{Top}_* \to \mathsf{Fun}(\mathsf{D}, \mathsf{Top}_*)$ is the left adjoint to Ev_d we constructed above, then the **generating cofibrations** are the maps $F_d(S^{n-1}_+ \to D^n_+)$ for each $n \geq 1$ and $d \in \mathsf{D}$, and the **acyclic cofibrations** are $F_d(D^n_+ \to (D_n \times I)_+)$ for each $n \geq 1$ and $d \in \mathsf{D}$.

Since all spaces are fibrant, all D-spaces are fibrant in the level model structure. The cofibrant objects are the retracts of **cellular objects**, which are built by iterated pushouts

$$\begin{tikzcd}
\bigvee F_d S^{n-1}_+ \arrow[r] \arrow[d] & X_n \arrow[d] \\
\bigvee F_d D^n_+ \arrow[r] & X_{n+1}. \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

While this is all nice, it’s not what we’re looking for, as it contains no information about stable phenomena. It’s like the category of spaces, just with more of them. For example, it’s not even true that $X \to \Omega\Sigma X$ is a weak equivalence, which is important if you want Ω and Σ to be homotopy inverses. We’ll define the correct model structure in the next section.

Recall that we defined a π_*-isomorphism of prespectra to be a map $f : X \to Y$ such that $\pi_q f : \pi_q X \to \pi_q Y$ is an isomorphism for all q. We’ll extend this to orthogonal spectra: let $U\colon \mathsf{Sp}^{\mathscr{I}} \to \mathsf{Sp}^{\mathbb{N}}$ be pullback by the map $[n] \mapsto \mathbb{R}^n$, i.e. $UX([n]) = X(\mathbb{R}^n)$. U is right adjoint to a left Kan extension $P\colon \mathsf{Sp}^{\mathbb{N}} \to \mathsf{Sp}^{\mathscr{I}}$.

Definition 2.3.16. A map of orthogonal spectra $f : X \to Y$ is a **π_*-isomorphism** if $Uf : UX \to UY$ is a π_*-isomorphism of prespectra.

We also defined an Ω-spectrum in prespectra, or an Ω-prespectrum, to be a prespectrum where the adjoints to the structure maps $X_n \xrightarrow{\cong} \Omega^m X_{n+m}$ are homeomorphisms. This is a pretty rigid condition, and so Ω-prespectra have nice properties.

Definition 2.3.17. Similarly, we define an **Ω-spectrum** in orthogonal spectra to be an orthogonal spectrum X such that the adjoints to the structure maps $X(U) \xrightarrow{\cong} \Omega^V X(U \oplus V)$ are homeomorphisms.

Classically, there were prespectra and then there were spectra (or Ω-spectra), and you would use some “spectrification” functor that took a prespectrum and produced a spectrum of the same homotopy type. Turning the adjoint maps into homeomorphisms looks difficult and is, as it involves some categorical and point-set wizardry. If you like this stuff, check out the appendix of [LMS86]. The first point-set symmetric monoidal model for the stable category [EKMM97] relies on this and even more magic, both clever and surprising.

2.4. Homotopy theory of diagram spectra

> *“But why point-set models?”*
> *“Are you serious, Derek? I just told you, like, a second ago!”*

In this section, we introduce the stable model structure on D-spectra, which provides stably homotopical information.

Since D-spaces are enriched over spaces, it’s possible to tensor with the interval and therefore define homotopies of D-spaces in the same way as for spaces. As usual, $[X, Y]$ will denote the set of homotopy classes of maps $X \to Y$.

Definition 2.4.1. A **stable equivalence** of prespectra is a map $f : X \to Y$ such that for all Ω-prespectra Z, the induced map $[Y, Z] \to [X, Z]$ is an isomorphism.

Theorem 2.4.2. *There are **stable model structures** on the categories of* D*-spaces, prespectra, and orthogonal spectra in which the weak equivalences are stable equivalences. For* $\mathsf{Sp}^{\mathbb{N}}$ *and* $\mathsf{Sp}^{\mathscr{I}}$*, the stable equivalences are the same as the* π_**-isomorphisms.*

Remark. You can make the same construction for symmetric spectra, but the stable equivalences are not the same as π_*-isomorphisms, and homotopy groups are consequently finickier. This ultimately comes from the fact that quotients $\mathrm{O}(n+k)/\mathrm{O}(n)$ get more highly connected as n and k grow in a way that quotients of symmetric groups don't. In any case, since symmetric spectra don't behave so well in the equivariant case, we won't use them. ⋖

The stable model structure does in fact manifest stable phenomena: the map $X \to \Omega\Sigma X$ is a weak equivalence.[12]

If X and Y are Ω-prespectra and $f : X \to Y$ is a level equivalence, then f is a π_*-isomorphism. That is, the weak equivalences in the stable model structure contain the weak equivalences in the level model structure, and it's possible to use **Bousfield localization** to obtain the stable model structure from the level model structure.

Let C be a model category and S be a set of maps in C.[13] Bousfield localization produces a new model structure $L_S\mathsf{C}$ on C in which the morphisms in S are weak equivalences. The homotopy category and homotopy (co)limits change, but all point-set phenomena remain the same. Localization is given by fibrant replacement.

Definition 2.4.3. Let C be a topologically enriched model category and S be as above.

- An S**-local object** in C is an object X such that for all $f : Y \to Z$ in S, the induced map
$$\mathrm{Map}_{\mathsf{C}}(Z,X) \xrightarrow{\simeq} \mathrm{Map}_{\mathsf{C}}(Y,X)$$
is a weak equivalence.
- A map $f : X \to Y$ is an S**-local equivalence** if for all S-local objects Z, the induced map
$$\mathrm{Map}_{\mathsf{C}}(Y,Z) \xrightarrow{\simeq} \mathrm{Map}_{\mathsf{C}}(X,Z)$$
is a weak equivalence.

The following theorem is due to many people, but Hirschhorn's formulation is particularly nice.

Theorem 2.4.4 ([Hir03]). *Let* C *be a cofibrantly generated,* Top*-enriched model category. Then, the Bousfield localization* $L_S\mathsf{C}$ *always exists. Moreover, the weak equivalences are the* S*-local equivalences, the fibrant objects are the* S*-local objects, and the cofibrations are exactly those in* C*.*

Example 2.4.5. One way this is used is to localize the category of spectra such that the S-local equivalences are detected by $-\wedge H\mathbb{Q}$ or $-\wedge H\mathbb{F}_p$. This is a slick way to construct the rationalization or p-completion, respectively, and in particular makes the localization map functorial. Thus one obtains the rational (or p-completed) stable homotopy category. ⋖

The stable model structure is obtained by Bousfield localization at the stable equivalences (Definition 2.4.1). It follows immediately that Ω-prespectra are local objects and stable equivalences are weak equivalences.

Proposition 2.4.6. *The stable model structure is stable, i.e.* $X \to \Omega\Sigma X$ *is a* π_**-isomorphism.*

Proof. On homotopy groups, this is asking for $\mathrm{colim}_n\, \pi_{q+n}X_n \to \mathrm{colim}_n\, \pi_{q+n}\Omega\Sigma X_n$ to be an isomorphism. But by the Freudenthal suspension theorem, these colimits stabilize to the same stable homotopy group. ⊠

This implies that when X is an Ω-prespectrum, $\pi_q X = \pi_q X_0$, and for $q < 0$, we can define $\pi_q X = \pi_0(X_{-q})$.

Theorem 2.4.7. *The adjunction* $P\colon \mathsf{Sp}^{\mathbb{N}} \rightleftarrows \mathsf{Sp}^{\mathscr{I}} : U$ *is a Quillen equivalence, and therefore* $\mathrm{Ho}(\mathsf{Sp}^{\mathbb{N}}) \cong \mathrm{Ho}(\mathsf{Sp}^{\mathscr{I}})$ *as triangulated categories.*

This homotopy category is called the stable category. It has a triangulated structure in which suspension Σ is the shift functor and the distinguished triangles are the cofiber sequences $X \xrightarrow{f} Y \to C_f$, where C_f is the homotopy cofiber of f.[14]

---

[12] If you like ∞-categories, you can say that the level model category is an ∞-category of presheaves, and this is different from the ∞-category of spectra, which is presented by the stable model category.

[13] We want S to contain the weak equivalences in C, but there are important set-theoretic issues. Often, one specifies that S contains a generating set (under filtered colimits) of the weak equivalences of C.

[14] You can also define the distinguished triangles in terms of fiber sequences.

Another sense in which the stable category is stable is that both fiber and cofiber sequences induce long exact sequences of homotopy groups, instead of just fiber sequences.

We'd like to construct a Quillen adjunction $\Sigma^\infty\colon \mathsf{Top}_* \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{Sp}^{\mathscr{I}} : \Omega^\infty$. Ω^∞ is just Ev_0, evaluating at the zero space. If F_d is the adjoint to Ev_d (so that $(F_dA)(e) = \mathrm{Map}_{\mathsf{D}}(d,e)_+ \wedge A$), then we can define $\Sigma^\infty A := F_0 A \wedge S$, where $S = S_{\mathbb{N}}$ for prespectra and $S = S_{\mathscr{I}}$ for orthogonal spectra.

If R is a monoid in D-spaces, the category of R-modules is equivalent to a category of diagram spaces over a more complicated diagram D_R. This is useful because diagrams are nice, and some things become less complicated. The recipe is that D_R is the category whose objects are the same as D and whose morphisms are

$$\mathrm{Map}_{\mathsf{D}_R}(d,e) = \mathrm{Map}_{\mathsf{Mod}_R}(F_d S^0 \wedge R, F_e S^0 \wedge R). \tag{2.4.8}$$

That is, we take the suspension spectrum of the sphere shifted by d and that of the sphere shifted by e.

Exercise 2.4.9. Show that for $\mathsf{D} = \mathbb{N}$, the structure maps for prespectra come out of (2.4.8) for $R = S_{\mathbb{N}}$. (This is an adjunction game.) The same is true for orthogonal spectra and $S_{\mathscr{I}}$.

This feels like a Spanier-Whitehead trick, but constructs the right category. In particular, in D_R-spaces, Σ^∞ is the left adjoint to evaluating at 0.

It's possible to bootstrap this to define model categories of ring spectra, i.e. algebras over $S_{\mathscr{I}}$.

Definition 2.4.10. A **monad** M on a category C is an endofunctor of C which is a monoid in the functor category $\mathsf{Fun}(\mathsf{C},\mathsf{C})$.

That is, there's a natural transformation $\mu\colon M^2 \to M$ and a unit, and μ is associative and unital in that the relevant diagrams commute.

There are lots of examples: we've already seen that if G is a group, the assignment $X \to G \times X$ is a monad. More generally, algebraic structures can usually be obtained monadically.

Definition 2.4.11. Let M be a monad on C. Then, the category $\mathsf{C}[M]$ of **algebras over** M is the category whose objects are pairs $X \in \mathsf{C}$ and structure maps $m\colon MX \to X$ satisfying associativity and unitality for M, and whose morphisms are the C-morphisms that are compatible with the structure maps.

The associativity diagram, for example, is

$$\begin{tikzcd}
M^2X \arrow[r, "m"] \arrow[d, "\mu"] & MX \arrow[d, "m"] \\
MX \arrow[r, "m"] & X.
\end{tikzcd}$$

We require this to commute.

Example 2.4.12. Let $\mathbb{T}\colon \mathsf{Sp}^{\mathscr{I}} \to \mathsf{Sp}^{\mathscr{I}}$ denote the **free associative algebra monad**, i.e.

$$\mathbb{T}X := \bigvee_{n\geq 0} X^{\wedge n},$$

so that the category of $\mathbb{T}$-algebras $\mathsf{Sp}^{\mathscr{I}}[\mathbb{T}]$ is the category of associative monoids in $\mathsf{Sp}^{\mathscr{I}}$. Similarly, let $\mathbb{P}\colon \mathsf{Sp}^{\mathscr{I}} \to \mathsf{Sp}^{\mathscr{I}}$ denote the **free commutative algebra monad**, so

$$\mathbb{P}X := \bigvee_{n\geq 0} X^{\wedge n}/\Sigma_n,$$

where Σ_n denotes the action of the symmetric group by permutations; thus, the category of $\mathbb{P}$-algebras $\mathsf{Sp}^{\mathscr{I}}[\mathbb{P}]$ is the category of commutative monoids in $\mathsf{Sp}^{\mathscr{I}}$ (i.e. commutative ring spectra). ◂

Lots of structures are monadic, e.g. groups are algebras over the **free group monad** in Set, and similarly for abelian groups, rings, etc. Monads very generally come from free structures in algebra; they also arise from adjunctions: an adjunction $F\colon \mathsf{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{D} : G$ defines a monad GF, with the structure map defined by the unit map $G(FG)F \to GF$. Many monads arise in this way.

There is always a free-forgetful adjunction $F\colon \mathsf{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{C}[M] : U$, which has to do with the Barr-Beck theorem. Suppose C is a model category. We'd like to lift this to a model structure on $\mathsf{C}[M]$ — when does $\mathsf{C}[M]$ have a model structure where the weak equivalences are determined by the forgetful functor U?[15] There are two issues.

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[15] This is not how we defined the model structure on G-spaces, but we'll use it to define model structures on rings and module spectra.

(1) Since C is complete, so is $\mathsf{C}[M]$: the arrows point the right way. But it's not always cocomplete.
(2) How do we define the model structure?

In order for $\mathsf{C}[M]$ to be cocomplete, we'll use a criterion about preserving certain colimits. Since monads tend to arise from adjunctions, the criterion definitely won't be true in general! There are seventeen versions of this criterion in Mac Lane, but we'll only need one, following Hopkins and McClure.

Definition 2.4.13. Let $f, g : X \rightrightarrows Y$ be two maps. A **reflexive coequalizer** is a coequalizer for f and g together with a simultaneous section $s : Y \to X$ for both f and g.

Exercise 2.4.14. Prove that if M preserves reflexive coequalizers, then $\mathsf{C}[M]$ is cocomplete. (This is hard, but worthwhile.)

Proposition 2.4.15 (Hopkins-McClure)**.** *Under very mild hypotheses, $\mathbb{T}$ and $\mathbb{P}$ preserve reflexive coequalizers.*

See [**EKMM97**] for a proof. $\mathsf{Sp}^{\mathscr{I}}$ satisfies these hypotheses, so we've addressed caveat (1).

Theorem 2.4.16 (Schwede-Shipley [**SS00**])**.** *Under mild hypotheses, $\mathsf{C}[M]$ inherits a model structure from C, where the fibrations are detected by U, and the generating cofibrations are M applied to the generating cofibrations of C.*

This is a very general theorem; the hard step is constructing a nice enough filtration on pushouts.

Warning! These hypotheses are met for the associative monad $\mathbb{T}$, but are *not* met by the commutative monad $\mathbb{P}$! This is another formulation of Lewis' paradox: for commutative ring spectra, you have to change the underlying homotopy theory.

Using these results, the stable model structure on $\mathsf{Sp}^{\mathscr{I}}$ induces one on $\mathsf{Sp}^{\mathscr{I}}[\mathbb{T}]$, in which the weak equivalences and fibrations are detected by those in $\mathsf{Sp}^{\mathscr{I}}$. For $\mathsf{Sp}^{\mathscr{I}}[\mathbb{P}]$, the quotient makes things harder: it has a model structure where the weak equivalences are **positive equivalences** (so those which are detected by **positive Ω-spectra**, i.e. those where $X_0 \to \Omega X_1$ need not be an equivalence). But this still doesn't behave very well. Namely, if you try to set the theory up for $\mathbb{P}$ to be the same as that for $\mathbb{T}$, then you get that $\Omega^\infty \Sigma^\infty S^0$ is a commutative topological monoid. It's a fact that any commutative topological monoid is a product of Eilenberg-Mac Lane spaces, and that $\Omega^\infty \Sigma^\infty S^0$ has nontrivial k-invariants. There's a short paper by Lewis which addresses this [**Lew91**], showing that there are five very reasonable axioms for the stable category that can't all be true! So people decided to forget about letting Σ^∞ be left adjoint to evaluation at 0, and it's okay, if not perfect.

REMARK. You might be used to thinking of these as the associative and commutative operads. Every operad determines a monad, and the operadic algebras become monadic algebras, but for $\mathbb{P}$ and $\mathbb{T}$, the explicit form of the monad makes it easier to analyze from the monadic viewpoint. ◃

Returning to diagram spectra, we've been putting a huge emphasis on strict symmetric monoidal structures, rather than just commutativity in the homotopy category. This is useful because it lets you do algebra: if R is a ring in the homotopy category, it's very hard to control the category of modules, e.g. the cofiber of a map of modules may not even be a module, the cyclic bar construction isn't a simplicial object, etc. Some people have tried to use operads to fix this, and this is extremely hard: operadic ring spectra are fine, but their modules are not. In a sense, Lurie's ∞-categorical machinery is designed to do this in a more modern way.

A lot of modern homotopy theory has been importing algebraic constructions about rings into homotopy theory, replacing tensor products with smash products. This has been very useful, yet can be hard, and everything is much more tractable with a point-set symmetric monoidal structure.

2.5. The equivariant stable category

"May your first talk be more peaceful."

In this section, we'll leverage the theory of diagram spectra developed over the past two sections in the equivariant setting, defining the equivariant stable category as the homotopy category of orthogonal G-spectra. In this context, we must talk about universes, making choices of which orbits G/H are dualizable. This is closely related to choosing representations V such that $G/H \hookrightarrow V$ equivariantly.

Definition 2.5.1. A **universe** is a countably infinite-dimensional real inner product space with a G-action through isometries. There is some collection R of irreducible representations such that U contains countably many copies of each irreducible in R, and R always has all of the trivial representations.[16]

Sometimes we'll ask for more structure, such as R being closed under tensor products. The inclusion of the trivial representation is what makes this a strict generalization of ordinary stable homotopy theory: for example, if $U = \mathbb{R}^\infty$ with the trivial G-action, the homotopy groups are the nonequivariant homotopy groups.[17] Another common choice of U is a **complete universe**, containing all irreducible representations infinitely often, so all orbits are dualizable.

G-prespectra were defined classically in a manner similar to prespectra. They mix point-set and homotopical data, which is a little odd.

Definition 2.5.2.

- A **G-prespectrum** X is the data of a G-space $X(V)$ for every finite-dimensional subspace $V \subset U$ and G-maps $S^W \wedge X(V) \to X(V \oplus W)$ for each pair V and W, such that the structure maps are associative.
- A G-prespectrum is an **Ω-prespectrum** if for all V and W, the adjoint to the structure map is a homeomorphism: $X(V) \xrightarrow{\cong} \Omega^W X(V \oplus W)$.
- The **homotopy groups** of a G-prespectrum are

$$\pi_q^H(X) := \operatorname*{colim}_V \pi_q^H \Omega^V X(V)$$
$$\pi_{-q}^H(X) := \operatorname*{colim}_{\substack{V \supset \mathbb{R}^m \\ m>0}} \pi_0^H \Omega^{V-\mathbb{R}^m} X(V).$$

Here $q \geq 0$, $\mathbb{R}^m$ denotes the trivial representation of dimension m, which we specified was in U, and $V - \mathbb{R}^n$ denotes the orthogonal complement.

Proposition 2.5.3. *There is a model structure on G-prespectra where the weak equivalences are the stable equivalences (equivalently, π_*-isomorphisms).*

The proof is identical to the nonequivariant case. One cool aspect of this is that homotopy groups are determined by trivial representations, and we'll see that for orthogonal G-spectra, this is also true at the point-set level. It's nice to not have to carry around the entire universe, just the trivial representations. We'll use this philosophy in the proof of Theorem 2.6.2.

Definition 2.5.4. The **equivariant stable homotopy category** (**structured by** U) is the homotopy category of G-prespectra.

This is the category in which we have Poincaré duality for the orbits G/H that embed in U. The functors $\Sigma^V := S^V \wedge -$ and $\Omega^V := \mathrm{Map}(S^V, -)$ are inverse equivalences.

REMARK. It would be nice to write down some sort of equivariant analogue of a triangulated structure on the equivariant stable category, but trying to get an $RO(G)$-grading (i.e. shifts by arbitrary representations of G) does not work except in ad hoc ways for $G = C_2$. However, you can keep track of the action of the Picard group $\mathrm{Pic}(\mathsf{Sp}^G)$ on the equivariant stable category. ◂

Orthogonal G-spectra. Let's build a point-set model of the equivariant stable category. Let V and V' be finite-dimensional irreducible G-representations (by which we mean finite-dimensional subspaces of U), and let $I_G(V, V')$ be the linear, G-equivariant isometries.[18]

Definition 2.5.5. The **complement bundle** $E(V, V')$ is the subbundle of the product bundle $I_G(V, V') \times V'$ consisting of pairs (f, x) such that $x \in V' - f(V)$. Let $\widetilde{I}_G(V, V')$ denote the Thom space of this bundle.

---

[16] Ideally, U should not contain other representations. Sometimes people are vague about this, where there may be other representations and you ignore them, but it's better to just not have them.

[17] More explicitly, this is a stabilization of the category of presheaves on BG. Depending on your definitions, this is more or less a tautology — stabilization in ∞-categories is defined by taking spectrum objects.

[18] Unlike the nonequivariant case, we're not taking isometric isomorphisms, so maps in $I_G(V, V')$ must be injective, but may have nontrivial cokernel.

We're trying to create spheres: $\widetilde{I}_G(V,V')$ should topologize $S^{V'-f(V)}$. For example, if $V\subset V'$, this is isomorphic to $\mathrm{O}(V')\wedge_{\mathrm{O}(V'-V)} S^{V'-V}$.

Let $\widetilde{I}_G$ denote the category whose objects are finite-dimensional subspaces V of U and whose morphisms are $\widetilde{I}_G(V,V')$.

Definition 2.5.6. An **orthogonal G-spectrum** is a Top-enriched functor $\widetilde{I}_G \to G\mathsf{Top}$. The category of orthogonal G-spectra is denoted Sp^G, or Sp^G_U if the universe needs to be explicit.

For the nonequivariant case, we defined D-spaces and spectra to be modules over a certain monoid in them, and then showed that you could think of them as diagram spaces for a more complicated diagram. Here, we've done this all at once — it's simpler to define $\widetilde{I}_G$ than find a module in a simpler category of diagram spaces.

There is a forgetful functor U from Sp^G to the category of G-prespectra; we define a map f in Sp^G to be a **π_*-isomorphism** if Uf is a π_*-isomorphism. Sp^G is symmetric monoidal under Day convolution, just as before.

Theorem 2.5.7. Sp^G *has a stable model structure in which the weak equivalences are π_*-isomorphisms, the fibrations are levelwise fibrations, and the generating cofibrations are*

$$F_V\big((G/H\times S^{n-1})_+ \longrightarrow (G/H\times D^n)_+\big).$$

Here, F_V is adjoint to evaluation at V, as before.

Moreover, the smash product is compatible with the model structure. If X is cofibrant, then $X\wedge Y$ computes Tor $X\wedge^{\mathbf{L}} Y$. Another way to say this is that if X is cofibrant, $X\wedge -$ preserves weak equivalences. The proof is where cofibrantly generated model categories shine: $X\wedge -$ is a colimit, so it commutes with colimits, and cofibrant objects are retracts of cell complexes, so you can reduce to the case where X is a single cell, which admits a direct proof. Once again the stable homotopy category behaves like homological algebra.

Example 2.5.8. Let $X\in\mathsf{Sp}$. Then, "letting G act trivially on X" defines a G-spectrum also denoted X, such that for any n-dimensional G-representation V, $X(V)$ is, as an O_n-space, equal to X_n. However, the G-action on $X(V)$ need not be trivial: V and its G-invariant inner product determine a map $G\to O(V)$, and G acts on $X(V)$ through this map. Since the structure maps respect the G-action, this defines an orthogonal G-spectrum.

One important example is the **equivariant sphere spectrum** $\mathbb{S}$: $\mathbb{S}(V)=S^V$. It is the unit of Sp^G. ◅

Example 2.5.9 (Suspension spectra)**.** Let X be a based G-space; then, the **suspension spectrum** of X, denoted $\Sigma^\infty X$, is the G-spectrum whose component on a G-representation V is

$$\Sigma^\infty X(V) := X\wedge S^V.$$

Given an equivariant map $V\to W$, the structure map $\Sigma^\infty X(V)\to\Sigma^\infty X(W)$ is its one-point compactification smashed with id_X.

In particular, if you give S^0 the trivial G-action, $\Sigma^\infty S^0\cong\mathbb{S}$. ◅

Example 2.5.10 (Real K-theory)**.** Real K-theory, as discovered by Atiyah [Ati66], is a less generic example. Throughout, $\mathbb{R}^{p,q}$ denotes the C_2-representation that's p copies of the trivial representation and q copies of the sign representation, and $\mathbb{C}^n\cong\mathbb{R}^{n,n}$ has a C_2-action by complex conjugation.

Definition 2.5.11. Let X be a C_2-space,[19] with $\varphi\colon X\to X$ the action of the nonzero element of C_2. A **real equivariant vector bundle** is a complex vector bundle $\pi\colon E\to X$ together with a C_2-action $\widetilde{\varphi}\colon E\to E$ such that

- π is a C_2-map, and
- for each $x\in X$, the map $\widetilde{\varphi}\colon E_x\to E_{\varphi(x)}$ is antilinear.

The Grothendieck group of real equivariant vector bundles on X is denoted $KR^0(X)$.

Theorem 2.5.12 (Real equivariant Bott periodicity)**.** *There's a natural isomorphism* $\beta\colon KR^0(S^{1,1}\wedge X)\cong KR^0(X)$.

Using this, one can define $KR^{p,q}(X) := KR^0(S^{p+n,q+n}\wedge X)$ for an $n>p,q$ as with ordinary K-theory, and obtain an $\mathrm{RO}(C_2)$-graded cohomology theory. We want to lift this to a C_2-spectrum.

Lemma 2.5.13. *In $\widetilde{I}_{C_2}$, the complex linear inclusions $\mathbb{C}^n\hookrightarrow\mathbb{C}^{n+1}$ are cofinal.*[20]

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[19] We require X to be compact and Hausdorff, as in ordinary K-theory, but are there any other assumptions?

[20] TODO: I'm not sure whether this is the right word, or whether this is why the construction works. The literature is terse as to why you can do this. I'd also like to be more explicit about this.

Hence, to define a C_2-spectrum E, it suffices to define structure maps $S^{1,1} \wedge E(\mathbb{C}^n) \to E(\mathbb{C}^{n+1})$.

For KR, let $KR(\mathbb{C}^n) := \mathbb{Z} \times BU$, with C_2 acting by complex conjugation on BU and trivially on $\mathbb{Z}$. Then, the structure map $S^{1,1} \wedge KR(\mathbb{C}^n) \to KR(\mathbb{C}^{n+1})$ is the map $\mathbb{Z} \times BU \to \mathbb{Z} \times BU$ which classifies Bott periodicity. Hence we obtain a C_2-spectrum KR, called **real equivariant K-theory**, and Bott periodicity tells us that $\Sigma^{1,1}KR \simeq KR$. ◄

We'll discuss another important example, equivariant Eilenberg-Mac Lane spectra, in Construction 3.4.7.

Functors on Sp^G. Let $H \subset G$ be a subgroup, so restriction defines a functor $i_H \colon \mathsf{Sp}^G \to \mathsf{Sp}^H$. Just as for G-spaces, i_H has a left adjoint $G_+ \wedge_H -$ and a right adjoint $F_H(G_+, -)$, and these are constructed space-wise.

Proposition 2.5.14. *$G_+ \wedge_H -$ and $F_H(G_+, -)$ are left (resp. right) Quillen functors.*

Remark. You might worry what universe you end up in after applying i_H — naïvely you get the H-representations that arise from restriction of G-representations, but you might want all H-representations. The way to address this is to use a change-of-universe functor, which we'll discuss in §4.6. This is a good thing to not be sloppy about, but is annoying, and is why people try to get the universe out of the point-set category. In any case, as in group cohomology, induction and coinduction are isomorphic through the **Wirthmüller isomorphism** (Theorem 2.6.2). ◄

One thing that's nice to do with G-spectra is take fixed points. There are *three* different ways to do this, namely homotopy fixed points, categorical fixed points, and geometric fixed points, and keeping them separate in your head is important.

Sp^G is a *closed* symmetric monoidal category, meaning it has internal function objects. This is true for D-spaces in general, with internal function objects

$$F_{\mathsf{D}}(X, Y)(n) := F(X, Y(n)).$$

This is exactly like the mapping complex of two chain complexes: you're looking at chain maps from X to shifts of Y.

Sp^G is tensored and cotensored over G-spaces, meaning we can smash G-spectra with G-spaces, and take function objects from G-spaces to G-spectra: if A is a G-space, tensoring with A is $A_+ \wedge -$ applied levelwise, and cotensoring with A is $F(A_+, -)$ applied levelwise.

Definition 2.5.15.

- The **homotopy fixed points** is the functor $X \mapsto F(EG_+, X)^G$. (Here, EG_+ is a G-space, so we're using the cotensor of G-spaces and G-spectra.)
- The **categorical fixed points** is the functor $(-)^H \colon X \mapsto F(G/H_+, X)$. As the notation suggests, this is analogous to $(-)^H$ on G-spaces.

Warning! You might hope that $(\Sigma^\infty A_+)^H = \Sigma^\infty A_+^H$, so that fixed points of spaces lead to fixed points of spectra, but this is *not* true. It has a more complicated description called **tom Dieck splitting** as a wedge of other pieces.

Definition 2.5.16. The **geometric fixed points**, denoted Φ^H or also X^{gH}, is the unique functor such that

(1) it's (derived) symmetric monoidal: if X and Y are cofibrant, $\Phi^H(X) \wedge \Phi^H(Y) \xrightarrow{\cong} \Phi^H(X \wedge Y)$.
(2) Φ^H preserves homotopy colimits.
(3) $\Phi^H \Sigma^\infty X \cong \Sigma^\infty X^H$.

Of course, uniqueness (up to a contractible space of choices) is a theorem, but it's true. We'll also give a preferred construction.

We'll prove that categorical and geometric fixed points detect weak equivalences.

Theorem 2.5.17.

(1) *A map $X \to Y$ in Sp^G is a weak equivalence iff $X^H \to Y^H$ is a weak equivalence for all subgroups $H \subset G$.*
(2) *A map $X \to Y$ in Sp^G is a weak equivalence iff $\Phi^H X \to \Phi^H Y$ is a weak equivalence for all subgroups $H \subset G$.*

This leads one to consider diagrams on categorical or geometric fixed points: because of this perspective, one can define Tate spectra using this philosophy, and this is particularly useful when applied to S^1-spectra. It was originally developed by John Greenlees, and has recently been repopularized and is being mined for applications, such as Nikolaus-Scholze's description of topological cyclic homology [NS17].

We're going to try to understand the equivariant stable category from a few other perspectives: the Wirthmüller isomorphism is an analogue of the isomorphism between finite products and coproducts in spectra, and we'll also learn more about tom Dieck splitting. You can also ask how the transfer maps we talked about characterize the equivariant stable category.

2.6. The Wirthmüller isomorphism

> *"What do homotopy theorists do when changing planes?"*
> *"Look for a transfer map."*

In this section, we'll conver the Wirthmüller isomorphism, the equivariant analogue of the nonequivariant statement that in the stable homotopy category Ho(Sp), finite sums and products are equivalent:

$$\bigvee_{i=1}^{n} X_i \xrightarrow{\cong} \prod_{i=1}^{n} X_i. \tag{2.6.1}$$

This is a backbone of the stable category: among other things, it ensures Ho(Sp) is additive. We'll assume G is a finite group; there's a statement for compact Lie groups, but it's more complicated.

Tom Dieck splitting is another important structural result about $\pi_G^*(\Sigma^\infty X)$. When $X = S^0$, this is used to compute the equivariant stable homotopy groups of the spheres (which is, of course, not completely known). This can be used to recover the Burnside category.

Transfers play a key role in both of these results, somewhat implicitly in the Wirthmüller isomorphism, but very explicitly in tom Dieck splitting.

Theorem 2.6.2 (Wirthmüller isomorphism)**.** *Let H be a subgroup of G and X be an object of Sp^H.*[21] *Then, there is a π_*-isomorphism $G \wedge_H X \xrightarrow{\simeq} F_H(G,X)$.*

Exercise 2.6.3. Adapt the proof of (2.6.1) for nonequivariant spectra to show that the natural map from finite sums to finite products in Sp^G is also an isomorphism.

However, if you only have (2.6.1), you've described the category of G-spectra structured by a universe with only trivial representations!

Applying Theorem 2.6.2 to $X = S^0$ computes the Spanier-Whitehead duals of orbits.

Corollary 2.6.4. *In the situation of Theorem 2.6.2, $\Sigma^\infty(G/H)_+ \cong F_H(G,S^0) \cong F((G/H)_+,S^0)$.*

That is, $\Sigma^\infty(G/H)_+$ is its own Spanier-Whitehead dual. The slogan is that orbits are self-dual in the equivariant stable category when G is finite.[22]

Recall that the forgetful map $i_H^* \colon \mathrm{Sp}^G \to \mathrm{Sp}^H$ has a left and a right adjoint, respectively $G \wedge_H -$ and $F_H(G,-)$. These spectrum-level constructions are induced by applying the space-level functors levelwise.

Exercise 2.6.5. Show that these definitions of $G \wedge_H -$ and $F_H(G,-)$ are consistent with the structure maps.

There are three approaches to proving Theorem 2.6.2.

1. One is a connectivity argument: show that on the space level, the maps get more and more highly connected.
2. Another approach is to construct an explicit inverse, using a transfer map.
3. One can also set up a Grothendieck six-functor formalism to prove it, which sets up a general theory for when a lax monoidal functor's left and right adjoints coincide. See [**FHM03**, **May03**, **BDS16**] for a proof using this approach; the first and the third papers set up general theory, and the second shows that it applies to the Wirthmüller map. This makes interesting contact with base-change theory in algebraic geometry.

Warning! The proof of Theorem 2.6.2 given in [**LMS86**] is incorrect, and the fix is nontrivial. It takes approach (2).

We'll use a connectivity argument, which will introduce some tools we'll find useful later.

PROOF OF THEOREM 2.6.2 (X CONNECTED). Though we assume X is connected, the same proof can be adapted when X is bounded-below. The theorem is true for general X, but this proof may not work in that case.

Let $X \in H\mathrm{Top}_*$. Then, there's a map $\theta \colon G \wedge_H X \to F_H(G,X)$ defined by

$$\theta(g_1,x)(g_2) = \begin{cases} g_2g_1x, & g_2g_1 \in H \\ *, & \text{otherwise.} \end{cases}$$

---

[21] Here, we're implicitly using the complete universe U to define Sp^G and Sp^H.

[22] When G is a compact Lie group, there's a degree shift arising from the tangent representation of G on the Lie algebra of H. TODO: double-check this.

Exercise 2.6.6. Show that θ is a G-map.

θ induces a map $\overline{\theta}\colon G\wedge_H X \to F_H(G,X)$ in Sp^G. We'll show that it's an equivalence by computing the connectivity of θ.

Let $K\subseteq G$, ρ be the regular representation of G, and $m\in\mathbb{N}$. Then, we'll compute the connectivity of

$$\theta_{m,K}\colon (G\wedge_H \Sigma^{m\rho}X)^K \longrightarrow (G_H(G,\Sigma^{m\rho}X))^K.$$

When G is finite, the sequence $(\rho,2\rho,3\rho,\dots)$ is cofinal (i.e. the colimits are the same) in the filtered diagram of finite-dimensional representations of the complete universe U. Thus, to understand $\operatorname{colim}_{V\subset U}\Omega^V\Sigma^V X$, it suffices to understand what happens when $V=m\rho$.

Exercise 2.6.7. The reason this is true is that when G is finite, the regular representation ρ contains a copy of every irreducible as a summand. Prove this by doing a character computation.

We'll show the connectivity of $\theta_{m,K}$ is increasing in m, which means that $\pi_*^K\overline{\theta}\colon \pi_*^K(G\wedge_H X)\to\pi_*^K(F_H(G,X))$ is an isomorphism.

The calculation itself will use the K,H double coset decomposition of G to identify the K-fixed points as a sum and as a product, and we understand the connectivity of both of these from nonequivariant homotopy theory.

Let $S\subset G$ be a set of representatives for the double coset partition of G, i.e. under the $K\times H$-action $(k,h)\cdot g = k^{-1}gh$. Then,

$$(G\wedge_H Z)^K = \left(\coprod_{g\in S}(KgH)_+\wedge_H Z\right)^K. \tag{2.6.8}$$

Either $g^{-1}Kg\subseteq H$ or it isn't.

- If $g^{-1}Kg\subseteq H$, then the action is only in Z, so $((KgH)_+\wedge_H Z)^K = Z^{g^{-1}Kg}$.
- If $g^{-1}Kg\not\subseteq H$, then the action is free, so $((KgH)_+\wedge_H Z)^K = *$.

In particular (2.6.8) simplifies to

$$(G\wedge_H Z)^K \cong \bigvee_{\substack{g\in S\\ g^{-1}Kg\subseteq H}} Z^{g^{-1}Kg}.$$

Next we look at $(F_H(G,Z))^K$. If S is a set of representatives of the double cosets, so is $S^{-1}:=\{g^{-1}\mid g\in S\}$, so we can decompose

$$\begin{aligned}(F_H(G,Z))^K &\cong \left(F_H\left(\coprod_{g\in S}Kg^{-1}H, Z\right)\right)^K\\ &\cong \prod_{g\in S}F_H(Kg^{-1}H,Z)^K\\ &\cong \prod_{g\in S}Z^{(gKg^{-1})\cap H}\cong\prod_{g\in S}Z^{(g^{-1}Kg)\cap H}.\end{aligned}$$

This identification sends $f\mapsto\{f(g)\}$; the last equivalence is by using S^{-1} as the set of representatives instead of S.

In particular, the space-level Wirthmüller isomorphism may be written

$$\bigvee_{\substack{g\in S\\ g^{-1}Kg\subseteq H}} Z^{g^{-1}Kg}\xrightarrow{\cong}\prod_{g\in S}Z^{(g^{-1}Kg)\cap H}. \tag{2.6.9}$$

This is nice-looking, but the indexing sets are slightly different. To overcome this, we'll factor (2.6.9) as

$$\bigvee_{\substack{g\in S\\ g^{-1}Kg\subseteq H}} Z^{g^{-1}Kg}\xrightarrow{\varphi_1}\prod_{\substack{g\in S\\ g^{-1}Kg\subseteq H}} Z^{(g^{-1}Kg)\cap H}\xrightarrow{\varphi_2}\prod_{g\in S}Z^{(g^{-1}Kg)\cap H},$$

where φ_1 is the natural map from the sum to the product and φ_2 is inclusion. Then, we will estimate the connectivity of φ_1 and φ_2 separately in the case when $Z=\Sigma^{m\rho}X$. Namely, we'd like to show that they're both $(m[G:K]+O(1))$-connected.

First, what are the fixed points in $(S^{m\rho})^{g^{-1}Kg}$? This is a sphere whose dimension has been shrunk by K, so smashing with it produces something $m[G:K]$-connected. Then, the usual argument about turning sums into products says that φ_1 is about $2m[G:K]$-connected.

For φ_2, what's the connectivity of the missing factors in the domain? In this case, the connectivity is about $m[G:K]$.

Thus, the connectivity of $\theta_{m,K}$ is $m([G,K]+O(1))$,[23] so the spectrum-level map $\overline{\theta}$ is a π_*^K-isomorphism. More precisely, the cofiber of $\theta\colon G\wedge_H X\to F_H(G,X)$ has trivial π_*^K, and there's a little bit to do here to check this. ⊠

Remark. Let's see how this connects to the transfer. Let V be a G-representation such that there's an embedding $G/H\hookrightarrow V$. Then, we obtain an H-map $G\wedge_H S^V\to S^V$ that takes points in $G\setminus H$ to the basepoint. The adjoint of this map is the G-map $G\wedge_H S^V\to F_H(G,S^V)$.

On the other hand, we have a Pontrjagin-Thom map $S^V\to G\wedge_H S^V$ induced as follows: $G/H\hookrightarrow V$ induces a map $G\wedge_H D(V)\to V$, and therefore a map $S^V\to G\wedge_H D(V)/(G\wedge_H S(V))\cong G\wedge_H S^V$. The observation is that the composition $S^V\to G\wedge_H S^V\to S^V$ is the identity (you can and should think about this: it's possible to write down an explicit homotopy inverse). This motivates the slogan that "the Wirthmüller isomorphism is the inverse of the transfer map," which we'll say more about later. ◄

You can use Corollary 2.6.4 to explicitly write down the transfer map: given subgroups $K\subseteq H\subseteq G$, we have the "right-way" map $G/H\to G/K$, and therefore a map $\Sigma^\infty(G/K_+)\cong D(G/K_+)\to D(G/H_+)\cong\Sigma^\infty(G/H_+)$. By the Wirthmüller isomorphism, this is the same as a map $F(G/K_+,S)\to F(G/H_+,S)$, which is exactly the transfer map!

This is a little circular; if you look carefully into the proof of Theorem 2.6.2, the transfer map is already there. But the point is more philosophical: the existence of transfers is the same thing as the Wirthmüller isomorphism.

Remark. Transfer maps can be studied in more generality, e.g. in the context of equivariant vector bundles on homogeneous spaces. Rothenberg wrote some stuff, but [LMS86] is probably the best source. ◄

2.7. Tom Dieck splitting

"Every tom Dieck and Harry works on this!"

In this section, we'll again assume that G is a finite group. Some of this content extends to compact Lie groups, but it's more complicated.

Tom Dieck splitting is a decomposition of the fixed points of a suspension G-spectrum, either of the statements

$$(\Sigma^\infty X)^G\cong\bigvee_{(H)\subseteq G} EWH_+\wedge_{WH}X^H \tag{2.7.1a}$$

$$\pi_*(\Sigma^\infty X)^G\cong\bigoplus_{(H)\subseteq G}\pi_*^{WH}\bigl(\Sigma^\infty EWH_+\wedge_{WH}X^H\bigr). \tag{2.7.1b}$$

By $(H)\subseteq G$, we mean indexing over the conjugacy classes of subgroups of G. Here $WH=NH/H$ as usual; if $H\trianglelefteq G$, then this is G/H. Tom Dieck splitting looks complicated; you might have hoped that Σ^∞ and $(-)^G$ commute, but they don't, and this is how they don't. But when $X=S^0$, this tells you a decomposition of $\pi_*^G(\mathbb{S})$.

Remark. Guillou-May [GM17] derive tom Dieck splitting as a consequence of the equivariant Barratt-Priddy-Quillen theorem. We'll give a more elementary proof. ◄

We have a cofiber sequence, called the **isotropy separation sequence**,

$$EG_+\longrightarrow S^0\longrightarrow\widetilde{EG},$$

which induces for any spectrum X a cofiber sequence

$$(EG_+\wedge X)^G\longrightarrow X^G\longrightarrow(\widetilde{EG}\wedge X)^G. \tag{2.7.2}$$

Take $X=\mathbb{S}=\Sigma^\infty S^0$, giving

$$(\Sigma^\infty EG_+)^G\longrightarrow\mathbb{S}^G\longrightarrow(\Sigma^\infty\widetilde{EG})^G.$$

There are three key observations; one is easy, one is harder, and the one was a major structural theorem.

- When X is a suspension spectrum, (2.7.2) splits.

---

[23] TODO: I want to run through the connectivity carefully and ensure I made no typos.

- The **Adams isomorphism** tells us the cofiber: $(\Sigma^\infty EG_+)^G \simeq \Sigma^\infty BG_+$.
- The last term is $(\widetilde{EG} \wedge X)^G \cong \Phi^G X$.

Tom Dieck proved tom Dieck splitting by inducting over isotropy sequences, adding one conjugacy class at a time. This is one of the two major methods of induction in equivariant homotopy theory (the other being induction over cells), and was a key step in the resolution of the Kervaire invariant 1 problem [HHR16].

There are several closely related versions of tom Dieck splitting, though going between them requires a deep result called the Adams isomorphism. We'll return to that point later.[24] Anyways, we'll prove the version in (2.7.1b).

When $X = S^0$, this "computes" the equivariant stable homotopy groups of the spheres, and one consequence is that

$$\pi_0^G(\Sigma^\infty S^0) \cong \bigoplus_{(H)\subseteq G} \mathbb{Z}, \tag{2.7.3}$$

i.e. it's free on the conjugacy classes of subgroups of G.

As a consequence of the proof of (2.7.1b), the generators are given by the Euler characteristics $S \to \Sigma^\infty G/H_+ \to S$.[25] This lets us identify homotopy classes of stable maps $G/H_1 \to G/H_2$; since we defined the Burnside category to be the full subcategory of Sp^G on $\Sigma^\infty G/H_+$, this amounts to the computation of $\pi_0 B_G$.

Define an abelian group A whose generators are diagrams of the form

$$\begin{tikzcd} G/H_1 & G/K \arrow[l, "\theta_1"'] \arrow[r, "\theta_2"] & G/H_2, \end{tikzcd} \tag{2.7.4}$$

and for which two diagrams $G/H_1 \leftarrow G/K \to G/H_2$ and $G/H_1 \leftarrow G/K' \to G/H_2$ are equivalent if there is an isomorphism $f : G/K \to G/K'$ such that the following diagram commutes.

$$\begin{tikzcd} & G/K \arrow[dl, "\theta_1"'] \arrow[dr, "\theta_2"] \arrow[dd, "\wr"', "f"] & \\ G/H_1 & & G/H_2. \\ & G/K' \arrow[ul, "\theta_1'"] \arrow[ur, "\theta_2'"'] & \end{tikzcd}$$

The point is that there's an isomorphism $\psi\colon A \xrightarrow{\cong} \pi_0^{H_1}(\Sigma_+ G/H_2)$, the group of stable maps $G/H_1 \to G/H_2$. We'll calculate these maps in the algebraic Burnside category $\pi_0(B_G)$. The isomorphism ψ takes a diagram of the form (2.7.4) to the composite $\Sigma_+^\infty G/H_1 \to \Sigma_+^\infty G/K \to \Sigma_+^\infty G/H_2$, where the first map is the transfer for θ_1 and the second is $\Sigma^\infty \theta_2$.

Exercise 2.7.5. Show that ψ is an isomorphism.

This is really nice: we have a nice combinatorial description of the stable maps coming from the structure of G. But what's the composition?

The above is more or less true for G a compact Lie group. But what follows really requires G to be finite.

It seems natural to play double coset games here, but this quickly turns into a notational mess. But if you know about spans, there's only one thing it should be, which is the pullback:

$$\begin{tikzcd} & & G/K \times_{G/H_2} G/K' \arrow[dl] \arrow[dr] & & \\ & G/K \arrow[dl] \arrow[dr] & & G/K' \arrow[dl] \arrow[dr] & \\ G/H_1 & & G/H_2 & & G/H_3. \end{tikzcd}$$

This is true, and we'll give a real proof later. The upshot is really cool: we have a complete combinatorial description of the algebraic Burnside category, including composition. This leads one to Mackey functors: equivariant stable homotopy theory is controlled by a pair of functors out of the orbit category, one covariant and one contravariant, and Mackey functors are nice examples of these.

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[24] TODO: I missed one version of the splitting.

[25] TODO: I didn't understand this.

REMARK. We've been tacitly assuming a complete G-universe for this, but what if you'd prefer to leave out certain suspensions Σ^V? In the case of an incomplete G-universe U, you only consider the orbits $G/H \hookrightarrow U$, and build the incomplete Burnside category and incomplete Mackey functors in this way. ◅

Now let's prove tom Dieck splitting.

PROOF OF (2.7.1b). We'll approach this inductively and look a single summand at a time, producing a map

$$\pi_*^{WH}\big(\Sigma^\infty EWH_+ \wedge X^H\big) \longrightarrow \pi_*^G(\Sigma^\infty X). \tag{2.7.6}$$

Recall that $WH = NH/H$, so we can define a sequence

$$\pi_*^{WH}\big(\Sigma^\infty EWH_+ \wedge X^H\big) \xrightarrow{\theta_1} \pi_*^{NH}\big(\Sigma^\infty EWH_+ \wedge X^H\big) \xrightarrow{\theta_2} \pi_*^{NH}(\Sigma^\infty EWH_+ \wedge X), \tag{2.7.7}$$

where θ_1 is induced by restriction $\pi_*^{WH} \to \pi_*^{NH}$, and θ_2 is induced from the inclusion $X^H \hookrightarrow X$. The Wirthmüller isomorphism induces an isomorphism on homotopy groups $\pi_*^H(X) \xrightarrow{\cong} \pi_*^G(G \wedge_H X)$ (by an adjunction game), so we can extend (2.7.7) with maps

$$\pi_*^{NH}(\Sigma^\infty EWH_+ \wedge X) \xrightarrow[\cong]{\theta_3} \pi_*^G(G \wedge_{NH} \Sigma^\infty EWH_+ \wedge X) \xrightarrow{\theta_4} \pi_*^G(\Sigma^\infty X).$$

Here, θ_4 is the adjoint of the NH-map $\Sigma^\infty EWH_+ \wedge X \to \Sigma^\infty X$ that crushes $EWH \to *$. Thus, our goal is to prove that $\theta_2 \circ \theta_1$ and θ_4 are isomorphisms.

Recall that a $\mathbb{Z}$-graded homology theory on G-spaces is a collection of functors $E_q\colon G\mathsf{Top} \to \mathsf{Ab}$ for $q \in \mathbb{Z}$ that are homotopy invariant and such that

$$E_q\Big(\bigvee_k X_k\Big) \cong \bigoplus_k E_q(X_k),$$

and a map $f : X \to Y$ induces a long exact sequence

$$\cdots \longrightarrow E_q(X) \longrightarrow E_q(Y) \longrightarrow E_q(Cf) \xrightarrow{\delta} E_{q-1}(X) \longrightarrow \cdots \tag{2.7.8}$$

The key observation is that *both sides of* (2.7.1b) *specify* $\mathbb{Z}$*-graded homology theories*.

Definition 2.7.9. An $X \in G\mathsf{Top}$ is **concentrated at a conjugacy class** H if for all K not in the conjugacy class of H in G, X^K is contractible.

Theorem 2.7.10. *Let $h\colon E_q \to \tilde{E}_q$ be a natural transformation of $\mathbb{Z}$-graded homology theories. If h is an isomorphism on all X concentrated at a conjugacy class, then h is an isomorphism.*[26]

This leads to a very innovative proof, invented by tom Dieck, whose pedagogical importance is just as significant as the theorem statement: we'll set up a sequence of families of subgroups of G

$$\{e\} = \mathscr{F}_0 \subseteq \mathscr{F}_1 \subseteq \mathscr{F}_2 \subseteq \cdots \subseteq \mathscr{F}_n = \{H \subseteq G\}$$

such that $\mathscr{F}_i$ is built from $\mathscr{F}_{i-1}$ by adding a conjugacy class not already present. Then, we can induct on i, and since G is finite, this must terminate. This is one of the key techniques of induction in equivariant homotopy theory, along with cellular induction.

PROOF OF THEOREM 2.7.10. Let's induct: suppose that $\mathscr{F}_i = \mathscr{F}_{i-1} \cup (K)$ for some $K \subseteq G$, and consider any G-space X. There is a natural map $f : X \wedge E\mathscr{F}_{i-1} \to X \wedge E\mathscr{F}_i$, and its cofiber $C(f)$ is concentrated at K!

Suppose that $E_q(X \wedge E\mathscr{F}_{i-1}) \xrightarrow{\cong} \tilde{E}_q(X \wedge E\mathscr{F}_{i-1})$. By hypothesis, $E_q(C(f)) \xrightarrow{\cong} \tilde{E}_q(C(f))$, since $C(f)$ is concentrated at K. Now, by the long exact sequence (2.7.8) and the five lemma, the map $E_q(X \wedge E\mathscr{F}_i) \to E_q(X \wedge E\mathscr{F}_i)$ is an isomorphism. ⊠

Now, to prove tom Dieck splitting, it suffices to show that the map we defined (the direct sum of (2.7.6) over all conjugacy classes) is an isomorphism when X is concentrated at a single conjugacy class (H).

First, we'll show that if X is concentrated at (H) and $K \notin (H)$, then (2.7.6) is trivial for K. Let's consider $\Sigma^\infty EWK_+ \wedge X^K$. Look at the space $S^V \wedge EWK_+ \wedge X^K$, and let's consider its L-fixed points for $L \subseteq K$. If $L = \{e\}$,

---

[26] The analogue of this theorem for $\mathbb{Z}$-graded cohomology theories is also true.

then $(X^K)^L$ is contractible, and otherwise, $(EWK_+)^L$ is contractible, so the smash product is always contractible. We conclude that $\Sigma^\infty EWK_+ \wedge X^K$ is contractible, and therefore that

$$\pi_*^{WK}\big(\Sigma^\infty EWK_+ \wedge X^K\big) = 0.$$

Now we've reduced to showing that (2.7.6) is an isomorphism when X is concentrated at H, i.e. that (2.7.7) is an isomorphism (its components aren't) and that θ_4 is an isomorphism.

Let's first tackle θ_4.[27] This uses an important piece of technology: suppose X and Y are G-spaces and $H \subseteq G$ is normal. Then, there is a **restriction map**

$$\rho_H\colon \operatorname{Map}^G(X,Y) \longrightarrow \operatorname{Map}^{G/H}(X^H, Y^H).$$

This map appears in an essential way in the definition of topological cyclic homology.[28]

Lemma 2.7.11. *Suppose Y is concentrated at (H); then, the restriction map $\rho_H\colon \operatorname{Map}^G(X,Y) \to \operatorname{Map}^{G/H}(X^H,Y^H)$ is an acyclic fibration.*

We'll prove this below with cellular induction; in any case, we only need that ρ_H is a weak equivalence.

Anyways, we want to show that (2.7.7) is an isomorphism, so we'll use the restriction map to define an inverse $\pi_*^{NH}(\Sigma^\infty EWH_+ \wedge X) \to \pi_*^{WH}(\Sigma^\infty EWH_+ \wedge X^H)$. That is, consider

$$[S^{n+m\rho_{NH}}, S^{m\rho_{NH}} \wedge EWH_+ \wedge X]$$

and take the H-fixed points on both sides; Lemma 2.7.11 ensures this is an isomorphism.

We've reduced to showing that the composite

$$\begin{tikzcd}
\pi_*^{WH}(\Sigma^\infty EWH_+ \wedge X^H) \arrow[r] \arrow[dr, "\theta_1"] & \pi_*^{NH}(\Sigma^\infty EWH_+ \wedge X^H) \arrow[d] & & \\
& \pi_*^{NH}(\Sigma^\infty EWH_+ \wedge X) \arrow[r, "\cong"] & \pi_*^G(G \wedge_H \Sigma^\infty EWH_+ \wedge X) \arrow[r, "\theta_2"] & \pi_*^G(X)
\end{tikzcd}$$

is an isomorphism when X is concentrated at the single conjugacy class (H) for G. The composite of the first two maps is denoted θ_1. We did this by showing that the two sides of the tom Dieck isomorphism are $\mathbb{Z}$-graded homology theories, and therefore are determined by their values on spaces concentrated at single conjugacy classes. We'll show separately that θ_1 and θ_2 are isomorphisms.

First let's look at θ_2, by looking at the spaces of the suspension spectrum for

$$\big(G \wedge_{NH} \big(EWH_+ \wedge X \wedge S^V\big)\big)^K.$$

Fixed points do not commute with suspension on the spectrum level (after all, this is the content of tom Dieck splitting), but they do commute with the smash product for spaces, so this is also

$$(G \wedge_H EWH_+)^K \wedge X^K \wedge S^{V^K}.$$

If $K \notin (H)$, then $X^K \simeq *$, since X is concentrated at (H). If $K \in (H)$, then $(EWH_+)^K \simeq *$, so either way what we obtain is an isomorphism for θ_2^K.

For θ_1, we needed to prove Lemma 2.7.11, that if $H \trianglelefteq G$ and Y is concentrated at (H), then the restriction map $\operatorname{Map}^G(X,Y) \to \operatorname{Map}^{G/H}(X^H, Y^H)$ is an isomorphism.

Proof of Lemma 2.7.11. The proof will proceed by cellular induction. Consider attaching a cell to a G-space Z to form a space $\tilde{Z}$, which is expressed by the pushout

$$\begin{tikzcd}
G/K_+ \wedge S^{n-1} \arrow[r] \arrow[d] & Z \arrow[d] \\
G/K_+ \wedge D^n \arrow[r] & \tilde{Z}. \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

---

[27] TODO: I got confused and should fix this.

[28] The new construction of *TC* by Nikolaus-Scholze [NS17] manages to avoid this in a way that is considerably more natural.

If we apply $\mathrm{Map}^G(-,Y)$ to this diagram, it becomes a pullback

$$\begin{tikzcd}
\mathrm{Map}^G(G/K_+ \wedge S^{n-1}, Y) & \mathrm{Map}^G(Z,Y) \arrow[l] \\
\mathrm{Map}^G(G/K_+ \wedge D^n, Y) \arrow[u, "\psi_1"] & \mathrm{Map}^G(\tilde{Z},Y). \arrow[l] \arrow[u, "\psi_2"]
\end{tikzcd}$$

Since $\mathrm{Map}^G(G/K_+ \wedge S^{n-1}, Y) \simeq \mathrm{Map}(S^{n-1}, Y^K)$, then ψ_1 is trivial, so ψ_2 is an isomorphism.

Let

$$X_0 := \{x \in X \mid G_x \nsubseteq H\} \subseteq X.$$

Then, $X \setminus (X_0 \cup X^H)$ is built by attaching cells $G/K_+ \wedge D^n$ with $K \notin (H)$.

The cell attachment computation we just made tells us that it suffices to understand $X_0 \cup X^H$, for which we use a Mayer-Vietoris argument associated to the pullback

$$\begin{tikzcd}
X_0 \cup X^H & X^H \arrow[l, "\phi_1"'] \\
X_0 \arrow[u] & X_0 \cap X^H \arrow[l, "\phi_2"'] \arrow[u]
\end{tikzcd}$$

We again apply $\mathrm{Map}^G(-,Y)$; this time, the same argument shows that ϕ_2^* is an isomorphism. Now the map we want factors as

$$\mathrm{Map}^G(X,Y) \longrightarrow \mathrm{Map}^G(X_0 \cup X, Y) \longrightarrow \mathrm{Map}^G(X^H, Y) \longrightarrow \mathrm{Map}^{G/H}(X^H, Y^H).$$

The last map comes from the adjunction, and in particular is a known weak equivalence. And by induction, the first two maps are weak equivalences. ⊠

We'll use Lemma 2.7.11 to check that the map

$$\pi_*^{WH}(\Sigma^\infty EWH_+ \wedge X^H) \xrightarrow{\theta_1} \pi_*^{NH}(\Sigma^\infty EWH_+ \wedge X)$$

is an isomorphism, by providing a map in the other direction. Specifically, this is a map of homotopy classes

$$[S^{n+m\rho}, \Sigma^{m\rho} EWH_+ \wedge X] \xrightarrow{\theta_1} [S^{n+m\rho}, \Sigma^{m\rho} EWH_+ \wedge X].$$

We've abused notation a bit here: on the left, ρ is the regular representation for NH, and on the right, it's the regular representation for WH. Lemma 2.7.11 implies that the map going the other way is an isomorphism when X is concentrated at (H):

$$[S^{n+m\rho}, S^{m\rho} \wedge EWH_+ \wedge X] \xrightarrow{\cong} [S^{n+m\rho^H}, S^{m\rho^H} \wedge EWH_+ \wedge X^H].$$

This is what we reduced the problem to, so we have completed the proof of tom Dieck splitting for $\pi_G^*(\Sigma^\infty X)$ on the level of homotopy groups. ⊠

As this is a statement about stable homotopy groups, it's desirable to have a spectrum-level equivalence which reduces to this isomorphism; the statement is (2.7.1a), but the proof is different.

2.8. Question-and-answer session II: 2/24/17

Question 2.8.1. Are orthogonal spectra the same as diagram G-spectra indexed on all finite-dimensional representations of G? If so, what are the advantages of working with only orthogonal representations? Is orthogonality needed to make sense of Ω-spectra?

Much of it is that the representations in question often come with a natural inner product. Also, it's useful to take orthogonal complements sometimes. Nonetheless, it's probably not strictly necessary.

Question 2.8.2. What conditions need to be put on a diagram category D such that D-spectra are Quillen equivalent to orthogonal spectra?

It's hard to be precise about this, but one thing that's common to all of them is an embedding of $\mathbb{N}$ into D producing an adjunction between the forgetful functor and the prolongation functor. We used the forgetful functor to define the homotopy groups. So if you can embed $\mathbb{N}$ into it, it's probably OK, but it's doubtful that a complete, explicit classification exists. Γ-spaces are kind of strange, though.

Question 2.8.3. There are geometric fixed-points, homotopy fixed-points, and categorical fixed-points. What's the intuition for what these three constructions are doing?

Recall that the categorical fixed points are $(-)^H = F(\Sigma_+ G/H, -)$, which tom Dieck splitting tells us is complicated and hard to compute in general. The homotopy fixed points $F(EG_+, -)^G$ are more computable: there's a spectral sequence (Theorem 5.3.5) arising from the filtration of BG. Geometric fixed points $\Phi^H X$, which satisfy the property $\Phi^H \Sigma^\infty X \cong \Sigma^\infty X^H$ (and are uniquely characterized by this, being strong symmetric monoidal, and preserving homotopy colimits), have nice properties.

You can also analyze them in terms of what they do to S^{-V}, i.e. the Spanier-Whitehead dual of S^V: geometric fixed points will send it to S^{-V^H}, whereas categorical fixed points will look more like a tom Dieck splitting.

It's possible to write every spectrum as a colimit of suspension spectra, and this "canonical presentation" is a good way to dissect the differences between these fixed points. We might talk more about this later.

Both the categorical and geometric fixed points detect weak equivalences, so you could use, e.g. the geometric fixed points to define homotopy groups.

Tate spectra fit into this story: the Tate spectrum X^{tG} of X fits into a homotopy pullback diagram (5.1.5) where the other three corners are the three kinds of fixed points. This looks like an arithmetic square, and this analogy can be made precise.

Question 2.8.4. Is there a similar story for coinvariants? Are there categorical, homotopical, and geometric coinvariants?

Not really, so far: Lewis and May remark that the construction you would write down is hard to control, and hard to derive. It's possible that new work on the foundations of equivariant homotopy theory could enable a better answer, but nothing's been done right now.

Question 2.8.5. Are there notions of naïve and genuine *G-spaces*? (This question was asked in the homotopy theory chatroom on MathOverflow.)

The terminology is less common than that of naïve and genuine G-spectra; in some sense, it's a little weird to be using terms like "genuine" and "naïve" in the first place to describe mathematical objects. In any case, by a genuine G-space, we mean a functor out of the orbit category, but a naïve G-space would be a functor out of the category containing just G/e, so capturing only the theory of spaces over BG.

On the subject of mathematical literature, the HULK smash product [**Man12**] is possibly the best mathematical joke of all time.

Question 2.8.6. You've said that Hill-Hopkins-Ravenel [**HHR16**] has significantly affected the direction of equivariant stable homotopy theory. Can you elaborate on this?

There's been some interest in S^1-equivariant methods, but a big piece has been a focus on computations, now that some computations look possible. The use of trace methods is also exciting. There's some work on relating equivariant homotopy theory to motivic homotopy theory, which looks exciting.

Another generalization is to global homotopy theory; Stefan Schwede has a program for setting this up. One of the eventual payoffs will be an equivariant *TMF*, which Lurie has sketched a construction of.

Question 2.8.7. Has the Tannakian formalism been applied to this context? Maybe instead of a universe of representations, it would be easier to take a category.

This hasn't really been considered; it might be interesting, and it might be hard to work with.

Question 2.8.8. If you like to think of spectra as infinite loop spaces, what changes in the equivariant case?

For G a finite group, everything is the same: you can use Γ-G-spaces or Γ_G-spaces (which are different notions for the same thing by Theorem 4.5.1); you can use functors from Γ to G-spaces, and a few other ones. Then, loop spaces come from a E_∞-operad action; for technical reasons you may need to assume there are trivial representations in the universe.

In the case of compact Lie groups, nobody really knows how to set up the theory of infinite loop spaces. In particular, being an E_∞-G-space is not the same as being the zero space of a G-spectrum.

One way to think about this is that on the space level, the E_∞ operad controls the transfers. The issue is that we only know how to handle finite-index transfers and norms; maybe you should put them in freely, but you get weird obstructions. There's lots of interesting questions to be asked here, but they are hard and likely not worth your time.

Question 2.8.9. What is a cyclotomic spectrum?

Fix a prime p; a **p-cyclotomic spectrum** is an S^1-spectrum X together with data of an equivalence with its C_p-geometric fixed points as S^1-spectra.

Recall that the **loop space** of X is $LX := \mathrm{Map}(S^1, X)$. Its suspension spectrum is an example of a cyclotomic spectrum: there is an identification $\Phi^{C_p}(\Sigma_+^\infty LX) \cong \Sigma_+^\infty (LX)^{C_p} \cong \Sigma_+^\infty LX$.

If X is an S^1-spectrum, we can take $X^{C_{p^n}} \cong (X^{C_p})^{C_{p^{n-1}}} \to (\Phi^{C_p} X)^{C_{p^{n-1}}} \to X^{C_{p^{n-1}}}$, and the homotopy limit of these maps is topological cyclic homology.

Question 2.8.10. What are cyclonic spectra?

It's not a standard term, and the professor didn't remember off the top of his head.

Question 2.8.11. Is there an analogue of the regular representation in stable homotopy theory?

The answer is, not really. This relates to the complicated structure of tom Dieck splitting — even in the case for compact Lie groups, where the proofs we gave don't completely generalize, there are versions of the Wirthmüller isomorphism and tom Dieck splitting that have shifts by the tangent representation at the identity coset of G/H. This shift pops up again and again.

Question 2.8.12. What is currently happening in EDAG? What roadblocks are currently in the way?

When you look at TMF, especially with level structure, it really looks like there should be some underlying equivariant spectrum, and it would be nice to have a way to make this explicit.

There are issues with localization behaving poorly, and ∞-operads were introduced to make things behave better. It's also conceptually interesting and cool how the action of an ∞-operad controls what transfers it has, and suspension converts the transfers into norms, and you get a ring spectrum: it's a nice conceptual picture that explains what's going on.

Question 2.8.13. What applications have there been besides [**HHR16**] and trace methods?

Both of those are good applications. A lot of the interesting work in chromatic homotopy theory regarding, e.g. homotopy fixed points of subgroups of the Morava stabilizer group, uses G-actions. Stable representation theory of finite groups also uses this material. Goodwillie calculus also uses equivariant methods: Nick Kuhn proved in [**Kuh04a**] that the vanishing of extensions in the Taylor tower is equivalent to the vanishing of the Tate spectrum.

There's an analogue of Goodwillie calculus for the equivariant setting, an "equivariant calculus;" since calculus is about passage from the unstable to the stable category, the equivariant stable category plays the analogous role in equivariant calculus.

Completion with respect to the $(\Sigma^\infty, \Omega^\infty)$-adjunction is important in calculus, in terms of a simplicial resolution, and it's a little weird that you get the same answer if you complete with the correct or the trivial choice.[29]

Question 2.8.14. In a vague sense, the classification of groups splits into two parts: find the simple pieces (finite simple groups) and figure out how to put then together (which is also hard). Algebraic K-theory throws out the second part: is there any way to understand the category of finitely-presented groups in this way?

Jeff Smith used a similar method in a different context, but it hasn't been extremely fruitful. People haven't looked into the K-theory of finitely presented groups, and it might be interesting.

Question 2.8.15. What did you mean by the generators of $\pi_0^G(S^0)$ being Euler characteristics as a consequence of tom Dieck splitting?

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[29] TODO: better word than "choice."

$\pi_0^G(S^0)$ is equivariant homotopy classes of maps from S^0 to itself. This is free on conjugacy classes — well, what maps generate? The answer is the composites of the maps $S \to \Sigma^\infty G/H_+$ and the crush map back to S: these are Euler characteristics, in that these are equivariant analogues of the maps whose degrees give Euler characteristics in the nonequivariant case?

Question 2.8.16. How much more bananas is the equivariant sphere than the sphere?

It's way more complicated: it has $\Omega^V S^V$ in it for all V, and the decomposition is not nice. Maybe for specific groups you can say something nice.

Question 2.8.17. Knowledge of facts about the sphere spectrum encodes information interesting in other fields. Does knowledge about the equivariant sphere spectrum tell us anything more than the equivariant generalizations of those results?

Presently, no such applications are known.

Question 2.8.18. Under certain conditions, you can detect the homotopy groups of a spectrum by looking at the 0 level. Can you do this for G-spectra by looking at the regular representation?

Better, in fact: in the case of an Ω-G-spectrum, you can compute the homotopy groups at the *trivial* representation.

Question 2.8.19. What is the status of Grothendieck's approach to homotopy theory?

That's a good question; it's not so clear. People like derivators, because they're 2-categorical and therefore easier to deal with, but they might go away once the higher-categorical foundations get standardized.

Question 2.8.20. In equivariant homotopy theory, there are multiple kinds of suspension functor. How does one then conclude that cofibers are the same as fibers up to some shift, if there are multiple choices of shift to make?

This is a great exercise for you to work out.

CHAPTER 3

Mackey functors and $\mathrm{RO}(G)$-graded cohomology

3.1. The Burnside category

Just as coefficient systems for Bredon cohomology were Ab-valued presheaves on the orbit category, Mackey functors are Ab-valued presheaves on the Burnside category. We'll begin discussing this in this section, provide another deifnition for the Burnside category which is more hands-on, and provide an example.

Let C be a small category with finite limits and finite coproducts. In C, consider the diagram

$$\begin{tikzcd}
X \arrow[r, "\varphi_1"] \arrow[d] & Y \arrow[d] & Z \arrow[l, "\varphi_2"'] \arrow[d] \\
X' \arrow[r] & X' \amalg Z' & Z'. \arrow[l]
\end{tikzcd}$$

Both of these squares are pullbacks iff $Y \cong X \amalg Z$ with φ_1 and φ_2 the universal maps. This tells us something about how coproducts interact with pullbacks.

Definition 3.1.1. Let $\mathsf{Span}(\mathsf{C})$ be the category whose objects are those of C and whose morphisms $\mathrm{Hom}_{\mathsf{Span}(\mathsf{C})}(X, Y)$ are equivalences classes of diagrams

$$\begin{tikzcd}
& A \arrow[dl] \arrow[dr] & \\
Z & & Y,
\end{tikzcd} \tag{3.1.2}$$

where $X \leftarrow A \rightarrow Y$ and $X \leftarrow A' \rightarrow Y$ are equivalent if there's an isomorphism $h\colon A \xrightarrow{\cong} A'$ such that the following diagram commutes.

$$\begin{tikzcd}
& A \arrow[dl] \arrow[dr] \arrow[dd, "h"] & \\
X & & Y \\
& A'. \arrow[ul] \arrow[ur] &
\end{tikzcd} \tag{3.1.3}$$

Composition is defined by pullback of spans

$$\begin{tikzcd}
& & A \times_Y A' \arrow[dl] \arrow[dr] & & \\
& A \arrow[dl] \arrow[dr] & & A' \arrow[dl] \arrow[dr] & \\
X & & Y & & Z.
\end{tikzcd}$$

Remark. If your category number[1] is at least 2, you can construct $\mathsf{Span}(\mathsf{C})$ as a 2-category, where the 1-morphisms between X and Y are spans (3.1.2), and the 2-morphisms between $X \leftarrow A \rightarrow Y$ and $X \leftarrow A' \rightarrow Y$ are the diagrams (3.1.3). ◀

There's a sum on $\mathrm{Hom}_{\mathsf{Span}(\mathsf{C})}(X, Y)$ given by the coproduct of $X \leftarrow A \rightarrow Y$ and $X \leftarrow A' \rightarrow Y$, which define a map $X \leftarrow A \amalg A' \rightarrow Y$. You might worry whether this is compatible with composition, which is why we took the morphisms to be equivalence classes of spans.

---

[1] Freed [**Fre09**] defines the **category number** of a mathematician to be the largest n such that he or she can think hard about n-categories for half an hour without getting a headache.

This sum does not make $\mathsf{Span}(\mathsf{C})$ into an additive category, so let $\mathsf{Span}^+(\mathsf{C})$ denote the **preadditive completion** of $\mathsf{Span}(\mathsf{C})$, i.e. $\mathrm{Hom}_{\mathsf{Span}^+(\mathsf{C})}(X,Y)$ is the Grothendieck group of $\mathrm{Hom}_{\mathsf{Span}(\mathsf{C})}(X,Y)$.

Exercise 3.1.4. Show that the category $G\mathsf{Set}^{\mathrm{fin}}$ of finite G-sets and G-equivariant maps has all finite limits and coproducts, so that we may apply the above formalism to it.

Definition 3.1.5. The **Burnside category** B_G is $\mathsf{Span}^+(G\mathsf{Set}^{\mathrm{fin}})$.

In Definition 2.2.2, we defined the Burnside category differently, as the full subcategory of Sp^G on $\Sigma^\infty G/H_+$; these two definitions are equivalent, and the content is that every finite G-set is a coproduct of orbits G/K_i.

Proposition 3.1.6. *Let $\tilde{B}_G$ denote the full subcategory of Sp^G spanned by finite G-sets. Then, B_G and $\pi_0\tilde{B}_G$ are equivalent.*

Proof. The functor in question sends $T \mapsto \Sigma^\infty T_+$ and a span

$$\begin{tikzcd} & Z \arrow[dl, "\tau"'] \arrow[dr, "\theta"] & \\ X & & Y \end{tikzcd}$$

to $\theta \circ \mathrm{tr}(\tau)$, where tr is the transfer map. Because the transfer behaves well under composition, this is a functor $\mathsf{Span}(G\mathsf{Set}^{\mathrm{fin}}) \to \pi_0\tilde{B}_G$, and therefore uniquely determines the functor $B_G \to \pi_0\tilde{B}_G$. This is essentially surjective (i.e. for objects), and we've already computed that it's an isomorphism on hom sets.[2] ⊠

Example 3.1.7. TODO: carefully construct the Burnside category of C_2. ◄

Exercise 3.1.8. Generalize the above example to C_p.

3.2. Mackey functors

In this section, we introduce Mackey functors, which play a role analogous to abelian groups for equivariant stable homotopy theory: given a Mackey functor $\underline{M}$, we can define an Eilenberg-Mac Lane G-spectrum $H\underline{M}$, which represents $RO(G)$-graded cohomology in M. Moreover, by remembering the transfer map, we can realize cohomology theories as valued in Mackey functors.

Definition 3.2.1. A **Mackey functor** is a functor $B_G^{\mathrm{op}} \to \mathsf{Ab}$.

Restricting to spans of the form $X \overset{\mathrm{id}}{\leftarrow} X \to Y$ defines a covariant functor $\mathscr{O}_G \to B_G$, and restricting to spans of the form $Y \leftarrow X \overset{\mathrm{id}}{\to} X$ defines a contravariant functor $\mathscr{O}_G^{\mathrm{op}} \to B_G$. Composing with a Mackey functor F defines a pair of functors $\mathscr{O}_G^{\mathrm{op}} \to \mathsf{Ab}$ and $\mathscr{O}_G \to \mathsf{Ab}$. We'll denote the image of an $f : X \to Y$ under these functors as f^* and f_*, and given a pullback diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "g"] \\ Y' \arrow[r, "f"] & Y, \end{tikzcd}$$

Mackey functors satisfy the **Chevalley condition** $g^* f_* = f'_*(g')^*$.

Proposition 3.2.2. *$\pi_*^H(-)$ defines a Mackey functor.*

The two functors $\mathscr{O}_G, \mathscr{O}_G^{\mathrm{op}} \to \mathsf{Ab}$ are evident, coming from the usual maps in the orbit category and the transfer maps, but there are axioms to check.

Let M be a Mackey functor. Then, there exists an object $HM \in \mathsf{Sp}^G$, called the **equivariant Eilenberg-Mac Lane spectrum** for M, whose homotopy groups π_G^* are determined by M.

Definition 3.2.3. The **cohomology** of an $X \in \mathsf{Sp}^G$ with coefficients in a Mackey functor M is

$$H^n(X;M) := [X, \Sigma^n HM],$$

---

[2] TODO: I think this was tom Dieck splitting, but want to make sure.

which defines a $\mathbb{Z}$-graded cohomology theory. Moreover, one can define an RO(G)-graded cohomology theory

$$H^V(X;M) := [X, \Sigma^V HM],$$

where V is a representation in U.

The definition of an RO(G)-graded cohomology theory requires some care, and we will investigate this; the naïve definition you might write down is likely to be wrong for subtle reasons. Nonetheless, it's still worth writing down.

The central algebraic analogue of the equivariant stable category is given by the category of Mackey functors. Mackey functors form a symmetric monoidal category under something called the box product; we'll define this later in (4.2.2), and it's familiar, arising as a Kan extension just like Day convolution.

The point is, it makes sense to talk about commutative monoids in the category of Mackey functors, which were first considered by Green [**Gre71**] and are hence called **Green functors**. These aren't quite the right objects, however: we need to account for stable transfer data, which is what makes the equivariant stable category act the way it does. The more sophisticated notion of a commutative ring in Mackey functors involves additional multiplicative transfer data; these are called **Tambara functors** or **TNR functors**, and were introduced by Tambara [**Tam93**]. We'll discuss Green functors in §4.2 and Tambara functors in §4.3.

Up to now, everything we've done was known to tom Dieck and can be found (perhaps not in this language) in [**LMS86**]. But this stuff is more modern: this perspective on Tambara functors was popularized by [**HHR16**]. Multiplicative transfers were originally considered by [**Eve63**] in representation theory, and [**GM95a**] also wrote about them, but [**HHR16**] innovated the norm and how it interacts with these multiplicative transfer maps.

This leads to an interesting question about calculations, which appears as soon as you try to start doing things.

Question 3.2.4. On the homotopy groups $\pi_*^{(-)}$, we have restrictions and transfer maps. What can we say about the composite of a restriction map and a transfer map, or a transfer map and a restriction map?

The answer involves double coset formulae, and can be thought of in a few ways: we'll start with the classical perspective and become more modern.

For the rest of this section, assume G is a finite group.

We'll let $G\mathsf{Set}$ denote the category of finite G-sets and equivariant maps. When we work with RO(G)-graded theories, it will be important to embed everything in some universe, an infinite G-set containing all finite G-sets up to isomorphism. It's okay to be lax about this at first, but there are subtleties to RO(G)-graded theories that make this important; [**GM13**] provided the first proof of the folklore result that G-spectra are spectral presheaves on the Burnside category, but there were important coherence issues to work out, for example.

We've seen that a Mackey functor $\underline{M}$ is specified by a pair of functors M_*, M^*, where $M_*\colon G\mathsf{Set} \to \mathsf{Ab}$ is covariant, $M^*\colon (G\mathsf{Set})^{\mathsf{op}} \to \mathsf{Ab}$ is contravariant, and M_* and M^* agree on objects, so it makes sense to write $\underline{M}(T)$ or $\underline{M}(G/H)$ (the latter may also be written $\underline{M}(H)$).

Mackey functors are determined by their restrictions to orbits, as $\underline{M}(A \amalg B) = \underline{M}(A) \oplus \underline{M}(B)$ and given a pullback

$$\begin{tikzcd}
W \arrow[r, "\alpha"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\gamma"] \\
Y \arrow[r, "\delta"'] & Z,
\end{tikzcd}$$

$M_*(\beta)M^*(\alpha) = M^*(\delta)M_*(\gamma)$. This resembles the six-functor formalism.

You should think of Mackey functors as abstracting double coset formulae, especially when reading the literature or in representation theory: they tell you something about how restriction and transfer maps commute.

Proposition 3.2.5. *The category of Mackey functors is abelian.*

We'll find this useful later, when we need to do homological algebra.

Suppose H and K are subgroups of G and $f : G/H \to G/K$ is a G-map. Then, we call $\mathrm{Res}_H^K := M^*(f)$ a **restriction map** and $\mathrm{Tr}_H^K := M_*(f)$ a **transfer map**.

Example 3.2.6.

(1) Let X be a G-space. Then, $\pi_n^{(-)}(X)$, the functor sending $G/H \mapsto \pi_n^H(X)$, is a Mackey functor: by the Wirthmüller isomorphism and the fact that orbits are self-dual,

$$\pi_n^H(X) \cong [S^n \wedge G/H_+, X] \cong [S^n, X \wedge G/H_+],$$

so it's clear how a map $G/H \to G/K$ induces both a right-way and a wrong-way map.

(2) Let $\underline{\mathbb{Z}}$ denote the **constant Mackey functor** in $\mathbb{Z}$, which assigns $\mathbb{Z}$ to every object. The restriction maps are all the identity, and the transfer $G/H \to G/K$ is multiplication by $|H/K|$.

(3) The **Burnside Mackey functor** $A(G)$ is defined by letting $A(G)(G/H)$ be the Grothendieck group of the symmetric monoidal category of finite H-sets under coproduct. The transfer and restriction maps come from induction and restriction, respectively, between $H\mathsf{Set}$ and $K\mathsf{Set}$.

(4) The **representation Mackey functor** $R(G)$ is defined by letting $R(G)(G/H)$ be the Grothendieck group of finite-dimensional H-representations, with transfer and restriction given by $\mathbb{Z}[K] \otimes_{\mathbb{Z}[H]} -$.[3]

(5) The Burnside category is self-dual, i.e. there's an equivalence $B_G \simeq B_G^{\mathrm{op}}$, which sends a span $G/H_1 \leftarrow G/K \to G/H_2$ to $G/H_2 \leftarrow G/K \to G/H_1$. If M is a Mackey functor, then $M^{\mathrm{op}}\colon B_G \to B_G^{\mathrm{op}} \to \mathsf{Ab}$ (reverse the arrows in B_G, then apply M) is also a Mackey functor, called the **opposite** of M. $\triangleleft$

Example 3.2.7 ([**Str12**])**.** The constant Mackey functor can be defined in greater generality: given a semigroup M with an action of G on M, let $\underline{M}\colon B_G^{\mathrm{op}} \to \mathsf{Ab}$ be the functor that on objects is $\underline{M}(G/H) := \mathrm{Map}^G(G/H, M) = M^H$, and on the span ω equal to

$$\begin{tikzcd} X & M \arrow[l, "p"'] \arrow[r, "q"] & Y, \end{tikzcd}$$

we define $\underline{M}(\omega)\colon \underline{M}(X) \to \underline{M}(Y)$ by sending a map $u\colon X \to A$ to

$$\underline{M}(\omega)(u)(y) = \sum_{q(a)=y} u(p(a)).$$

This Mackey functor is called the **constant Mackey functor** $\underline{M}$ for M. $\triangleleft$

Exercise 3.2.8. Check that this is a functor.

Furthermore, if the action of G on M is trivial, then we have that the restriction maps are identities, hence the term constant Mackey functor. However, the transfers are not identities: Tr_H^K is multiplication by $|H|/|K|$.

The representation Mackey functor contains the information of double coset formulae in representation theory. Namely, suppose H and K are subgroups of G contained in a common subgroup $J \subseteq G$. The universal property of the product gives us a pullback diagram

$$\begin{tikzcd} J/H \times J/K \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & J/K \arrow[d] \\ J/H \arrow[r] & J/J. \end{tikzcd}$$

If you apply induction to this, you obtain

$$\begin{tikzcd} G \times_J (J/H \times J/K) \arrow[r] \arrow[d] & G \times_J (J/K) \cong G/K \arrow[d] \\ G/H \arrow[r] & G/J. \end{tikzcd} \tag{3.2.9}$$

As a finite G-set, $J/H \times J/K$ is a coproduct of orbits, determined by their stabilizers.

Exercise 3.2.10. Compute the stabilizer of a point in $G/H \times G/K$, therefore proving that

$$J/H \times J/K \cong \coprod_{x \in [H \backslash J/K]} G/(H \cap xKx^{-1}).$$

This is good practice for working with double cosets, and how group-theoretically, they behave much like ordinary cosets.

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[3] TODO: what's the base ring? It should be possible to define the representation ring Mackey functor for representations valued in any commutative ring.

Thus (3.2.9) becomes

(3.2.11)
$$\begin{tikzcd}
\coprod_{X\in[H\backslash J/K]} G/(H\cap xKx^{-1}) \arrow[r, "f_2"] \arrow[d, "f_1"] & G/K \arrow[d] \\
G/H \arrow[r] & G/J.
\end{tikzcd}$$

The point is that explicitly determining what f_1 and f_2 are will produce the double coset formulae. Since $H\cap xKx^{-1}$ is a subgroup of K, we have a map $G/(H\cap xKx^{-1})\to G/H$, and taking the coproduct over $x\in[K\backslash J/H]$ we obtain f_1.

f_2 is harder, but not much harder: after conjugating by x, we obtain a map $G/(x^{-1}Hx\cap K)\to G/K$, so f_2 is conjugation followed by the natural map.

This means that you can compute how Res^k_H and Tr^K_H interact by looking at the two ways around the diagram from G/H to G/K, using the restriction and transfer maps for contravariance and covariance, respectively.

(3.2.12)
$$\mathrm{Res}^G_K \mathrm{Tr}^G_H = \bigoplus_{x\in H\backslash G/K} \mathrm{Tr}^K_{H\cap xKx^{-1}} c_x \mathrm{Res}^H_{H\cap xK^{-1}x},$$

where c_x is the conjugation map.

REMARK. Some treatments define a Mackey functor directly in terms of the restriction and transfer maps with axioms including the double coset formula (3.2.12). The more abstract approach we took follows [**Dre71**], while older treatments take the axiomatic approach. In general, the proofs of properties of Mackey functors are sketched or left out in the literature, but [**Web00**] and [CITE ME: Devinatz] are pretty good references. ◂

3.3. Brown representability and RO(G)-graded cohomology theories

"I thought it was obvious, but it's obviously not obvious."

In the back of your mind, you should be wondering about the connection between G-spectra, equivariant cohomology theories, and Mackey functors. In the nonequivariant setting, Brown representability establishes an (almost) equivalence between spectra and cohomology theories. Equivariantly, things are very similar: G-spectra play the role of spectra, equivariant cohomology theories play the role of cohomology theories, and, less tautologically, Mackey functors play the role of abelian groups. We'll talk about rings later.

First, we should discuss RO(G), the ring we'll index cohomology groups on.

Definition 3.3.1. Let G be a group and R be a ring. Then, the **representation ring** of G over R is the Grothendieck construction applied to the category of finite-rank G-representations over R, i.e. the ring generated by isomorphism classes of finite-rank G-representations over R, with the relations $[V\oplus W]=[V]\oplus[W]$ and $[V\otimes W]=[V]\cdot[W]$.[4] By *the* representation ring, we'll mean $\mathrm{RO}(G):=\mathbb{R}(G)$, the ring of real representations.

We can think of $\mathbb{R}(G)$ and $\mathbb{C}(G)$ as rings of characters: given a representation V, its **character** is the map $\chi_V\colon G\to\mathbb{R}$ (or to $\mathbb{C}$) where $\chi_V(g)=\mathrm{tr}(g\cdot)$. Since $\chi_{V\oplus W}=\chi_V+\chi_W$, $\chi_{V\otimes W}=\chi_V\cdot\chi_W$, and for irreducible representations, $\chi_V=\chi_W$ iff $V\cong W$, then the characters detect isomorphism classes, so $\mathbb{R}(G)$ and $\mathbb{C}(G)$ can be thought of as the rings of functions $G\to\mathbb{R}$ (resp. $\mathbb{C}$) that are characters.

Exercise 3.3.2.

(1) Compute $\mathbb{C}(G)$ for $G=C_p$, and then for C_m where m isn't prime.
(2) Harder: compute $\mathbb{R}(G)$ for the same G.

Roughly speaking, an RO(G)-graded cohomology theory should be some collection of functors E^α indexed by $\alpha\in\mathrm{RO}(G)$ such that for any G-representation V, $E^\alpha(X)\cong E^{\alpha+V}(\Sigma^V X)$, and any fixed α should satisfy the wedge and cofiber axioms, taking wedge products to coproducts and cofiber sequences to long exact sequences.

The (well, an) issue that makes this tricky is that you can't really work with isomorphism classes of representations, even though you need that to define RO(G). As a workaround, let's fix a G-universe U and define $\mathsf{RO}(G;U)$ to be the category whose objects are finite-dimensional representations V that embed equivariantly in U

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[4] Not every element of $R(G)$ is the class of a representation: some elements are formal differences of two representations. These elements are sometimes called **virtual representations**.

and whose morphisms $V \to W$ are G-equivariant isometric isomorphisms.[5] We say that two maps $f, g\colon V \to W$ are **homotopic** if their one-point compactifications $\overline{f}, \overline{g}\colon S^V \to S^W$ are stably homotopic. We don't have enough structure to set up a model category or anything, but we can define the homotopy category $\mathrm{Ho}(\mathsf{RO}(G;U))$ by passing to homotopy classes of maps.

A finite-dimensional G-representation defines a suspension functor

$$\Sigma^W\colon \mathrm{Ho}(\mathsf{RO}(G;U)) \times \mathrm{Ho}(G\mathsf{Top}) \longrightarrow \mathrm{Ho}(\mathsf{RO}(G;U)) \times \mathrm{Ho}(G\mathsf{Top})$$

which sends

$$(V,X) \longmapsto (V \oplus W, \Sigma^W X).$$

This feels a lot like ordinary S^W-suspension, but we're keeping track of the automorphisms of W.

Definition 3.3.3. An $\mathrm{RO}(G)$**-graded cohomology theory** is a functor $E\colon \mathrm{Ho}(\mathsf{RO}(G;U)) \times \mathrm{Ho}(G\mathsf{Top})^{\mathrm{op}} \to \mathsf{Ab}$, where $E(V,X)$ is generally written $E^V(X)$, together with isomorphisms

$$\sigma_W\colon E^V(X) \longrightarrow E^{V\oplus W}(\Sigma^W X),$$

such that for each V, $E^V(\text{–})$ satisfies the wedge and cofiber axioms and for each isometric isomorphism $\alpha\colon W \to W'$, the map

$$\begin{tikzcd}
E^V(X) \arrow[r, "\sigma_W"] \arrow[d, "\sigma_{W'}"] & E^{V\oplus W}(\Sigma^W X) \arrow[d, "(\mathrm{id}\oplus\alpha, \mathrm{id})"] \\
E^{V\oplus W'}(\Sigma^{W'} X) \arrow[r, "(\mathrm{id}\oplus\mathrm{id}, \alpha)"] & E^{V\oplus W'}(\Sigma^W X).
\end{tikzcd}$$

We further require that $\sigma_0 = \mathrm{id}$ and the $\{\sigma_W\}$ are transitive: $\sigma_W \circ \sigma_V = \sigma_{V\oplus W}$.

Definition 3.3.4. The **formal differences** of objects in $\mathsf{RO}(G;U)$ are equivalence classes of pairs (V,W) of objects in $\mathsf{RO}(G;U)$, where $(V,W) \sim (V',W')$ if $V \oplus W' \cong V' \oplus W$. The pair (V,W) is also denoted $V \ominus W$.

We can extend an $\mathsf{RO}(G;U)$-graded cohomology theory to formal differences by $E_{V\ominus W}(X) := E_V(\Sigma^W X)$.

Let $\tau\colon V \oplus W \to W \oplus V$ be the transposition natural isomorphism. Then, the coherence condition on the $\sigma_{(-)}$ means that the diagram

$$\begin{tikzcd}
E_V(\Sigma^W X) \arrow[r] \arrow[d] & E_{V\oplus W'}(\Sigma^{W\oplus W'} X) \arrow[d, "\tau_*"] \\
E_{V'}(\Sigma^{W'} X) \arrow[r] & E_{V'\oplus W}(\Sigma^{W'\oplus W} X)
\end{tikzcd}$$

commutes.

Remark. When you add ring structures to this story, the coherence conditions become gnarlier: there's a cocycle arising from the pentagon diagram for associativity. Unfortunately, this is not treated well in the literature; Lewis' thesis [**Lew78**] discusses it, but was not fully published. [**LM06**, Appendix A] talks about it, and [CITE ME: Dugger] worked it out in the abstract in the motivic setting. ◂

Definition 3.3.5. When G is finite, it's possible to extend an $\mathrm{RO}(G)$-graded cohomology theory E to a Mackey-functor-valued theory $\underline{E}$.

Let X be a G-space and $V \in \mathrm{RO}(G)$. Then, the **Mackey functor-valued E-cohomology** of X, denoted $\underline{E}^V(X)$, is the Mackey functor defined as follows:

- On an orbit G/H,
$$\underline{E}^V(X)(G/H) := E^V(G/H \times X).$$
- Given subgroups $H \subseteq K$, we get a covering map $\pi\colon G/H \times X \to G/K \times X$. The restriction map for $\underline{E}^V(X)$ is the pullback $\pi^*\colon E^V(G/K \times X) \to E^V(G/H \times X)$ and the transfer map is the Gysin map $\pi_!\colon E^V(G/H \times X) \to E^V(G/K \times X)$.[6]

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[5] The data of the embedding $V \hookrightarrow U$ is not used in $\mathsf{RO}(G;U)$. When we discuss multiplicative structures, this will change.

[6] TODO: double-check to make sure this is right. Why does the Gysin map exist? Are all the arrows in the right direction?

There's a similar construction for E-homology. We'll see this structure explicitly in examples in §§3.5 and 3.6.

We'll also need to talk about the equivalence between Sp^G and RO(G)-graded cohomology theories; one direction is Brown representability, and the other is more or less straightforward.

You may be asking why we use RO(G)-graded theories at all. One great reason is that there are known examples of $\mathbb{Z}$-graded cohomology theories that are extremely chaotic, but the RO(G)-graded theories have nice patterns. Another, more recent, realization is that in many cases the RO(G)-graded theories carry more useful information. However, of course, they're extremely hard to compute.

One of the important connections in ordinary stable homotopy theory is that cohomology theories are closely related to spectra: every spectrum determines a cohomology theory, and up to a very small ambiguity, a cohomology theory is represented by a spectrum.

Definition 3.3.6. Let E be a G-spectrum. Then, we define E**-cohomology** to be the functor $E_G^V(X) := [S^V \wedge X, E]_G$.

This is an RO(G)-graded cohomology theory, but this is something to check. If you want to restrict your universe, there are some nuances that have to be overcome.

If you put in finite G-sets (basically points), what you get is a Mackey functor: considering maps out of X ensures the transfer maps have the correct variance.

That was the easy direction: the other direction requires Brown representability. This is originally due to [**Bro62**], and we'll give Neeman's interpretation [**Nee96**]. It works in any triangulated category, and is very close to the small object argument, which already makes it a good thing.

Fix a triangulated category C that has small coproducts. The stable homotopy categories Ho(Sp) and Ho(Sp^G) are the examples to keep in mind.

Definition 3.3.7. An $x \in \mathsf{C}$ is **compact** if for all countable coproducts over $y_i \in \mathsf{C}$,

$$\mathrm{Map}_{\mathsf{C}}\left(x, \coprod_i y_i\right) \cong \coprod_i \mathrm{Map}_{\mathsf{C}}(x, y_i).$$

This is not the usual definition of compactness in category theory, which uses filtered colimits, but in the stable setting these are the same as coproducts, motivating our definition. For example, this definition does not characterize compact topological spaces.

Definition 3.3.8.

- A **generating set** of a category C is a set $T \subseteq \mathrm{ob}(\mathsf{C})$ that **detects zero**, i.e. for all $x \in \mathsf{C}$, $x = 0$ iff $\mathrm{Map}_{\mathsf{C}}(z, x) = 0$ for all $z \in T$. If C is triangulated, we additionally require T to be closed under shift.
- C is **compactly generated** if it has a generating set consisting of compact objects.

We introduce these to skate around set-theoretic issues: at some point, we'd like to take a coproduct over all objects in the category, but that's too large. Instead, taking the coproduct over all generators will have the same power, and is actually well-defined.

For example, dualizable spectra (resp. dualizable G-spectra) are a compact generating set for Ho(Sp) (resp. Ho(Sp^G)).

Theorem 3.3.9 (Brown representability [**Bro62**, **Nee96**])**.** *Let C be a compactly generated triangulated category and $H\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ be a functor such that*

(1)
$$H\left(\coprod_i X_i\right) \cong \prod_i H(X_i)$$
and

(2) *H sends exact triangles $X \to Y \to Z \to X[1]$ to long exact sequences*

$$H(X) \longrightarrow H(Y) \longrightarrow H(Z) \longrightarrow H(X[1]) \longrightarrow H(Y[1]) \longrightarrow H(Z[1]) \longrightarrow H(X[2]) \longrightarrow \cdots$$

Then, H is ***representable****, i.e. there's an $X \in \mathsf{C}$ and a natural isomorphism $\mathrm{Hom}_{\mathsf{C}}(-, X) \to H$.*

Proof. We're going to build X inductively. Fix a generating set T for C of compact objects.

In the base case, let

$$U_0 := \coprod_{x \in T} H(x) \qquad \text{and} \qquad X_0 := \coprod_{\substack{(\alpha,t)\in U_0 \\ \alpha \in H(t)}} t.$$

Thus,

$$H(X_0) = H\left(\coprod_{(\alpha,t)\in U_0} t\right) \cong \prod_{(\alpha,t)\in U_0} H(t),$$

and in particular there is a distinguished element $\alpha_0 \in H(X_0)$ which is α at the (α, t) factor. By the Yoneda lemma, α_0 specifies a natural transformation $\theta_0 \colon \mathrm{Map}_{\mathsf{C}}(-,X_0) \to H$, and by construction, θ_0 is surjective for each $t \in T$.

Now we induct: assume we have X_i and α_i specifying a natural transformation $\theta_i \colon \mathrm{Map}_{\mathsf{C}}(-,X_i) \to H$. Then, define

$$U_{i+1} := \coprod_{t\in T} \ker(\theta_i(t) \colon \mathrm{Map}_{\mathsf{C}}(t,X_i) \to H(t)). \qquad \text{and} \qquad K_{i+1} := \coprod_{(f,t)\in U_{i+1}} t.$$

There's a natural map $K_{i+1} \to X_i$ which applies f; let X_{i+1} be its cofiber. Applying H, this produces a map

$$H(X_{i+1}) \longrightarrow H(X_i) \longrightarrow H(K_{i+1}),$$

and by construction, $\alpha_i \mapsto 0$, so by exactness, we can lift to $\alpha_{i+1} \in H(X_{i+1})$. This α_{i+1} specifies a natural transformation $\theta_{i+1} \colon \mathrm{Map}_{\mathsf{C}}(-,X_{i+1}) \to H$, and the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Map}_{\mathsf{C}}(-,X_i) \arrow[r, "\theta_i"] \arrow[d] & H \\
\mathrm{Map}_{\mathsf{C}}(-,X_{i+1}). \arrow[ur, "\theta_{i+1}"'] &
\end{tikzcd}$$

Thus we have a tower $\mathbb{N} \to \mathsf{C}$ sending $i \mapsto X_i$, and we can define

$$X := \operatorname*{hocolim}_i X_i,$$

which we'll show represents H. The **homotopy colimit** in a triangulated category is TODO.

Remark. Though triangulated categories are a good language to know, as they're historically interesting and often useful, they have drawbacks: the octahedral axiom is awkward, for example, and sometimes the triangulated structure works against you.

For this reason, it can be useful to remember that triangulated categories arise as the homotopy categories of stable ∞-categories, where there's additional versatility simplifying some arguments. For this reason, we use words such as "cofiber" and "homotopy colimit," because they're secretly the same thing. ◂

Exercise 3.3.10. The homotopy colimit is also called the **telescope**. Relate this to the usual definition of a telescope.

Anyways, we'll construct a natural transformation $\theta \colon \mathrm{Map}_{\mathsf{C}}(-,X) \to H$ as follows. Since H sends cofiber sequences to long exact sequences, applying it to

$$\coprod_i X_i \longrightarrow \coprod_i X_i \longrightarrow X$$

produces a long exact sequence

$$\cdots \longrightarrow H(X) \longrightarrow \prod_i H(X_i) \longrightarrow \prod_i H(X_i) \longrightarrow \cdots,$$

so in particular we can lift the $\alpha_i \in X_i$ to an $\alpha \in X$, which defines θ as before, and there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathsf{C}}(-,X_0) \arrow[r, "\theta_0"] \arrow[d] & H. \\
\mathrm{Map}_{\mathsf{C}}(-,X) \arrow[ur, "\theta"'] &
\end{tikzcd}$$

In particular, θ is surjective on T (namely, $\mathrm{Map}_{\mathsf{C}}(t,X) \to H(t)$ is surjective for $t \in T$).

To see that θ is injective on T, we use compactness. Let $f \in \mathrm{Map}_{\mathsf{C}}(t,X)$ be such that $\theta(f) = 0$; we wish to show that $f = 0$. Since t is compact, f lies in some $\mathrm{Map}_{\mathsf{C}}(t,X_i)$, i.e. $f \in \ker(\mathrm{Map}_{\mathsf{C}}(t,X_i) \to H(t))$. Therefore, $f \in U_{i+1}$, so it's killed in X_{i+1}, and thus is 0 in the colimit. Therefore θ is injective on T, hence and isomorphism.

There exists a largest full triangulated subcategory C' of C that's closed under small coproducts and such that $\theta|_{\mathsf{C}'}$ is a natural isomorphism; we'll show that $\mathsf{C}' = \mathsf{C}$.[7]

Running the whole argument again,[8] we obtain a $Z \in \mathsf{C}'$ and a natural transformation $\widetilde{\theta}\colon \operatorname{Map}_{\mathsf{C}'}(-,Z) \to H$. We know $\operatorname{Map}_{\mathsf{C}'}$ and $\operatorname{Map}_{\mathsf{C}}$ are isomorphic, and proved that θ and $\widetilde{\theta}$ are isomorphisms on the generating set, we can consider the cofiber of

$$\operatorname{Map}_{\mathsf{C}}(-,Z) \longrightarrow \operatorname{Map}_{\mathsf{C}}(-,X).$$

By the Yoneda lemma, such natural transformations are naturally identified with $\operatorname{Map}_{\mathsf{C}}(Z,X)$, and you can compute the cofiber of $Z \to X$ in terms of these maps. This is 0, so $Z \cong X$.

So we've proven that H is representable when restricted to the objects in the generating set T in C. Furthermore, both H and $\operatorname{Hom}_{\mathsf{C}}(-,X)$ take coproducts to products and take cofiber sequences to long exact sequences. As a consequence, we can conclude that the full subcategory C' of C on which the natural transformation $\theta\colon \operatorname{Hom}_{\mathsf{C}}(-,X) \to H$ is an isomorphism is a full triangulated subcategory of C which is closed under small coproducts (in C) and contains T.[9]

Therefore, it suffices to show that for any compactly generated triangulated category C, a full triangulated subcategory C' that contains T and has small coproducts must in fact be all of C. We show this as follows. Without loss of generality assume that C' is the smallest full triangulated subcategory containing T and small coproducts; that is, take C' to be the smallest localizing subcategory containing T.

Now fix an object $c \in \mathsf{C}$ and consider the functor $\operatorname{Hom}_{\mathsf{C}}(-,c)$. Applying the construction from above, we obtain an object z and a natural transformation $\operatorname{Hom}_{\mathsf{C}}(-,z) \to \operatorname{Hom}_{\mathsf{C}}(-,c)$. By construction, the object $z \in \mathsf{C}'$ — each X_i is in C', and so the homotopy colimit is too. Moreover, the map $\operatorname{Hom}_{\mathsf{C}}(t,z) \to \operatorname{Hom}_{\mathsf{C}}(t,c)$ is an isomorphism for all $t \in T$.

The Yoneda lemma now implies that we have a map $z \to c$ which corresponds to the natural transformation $\operatorname{Hom}_{\mathsf{C}}(-,z) \to \operatorname{Hom}_{\mathsf{C}}(-,c)$. Consider the cofiber c/z. By the previous paragraph and the fact that $\operatorname{Hom}_{\mathsf{C}}(t,-)$ takes triangles to long exact sequences, $\operatorname{Hom}_{\mathsf{C}}(t,c/z) = 0$ for all $t \in T$, and so $c/z = 0$. Therefore c and z are isomorphic. Since z is in C', $c \in \mathsf{C}'$. As c was chosen arbitrarily, we conclude that C and C' coincide. ⊠

In the next section, we'll discuss what Brown representability means for G-spectra, and use it to define Eilenberg-Mac Lane spectra as the spectra representing certain cohomology theories. One fun fact about G-spectra is that every $\mathbb{Z}$-graded cohomology theory uniquely extends to an $RO(G)$-graded cohomology theory by taking shifts, but this is not true for G-spaces!

3.4. Eilenberg-Mac Lane spectra

> *"The large print giveth and the small print taketh away."*

The goal in this section is to construct Eilenberg-Mac Lane spectra: if $\underline{M}$ is a Mackey functor, we'll produce a G-spectrum $H\underline{M}$ which represents ordinary cohomology with coefficients in $\underline{M}$. This will take some work, but will have some nice payoffs. In subsequent sections, we'll run some computations: for $G = C_2$, we'll compute the $RO(G)$-graded cohomology $H_{C_2}(*;\underline{\mathbb{Z}})$ and $H_{C_2}(*;A)$, where $\underline{\mathbb{Z}}$ is the constant $\mathbb{Z}$-valued Mackey functor and A is the Burnside Mackey functor (Example 3.2.6).

In the previous section, we proved Brown representability (Theorem 3.3.9) for triangulated categories, which are stable. We also want an unstable variant.

Theorem 3.4.1 (Unstable Brown representability)**.** *Let C denote the category of based, G-connected G-CW complexes and $H\colon \operatorname{Ho}(\mathsf{C})^{\mathrm{op}} \to \mathsf{Set}$ be a functor such that*

1. *H satisfies the wedge axioms, and*
2. *H takes homotopy pushouts to (weak) pullbacks.*

Then, H is represented, and the converse is true.

Though we stated this theorem for H going to Set, it turns out all such functors factor through the forgetful functor $\mathsf{Ab} \to \mathsf{Set}$, and therefore can be taken to be Ab-valued.

Warning! The basepoint and connectivity hypotheses are important: Brown representability does not hold in the unstable case if you fail to impose those requirements.

---

[7] Dissecting what "largest full triangulated subcategory" means requires a little care, but such a C' exists.

[8] It's important that we use the same generating set T for this.

[9] Terminology: a triangulated subcategory that is closed under small coproducts (taken in the ambient category) is called **localizing**.

We can use Brown representability to prove the following result.

Proposition 3.4.2. *Any* RO(G)*-graded cohomology theory is represented by a* G*-spectrum.*

PROOF. Let E be an RO(G)-graded cohomology theory. We can apply Brown representability to $E_G^V(-)$ for each V, and the identification $E_G^V(\Sigma^W X) \cong E_G^{V\oplus W}(X)$ gives rise to a homotopy class of structure maps on representing objects, but there's a coherence issue: we constructed G-spectra with actual structure maps, not homotopy classes of maps.

The usual way to solve this is with a trick: restrict attention to a cofinal indexing sequence $V_n \hookrightarrow V_{n+1} \hookrightarrow V_{n+2} \hookrightarrow \dots$, and then observe that the homotopy theory of G-prespectra on this sequence is equivalent to $\mathrm{Ho}(\mathsf{Sp}^G)$, which is guaranteed by cofinality: limits are preserved, and the homotopy groups are the same. This fixes the coherence problem: we can pick a representative for each map and force transitivity. Then, we can left Kan extend back to an orthogonal G-spectrum. ⊠

This would be an exercise, except that maybe it's a bit too hard to be one.

Question 3.4.3. Can we build an object of Sp^G directly? It would be nice not to worry about prespectra on a cofinal sequence.

There are different ways to think about doing this. One approach goes through the model of the G-stable category as spectral presheaves.

This provides the connection between cohomology theories on G-spaces and G-spectra, just like in the nonequivariant world. They're almost the same, except for the existence of phantom maps,[10] just like in the nonequivariant case. These are poorly understood, but of course we also want to understand them better in the nonequivariant case.

Corollary 3.4.4. *A* $\mathbb{Z}$*-graded cohomology theory on* Sp^G *is represented by an object of* Sp^G.

The proof is that Σ^V and Ω^V are inverse equivalences in $\mathrm{Ho}(\mathsf{Sp}^G)$, so we can use them to extend to RO(G)-graded theories. Corollary 3.4.4 feels like a weird coincidence, but it's a key technical result that we'll use more than once. The unstable case is different, though.

Corollary 3.4.5. *A* $\mathbb{Z}$*-graded cohomology theory on* $G\mathsf{Top}$ *with coefficients in a coefficient system* M *extends to an* RO(G)*-graded theory iff* M *extends to (the contravariant part of) a Mackey functor.*

Now we can build Eilenberg-Mac Lane spectra for a Mackey functor $\underline{M}$. There will be two constructions: one may be more satisfying. You might hope to use obstruction theory to build Eilenberg-Mac Lane spaces and chain them together, but this is a mess. We'll adopt one unfamiliar-looking approach, which always works, and also see a more familiar one.

Construction 3.4.6. Given $\underline{M}$, we'll construct a $\mathbb{Z}$-graded cohomology theory on Sp^G; by Corollary 3.4.5, this is represented by an object of Sp^G, which will be $H\underline{M}$.

Let X be a G-CW complex, and define $\underline{C}_n(X) := \underline{\pi}_n(X_n/X_{n-1})$ (where X_n is the n-skeleton of X). Now we'll replicate the construction of Bredon cohomology; in particular, the long exact sequence for the triple (X_n, X_{n-1}, X_{n-2}) induces a differential $\underline{C}_n(X) \to \underline{C}_{n-1}(X)$. If Mac denotes the category of Mackey functors, consider the cochain complex

$$\mathrm{Hom}_{\mathsf{Mac}}(\underline{C}_n(X), \underline{M}),$$

and take the cohomology of this cochain complex.

This is represented by an **Eilenberg-Mac Lane spectrum** $H\underline{M} \in \mathsf{Sp}^G$, and one can calculate that

$$\pi_k^H(H\underline{M}) = \begin{cases} 0, & k \neq 0 \\ \underline{M}(G/H), & k = 0. \end{cases}$$

From this perspective, it's easy to show that H is functorial, which is nice. But you might wonder if it's symmetric monoidal: in the nonequivariant case, $H(R \otimes S) \simeq HR \wedge HS$. This is false, and is one of the unfortunate things about the equivariant setting.

---

[10] A **phantom map** is a nonzero morphism between spectra that's not detected by any compact object. In particular, this means that as a natural transformation of cohomology theories, it's 0, but arises from a nonzero morphism. These encode the non-uniqueness of Brown representability.

Though the cohomology theory $H\underline{M}$ represents is explicit, the spectrum itself is not. Brown's original proof of Brown representability [**Bro62**] looks a lot like Postnikov induction, which closely resembles one classical construction of nonequivariant Eilenberg-Mac Lane spaces, which suggests a more explicit approach is possible.

Construction 3.4.7. One can build some Eilenberg-Mac Lane spectra via the equivariant Dold-Thom construction; this approach only works in certain settings (we'll say which ones in a bit), but is interesting enough to mention.

First let's discuss the nonequivariant case. Let $\mathrm{Sp}^\infty\colon \mathsf{Top}_* \to \mathsf{TopCMon}$ be the free functor from based spaces to topological abelian monoids (i.e. adjoint to the forgetful functor). The proof of Dold-Thom amounts to check that Sp^∞ is a fibrant $\mathscr{W}$-space, which means checking that it takes homotopy pushouts to homotopy pullbacks. We can identify this with $H\mathbb{Z}$.

We'll now introduce the **McCord construction**. McCord's original paper [**McC69**] is a good reference. Nick Kuhn has a talent for clarifying constructions and finding cool uses for them, and his paper [**Kuh04b**] on the McCord construction is also great.

Let A be an abelian group and X be a based space. Let $A \otimes X$ denote the free abelian topological monoid on pairs (a, x), written x^a, subject to the relations

1. $*^a = *$,
2. $x^0 = *$, and
3. $x^a x^b = x^{a+b}$.

That is, Y inherits the quotient topology from the free abelian topological monoid.

Remark. The McCord construction can be carried out very generally, e.g. Kuhn [**Kuh04b**] uses a closely related model to analyze the categorical tensor product in ring spectra. ⋖

One can prove a version of the Dold-Thom theorem that implies that $\{A \otimes -\}$ is a model for HA. This is a $\mathscr{W}$-space, and we can think of this as $\{A \otimes S^n\}$. There's also a Γ-space $\underline{n} \mapsto \underline{n} \otimes A$, and one can check this is fibrant, i.e. very special.

In our case, suppose M is a $\mathbb{Z}[G]$-module, and that G is finite. We can extract a Mackey functor $\underline{M}(G/H) := M^H$, where the transfer Tr_H^K sends

$$x \longmapsto \sum_{hH \in K/H} hx.$$

You can check this is a Mackey functor. The equivariant McCord construction produces Eilenberg-Mac Lane spectra for the Mackey functors that arise in this way; see dos Santos [**dS03**], dos Santos-Nie [**dSN09**], or Shimakawa [**Shi89**].

If A is a based G-space and M is a $\mathbb{Z}[G]$-module, then $A \otimes M$ is a G-space using the diagonal G-action. What you obtain is an equivariant Γ-space. We haven't talked much about these, and we won't, but they're interesting. Segal's preprint on equivariant Γ-spaces [**Seg78**] is highly influential and widely circulated, but the statement has a mistake (though the proof is correct).

Exercise 3.4.8. Find the mistake, and fix it. At this point, you have the background to do so.

[**Shi89**] wrote up the correction, and [**Blu06**] answers a related question about equivariant $\mathscr{W}$-spaces.

Theorem 3.4.9. *$\{S^V \otimes M\}$ is a model of $H\underline{M}$.*

So this construction is nice and explicit, but only works when G is finite, and for Mackey functors that arise from $\mathbb{Z}[G]$-modules.

Understanding the Dold-Thom construction is equivalent to understanding infinite loop space theory, and for general compact Lie groups we understand neither.

3.5. Calculation of $\underline{H}_{C_2}^{p,q}(*;\underline{\mathbb{Z}})$

> *"I'm picking up good fibrations,*
> *they're giving me computations."*

In the next two sections, we'll use the formalism we developed and actually calculate things. There aren't many examples of calculations in RO(G)-graded cohomology, except for what people needed for other things: Stong has some lecture notes which might be published, and the motivic homotopy theorists (specifically Dugger and Voevodsky) used the C_2-equivariant category as a model for the motivic setting, and wrote up some calculations.

See, e.g., [**Lew88**, **Dug15**]. Holler and Kriz [**HK17**] compute the $\mathrm{RO}(C_2^n)$-graded cohomology $H^*_{C_2^n}(*;\underline{\mathbb{Z}/2})$, and Zeng [**Zen17**] computes the $\mathrm{RO}(C_{p^2})$-graded cohomology $H^*(*;\underline{\mathbb{Z}})$.

The takeaway is that, even for groups such as C_{p^n}, calculating the Bredon cohomology of a point would almost be a paper. People probably know what it is, but it would be interesting to understand. The equivariant Steenrod and Dyer-Lashof algebras are also not understood well; Hill-Hopkins-Ravenel figured some of it out, but might not have written all of it down. This would be hard to attack via a Cartan-style seminar, unfortunately.

Thus, it's not at all trivial that we're computing $H^*_{C_2}(*;\underline{\mathbb{Z}})$. Recall that $\underline{\mathbb{Z}}$ comes from the coefficient system

$$\begin{tikzcd}
M(C_2/e) \arrow[loop above] \\
M(C_2/C_2) \arrow[u]
\end{tikzcd}$$

where both objects are $\mathbb{Z}$ and the restriction map is the identity, but as a Mackey functor, it also has a transfer map f_*:

$$\begin{tikzcd}
M(C_2/e) \arrow[loop above, "\gamma"] \arrow[d, bend right, "f_*"'] \\
M(C_2/C_2). \arrow[u, bend right, "f^*"']
\end{tikzcd}$$

Exercise 3.5.1. Show that the double coset formulae force $f_*f^* = 1+\gamma$ and $f^*f_* = 2$.[11]

For $\underline{\mathbb{Z}}$,

- The coefficients are $M(C_2/C_2) = \mathbb{Z}$ and $M(C_2/e) = \mathbb{Z}$.
- $f^* = \mathrm{id}$, f_* is multiplication by 2, and $\gamma = \mathrm{id}$.

As rings, $\mathrm{RO}(C_2) \cong \mathbb{Z}[\sigma]/(\sigma^2-1)$, where σ denotes the sign representation. Hence, every virtual representation can be written $p+q\sigma$. Since the additive structure is just $\mathbb{Z}^2$, we'll introduce a bigrading on $H^*_{C_2}(\text{–})$, where $H^{p,q}_{C_2}(\text{–}) := H^{p+q\sigma}_{C_2}(\text{–})$. The index p is called the **fixed dimension**, and q is called the **sign dimension**.

REMARK. There's an alternative **motivic indexing** $H^{p+q,q}_{C_2}$, where q counts the number of copies of σ and $p+q$ counts the total underlying dimension. We're not going to use this grading. $\triangleleft$

We're going to compute $\underline{H}^{*,*}_{C_2}(*;\underline{\mathbb{Z}})$, i.e. as an $\mathrm{RO}(C_2)$-graded Mackey functor as in Definition 3.3.5. There are three steps:

1. First, compute the value at C_2/C_2, which is the abelian-group-valued cohomology $H^{*,*}_{C_2}(C_2;\underline{\mathbb{Z}})$.
2. Next, compute the value at C_2/e, which is the abelian-group-valued cohomology $H^{*,*}_{C_2}(*;\underline{\mathbb{Z}})$.
3. Finally, we must determine the restriction and transfer maps.

Step 1.

Lemma 3.5.2.

$$H^{p,q}_{C_2}(C_2;\underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z}, & p+q=0 \\ 0, & \text{otherwise.} \end{cases}$$

PROOF. By adjunction, $H^{p,q}_{C_2}(C_2;\underline{\mathbb{Z}}) \cong H^{p+q}(*;\mathbb{Z})$ (nonequivariant cohomology), because $[C_{2+}\wedge X, Y]_{C_2} \cong [X,Y]$, and we know what the nonequivariant cohomology of a point is. ⊠

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[11] TODO: I'd like to double-check that I got this correct.

Step 2. We'll compute $H^{*,*}_{C_2}(*;\underline{\mathbb{Z}})$ by identifying it with Bredon homology or cohomology of spaces with free C_2-actions, hence with the nonequivariant cohomology of the quotient. Let $q\sigma$ denote the direct sum of q copies of the sign representation, so that $S^{q\sigma}$ denotes its one-point compactification.

Let $q > 0$. Then,

$$\begin{aligned} H^{p,-q}_{C_2}(*) &\cong \widetilde{H}^{p,-q}_{C_2}(S^0) \\ &\cong \widetilde{H}^{p,0}_{C_2}(S^{q\sigma}). \end{aligned}$$

If $S(q\sigma)$ denotes the unit sphere inside $q\sigma$ and $D(q\sigma)$ denotes the unit disc, then there is a cofiber sequence

$$S(q\sigma) \longrightarrow D(q\sigma) \longrightarrow S^{q\sigma}. \tag{3.5.3}$$

Exercise 3.5.4. Show that $D(q\sigma)$ is equivariantly contractible, and hence

$$\begin{aligned} \widetilde{H}^{p,0}_{C_2}(S^{q\sigma}) &\cong \widetilde{H}^{p,0}_{C_2}(\Sigma S(q\sigma)) \\ &\cong \widetilde{H}^{p-1,0}_{C_2}(S(q\sigma)). \end{aligned}$$

The C_2-action on $S(q\sigma)$ is the antipodal action, hence free. Thus, the cohomology of $S(q\sigma)$ at a trivial representation is the nonequivariant cohomology of the quotient:

$$\begin{aligned} \widetilde{H}^{p-1,0}_{C_2}(S(q\sigma)) &\cong \widetilde{H}^{p-1}(S(q\sigma)/C_2) \\ &\cong \widetilde{H}^{p-1}(\mathbb{RP}^{q-1}) \end{aligned}$$

TODO: fill in the rest of the details.

For the other half of the plane, we use equivariant Spanier-Whitehead duality:

$$H^{p,q}_{C_2}(*) \cong \widetilde{H}^{p,q}_{C_2}(S^0) \cong \widetilde{H}^{C_2}_{-p,-q}(S^0) \cong \widetilde{H}_{-p,0}(S^{\sigma q}).$$

Once again we'll apply $\widetilde{H}_{*,0}$ to the cofiber sequence, and conclude when $k \neq 0, 1$, $\widetilde{H}_{k,0}(S^{\sigma q}) \cong \widetilde{H}_{k-1,0}(S(\sigma q))$, which will again relate to projective spaces; for $k = 0, 1$, we'll have to open up the long exact sequence.

Inducting over the cofiber sequence is a common trick, and is a good one to carry with you.

Let's recall some Mackey functors from Example 3.2.6. We'll write them out carefully so as to not make a mistake. The constant Mackey functor $\underline{\mathbb{Z}}$ is

$$\begin{tikzcd} \mathbb{Z} \arrow[loop above, "\mathrm{id}"] \arrow[d, bend right, "2"'] \\ \mathbb{Z}, \arrow[u, bend right, "\mathrm{id}"'] \end{tikzcd}$$

where the top entry corresponds to the orbit C_2/e, and the bottom to the orbit C_2/C_2. There's another Mackey functor, denoted $\underline{\mathbb{Z}}^{\mathrm{op}}$, given by switching the restriction and transfer:

$$\begin{tikzcd} \mathbb{Z} \arrow[loop above, "\mathrm{id}"] \arrow[d, bend right, "\mathrm{id}"'] \\ \mathbb{Z}. \arrow[u, bend right, "2"'] \end{tikzcd}$$

If $\pi\colon C_2 \times X \to X$ is projection onto the second factor and M is a Mackey functor, we get maps going both ways:

$$\begin{tikzcd} H^{p,q}_{C_2}(X;M) \arrow[r, shift left, "\pi^*"] & H^{p,q}_{C_2}(C_2 \times X;M). \arrow[l, shift left, "\pi_!"] \end{tikzcd}$$

Finally, it'll be good to remember a fact about the dimension axiom: the transfers in a Mackey functor M coincide with the transfer maps in cohomology induced by maps between orbits.

TODO: streamline this, and explicitly state the final answer.

To compute $H^{p,q}_{C_2}(*)$, we use duality:

$$H^{p,q}_{C_2}(*) \cong \widetilde{H}^{p,q}_{C_2}(S^0) \cong \widetilde{H}_{-p,-q}(S^0) \cong \widetilde{H}_{-p,0}(S^{\sigma q}),$$

and that's where we left off. To continue, we'll use the cofiber sequence (3.5.3), as well as its based version

$$S(\sigma q)_+ \longrightarrow D(\sigma q)_+ \longrightarrow S^{\sigma q}, \tag{3.5.5}$$

and it's worth noting that the middle space is equivalent to S^0, which is sometimes implicit in the literature.

If $k \neq 0, 1$, apply $\widetilde{H}_{k,0}$ to (3.5.3) to conclude that

$$\widetilde{H}_{k,0}(S^{q\sigma}) \cong \widetilde{H}_{k-1}(S(\sigma q))$$

and therefore that

$$\widetilde{H}^{p,q}_{C_2}(*) \cong \widetilde{H}_{-p-1}(\mathbb{RP}^{q-1}).$$

Recall that the cohomology of $\mathbb{RP}^m$ is

$$H^i(\mathbb{RP}^m) = \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z}, & i = m \text{ and } m \text{ is odd} \\ \mathbb{Z}/2, & 0 < i < m \text{ and } i \text{ is odd} \\ 0, & \text{otherwise,} \end{cases}$$

so we get a bunch of copies of $\mathbb{Z}$ and $\mathbb{Z}/2$.

That leaves $k = 0, 1$, for which we must look at the long exact sequence

$$0 \longrightarrow \widetilde{H}_{1,0}(S^{q\sigma}) \longrightarrow \widetilde{H}_{0,0}(S(q\sigma)) \xrightarrow{\gamma} \mathbb{Z} \longrightarrow \widetilde{H}_{0,0}(S^{q\sigma}) \longrightarrow 0.$$

Claim 3.5.6. *γ is multiplication by 2, and therefore $\widetilde{H}_{1,0}(S^{q\sigma}) \cong 0$ and $\widetilde{H}_{0,0}(S^{q\sigma}) \cong \mathbb{Z}/2$.*

PROOF. The idea is to identify the map γ as the transfer map for the Mackey functor $\underline{\mathbb{Z}}$. Said another way, we want to see γ as coming from the map $H_{0,0}(C_2) \to H_{0,0}(\mathrm{pt})$ associated to the map $C_2 \to *$. This follows from the dimension axiom and the fact that $S(\sigma) \cong C_2$. ⊠

So now, how do we express this in terms of Mackey functors? This is only nontrivial in the C_2 orbit and when the cohomology is nonzero (the zero Mackey functor is not hard to define). Thus, we care about the case $p + q = 0$.

Consider the long exact sequence

$$\begin{tikzcd}
H^{-(2n-1),2n-1}_{C_2}(*) & H^{-2n,2n}_{C_2}(C_2) \arrow[l] & \\
 & H^{-2n,2n}_{C_2}(C_2) \arrow[u, "\zeta"'] & H^{-2n,2n-1}_{C_2}(*), \arrow[l]
\end{tikzcd}$$

which comes from (3.5.3) when $q = 1$, i.e. the cofiber sequence $C_2 \to S^0 \to S^\sigma$. The map ζ is the transfer in the Mackey functor.

Exercise 3.5.7. Show, using the double coset formula, that when $n > 0$, ζ is an isomorphism, and when $n < 0$, it's multiplication by 2.

That is, when $n > 0$, the cohomology, as a Mackey functor, is $\underline{\mathbb{Z}}$, and when $n < 0$, it's $\underline{\mathbb{Z}}^{\mathrm{op}}$.

3.6. Calculation of $\underline{H}^{p,q}_{C_2}(*; A_{C_2})$

"I came here to chew bubblegum and compute cohomology groups, and I'm all out of bubblegum."

Now we'll repeat this all in the slightly harder case of the Burnside Mackey functor A_{C_2}. The published source is [**Lew88**], building on unpublished work of Stong. The answer is somewhat crazy, and we'll have to define a whole bunch of Mackey functors along the way. Recently, Basu and Ghosh [**BG16**] generalized to $G = C_{pq}$. There are also some computations for nontrivial spaces, e.g. Megan Shulman's thesis [**Shu10**] computes the $\mathrm{RO}(C_p)$-graded cohomology of BO and BSO.

Bestiary 3.6.1. We use Lewis' notation [**Lew88**] for the various Mackey functors that will arise in the computation.[12]

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[12] TODO: However, for consistency with the notation for the orbit category in previous sections, we've switched which of C_2/e and C_2/C_2 is on top. Should this be changed? Whichever we choose, it should be consistent.

(1) The Burnside Mackey functor A_{C_2} takes the form[13]

$$\begin{tikzcd}
\mathbb{Z} \arrow[loop above, "\mathrm{id}"] \arrow[d, bend right, "1\mapsto x"'] \\
\mathbb{Z}[x]/(x^2-2x). \arrow[u, bend right, "{1\mapsto 1,\ x\mapsto 2}"']
\end{tikzcd} \tag{3.6.2}$$

(2) For any abelian group G, there's a Mackey functor $\langle G\rangle$ for which all of the maps are 0:

$$\begin{tikzcd}
0 \arrow[loop above, "0"] \arrow[d, bend right, "0"'] \\
G. \arrow[u, bend right, "0"']
\end{tikzcd}$$

(3) Given a C_2-Mackey functor, we can forget to a $\mathbb{Z}[C_2]$-module by evaluating at C_2/e. This has both left and right adjoints: if M is a $\mathbb{Z}[C_2]$-module, the left adjoint, denoted $L(M)$, is

$$\begin{tikzcd}
M \arrow[loop above, "\tau"] \arrow[d, bend right, "\pi"'] \\
M/C_2. \arrow[u, bend right, "\overline{\mathrm{tr}}"']
\end{tikzcd}$$

where π is projection, τ is the action of the nontrivial element of C_2, and $\overline{\mathrm{tr}}([x]) = \sum_{g\in G} g\cdot x$ for any x in the coset $[x]$. The right adjoint, denoted $R(M)$, is

$$\begin{tikzcd}
M \arrow[loop above, "\tau"] \arrow[d, bend right, "\mathrm{tr}"'] \\
M^{C_2}. \arrow[u, bend right, "\iota"']
\end{tikzcd}$$

where ι is inclusion and $\mathrm{tr}(x) = \sum_{g\in G} g\cdot x$.[14]

(4) Regarding $\mathbb{Z}$ as a trivial $\mathbb{Z}[C_2]$-module, we obtain $L := L(\mathbb{Z})$ and $R := R(\mathbb{Z})$, which are explicitly

$$\begin{tikzcd}
\mathbb{Z} \arrow[loop above, "\mathrm{id}"] \arrow[d, bend right, "\mathrm{id}"'] \\
\mathbb{Z} \arrow[u, bend right, "\cdot 2"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathbb{Z} \arrow[loop above, "\mathrm{id}"] \arrow[d, bend right, "\cdot 2"'] \\
\mathbb{Z}, \arrow[u, bend right, "\mathrm{id}"']
\end{tikzcd}$$

respectively.

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[13] For $G = C_p$, the Burnside Mackey functor looks very similar: $A_{C_p}(C_p/C_p) = \mathbb{Z}[x]/(x^2 - px)$, and the restriction map sends $1 \mapsto 1$ and $x \mapsto p$.

[14] These generalize *mutatis mutandis* to $G = C_p$.

(5) If $\mathbb{Z}\sigma$ denotes $\mathbb{Z}$ as a $\mathbb{Z}[C_2]$-module with the sign action, we obtain $L_- := L(\mathbb{Z}\sigma)$ and $R_- := R(\mathbb{Z}\sigma)$, which are explicitly

$$\begin{tikzcd} \mathbb{Z} \arrow[loop above, "-1"] \arrow[d, bend right, "\pi"'] \\ \mathbb{Z}/2 \arrow[u, bend right, "0"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbb{Z} \arrow[loop above, "-1"] \arrow[d, bend right, "0"'] \\ 0, \arrow[u, bend right, "0"'] \end{tikzcd}$$

respectively.

Exercise 3.6.3. Show that $L(-)$ and $R(-)$ are really left and right adjoints to the forgetful map $\mathsf{Mac} \to \mathsf{Mod}_{\mathbb{Z}[C_2]}$.

It's possible to write down lots of Mackey functors, and it's entertaining to do so. But the key examples arise as $\pi_0 X$, where X is a G-spectrum. You might ask if all Mackey functors arise this way — and for once we know the answer: we constructed Eilenberg-Mac Lane spectra for every Mackey functor, so the answer is yes!

The next question: do functors on the category of Mackey functors arise as π_0 of functors on Sp^G? You can also ask lots of questions from commutative algebra, e.g. how to think of rings and modules in this context, and the answers are generally harder than in the purely algebraic case.

Anyways, using the bestiary, we can state the answer. It's complicated, but the dimension axiom is complicated in equivariant cohomology too.

Theorem 3.6.4.

$$H^{p,q}_{C_2}(*;A_{C_2}) = \begin{cases} A, & p=q=0 \\ R, & p+q=0,\ p<0,\ p \text{ even} \\ R_-, & p+q=0,\ p\le 1,\ p \text{ odd} \\ L, & p+q=0,\ p>0,\ p \text{ even} \\ L_-, & p+q=0,\ p<1,\ p \text{ odd} \\ \langle\mathbb{Z}\rangle, & p+q\neq 0, p=0 \\ \langle\mathbb{Z}/2\rangle & p+q>0,\ p<0,\ p \text{ even} \\ \langle\mathbb{Z}/2\rangle & p+q<0,\ p>1,\ p \text{ even} \\ 0, & \text{otherwise.} \end{cases}$$

REMARK. The description of $H^*_{C_2}(*;A_{C_2})$ in [**Lew88**, Thm. 2.1] uses a different grading, indexing by $(p+q,p)$. $\triangleleft$

So the good news is, the answer is determined by the fixed and total dimensions. This also generalizes nicely to C_p when p is odd. Though this looks kind of frightening, Lewis [**Lew88**] proved that if X has even-dimensional cells, then its cohomology is free over that of a point, which is good. Moreover, the $\mathrm{RO}(C_2)$-grading makes these statements cleaner; things like this just aren't true for the $\mathbb{Z}$-graded theories.

The computation follows by analyzing the following cofiber sequence(s),[15] and proceeds in a similar way as for $\underline{\mathbb{Z}}$, just with a harder answer.[16]

$$(C_2)_+ \cong S(\sigma)_+ \longrightarrow D(\sigma)_+ \longrightarrow S^\sigma \tag{3.6.5}$$

Here σ is the sign representation, and we'd like η to be the nontrivial irreducible complex C_2-representation, regarded as a two-dimensional real representation.

Recall that we defined a zoo of Mackey functors for C_2 in Bestiary 3.6.1: we will continue to use that notation here. We stated the answer to the calculation of $\underline{\widetilde{H}}^{p,q}_{C_2}(S^0;A_{C_2})$ (the underline means calculating the Mackey functors) in Theorem 3.6.4, and it may help to see it as a plot indexed in $x=p$ (fixed dimension) and $y=p+q$ (total dimension), as in Figure 1.

The approach we'll use to prove Theorem 3.6.4, which is a general strategy for understanding $H^*_G(S^0)$, is to calculate $H^*_0(S^V)$ and use the Künneth spectral sequence to combine $S^{V\oplus W}$. However, this has only been done for

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[15] In the nonequivariant case, you probably wouldn't dwell on this so explicitly, just like we mentioned Spanier-Whitehead duality for a point to make it clearer what's going on.

[16] TODO: some confusion as to which cofiber sequences work out correctly.

$$\begin{array}{ccccccccccccc}
 & & \vdots & & \vdots & & \vdots & & & & & & \\
 & & \langle\mathbb{Z}/2\rangle & & \langle\mathbb{Z}/2\rangle & & \langle\mathbb{Z}\rangle & & & & & & \\
 & & \langle\mathbb{Z}/2\rangle & & \langle\mathbb{Z}/2\rangle & & \langle\mathbb{Z}\rangle & & & & & & \\
 & & \langle\mathbb{Z}/2\rangle & & \langle\mathbb{Z}/2\rangle & & \langle\mathbb{Z}\rangle & & & & & & \\
 & & \langle\mathbb{Z}/2\rangle & & \langle\mathbb{Z}/2\rangle & & \langle\mathbb{Z}\rangle & & & & & & \\
\cdots & R^- & R & R^- & R & R^- & A & R^- & L & L^- & L & L^- & \cdots \\
 & & & & & & \langle\mathbb{Z}\rangle & & \langle\mathbb{Z}/2\rangle & & \langle\mathbb{Z}/2\rangle & & \\
 & & & & & & \langle\mathbb{Z}\rangle & & \langle\mathbb{Z}/2\rangle & & \langle\mathbb{Z}/2\rangle & & \\
 & & & & & & \langle\mathbb{Z}\rangle & & \langle\mathbb{Z}/2\rangle & & \langle\mathbb{Z}/2\rangle & & \\
 & & & & & & \langle\mathbb{Z}\rangle & & \langle\mathbb{Z}/2\rangle & & \langle\mathbb{Z}/2\rangle & & \\
 & & & & & & \vdots & & \vdots & & \vdots & &
\end{array}$$

FIGURE 1. $\underline{H}^{p,q}_{C_2}(S^0;A_{C_2})$, indexed by $(p,p+q)$.

cyclic groups and D_3; if you do it for pretty much any other group, it would be novel (and publishable). If you like spectral sequences and homological algebra, you could work on it — but it might not be interesting. On the other hand, the engine that lets [**HHR16**] prove the Kervaire invariant 1 problem is a computation of RO($(C_2)^n$)-graded cohomology of a point with coefficients in constant Mackey functors, which leads to the gap theorem, using a method Doug Ravenel suggested in the 1970s, so who knows what you could discover?

In our calculation, we'll exploit the cofiber sequence (3.6.5) heavily. The first example is the calculation of $H^{p,q}_{C_2}((C_2)_+)$.

Proposition 3.6.6.

$$H^{p,q}_{C_2}((C_2)_+) = \begin{cases} A_{C_2}, & p+q=0 \\ 0, & \text{otherwise.} \end{cases}$$

PROOF SKETCH. Let M be an abelian group and M^2 denote the $\mathbb{Z}[C_2]$-module $M\oplus M$, where the nontrivial element of C_2 acts by switching the factors.[17]

Exercise 3.6.7. Show that $L(M^2)=R(M^2)$.

We'll let M_{C_2} denote $L(M^2)$. If M is a Mackey functor, the notation M_{C_2} means $L(M(e)^2)$.

Observe that $\underline{H}^{p,q}_{C_2}((C_2)_+)\cong(\underline{H}^{p,q}_{C_2}(S^0)(e))_{C_2}$. ⊠

Using the long exact sequence associated to (3.6.5), the map

$$H^{p,q-1}_{C_2}(S^0)\cong H^{p,q}_{C_2}(S^\sigma)\longrightarrow H^{p,q}_{C_2}(S^0)$$

is injective if $p+q\neq 1$ and surjective when $p+q\neq 0$, hence an isomorphism when $p+q\neq 0,1$.

Therefore, we conclude that there are 0s in the upper right and lower left quadrants (indexing by $(p,p+q)$ as in Figure 1): when $H^{p,q-1}_{C_2}(S^0)\cong H^{p,q}_{C_2}(S^0)$, we can pull the σ factors out of the representation and use the dimension axiom.

Proposition 3.6.8. *When $p+q=1$, we have*

$$\underline{H}^{p,q}_{C_2}(S^0)\cong\operatorname{coker}\Bigl(\underline{H}^{p,q-1}_{C_2}((C_2)_+)\longrightarrow\underline{H}^{p,q-1}_{C_2}(S^0)\Bigr)$$

---

[17] TODO: I didn't follow this proof at all.

and when $p+q=-1$*,*

$$\underline{H}^{p,q}_{C_2}(S^0)\cong \ker\Big(\underline{H}^{p,q+1}_{C_2}(S^0)\longrightarrow \underline{H}^{p,q+1}_{C_2}((C_2)_+)\Big).$$

We also have

$$(\underline{H}^{p,q}_{C_2}(S^0))(e)=0.$$

These follow from the long exact sequences for (3.6.5) in cohomology and homology, using the dualities

$$\underline{H}^{p,q}_{C_2}(S^0)\cong \underline{H}^{C_2}_{-p,-q}(S^0) \qquad\text{and}\qquad \underline{H}^{p,q}_{C_2}((C_2)_+)\cong \underline{H}^{C_2}_{-p,-q}((C_2)_+).$$

Lemma 3.6.9. $\underline{H}^{1,-1}_{C_2}(S^0)\cong R^-$.

PROOF. Once again we use the long exact sequence

$$\begin{tikzcd} H^{0,0}_{C_2}(S^0) \arrow[r] & H^{0,0}_{C_2}((C_2)_+) \arrow[r] & H^{1,0}_{C_2}(S^\sigma) \arrow[r] & H^{1,0}_{C_2}(S^0). \end{tikzcd}$$

Since $H^{0,0}_{C_2}(S^0)=A$, $H^{0,0}_{C_2}((C_2)_+)=A_{C_2}$, $H^{1,0}_{C_2}(S^\sigma)\cong H^{1,-1}_{C_2}(S^0)$, and $H^{1,0}_{C_2}(S^0)=0$, then this long exact sequence is actually

$$\begin{tikzcd} A \arrow[r] & A_{C_2} \arrow[r] & H^{1,-1}_{C_2}(S^0) \arrow[r] & 0, \end{tikzcd}$$

from which the result follows. ⊠

Thus, the only piece left is the case when $p+q=0$.

Exercise 3.6.10. Compute $\underline{H}^{p,q}_{C_2}(S^0)$ when $p+q=0$. This will again involve a manipulation of the long exact sequence.

Exercise 3.6.11. What happens for other primes? Though it's more difficult, it isn't that different from the case where $p=2$, and H^α still only depends on $|\alpha|$ and $|\alpha^{C_p}|$. The key will be to use two cofiber sequences associated to an irreducible C_p-representation V:

(3.6.12a)
$$\begin{tikzcd} (C_p)_+ \arrow[r] & S^0 \arrow[r] & S^V \end{tikzcd}$$

(3.6.12b)
$$\begin{tikzcd} (C_p)_+ \arrow[r] & S(V) \arrow[r] & \Sigma(C_p)_+. \end{tikzcd}$$

If you like computations, working through this will be enlightening.

REMARK. This RO(G)-graded structure is a lot of extra data and pain, but there are good reasons for it: the nice symmetry present in Figure 1 collapses into confusing data in the $\mathbb{Z}$-graded theory, and freeness results that ought to be true require the RO(G)-grading. ◄

There are a few other cases for which the RO(G)-graded cohomology of a point has been calculated: Ferland and Lewis [**FL04**] worked out a formula for an arbitrary Mackey functor over C_p, and explicate it for the cases $H^{*,*}_{C_p}(*;\langle G\rangle)$, $H^{*,*}_{C_p}(*;L)$, and $H^{*,*}_{C_p}(*;R)$.

CHAPTER 4

Multiplicative structures in the equivariant stable category

4.1. Operadic multiplication and N_∞ operads

"If you liked it then you should have put a ring on it."

TODO: this section may need to be reorganized.

Now, we'll shift gears and talk about ring structures on G-spectra. This will lead eventually to multiplicative structures on RO(G)-graded cohomology theories, through Green functors (simpler) and Tambara functors (more sophisticated), but the topology comes from the algebra, so let's start with the algebra.

By construction, Sp^G is symmetric monoidal: it's a diagram category, and as such has a symmetric monoidal Day convolution (2.3.2). By a **ring spectrum** we mean a monoid in Sp^G, and by a **commutative ring spectrum** we mean a commutative monoid.

One issue with this approach: why is there only one kind of ring? We found lots of different kinds of abelian groups in Sp^G: different subcategories of $\mathscr{O}_G$ or different universes U lead to different definitions. In other words, a choice of abelian group structure amounts to a choice of compatible transfers. We want a parallel story for rings, meaning we'll want some sort of "multiplicative transfer" maps.

We'll say this in two ways: first in an impressionistic way, then more formally.

Exercise 4.1.1. Show that in any symmetric monoidal category C (e.g. Sp, Sp^G, Ab), the coproduct in the category of commutative monoid objects in C is the same as the monoidal product in C.

This implies, for example, that the tensor product is the coproduct for rings. Commutative rings are tensored over finite sets, and multiplication is encoded by maps of sets: given a finite set $\underline{n}$, we let $R \otimes \underline{n} \cong R^{\vee n}$, and the multiplication map is the universal map $R^{\vee n} \to R$, i.e. the coproduct. The point is that this structure determines a ring structure on an abelian group, so we can use it in less familiar settings.

In Sp^G, this says that equivariant multiplication should be controlled by tensoring with G-sets (at least when G is a finite group): that is, if T and S are finite G-sets, we want a ring structure to be defined by maps

$$\bigvee_T R \longrightarrow R \qquad \text{and} \qquad \bigvee_T R \longrightarrow \bigvee_S R.$$

This is where the other notions of rings come from: it's reasonable to restrict to certain subcollections of finite G-sets. In G-spaces, this amounts to choosing transfers, and for G-spectra, this amounts to choosing multiplicative transfers, which are called norms; suspension will send transfers to norms.

Operadic multiplication will make this precise. Sp^G and $G\mathsf{Top}$ are enriched over $G\mathsf{Top}$, so we'll work with operads in $G\mathsf{Top}$.[1]

Definition 4.1.2. A ***G*-operad** is a collection of $G \times \Sigma_n$-spaces $\{\mathcal{O}(n)\}$ along with structure maps

$$s_{k,\mathbf{n}} \colon \mathcal{O}(k) \times \mathcal{O}(n_1) \times \mathcal{O}(n_2) \times \cdots \times \mathcal{O}(n_k) \longrightarrow \mathcal{O}(n_1 + \cdots + n_k)$$

which are suitably equivariant, i.e. G-equivariant and equivariant under the permutations in $\Sigma_{n_1+\cdots+n_k}$ that arise as block permutations in $(n_1, \dots, n_k)$, and are associative and unital.

You should think of the structure map $s_{k,\mathbf{n}}$ as grafting leaves from $\mathcal{O}(n_1), \dots, \mathcal{O}(n_k)$ onto a tree $\mathcal{O}(k)$.

Definition 4.1.3. Let $\mathcal{O}$ be a G-operad. An [algebra!over a G-operad]$\mathcal{O}$-algebra, in either $G\mathsf{Top}$ or Sp^G, is an object X with structure maps

$$\mathcal{O}(n) \times_{\Sigma_n} X^{\times n} \longrightarrow X$$

compatible with the operadic multiplication. Here, $\times$ denotes the product in $G\mathsf{Top}$ or Sp^G.

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[1] There are more general settings you can work in, e.g. using operads in Sp^G or Sp in the stable setting. This is reasonable, but we won't need to do this.

Example 4.1.4.

1. Any operad in Top, such as the associative or commutative operads, defines a G-operad with trivial G-action.
2. Let U be a G-universe, and $O(n)$ be the space of linear isometries from $U^{\oplus n}$ to U; the structure maps come from composition.[2] This operad is called $\mathscr{L}_U$, the **linear isometries operad**. ◂

Definition 4.1.5 (Blumberg-Hill [BH15]). An N_∞**-operad**[3] is a G-operad $O = \{O(n)\}$ such that

1. Σ_n acts freely on $O(n)$ (one says $O(n)$ is Σ_n**-free**), and
2. $O(n)$ is a universal space for some family $\mathscr{F}_n$ of subgroups of $G \times \Sigma_n$ containing $H \times \{1\}$ for all subgroups $H \subseteq G$. That is, if $\Gamma \subset G \times \Sigma_n$,
$$O(n)^\Gamma = \begin{cases} *, & \Gamma \in \mathscr{F}_n \\ \varnothing, & \text{otherwise.} \end{cases}$$
3. $O(n)$ is nonequivariantly contractible.[4]

That $O(n)$ is Σ_n-free places strong constraints on $\{\mathscr{F}_n\}$. For example, if $\Gamma \in \mathscr{F}_n$, then $\Gamma \cap (\{1\} \times \Sigma_n) = \{1\}$.

Lemma 4.1.6. *Let $\Gamma \subseteq G \times \Sigma_n$ be a subgroup. Then, the following are equivalent.*

1. $\Gamma \cap (\{1\} \times \Sigma_n) = \{1\}$.
2. Γ *is the graph of a homomorphism, i.e. there's an $H \subseteq G$ and a group homomorphism $p\colon H \to \Sigma_n$ such that* $\Gamma = \{(h, p(h)) \mid h \in H\}$.

The proof is two lines, neither of which is too hard. In the context of a family $\mathscr{F}_n$, subgroups therefore define maps $p\colon H \to \Sigma_n$ and therefore H-set structures on $\underline{n}$; such H-sets will be called **admissible**.

Example 4.1.7. The operad $\mathscr{L}_U$ from Example 4.1.4 is N_∞, which is a nontrivial fact: unlike for finite-dimensional spaces, the space of linear isometries from U^n to U is contractible. This is a nice thing to think through. The rest of the axioms are easier. ◂

You might wonder whether the little discs operad appears in this context; it turns out that fitting everything together into a colimit behaves poorly, and so one uses something else called the Steiner operad, which maybe will appear in these notes.

In the nonequivariant case, the linear isometries operad is a model for the E_∞-operad. But in the equivariant case, there are N_∞-operads that aren't the linear isometries operad, and this is a disturbing representation-theoretic fact. The goal is to figure out what the right notion of a multiplicative RO(G)-graded cohomology theory is — in the nonequivariant case, ring spectra are the objects that represent multiplicative cohomology theories, and we want to discover the equivariant analogue.

Operadic multiplication is one way to approach this, an algebraic (or infinite loop space) approach to this by thinking of Sp^G as algebras in spaces.

In the nonequivariant case, Ω^∞ defines an equivalence between connective spectra and grouplike E_∞-spaces (X is **grouplike** if $\pi_0(X)$ is a group). These are spaces that have homotopy coherent abelian group structures, and is a fancy way of saying that spectra behave like abelian groups. We want to do this equivariantly, and the big question therein is: *how do we operadically encode transfers?*

Let's first recall what's going on in the nonequivariant case. We'll think of connective spectra as **infinite loop spaces**: if X is an Ω-spectrum, $X_0 \cong \Omega X_1 \cong \Omega^2 X_2 \cong \cdots$, and therefore is an n-fold loop space for all n.

We want to recognize n-fold loop spaces: how do you know when $X \simeq \Omega^n Y$ for some Y? One way to approach this is to define a delooping construction B, and be able to use it n times. Beck (of Barr-Beck fame) provided an answer in [Bec69]: (Σ^n, Ω^n) define an adjunction $\mathsf{Top} \rightleftarrows \mathsf{Top}$, whose associated comonad is $\Sigma^n\Omega^n$ and whose associated monad is $\Omega^n\Sigma^n$.

What Beck observed is that if $X = \Omega^n Y$, X admits an action of the monad $\Omega^n\Sigma^n$:

$$\Omega^n\Sigma^n(\Omega^n Y) = \Omega^n(\Sigma^n\Omega^n)Y \xrightarrow{\eta} \Omega^n Y,$$

where $\eta\colon \Sigma^n\Omega^n \to \mathrm{id}$ is the counit of the adjunction.

---

[2] Interestingly, there's something important about spaces here: if you take chains, the operadic axioms fail.

[3] The N in N_∞ stands for "norm."

[4] This means that if you forget from G-spaces to spaces, N_∞-operads become E_∞-operads.

Theorem 4.1.8 (Beck [**Bec69**]). *If $B(\Sigma^m, \Omega^n\Sigma^n, \Omega^n Y)$ is the simplicial bar construction, i.e. the simplicial space*

$$[q] \longmapsto (\Sigma^m)(\Omega^n\Sigma^n)\Omega^n Y,$$

then its geometric realization is equivalent to $\Omega^{n-m}Y$:

$$|B(\Sigma^m, \Omega^n\Sigma^n, \Omega^n Y)| \simeq \Omega^{n-m}Y.$$

This recognition principle is nice: it means that if Z is any algebra over $\Omega^n\Sigma^n$, then Z is an n-fold loop space. Great! Except that we don't know any $\Omega^n\Sigma^n$-algebras other than $\Omega^n Y$: they don't arise in nature.

Boardman-Vogt and May realized it's convenient to work with operads C_n such that there's a map of monads from $\overline{C}_n \to \Omega^n\Sigma^n$.

Definition 4.1.9. Let O be an operad. Then, $\overline{O}$ denotes the associated monad

$$X \longmapsto \bigvee_n O(n) \times_{\Sigma_n} X^n.$$

The point is that algebras for the monad $\overline{O}$ are the same thing as algebras for the operad O, but the monadic category is easier to deal with. For example, you can use this construction to show that Alg_O is complete and cocomplete under mild hypotheses on O.

Example 4.1.10. The **little n-cubes operad** C is the operad such that $C_n(m)$ is the space of configurations of m disjoint closed cubes in an n-dimensional cube. There is a map $\overline{C}_n \to \Omega^n\Sigma^n$ which arises from an action of $\overline{C}_n$ on

$$C_n(m) \times (\Omega^n X)^m \to \Omega^n X$$

The idea[5] is, given m n-cubes inside a big n-cube, crush the boundary of each cube to a point, and send everything outside the little cubes to the basepoint. It's not too hard to write this out explicitly. ◂

This is a fattened-up version of a configuration space, and so the topology of $C_n(m)$ is already very interesting (and why the little n-cubes operad can be used to define E_n-algebras).

Now, given a connected C_n-space X, it's possible to deloop through the bar construction:

$$X \xleftarrow{\simeq} B(C_n, C_n, X) \xrightarrow{\theta_1} B(\Omega^n\Sigma^n, C_n, X) \xrightarrow{\theta_2} \Omega^n B(\Sigma^n, C_n, X).$$

The map $B(C_n, C_n, X) \to X$ is the easy part: it's an equivalence by standard methods, and should be thought of as the free C_n-resolution of X. The map θ_1 comes from the map $C_n \to \Omega_n\Sigma_n$, and is an equivalence. But θ_2 is hard: it's a theorem that geometric realization commutes with homotopy limits in this context, and uses the fact that X is connected. But the point is, with all this setup, C_n-algebras can be delooped. This approach is studied in [**May72**].

Now, taking the sequential colimit along $C_n \to C_{n+1} \to \cdots$, we get an E_∞-operad whose algebras are called **E_∞-spaces**.

Definition 4.1.11. An operad O is an **E_∞-operad** if $O(n)$ is contractible for all n and Σ_n acts freely on each $O(n)$.

Theorem 4.1.12. *If X is an E_∞-space and $\pi_0(X)$ is a group, then $X \cong \Omega^\infty Z$ for a spectrum Z.*

That is, we can deloop X arbitrarily many times. This is the classical way to understand infinite loop spaces.

Suppose $f : O \to O'$ is a map of operads. Then, we get functors between algebras in both directions: there's a pullback $f^*\colon \mathsf{Top}[O'] \to \mathsf{Top}[O]$ and a **change of scalars** operad $B(O', O, –)\colon \mathsf{Top}[O] \to \mathsf{Top}[O']$; the bar construction model for this is a nice thing to have around.

When do these induce an equivalence on the categories of algebras? The answer for E_∞-operads, is when $f_n\colon O(n) \to O'(n)$ is a (nonequivariant) equivalence.[6] But there's always a map, given by the push-pull construction on

$$\begin{tikzcd} & O \times O' \arrow[dl] \arrow[dr] & \\ O & & O'. \end{tikzcd}$$

So it really doesn't matter which E_∞-operad you use.

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[5] TODO: I don't understand this.

[6] The **commutative operad**, for which $O(n) = *$ for all n, parametrizes commutative algebras, and it's not E_∞! This is a source of much frustration; for example, it's the reason that you generally have to use a different model structure on categories of commutative ring spectra.

The equivariant case. What does this look like equivariantly? We defined N_∞-operads to be those for which $O(0) \simeq *$, the Σ_n-action on $O(n)$ is free, and $O(n) \simeq E\mathscr{F}_n$ for some family $\mathscr{F}_n$ of $G \times \Sigma_n$ that contains $H \times \{1\}$ (see Definition 4.1.5).

We're going to show that algebras for an N_∞-operad O in Top will correspond to some kinds of G-spectrum with transfer structures by $\{\mathscr{F}_n\}$. If $O = \mathscr{L}_U$ (the linear isometries operad in Example 4.1.4), then we get G-spectra dictated by the universe U, but these are not the only examples.

This leaves a few questions.

1. Why do we get transfer maps? How can we see this from the operadic structure?
2. What can we say about the collection $\{\mathscr{F}_n\}$? You can't just pick any of them: they have complicated interrelationships.

Looking at the second question, suppose $\Gamma \subset G \times \Sigma_n$ is such that $\Gamma \cap (\{1\} \times \Sigma_n) = \{1\}$. We proved in Lemma 4.1.6 that this is equivalent to Γ being the graph of a homomorphism $\rho : H \to \Sigma_n$, which is equivalent to the data of an H-set of cardinality n. We called such H-sets admissible.

Definition 4.1.13. Let Cat denote the category of small categories and functors, and $\otimes$Cat denote the category of small symmetric monoidal categories and strong monoidal functors.

- A **categorical coefficient system** is a functor $\mathscr{O}_G^{\mathrm{op}} \to \mathsf{Cat}$.
- A **symmetric monoidal categorical coefficient system** (SMCCS) is a functor $\mathscr{O}_G^{\mathrm{op}} \to \otimes\mathsf{Cat}$.

Example 4.1.14. The most important example is $\underline{\mathsf{Set}}$, the functor sending $G/H \mapsto H\mathsf{Set}$, the category whose objects are H-sets, morphisms are H-maps, and monoidal product is disjoint union. $\underline{\mathsf{Set}}$ is a symmetric monoidal categorical coefficient system.

Similarly, $\underline{\mathsf{Top}}$ denotes the functor sending $G/H \mapsto H\mathsf{Top}$, with monoidal product disjoint union. This is also a symmetric monoidal categorical coefficient system. ◄

Lemma 4.1.15. *Let $\{\mathscr{F}_n\}$ be a collection of families of subgroups of G. Then, $\{\mathscr{F}_n\}$ arises from an N_∞-operad iff it's a sub-SMCCS of $\underline{\mathsf{Set}}$.*[7]

That is, these are exactly the functors sending G/H to admissible H-sets.

Definition 4.1.16. An **indexing system** $\underline{I}$ is a sub-SMCCS of $\underline{\mathsf{Top}}$ which is

1. closed under self-induction,
2. closed under Cartesian product,
3. closed under passage to self-objects, and
4. contains all trivial sets.

Here, **self-induction** means that if $H/K \in \underline{I}(H)$ and $T \in \underline{I}(K)$, then $H \times_K T \in \underline{I}(H)$.

These axioms appear somewhat arbitrary, but there's a nice categorical description in terms of bispans, which we'll discuss later.

Definition 4.1.17. Let O be an N_∞-operad, so that it defines a collection of families $\{\mathscr{F}_n\}$ of subgroups of G. Let $\underline{O}$ denote the functor sending G/H to the collection of admissible H-sets for $\{\mathscr{F}_n\}$.

Proposition 4.1.18. *Let O be an N_∞-operad; then, $\underline{O}$ is an indexing system.*

We say that a map $f : O \to O'$ of N_∞-operads is a **homotopy equivalence** if it defines a $G \times \Sigma_n$-homotopy equivalence $O(n) \to O'(n)$ for all n.

Theorem 4.1.19. *The assignment $O \mapsto \underline{O}$ is an equivalence of categories from the homotopy category of N_∞-operads to the poset of indexing systems.*

That this is fully faithful is proven in [**BH15**]; the converse was proven independently by Bonventre-Pereira [**BP17**], Gutiérrez-White [**GW17**], and Rubin [**Rub17**].

Partial proof of Proposition 4.1.18. It's clear that trivial sets are admissible for an N_∞-operad O.[8]

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[7] TODO: did I get this right?

[8] As a corollary, this means that if O is an N_∞-operad, then $i_e^* O$ is an E_∞-operad, where $i_e : \mathsf{Top} \to G\mathsf{Top}$ prescribes the trivial action.

We want to show that $\underline{O}$ is closed under coproduct: if S and T are admissible H-sets, we'd like for $S \amalg T$ to be admissible. Well, S is the graph of a $p_S\colon H \to \Sigma_{|S|}$ and T is the graph of $P_T\colon H \to \Sigma_{|T|}$. Let Γ_S be the associated subgroup of $G \times \Sigma_{|S|}$, and define Γ_T in the same way.

Let

$$\Gamma := \{(h, p_S(h) \amalg p_T(h)) \mid h \in H\}.$$

This acts on $S \amalg T$, and gives it the structure of an admissible H-set. Given the map

$$O(2) \times O(|S|) \times O(|T|) \longrightarrow O(|S| + |T|),$$

we'd like to show that the Γ-fixed point set of $O(|S|+|T|)$ is nonempty. But by hypothesis, $(O(2) \times O(|S|) \times O(|T|)^\Gamma$ is nonempty, so we're done.

The pattern of proof is the same for the rest of the conditions. ⊠

We've been talking about transfers, so let's see why algebras over an N_∞-operad have transfers.

Let T be an admissible H-set. Then, the space of maps

$$G \times \Sigma_{|T|}/\Gamma_T \longrightarrow O(|T|)$$

is contractible.

Lemma 4.1.20.

$$(G \times \Sigma_{|T|}/\Gamma_T) \times_{\Sigma_{|T|}} X^{|T|} \cong G \times_H F(T, X).$$

Thus, if X is an algebra over an N_∞-operad O, we have maps

$$G \times_H F(T, X) \longrightarrow (G \times \Sigma_{|T|}/\Gamma_T) \times \Sigma_{|T|} X^{|T|} \longrightarrow O(|T|) \times_{\Sigma_{|T|}} X^{|T|} \longrightarrow X,$$

and this is where the transfer maps on $\pi_0(X)$ come from.

If X is a G-space and T is an H-set, let

$$N^T X := G \times_H F(T, i_H^* X).$$

Lemma 4.1.21. *A map $S \to T$ of H-sets induces a map $N^S X \to N^T X$.*

The proof is the same as for the construction of the transfer maps $H/K \to H/H$: build this transfer, and then induce up. In general, an H-map $S \to T$ induces a map

$$F(T, i_H^* X) \longrightarrow F(S, i_H^* X).$$

Passing to fixed points and taking π_0, this is precisely a system of transfers on π_0.

REMARK. One consequence of this is a conceptual description of the double coset formula: the restriction to K' of a transfer on T is just the transfer on $i_{K'} T$. We'll unpack this in the next section. ⋖

Another conceptual takeaway is that N_∞-operads are the right way to discuss transfers in the equivariant setting. So we may as well port them to G-spectra, since they're also enriched over G-spaces. But when you do this, you stumble upon one of the basic conundrums of equivariant homotopy theory.

Let X be a space. Then, X^n is a C_n-space, where C_n acts by permuting the factors. The diagonal map $\Delta\colon X \to X^n$ induces an equivalence $X \cong (X^n)^{C_n}$. The problem is to do this with spectra: we want a map $\mathrm{Sp} \to \mathrm{Sp}^{C_n}$ such that we have a diagonal.

The naïve approach is to send $X \mapsto X^{\wedge n}$, but what universe does $X^{\wedge n}$ live in? Naïvely, we're in U^n, which maps back down to U. But in U^n, we have good fixed-point behavior for representations of the form $(V, V, \dots, V)$ — and only for representations of this form. So it's not clear how to make this work.

Bökstedt's construction of *THH* [**Bök87**] needed something like this, and so Bökstedt did the minimal amount necessary to get something to work. The technical solution to this uses ideas from group cohomology, specifically the Evens norm, which we'll discuss in §4.4. Figuring out how to make this work in homotopy theory was one of the important new ideas in [**HHR16**].

We'll eventually get to ring spectra and spectra with norm maps, which encode a kind of twisted multiplication.

4.2. Green functors

"What does the box say?"

We're going to pick up where we left off, discussing multiplicative structures on equivariant spectra. As foreshadowed, we'll describe them using N_∞-operads, the natural equivariant generalizations of E_∞-operads. This section, though, is about the underlying algebra. For example, if X is a G-spectrum, we know $\pi_0(X)$ is a Mackey functor, and in fact $\pi_*(X)$ is a graded Mackey functor. Thus Mackey functors are our replacement for abelian groups in equivariant stable homotopy theory. There's a very natural followup question.

Question 4.2.1. Suppose R is a commutative ring in orthogonal G-spectra. What structure does $\pi_0(R)$ have?

Recall that the category Mac_G of Mackey functors can be defined as the functor category $\mathsf{Fun}(B_G^{\mathrm{op}}, \mathsf{Ab})$ (i.e. the enriched functor category; in particular, we ask for additive functors). The Burnside category B_G admits several descriptions, but we described it in Definition 2.2.2 as the category whose objects are orbits G/H and whose morphisms are the homotopy classes of stable maps:

$$\mathrm{Hom}_{B_G}(G/H, G/K) := \pi_0 F(\Sigma^\infty G/H_+, \Sigma^\infty G/K_+).$$

One advantage of this approach is that functor categories of this sort automatically have a symmetric monoidal structure defined by Day convolution (2.3.2). Namely, if C and D are symmetric monoidal categories, then $\mathsf{Fun}(\mathsf{C},\mathsf{D})$ admits a symmetric monoidal product whose multiplication is left Kan extension of the product in D along the product in C. Ab is symmetric monoidal under tensor product, and B_G^{op} is symmetric monoidal under Cartesian product, so Mac_G inherits a symmetric monoidal structure.

The Day convolution for Mackey functors is also called the **box product**, and has an explicit coend formula: if $\underline{M}$ and $\underline{N}$ are Mackey functors, their box product is

$$\underline{M} \boxtimes \underline{N}(X) = \int^{(Y,Z)\in B_G^{\mathrm{op}}\times B_G^{\mathrm{op}}} \underline{M}(Y)\otimes \underline{M}(Z)\otimes B_G^{\mathrm{op}}(X, Y\times Z). \tag{4.2.2}$$

Coends are examples of coequalizers: in general, the coend of two functors $F, G\colon \mathsf{C}\to\mathsf{D}$ is the coequalizer of the diagram

$$\bigvee_{f\colon x\to y\in\mathsf{C}} F(x)\otimes G(y) \overset{\mathrm{id}\otimes f}{\underset{f\otimes\mathrm{id}}{\rightrightarrows}} \bigvee_{c\in\mathsf{C}} F(c)\otimes G(c).$$

This implies formally that the unit for the symmetric monoidal structure on Mac_G is the Burnside Mackey functor A_G, which is the functor $\mathrm{Map}_{B_G^{\mathrm{op}}}(-, G/G)$.

We also obtained the morphisms $G/H\to G/K$ in the Burnside category as the Grothendieck group of a category of spans, which is equivalent to the category of G-sets over G/H.[9] But this category is equivalent to the category of H-sets.

Now we can do the usual thing to define rings, just as we did in orthogonal (nonequivariant) spectra.

Definition 4.2.3.

- A **(commutative) Green functor** is a commutative monoid in $(\mathsf{Mac}_G, \boxtimes, A_G)$.
- A **module** $\underline{M}$ over a Green functor $\underline{R}$ is a Mackey functor together with an associative, unital action map $\underline{R}\boxtimes\underline{M}\to\underline{M}$.

Green functors are our first guess for the algebraic analogue of a commutative ring in Sp^G, and were originally considered in [Gre71]. They're a plausible guess: if $\mathcal{O}$ is an E_∞-operad, we can regard it as a G-trivial G-operad, and $\mathsf{Sp}^G[\mathcal{O}]$ is the category of G-spectra whose π_0 is naturally a Green functor.

So what kind of structure do we get from this definition?

Proposition 4.2.4. *A Mackey functor $\underline{R}$ is a Green functor iff all of the following are true:*

1. *$\underline{R}(G/H)$ is a commutative ring.*
2. *The restriction map $\mathrm{res}^H_K\colon \underline{R}(G/H)\to\underline{R}(G/K)$ is a ring homomorphism, and hence $\underline{R}(G/K)$ is an $\underline{R}(G/H)$-module.*
3. *The transfer map $\mathrm{tr}^H_K\colon \underline{R}(G/K)\to\underline{R}(G/H)$ is a map of $\underline{R}(G/H)$-modules.*

---

[9] TODO: this should depend on K, right? I'm probably missing something.

Condition (3) is also called **Frobenius reciprocity**, and implies the **push-pull formula**

$$\mathrm{tr}_K^H(x) \cdot y = \mathrm{tr}_K^H(x \cdot \mathrm{res}_K^H(y)).$$

Example 4.2.5. Let $G = C_p$ for concreteness.

(1) The constant Mackey functor $\underline{\mathbb{Z}}$ is a Green functor with the usual ring structure on each copy of $\mathbb{Z}$, and maps

$$\begin{tikzcd} \mathbb{Z} \arrow[d, bend right, "\mathrm{id}"'] \\ \mathbb{Z}. \arrow[u, bend right, "p"'] \end{tikzcd}$$

The restriction map is on the right, and is a ring homomorphism, and the transfer, multiplication by p, is $\mathbb{Z}$-linear.

(2) The Burnside Mackey functor A_G is also a Green functor, which follows formally because it's the unit for $\square$. When $G = C_2$, its additive structure is given in (3.6.2). So what ring structure do we get on $\mathbb{Z} \oplus \mathbb{Z}$? There are two isomorphism classes in the Grothendieck group, $[C_2/C_2]$ and $[C_2/e]$. The product tells us $[C_2/e] \cdot [C_2/e] = 2[C_2/e]$, so the ring structure is $\mathbb{Z}[x]/(x^2 - 2x)$. The restriction map $\mathbb{Z}[x]/(x^2 - 2x) \to \mathbb{Z}$ sends $1 \mapsto 1$ and $x \mapsto 2$, and the transfer map $\mathbb{Z} \to \mathbb{Z}[x]/(x^2 - 2x)$ sends $1 \mapsto x$, which is linear. The push-pull formula tells us that in the Grothendieck group of finite G-sets,

$$[(H \times_K X) \times Y] = [H \times_K (X \times i_K^* Y)].$$

Since A_G is the unit, every Mackey functor is an A_G-module. Can we describe this action explicitly on a general Mackey functor $\underline{M}$? We want compatible actions of $A_G(G/H)$ on $\underline{M}(G/H)$, and this is a restriction-and-transfer construction: an element of $A_G(G/H)$ is (the class of) a $G/K \in G\mathsf{Set}_{/(G/H)}$, which comes with restriction and transfer maps between G/K and G/H. Thus, for an $x \in \underline{M}(G/H)$, we can restrict it to $\underline{M}(G/K)$ and transfer it back to $\underline{M}(G/H)$ using those maps, and this defines the $A_G(G/H)$-action. ◂

In the next section, we'll upgrade this structure into something with multiplicative analogues of transfer maps, called a Tambara functor.

4.3. Tambara functors

Green functors are pretty cool, but we can and should expect more. Here's one reason. Suppose R is an ordinary ring, or even a semiring, and let's look at the structure on $\{R^X\}$ (here, $R^X := \mathrm{Map}(X, R)$) for some sets X. A map $X \to Y$ produces two kinds of maps $t, n \colon R^X \to R^Y$. The first, a transfer-like map, is defined by the formula

$$t(f)(y) := \sum_{f(x)=y} r(x),$$

but we could also do

$$n(f)(y) := \prod_{f(x)=y} r(x),$$

and there's no analogue of this in Green functors. The replacements are called **Tambara functors** (or **TNR functors**).[10] Strickland's paper [**Str12**] on Tambara functors is really nice: it's complete and careful. Lewis has unpublished notes [**Lew80**] on Green functors, which are also good; you can find them, along with almost everything else in equivariant homotopy theory, on Doug Ravenel's website. Lewis' notes owe a debt to McClure's unpublished notes, which have been lost to history.

REMARK. This is related to the theory of **polynomial functors**, which we won't talk about. You can set this up in great abstraction; if you torture it, you can get it to output actual polynomials. Tambara functors are an example of polynomial functors. ◂

Definition 4.3.1. Let C be a locally Cartesian closed category and $X, Y \in \mathsf{C}$. Then, the **category of bispans** from X to Y, denoted $\mathsf{Bispan}_{\mathsf{C}}(X, Y)$ is the category whose objects are diagrams

$$X \longleftarrow S \longrightarrow T \longrightarrow Y \tag{4.3.2}$$

---

[10] TNR stands for "transfer, norm, restriction."

and whose morphisms are commutative diagrams

$$\begin{tikzcd}
& S \arrow[r] \arrow[ld] \arrow[dd, "\cong"] & T \arrow[rd] \arrow[dd, "\cong"] & \\
X & & & Y \\
& S' \arrow[r] \arrow[lu] & T'. \arrow[ru] &
\end{tikzcd}$$

Example 4.3.3. We'll care in particular about a few suggestively named bispans associated to a C-morphism $f : S \to T$:

(1) Let R_f denote the bispan

$$\begin{tikzcd}
T & S \arrow[l, "f"'] \arrow[r, "\mathrm{id}"] & S \arrow[r, "\mathrm{id}"] & S.
\end{tikzcd}$$

(2) Let N_f denote the bispan

$$\begin{tikzcd}
S & S \arrow[l, "\mathrm{id}"'] \arrow[r, "f"] & T \arrow[r, "\mathrm{id}"] & T.
\end{tikzcd}$$

(3) Let T_f denote the bispan

$$\begin{tikzcd}
S & S \arrow[l, "\mathrm{id}"'] \arrow[r, "\mathrm{id}"] & S \arrow[r, "f"] & T.
\end{tikzcd}$$

R_f and T_f will encode the restriction and transfer maps, respectively, in the underlying Mackey functor of a Tambara functor.

One interesting question about bispans is when you can compose in the objects: is there a "composition map"

$$\mathsf{Bispan}_{\mathsf{C}}(Y,Z) \times \mathsf{Bispan}_{\mathsf{C}}(X,Y) \longrightarrow \mathsf{Bispan}_{\mathsf{C}}(X,Z)? \tag{4.3.4}$$

To get this, we'll need to actually look at what being a locally Cartesian closed category means. Specifically, it means that any map $f : X \to Y$ in C induces a pullback $f^* : \mathsf{C}_{/Y} \to \mathsf{C}_{/X}$, and we require that this map has a left and a right adjoint, respectively called the **dependent sum** Σ_f and the **dependent product** Π_f, respectively. These can be thought of (e.g. in Set) as coproducts (resp. products) indexed by the fibers.

REMARK. There's an interesting connection to homotopy type theory: one of the early big theorems in type theory is that propositional calculus and lambda calculus are formally equivalent to the theory of locally Cartesian closed categories, so there was a lot of research into importing these categorical notions into type theory.

The dependent sum and product are examples of base-change functors $f_!$ and f_*; the given notation may be unfamiliar, but is standard in the category-theoretic literature. ⋖

Example 4.3.5. Let's see what this means in GSet. For a map $f : X \to Y$, we want to compute Π_f. If $q : A \to Y$ is an object of $G\mathsf{Set}_{/X}$, then its dependent product is $q' : \Pi_f A \to Y$, where

$$\Pi_f A = \{(y, s) \mid y \in Y, s : f^{-1}(y) \to A \text{ such that } q \circ s = \mathrm{id}\}.$$

That is, it's elements of y along with sections of q above the fiber $f^{-1}(y)$. ⋖

Tambara functors will be associated to a particular kind of diagram.

Definition 4.3.6. An **exponential diagram** in a locally Cartesian closed category C is any diagram isomorphic[11] to one of the form

$$\begin{tikzcd}
X \arrow[d, "f"] & A \arrow[l, "q"'] & X \times_Y \Pi_f A \arrow[l, "ev"'] \arrow[d, "\pi_2"] \\
Y & \Pi_f A \arrow[l, "q'"'] \arrow[r, equal] & \Pi_f A.
\end{tikzcd} \tag{4.3.7}$$

This commutes.

---

[11] This means in the category of diagrams of this shape, i.e. there are isomorphisms for every object in the diagram that commute with the maps in the diagram.

We can use this to build a diagram associated to two spans $X \leftarrow A \to B \to Y$ and $Y \leftarrow C \to D \to Z$: start with

$$\begin{tikzcd}
X & A \arrow[l] \arrow[d] & \\
& B \arrow[d] & \\
& Y & C \arrow[l] \arrow[d] \\
& Z & D. \arrow[l]
\end{tikzcd} \tag{4.3.8}$$

TODO: in class, we got confused and didn't finish writing this diagram down. Anyways, you can use this (and in particular the exponential map) to define the composition operation we asked for in (4.3.4). If we let 0 denote the bispan

$$\begin{tikzcd}
X & \varnothing \arrow[l] \arrow[r] & \varnothing \arrow[r] & Y
\end{tikzcd}$$

and 1 denote the bispan

$$\begin{tikzcd}
X & \varnothing \arrow[l] \arrow[r] & Y \arrow[r, "\mathrm{id}"] & Y
\end{tikzcd}$$

we get a semiring structure on the set of isomorphism classes of bispans. The sum of $[X \leftarrow S \to T \to Y]$ and $[X \leftarrow S' \to T' \to Y]$ is

$$\begin{tikzcd}
X & S \amalg S' \arrow[l] \arrow[r] & T \amalg T' \arrow[r] & Y,
\end{tikzcd}$$

and their product is

$$\begin{tikzcd}
X & (S \times_Y T') \amalg (S' \times_Y T) \arrow[l] \arrow[r] & T \times_Y T' \arrow[r] & Y,
\end{tikzcd}$$

and you can check 0 and 1 are the identities for the sum and product, respectively. When you see a semiring, you might think to take its Grothendieck group, and we will do this. We'll eventually also consider "bispans with coefficients," i.e. those bispans $X \leftarrow S \xrightarrow{f} T \to Y$ where f is constrained to a subcategory D of C. You need a condition on D for this to be a category,[12] which is exactly the same as the condition for an indexing system! So there's N_∞-operads floating around, and hence from a categorical perspective, we're forced to consider this notion of multiplication.

Remark. Spans (sometimes also called correspondences) are useful in many different contexts. That bispans come up is weirder: they're certainly less ubiquitous. On the other hand, when you want one wrong-way map and two right-way maps, it does seem like a good idea. ◀

One can also use the category of bispans can be used to understand polynomial functors, but the papers on polynomial functors are written in the language of type theory and things internal to the relevant topos. This is fine, but it may be unfamiliar.

Given the bispan (4.3.2) in the category of sets, we obtain a sequence of functors

$$\begin{tikzcd}
\mathsf{Set}_{/X} \arrow[r, "f^*"] & \mathsf{Set}_{/S} \arrow[r, "\Pi_g"] & \mathsf{Set}_{/T} \arrow[r, "\Sigma_h"] & \mathsf{Set}_{/Y}.
\end{tikzcd}$$

Their composition is called a **polynomial functor**: it's a sum of a product of "monomials." The sum and product are the dependent ones we defined above and can be a little confusing, but f^* is pullback and Σ_h is composition, so that's not so bad.

We'd like to compose bispans. Last time, we defined exponential diagrams to be those isomorphic to diagrams of the form (4.3.7).

Exercise 4.3.9. Show that an exponential diagram is a pullback diagram.

---

[12] The condition is that D is **wide** (i.e. contains all objects in C), closed under pullbacks, and closed under coproducts. TODO: I might have gotten an axiom wrong.

We can use this to define the composition of two bispans $[Y \leftarrow C \xrightarrow{f} D \to Z] \circ [X \leftarrow A \to B \to Y]$. We start by superimposing the two diagrams as in (4.3.8), then letting $B' := B \times_Y C$ and $A' := A \times_B B'$:

$$\begin{tikzcd}
X & A \arrow[l] \arrow[d] & A' \arrow[l] \arrow[d] \\
 & B \arrow[d] & B' \arrow[l] \arrow[d] \\
 & Y & C \arrow[l] \arrow[d, "f"] \\
 & Z & D. \arrow[l]
\end{tikzcd}$$

Now we can form the exponential diagram for $B' \to C \to D$:

$$\begin{tikzcd}
X & A \arrow[l] \arrow[d] & A' \arrow[l] \arrow[d] & \\
 & B \arrow[d] & B' \arrow[l] \arrow[d] & C \times_D \Pi_f B' \arrow[l] \arrow[dd] \\
 & Y & C \arrow[l] \arrow[d, "f"] & \\
 & Z & D \arrow[l] & \Pi_f B'. \arrow[l]
\end{tikzcd}$$

Finally, let $\widetilde{A} := A' \times_{B'} (C \times_D \Pi_f B')$:

$$\begin{tikzcd}
X & A \arrow[l] \arrow[d] & A' \arrow[l] \arrow[d] & \widetilde{A} \arrow[l] \arrow[d] \\
 & B \arrow[d] & B' \arrow[l] \arrow[d] & C \times_D \Pi_f B' \arrow[l] \arrow[dd] \\
 & Y & C \arrow[l] \arrow[d, "f"] & \\
 & Z & D \arrow[l] & \Pi_f B'. \arrow[l]
\end{tikzcd}$$

Now, we can define the **composition** of these two bispans to be the bispan $[X \leftarrow \widetilde{A} \to \Pi_f B' \to Z]$.

We'd like this to actually behave like a composition.

Lemma 4.3.10. *Composition with the **identity bispan***

$$Y \xleftarrow{\mathrm{id}} Y \xrightarrow{\mathrm{id}} Y \xrightarrow{\mathrm{id}} Y$$

is the identity, up to natural isomorphism.

Partial proof. Let $[X \xleftarrow{f} A \xrightarrow{g} B \xrightarrow{h} Y]$ be some other bispan. Then, $\Pi_{\mathrm{id}} B \cong B$, and pulling back by the identity acts as the identity, so $B' = B$ and $A' = A$. Thus, $Y \times_Y \Pi_{\mathrm{id}} B \cong \Pi_{\mathrm{id}} B = B$ and $\widetilde{A} = A$ (again, we're pulling back by the identity map), so we obtain the diagram

$$\begin{tikzcd}
X & A \arrow[l, "f"'] \arrow[d, "g"] & A \arrow[l, "\mathrm{id}"'] \arrow[d, "g"] & A \arrow[l, "\mathrm{id}"'] \arrow[d, "g"] \\
 & B \arrow[d, "h"] & B \arrow[l, "\mathrm{id}"'] \arrow[d, "h"] & B \arrow[l, "\mathrm{id}"'] \arrow[dd] \\
 & Y & Y \arrow[l, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"] & \\
 & Y & Y \arrow[l, "\mathrm{id}"'] & B. \arrow[l, "h"']
\end{tikzcd}$$

and therefore the composition is $[X \leftarrow A \rightarrow B \rightarrow Y]$ again, at least up to natural isomorphism. ⊠

Exercise 4.3.11. Finish the proof by checking composition with the identity on the right.

REMARK. We're working towards showing that isomorphism classes of bispans are the morphisms in a category. If you don't like taking isomorphism classes, you can use bispans on the nose, in which case you'll get a bicategory. If you like this, it's an exercise to write this out carefully. In general, since working through this argument (at either categorical level) is a useful exercise, it's not written down explicitly: if you work through this stuff, feel free to add it to the notes.

That said, associativity is going to be a chore no matter how you go about it. ◅

Last time in Example 4.3.3, we defined three bispans R_f, N_f, and T_f associated with a map $f : S \rightarrow T$.

Lemma 4.3.12. *$[X \xleftarrow{f} A \xrightarrow{g} B \xrightarrow{h} Z]$ is isomorphic to the composition $T_h \circ N_g \circ R_f$.*

PARTIAL PROOF. Recall $R_f = [X \xleftarrow{f} S \xrightarrow{\mathrm{id}} S \xrightarrow{\mathrm{id}} S]$ and $N_g = [S \xleftarrow{\mathrm{id}} S \xrightarrow{g} T \xrightarrow{\mathrm{id}} T]$. We'll show their composition is

$$X \xleftarrow{f} S \xrightarrow{g} T \xrightarrow{\mathrm{id}} T. \tag{4.3.13}$$

Pullbacks by the identity are the identity, so we can start with

$$\begin{tikzcd}
X & S \arrow[l, "f"'] \arrow[d, "\mathrm{id}"] & S \arrow[l, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"] \\
 & S \arrow[d, "\mathrm{id}"] & S \arrow[l, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"] \\
 & S & S \arrow[l, "\mathrm{id}"'] \arrow[d, "g"] \\
 & T & T. \arrow[l, "\mathrm{id}"']
\end{tikzcd}$$

We claim $\Pi_g S \cong T$, which we'll check momentarily; under this assumption, $S \times_T \Pi_g S \cong S$, so the full diagram is

$$\begin{tikzcd}
X & S \arrow[l, "f"'] \arrow[d, "\mathrm{id}"] & S \arrow[l, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"] & S \arrow[l, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"'] \\
 & S \arrow[d, "\mathrm{id}"] & S \arrow[l, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"] & S \times_T \Pi_g S \cong S \arrow[l, "\mathrm{id}"'] \arrow[dd, "g"] \\
 & S & S \arrow[l, "\mathrm{id}"'] \arrow[d, "g"] & \\
 & T & T \arrow[l, "\mathrm{id}"'] & \Pi_g S \cong T, \arrow[l, "\mathrm{id}"']
\end{tikzcd}$$

and the composition is (4.3.13), as desired.

Now let's prove the claim. Given an $f : X \to Y$ and an $A \to X$, $\Pi_f A$ fits into a pullback

(4.3.14)
$$\begin{tikzcd}
\Pi_f A \arrow[r] \arrow[d] & \mathrm{Map}_T(X,X) \arrow[d] \\
Y \arrow[r] & \mathrm{Map}_T(A,X).
\end{tikzcd}$$

From (4.3.14), $\Pi_g S$ is the space of functions $s\colon g^{-1}(t) \to S$ such that $\mathrm{id} \circ s(x) = x$, and this space can be identified with T.

The second step of the proof, composing with T_h, is analogous. ⊠

Proposition 4.3.15.

(1) $N_g \circ N_{g'} = N_{gg'}$.
(2) $T_h \circ T_{h'} = T_{hh'}$.
(3) $R_f \circ R_{f'} = R_{f'f}$.

Proof of part (2). Explicitly, we want to compose $X \overset{\mathrm{id}}{\leftarrow} X \overset{g'}{\rightarrow} Y \overset{\mathrm{id}}{\rightarrow} Y$ and $X \overset{\mathrm{id}}{\leftarrow} X \overset{g}{\rightarrow} Y \overset{\mathrm{id}}{\rightarrow} Y$. We can fill in the pullbacks on the left immediately:

$$\begin{tikzcd}
X & X \arrow[l, "\mathrm{id}"'] \arrow[d, "g'"] & X \arrow[l, "\mathrm{id}"'] \arrow[d, "g'"] \\
& Y \arrow[d, "\mathrm{id}"] & Y \arrow[l, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"] \\
& Y & Y \arrow[l, "\mathrm{id}"'] \arrow[d, "g"] \\
& Z & Z. \arrow[l, "\mathrm{id}"']
\end{tikzcd}$$

To form the exponential diagram, observe that by the same argument as in the proof of Lemma 4.3.12, $\Pi_g Y \cong Z$, and therefore $Y \times_Z \Pi_g Y \cong Y$. Thus, the finished diagram is

$$\begin{tikzcd}
X & X \arrow[l, "\mathrm{id}"'] \arrow[d, "g'"] & X \arrow[l, "\mathrm{id}"'] \arrow[d, "g'"] & X \arrow[l, "\mathrm{id}"'] \arrow[d, "g'"] \\
& Y \arrow[d, "\mathrm{id}"] & Y \arrow[l, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"] & Y \arrow[l, "\mathrm{id}"'] \arrow[dd, "g"] \\
& Y & Y \arrow[l, "\mathrm{id}"'] \arrow[d, "g"] & \\
& Z & Z \arrow[l, "\mathrm{id}"'] & Z, \arrow[l, "\mathrm{id}"']
\end{tikzcd}$$

so the composition really is $T_{hh'}$. ⊠

Again, it's instructive to fill in the other two parts, which have similar proofs.

The next proposition is harder.

Proposition 4.3.16. *Given a pullback diagram*

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"] & Y,
\end{tikzcd}$$

then

(1) $R_f \circ N_g = N_{g'} \circ R_{f'}$, *and*
(2) $R_f \circ T_g = T_{g'} \circ R_{f'}$.

Proposition 4.3.17. *Given an exponential diagram*

$$\begin{tikzcd}
X \arrow[d, "g"] & A \arrow[l, "h"'] & X \times_Y \Pi_g A \arrow[l, "f'"'] \arrow[d, "g'"] \\
Y & & \Pi_g A, \arrow[ll, "h'"']
\end{tikzcd}$$

$N_g \circ T_h = T_{h'} \circ N_{g'} \circ R_{f'}$.

With all of these in place, we get what we're looking for.

Corollary 4.3.18. *The category $P(\mathsf{C})$ whose objects are objects of C and morphisms are isomorphism classes of bispans really is a category, in that composition of bispans satisfies the necessary axioms.*

TODO: I missed the thing right after this.

For a reference on this stuff, check out Tambara's original paper [**Tam93**]. This is really recent, from the early 1990s, and this business about using bispans to encode right-way and wrong-way maps is not as well explored as they could be.

REMARK. This category of bispans is reminiscent of the Dwyer-Kan hammock localization [**DK80**], and it may be possible to obtain bispans from the hammock localization of a particular category with weak equivalences. Is this useful? Maybe. ◁

If $\mathsf{C} = G\mathsf{Set}$, then we'll let $\tilde{P}^G := P(\mathsf{C})$. The hom sets of $\tilde{P}^G$ are commutative monoids under direct sum. Taking the Grothendieck group of each hom-set (as we did with Mackey functors), we obtain a category P^G, sometimes called the **bispan category**.

Definition 4.3.19. A **Tambara functor** is an additive functor $P^G \to \mathsf{Ab}$.

REMARK.

(1) Just as you can think of G-spectra as spectrally enriched Mackey functors, you can think of G-ring spectra as spectrally enriched Tambara functors. This is harder to make precise, but is still a useful guiding analogy. For connective spectra, this is discussed in [**Hoy14**].
(2) The algebra of Tambara functors is still under construction. Nakaoka has a few recent papers [**Nak12a**, **Nak12b**, **Nak14**] about, e.g. ideals and modules for Tambara functors. This is closely related to the project of equivariant derived algebraic geometry, which would relate to the algebraic geometry of Tambara functors. The questions are interesting and complicated — how do you localize? What happens when you do? There are probably algebraic questions in this area that aren't too difficult to pose and answer, simply because there aren't that many people looking.
(3) Just as we described Mackey functors as a pair of a covariant and a contravariant functor satisfying a push-pull axiom, there's a similar (more complicated) definition for Tambara functors, with three functors satisfying some interoperability conditions. We won't need this for the time being, so we're not going to write it out.
(4) If you forget the norm map, a Tambara functor defines a Green functor. A Green functor may extend to a Tambara functor, but such an extension need not exist or be unique [**Maz13**]. ◁

Example 4.3.20. Let R be a ring with a G-action. Then, $\mathrm{Map}_G(-, R)$ is a Tambara functor. In this case, given a map $f : X \to Y$, the descriptions of f^*, Σ_f, and Π_f are pretty concrete:

$$\begin{aligned}
f^*(\theta)(x) &= \theta(f(x)) \\
\Sigma_f(\theta)(x) &= \sum_{x \in f^{-1}(y)} \theta(x) \\
\Pi_f(\theta)(x) &= \prod_{x \in f^{-1}(y)} \theta(x).
\end{aligned}$$

◁

Example 4.3.21 (Burnside Tambara functor)**.** The Burnside ring continues to keep giving. Let $B_G(X)$ denote the category of G-sets over X. Given a map $f : X \to Y$ of sets, $f^*(B \to Y)$ is $X \times_Y B$, $\Sigma_f(A \to X)$ is $A \to X \to Y$, and $\Pi_f(A \to X) = \Pi_f A \to X$. ◂

Example 4.3.22.

(1) Group cohomology is a Tambara functor: the transfer and restriction maps are what we've seen before, and the norm is the **Evens multiplicative transfer**, which we'll talk about in §4.4.
(2) Let X be a commutative ring object in Sp^G. Then, $\pi_0(X)$ is a Tambara functor. The idea is that it'll be an algebra over an N_∞ operad over a "complete" indexing system (again, this will be elaborated on).
(3) Let $\mathsf{D} \subseteq G\mathsf{Set}$ be a wide subcategory, meaning it has all the objects (but perhaps not all the morphisms). Then, let P^G_{D} denote the category of bispans $X \xleftarrow{f} A \xrightarrow{g} B \xrightarrow{h} Y$ where g is in D. We'd like P^G_{D} to be a category, and there's a criterion that's not too hard to check. ◂

Proposition 4.3.23. *P^G_{D} is a category if D is* ***stable under pullback****, i.e. for any pullback diagram*

$$\begin{tikzcd}
Q \arrow[r] \arrow[d, "f"] & Z \arrow[d, "g"] \\
X \arrow[r] & Y,
\end{tikzcd}$$

if $g \in \mathsf{D}$, then $f \in \mathsf{D}$.

One also says that D is **pullback stable** or **closed under base change**.

Proof sketch. The argument amounts to showing that $(T_h N_g R_f)(T_{h'} N_{g'} R_{f'})$ can be put in some normal form where the N piece is obtained from something in D. We mostly only have to worry about interchanging N with things: since D is a category, $N_g N_{g'} = N_{gg'}$ keeps us inside P^G_{D}.

We use the pullback stability to address the exponential diagram

$$\begin{tikzcd}
X \arrow[d, "f"] & A \arrow[l] & X \times_Y \Pi_f A \arrow[l] \arrow[d, "\varphi"] \\
Y & & \Pi_f A. \arrow[ll]
\end{tikzcd}$$

This is a pullback diagram, so if $f \in \mathsf{D}$, then so is φ. ⊠

Since G-sets are sums of orbits, we want to be able to work with sums of orbits as we've done before. Let $\mathsf{Orb}_{\mathsf{D}}$ be the full subcategory of D on the orbits $\{G/H\}$.

Definition 4.3.24. We say that D is **coproduct complete** if coproducts in D are the same as those in C, and D is closed under coproducts.

Lemma 4.3.25. *If D is coproduct complete, then D is the coproduct completion of $\mathsf{Orb}_{\mathsf{D}}$.*

That is, we start with orbits, throw in coproduct, and obtain everything, which is nice.

Definition 4.3.26. Let D be a wide, pullback stable, coproduct complete subcategory of C. Then, a D**-Tambara functor** is a functor $P^G_{\mathsf{D}} \to \mathsf{Ab}$. In [BH16], these are called **incomplete Tambara functors**.

We won't be able to delve into the following proof.

Theorem 4.3.27 ([BH16])**.** *There is an equivalence of categories between the poset of wide, pullback stable, coproduct complete subcategories of $G\mathsf{Set}$ and the category of indexing systems for G.*

This tells us that all of the conditions imposed on indexing systems are inevitably imposed on us if we want equivariant ring spectra.

Looking forward, we'll discuss the Evens norm and π_0 of ring spectra as promised in Example 4.3.22, then proceed to the norm as defined in [HHR16]: what its structure is, what you do with it, and why it was useful in their argument. Once we've finished with that, we've almost arrived at the frontier of equivariant stable homotopy theory.

To the degree that you can fill in the details, this is a large subset of what people know about equivariant stable homotopy theory. Fifteen years ago, very few computations were known, and the computations were hard.

But the development of the slice filtration and slice spectral sequence is a recent innovation, used to attack the Kervaire invariant one problem, and has led to lots of computations. We won't discuss this in detail, but it will probably lead to further useful and interesting computations.

4.4. The Evens norm

> *"Speaking as someone who spent three years thinking about this as a graduate student, this is hard."*

In this section, we'll cover the Evens norm in group cohomology and tensor induction. All of this is about motivating the norm structure on Tambara functors, an extra covariant map N_H^G. We're going to explain where this is coming from in the context of topology and/or group cohomology. We've seen one approach already, using the dependent sum and dependent product, but that was kind of abstract, and it's always useful to have more than one approach.

Even if you don't care about equivariant ring spectra, the Evens norm is a cool construction, less well known than it should be, and is a useful tool for proving things. It was introduced by Evens in [CITE ME: Evens] and also written about in [CITE ME: Greenlees-May].

By H we'll mean a finite-index subgroup of G. Of course, if G is finite, any subgroup will do.

Let's recall first the definition of the transfer map $H^*(H;M) \to H^*(G;M)$. Here, the H-action on the $\mathbb{Z}[G]$-module M is through restriction i_H^*. We can also write $H^*(H;M) \cong \operatorname{Ext}_{\mathbb{Z}[H]}(\mathbb{Z}, M)$, the derived functors of M^H.

The right adjoint of restriction is $\operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[H], –)$, but the Wirthmüller isomorphism, in this group-theoretic context, says this is the same as the left adjoint $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} –$. Thus, we get a map

$$H^*(H; i^*M) \cong H^*(G; \operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[H], M)) \cong H^*(G, \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} i_H^* M) \longrightarrow H^*(G;M),$$

where the map is induced by the counit $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} i_H^* M \to M$.

Another way to describe the transfer is through the map $M^H \to M^G$ sending

$$x \longmapsto \sum_{g \in G/H} gx. \tag{4.4.1}$$

Then, the transfer is the induced map between derived functors. It's not hard to show these are the same: since both of them are effaceable δ-functors, it suffices to check on degree 0, where it's straightforward. The idea is that information about derived functors can be obtained from their degree-0 terms.

In (4.4.1), we added. But if M is a $\mathbb{Z}[G]$-algebra, then we should also be able to multiply. This will give us a variant description of

$$\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M \cong \bigoplus_{g \in G/H} g \otimes M,$$

where $g \otimes M$ denotes M with the action twisted by g.

We want to describe this more explicitly, and will do so with a wreath product.

Definition 4.4.2. The **wreath product** of Σ_n and H is the group

$$\Sigma_n \wr H := \{(\sigma, h_1, \dots, h_n) \mid \sigma \in \Sigma_n, h_i \in H\}$$

with the multiplication

$$(\sigma, h_1, \dots, h_n)(\tau, \overline{h}_1, \dots, \overline{h}_n) = (\sigma\tau, h_{\tau(1)}\overline{h}_1, h_{\tau(2)}\overline{h}_2, \dots, h_{\tau(n)}\overline{h}_n).$$

This is a kind of semidirect product. If $n = [G : H]$, there's a natural homomorphism $\theta\colon G \to \Sigma_n \wr H$, because G acts by permuting the elements of H. Namely, if $g_1, \dots, g_n$ is a set of coset representatives for G/H, let $\pi_g \in \Sigma_n$ and $h_i(g) \in H$ be such that $gg_i = g_{\pi(i)} h_i(g)$. Then, θ sends $g \mapsto (\pi, h_1(g), \dots, h_n(g))$.

Proposition 4.4.3.

1. *θ is a group homomorphism.*
2. *A different choice of coset representatives determines a map conjugate to θ.*
3. *θ induces a natural G-action on $\bigoplus_{i=1}^n M$, which is naturally a $\Sigma_n \wr H$-object.*
4. *The $\mathbb{Z}[G]$-module $\bigoplus_{i=1}^n M$ induced by θ is precisely $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M$.*

Partial proof. For (1),

$$\begin{aligned}(\gamma_1\gamma_2)g &= \gamma_1(\gamma_2 g_i)\\ &= \gamma_1(g_{\pi_1(i)}h_i(\gamma_2))\\ &= g_{\pi_2(\pi_1(i))}h_{\pi_1(i)}(\gamma_1)h_i(\gamma_2).\end{aligned}$$

This plays well with the group structure on $\Sigma_n \wr H$, and implies that θ is a homomorphism. (2) follows from a similar calculation.

The action of $\Sigma_n \wr H$ on $\bigoplus_{i=1}^n M$ is

$$(\sigma, h_1, \dots, h_n)\cdot(m_1,\dots,m_n) = (h_{\sigma^{-1}(1)}m_{\sigma^{-1}(1)}, h_{\sigma^{-1}(2)}m_{\sigma^{-1}(2)}, \dots, h_{\sigma^{-1}(n)}m_{\sigma^{-1}(n)}). \qquad \boxtimes$$

The key observation is that you can do this in any symmetric monoidal category: we haven't used anything specific about the symmetric monoidal structure, just the fact that there's an Σ_n-action on an n-fold monoidal product.

Definition 4.4.4. Let M be an object in a symmetric monoidal category C. Its **tensor induction** is the G-object

$$N_H^G M := \theta^*\left(\bigotimes_{i=1}^n M\right),$$

where we give the iterated tensor product the $\Sigma_n \wr H$-action described above.

There are a lot of interesting properties of tensor induction, which you can work out.

Proposition 4.4.5.

(1) *Let $H_2 \subseteq H_1 \subseteq G$. Then, $N_{H_1}^G N_{H_2}^{H_1} \cong N_{H_2}^G$.*

(2) *$N_H^G(M_1 \oplus M_2)$ is isomorphic to $N_H^G M_1 \oplus N_H^G M_2$ plus some norms to K, where K is a proper subgroup containing $\bigcap {}^g H$.*

The Evens norm is a kind of multiplicative transfer. Segal [CITE ME: Segal] made this literal, writing down an equivariant cohomology theory whose transfer map is the Evens norm.

Theorem 4.4.6 (Evens). *If R is a Noetherian ring, then $H^*(G;R)$ is finitely generated over R.*

Evens wrote a book [**Eve82**] on group cohomology which adopts the Evens norm as an organizing principle, and this theorem is probably proved in there.

Definition 4.4.7. The **Evens norm** is a map $H^p(H;R) \to H^{p^n}(G;R)$ (where R is a $\mathbb{Z}[G]$-algebra) which sends $\alpha \in H^p(H;R)$ to $\alpha^{\otimes n}$ in $N_H^G R$ (with its $\Sigma_n \wr H$-action), then pulls back by $\theta^*(\alpha^{\otimes n})$ to define the Evens norm of α.

We will assume α is in even degree: otherwise, there are factors (-1) to the sign of a permutation. The trick is that instead of using the counit $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} i_H^* M \to M$, we use the twisted counit

$$\bigotimes_{i=1}^n R \longrightarrow R$$

coming from multiplication. The interesting piece is that this plays well with the G-action: it's really a map $N_H^G(i_H^* R) \to R$. In other words, it's easy to get a map $H^p(H;M) \to H^{p^n}(G;R^n)$, and we use the ring structure to get back to R. There's a nice analogy of the norm as the multiplicative transfer, where we've replaced one counit with another. From this perspective, the definition of an N_∞-operad makes more sense.

More explicitly, let $P_\bullet \to R$ be a projective resolution of R as a $\mathbb{Z}[H]$-module, so we obtain a resolution $P_\bullet^{\otimes n} \to R^{\otimes n} \to R$. Then, let $\varepsilon\colon Q_\bullet \to R$ be a resolution of R as a $\mathbb{Z}[\Sigma_n]$-object (induced from the $\Sigma_n \wr H$-action).[13] We can join these into a resolution

$$Q_\bullet \otimes_R P_\bullet^{\otimes n} \longrightarrow R \otimes_R R^{\otimes n} \longrightarrow R,$$

and given some cocycle $f : P_p \to R$, $\varepsilon \otimes f^n$ is a cocycle for H^{pn} whose cohomology class is the Evens norm of f. This is far from obvious, but comes from the fact that

$$(P_\bullet^{\otimes n})_{np} \longrightarrow P_p^{\otimes n} \longrightarrow R$$

---

[13] Don't we need to act on $R^{\oplus n}$, rather than R?

produces cocycles for $H^{np}(H;R)$.[14]

Remark.

(1) The Evens norm satisfies a double coset formula, similarly to the transfer map.
(2) The Evens norm is part of a Tambara functor structure where

$$G/H \longmapsto \bigoplus_{i=0} H^{2i}(H;-).$$

(3) Multiplicative transfers, e.g. power operations, are ubiquitous, e.g. the Steenrod operations. In fact, one can realize the Steenrod operations as a homotopical version of this norm construction, and this is a fruitful approach to research for some homotopy theorists. The analogue of the twisted action is a map $E\Sigma_n \times_{\Sigma_n} X^n \to X$, which is used to construct the Steenrod operations. ⋖

We'll soon bring this from group cohomology into equivariant stable homotopy theory, where it becomes the Hill-Hopkins-Ravenel norm.

4.5. Equivariant Γ-spaces

In this section, we'll take a historical detour and discuss the difference between Γ-G-spaces and Γ_G-spaces. This arises from a reasonable question: what's the analogue of a Γ-space in the equivariant context?

Recall that Γ is the category of based finite sets and based maps, and the category of Γ-spaces is the functor category $\mathsf{Fun}(\Gamma, \mathsf{Top}_*)$, as we discussed in Example 2.3.6. A special Γ-space X is the analogue of an abelian group: we ask that the induced map

$$X(n) \longrightarrow \prod_{i=1}^{n} X(1)$$

is an equivalence.

In the equivariant setting, there are two things we can do:

- **Γ-G-spaces** are Γ-objects in $G\mathsf{Top}_*$. These were introduced by Segal [**Seg78**].
- Let Γ_G be the category of finite based G-sets and all based maps, so it's G-enriched, where G acts by conjugation on the mapping space. Then, **Γ_G-spaces** are the enriched functors $\Gamma_G \to G\mathsf{Top}_*$. These were introduced by Shimakawa [**Shi89**].

Which of these is right? Turns out both are.

Theorem 4.5.1 (Shimakawa [**Shi91**]). *These two categories are equivalent.*

We can also line up the special conditions in a nice way. The proof uses that one weird trick in diagram spectra. The manifestation in orthogonal spectra means that the category of orthogonal spectra on the trivial universe is equivalent (on the point-set level) to orthogonal spectra on the complete universe! And in fact this is true for any universe.

The reason this happens is that diagram spectra mix the point-set and homotopical data in a way that can be confusing. Of course, the homotopy groups depend on the universe, so the homotopy theory and homotopy category are genuinely different.

Remark. You might be thinking that this is an artifact of one's desire for concreteness and model categories, but the ∞-categorical view isn't necessarily easier: yes, you don't have to worry as much about point-set phenomena, and all functors are derived, but it's a lot harder to infer the homotopy theory from first principles and you can write stuff down, but it's not at all clear how to prove it's the right thing.

To be sure, some of the difficulties in writing down the correct norm map were point-set in nature; but not all of them were. ⋖

Since finite based sets are finite based G-sets (with G acting trivially), there is an inclusion $\Gamma \to \Gamma_G$, which induces a functor

$$R\colon \mathsf{Fun}(\Gamma_G, G\mathsf{Top}_*) \longrightarrow \mathsf{Fun}(\Gamma, G\mathsf{Top}_*).$$

It has a left adjoint

$$L\colon \mathsf{Fun}(\Gamma, G\mathsf{Top}_*) \longrightarrow \mathsf{Fun}(\Gamma_G, G\mathsf{Top}_*),$$

---

[14] TODO: I didn't follow this.

which is a Kan extension, explicitly given by the coend

$$LX = \int^{\Gamma} \operatorname{Hom}(n,\text{–}) \times X(\text{–}) = \coprod_{n\ge 0} \operatorname{Map}(n,S)\times X(n)/\sim,$$

where $(sf,x)\sim(s,X(f)x)$ for any Γ-morphism $f: n\to m$. The G-action is $g[s,x]=[gs,gx]$.

Now, L is an equivariant functor: if $f: S\to T$ is a Γ_G-morphism,

$$\begin{aligned} LX(gfg^{-1})[s,x] &= [gfg^{-1}s,x] \\ &= g[fg^{-1}s,g^{-1}x] \\ &= gLX(f)[g^{-1}s,g^{-1}x]. \end{aligned}$$

Exercise 4.5.2. Check that L and R are adjoint. This follows from essentially formal reasons.

Proposition 4.5.3. (L,R) *are adjoint equivalences.*

Proof. We need to check that the natural transformations $\mathrm{id}\to RL$ and $LR\to\mathrm{id}$ are natural isomorphisms. The first is formal: we're Kan extending along a full subcategory, and in this case $\mathrm{id}\to RL$ is always an equivalence.

Now, we want to recover $X(S)$ from $X(n)$, where $|S|=n$.[15] Choose a bijection $f: S\to n$. The key is that this defines a map $\rho: G\to \operatorname{Hom}(n,n)\cong\Sigma_n$, through the G-action on S. But now this looks familiar from the story of G-operads. Why were we able to get away with indexing them on finite sets rather than on finite G-sets? Well, from the ∞-categorical perspective, maybe it would be better, but we were able to get away with using just finite sets by the same trick!

So now we have a commutative diagram

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "g"] & n \arrow[d, "\rho(g)"] \\ S \arrow[r, "f"'] & n. \end{tikzcd}$$

Endow $X(n)$ with a new G-action[16] via the map $G\times X(n)\to X(n)$ sending

$$g,x \longmapsto X(\rho(g))(gx).$$

We'll call this G-space $X(n)_\rho$. This is another manifestation of a common principle from this class: just like finite sums and finite products are the same in the stable setting, in the equivariant stable setting, finite G-indexed sums and finite G-indexed products are the same. Since $X(n)$ is a product of n copies of $X(1)$, this is a G-indexed product.

Now, we claim that as G-spaces, $X(S)\cong X(n)_\rho$.

Exercise 4.5.4. Prove this. There's an obvious map, and you just have to check that it's G-equivariant, like we did for L above.

Assuming this exercise, we get that $LR\to\mathrm{id}$ is a natural isomorphism. ⊠

Now, what does it mean to be special in the equivariant setting?

Definition 4.5.5.

- A **special Γ_G-space** is one for which the natural map

$$X(S)\xrightarrow{\cong}\prod_S X(1)$$

is an equivalence.

---

[15] We're using n to denote both the set $\{1,\dots,n\}$ and its cardinality; hopefully they can be distinguished from context.

[16] $X(n)$ was already a G-space, but we're putting a different G-action on it.

- A **special Γ-G-space** is one for which the natural map
$$X(S) \xrightarrow{\cong} \prod_{i=1}^{n} X(1)$$
is an equivalence[17] — not just of G-spaces, but for the family of subsets $H \subset G \times \Sigma_n$ such that $H \cap \{(e,\sigma)\} = \varnothing$.[18]

The definition for Γ-G-spaces again is reminiscent to the operadic story: we're asking for homomorphisms.

Exercise 4.5.6. Show that under the equivalence of Proposition 4.5.3, these two notions of special objects are identified.

REMARK.

(1) You can run the rest of Segal's story: there's a notion of a very special Γ-G-space or Γ_G-space, which is a condition on fixed points, and using similar machinery as before, you can turn very special objects into connective genuine G-spectra.
(2) We didn't discuss the homotopy theory of these two categories: the natural notions of fibrancy for Γ_G-spaces and G-Γ-spaces are different. This is analogous to the difference between the naïve homotopy groups of G-spectra on the trivial universe, and the more sophisticated homotopy groups that you could define on them, and which agree with the homotopy groups of G-spectra on the complete universe. This is really saying something about Kan extensions and diagram spaces. ⊲

4.6. The HHR norm

"Do I dare disturb the universe?"

In this section, we discuss the construction and some properties of the HHR norm $N_H^G \colon \mathsf{Sp}^H \to \mathsf{Sp}^G$ and $N_H^G \colon \mathsf{Sp}^H[\mathbb{P}] \to \mathsf{Sp}^G[\mathbb{P}]$. This is the left adjoint to i_H^* (for commutative algebras), and is the basis for multiplicative induction in the spectral setting. Thus it specializes to the norm in algebra.

We know what we want to do, but the technicalities are tricky.

"Definition" 4.6.1. The **norm map** $N_H^G \colon \mathsf{Sp}^H \to \mathsf{Sp}^G$ sends
$$M \longmapsto \theta^*\left(\bigwedge_{g\in G/H} M\right),$$
which is the indexed smash product, in direct analogue with tensor induction.

The reason we use the smash product instead of the wedge sum is that spectra are already abelian groups. The additive analogue of this construction is the Wirthmüller isomorphism.

What makes this construction tricky is figuring out how to give $\bigwedge_{g\in G/H} M$ a $\Sigma_n \wr H$-action. The issue is the universe.

Example 4.6.2. Let $X \in \mathsf{Sp}$, and let's consider $N_e^{C_2}X$. If U is the universe we started with, $X \wedge X$ is in $U \oplus U$, and the smash product is a Kan extension along $U \oplus U \to U$.

If X is a space, C_2 acts on $X \times X$ by switching the pairs. And we know the diagonal map $x \mapsto (x,x)$ defines an isomorphism $X \xrightarrow{\cong} (N_e^{C_2}X)^{C_2}$. But if you have spectra instead, and the two universes are different, i.e. $U_1 \oplus U_2$ instead of $U \oplus U$, which makes it hard to resolve. ⊲

The solution is a trick, due to unpublished work of Lydakis and elucidated by [HHR16]. Consider X in the trivial H-universe, i.e. X is an H-object in Sp. Then there's no problem: $N_e^{C_2}X$ is a C_2-object in Sp, or Sp^{C_2} with the trivial universe. If we're not keeping track of the equivariant universe, it's easy to make sense of the map $N_H^G \colon H\mathsf{Sp} \to G\mathsf{Sp}$.

The trick is the following point, which is extremely subtle: we can define $N_G^H \colon \mathsf{Sp}^H \to \mathsf{Sp}^G$ by changing the universe from Sp^H to $H\mathsf{Sp}$ (i.e. the trivial universe), then using the tensor induction to get to $G\mathsf{Sp}$ (i.e. the trivial universe), then changing universe again to get back to Sp^G. As a point-set construction, well, sure, this is a functor.

---

[17] TODO: I might have gotten this criterion wrong.

[18] This is what Segal missed.

But the crazy thing is that this works: you can make this a homotopical functor such that its derived functor is the right thing.

Recall that $\mathsf{Sp}^G[\mathbb{P}]$ denotes the category of commutative ring objects in G-spectra (sometimes also written Comm_H). We'll show that $N_H^G\colon \mathsf{Sp}^H[\mathbb{P}] \to \mathsf{Sp}^G[\mathbb{P}]$ is left adjoint to i_H^*, and hence, if R is a commutative ring G-spectrum, we have a counit $N_H^G i_H^* R \to R$ as we did in group cohomology.

The principal player in [**HHR16**] is obtained from this norm and a concrete C_2-spectrum: in Definition 4.7.5, we'll define real bordism MR, the cobordism theory of stably almost complex manifolds with a C_2-action by complex conjugation (there are some nuances to writing this down). Then, they consider the C_8-theory $N_{C_2}^{C_8} MR$, and do a lot of hard work to show this detects the Kervaire invariant in the stable homotopy groups of spheres. People are still working out the applications of this spectrum and this idea more generally, and you could count yourself among their numbers.

Now we'll construct the norm map. This will involve some work with orthogonal spectra, so let's recall how orthogonal G-spectra are defined, especially since we covered it rather quickly.[19]

Recall that if V and V' are real G-representations on inner product spaces, we defined $I_G(V,V')$ to be the G-space of linear isometries $V \to V'$. We defined $J_G(V,V')$ to be the Thom space for $E \subseteq I_G(V,V') \times V' \to I_G(V,V')$, where E is the **orthogonal complement**

$$E := \{(f,x) \mid x \in V - fV\}.$$

When $V \subseteq V'$, you can write this explicitly as

$$J_G(V,V') = \mathrm{O}(V')_+ \wedge_{\mathrm{O}(V'-V)} S^{V'-V}.$$

Now, let J_G be the category whose objects are G-representations on finite-dimensional real inner product spaces and whose morphism spaces are $J_G(V,V')$ as we defined them. In particular, there's a composition law $J_G(V',V'') \times J_G(V,V') \to J_G(V,V'')$ and a symmetric monodial product

$$(f,x) + (f',x') = (f+f', x+x')$$
$$(g,y)(f,x) = (gf, gx+y).$$

The unit is $(\mathrm{id}, 0)$.

Now, we were able to define $\mathsf{Sp}^G = \mathsf{Fun}(J_G, G\mathsf{Top}_*)$, and we saw this has a symmetric monoidal structure.

One important point is that the Yoneda lemma implies

$$\bigvee_{V,W} S^{-W} \wedge J_G(V,W) \wedge X(V) \rightrightarrows \bigvee_{V} S^{-V} \wedge X(V) \longrightarrow X \tag{4.6.3}$$

is a coequalizer of X in terms of representable objects. This is true in general for presheaves, and in this context, [**HHR16**] call it the **tautological presentation** of X.

Now we'll introduce the point-set change-of-universe functor $\mathsf{Sp}^G[U'] \to \mathsf{Sp}^G[U]$, where U and U' are G-universes. Let

$$I_{U'}^U X(V) := J_G(\mathbb{R}^n, V) \wedge_{\mathrm{O}(n)} X(\mathbb{R}^n),$$

where $\dim V = n$. If W is a p-dimensional G-representation, the action of $J_G(V,W)$ on this comes through the diagram

$$\begin{tikzcd}
J_G(V,W) \wedge J_G(\mathbb{R}^n,V) \wedge_{\mathrm{O}(n)} X(\mathbb{R}^n) \arrow[d] & & \\
J_G(\mathbb{R}^n,W) \wedge_{\mathrm{O}(n)} X(\mathbb{R}^n) \arrow[r] & J_G(\mathbb{R}^p,W) \wedge_{\mathrm{O}(p)} J_G'(\mathbb{R}^n,\mathbb{R}^p) \wedge_{\mathrm{O}(n)} X(\mathbb{R}^n) \arrow[r] & J_G(\mathbb{R}^p,W) \wedge_{\mathrm{O}(p)} X(\mathbb{R}^p).
\end{tikzcd}$$

Exercise 4.6.4. Show that $I_{U'}^U I_U^{U'} = \mathrm{id}$. This is not hard. Also, check that $I_{U'}^U$ is strong symmetric monoidal.

Thus, for any U and U', $\mathsf{Sp}^G[U]$ and $\mathsf{Sp}^G[U']$ are equivalent! This is true for the same reason as Proposition 4.5.3.

REMARK. It's also quick to check that $I_U^{\mathbb{R}^\infty}$ is the forgetful functor. ◀

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[19] TODO: there may be notational inconsistencies with the first time orthogonal spectra were covered, which I should fix at some point.

Let U be a complete H-universe and $\widetilde{U}$ be a complete G-universe. Then, we'll define $N_H^G\colon \mathsf{Sp}_U^H \to \mathsf{Sp}_{\widetilde{U}}^G$ by

$$X \longmapsto I_{\mathbb{R}^\infty}^{\widetilde{U}}\,\theta^*\Bigl(\bigwedge_{G/H} X\Bigr) I_U^{\mathbb{R}^\infty} X.$$

Here θ is the wreath action we introduced in §4.4. This functor is clearly strong symmetric monoidal, and turns out to be left-adjoint to $i_H^*\colon \mathsf{Sp}_{\widetilde{U}}^G[\mathbb{P}] \to \mathsf{Sp}_U^H[\mathbb{P}]$, i.e. on the categories of commutative algebras.

The miracle is that this can be derived in the usual model structure by cofibrant replacement, so it's homotopically the right thing. We won't prove this; it's one of the technically hardest parts of [**HHR16**]: it's not a left adjoint on modules, so it doesn't commute with colimits, and one has to carefully analyze what it does to pushouts.

There are a few key facts:

- N_H^G is symmetric monoidal.
- $N_H^G S^{-V} = S^{-\mathrm{Ind}_H^G V}$.
- There's a diagonal map $\Phi^H X \to \Phi^G N_H^G X$, and it's an equivalence. In fact, when X is cofibrant, this is a point-set equivalence!

We'll begin showing this by describing Φ^G, which is a strong symmetric monoidal functor such that if Z is a space, $\Phi^G(S^{-V}\wedge Z) \cong S^{-V^G}\wedge Z^G$. In the ∞-categorical sense, it commutes with colimits, but this is not true in the point-set case; in any case, mirroring the tautological presentation (4.6.3), $\Phi^G X$ is the coequalizer

$$\bigvee_{V,W} S^{-W^G}\wedge J_G(V,W)^G \wedge X(V)^G \rightrightarrows \bigvee_{V} S^{-V^G}\wedge X(V)^G \longrightarrow \Phi^G X.$$

It's not too hard to check this is the same as the model described in [CITE ME: Mandell-May]. We'll use this description to construct the diagonal map. Using the fact that $Z^H \cong (N_H^G Z)^G$,

$$\Phi^G N_H^G S^{-V} \cong \Phi^G\Bigl(S^{-\mathrm{Ind}_H^G V}\Bigr) \cong S^{-V^H} \cong \Phi^H S^{-V}.$$

Therefore we have an isomorphism

$$\Phi^H(S^{-V}\wedge Z) \xrightarrow{\cong} \Phi^G N_H^G(S^{-V}\wedge Z).$$

Applying this termwise to the tautological construction, we obtain a map

$$\bigvee_{V,W} S^{-W^H}\wedge I_H(V,W)^G \wedge X(V)^H \rightrightarrows \bigvee_{V} S^{-V^G}H\wedge X(V)^H \longrightarrow \Phi^G N_H^G X,$$

hence by the universal property of the coequalizer, a map $\Phi^H G \to \Phi^G N_H^G X$.

4.7. Consequences of the construction of the norm map

"The genuine equivariant model is always the most painful of the two options."

In this section, we apply the norm map from the previous section to learn why π_0 of an E_∞-ring spectrum is a Tambara functor and how the norm construction interacts with the resolution of the Kervaire invariant 1 problem.

Over the summer, there will be more work on the notes: if you did any exercises while working this material out, feel free to include them, even if they're very rough. If anything is unclear or not useful or you have any other comments, feel free to leave them in the notes.

In this section, R will be a commutative ring in orthogonal G-spectra, i.e. an object of $\mathsf{Sp}^G[\mathbb{P}]$. The forgetful functor from G-ring spectra to H-ring spectra is lax symmetric monoidal and has a left adjoint, which is the HHR norm; it was known that this left adjoint existed for a long time, and that it wasn't the same adjoint as for G-spectra without a ring structure, but it wasn't understood how to think of it until recently.

One consequence is the existence of the counit $\eta\colon N_H^G i_H^* R \to R$. If $X \in \mathsf{Sp}^H$, then $R_H^0(X) = [X, i_H^* R]_H$, so the norm provides us with a map $R_H^0(X) \to R_G^0(N_G^H X)$: given a map $f : X \to i_H^* R$, norm it and apply the counit: $\varepsilon \circ N_H^G(f)\colon N_H^G X \to N_H^G i_H^* R \to R$. This and a few similar constructions are reminiscent of the Evens norm: if V is an H-representation and $X = S^V$, this is a map $R_H^0(S^V) \to R_G^0(N_H^G S^V) = R_G^0(S^{\mathrm{Ind}_H^G V})$, and these are homotopy groups: $R_H^0(S^V) = [S^V, i_H^* R] = \pi_V(i_H^* R)$, so we obtain a map $\pi_V(i_H^* R) \to \pi_{\mathrm{Ind}_H^G V}(R)$.

For example, if A is a G-space, given a map $f : \Sigma^\infty A \to i_H^* R$, we can construct a map $R_H^0(i_H^* A) \to R_G^0(A)$ sending $\Sigma^\infty A \to \Sigma^\infty N_H^G A$, which is homeomorphic to $N_H^G \Sigma^\infty A$ (here this is the norm map on spaces); then, using f, map to $N_H^G i_H^* R$ then to R by the counit.

Exercise 4.7.1. This is nearly saying that $\pi_0 R = R^0_{(-)}$ is a Tambara functor; check that this is really the case.

There's lots of things to check: figuring out why the double coset formula holds is work, though not too conceptually hard if you've followed everything so far. The original papers are complicated, because this check is inevitably notation-heavy.

On the other hand, there's a converse:

Theorem 4.7.2 (Ullman [Ull12])**.** *All Tambara functors arise as $\pi_0 R$ for some ring spectrum R, e.g. an Eilenberg-Mac Lane spectrum. However, there is no Eilenberg-Mac Lane functor that is symmetric monoidal.*

REMARK. There's a cool connection between the structures of Tambara functors and Witt vectors. For example, [HM97] identify $\pi_0 THH$ and $\pi_0 THH^{C_{p^n}}$ with certain rings of Witt vectors. This fits into a bigger picture. ◅

N_∞-operads come back. We used them to specify which transfer maps you have, and here they will control which norm maps you get. There are some complicated questions here: distributivity of norm and transfer maps is intricate, and the interactions between two different N_∞-operads, neither of which is complete, is not written down.

The rough idea is that any commutative G-ring spectrum should be characterized by having certain norm maps $N_H^G i_H^* X \to X$.

Lemma 4.7.3. *Let T be an admissible H-set for an N_∞-operad O. Then,*

$$(G \times \Sigma_{|T|}/\Gamma_T)_+ \wedge_{\Sigma_{|T|}} X^{|T|} \cong G_+ \wedge_H \bigwedge_i N_{K_i}^H i_{K_i}^* X.$$

The idea is that we can write T as a coproduct of orbits H/K_i.

Using this lemma, since T is admissible, we get a map $G \times \Sigma_{|T|}/\Gamma_{|T|} \to O(|T|)$, and therefore if X is an O-algebra (i.e. is in $\mathsf{Sp}^G[O]$), this becomes a map

$$G_+ \wedge_H \bigwedge N_{K_i}^H i_{K_i}^* X \longrightarrow O(|T|) \wedge_{\Sigma_{|T|}} X^{|T|} \longrightarrow X,$$

so for X an N_∞-algebra over O, $N_H^G : \mathsf{Sp}^H[O] \rightleftarrows \mathsf{Sp}^G[O] : i_H^*$ is homotopical whenever G/H is admissible. In this case, $\pi_0 X$ is an incomplete Tambara functor: we asked for it to contain the norm maps for O, and we have to accept certain other ones.

If instead A is an N_∞-algebra in G-spaces, then $\Sigma_+^\infty A$ is an N_∞-algebra in Sp^G, and is in fact the spectral group ring (also denoted $\mathbb{S}[A]$). The multiplication comes from the isomorphism

$$\mathrm{O}(n) \times_{\Sigma_n} (\Sigma_+^\infty A)^n \cong \Sigma_+^\infty(\mathrm{O}(n) \times_{\Sigma_n} A^n).$$

REMARK. The nonequivariant analogue of this is called the **spherical group ring** of a space M, $\mathbb{S}[\Omega M] := \Sigma_+^\infty \Omega M$. It's worth thinking about the similarities between these two cases. ◅

What ends up happening is that Σ_+^∞ takes transfers to norms, which is a good slogan to remember.

We can also look at what happens to the units of a commutative ring spectrum. Nonequivariantly, let R be a commutative ring spectrum. Its **group of units** was initially defined as the space $\mathrm{GL}_1 R$ that fits into the pullback diagram

(4.7.4)
$$\begin{tikzcd}
\mathrm{GL}_1 R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega^\infty R \arrow[d] \\
(\pi_0 \Omega^\infty R)^\times \arrow[r] & \pi_0 \Omega^\infty R.
\end{tikzcd}$$

This definition makes sense for any A_∞-ring spectrum, but behaves better when R is commutative.

FACT.

(1) If R is an E_∞-ring spectrum, then $\mathrm{GL}_1 R$ is an E_∞-space, and therefore extends to a spectrum called $\mathrm{gl}_1 R$.

(2) GL_1 is (homotopically) the right adjoint to Σ^∞_+.[20] ◀

This relates to the theory of orientations, and in fact this was originally considered for this purpose by Dennis Sullivan, and expanded on by [**May77**]. The modern approach was written down in [**ABG⁺14a**, **ABG⁺14b**]. Peter May has three articles [**May09a**, **May09b**, **May09c**] from the 2009 Banff conference about these things that are particularly clear.

Another characterization of $\mathrm{GL}_1 R$ is as the homotopy automorphisms of R in the category of R-modules: $\mathrm{GL}_1 R = \mathbf{L}\mathrm{Aut}_{\mathrm{Mod}_R}(R)$.

Anyways, there's an analogous construction in the equivariant setting [CITE ME: ???], where you have to work out (4.7.4) for all fixed points simultaneously. But this works out, and you can show that GL_1 sends norms to transfers, which makes sense.

An outline of the Kervaire invariant 1 proof. Of course, this is the proof in [**HHR16**]. Kervaire [**Ker60**] connected his invariant to the stable homotopy groups of spheres, and therefore was able to show that every 10-dimensional smooth manifold has Kervaire invariant zero, then produced a PL manifold with Kervaire manifold 1 (hence a nonsmoothable manifold).

Browder [**Bro69**] managed to connect this to elements $\theta_j \in \pi^s_{2^{j+1}-2}(\mathbb{S})$, so that a manifold of Kervaire invariant one exists in a certain dimension iff there's a differential in the Adams spectral sequence showing that $\theta_j \neq 0$.

The proof idea was concocted by Ravenel in the 1970s, but the tools to implement it weren't available until very recently. The idea is to construct a designer cohomology theory Ω meeting the following criteria.

(1) Ω is multiplicative, therefore admitting a map $\mathbb{S} \to \Omega$ such that the image of θ_j is nonzero.
(2) The homotopy groups $\pi_i\Omega$ are zero in degrees $0 < i < 4$.
(3) $\pi_*\Omega \cong \pi_{*+256}\Omega$.

These suffice: the idea is that for $j \geq 7$, θ_j is detected by Ω, but lands in degree between 0 and 4 mod 256, so must be zero.

REMARK. For a while, people suspected there were infinitely many dimensions in which manifolds with Kervaire invariant 1 existed, which would have simplified a lot of Adams spectral sequence calculations. One manifestation of this was Cohen's "doomsday conjecture" [**Mil71**], which Mahowald [**Mah77**] showed was false. ◀

Then, Hill, Hopkins, and Ravenel constructed Ω as $N_{C_2}^{C_8} MR[\kappa^{-1}]$: take real bordism, norm it up from C_2 to C_8, and then invert something (an analogue of the Bott element), and they showed it meets the desired criteria.

Definition 4.7.5 (Real cobordism)**.** Real cobordism MR is a C_2-spectrum originally defined by Landweber [**Lan67**]. We'll provide a construction for it as an orthogonal C_2-spectrum due to Schwede [**Sch16**].

Consider the space

$$MU_n := E\mathrm{U}(n)_+ \wedge_{\mathrm{U}(n)} S^{\mathbb{C}^n},$$

where $\mathrm{U}(n)$ is the group of unitary matrices. Then, MU_n is a C_2-space, where the action is by complex conjugation.[21]

We'll use these spaces to define a C_2-ring spectrum. First, the decomposition $\mathbb{C}^{n+m} \cong \mathbb{C}^n \oplus \mathbb{C}^m$ induces a multiplication map $MU_n \wedge MU_m \longrightarrow MU_{n+m}$, and the map $\mathrm{U}(n) \to B(\mathrm{U}(n), \mathrm{U}(n), *)$ defines a map $S^{\mathbb{C}^n} \to E\mathrm{U}(n) \times_{\mathrm{U}(n)} S^{\mathbb{C}^n}$; together, these define C_2-equivariant structure maps $S^{\mathbb{C}^n} \wedge MU_m \to MU_{n+m}$.

This is very close to the definition of an orthogonal ring spectrum, but indexed on complex representations rather than real ones. One of the lessons of diagram spectra is that the diagram you index spectra by doesn't matter all that much, and there's a way to turn these **unitary spectra** into orthogonal ones.

Namely, using the decomposition $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$, we can define

$$MR_n := \mathrm{Map}(i\mathbb{R}^n, MU_n),$$

where O_n and C_2 both act by conjugation. The structure maps for MU_n factor as

$$S^n \wedge S^{i\mathbb{R}^n} \wedge MU_k \longrightarrow MU_{n+k},$$

where C_2 acts on $\mathbb{R}^n$ by the sign representation and on MU_N by complex conjugation.

Now, we get an object $MR \in \mathsf{Sp}^{C_2}$ called **real cobordism**.

$$MR_V := \mathrm{Map}(S^{iV}, E\mathrm{U}(V_{\mathbb{C}})_+ \wedge_{\mathrm{U}(V(\mathbb{C}))} S^{V_{\mathbb{C}}}),$$

where $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$.

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[20] Maybe we should call this a factoid, as it's a fact with multiple objects.

[21] Here, $E\mathrm{U}(n) = B(\mathrm{U}(n), \mathrm{U}(n), *)$ is a contractible space with a free $\mathrm{U}(n)$-action, and the quotient is $B\mathrm{U}(n) = B(*, \mathrm{U}(n), *)$.

Real cobordism relates to usual cobordism spectra in nice ways: the geometric fixed points $\Phi^{C_2}MR \cong MO$, and $i_{C_2}^* MR = MU$.

From here, one uses spectral sequences to compute that Ω satisfies the various criteria, including the homotopy fixed point spectral sequence and the slice spectral sequence (which we'll discuss in Theorem 5.3.5 and (5.2.6), respectively).

CHAPTER 5

Spectral sequences

5.1. Tate spectra

"One shift (Σ), two shift (Σ^2), red shift (K), blue shift ($(-)^{tG}$)."

Tate spectra are the spectral analogue of Tate cohomology in group cohomology. This is yet another instance of a fruitful phenomenon: *group cohomology is an important source of inspiration in equivariant homotopy theory*. By looking at group cohomology in the right way, we found Mackey functors, the Wirthmüller isomorphism, norms, Tambara functors, and more (though some of these were hard to see from group cohomology first). Group cohomology is a nice testing ground, because of its concreteness: there are elements and you can add and multiply things!

REMARK. The definitive reference on Tate spectra is Greenlees-May [**GM95b**], which builds on [**ACD89**]. The latter is lower-tech, but is more direct and has nice ideas. Another good reference is [**HM03**, §4], a section of Hesselholt-Madsen's paper about the algebraic K-theory of local fields.

The reason you can find Tate spectra in an *a priori* unrelated paper is that they are important for computing algebraic K-theory via trace methods: it's possible to compute algebraic K-theory from topological cyclic homology and topological Hochschild homology, and the computation passes through iterated Tate spectral sequences. The best examples for the use of Tate spectra were until quite recently from the literature on trace methods, e.g. almost all of Hesselholt-Madsen's papers.

There's also a connection between Tate spectra and chromatic homotopy theory: like the redshift conjecture asserts that algebraic K-theory raises chromatic height, Tate spectra seem to have blueshift, lowering by a chromatic level. ⋖

Let M be a $k[G]$-module. Then, its **algebraic orbits** are $M_G := k \otimes_{k[G]} M$, and its **algebraic fixed points** are $M^G := \mathrm{Hom}_{k[G]}(k, M)$. We'll define a map $N\colon M_G \to M^G$, which is called a norm map, but it's more like a transfer map. Namely, there's a distinguished element in $k[G]$, $N_G := \sum_{g\in G} g$, and N is multiplication by N_G:

$$N\colon \overline{x} \longmapsto \sum_{g\in G} gx.$$

This lands in M, but is clearly a fixed point.

Exercise 5.1.1. Show that this is independent of the choice of representative for $\overline{x} \in M_G$.

This is slightly less obvious, but not hard.

We'll think of N as a map $H_0(G;M) \to H^0(G;M)$, which "sews together" group homology and group cohomology with coefficients in M. Nonetheless, it's an interesting object in its own right. What we'll get is

(5.1.2a)
$$\widehat{H}^i(G;M) = \begin{cases} H^i(G;M), & i \geq 1 \\ H_{-i-1}(G;M), & i \leq -2, \end{cases}$$

and for $i = 0, 1$ there's an exact sequence

(5.1.2b)
$$0 \longrightarrow \widehat{H}^{-1}(G;M) \longrightarrow H_0(G;M) \xrightarrow{N} H^0(G;M) \longrightarrow \widehat{H}^0(G;M) \longrightarrow 0.$$

You can build this by forming **complete resolutions**: beginning with a pair $P_* \to k$ and $k \to I_*$, respectively a projective and an injective resolution of k as a $k[G]$-module, and you can sew these together by truncating I_* and dualizing to $k \to \overline{P}_*$, then connecting them along k.[1]

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[1] TODO: make precise.

Another construction is to let $\widetilde{P}_*$ be the mapping cone of a projective resolution $P_* \to k$ of k as a $k[G]$-module. This extends to a sequence

$$\begin{tikzcd} P_* \arrow[r] & k \arrow[r] & \widetilde{P}_* \arrow[r] & \Sigma P_* \arrow[r] & \cdots \end{tikzcd}$$

Definition 5.1.3. Using this, define the **Tate cohomology** $\widehat{H}^*(G;M)$ to be the cohomology of the complex

$$(\widetilde{P}_* \otimes \mathrm{Hom}(P_*, M))^G.$$

To see that this agrees with (5.1.2a) and (5.1.2b), use the long exact sequence and a comparison

$$(P_* \otimes \mathrm{Hom}(P_*, M))^G \cong (P_* \otimes M)_G \cong (P_* \otimes M)^G.$$

We'd like to imitate this construction topologically. The analogue of the projective resolution of k is $EG_+ \to S^0$ which collapses everything to the non-basepoint. Let $\widetilde{EG}$ be the cofiber of this map. Thus we get a map

$$X \cong F(S^0, X) \longrightarrow F(EG_+, X),$$

and thus obtain a diagram of cofiber sequences:

$$\begin{tikzcd} EG_+ \wedge X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \widetilde{EG} \wedge X \arrow[d] \\ EG_+ \wedge F(EG_+, X) \arrow[r] & F(EG_+, X) \arrow[r] & \widetilde{EG} \wedge F(EG_+, X). \end{tikzcd} \tag{5.1.4}$$

Using the Adams isomorphism,[2] you can check that

$$(EG_+ \wedge X)^G \cong X_{hG}.$$

Now, apply $(-)^G$ to (5.1.4), and you obtain

$$\begin{tikzcd} X_{hG} \arrow[r] \arrow[d, "\cong"] & X^G \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \Phi^G X \arrow[d] \\ X_{hG} \arrow[r, "N"] & X^{hG} \arrow[r] & X^{tG}. \end{tikzcd} \tag{5.1.5}$$

The map N is the canonical orbits-to-fixed-points map. The spectrum in the lower right, X^{tG}, is called the **Tate spectrum** of X.

Remark.

(1) The fact that the right-hand square is a pullback diagram means that you can recover X^G from $\Phi^G X$. For example, if $G = C_p$, so there's only one nontrivial subgroup, you can describe the homotopy theory of G-spectra diagramatically, since geometric fixed points capture weak equivalences. This philosophy is put to work in Saul Glassman's papers.

(2) The right-hand square looks like an arithmetic square or fracture square, and this is a perspective worth taking seriously. ◂

Definition 5.1.6. The **free homotopy type** or **Borel homotopy type** of a G-spectrum X is the homotopy type of $EG_+ \wedge X$.[3]

If you make the action free, you lose some information, but often enough remains to be useful. In particular, the bottom row of (5.1.5) is a cofiber sequence, so X^{tG} only depends on the free homotopy type of X.

Remark. Since EG_+ is a space, it has a diagonal, and $(EG \times EG)_+ \cong EG_+$, so it's a ring, and therefore $(-)^{tG}$ is lax monoidal. It's a bit of a chore to make this precise.

The Tate spectrum construction is not functorial in G; however, it does define a Mackey functor [GM95b]. ◂

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[2] We haven't discussed the Adams isomorphism, but you can also show this directly: the idea is that smashing with EG_+ makes things free, hence you just have to check on underlying spaces, and taking $F(EG_+, -)$ makes it cofree, so you can work with that.

[3] Borel homotopy types capture the naïve notion of equivariant spectra, namely spectra with a G-action or spectra over BG. They have some nice properties, e.g. their homotopy fixed points are the same as their ordinary fixed points [MNN17, Prop. 6.19], and are often useful outside homotopy theory, such as in [FH16].

This construction now leads to the Tate spectral sequence.[4] The filtrations on $\widetilde{EG}$ and EG combine to produce a filtration on $\widetilde{EG} \wedge F(EG_+, X)$.[5] Namely, let $\{E_r\}$ be a CW filtration of EG and $\{\widetilde{E}_r\}$ be a CW filtration of $\widetilde{EG}$. Let

$$X_{r,s} := \widetilde{E}_r \wedge F(E/E_{-s-1}, X)$$
$$\overline{X}_{r,s} := \operatorname*{hocolim}_{\substack{0 \le x \le r \\ 0 \le y \le s}} X_{x,y}$$
$$\overline{X}_t := \bigcup_{r+s=t} \overline{X}_{r,s}.$$

Definition 5.1.7. The **Tate spectral sequence** is the spectral sequence induced from $\{\overline{X}_t\}$. It is a *conditionally convergent* spectral sequence

$$E^2_{s,t} = \widehat{H}^s(G; \pi_t X) \Longrightarrow \pi_{s+t} X^{tG}.$$

REMARK. Conditionally convergent spectral sequences may be somewhat unfamiliar, and are less well behaved. Boardman's "Conditionally convergent spectral sequences" [**Boa99**] is a great paper separating the information of a spectral sequence into a structural piece and a calculational piece (using homotopy limits). Conditional convergence means the structural part exists. Boardman also proves a spectral sequence comparison theorem for conditionally convergent spectral sequences! Hesselholt-Madsen [**HM92**] make heavy use of a nonconvergent spectral sequence to make calculations about the S^1-Tate spectrum of the image-of-J spectrum, which is a really interesting use of nonconvergence. Their papers in general are a masterclass in thinking with spectral sequences, and this one is particularly accessible. ◂

The Tate spectral sequence has a relationship to the homotopy fixed-point spectral sequence, which you might expect.

There's an alternate construction of the Tate spectral sequence using the **Greenlees filtration** of $\widetilde{EG}$:

$$E'_r := \begin{cases} E_r, & r \ge 0 \\ DE_r, & r \le 0. \end{cases}$$

(Here, $DE_r = \mathrm{Map}(E_r, \mathbb{S})$.) Then, you can smash using E'_r instead of E_r, and this was Greenlees' original construction of the Tate spectral sequence.

Let's analyze the E^1 term a bit more. There's an isomorphism

$$\overline{X}_r / \overline{X}_{r-1} \cong \bigvee_s W_{r,s},$$

where

$$W_{r,s} := (\widetilde{E}_r / \widetilde{E}_{r-1}) \wedge F(E_{-s}/E_{-s-1}, X).$$

Now, taking G-fixed points,

$$\pi_{r+s+t}((\widetilde{E}_r/\widetilde{E}_{r-1}) \wedge F(E_{-s}/E_{-s-1}, X)^G) \cong \pi_r(\widetilde{E}_r/\widetilde{E}_{r-1}) \otimes \pi_{s+t}(F(E_{-s}/E_{-s-1}, X)),$$

and

$$\pi_r(\widetilde{E}_s/\widetilde{E}_{r-1}) \otimes \mathrm{Hom}(\pi_s(E_{-s}, E_{-s-1}), \pi_t X) \cong H_r(-) \otimes \mathrm{Hom}(H_s(-), \pi_t X),$$

which is the Tate resolution.

REMARK. As mentioned, there are connections between Tate spectra and chromatic phenomena. This is the subject of Hopkins-Lurie's work on ambidexterity [**HL16**], the astounding fact that $K(n)$-locally, BG (and more generally any orbifold) is dualizable. This was first noticed computationally by Ravenel and Wilson [**RW80**], but can be understood as the $K(n)$-local Tate spectrum vanishing.

There are also interesting connections to the stable module category and modular representation theory. ◂

5.2. The slice spectral sequence

In this section, we discuss the slice spectral sequence, the equivariant version of the Postnikov spectral sequence. Very little is known about the slice spectral sequence, so you could pursue it in your research. For example, its use in [**HHR16**] depended on an explicit identification of the slices with something else, which does not generalize.

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[4] In this argument, we'll make nice (co)fibrancy assumptions, e.g. that X is fibrant and EG_+ is cofibrant; a careful treatment would make the (co)fibrant replacements explicit.

[5] One unpleasant aspect of this construction is that, though these filtrations are both cellular, and the maps in (5.1.4) can be made cellular, it's extremely hard to see the multiplicative structure if you do this. This is addressed in [**BM17**].

Postnikov towers. For this section, we work in the category Top of spaces. We want a collection of functors $P_n\colon \mathsf{Top} \to \mathsf{Top}$, called **Postnikov sections**, together with natural transformations $P_n \to P_{n-1}$, such that

- $$\pi_i(P_nX) = \begin{cases} \pi_i(X), & 0 \le i \le n \\ 0, & \text{otherwise;} \end{cases}$$
- $X \cong \operatorname{holim} P_nX$; and
- the homotopy fiber of $P_nX \to P_{n-1}X$ is weakly equivalent to $K(\pi_n(X), n)$.

So we obtain a **Postnikov tower** $\cdots \to P_2X \to P_1X \to P_0X$, which is in a sense dual to the cellular filtration.

Remark. The axiomatization of the idea of a Postnikov tower in a triangulated category is called a ***t*-structure** [**BBD82**]. The dual notion of a cellular filtration is axiomatized as a **weight structure** [**Bon10**]. ◄

These P_nX have only finitely many nonzero homotopy groups, hence must be very large spaces. There are multiple different point-set models for them. Here's one model.

Choose an $\alpha \in \pi_{n+1}(X)$, which defines a homotopy class of maps $f_\alpha\colon S^{n+1} \to X$. Let $\widetilde{X}$ be the pushout

$$\begin{tikzcd} S^{n+1} \arrow[r, "f_\alpha"] \arrow[d] & X \arrow[d] \\ D^{n+2} \arrow[r] & \widetilde{X}. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

This kills α, and since S^{n+1} is n-connected, the induced map $\pi_k(X) \to \pi_k(\widetilde{X})$ is an isomorphism for $k < n+1$. Then we can iterate.

This has a fatal flaw: it's not functorial. As usual, we fix this with the small object argument. Consider *all* of the maps $S^{n+1} \to X$ and take the pushout

$$\begin{tikzcd} \bigvee S^{n+1} \arrow[r] \arrow[d] & X \arrow[d] \\ \bigvee D^{n+2} \arrow[r] & X_{n+1}. \end{tikzcd}$$

Continue this way for maps $S^N \to X$ for $N \ge n+1$ and let $P_nX := \operatorname{colim}_N X_N$. This is functorial.

The functor P_n can be described as localization, e.g. in [**MS02**], which allows for some slick high-tech setups: if you work with presentable ∞-categories, you can describe P_n as adjoint to the inclusion of spaces with homotopy groups within $[0, n]$, as in [**Lur09a**, §5.5.6].

The Postnikov tower leads to the **Atiyah-Hirzebruch spectral sequence** (AHSS), which for a generalized homology theory E, has signature

$$E^2_{p,q} = H_p(X; E_q(*)) \Longrightarrow E_{p+q}(X).$$

Exercise 5.2.1 (Maunder [**Mau63**])**.** If you play the same game with the CW filtration, you obtain an isomorphic spectral sequence. It's a good exercise to work this out yourself.

This works in more than just spaces: you can set it up for spectra, and (with a little technical work) for (commutative) ring spectra.

The equivariant case. We'd like to do this in Sp^G. The slice spectral sequence will be an analogue to the Atiyah-Hirzebruch spectral sequence. One difficulty will be understanding the associated graded, which is considerably more complicated than the nonequivariant Postnikov or CW associated graded complexes, and this is ultimately because there are more spheres around.

The first piece of the slice spectral sequence was worked out by Dugger [**Dug05**]. His motivation was the analogy between C_2-equivariant homotopy theory and motivic homotopy theory. Following Bloch-Lichtenbaum [**BL95**] and Grayson [**Gra92**], the goal was to approach the Quillen-Lichtenbaum conjecture [**Qui74**], the existence of a spectral sequence

$$H^p(X; \mathbb{Z}(-\varepsilon/2)) \Longrightarrow K^{p+q}(X).$$

Here, $H^*(-, \mathbb{Z}(-\varepsilon/2))$ is "motivic cohomology," which was not well-understood when the conjecture was formulated. The $-\varepsilon/2$ is a **Tate twist**, which is akin to Bott periodicity for algebraic K-theory of finite fields. See [**BL95**, **Gra92**] for details or Mitchell [**Mit94**] for an exposition. Dugger replaced this with a conditionally convergent spectral sequence

$$E_2^{p,q} = H^{p,r-q/2}(X;\underline{\mathbb{Z}}) \Longrightarrow KR^{p+q,r}(X),$$

where

- X is any C_2-space,
- $\underline{\mathbb{Z}}$ is the constant Mackey functor valued in $\mathbb{Z}$, and
- KR is Atiyah's KR-theory [**Ati66**].

If q is odd, we take $E_2^{p,q} = 0$.

The general formulation of the slice spectral sequence was worked out in [**HHR16**]; see also Hu-Kriz-Ormsby [**HKO11**]. The exposition in [**HHR16**] is pretty good, and you should also check out Mike Hill's introduction [**Hil12**]. There are also some worked-out computations with the slice spectral sequence due to Hill [**Hil15**], Yarnall [**Yar15**], Hill-Hopkins-Ravenel [**HHR17a, HHR17b**], Hill-Meier [**HM16**], Hill-Yarnall [**HY17**], and Greenlees [**Gre17**].

To get at the spectral sequence, we first approach the filtration. The motivation is to have a Postnikov section P_H for the regular representation ρ_H of H. The **slice cells** will be the cells

$$\{G_+ \wedge_H S^{m\rho_H}, \Sigma^{-1} G_+ \wedge_H S^{m\rho_H}\}.$$

The **dimensions** of these slice cells are $m|H|$, resp. $m|H|-1$.

Slice cells are well-behaved under the "change functors" i_K^*, $G_+ \wedge_K -$, and N_K^G: all of these preserve slice cells.

Definition 5.2.2. Let $X \in \mathsf{Sp}^G$.

- X is **slice n-null** if $\mathrm{Map}(\widehat{S}, X)$ is contractible (as a G-space) for all slice cells $\widehat{S}$ of dimension greater than n. One also says X is **slice** $< n$ or **slice** $\le n-1$.
- X is **slice n-positive** (also **slice** $> n$ or **slice** $\ge n+1$) if $\mathrm{Map}(\widehat{S}, X)$ is contractible (as a G-space) for all slice cells $\widehat{S}$ of dimension at most n.

Exercise 5.2.3 ([**HHR16**], Prop. 4.11)**.** Let X be a G-spectrum.

1. Show that X is slice 0-positive iff it's -1-connected.[6]
2. Show that X is slice -1-positive iff it's -2-connected.
3. Show that X is slice 0-null iff it's 0-coconnected.
4. Show that X is slice -1-null iff it's -1-coconnected.

Proposition 5.2.4 ([**HHR16**], Prop. 4.15)**.** *X is slice n-positive iff up to weak equivalence, there's a filtration $X_0 \subseteq X_1 \subseteq \cdots \subseteq X$ such that X_i/X_{i-1} is a wedge of slice cells of dimensions greater than n.*

We localize at the slices of dimension greater than n to obtain $P^n X$, the **n-slice section** of X. This is a bit tricky, because there are maps $S^V \to S^W$ when $V \subseteq W$ that aren't null-homotopic, frustrating our approach to defining the Postnikov tower, but the key is that if V contains a trivial representation, all such maps are null-homotopic.

Localization means there's a map $X \to P^n X$ by fiat, and one can show that the homotopy fiber of this map is slice n-positive. We obtain a tower $\cdots \to P^n X \to P^{n-1} X \to \cdots$. Let $P_n^n X$ denote the homotopy fiber of $P^n X \to P^{n-1} X$; $P_n^n X$ is called the **n-slice** of X.

Exercise 5.2.5 ([**HHR16**], Prop. 4.20)**.** Show that the -1-slice of X is

$$P_{-1}^{-1} X = \Sigma^{-1} H(\underline{\pi}_{-1} X),$$

where again $\underline{\pi}_{-1}$ means to take the homotopy group as a Mackey functor. (Hint: use Exercise 5.2.3.)

[**HHR16**] heavily use the fact that the regular representation contains a copy of all irreducibles to understand what's going on, which is great for finite groups, but doesn't work for compact Lie groups. There's a different perspective adopted by Dugger, that $G/H_+ \wedge S^n$ detects homotopies, so we can restrict to slice cells $G/H_+ \wedge S^V$ where V contains a copy of the trivial representation (and Σ^{-1} of these cells). The slice filtration has some other nice properties: $\operatorname{holim} P^n X \cong X$, and $\operatorname{colim} P^n X \simeq *$.

These localizations are controlled by the subcategory of Sp^G determined by positive shifts and cofibers of slice cells. Negative shifts are not allowed, so when passing to the homotopy category, this isn't a triangulated subcategory.

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[6] By n-**connected** we mean the Mackey functor $\underline{\pi}_k X = 0$ for $k \le n$, and by n-**coconnected** we mean $\underline{\pi}_k X = 0$ for $k \ge n$.

This slice filtration produces the **slice spectral sequence**, which is a strongly convergent spectral sequence

$$E_2^{s,t} = \pi_s^G P_t^t X \Longrightarrow \pi_{s-t}^G X. \tag{5.2.6}$$

Here we use the Adams grading, so this may look funny compared to the usual grading on, e.g. the Serre spectral sequence. You can refine the slice spectral sequence into a spectral sequence of Mackey functors, and there is an $RO(G)$-graded version. If you think about it, you'll see this is a first- and third-quadrant spectral sequence.

The key to understanding the slice spectral sequence is understanding the slices $P_t^t X$, and this is hard. In [HHR16], they use the fact that real bordism has nice slices, and [Dug05] works out some computations for C_2, but there are few calculations in the literature, and more would be welcome.

These notes will hopefully be a living document; feel free to update with better explanations, or to fix mistakes, or to add examples, or anything like that. There are relatively few references for equivariant homotopy theory, and it would be great for the notes to be one more, hopefully in a presentable way.

5.3. The homotopy fixed point spectral sequence

This section was not part of the class: it was delivered by Richard Wong as part of a mini-course on spectral sequences in equivariant stable homotopy theory. See `https://www.ma.utexas.edu/users/richard.wong/2017/Resources.html` for more information.

We'll start with the Bousfield-Kan spectral sequence (BKSS). One good reference for this is Guillou's notes [Gui07], and Hans Baues [Bau89] set it up in a general model category.

We'll work in sSet, so that everything is connective. Consider a tower of fibrations

$$\begin{tikzcd} \cdots \arrow[r] & Y_s \arrow[r, "p_s"] & Y_{s-1} \arrow[r, "p_{s-1}"] & Y_{s-2} \arrow[r] & \cdots \end{tikzcd} \tag{5.3.1}$$

for $s \geq 0$, and let $Y := \varprojlim Y_s$. Let F_s be the fiber of p_s.

Theorem 5.3.2 (Bousfield-Kan [BK72a])**.** *In this situation, there is a spectral sequence, called the* ***Bousfield-Kan spectral sequence****, with signature*

$$E_1^{s,t} = \pi_{t-s}(F_s) \Longrightarrow \pi_{t-s}(Y).$$

If everything here is connective (which is not always the case in other model categories, as in one of our examples), this is first-quadrant. One common convention is to use the **Adams grading** $(t-s, s)$ instead of (s,t).

We can extend (5.3.1) into a diagram

$$\begin{tikzcd} \cdots \arrow[r] & Y_s \arrow[r, "p_s"] & Y_{s-1} \arrow[r, "p_{s-1}"] & Y_{s-2} \arrow[r] & \cdots \\ & F_s \arrow[u, "i_s"'] & F_{s-1} \arrow[u, "i_{s-1}"'] & F_{s-2}, \arrow[u, "i_{s-2}"'] & \end{tikzcd}$$

and hence into an exact couple

$$\begin{tikzcd} \cdots \arrow[r] & \pi_*(Y_{s+1}) \arrow[r] & \pi_*(Y_s) \arrow[r] \arrow[dl, "\delta"'] & \pi_*(Y_{s-1}) \arrow[r] \arrow[dl, "\delta"'] & \cdots \\ & \pi_*(F_s) \arrow[u, "i_{s*}"] & \pi_*(F_{s-1}) \arrow[u, "i_{(s-1)*}"] & \pi_*(F_{s-2}), \arrow[u, "i_{(s-2)*}"] & \end{tikzcd}$$

and the differentials are the compositions of the maps $\pi_*(F_s) \to \pi_*(Y_s) \to \pi_*(F_{s+1})$:

$$\begin{tikzcd} \cdots \arrow[r] & \pi_*(Y_{s+1}) \arrow[r] & \pi_*(Y_s) \arrow[r] \arrow[dl, "\delta"'] & \pi_*(Y_{s-1}) \arrow[r] \arrow[dl, "\delta"'] & \cdots \\ {} & \pi_*(F_s) \arrow[u, "i_{s*}"] \arrow[l] & \pi_*(F_{s-1}) \arrow[u, "i_{(s-1)*}"] \arrow[l, "d_1"] & \pi_*(F_{s-2}). \arrow[u, "i_{(s-2)*}"] \arrow[l, "d_1"] & {} \arrow[l] \end{tikzcd}$$

Taking homology, we'll get a differential d_2 that jumps two steps to the left, then d_3 three steps to the left, and so on. After you check that $\operatorname{Im}(d_r) \subset \ker(d_r)$, you can define $E_r^{s,s+1} := \ker(d_r)/\operatorname{Im}(d_r)$. Let $A_s := \operatorname{Im}(\pi_0(Y_{s+r}) \to \pi_0(Y_s))$, and let $Z_r^{s,s} := (i_s)_*^{-1}(A_s)$. Then, $E_{r+1}^{s,s} = Z_r^{s,s}/d_r(E_r^{s-r,s-r+1})$.

Remark. One important caveat is that for $i \leq 2$, π_i does not produce abelian groups, but rather groups or just sets! This means that a few of the columns of this spectral sequence don't quite work, but the rest of it is normal, and the degenerate columns can still be useful. This is an example of a **fringed spectral sequence**. ◂

Bousfield and Kan cared about this spectral sequence because it allowed them to write down a useful long exact sequence, the r^{th} **derived homotopy sequence**: let $\pi_i Y^{(r)} := \mathrm{Im}(\pi_i(Y_{n+r}) \to \pi_i(Y_n))$; then, there's a long exact sequence

$$\begin{tikzcd}\cdots \arrow[r] & \pi_{t-s-1} Y^{(r)}_{s-r-1} \arrow[r] & E^{s,t}_{r+1} \arrow[r] & \pi_{t-s} Y^{(r)}_{s} \arrow[r, "\delta"] & \pi_{t-s} Y^{(r)}_{s-1} \arrow[r] & E^{s+r,t+r+1}_{r+1} \arrow[r] & \pi_{t-s+1} Y^{(r)}_{s} \arrow[r] & \cdots\end{tikzcd}$$

You can do something like this in general given a spectral sequence, though you need to know how to obtain it from the exact couple.

Remark. When $r = 0$, $E_1^{s,t} = \pi_{t-s}(F_s)$, and so the first derived homotopy sequence is the long exact sequence of homotopy groups of a fibration. ◂

One nice application is to **Tot towers** ("Tot" for totalization).

Definition 5.3.3. Let $X^\bullet$ be a cosimplicial object in sSet. Then, its **totalization** is the complex

$$\mathrm{Tot}(X^\bullet) := \mathrm{sSet}(\Delta^\bullet, X^\bullet),$$

i.e.

$$\mathrm{Tot}_n(X^\bullet) := \mathrm{sSet}(\mathrm{sk}_n \Delta^\bullet, X^\bullet).$$

Here $(\mathrm{sk}_n \Delta^\bullet)^n := \mathrm{sk}_n \Delta^m$.

Then

$$\varprojlim \mathrm{Tot}_n(X^\bullet) = \mathrm{Tot}(X^\bullet),$$

reconciling the two definitions.

Exercise 5.3.4. In the Reedy model structure, $\mathrm{Tot}_n(X^\bullet) \to \mathrm{Tot}_{n-1}(X^\bullet)$ is a fibration.

Assuming this exercise, we can apply the Bousfield-Kan spectral sequence.

One place this pops up is that if $C, D \in \mathsf{C}$ and $X_\bullet \to C$ is a simplicial resolution in a simplicial category C,[7] then $\mathrm{Hom}_{\mathsf{C}}(X_\bullet, D)$ is a cosimplicial object, and this spectral sequence can be used to compute homotopically meaningful information about $\mathrm{sSet}(C, D)$.

We can use this formalism to derive the homotopy fixed point spectral sequence. Recall (Definition 2.5.15) that if X is a G-spectrum, its homotopy fixed point spectrum is the nonequivariant spectrum $X^{hG} := F((EG)_+, X)^G$, i.e. the spectrum of G-equivariant maps $(EG)_+ \to X$.[8] The bar construction gives us a simplicial resolution of $(EG)_+$, producing a cosimplicial object that can be plugged into the Bousfield-Kan spectral sequence. Specifically, we write $EG = B^\bullet(G, G, *)$, add a disjoint basepoint, and then take maps into X.

Theorem 5.3.5. *If X is a G-spectrum, there's a spectral sequence, called the **homotopy fixed point spectral sequence**, with signature*

$$E_2^{p,q} = H^p(G, \pi_q(X)) \Longrightarrow \pi_{q-p}(X^{hG}).$$

TODO: discuss the multiplicative structure.

Example 5.3.6. The first example is really easy. Let k be a field, and consider the Eilenberg-Mac Lane spectrum Hk. Let G act trivially on k; we want to understand $\pi_*(Hk^{hG})$. The homotopy fixed-points spectral sequence is particularly simple:

$$E_2^{p,q} = H^p(G; \pi_q(Hk)) = \begin{cases} H^p(G; k), & q = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Since this is a single row,[9] all differentials vanish, and this is also the E_∞ page. So we just have to compute $H^p(G; k)$ for $k \geq 0$.

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[7] Meaning that after geometrically realizing, there's an equivalence.

[8] Notationally, this is the function spectrum of maps from $\Sigma^\infty(EG)_+$ to X, or you can use the fact that spectra are cotensored over spaces.

[9] It's a single row in the usual grading, and a single diagonal line with slope -1 in the Adams grading.

For example, if $G = \mathbb{Z}/2$ and $k = \mathbb{F}_2$, then $H^*(\mathbb{Z}/2;\mathbb{F}_2) = H^*(\mathbb{RP}^\infty;\mathbb{F}_2) = \mathbb{F}_2[x]$, $|x| = 1$. There are no extension issues, since there's only one nonzero term in each total degree. Thus,

$$\pi_{-p}(Hk^{hG}) = H^p(G;k).$$

If you let $G = \mathbb{Z}/2$ and k be any field of odd characteristic, then $H^*(\mathbb{Z}/2;k) = k$ in degree 0, so the homotopy groups of $Hk^{h\mathbb{Z}/2}$ are all trivial except for π_0, which is k. ◀

Remark. The homotopy fixed points of X are related to the Tate spectrum through a fiber sequence (5.1.5), and the homotopy fixed point spectral sequence has a similar-looking signature to the Tate spectral sequence (Definition 5.1.7):

$$E_2^{p,q} = H^p(G,\pi_q(X)) \Longrightarrow \pi_{q-p}(X^{hG})$$
$$E_2^{p,q} = \widehat{H}^p(G;\pi_q(X)) \Longrightarrow \pi_{q-p}(X^{tG}).$$

As Tate cohomology puts group homology and group cohomology together, you might expect there's a third similar-looking spectral sequence, and you're right:

$$E^2_{p,q} = H_p(G,\pi_q(X)) \Longrightarrow \pi_{q-p}(X_{hG})$$

This is called the **homotopy orbit spectral sequence**. ◀

TODO: the cofiber sequence $X_{hG} \to X^{hG} \to X^{tG}$ induces a morphism of multiplicative spectral sequences from the homotopy fixed point spectral sequence to the Tate spectral sequence. You can use this, as in [HS14, Prop. 11], to show that $KR^{tC_2} \simeq *$ and therefore $KR^{hC_2} \simeq KR_{hC_2}$. Does the homotopy orbit spectral sequence fit into this? Would we have to grade it cohomologically?

Example 5.3.7. Let C_2 act on S^1 by reflection. Then, $\pi_i(S^1)$ is trivial unless $i = 1$, in which case we get $\mathbb{Z}$. Hence,

$$E_2^{p,q} = \begin{cases} H^p(C_2;\mathbb{Z}), & q = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Under the isomorphism $\mathbb{Z}[C_2] \cong \mathbb{Z}[x]/(x^2-1)$, the $\mathbb{Z}[C_2]$-module structure on $\mathbb{Z}$ is the map $\mathbb{Z}[C_2] \to \mathbb{Z}$ sending $x \mapsto -1$, i.e. C_2 acts on $\mathbb{Z}$ through the nontrivial action. We'll let $\mathbb{Z}_\sigma$ denote $\mathbb{Z}$ with this action, and $\mathbb{Z}$ denote the integers with the trivial C_2-action. To compute the group cohomology, we need to compute a free resolution $P_\bullet \to \mathbb{Z}$ as a *trivial* $\mathbb{Z}[C_2]$-module:[10]

$$\cdots \longrightarrow \mathbb{Z}[C_2] \xrightarrow{\cdot(x-1)} \mathbb{Z}[C_2] \xrightarrow{\cdot(x+1)} \mathbb{Z}[C_2] \xrightarrow{\cdot(x-1)} \mathbb{Z}[C_2] \xrightarrow{x\mapsto 1} \mathbb{Z} \longrightarrow 0.$$

Now we compute $\mathrm{Hom}_{\mathbb{Z}[C_2]}(P_\bullet,\mathbb{Z}_\sigma)$:

$$\cdots \longleftarrow \mathbb{Z} \xleftarrow{-2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{-2} \mathbb{Z}.$$

Taking homology, we conclude that

$$H^p(C_2,\mathbb{Z}_\sigma) = \begin{cases} \mathbb{Z}/2, & p > 0 \text{ odd} \\ 0, & \text{otherwise.} \end{cases}$$

The spectral sequence degenerates at page 2, but we haven't yet calculated $\pi_*((S^1)^{hC_2})$ — $(S^1)^{hC_2}$ is a space, so cannot have negative-degree homotopy groups. But since this is a fringed spectral sequence, the stuff in negative degrees doesn't apply to the calculation of homotopy groups, and $q - p = 0, 1$ mixes together in a complicated way. In this case, it tells us that $\pi_0((S^1)^{hC_2}) = \mathbb{Z}/2$ and the higher homotopy groups vanish.[11] ◀

Example 5.3.8. Atiyah [Ati66] showed that complex conjugation places a C_2-equivariant structure on KU; the resulting C_2-spectrum is called ***KR*-theory**, and denoted KR. Its homotopy fixed points are real (nonequivariant) K-theory:

Theorem 5.3.9. *$KR^{hC_2} = KO$.*

Playing with the homotopy fixed-point spectral sequence for KR, as in [HS14], is a good way to become more familiar with it. TODO: let's say more about this. ◀

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[10] TODO: why was this again?

[11] TODO: can we calculate $(S^1)^{hC_2}$ explicitly and see this?

Question 5.3.10. It should be possible to do something similar with KSp and $i \mapsto -i$, $j \mapsto -j$, and $k \mapsto -k$. If we take homotopy fixed points with respect to the last two, do we end up with KU? If we take homotopy fixed points with respect to all three, do we get KO?

Exercise 5.3.11. The analogous result doesn't hold for Eilenberg-Mac Lane spectra. Let C_2 act on $\mathbb{Z}[i]$ by complex conjugation, making $\mathbb{Z}[i]$ into a $\mathbb{Z}[C_2]$-module. This data determines a Mackey functor $\underline{\mathbb{Z}[i]}$ and therefore an Eilenberg-Mac Lane C_2-spectrum $H\underline{\mathbb{Z}[i]}$ as in Theorem 3.4.9.

Though $\mathbb{Z}[i]^{C_2} = \mathbb{Z}$, run the homotopy fixed-point spectral sequence to show that $H\underline{\mathbb{Z}[i]}^{hC_2} \not\simeq H\mathbb{Z}$.

There are some other examples of calculations with this spectral sequence, including Bruner-Greenlees [**BG10**], Hill-Hopkins-Ravenel [**HHR15**, **HHR16**], and Hahn-Shi [**HS17**]. Greenlees [**Gre17**], working over C_2, introduces an $RO(C_2)$-graded version of the homotopy fixed point spectral sequence and applies it to $H\underline{\mathbb{Z}}$ and connective real K-theory kr.

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