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Localizing the E_2 page of the Adams spectral sequence

EVA BELMONT

There is only one nontrivial localization of $\pi_*S_{(p)}$ (the chromatic localization at $v_0 = p$), but there are infinitely many nontrivial localizations of the Adams E_2 page for the sphere. The first nonnilpotent element in the E_2 page after v_0 is $b_{10} \in \mathrm{Ext}_A^{2,2p(p-1)}(\mathbb{F}_p, \mathbb{F}_p)$. We work at $p=3$ and study $b_{10}^{-1}\mathrm{Ext}_P^{*,*}(\mathbb{F}_3, \mathbb{F}_3)$ (where P is the algebra of dual reduced powers), which agrees with the infinite summand $\mathrm{Ext}_P^{*,*}(\mathbb{F}_3, \mathbb{F}_3)$ of $\mathrm{Ext}_A^{*,*}(\mathbb{F}_3, \mathbb{F}_3)$ above a line of slope $\frac{1}{23}$. We compute up to the E_9 page of an Adams spectral sequence in the category $\mathrm{Stable}(P)$ converging to $b_{10}^{-1}\mathrm{Ext}_P^{*,*}(\mathbb{F}_3, \mathbb{F}_3)$, and conjecture that the spectral sequence collapses at E_9. We also give a complete calculation of $b_{10}^{-1}\mathrm{Ext}_P^{*,*}(\mathbb{F}_3, \mathbb{F}_3[\xi_1^3])$.

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1 Introduction

For a p–local finite spectrum X, the Adams spectral sequence

$$E_2^{**} = \mathrm{Ext}_A^{*,*}(\mathbb{F}_p, H_*X) \Rightarrow \pi_*X_p^\wedge$$

is one of the main tools for computing (the p–completion of) the homotopy groups of X. If one understands the A–comodule structure of H_*X, it is possible to compute

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the E_2 page algorithmically in a finite range of dimensions. However, for many spectra X of interest such as the sphere spectrum S, there is no chance of determining the E_2 page completely. The motivating goal behind this work is to compute an infinite part of the Adams E_2 page $\mathrm{Ext}_A^{*,*}(\mathbb{F}_p,\mathbb{F}_p)$ for the sphere at $p=3$. Specifically, we wish to compute the b_{10}–periodic part, where $b_{10}\in\mathrm{Ext}_A^{2,2p(p-1)}(\mathbb{F}_p,\mathbb{F}_p)$ converges to $\beta_1\in\pi_{2p(p-1)-2}S$. We show that there is a plane above which $\mathrm{Ext}_A^{*,*}(\mathbb{F}_p,\mathbb{F}_p)$ is b_{10}–periodic, where the third grading f (in addition to internal degree t and homological degree s) is related to the collapse of the Cartan–Eilenberg spectral sequence at odd primes p (see (1-2)). The b_{10}–periodic region of the $f=0$ summand of $\mathrm{Ext}_A^{*,*}(\mathbb{F}_3,\mathbb{F}_3)$ (the main focus of study in this paper) is the region lying above the red line in Figure 1.

The only localization of the Adams E_2 page for the sphere that has been completely computed is

$$a_0^{-1}\,\mathrm{Ext}_A^{*,*}(\mathbb{F}_p,\mathbb{F}_p)\cong\mathbb{F}_p[a_0^{\pm1}], \tag{1-1}$$

where $a_0=[\tau_0]$ converges to $p\in\pi_0S$; this follows from Adams' fundamental work [1] on the structure of the E_2 page. This localization agrees with $\mathrm{Ext}_A^{*,*}(\mathbb{F}_p,\mathbb{F}_p)$ above a line of slope $1/(2p-2)$ (in the $(t-s,s)$ grading). Our proposed localization $b_{10}^{-1}\,\mathrm{Ext}_A^{*,*}(\mathbb{F}_p,\mathbb{F}_p)$ agrees with $\mathrm{Ext}_A^{*,*}(\mathbb{F}_p,\mathbb{F}_p)$ above a plane whose fixed-f cross-section is a line of slope $1/(p^3-p-1)$. While the only a_0–periodic elements lie in the zero-stem (corresponding to chromatic height zero), the b_{10}–periodic region encompasses nonzero classes in arbitrarily high stems, including some elements in chromatic height 2, such as b_{10} itself. Though we do not give a complete calculation of $b_{10}^{-1}\,\mathrm{Ext}_A^{*,*}(\mathbb{F}_p,\mathbb{F}_p)$, we will see that it is much more complicated than $a_0^{-1}\,\mathrm{Ext}_A^{*,*}(\mathbb{F}_p,\mathbb{F}_p)$. Thus, in some sense, one may think of $b_{10}^{-1}\,\mathrm{Ext}_A^{*,*}(\mathbb{F}_p,\mathbb{F}_p)$ as a richer and more revealing version of the classical calculation.

In a different sense, however, these two localizations come from different worlds. Inverting a_0 is the Adams E_2 avatar of p–localization on (p–local) homotopy (rationalization). Equivalently, the sphere has chromatic type zero, and a_0 is just the algebraic name for the chromatic height-0 operator v_0. On the other hand, inverting b_{10} is not the shadow of any homotopy-theoretic localization: by the Nishida nilpotence theorem, β_1 is nilpotent in homotopy, so $\beta_1^{-1}\pi_*S=0$. While $v_0=p$ is the only chromatic periodicity operator acting on the sphere, a_0 and b_{10} are just the first two out of infinitely many nonnilpotent elements in $\mathrm{Ext}_A^{*,*}(\mathbb{F}_p,\mathbb{F}_p)$. Palmieri [14] describes a more complicated analogue of the classical theory of periodicity and nilpotence that

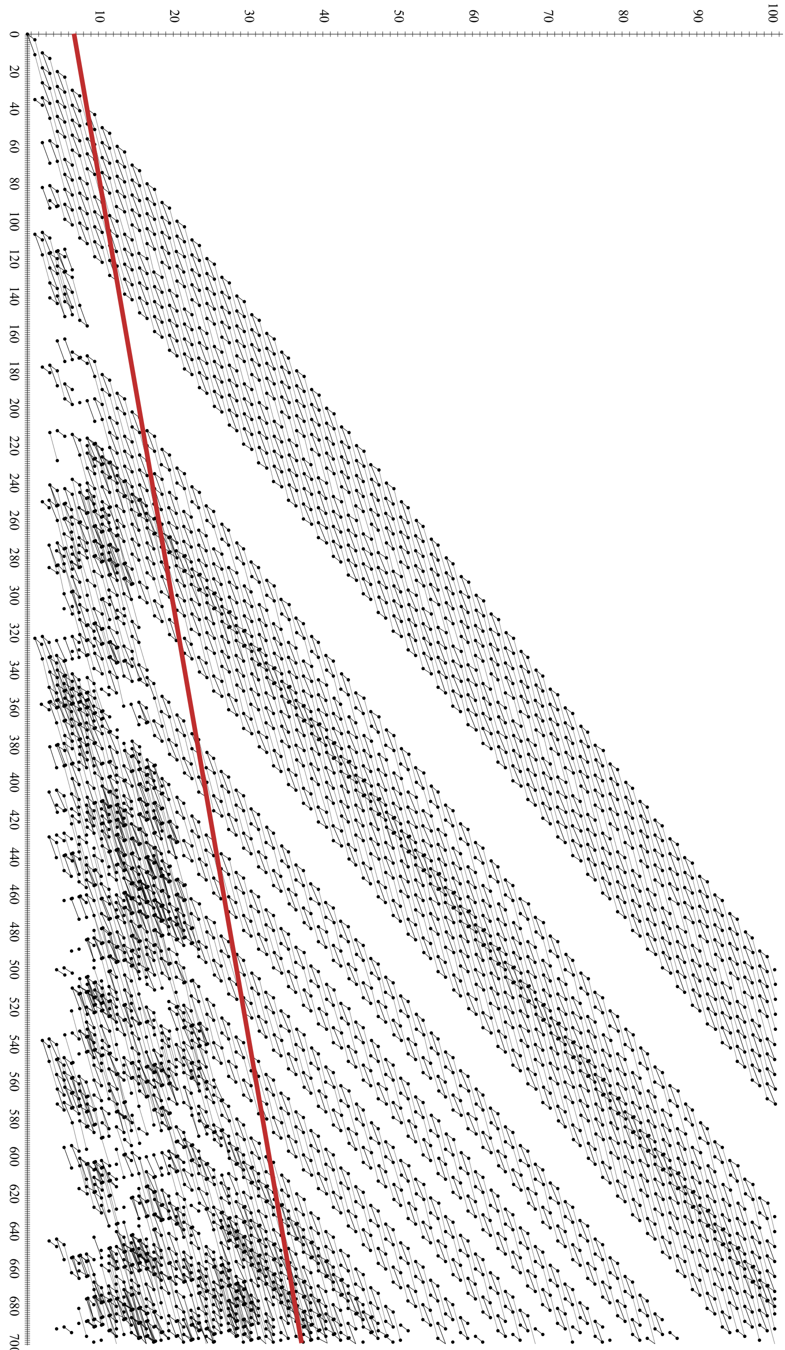

Figure 1: Chart of $\mathrm{Ext}_P^{*,*}(\mathbb{F}_3, \mathbb{F}_3)$ with the line of Proposition 3.1 drawn in red: classes above the line are b_{10}–periodic.

operates only on Adams E_2 pages, almost all of which (except the v_n operators) is destroyed by the time one reaches the Adams E_∞ page.

Recall that the odd-primary dual Steenrod algebra has a presentation

$$A = \mathbb{F}_p[\xi_1, \xi_2, \dots] \otimes E[\tau_0, \tau_1, \dots]$$

where $E[x] = \mathbb{F}_p[x]/x^2$ denotes an exterior algebra, $|\xi_n| = 2(p^n - 1)$ and $|\tau_n| = 2p^n - 1$. Let $P = \mathbb{F}_p[\xi_1, \xi_2, \dots]$ be the Steenrod dual reduced powers algebra, and let E be the quotient Hopf algebra $E[\tau_0, \tau_1, \dots]$. If M is an evenly graded A–comodule, there is an isomorphism

$$\mathrm{Ext}_A^{s,t}(\mathbb{F}_p, M) \cong \mathrm{Ext}_P^{s,t-f}(\mathbb{F}_p, \mathrm{Ext}_E^{f,*}(\mathbb{F}_p, M)) \tag{1-2}$$

which arises from the collapse of the Cartan–Eilenberg spectral sequence at odd primes p. In light of this, we recast our goal as follows:

Goal 1.1 Compute $b_{10}^{-1}\,\mathrm{Ext}_P^{*,*}(\mathbb{F}_p, M)$ for P–comodules M.

In particular, we are most interested in $M = \mathrm{Ext}_E^{*,*}(\mathbb{F}_p, \mathbb{F}_p)$. In this paper, we focus on the $f = 0$ summand $\mathrm{Ext}_E^{0,*}(\mathbb{F}_p, \mathbb{F}_p) \cong \mathbb{F}_p$ and set $p = 3$. We show the following:

Theorem 1.2 *Let $D = \mathbb{F}_3[\xi_1]/\xi_1^3$ and let $R = b_{10}^{-1}\,\mathrm{Ext}_D^{*,*}(\mathbb{F}_3, \mathbb{F}_3) = E[h_{10}] \otimes \mathbb{F}_3[b_{10}^{\pm 1}]$. There is a convergent spectral sequence with E_2 term*

$$E_2^{s,t,u} \cong R[w_2, w_3, \dots] \Rightarrow b_{10}^{-1}\,\mathrm{Ext}_P^{s+t,u}(\mathbb{F}_3, \mathbb{F}_3),$$

where $w_n \in E_2^{1,1,2(3^n+1)}$, and differentials

$$d_r \colon E_r^{s,t,u} \to E_r^{s+r,t-r+1,u}.$$

We have $d_r = 0$ for $r \geq 2$ unless $r \equiv 4 \pmod 9$ or $r \equiv 8 \pmod 9$. The first nontrivial differential is

$$d_4(w_n) = b_{10}^{-4} h_{10} w_2^2 w_{n-1}^3$$

for $n \geq 3$.

The class w_2 is a permanent cycle, which converges to $g_0 = \langle h_{10}, h_{10}, h_{11} \rangle \in b_{10}^{-1}\,\mathrm{Ext}_P^{*,*}(\mathbb{F}_p, \mathbb{F}_p)$. For $n \geq 3$, w_n is not a permanent cycle, but is represented in the E_1 page by $\langle h_{10}, h_{10}, h_{n-1,1} \rangle$. Theorem 1.2 completely describes the d_4 differentials. In Proposition 7.1 we give a complete description of the d_8's. We conjecture that the spectral sequence collapses at E_9; computer calculations using the software [13] verify that this is true for stems ≤ 700. See Figures 2 and 3 for a picture of the $E_2 = E_4$,

E_8, and E_∞ pages of this spectral sequence. The charts are drawn in the (u', s) grading, defined in the introduction to Section 2 so that b_{10} has degree $(0,0)$ and the y–axis represents filtration in the spectral sequence. In particular, each dot represents a b_{10} tower in Figure 1, and b_{10} multiplication goes into the page. The class h_{10} is abbreviated as h.

Conjecture 1.3 There is an isomorphism

$$b_{10}^{-1}\operatorname{Ext}_P^{*,*}(\mathbb{F}_3, \mathbb{F}_3) \cong b_{10}^{-1}\operatorname{Ext}_D^{*,*}(\mathbb{F}_3, \widetilde{W}),$$

where $\widetilde{W} = \mathbb{F}_3[\widetilde{w}_2, \widetilde{w}_3, \dots]$ with $|\widetilde{w}_n| = 2(3^n - 5)$ and coaction given by

$$\psi(\widetilde{w}_n) = 1 \otimes \widetilde{w}_n + \xi_1 \otimes \widetilde{w}_2^2 \widetilde{w}_{n-1}^3 \text{ for } n \geq 3.$$

(These generators are related to the generators of Theorem 1.2 by $\widetilde{w}_n = b_{10}^{-1} w_n$.)

Adams' theorem (1-1) has the more general form

$$a_0^{-1}\operatorname{Ext}_A^{*,*}(\mathbb{F}_p, M) \cong a_0^{-1}\operatorname{Ext}_{E[\tau_0]}^{*,*}(\mathbb{F}_p, M)$$

for an A–comodule M (see also May and Milgram [9]). In particular, the localized cohomology depends only on the $E[\tau_0]$–comodule structure on M. The analogous statement for b_{10}–localization (that $b_{10}^{-1}\operatorname{Ext}_P^{*,*}(\mathbb{F}_p, M)$ depends only on the D–comodule structure of M) is not true. In general, we propose the following:

Conjecture 1.4 At $p = 3$, there is a functor $\mathscr{E}$: $\mathrm{Comod}_P \to \mathrm{Comod}_D$ such that

$$b_{10}^{-1}\operatorname{Ext}_P^{*,*}(\mathbb{F}_3, M) \cong b_{10}^{-1}\operatorname{Ext}_D^{*,*}(\mathbb{F}_3, \mathscr{E}(M))$$

and, as vector spaces, $R \otimes \mathscr{E}(M)$ agrees with the E_2 page of the spectral sequence described below in Theorem 1.6 with $\Gamma = P$.

Our best complete result is the following; it is proved in Section 8 using different methods.

Theorem 1.5 *There is an isomorphism*

$$b_{10}^{-1}\operatorname{Ext}_P^{*,*}(\mathbb{F}_3, \mathbb{F}_3[\xi_1^3]) \cong b_{10}^{-1}\operatorname{Ext}_D^{*,*}(\mathbb{F}_3, \mathbb{F}_3[h_{20}, b_{20}, w_3, w_4, \dots]/h_{20}^2),$$

where D acts trivially on all the generators on the right.

1.1 Main tool

Our main tool (the spectral sequence mentioned in Theorem 1.2) is as follows. It is a special case of the construction discussed in Belmont [3].

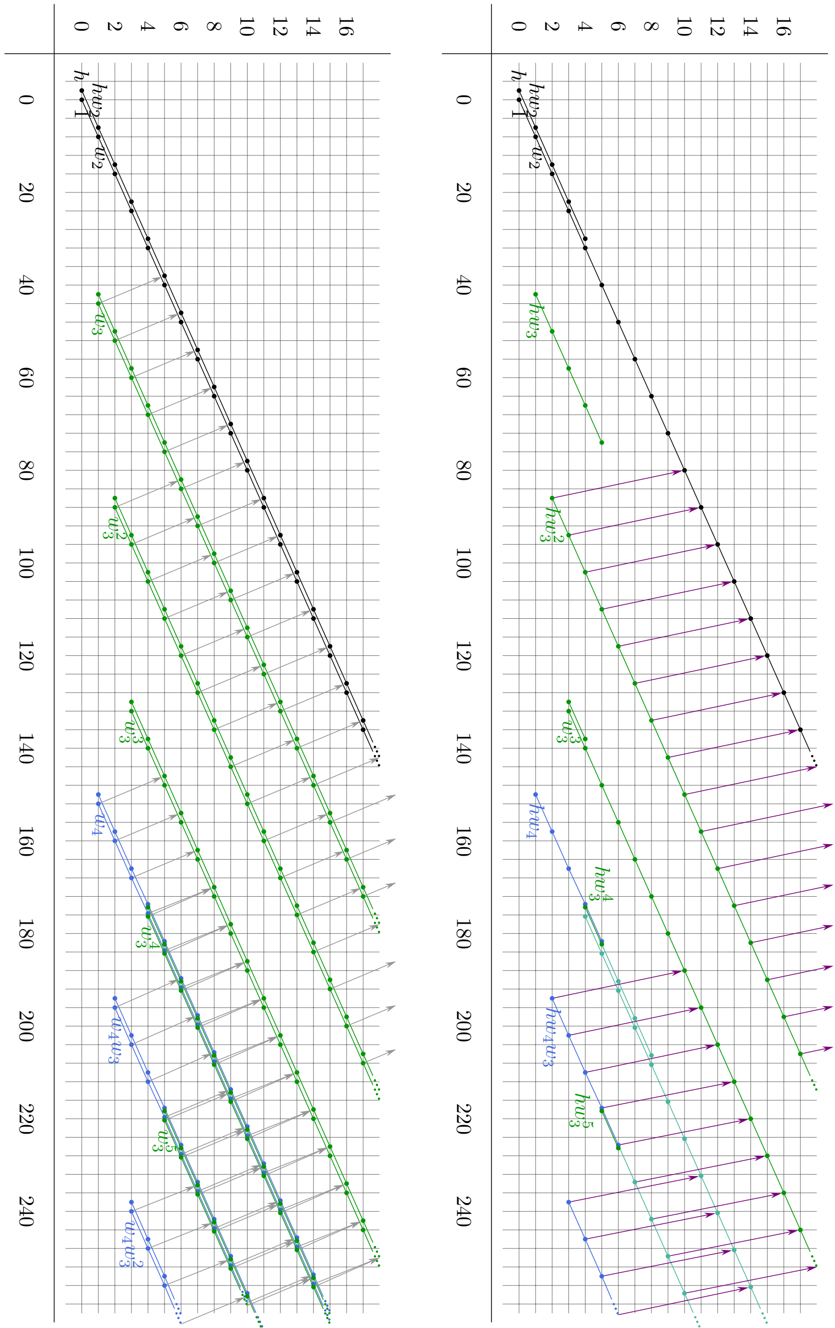

Figure 2: Left: E_4 page of the $K(\xi_1)$–based MPASS, with d_4 differentials shown. Right: E_8 page of the $K(\xi_1)$–based MPASS. The grading is (u', s) (see the introduction to Section 2 for details).

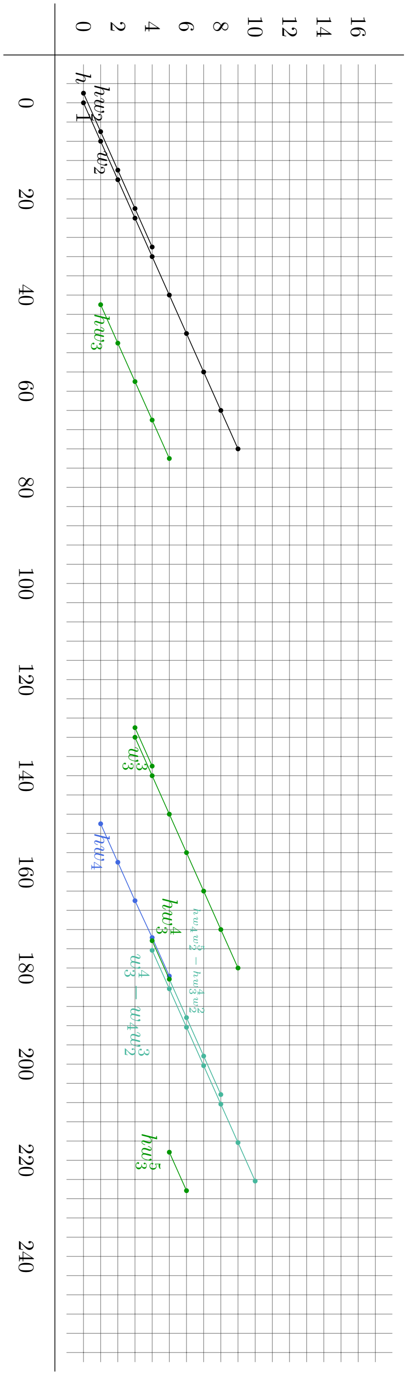

Figure 3: E_∞ page of the $K(\xi_1)$–based MPASS.

Theorem 1.6 *Let $D = \mathbb{F}_p[\xi_1]/\xi_1^p$ and let Γ be a Hopf algebra over $\mathbb{F}_p$ with a surjection of Hopf algebras $\Gamma \to D$. Let $B_\Gamma = \Gamma \,\square_D\, \mathbb{F}_p$. For a Γ–comodule M, there is a spectral sequence*

$$E_1^{s,t} \cong b_{10}^{-1}\operatorname{Ext}_D^{t,*}(\mathbb{F}_p, \bar{B}_\Gamma^{\otimes s} \otimes M) \Rightarrow b_{10}^{-1}\operatorname{Ext}_\Gamma^{*,*}(\mathbb{F}_p, M) \tag{1-3}$$

(*where $\bar{B}_\Gamma$ is the coaugmentation ideal* $\operatorname{coker}(\mathbb{F}_p \to B_\Gamma)$).

At $p = 3$, $b_{10}^{-1}\operatorname{Ext}_D^{*,*}(\mathbb{F}_p, B_\Gamma)$ is flat as a $b_{10}^{-1}\operatorname{Ext}_D^{*,*}(\mathbb{F}_p, \mathbb{F}_p)$–module, and

$$E_2^{**} \cong b_{10}^{-1}\operatorname{Ext}^{*,*}_{b_{10}^{-1}\operatorname{Ext}_D^{*,*}(\mathbb{F}_p, B_\Gamma)}(R, b_{10}^{-1}\operatorname{Ext}_D^{*,*}(\mathbb{F}_p, M)). \tag{1-4}$$

We work at $p = 3$ throughout. The main focus is the case $\Gamma = P$ and $B_P = P \,\square_D\, \mathbb{F}_3 =: B$; this is the spectral sequence of Theorem 1.2. We also apply this for two quotients of P — for a spectral sequence comparison argument in Section 6 and for the proof of Theorem 1.5 in Section 8. Convergence is proved in the appendix in the case that Γ is a quotient of P.

In [3], we show that the following three constructions of (1-3) coincide at the E_1 page:

(1) The first construction is a b_{10}–localized Cartan–Eilenberg-type spectral sequence associated to the sequence of P–comodule algebras $B \to P \to D$. (Note that the inclusion $B \to P$ is not a map of coalgebras; see [3, Section 2.3] for a precise construction in this case.)

(2) The second construction is an Adams spectral sequence internal to the category $\mathrm{Stable}(P)$. See Margolis [7, Chapter 14] or Hovey, Palmieri and Strickland [4, Section 9.6] for a definition of $\mathrm{Stable}(\Gamma)$ for a Hopf algebra Γ over $\mathbb{F}_p$, or Barthel, Heard and Valenzuela [2, Section 4] for a more modern viewpoint; the idea is that it is a variation of the derived category of Γ–comodules designed to satisfy $\operatorname{Hom}_{\mathrm{Stable}(\Gamma)}(\mathbb{F}_p, x^{-1}M) = x^{-1}\operatorname{Hom}_{\mathrm{Stable}(\Gamma)}(\mathbb{F}_p, M)$. In particular, if M is a Γ–comodule, then $\operatorname{Hom}_{\mathrm{Stable}(\Gamma)}(\mathbb{F}_p, M) = \operatorname{Ext}_\Gamma^{*,*}(\mathbb{F}_p, M)$. The Adams spectral sequence in this setting was first studied by Margolis [7] and Palmieri [14], and so we call this the *Margolis–Palmieri Adams spectral sequence* (MPASS).

In particular, let $K(\xi_1) := b_{10}^{-1}B = \operatorname{colim}(B \xrightarrow{b_{10}} B \xrightarrow{b_{10}} \cdots)$ (where the colimit is taken in $\mathrm{Stable}(P)$); then our spectral sequence is the $K(\xi_1)$–based Adams spectral sequence.

(3) The third construction is obtained by b_{10}–localizing the filtration spectral sequence on the normalized P–cobar complex $C_P^*(\mathbb{F}_p, \mathbb{F}_p) := \bar{P}^{\otimes *}$ in which

$[a_1 \,|\, \cdots \,|\, a_n] \in F^s C_P^*$ if at least s of the a_i lie in $\ker(P \to D) = \bar{B}P$. (Here $\bar{B}$ denotes the augmentation ideal of B.)

Our dominant viewpoint will be via the framework of (2), but the other two formulations will be useful at key moments. By a "b_{10}–localized" spectral sequence, we mean the spectral sequence whose E_r page is obtained by b_{10}–localizing the original E_r page. It is not automatic that this converges to the b_{10}–localization of the original spectral sequence; this is what is checked in the appendix.

Remark 1.7 The essential reason we focus on $p = 3$ is that for the analogous construction at $p > 3$, the flatness condition does not hold. (This comes from the Adams spectral sequence flatness condition applied in the setting of (2).)

Outline

In Section 2, we prove some basic results about the structure of the spectral sequence converging to $b_{10}^{-1} \operatorname{Ext}_P^{*,*}(\mathbb{F}_3, \mathbb{F}_3)$ and introduce definitions and notation that will be used extensively in the computational sections. In Section 3, we apply vanishing line results to describe a region in which $b_{10}^{-1} \operatorname{Ext}_P^{*,*}(\mathbb{F}_p, M)$ agrees with $\operatorname{Ext}_P^{*,*}(\mathbb{F}_p, M)$ for an arbitrary odd prime p. Sections 4 and 5 are devoted to computing the E_2 page of the $K(\xi_1)$–based MPASS at $p = 3$ converging to $b_{10}^{-1} \operatorname{Ext}_P^{*,*}(\mathbb{F}_3, \mathbb{F}_3)$. In Section 6 we determine d_4, the first nontrivial differential after the E_2 page. In Section 7, we determine d_8 and show that our conjecture that the spectral sequence collapses at E_9 would imply the desired form of $b_{10}^{-1} \operatorname{Ext}_P^{*,*}(\mathbb{F}_3, \mathbb{F}_3)$ in Conjecture 1.3. In Section 8 we prove Theorem 1.5. In the appendix we show convergence of the MPASS in the cases of interest, and also show convergence of an auxiliary spectral sequence needed for Section 6.

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2 Overview of the MPASS converging to $b_{10}^{-1} \operatorname{Ext}_P^{*,*}(\mathbb{F}_3, \mathbb{F}_3)$

In every section except Sections 3 and 4 we will set $p = 3$ and let $k = \mathbb{F}_3$. We will denote exterior and truncated polynomial algebras, respectively, by $E[x] = k[x]/x^2$ and $D[x] = k[x]/x^p$. Let $D = D[\xi_1]$.

If M is a P–comodule and $E = b_{10}^{-1}M$, we adopt the notation of [14] and write

$$\begin{aligned}\pi_{**}(M) := M_{**} &:= \operatorname{Hom}^{**}_{\mathrm{Stable}(P)}(k, M) = \operatorname{Ext}_P^{*,*}(k, M),\\ M_{**}M &:= \operatorname{Hom}^{**}_{\mathrm{Stable}(P)}(k, M \otimes M) = \operatorname{Ext}_P^{*,*}(k, M \otimes M),\\ \pi_{**}(E) := E_{**} &:= \operatorname{Hom}^{**}_{\mathrm{Stable}(P)}(k, E) = b_{10}^{-1}\operatorname{Ext}_P^{*,*}(k, M),\\ E_{**}E &:= \operatorname{Hom}^{**}_{\mathrm{Stable}(P)}(k, E \otimes E) = b_{10}^{-1}\operatorname{Ext}_P^{*,*}(k, M \otimes M).\end{aligned}$$

Here $M \otimes M$ is given the diagonal P–comodule structure $\psi(a \otimes b) = \sum a'b' \otimes a'' \otimes b''$, where $\psi(a) = \sum a' \otimes a''$ and $\psi(b) = \sum b' \otimes b''$. Define

$$B := P \,\square_D\, k, \quad K(\xi_1) := b_{10}^{-1}B := \operatorname{colim}(B \xrightarrow{b_{10}} B \xrightarrow{b_{10}} \cdots),$$

where the colimit is taken in $\mathrm{Stable}(P)$. Due to the general machinery of Adams spectral sequences in $\mathrm{Stable}(P)$ (see [14, Section 1.4]), we have a $K(\xi_1)$–based spectral sequence

$$E_1^{s,t,u} = \pi_{t,u}(K(\xi_1) \otimes \overline{K(\xi_1)}^{\otimes s}) = b_{10}^{-1}\operatorname{Ext}_P^{*,*}(k, B \otimes \bar{B}^{\otimes s}) \Rightarrow b_{10}^{-1}\operatorname{Ext}_P^{*,*}(k,k),$$

which we call the $K(\xi_1)$*–based Margolis–Palmieri Adams spectral sequence* (MPASS). Here $\overline{(\cdot)}$ denotes the coaugmentation ideal. By the shear isomorphism (Lemma 5.16) and the change-of-rings theorem, we may write $E_1^{s,t,u} = b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k, \bar{B}^{\otimes s})$. If $K(\xi_1)_{**}K(\xi_1)$ is flat over $K(\xi_1)_{**}$, then the E_2 page (2-1) has the form

$$\operatorname{Ext}^{*,*,*}_{K(\xi_1)_{**}K(\xi_1)}(K(\xi_1)_{**}, K(\xi_1)_{**}). \tag{2-1}$$

The differential d_r is a map $E_r^{s,t,u} \to E_r^{s+r,t-r+1,u}$. Here s is the MPASS filtration, t is the internal homological degree and u is the internal topological degree. Furthermore, we will often find it convenient to work with the degree

$$u' := u - 6(s+t),$$

which has the property that $u'(b_{10}) = 0$. In this grading, the differential d_r is a map $E_r^{s,u'} \to E_r^{s+r,u'-6}$.

The coefficient ring $K(\xi_1)_{**}$ is easy to compute using the change-of-rings theorem:

$$\begin{aligned}K(\xi_1)_{**} &= b_{10}^{-1}\operatorname{Ext}_P^{*,*}(k, B) = b_{10}^{-1}\operatorname{Ext}_P^{*,*}(k, P \,\square_D\, k)\\ &= b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k,k) = E[h_{10}] \otimes k[b_{10}^{\pm 1}],\end{aligned}$$

where h_{10} is in homological degree 1 and b_{10} is in homological degree 2. It will be useful to have notation for this coefficient ring:

$$R := E[h_{10}] \otimes k[b_{10}^{\pm}]. \tag{2-2}$$

Using the shear isomorphism (Lemma 5.16) and the change-of-rings theorem, we have

$$K(\xi_1)_{**}K(\xi_1) \cong b_{10}^{-1}\operatorname{Ext}_P^{*,*}(k, B\otimes B) \cong b_{10}^{-1}\operatorname{Ext}_P^{*,*}(k, P\,\square_D\, B) \cong b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k, B). \tag{2-3}$$

Notation 2.1 We have chosen to define B as a left P–comodule. It can be written explicitly as $\mathbb{F}_p[\overline{\xi}_1^p, \overline{\xi}_2, \overline{\xi}_3, \dots]$. To simplify the notation, from now on we will redefine the symbol ξ_n to mean the antipode of the usual ξ_n. Thus, going forward, we will have $\Delta(\xi_n) = \sum_{i+j=n} \xi_i \otimes \xi_j^{p^i}$, and

$$B = \mathbb{F}_p[\xi_1^p, \xi_2, \xi_3, \dots].$$

In Section 5, we will show that the flatness condition holds and $K(\xi_1)_{**}K(\xi_1)$ is isomorphic, as a Hopf algebra over R, to an exterior algebra on generators

$$e_n = [\xi_1]\xi_n - [\xi_1^2]\xi_{n-1}^3 \in \operatorname{Ext}_D^{*,*}(k, B).$$

This implies that the E_2 page is isomorphic to a polynomial algebra over R on classes $w_n := [e_n]$ of degree $(s,t,u) = (1, 1, 2(3^n+1))$.

The generator w_2 is a permanent cycle, and converges to $g_0 = \langle h_{10}, h_{10}, h_{11}\rangle \in \operatorname{Ext}_P^{*,*}(k,k)$. We will see in Section 6 that the other w_n support differentials, so it is less easy to see how these generators connect to familiar elements in the Adams E_2 page. One heuristic comes from looking at the images of these classes in $P/(\xi_1^3, \xi_2^9, \xi_3^9, \dots)$: in that setting, $w_n = \langle h_{10}, h_{10}, h_{n-1,1}\rangle$ and $h_{10}w_n = b_{10}h_{n-1,1}$.

Let $W_+ = k[b_{10}^{\pm 1}][w_2, w_3, \dots]$ and $W_- = W_+\{h_{10}\}$. Then $E_2 = W_+ \oplus W_-$, and, using simple degree arguments, we will show that higher differentials take W_+ to W_- and vice versa.

Lemma 2.2 *Suppose $x \in E_2^{s(x),u'(x)}$ is nonzero. If $u'(x) \equiv 0 \pmod 4$, then $x \in W_+$ and $s \equiv -u' \pmod 9$. Otherwise, $u'(x) \equiv 2 \pmod 4$, in which case $x \in W_-$ and $s \equiv 7 - u' \pmod 9$.*

Proof This can be read off the following table of degrees:

element	s	u'	t
h_{10}	0	-2	1
b_{10}	0	0	2
w_n	1	$2(3^n-5)$	1

□

Proposition 2.3 *If $r \geq 2$ with $r \not\equiv 4 \pmod 9$ and $r \not\equiv 8 \pmod 9$, then $d_r = 0$. Furthermore,*

$$d_{4+9n}(W_+) \subseteq W_-, \quad d_{4+9n}(W_-) = 0, \qquad d_{8+9n}(W_-) \subseteq W_+, \quad d_{8+9n}(W_+) = 0.$$

Proof This is a degree argument, so we simplify to considering $d_r(x)$ where x is a monomial. First notice that $s(d_r(x)) + t(d_r(x)) = s(x) + t(x) + 1$. If $x \in W_+$, then $s+t$ is even; if $x \in W_-$, then $s+t$ is odd. Thus $d_r(W_+) \subseteq W_-$ and $d_r(W_-) \subseteq W_+$.

If $x \in W_+^{s,u'}$, then $d_r(x) \in W_-^{s+r,u'-6}$. If $d_r(x) \neq 0$, Lemma 2.2 implies $s+u' \equiv 0 \pmod 9$ and $s+r+u'-6 \equiv 7 \pmod 9$, so $r \equiv 4 \pmod 9$. Similarly, if $x \in W_-^{s,u'}$, then $d_r(x) \in W_+^{s+r,u'-6}$, which implies $r \equiv 8 \pmod 9$ if $d_r(x) \neq 0$. □

In Section 7, we show that if $d_r(x) = h_{10}y$ is the first nontrivial differential on $x \in W_+$, and $d_4(y) = h_{10}z$, then $d_{r+4}(h_{10}x) = b_{10}z$. Combined with our complete calculation of d_4 in Section 6, this determines the spectral sequence through the E_9 page. We conjecture that the spectral sequence collapses at E_9. The idea is that there is an operator $\partial\colon W_+ \to W_+$ defined by $\partial(x) = \frac{1}{h_{10}} d_r(x)$, where $d_{r'}(x) = 0$ for $r < r'$, and that the spectral sequence essentially operates by taking Margolis homology of this operator: if $x \in E_2$ supports a nontrivial d_r, then $d_r(x) = h_{10}\partial(x)$, and $d_{r+4}(h_{10}x) = b_{10}\partial^2(x)$.

Remark 2.4 It is tempting to expect that Conjecture 1.3 comes from a map $k \to P \,\square_D\, \widetilde{W}$, which would induce a map $b_{10}^{-1}\operatorname{Ext}_P^{*,*}(k,k) \to b_{10}^{-1}\operatorname{Ext}_P^{*,*}(k, P\,\square_D\,\widetilde{W}) \cong b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k,\widetilde{W})$ by the change-of-rings theorem. However, this is not the case: $k \to P\,\square_D\,\widetilde{W}$ would factor through $P\,\square_D\,k$, which would mean that the map in $b_{10}^{-1}\operatorname{Ext}_P^{*,*}(k,-)$ would factor through $b_{10}^{-1}\operatorname{Ext}_P^{*,*}(k, P\,\square_D\,k) \cong R$.

3 Identifying the b_{10}–periodic region

In this section, let p be an odd prime and let $k = \mathbb{F}_p$. The following characterization of a b_{10}–periodic region in Ext is a consequence of results of Palmieri that generalize the vanishing line theorems of Miller and Wilkerson [11] to the stable category of comodules.

Proposition 3.1 *The localization map $\operatorname{Ext}_P^{s,t}(k,M) \to b_{10}^{-1}\operatorname{Ext}_P^{s,t}(k,M)$ is an isomorphism in the range $s > (1/(p^3-p-1))(t-s) + c'$ for some constant c'.*

Our main input is the following theorem, which Palmieri states for the Steenrod dual algebra A instead of the algebra P of dual reduced powers, as we do below. The necessary changes in the case of P follow immediately from the discussion in [14, Section 2.3.2].[1]

Following Palmieri [14, Notation 2.2.8], define the *slope* of $\xi_t^{p^s}$ to be

$$s(\xi_t^{p^s}) = \tfrac{1}{2}p|\xi_t^{p^s}| = p^{s+1}(p^t-1).$$

Let $D[x] = k[x]/x^p$. We have $\mathrm{Ext}^{*,*}_{D[\xi_t^{p^s}]}(k,k) = E[h_{ts}] \otimes k[b_{ts}]$. Let $K(\xi_t^{p^s}) = b_{ts}^{-1}(P \,\Box_{D[\xi_t^{p^s}]} k)$, where the localization is defined by taking a colimit of multiplication by b_{ts} in $\mathrm{Stable}(P)$.

Theorem 3.2 [14, Theorem 2.3.1] *Suppose X is an object in* $\mathrm{Stable}(P)$ *satisfying the following conditions*:

(1) *There exists an integer i_0 such that $\pi_{i,*}X = 0$ if $i < i_0$,*

(2) *There exists an integer j_0 such that $\pi_{i,j}X = 0$ if $j - i < j_0$,*

(3) *There exists an integer i_1 such that the homology of the cochain complex X vanishes in homological degree $> i_1$. (In particular, this is satisfied if X is the resolution of a bounded-below comodule.)*

Suppose $d = s(\xi_{t_0}^{p^{s_0}})$ (with $s_0 < t_0$) has the property that $K(\xi_t^{p^s})_{**}(X) = 0$ for all (s,t) with $s < t$ and $s(\xi_t^{p^s}) < d$. Then $\pi_{**}X$ has a vanishing line of slope d: for some c, $\pi_{i,j}X = 0$ when $j < di - c$.

Proof of Proposition 3.1 Let M/b_{10} denote the cofiber in $\mathrm{Stable}(P)$ of $b_{10} \in \mathrm{Ext}_P^{2,12}(k,k)$, thought of as a map $k \to k$ in $\mathrm{Stable}(P)$. It is not hard to check the conditions (1)–(3) of Theorem 3.2 for M/b_{10}. We will apply the theorem with $d = s(\xi_2) = p^3 - p$; note that ξ_2 is the next $\xi_t^{p^s}$ with $s < t$ and higher slope than ξ_1, so we just have to check $K(\xi_1)_{**}(M/b_{10}) = 0$. This follows because the cofiber sequence

$$M \xrightarrow{b_{10}} M[2] \to M/b_{10}[2] \tag{3-1}$$

gives rise to a long exact sequence in $K(\xi_1)_{**}$, and multiplication by b_{10} is an isomorphism on $K(\xi_1)_{**}(M)$ by construction. So the theorem implies that there exists some c such that $\pi_{s,t}(M/b_{10}) = 0$ when $t < (p^3-p)s - c$.

[1] The only difference is that, over A, one must also take into account the objects $Z(n)$ corresponding to the τ_n as opposed to the $\xi_t^{p^s}$, which do not come into play over P.

Applying Ext to (3-1), we obtain

$$\mathrm{Ext}_P^{s+1,t+|b_{10}|}(k, M/b_{10}) \to \mathrm{Ext}_P^{s,t}(k,k) \to \mathrm{Ext}_P^{s+2,t+|b_{10}|}(k,k) \\ \to \mathrm{Ext}_P^{s+2,t+|b_{10}|}(k, M/b_{10}),$$

where $|b_{10}| = 2p(p-1)$. Applying the vanishing condition for M/b_{10} directly gives a region in which multiplication by b_{10} is an isomorphism. □

In particular, at $p = 3$, $b_{10}^{-1}\,\mathrm{Ext}_P^{*,*}(k,k)$ agrees with $\mathrm{Ext}_P^{*,*}(k,k)$ above a line of slope $\frac{1}{23}$ in the $(s, t-s)$ grading (see Figure 1). In [14, Remark 2.3.5(c)], Palmieri gives an explicit expression for the constant, which allows us to calculate the y–intercept to be $c' \approx 6.39$.

4 R–module structure of $K(\xi_1)_{**}K(\xi_1)$ at $p > 2$

In this section, we work at an arbitrary odd prime, and let $k = \mathbb{F}_p$ and $D = \mathbb{F}_p[\xi_1]/\xi_1^p$. In preparation for studying the E_2 page $\mathrm{Ext}^{*,*,*}_{K(\xi_1)_{**}K(\xi_1)}(R, R)$, our goal for the next two sections is to study the Hopf algebra $K(\xi_1)_{**}K(\xi_1)$, which in (2-3) we showed is isomorphic to $b_{10}^{-1}\,\mathrm{Ext}_D^{*,*}(k, B)$. Most of this section is devoted to giving an expression for B as a D–comodule. In the next section, we will obtain a more explicit description at $p = 3$, given which we calculate the E_2 page.

4.1 D–comodule structure of B

Note that B is an algebra and a P–comodule, but not a coalgebra. Let ψ denote the D–coaction $B \to D \otimes B$ that comes from composing the P–coaction $B \to P \otimes B$ with the surjection $P \to D$.

Definition 4.1 If we write

$$\psi(x) = 1 \otimes x + \xi_1 \otimes a_1 + \xi_1^2 \otimes a_2 + \cdots + \xi_1^{p-1} \otimes a_{p-1}$$

for some a_i, define

$$\partial(x) := a_1.$$

For example, since $\Delta(\xi_n) = 1 \otimes \xi_n + \xi_1 \otimes \xi_{n-1}^p + \cdots$ (using the convention of Notation 2.1), we have $\partial(\xi_n) = \xi_{n-1}^p$, and $\partial(\xi_{n-1}^p) = 0$. One can show using coassociativity that $a_k = \frac{1}{k!}\partial^{k-1}a_1$. As ξ_1 is dual to P_1^0 in the Steenrod algebra, the operator $\partial\colon P \to P$ is dual to the operator on the reduced powers subalgebra of the Steenrod algebra given by left P_1^0–multiplication. In particular, $(P_1^0)^p = 0$ implies $\partial^p = 0$.

Lemma 4.2 *$\mathrm{Ext}_D^{*,*}(k,M)$ is the cohomology of the chain complex*

$$0 \to M \xrightarrow{\partial} M \xrightarrow{\partial^2} M \xrightarrow{\partial} M \to \cdots,$$

and $b_{10}^{-1}\,\mathrm{Ext}_D^{*,*}(k,M)$ is the cohomology of the unbounded chain complex

$$\cdots \to M \xrightarrow{\partial} M \xrightarrow{\partial^2} M \xrightarrow{\partial} M \to \cdots.$$

Lemma 4.3 *We have $\partial(xy) = \partial(x)y + x\partial(y)$.*

Proof We have

$$\begin{aligned}\Delta(xy) = \Delta(x)\Delta(y) &= (1\otimes x + \xi_1\otimes\partial x + \cdots)(1\otimes y + \xi_1 \otimes \partial y + \cdots)\\ &= 1\otimes xy + \xi_1\otimes(y\partial x + x\partial y) + \cdots. \end{aligned}$$ □

The structure theorem for modules over a PID says that modules over $D^\vee \cong D$ decompose as sums of modules isomorphic to $\mathbb{F}_p[\xi_1]/\xi_1^i$ for $1 \le i \le p$. Dually, we have the following:

Lemma 4.4 *Let $M(n)$ denote the D–comodule $\mathbb{F}_p[\xi_1]/\xi_1^{n+1}$. Every D–comodule splits uniquely as a direct sum of D–comodules isomorphic to $M(n)$ for $n \le p-1$.*

Note that $M(0) \cong \mathbb{F}_p$ and $M(p-1) \cong D$.

Remark 4.5 Since $\mathrm{Ext}_D^{*,*}(k,D)$ is a 1–dimensional vector space in homological degree 0 and zero otherwise, $b_{10}^{-1}\,\mathrm{Ext}_D^{*,*}(k,F) = 0$ for any free D–comodule F. If $0 \le i \le p-2$, then $\mathrm{Ext}_D^{*,*}(k, M(i))$ is 1–dimensional in every homological degree.

The goal is to prove the following proposition:

Proposition 4.6 *Define the indexing set $\mathscr{B}$ to be the set of monomials of the form $\prod_{j=1}^n \xi_{i_j}^{e_j}$ such that $1 \le e_j \le p-2$, and for $X \in \mathscr{B}$, write $x_j(X) := \xi_{i_j}^{e_j}$ and $e_j(X) := e_j$. Then there is a D–comodule isomorphism*

$$B \cong \bigoplus_{X\in\mathscr{B}} \bigotimes_{j=1}^{n} M(e_j(X))_{x_j(X)} \oplus F,$$

where F is a free D–comodule, the tensor product is endowed with the diagonal D–comodule structure, and $M(e)_{\xi_i^e} := \mathbb{F}_p\{\xi_i^e, \partial\xi_i^e, \dots, \partial^e\xi_i^e\} \cong M(e)$.

Corollary 4.7 *We have an R–module isomorphism*

$$b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k,B) \cong b_{10}^{-1}\operatorname{Ext}_D^{*,*}\left(k, \bigoplus_{X\in\mathscr{B}}\bigotimes_{j=1}^{n} M(e_j(X))_{x_j(X)}\right). \tag{4-1}$$

Remark 4.8 There is a formula due to Renaud [15, Theorem 1] that allows one to decompose the tensor products $\bigotimes M(e_i)$ into a sum of the basic comodules $M(n)$, but in general it is rather complicated; instead we will do this in the next section only at $p=3$.

If $e \le p-1$ then $M(e)_{\xi_n^e}$ is a sub-D–comodule of B with dimension $e+1$. By the Leibniz rule (Lemma 4.3) we have

$$M(e+pf)_{\xi_n^{e+pf}} = \mathbb{F}_p\{\xi_n^e\xi_n^{pf}, \partial(\xi_n^e)\xi_n^{pf}, \dots, \partial^e(\xi_n^e)\xi_n^{pf}\} = M(e)_{\xi_n^e}\otimes\mathbb{F}_p\{\xi_n^{pf}\}$$

for $e \le p-1$. For any collection of $e_i \in \mathbb{N}$, define

$$T(\xi_{n_1}^{e_1}\cdots\xi_{n_d}^{e_d}) := M(e_1)_{\xi_{n_1}^{e_1}} \otimes\cdots\otimes M(e_d)_{\xi_{n_d}^{e_d}}. \tag{4-2}$$

This is a sub-D–comodule spanned (as a vector space) by monomials of the form $\partial^{k_1}(\xi_{n_1}^{e_1})\cdots\partial^{k_d}(\xi_{n_d}^{e_d})$. Clearly, $B = \sum_{\text{monomials }\prod\xi_{n_i}^{e_i}\in B} T(\xi_{n_1}^{e_1}\cdots\xi_{n_d}^{e_d})$, but this is not a direct sum decomposition — any given monomial appears in many different summands. To fix this, we will study the poset of $T(X)$'s, and find that B is a direct sum of the maximal elements of that poset.

Notation 4.9 Define

$$\left\langle\prod_{i\ge1}\xi_i^{e_i};\prod_{i\ge2}\xi_i^{f_i}\right\rangle := \prod\xi_i^{e_i}\prod\xi_{i-1}^{pf_i}.$$

(These are not formal products; they only make sense if $e_i = 0 = f_i$ for all but finitely many i.) For example, we have $\langle X;1\rangle = X$ for any monomial X, and $\langle 1;\xi_n\rangle = \xi_{n-1}^p = \partial(\xi_n)$. Expressions $\left\langle\prod_{i\ge2}\xi_i^{e_i};\prod_{i\ge2}\xi_i^{f_i}\right\rangle$ represent elements of $B\subset P$, and conversely every element of B has a representation of this form (note that $\xi_1^p = \langle 1;\xi_2\rangle$). Monomials in B do not have unique expressions of the form $\langle X;Y\rangle$: for example, $\langle\xi_{n-1}^p;1\rangle = \langle 1;\xi_n\rangle$.

Lemma 4.10 *There is a bijection*

$$\{\text{monomials in } B\} \leftrightarrow \left\{\left\langle\prod_{i\ge2}\xi_i^{e_i};\prod_{i\ge2}\xi_i^{f_i}\right\rangle : e_i \le p-1\right\}. \tag{4-3}$$

Say that a bracket expression is *admissible* if it is of the form on the right-hand side.

Proof Given a monomial, the admissible bracket expression is the one with the greatest number of terms on the right-hand side. $\square$

Lemma 4.11 *If X is a monomial with admissible bracket expression $\langle \prod \xi_i^{e_i}; \prod \xi_i^{f_i} \rangle$ and Y is a monomial in $T(X)$, then Y (up to invertible scalar) has admissible expression $\langle \prod \xi_i^{e_i - c_i}; \prod \xi_i^{f_i + c_i} \rangle$ for a set of $c_i \geq 0$ that are zero for all but finitely many i.*

The idea is that Y is obtained from X by moving terms from the left to the right.

Proof If $e \leq p-1$ then we have

$$\partial^i(\xi_n^e) = \frac{e!}{(e-i)!} \xi_n^{e-i} \xi_{n-1}^{pi}.$$

By definition, $X = \prod_{i \geq 1} \xi_i^{e_i + pf_{i+1}}$, where $e_1 = 0$, and

$$Y = \prod \partial^{k_i} \xi_i^{e_i + pf_{i+1}} = \prod (\partial^{k_i} \xi_i^{e_i}) \xi_i^{pf_{i+1}} = \prod \frac{e_i!}{(e_i - k_i)!} \xi_i^{e_i - k_i + pk_{i+1}} \xi_i^{pf_{i+1}}$$
$$= \left\langle \prod \frac{e_i!}{(e_i - k_i)!} \xi_i^{e_i - k_i}; \prod \xi_i^{k_i + f_i} \right\rangle$$

using the fact that $\partial \xi_i^p = 0$. So we can take $c_i = k_i$ in the lemma statement. $\square$

Definition 4.12 For monomials X and Y, write $X \geq Y$ if $Y \in T(X)$.

It is easy to check that this makes the set of monomials into a poset, and that $X \geq Y$ if and only if $T(X) \supseteq T(Y)$.

Lemma 4.13 *Suppose W is a monomial with the admissible bracket expression $\langle \prod \xi_i^{e_i}; \prod \xi_i^{f_i} \rangle$. Let $\widetilde{W} = \langle \prod \xi_i^{c_i}; \prod \xi_i^{d_i} \rangle$, where $c_i = \min\{e_i + f_i, p-1\}$ and $d_i = f_i - (c_i - e_i)$. Then $\widetilde{W}$ is the maximal object $\geq W$.*

Proof Let X be an arbitrary monomial, written in its unique admissible bracket expression. Then $X \geq W$ if and only if X can be obtained from W by moving terms in W from the right to the left side of the bracket expression. Note that $\widetilde{W}$ is the bracket expression obtained by moving as many terms to the left as possible while still keeping the resulting expression admissible. This implies $\widetilde{W}$ is maximal. $\square$

Define an equivalence relation on monomials where $X \sim Y$ if $\widetilde{X} = \widetilde{Y}$.

Lemma 4.14 *There is a direct sum decomposition* $B \cong \bigoplus_{\text{eq. class reps. } X} T(\widetilde{X})$.

Proof I claim that $T(\widetilde{X}) = \mathbb{F}_p\{Y : X \sim Y\}$; this follows from the fact that, by definition, $T(\widetilde{X})$ is generated by Y such that $Y \leq \widetilde{X}$. So the direct sum decomposition comes from partitioning monomials into their equivalence classes. □

Let $\mathscr{I}$ be the set of admissible bracket expressions X such that $\widetilde{X} = X$. By Lemma 4.13 we have the following:

Lemma 4.15 $\mathscr{I}$ *is the set of admissible bracket expressions* $\langle \prod \xi_i^{e_i}; \prod \xi_i^{f_i} \rangle$ *such that* $e_i \leq p-1$ *and if* $e_i < p-1$ *then* $f_i = 0$.

Lemma 4.16 *If* $X = \langle \prod \xi_i^{e_i}; \prod \xi_i^{f_i} \rangle$ *is an admissible expression, there is an isomorphism of* D*–comodules* $T(\langle \prod \xi_i^{e_i}; 1 \rangle) \cong T(X)$.

Proof By Lemma 4.11, every Y in $T(X)$ has a bracket expression obtained from X by moving terms from the left to the right, so the right-hand side of the bracket expression for Y is divisible by $\prod \xi_i^{f_i}$, and so Y is divisible by $u := \langle 1; \prod \xi_i^{f_i} \rangle = \prod \xi_{i-1}^{pf_i}$. So multiplication by u gives a map $T(\langle \prod \xi_i^{e_i}; 1 \rangle) \to T(X)$, and moreover from the above description of $Y \in T(X)$ it is easy to see that this is a bijection. Finally, since $\partial(u) = 0$, this is an isomorphism of D–comodules. □

Lemma 4.17 *If* $X = \langle \prod \xi_i^{e_i}; \prod \xi_i^{f_i} \rangle$ *is an admissible expression such that* $e_k = p-1$ *for some* k, *then* $T(X)$ *is a free* D*–comodule.*

Proof By definition, we have $T(X) = \bigotimes M(e_i)_{\xi_{n_i}^{e_i}}$, where the tensor product is endowed with the diagonal D–comodule structure and $M(e_k)_{\xi_{n_k}^{e_k}} \cong M(p-1) \cong D$ by assumption. After rearranging terms, it suffices to show that, for any D–comodule M, there is a D–comodule isomorphism $D \otimes M \to D \otimes M$ where the left-hand side has a diagonal D–coaction and the right-hand side has a left coaction $\big(\psi(d \otimes m) = \sum d'm' \otimes d'' \otimes m''$ vs $\psi(d \otimes m) = \sum d' \otimes d'' \otimes m$, where $\Delta(d) = \sum d' \otimes d''$ and $\psi(m) = \sum m' \otimes m''\big)$. This isomorphism is a variant of the shear isomorphism of Lemma 5.16, and is given by $d \otimes m \mapsto \sum dm' \otimes m''$. □

By Lemmas 4.16 and 4.17, we have:

Corollary 4.18 *If* $X = \langle \prod \xi_i^{e_i}; \prod \xi_i^{f_i} \rangle$ *is an admissible bracket expression in* $\mathscr{I}$ *such that* $f_i \neq 0$ *for any* i, *then* $T(X)$ *is free as a* D*–comodule.*

Proof of Proposition 4.6 From Lemma 4.14 we have $B \cong \bigoplus_{X \in \mathscr{I}} T(X)$, and by Corollary 4.18 there are free D–comodules F and F' such that

$$\begin{aligned} B &\cong \bigoplus_{\langle X;1\rangle \in \mathscr{I}} T(\langle X;1\rangle) \oplus F = \bigoplus_{\langle X;1\rangle \in \mathscr{I}} T(X) \oplus F \\ &\cong \bigoplus_{\substack{\langle X;1\rangle \text{ s.t.} \\ e_i(X) \le p-2}} T(X) \oplus F' = \bigoplus_{X \in \mathscr{B}} T(X) \oplus F' \\ &\cong \bigoplus_{X \in \mathscr{B}} \bigotimes_i M(e_i(X))_{x_i(X)} \oplus F'. \end{aligned}$$ □

We conclude this section with a useful lemma that simplifies checking relations in certain b_{10}–local Ext groups of interest.

Lemma 4.19 *Let $I(n) = (\xi_1^{pn}, \xi_2^{pn}, \dots)B$. Then $I(p-1)$ is contained in the free part of B according to the decomposition in Proposition 4.6. In particular, if $x \in \operatorname{Ext}_P^{*,*}(k, P \,\square_D\, I(p-1))$ then $x = 0$ in $b_{10}^{-1} \operatorname{Ext}_P^{*,*}(k, P \,\square_D\, B)$.*

Proof Consider an arbitrary monomial $q = \xi_n^{(p-1)p} X$ in $I(p-1)$. If X has an admissible expression $\langle \prod \xi_i^{e_i}; \prod \xi_i^{f_i} \rangle$ then q has an admissible expression $\langle \prod \xi_i^{e_i}; \xi_{n+1}^{p-1} \prod \xi_i^{f_i} \rangle$. By Lemmas 4.14 and 4.17, it suffices to show that $\widetilde{q} = \langle \prod \xi_i^{c_i}; \prod \xi_i^{d_i} \rangle$ satisfies $c_k = p-1$ for some k. Using the formula for $\widetilde{q}$ in Lemma 4.13, we have $c_{n+1} = p-1$. □

Corollary 4.20 *Let $I(n)$ be as in Lemma 4.19. If $x \in \operatorname{Ext}_P^{*,*}\big(k, P \,\square_D\, (P \,\square_D\, I(p-1))\big)$, then x is zero in $b_{10}^{-1} \operatorname{Ext}_P^{*,*}\big(k, P \,\square_D\, (P \,\square_D\, I(p-1))\big)$.*

5 Hopf algebra structure of $K(\xi_1)_{**}K(\xi_1)$ at $p = 3$

Henceforth we will work at $p = 3$. This assumption will allow us to simplify the formula for $K(\xi_1)_{**}K(\xi_1)$ obtained in Corollary 4.7 and show that $K(\xi_1)_{**}K(\xi_1)$ is flat over $K(\xi_1)_{**}$ (this is not true at higher primes), enabling us to calculate the E_2 page (2-1) of the $K(\xi_1)$–based MPASS. In particular, our goal is to show the following:

Theorem 5.1 *At $p = 3$, the ring of co-operations $K(\xi_1)_{**}K(\xi_1)$ is flat over $K(\xi_1)_{**}$, and moreover there is an isomorphism of Hopf algebras*

$$K(\xi_1)_{**}K(\xi_1) = R \otimes E[e_2, e_3, \dots]$$

for generators $e_n \in b_{10}^{-1} \operatorname{Ext}_D^{1,2(3^n+1)}(k, B)$. That is, e_n is primitive, and $K(\xi_1)_{**}K(\xi_1)$ is exterior as a Hopf algebra over $R = K(\xi_1)_{**}$.

Plugging this into the expression (2-1) for the E_2 page, we obtain:

Corollary 5.2 *The E_2 page of the $K(\xi_1)$–based MPASS for computing $\pi_{**}(b_{10}^{-1}k)$ is*

$$E_2^{**} \cong R[w_2, w_3, \dots],$$

where $w_n = [e_n]$.

Remark 5.3 As B is a P–comodule algebra, there is a Hopf algebroid $(B, B \otimes B)$ in $\mathrm{Stable}(P)$, where $B \otimes B$ carries the diagonal coaction of P (see Section 2) and the comultiplication is given by

$$B \otimes B \xrightarrow{-\otimes\eta\otimes-} B \otimes B \otimes B \cong (B \otimes B) \otimes_B (B \otimes B).$$

The Hopf algebroid above is given by applying $b_{10}^{-1}\pi_{**}(-) = b_{10}^{-1}\,\mathrm{Ext}_P^{*,*}(k,-)$ to this one.

5.1 Vector space structure of $K(\xi_1)_{**}K(\xi_1)$ at $p = 3$

In the $p = 3$ case, Corollary 4.7 reads

$$K(\xi_1)_{**}K(\xi_1) \cong b_{10}^{-1}\,\mathrm{Ext}_D^{*,*}(k, B) \cong b_{10}^{-1}\,\mathrm{Ext}_D^{*,*}\Bigg(k, \bigoplus_{\substack{\text{monomials}\\ \xi_{n_1}\cdots\xi_{n_d}\\ n_i \neq n_j}} \bigotimes_{i=1}^{d} M(1)_{\xi_{n_i}}\Bigg),$$

where the tensor product has a diagonal D–coaction. It is easy to see directly that $M(1) \otimes M(1) \cong D \oplus \Sigma^{0,|\xi_1|}k$. (Here we use bigraded notation for the shift for consistency with viewing these objects in $\mathrm{Stable}(D)$, so $\Sigma^{0,|\xi_1|}$ denotes a shift of 0 in the homological dimension and $|\xi_1|$ in internal degree). In particular,

$$k\{x, \partial x\} \otimes k\{y, \partial y\} \cong k\{xy, \partial(x)y + x\partial(y), \partial(x)\partial(y)\} \oplus k\{\partial(x)y - x\partial(y)\}.$$

After inverting b_{10}, free comodules become zero, and the only basic types of comodules are $M(0) = k$ and $M(1)$.

Lemma 5.4 *In $\mathrm{Stable}(D)$, we have an isomorphism*

$$b_{10}^{-1}M(1) \cong \Sigma^{-1,2|\xi_1|}b_{10}^{-1}M(0).$$

Proof A representative for $M(1)$ as an object of $\mathrm{Stable}(D)$ (ie an injective resolution for the comodule $M(1)$) is

$$0 \to D \xrightarrow{\partial^2} \Sigma^{2|\xi_1|}D \xrightarrow{\partial} \Sigma^{3|\xi_1|}D \xrightarrow{\partial^2} \Sigma^{5|\xi_1|}D \to \cdots,$$

where the degree shift in the terms of the complex denotes shift in internal degree. So $b_{10}^{-1}M(1) := \operatorname{colim}(M(1) \xrightarrow{b_{10}} \Sigma^{2,-|b_{10}|}M(1) \to \cdots)$ is represented by the injective resolution

$$\cdots \to \Sigma^{-|\xi_1|}D \xrightarrow{\partial} \Sigma^0 D \xrightarrow{\partial^2} \Sigma^{2|\xi_1|}D \xrightarrow{\partial} \Sigma^{3|\xi_1|}D \to \cdots$$

with $\Sigma^0 D$ in homological degree zero. Similarly, $b_{10}^{-1}M(0)$ is represented by

$$\cdots \to \Sigma^{-2|\xi_1|}D \xrightarrow{\partial^2} \Sigma^0 D \xrightarrow{\partial} \Sigma^{|\xi_1|}D \xrightarrow{\partial^2} \Sigma^{3|\xi_1|}D \to \cdots,$$

with $\Sigma^0 D$ in homological degree zero, and so there is a degree-preserving isomorphism $b_{10}^{-1}M(1) \to \Sigma^{-1,2|\xi_1|}b_{10}^{-1}M(0)$. $\square$

(At arbitrary primes, the formula $b_{10}^{-1}M(n) \cong \Sigma^{-1,(p-1)|\xi_1|}b_{10}^{-1}M(p-2-n)$ holds for the same reason.) Therefore, if M is a D–comodule, then $b_{10}^{-1}M \in \mathrm{Stable}(D)$ is a sum of shifts of the unit object $k \cong M(0)$. Remembering that $\mathrm{Stable}(D)$ was constructed so that $\mathrm{Hom}_{\mathrm{Stable}(D)}(k, b_{10}^{-1}M) = b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k, M)$, we obtain the following Künneth isomorphism:

Lemma 5.5 (Künneth isomorphism for $b_{10}^{-1}\operatorname{Ext}_D^{*,*}(\mathbb{F}_3, -)$) *If M and N are D–comodules, then*

$$b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k, M \otimes N) \cong b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k, M) \otimes b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k, N).$$

This only works at $p = 3$, and is the essential reason we have made the simplification of working at $p = 3$.

Applying this to (4-1) we have the following:

Corollary 5.6 *We have an isomorphism*

$$b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k, B) \cong \bigoplus_{\substack{\text{monomials}\\ \xi_{n_1}\cdots\xi_{n_d}}} b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k, \Sigma^{-d,2|\xi_1|}k_{\xi_{n_1}\cdots\xi_{n_d}}),$$

where $\Sigma^{-d,2d|\xi_1|}k_{\xi_{n_1}\cdots\xi_{n_d}}$ is the copy of $\Sigma^{-d,2d|\xi_1|}k$ isomorphic to $\bigotimes_{i=1}^d M(1)_{\xi_{n_i}}$ under Lemma 5.4. In particular, $K(\xi_1)_{**}K(\xi_1) = b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k, B)$ is free over $K(\xi_1)_{**} = b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k, k)$.

So $b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k, B)$ has R–module generators in bijection with monomials of the form $\xi_{n_1}\cdots\xi_{n_d}$ (where $n_i \neq n_j$ if $i \neq j$). Now we will be more precise in choosing these generators.

Lemma 5.7 *Suppose N is a D–comodule algebra with sub-D–comodules $k\{x, \partial x\} \cong M(1)$ and $k\{y, \partial y\} \cong M(1)$. Then:*

(1) *The image of $\mathrm{Ext}^{1,*}_D(k, k\{x, \partial x\})$ in $\mathrm{Ext}^{1,*}_D(k, N)$ is generated by*

$$e(x) := [\xi_1]x - [\xi_1^2]\partial x.$$

(2) *We have*

$$e(x) \cdot e(y) = b_{10}(y\partial x - x\partial y)$$

in the multiplication $\mathrm{Ext}^{*,*}_D(k, N) \otimes \mathrm{Ext}^{*,*}_D(k, N) \to \mathrm{Ext}^{*,*}_D(k, N)$ induced by the product structure on N. In particular, $e(x)^2 = 0$.

(3) *If the multiplication map embeds $k\{x, \partial x\} \otimes k\{y, \partial y\}$ in N injectively, then*

$$b_{10}^{-1} \mathrm{Ext}^{2,*}_D(k, k\{x, \partial x\} \otimes k\{y, \partial y\}) \subset b_{10}^{-1} \mathrm{Ext}^{2,*}_D(k, N)$$

is a 1–dimensional vector space with generator $e(x) \cdot e(y)$.

Since $\mathrm{Ext}^{i,*}_D(k, M) = b_{10}^{-1} \mathrm{Ext}^{i,*}_D(k, M)$ for $i > 0$, note that this also gives a generator of $b_{10}^{-1} \mathrm{Ext}^{1,*}_D(k, N)$.

Proof Since $\mathrm{Ext}^{1,*}_D(k, M(1))$ is a 1–dimensional k–vector space, for (1) it suffices to show that $e(x)$ is a cycle that is not a boundary. Indeed, since $dx = [\xi_1]\partial x$ and $d(\partial x) = 0$, we have $d(e(x)) = -[\xi_1 \mid \xi_1]\partial x + [\xi_1 \mid \xi_1]\partial x = 0$, and $e(x)$ is not in $d(C^0_D(k, k\{x, \partial x\})) = d(k\{x, \partial x\})$.

For (2), we use a special case of the cobar complex multiplication formula in [10, Proposition 1.2]:

Fact 5.8 *The multiplication $C^1_D(k, M) \otimes C^1_D(k, N) \to C^2_D(k, M \otimes N)$ is given by*

$$[\xi]m \otimes [\omega]n \mapsto \sum [\xi \otimes m'\omega](m'' \otimes n).$$

Thus the product $C^1_D(k, N) \otimes C^1_D(k, N) \to C^2_D(k, N \otimes N) \xrightarrow{\mu} C^2_D(k, N)$ takes $[\xi]m \otimes [\omega]n \mapsto \sum [\xi \otimes m'\omega]m''n$. Using this formula, we have

$$e(x) \cdot e(y) = [\xi_1|x] \cdot [\xi_1|y] - [\xi_1|x] \cdot [\xi_1^2|\partial y] - [\xi_1^2|\partial x] \cdot [\xi_1|y] + [\xi_1^2|\partial x] \cdot [\xi_1^2|\partial y],$$

$$[\xi_1|x] \cdot [\xi_1|y] = \sum [\xi_1|x'\xi_1]x''y = [\xi_1|\xi_1]xy + [\xi_1|\xi_1^2](\partial x)y,$$

$$[\xi_1|x] \cdot [\xi_1^2|\partial y] = \sum [\xi_1|x'\xi_1^2]x''\partial y = [\xi_1|\xi_1^2]x\partial y,$$

$$[\xi_1^2|\partial x] \cdot [\xi_1|y] = \sum [\xi_1^2|(\partial x)'\xi_1](\partial x)''y = [\xi_1^2|\xi_1](\partial x)y,$$

$$[\xi_1^2|\partial x] \cdot [\xi_1^2|\partial y] = \sum [\xi_1^2|\xi_1^2(\partial x)'](\partial x)''\partial y = [\xi_1^2|\xi_1^2]\partial x\partial y,$$

$$
\begin{aligned}
d([\xi_1^2]xy) &= 2[\xi_1|\xi_1]xy - [\xi_1^2|\xi_1](\partial x)y - [\xi_1^2|\xi_1]x\partial y - [\xi_1^2|\xi_1^2]\partial x\partial y,\\
e(x)\cdot e(y) + d([\xi_1^2]xy) &= [\xi_1|\xi_1^2](\partial x)y + [\xi_1^2|\xi_1](\partial x)y - [\xi_1|\xi_1^2]x\partial y - [\xi_1^2|\xi_1]x\partial y\\
&= b_{10}((\partial x)y - x\partial y).
\end{aligned}
$$

For (3), note that there is a decomposition of D–comodules

$$
\begin{aligned}
k\{x,\partial x\}\otimes k\{y,\partial y\} &\xrightarrow[\cong]{\mu} k\{xy, x\partial y, (\partial x)y, (\partial x)(\partial y)\}\\
&= k\{xy, (\partial x)y + x\partial y, (\partial x)(\partial y)\} \oplus k\{(\partial x)y - x(\partial y)\}
\end{aligned}
$$

and, since $\mathrm{Ext}_D^{s,t}(k,D) = 0$ for $s>0$, the quotient map

$$
b_{10}^{-1}\,\mathrm{Ext}_D^{2,*}(k, k\{x,\partial x\}\otimes k\{y,\partial x\}) \cong b_{10}^{-1}\,\mathrm{Ext}_D^{2,*}\big(k, k\{x\partial y - (\partial x)y\}\big)
$$

is an isomorphism. By (2), $e(x)\cdot e(y)$ is a generator of the latter Ext group. $\square$

Lemma 5.9 *Suppose N is a D–comodule algebra with sub-D–comodules $k\{x,\partial x\}\cong M(1)$ and $k\{y\}\cong k$.*

(1) *The image of $\mathrm{Ext}_D^{0,*}(k,k\{y\})$ in $\mathrm{Ext}_D^{0,*}(k,N)$ is generated by y.*

(2) *We have*
$$
e(x)\cdot y = [\xi_1]xy - [\xi_1^2](\partial x)y = y\cdot e(x).
$$

(3) *If the multiplication map embeds $k\{x,\partial x\}\otimes k\{y\}$ in N injectively, then $e(x)\cdot y$ is a generator of the 1–dimensional vector space $b_{10}^{-1}\,\mathrm{Ext}_D^{1,*}(k,k\{x,\partial x\}\otimes k\{y\})$.*

Proof (1) is clear. (2) follows from the cobar complex multiplication formulas

$$
\begin{aligned}
C_D^0(k,M)\otimes C_D^1(k,N) &\to C_D^1(k,M\otimes N), \quad m\otimes[\xi]n \mapsto [\xi](m\otimes n),\\
C_D^1(k,M)\otimes C_D^0(k,N) &\to C_D^1(k,M\otimes N), \quad [\xi]n\otimes m \mapsto [\xi](n\otimes m).
\end{aligned}
$$

For (3), note that $k\{x,\partial x\}\otimes k\{y\} = k\{xy,(\partial x)y\}$. Note that $(\partial x)y = \partial(xy)$. From Lemma 5.7, $b_{10}^{-1}\,\mathrm{Ext}_D^{1,*}(k,k\{xy,\partial(xy)\})$ is generated by $e(xy) = [\xi_1]xy - [\xi_1^2]\partial(xy) = e(x)\cdot y$. $\square$

Definition 5.10 Define $e_n := e(\xi_n) = [\xi_1]\xi_n - [\xi_1^2]\xi_{n-1}^3$ to be the chosen generator of $b_{10}^{-1}\,\mathrm{Ext}_D^{1,2(3^n+1)}(k, M(1)_{\xi_n})$.

Lemma 5.11 *Under the change-of-rings isomorphism*

$$
b_{10}^{-1}\,\mathrm{Ext}_D^{*,*}(k,B) \cong b_{10}^{-1}\,\mathrm{Ext}_P^{*,*}(k, P\,\square_D\,B),
$$

the image of $e(x)$ in $\mathrm{Ext}_P^{1,|x|+4}(k,P\,\square_D\,B)$ has cobar representative

$$
[\xi_1](1\,|\,x) - [\xi_1^2](1\,|\,\partial x) + [\xi_1](\xi_1\,|\,\partial x) \in \overline{P}\otimes(P\,\square_D\,B).
$$

Proof The change-of-rings isomorphism $\mathrm{Ext}_D^{*,*}(k, M) \cong \mathrm{Ext}_P^{*,*}(k, P \,\square_D\, M)$ works as follows: since P is free over D, the functor $P \,\square_D -$ is exact, and so given an injective D–resolution $M \to X^\bullet$ for M, the complex $P \,\square_D\, M \to P \,\square_D\, X^\bullet$ is an injective P–resolution. So we have $\mathrm{Ext}_D^{i,*}(k, M) \cong \mathrm{Cotor}_D^i(k, M) = H^i(k \,\square_D\, X^\bullet)$, which agrees with $\mathrm{Ext}_P^{i,*}(k, P \,\square_D\, M) \cong \mathrm{Cotor}_P^i(k, P \,\square_D\, M) = H^i(k \,\square_P\, (P \,\square_D\, X^\bullet)) \cong H^i(k \,\square_D\, X^\bullet)$.

In particular, $\mathrm{Ext}_P^{*,*}(k, P \,\square_D\, B)$ can be computed by applying $k \,\square_P -$ to the resolution

$$P \,\square_D\, C_D(k, B) = (P \,\square_D\, B \to P \,\square_D\, (D \otimes B) \to P \,\square_D\, (D \otimes \bar{D} \otimes B) \to \cdots). \tag{5-1}$$

By Lemma 5.7, $e(x)$ has representative $[1 \mid \xi_1]x - [1 \mid \xi_1^2]\partial x \in D \otimes \bar{D} \otimes B$ in the D–cobar *resolution* for B, and so its representative in (5-1) is $1 \mid 1 \mid \xi_1 \mid x - 1 \mid 1 \mid \xi_1^2 \mid \partial x$.

But we wanted a representative in the cobar complex $C_P(k, P \,\square_D\, B)$, so we will write down part of an explicit map from the P–cobar resolution for $P \,\square_D\, B$ to (5-1):

$$\begin{tikzcd}
P \,\square_D\, B \arrow[r, equal] \arrow[d] & P \,\square_D\, B \arrow[d] \\
P \otimes (P \,\square_D\, B) \arrow[r, "f^0"] \arrow[d] & P \otimes B \arrow[d] \\
P \otimes \bar{P} \otimes (P \,\square_D\, B) \arrow[r, "f^1"] \arrow[d] & P \otimes \bar{D} \otimes B \arrow[d] \\
P \otimes \bar{P}^{\otimes 2} \otimes (P \,\square_D\, B) \arrow[r] \arrow[d] & P \otimes \bar{D}^{\otimes 2} \otimes B \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

By basic homological algebra, the map f^* exists and is unique, so to find f^0 and f^1 it suffices to find P–comodule maps that make the first two squares commute. In particular, one can check that the maps

$$f^0(a \mid b \mid c) = \varepsilon(b) a \mid c, \quad f^1(a \mid b \mid c \mid d) = \varepsilon(c) a \mid b \mid d$$

make the diagram commute, and $z := [1 \mid \xi_1](1 \mid x) + [1 \mid \xi_1](\xi_1 \mid \partial x) - [1 \mid \xi_1^2](1 \mid \partial x)$ is a cycle in $P \otimes \bar{P} \otimes (P \,\square_D\, B)$ such that $(k \,\square_P\, f)(z) = e(x)$. $\square$

5.2 Multiplicative structure

Proposition 5.12 *The summand*

$$b_{10}^{-1}\operatorname{Ext}_D^{d,*}(k, M(1)_{\xi_{n_1}}\otimes\cdots\otimes M(1)_{\xi_{n_d}})\subset b_{10}^{-1}\operatorname{Ext}_D^{d,*}(k,B)$$

is generated by the product $e_{n_1}\cdots e_{n_d}$.

Proof Since

$$b_{10}^{-1}\operatorname{Ext}_D^{d,*}\Big(k,\bigotimes M(1)_{\xi_{n_i}}\Big)=\begin{cases}\Sigma^{d,0}b_{10}^{-1}\operatorname{Ext}_D^{0,*}\big(k,\bigotimes M(1)_{\xi_{n_i}}\big) & \text{if } d \text{ is even,}\\ \Sigma^{d-1,0}b_{10}^{-1}\operatorname{Ext}_D^{1,*}\big(k,\bigotimes M(1)_{\xi_{n_i}}\big) & \text{if } d \text{ is odd,}\end{cases}$$

it suffices to show that $b_{10}^{-1}\operatorname{Ext}_D^{0,*}(k, M(1)_{\xi_{n_1}}\otimes\cdots\otimes M(1)_{\xi_{n_d}})$ is generated by $b_{10}^{-d/2}e_{n_1}\cdots e_{n_d}$ when d is even, and $b_{10}^{-1}\operatorname{Ext}_D^{1,*}(k, M(1)_{\xi_{n_1}}\otimes\cdots\otimes M(1)_{\xi_{n_d}})$ is generated by $b_{10}^{-(d-1)/2}e_{n_1}\cdots e_{n_d}$ when d is odd. We proceed by induction on d. The base case $d=1$ is by definition.

Case 1 (d is even) The tensor product $M(1)_{\xi_{n_1}}\otimes\cdots\otimes M(1)_{\xi_{n_{d-1}}}$ is isomorphic to $M(1)\oplus F$ for a free summand F. By Lemma 5.7,

$$b_{10}^{-1}\operatorname{Ext}_D^{2,*}\big(k,(M(1)_{\xi_{n_1}}\otimes\cdots\otimes M(1)_{\xi_{n_{d-1}}})\otimes M(1)_{\xi_{n_d}}\big)$$

is generated by $e(x)\cdot e_{n_d}$, where $e(x)$ is a generator of

$$b_{10}^{-1}\operatorname{Ext}_D^{1,*}(k, M(1)_{\xi_{n_1}}\otimes\cdots\otimes M(1)_{\xi_{n_{d-1}}}).$$

By the inductive hypothesis, we can take $e(x)=b_{10}^{-(d-2)/2}e_{n_1}\cdots e_{n_{d-1}}$. So then $b_{10}^{-1}e(x)e_{n_d}=b_{10}^{-d/2}e_{n_1}\cdots e_{n_d}$ is a generator for

$$b_{10}^{-1}\operatorname{Ext}_D^{0,*}(k, M(1)_{\xi_{n_1}}\otimes\cdots\otimes M(1)_{\xi_{n_d}}).$$

Case 2 (d is odd) In this case, $M(1)_{\xi_{n_1}}\otimes\cdots\otimes M(1)_{\xi_{n_{d-1}}}$ is isomorphic to $k\oplus F$ for a free summand F. By Lemma 5.9,

$$b_{10}^{-1}\operatorname{Ext}_D^{1,*}\big(k,(M(1)_{\xi_{n_1}}\otimes\cdots\otimes M(1)_{\xi_{n_{d-1}}})\otimes M(1)_{\xi_{n_d}}\big)$$

is generated by $y\cdot e_{n_d}$, where y is a generator of

$$b_{10}^{-1}\operatorname{Ext}_D^{0,*}(k, M(1)_{\xi_{n_1}}\otimes\cdots\otimes M(1)_{\xi_{n_{d-1}}}).$$

By the inductive hypothesis, we can take $y=b_{10}^{-(d-1)/2}e_{n_1}\cdots e_{n_{d-1}}$. □

Recall we defined $R=b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k,k)=E[h_{10}]\otimes k[b_{10}^{\pm 1}]$.

Corollary 5.13 *There is an R–module isomorphism*

$$b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k, M(1)_{\xi_{n_1}}\otimes\cdots\otimes M(1)_{\xi_{n_d}})\cong R\{e_{n_1}\cdots e_{n_d}\},$$

where the generator $e_{n_1}\cdots e_{n_d}$ is in degree d.

Corollary 5.14 *The map $R\otimes E[e_2, e_3, \dots]\to b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k, B)$ is an isomorphism of R–algebras.*

5.3 Antipode

The antipode is the map induced on Ext by the swap map $\tau\colon B\otimes B\to B\otimes B$. In order to get a useful formula for this map, we will need the following basic properties of Hopf algebras:

Fact 5.15 *Denote the coproduct on an element x of a Hopf algebra by $\Delta(x) = \sum x'\otimes x''$.*

(1) ***Coassociativity*** $\sum x'\otimes(x'')'\otimes(x'')'' = \sum(x')'\otimes(x')''\otimes x''$.

(2) $\sum c(x)'\otimes c(x)'' = \sum c(x'')\otimes c(x')$.

(3) $\sum c(x')x'' = \varepsilon(x)$.

(4) $\sum \varepsilon(x')\otimes x'' = 1\otimes x$.

Lemma 5.16 (shear isomorphism) *Suppose M is a left P–comodule, and $B\otimes M$ is given the diagonal P–coaction $\psi(b\otimes m) = \sum b'm'\otimes b''\otimes m''$ (where $\psi(b) = \sum b'\otimes b''$ and $\psi(m) = \sum m'\otimes m''$). Then there is an isomorphism $S_M\colon B\otimes M\to P\,\square_D\,M$ (where P coacts on the left on $P\,\square_D\,M$) sending $b\otimes m\mapsto\sum bm'\otimes m''$. It has an inverse $S_M^{-1}\colon b\otimes m\mapsto\sum bc(m')\otimes m''$.*

In order to be able to apply Lemma 4.19, we now obtain an explicit formula for the induced map $\tau' := S_B\circ\tau\circ S_B^{-1}\colon P\,\square_D\,B\to P\,\square_D\,B$. This map is

$$\begin{tikzcd} B\otimes B \arrow[r, "\tau"] & B\otimes B \arrow[d, "S_B"] \\ P\,\square_D\,B \arrow[u, "S_B^{-1}"] \arrow[r, dotted, "\tau'"] & P\,\square_D\,B \end{tikzcd}
\qquad
\begin{tikzcd} \sum xc(y')\mid y'' \arrow[r, mapsto] & \sum y''\mid xc(y') \arrow[d, mapsto] \\ x\mid y \arrow[u, mapsto] & \sum y''\cdot x'c(y')'\mid x''c(y')'' \end{tikzcd}$$

Using Fact 5.15, we have

$$\begin{aligned}\tau'(x\otimes y) &= \sum y''\cdot x'c(y')' \mid x''c(y')'' \\ &= \sum x'y''c((y')'') \mid x''c((y')') \\ &= \sum x'(y'')''c((y'')') \mid x''c(y') \quad \text{(coassociativity)} \\ &= \sum x'\varepsilon(y'') \mid x''c(y') \\ &= \sum x' \mid x''c(y).\end{aligned}$$

Since $(K(\xi_1)_{**}, K(\xi_1)_{**}K(\xi_1))$ is a Hopf algebroid, the antipode is multiplicative, so to determine it, it suffices to show:

Proposition 5.17 *We have*

(1) $c(h)=h$,

(2) $c(e_n)=-e_n$.

Proof The antipode is given by the map $\tau'_*\colon \mathrm{Ext}^{*,*}_P(k, P\,\Box_D\, B)\to \mathrm{Ext}^{*,*}_P(k, P\,\Box_D\, B)$ induced by τ', defined so that $\tau'_*([x_1\mid\cdots\mid x_s]m)=[\xi_1\mid\cdots\mid x_s]\tau'(m)$. Since $h=[\xi_1](1\mid 1)\in \mathrm{Ext}^{1,*}_P(k, P\,\Box_D\, B)$, we have $c(h)=\tau'_*(h)=h$. For (2), we need an explicit formula for the antipode in the dual Steenrod algebra:

Fact 5.18 [12, Lemma 10] *Let* $\mathrm{Part}(n)$ *be the set of ordered partitions of* n, $\ell(\alpha)$ *the length of the partition* α, *and* $\sigma_i(\alpha)=\sum_{j=1}^{i}\alpha_j$ *the partial sum. Then*

$$c(\xi_n)=\sum_{\alpha\in\mathrm{Part}(n)}(-1)^{\ell(\alpha)}\prod_{i=1}^{\ell(\alpha)}\xi_{\alpha_i}^{p^{\sigma_{i-1}(\alpha)}}.$$

In particular, if $n\geq 2$ *then* $c(\xi_n)\equiv -\xi_n+\xi_1\xi_{n-1}^p \pmod{\overline{P}^{p^2}P}$ *and* $c(\xi_{n-1}^p)\equiv -\xi_{n-1}^p \pmod{\overline{P}^{p^2}P}$.

Recall (Notation 2.1) that we have defined ξ_n to be the antipode of its usual definition, so here we have $\Delta(\xi_n)=\sum_{i+j=n}\xi_i\otimes\xi_j^{p^i}$. (Since the antipode is a ring homomorphism, the formula in Fact 5.18 is the same in either case.)

Combining this antipode formula with the formula for e_n in Lemma 5.11, we have

$$\begin{aligned}\tau'_*(e_n) &= \tau'_*([\xi_1](1\mid\xi_n)-[\xi_1^2](1\mid\xi_{n-1}^3)+[\xi_1](\xi_1\mid\xi_{n-1}^3)) \\ &= [\xi_1](1\mid c(\xi_n))-[\xi_1^2](1\mid c(\xi_{n-1}^3))+[\xi_1](\xi_1\mid c(\xi_{n-1}^3)+1\mid\xi_1c(\xi_{n-1}^3))\end{aligned}$$

$$\begin{aligned}&= [\xi_1](-1 \mid \xi_n + 1 \mid \xi_1\xi_{n-1}^3 + 1 \mid A) - [\xi_1^2](-1 \mid \xi_{n-1}^3 + 1 \mid B) \\ &\qquad + [\xi_1](-\xi_1 \mid \xi_{n-1}^3 + \xi_1 \mid C - 1 \mid \xi_1\xi_{n-1}^3 + 1 \mid D) \\ &= -e_n + [\xi_1](1 \mid A + \xi_1 \mid C + 1 \mid D) - [\xi_1^2](1 \mid B)\end{aligned}$$

for A, B, C and D in $\overline{P}^9 P = I(3)$. By Lemma 4.19 these terms are zero in b_{10}–local cohomology, and $c(e_n) = \tau'_*(e_n) = -e_n$. □

Corollary 5.19 *We have $\eta_L = \eta_R$, ie the Hopf algebroid $(K(\xi_1)_{**}, K(\xi_1)_{**}K(\xi_1))$ is, in fact, a Hopf algebra.*

Proof One of the axioms of a Hopf algebroid is $c \circ \eta_R = \eta_L$. Since η_L is just the inclusion of R into $b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k, B)$, its image is invariant under the antipode c. □

5.4 Comultiplication

To define the comultiplication map

$$b_{10}^{-1}\operatorname{Ext}_P^{*,*}(k, B \otimes B) \to b_{10}^{-1}\operatorname{Ext}_P^{*,*}(k, B \otimes B)^{\otimes 2},$$

first consider the maps

$$\operatorname{Ext}_P^{*,*}(k, B \otimes B) \xrightarrow{\alpha_*} \operatorname{Ext}_P^{*,*}(k, B \otimes B \otimes B) \xleftarrow{\beta} \operatorname{Ext}_P^{*,*}(k, B \otimes B) \otimes \operatorname{Ext}_P^{*,*}(k, B \otimes B),$$

where α_* is the map on Ext induced by $\alpha\colon B^{\otimes 2} \to B^{\otimes 3}$ with $\alpha\colon a \otimes b \mapsto a \otimes 1 \otimes b$, and β is defined as the map in the factorization

$$\begin{tikzcd} \operatorname{Ext}_P^{*,*}(k, B^{\otimes 2}) \otimes \operatorname{Ext}_P^{*,*}(k, B^{\otimes 2}) \arrow[r, "\text{Künneth}"] \arrow[dr] & \operatorname{Ext}_P^{*,*}(k, B^{\otimes 2} \otimes B^{\otimes 2}) \arrow[r, "-\otimes\mu\otimes-"] & \operatorname{Ext}_P^{*,*}(k, B^{\otimes 3}) \\ & \operatorname{Ext}_P^{*,*}(k, B^{\otimes 2}) \otimes_{\operatorname{Ext}_P^{*,*}(k,B)} \operatorname{Ext}_P^{*,*}(k, B^{\otimes 2}) \arrow[ur, dotted, "\beta"'] & \end{tikzcd} \tag{5-2}$$

It follows from the shear isomorphism (Lemma 5.16) and the change-of-rings theorem that $\operatorname{Ext}_P^{*,*}(k, B \otimes M) \cong \operatorname{Ext}_P^{*,*}(k, P \,\square_D\, M) \cong \operatorname{Ext}_D^{*,*}(k, M)$, and the Künneth isomorphism for b_{10}–local cohomology over D (Lemma 5.5) implies that β is an isomorphism after inverting b_{10}. We define the comultiplication map on $b_{10}^{-1}\operatorname{Ext}_P^{*,*}(k, B \otimes B)$ by $\Delta := \beta^{-1} \circ \alpha_*$.

In particular, flatness of $K(\xi_1)_{**}K(\xi_1)$ over $K(\xi_1)_{**}$ implies that

$$(K(\xi_1)_{**}, K(\xi_1)_{**}K(\xi_1))$$

is a Hopf algebroid using the definitions of comultiplication, antipode, counit and unit above. In a Hopf algebroid, the comultiplication is a homomorphism, and so to determine Δ explicitly it suffices to determine $\Delta(e_n)$. We prove this in Proposition 5.21. Lemma 5.11 gives an expression for e_n in $\mathrm{Ext}_P^{1,*}(k, P\,\square_D\,B)$, so we prefer to calculate $\Delta\colon b_{10}^{-1}\,\mathrm{Ext}_P^{*,*}(k, B\otimes B) \to b_{10}^{-1}\,\mathrm{Ext}_P^{*,*}(k, B\otimes B)^{\otimes 2}$ after composing with the shear isomorphism; that is, there is a commutative diagram

$$\begin{tikzcd}
b_{10}^{-1}\,\mathrm{Ext}_P^{*,*}(k, B\otimes B) \arrow[r, "\alpha_*"] \arrow[d, "(S_B)_*"'] & b_{10}^{-1}\,\mathrm{Ext}_P^{*,*}(k, B\otimes B\otimes B) \arrow[d, "((\mathrm{Id}\otimes S_B)\circ S_{B\otimes B})_*"] & b_{10}^{-1}\,\mathrm{Ext}_P^{*,*}(k, B\otimes B)^{\otimes 2} \arrow[l, "\beta"'] \arrow[d, "S_B\otimes S_B"] \\
b_{10}^{-1}\,\mathrm{Ext}_P^{*,*}(k, P\square_D B) \arrow[r, "\alpha'_*"] & b_{10}^{-1}\,\mathrm{Ext}_P^{*,*}(k, P\square_D(P\square_D B)) & b_{10}^{-1}\,\mathrm{Ext}_P^{*,*}(k, P\square_D B)^{\otimes 2} \arrow[l, "\beta'"']
\end{tikzcd}$$

and we will show that $\alpha'_*(e_n) = \beta'(1\otimes e_n + e_n\otimes 1)$ in $b_{10}^{-1}\,\mathrm{Ext}_P^{*,*}(k, P\,\square_D\,(P\,\square_D\,B))$. (We have chosen to use an extra application of the shear isomorphism on the middle term in order to apply Corollary 4.20.)

Lemma 5.20 *If $a \in \mathrm{Ext}_P^{*,*}(k, P\,\square_D\,B)$ has cobar representative $[a_1\,|\cdots|\,a_s](p\,|\,q)$, we have*

$$\alpha'_*(a) = \sum [a_1\,|\cdots|\,a_s](p\,|\,q'\,|\,q''),$$
$$\beta'(1\otimes a + a\otimes 1) = [a_1\,|\cdots|\,a_s]\Big(\sum p'\,|\,p''\,|\,q + p\,|\,q\,|\,1\Big)$$

in $\mathrm{Ext}_P^{*,*}(k, P\,\square_D\,(P\,\square_D\,B))$.

So, to check that a is primitive after inverting b_{10}, it suffices to check

$$\sum [a_1\,|\cdots|\,a_s](p\,|\,q'\,|\,q'') - [a_1\,|\cdots|\,a_s]\Big(\sum p'\,|\,p''\,|\,q + p\,|\,q\,|\,1\Big) = 0 \tag{5-3}$$

in $b_{10}^{-1}\,\mathrm{Ext}_P^{*,*}(k, P\,\square_D\,(P\,\square_D\,B))$.

Proof By definition, α' is the map induced on Ext by the composition

$$P\square_D B \xrightarrow{S_B^{-1}} B\otimes B \xrightarrow{-\otimes\eta\otimes -} B\otimes B\otimes B \xrightarrow{S_{B\otimes B}} P\square_D(B\otimes B) \xrightarrow{P\square_D S_B} P\square_D(P\square_D B).$$

On elements, we have

$$\begin{aligned} x\,|\,y &\mapsto \sum xc(y')\,|\,y'' \mapsto \sum xc(y')\,|\,1\,|\,y'' \mapsto \sum xc(y')(y'')'\,|\,1\,|\,(y'')'' \\ &\mapsto \sum xc(y')(y'')'\,|\,((y'')'')'\,|\,((y'')'')'' = \sum x\,|\,y'\,|\,y'', \end{aligned}$$

where the last equality is a coassociativity argument similar to the one at the beginning of Section 5.3. That is, we have $\alpha'(x\otimes y)=\sum x\otimes y'\otimes y''$, which implies

$$\alpha'_*([a_1\,|\cdots|\,a_s](p\,|\,q))=\sum[a_1\,|\cdots|\,a_s](p\,|\,q'\,|\,q'').$$

The map β' comes from the bottom composition in

$$\begin{tikzcd}
\mathrm{Ext}_P^{*,*}(k,B^{\otimes 2})^{\otimes 2} \arrow[r, "\text{Künneth}"] \arrow[d, "(S_B)_*\otimes(S_B)_*"] & \mathrm{Ext}_P^{*,*}(k,B^{\otimes 2}\otimes B^{\otimes 2}) \arrow[r, "(-\otimes\mu\otimes-)_*"] \arrow[d, "(S_B\otimes S_B)_*"] & \mathrm{Ext}_P^{*,*}(k,B^{\otimes 3}) \arrow[d, "(S_{B\otimes B})_*"] \\
\mathrm{Ext}_P^{*,*}(k,P\,\square_D\,B)^{\otimes 2} \arrow[r, "\text{Künneth}"] & \mathrm{Ext}_P^{*,*}(k,(P\,\square_D\,B)\otimes(P\,\square_D\,B)) \arrow[r, "\gamma_*"] & \mathrm{Ext}_P^{*,*}(k,P\,\square_D(P\,\square_D\,B))
\end{tikzcd}$$

We will only give an explicit expression for β' on elements of the form $1\otimes a$ and $a\otimes 1$, where 1 denotes the unit $1\otimes 1\in\mathrm{Ext}_P^{0,*}(k,P\,\square_D\,B)$ and $a=[a_1\,|\cdots|\,a_s](p\otimes q)\in\mathrm{Ext}_P^{s,*}(k,P\,\square_D\,B)$. In [10], there is a full description of the Künneth map K on the level of cochains, but here all we need are the maps $K\colon C_P^0(k,M)\otimes C_P^s(k,N)\to C_P^s(k,M\otimes N)$ and $K\colon C_P^s(k,N)\otimes C_P^0(k,M)\to C_P^s(k,M\otimes N)$. The former sends $m\otimes[a_1\,|\cdots|\,a_s]n\mapsto[a_1\,|\cdots|\,a_s](m\otimes n)$ and the latter sends $[a_1\,|\cdots|\,a_s]n\otimes m\mapsto[a_1\,|\cdots|\,a_s](n\otimes m)$. In particular, we have

$$K(1\otimes a)=[a_1\,|\cdots|\,a_s](1\,|\,1\,|\,p\,|\,q),\qquad K(a\otimes 1)=[a_1\,|\cdots|\,a_s](p\,|\,q\,|\,1\,|\,1)$$

in $\mathrm{Ext}_P^{s,*}(k,(P\,\square_D\,B)\otimes(P\,\square_D\,B))$.

To determine β', it remains to determine the map $\gamma\colon(P\,\square_D\,B)\otimes(P\,\square_D\,B)\to P\,\square_D\,(P\,\square_D\,B)$ induced by $-\otimes\mu\otimes-$. This is accomplished by calculating the effect of shear isomorphisms as follows:

$$\begin{tikzcd}
(B\otimes B)\otimes(B\otimes B) \arrow[rr, "-\otimes\mu\otimes-"] & & B^{\otimes 3} \arrow[d, "S_{B\otimes B}"] \\
(P\,\square_D\,B)\otimes(P\,\square_D\,B) \arrow[u, "S_B^{-1}\otimes S_B^{-1}"] & & P\,\square_D\,(B\otimes B) \arrow[r, "P\square_D S_B"] & P\,\square_D\,(P\,\square_D\,B)
\end{tikzcd}$$

$$\begin{tikzcd}
\sum xc(y')\,|\,y''\otimes zc(w')\,|\,w'' \arrow[r, mapsto] & \sum xc(y')\,|\,y''zc(w')\,|\,w'' \arrow[d, mapsto] \\
x\,|\,y\otimes z\,|\,w \arrow[u, mapsto] & \begin{array}{l}\sum xc(y')(y'')'z'c(w')'(w'')' \\ \quad\otimes(y'')''z''c(w')''\otimes(w'')'' \\ \qquad=\sum xz'\,|\,yz''c(w')\,|\,w''\mapsto\sum xz'\,|\,yz''\,|\,w\end{array}
\end{tikzcd}$$

That is, $\gamma(x \mid y \otimes z \mid w) = \sum xz' \mid yz'' \mid w$, which implies

$$\begin{aligned}\beta'(1 \otimes a + a \otimes 1) &= \gamma_* K(1 \otimes a + a \otimes 1)\\ &= \gamma_*([a_1 \mid \cdots \mid a_s](1 \mid 1 \mid p \mid q + p \mid q \mid 1 \mid 1))\\ &= [a_1 \mid \cdots \mid a_s]\gamma(1 \mid 1 \mid p \mid q + p \mid q \mid 1 \mid 1)\\ &= [a_1 \mid \cdots \mid a_s]\Big(\sum p' \mid p'' \mid q + p \mid q \mid 1\Big). \qquad \square\end{aligned}$$

Proposition 5.21 *The element e_n is primitive.*

Proof We need to check the criterion (5-3) for $a = e_n$. Recall we had the formula

$$e_n = [\xi_1](1 \mid \xi_n) - [\xi_1^2](1 \mid \xi_{n-1}^3) + [\xi_1](\xi_1 \mid \xi_{n-1}^3) \in C_P^1(P \,\square_D\, B)$$

from Lemma 5.11. It suffices to check that $\alpha'_*(e_n) - \beta'_*(1 \otimes e_n + e_n \otimes 1)$ is zero as an element of $b_{10}^{-1}\operatorname{Ext}_P^{*,*}(k, P \,\square_D\, (P \,\square_D\, B))$. Using Lemma 5.20, we have

$$\begin{aligned}&\alpha'_*(e_n) - \beta'_*(1 \otimes e_n + e_n \otimes 1)\\ &= \big([\xi_1](1 \mid \Delta\xi_n) - [\xi_1^2](1 \mid \Delta\xi_{n-1}^3) + [\xi_1](\xi_1 \mid \Delta\xi_{n-1}^3)\big)\\ &\qquad - \big([\xi_1](1 \mid 1 \mid \xi_n + 1 \mid \xi_n \mid 1) - [\xi_1^2](1 \mid 1 \mid \xi_{n-1}^3 + 1 \mid \xi_{n-1}^3 \mid 1) + [\xi_1](1 \mid \xi_1 \mid \xi_{n-1}^3\\ &\qquad\qquad + \xi_1 \mid 1 \mid \xi_{n-1}^3 + \xi_1 \mid \xi_{n-1}^3 \mid 1)\big)\\ &= [\xi_1] \sum_{\substack{i+j=n\\ 2\le i\le n-1}} 1 \mid \xi_i \mid \xi_j^{3^i} - [\xi_1^2] \sum_{\substack{i+j=n-1\\ 1\le i\le n-2}} 1 \mid \xi_i^3 \mid \xi_j^{3^{i+1}} + [\xi_1] \sum_{\substack{i+j=n-1\\ 1\le i\le n-2}} \xi_1 \mid \xi_i^3 \mid \xi_j^{3^{i+1}}.\end{aligned}$$

But all the remaining terms in the difference are in $C_P\big(P \,\square_D\, (P \,\square_D\, I(3))\big)$, so by Corollary 4.20 they are zero in b_{10}–local cohomology. $\square$

Proof of Theorem 5.1 The flatness assertion was proved in Corollary 5.6. Putting together Corollary 5.14, Proposition 5.17, Corollary 5.19 and Proposition 5.21, we see that the map $R \otimes E[e_2, e_3, \dots] \to b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k, B)$ is an isomorphism of Hopf algebras. $\square$

6 Computation of d_4

6.1 Overview of the computation

In the previous section, we have shown that the $K(\xi_1)$–based MPASS computing $b_{10}^{-1}\operatorname{Ext}_P^{*,*}(k,k)$ has the form

$$E_2^{**} = E[h_{10}] \otimes k[b_{10}^{\pm 1}, w_2, w_3, \dots] \Rightarrow b_{10}^{-1}\operatorname{Ext}_P^{*,*}(k,k),$$

where w_n is represented in $E_1^{1,2(3^n+1)}$ by

$$e_n = [\xi_1]\xi_n - [\xi_1^2]\xi_{n-1}^3 \in b_{10}^{-1}\,\mathrm{Ext}_D^{1,2(3^n+1)}(k, \bar{B}).$$

Recall that d_r is a map

$$E_r^{s,t,u} \to E_r^{s+r,t-r+1,u},$$

w_n has degree $(s,t,u) = (1,1,2(3^n+1))$, h_{10} has degree $(0,1,4)$ and b_{10} has degree $(0,2,12)$. Furthermore, $u'(w_n) = 2(3^n-5)$, $u'(h_{10}) = -2$, and $u'(b_{10}) = 0$. In Proposition 2.3, we have shown that the next nontrivial differential is d_4. In this section we will completely determine this differential. We begin by recording some d_4's in low degrees.

Proposition 6.1 *We have the following:*

$$\begin{aligned} d_r(h_{10}) &= 0 \quad \textit{for } r \geq 2, \\ d_r(w_2) &= 0 \quad \textit{for } r \geq 2, \\ d_4(w_3) &= \pm b_{10}^{-4} h_{10} w_2^5, \\ d_4(w_4) &= \pm b_{10}^{-4} h_{10} w_2^2 w_3^3. \end{aligned}$$

Proof The first two facts can be seen directly in the cobar complex $C_P(k,k)$, using the cobar representatives $h_{10} = [\xi_1]$ and $w_2 = [\xi_1 \,|\, \xi_2] - [\xi_1^2 \,|\, \xi_1^3]$, which are permanent cycles.

The differentials on w_3 and w_4 were deduced from the chart of $\mathrm{Ext}_P^{*,*}(k,k)$ up to the 700 stem that appears as Figure 1 (generated by the software [13]). In Proposition 3.1, we show that $\mathrm{Ext}_P^{*,*}(k,k)$ agrees with $b_{10}^{-1}\,\mathrm{Ext}_P^{*,*}(k,k)$ in the range of dimensions depicted in the chart. Thus we know which classes in $E_2 = R[w_2, w_3, \dots]$ in this range of dimensions die in the spectral sequence, and, using multiplicativity of the spectral sequence, this forces the differentials above. □

The goal of this section is to prove the following:

Theorem 6.2 *For $n \geq 5$, there is a differential in the MPASS*

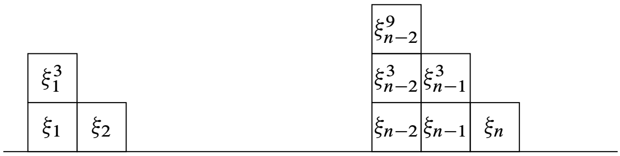
$$d_4(w_n) = \pm b_{10}^{-4} h_{10} w_2^2 w_{n-1}^3.$$

Since the spectral sequence is multiplicative, this determines d_4.

The main idea is to use comparison with a spectral sequence computing $b_{10}^{-1}\,\mathrm{Ext}_{P_n}^{*,*}(k,k)$, where

$$P_n = k[\xi_1, \xi_2, \xi_{n-2}, \xi_{n-1}, \xi_n]/(\xi_1^9, \xi_2^3, \xi_{n-2}^{27}, \xi_{n-1}^9, \xi_n^3).$$

(The idea is that this is the smallest algebra in which the desired differential can be seen.) This is a quotient Hopf algebra of P by the classification of such (see Theorem 2.1.1(a) of [14]). Here's a picture:



Recall $B = P \,\square_D\, k$; let $B_n = P_n \,\square_D\, k$. We will refer to the spectral sequence of Theorem 1.6 with $\Gamma = P_n$ as the $b_{10}^{-1} B_n$–based MPASS computing $b_{10}^{-1}\operatorname{Ext}_{P_n}^{*,*}(k,k)$, and use $E_r(k, B_n)$ to denote its E_r page. For example,

$$E_1(k, B_n) = b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k, \overline{B}_n^{\otimes *}). \tag{6-1}$$

Let $E_r(k, B)$ denote the $b_{10}^{-1}B$–based MPASS for $b_{10}^{-1}\operatorname{Ext}_{P_n}^{*,*}(k,k)$ we have been focusing on. Then the diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & D \arrow[d, equal] \\ B_n \arrow[r] & P_n \arrow[r] & D \end{tikzcd}$$

shows there is a map of spectral sequences $E_r(k, B) \to E_r(k, B_n)$.

Lemma 6.3 *It suffices to show that $d_4(w_n) \neq 0$ in $E_4(k, B)$.*

Proof Since $s(d_4(w_n)) = 4 + s(w_n) = 5$, we know that $d_4(w_n)$ is a linear combination of terms of the form $b_{10}^N h_{10} w_{k_1} \cdots w_{k_5}$. We have

$$\begin{aligned} u'(w_n) &= u'(b_{10}^N h_{10} w_{k_1} \cdots w_{k_5}) + 6, \\ 2(3^n - 5) &= -2 + \sum_{i=1}^{5} 2(3^{k_i} - 5) + 6, \\ 3^n + 18 &= \sum_{i=1}^{5} 3^{k_i}. \end{aligned}$$

Note that $k_i \geq 2$. Looking at this mod 27, we see that (at least) two of the k_i have to equal 2, say k_1 and k_2. Then we have $3^n = 3^{k_3} + 3^{k_4} + 3^{k_5}$. The only possibility is $n - 1 = k_3 = k_4 = k_5$. So if $d_4(w_n) \neq 0$ then $d_4(w_n) = b_{10}^N h_{10} w_2^2 w_{n-1}^3$, and checking internal degrees shows $N = -4$. $\square$

When we discuss $E_r(k, B_n)$ it will be easy to see that there is a class $w_n \in E_2(k, B_n)$ which is the target of $w_n \in E_2(k, B)$ along the quotient map

$$\begin{tikzcd} E_4(k,B) \arrow[r, "d_4"] \arrow[d] & E_4(k,B) \arrow[d] \\ E_4(k,B_n) \arrow[r, "d_4"] & E_4(k,B_n) \end{tikzcd}$$

Lemma 6.3 says that it suffices to show $d_4(w_n) \neq 0$ in $E_4(k, B_n)$, but it turns out to be the same amount of work to show the following more attractive statement:

Claim 6.4 *There is a differential* $d_4(w_n) = \pm b_{10}^{-4} h_{10} w_2^2 w_{n-1}^3$ *in* $E_r(k, B_n)$.

Using the same argument as Proposition 2.3, we know that $d_2 = 0 = d_3$ in $E_r(k, B_n)$, so $h_{10} w_2^2 w_{n-1}^3$ is not the target of an earlier differential. We will use the following strategy to show the desired differential in $E_r(k, B_n)$:

(1) Calculate $E_2(k, B_n)$ in a region and identify classes w_2, w_{n-1} and w_n that are the targets of their namesake classes under the quotient map $E_2(k, B) \to E_2(k, B_n)$.

(2) Show that $b_{10}^{-1} \operatorname{Ext}_{P_n}^{*,*}(k, k)$ is zero in the stem of $b_{10}^{-4} h_{10} w_2^2 w_{n-1}^3$. This implies that $b_{10}^{-4} h_{10} w_2^2 w_{n-1}^3$ either supports a differential or is the target of a differential.

(3) Show that $b_{10}^{-4} h_{10} w_2^2 w_{n-1}^3$ is a permanent cycle in the MPASS (so it must be the target of a differential) and show that, for degree reasons, w_n is the only element that can hit it. By looking at filtrations, we see this differential is a d_4.

For (2), we introduce another spectral sequence for calculating $b_{10}^{-1} \operatorname{Ext}_{P_n}^{*,*}(k, k)$, the Ivanovskii spectral sequence (ISS) [6]. This is the (b_{10}–localized version of the) dual of the May spectral sequence; that is, it is the spectral sequence obtained by filtering the cobar complex on P_n by powers of the augmentation ideal. (For example, $[\xi_1 \xi_2 \mid \xi_{n-1}^3]$ has filtration $2 + 3 = 5$.)

In Section 6.2 we will introduce notation and record facts about gradings. In Section 6.3 we will compute $E_1(k, B_n)$ and the relevant part of $E_2(k, B_n)$, and show (1) and (3) assuming (2). In Section 6.4 we will calculate the relevant part of the ISS and show (2). Convergence of the localized ISS is discussed in Section A.2.

6.2 Notation and gradings

Since much of the work in this section consists of degree-counting arguments, we will now record how differentials and convergence affect the various gradings at play. We emphasize a change of coordinates on degrees that simplifies degree arguments by putting b_{10} in degree zero.

MPASS gradings In Section 2, we introduced the gradings (s,t,u). The differential has the form

$$d_r\colon E_r^{s,t,u} \to E_r^{s+r,t-r+1,u}$$

and a permanent cycle in $E_r^{s,t,u}$ converges to an element in $b_{10}^{-1}\operatorname{Ext}_P^{s+t,u}(k,k)$. We also introduced $u' := u - 6(s+t)$. We prefer to track (u',s) instead of (s,t,u), because $u'(b_{10}) = 0 = s(b_{10})$, so all classes in a b_{10}–tower have the same (u',s) degree. The differential under the change of coordinates has the form

$$d_r\colon E_r^{u',s} \to E_r^{u'-6,s+r}$$

and a permanent cycle in $E_r^{u',s}$ converges to an element in $b_{10}^{-1}\operatorname{Ext}_P^{a,b}(k,k)$ (where b is internal topological degree and a is homological degree) with $b-6a = u'$.

Definition 6.5 Let *stem* in $b_{10}^{-1}\operatorname{Ext}_P^{a,b}(k,k)$ denote the quantity $b-6a$. Then a permanent cycle in $E_r^{u',s}$ converges to an element in the u' stem.

Finally, define

$$u'' := u - 6t.$$

This is only useful for looking at the E_1 page of the MPASS, as d_1 fixes u''.

ISS gradings The *Ivanovskii spectral sequence* computing $b_{10}^{-1}\operatorname{Ext}_{P_n}^{*,*}(k,k)$ is the spectral sequence obtained by filtering the cobar complex on P_n by powers of the augmentation ideal. Let E_r^{ISS} denote the E_r page of the Ivanovskii spectral sequence.

We use slightly different grading conventions: classes have degree (s,t,u), where s is the ISS filtration, t denotes degree in the cobar complex and u denotes internal topological degree (as in the MPASS). The differential has the form

$$d_r^{\mathrm{ISS}}\colon E_r^{s,t,u} \to E_r^{s+r,t+1,u}$$

and a permanent cycle in $E_r^{s,t,u}$ converges to an element in $b_{10}^{-1}\operatorname{Ext}_P^{t,u}(k,k)$.

We will use the change of coordinates

$$u' := u - 6t,$$

which is designed so that $u'(b_{10}) = 0$. (This has a different formula from the MPASS change of coordinates simply because (s,t,u) correspond to different parameters here.) The differential has the form

$$d_r^{\mathrm{ISS}}\colon E_r^{u',s} \to E_r^{u'-6,s+r}$$

and a permanent cycle in $E_r^{u',s}$ converges to an element in $b_{10}^{-1}\operatorname{Ext}_P^{a,b}(k,k)$ with $u' = b - 6a$, ie an element in the u' stem.

Note that u' has different formulas for the MPASS and ISS, but in both spectral sequences u' corresponds to stem, with the definition given above. Now we will introduce another grading on P_n (for $n \geq 5$) preserved by the comultiplication.

Extra grading on $\boldsymbol{P_n}$ Let $P_n' = k[\xi_1, \xi_2, \xi_{n-2}^3, \xi_{n-1}, \xi_n]/(\xi_1^9, \xi_2^3, \xi_{n-2}^{27}, \xi_{n-1}^9, \xi_n^3)$. Note that every monomial in P_n can be written as $\xi_{n-2}^e x$, where $e \in \{0,1,2\}$ and $x \in P_n'$.

Lemma 6.6 *For $n \geq 5$, P_n' is a subcoalgebra of P_n.*

Proof This is clear from the comultiplication formulas

$$\begin{aligned}\Delta(\xi_n) &= 1 \otimes \xi_n + \xi_1 \otimes \xi_{n-1}^3 + \xi_2 \otimes \xi_{n-2}^9,\\ \Delta(\xi_{n-1}) &= 1 \otimes \xi_{n-1} + \xi_1 \otimes \xi_{n-2}^3 + \xi_{n-1} \otimes 1,\\ \Delta(\xi_{n-2}^3) &= 1 \otimes \xi_{n-2}^3 + \xi_{n-2}^3 \otimes 1,\end{aligned} \tag{6-2}$$

and the assumption $n \geq 5$ guarantees that $\xi_1, \xi_2 \neq \xi_{n-2}$. □

Proposition 6.7 *Let $n \geq 3$. There is an extra grading α on P_n that respects the comultiplication, defined by the property that it is multiplicative on P_n', and*

$$\begin{aligned}\alpha(\xi_1) = \alpha(\xi_2) &= \alpha(\xi_{n-2}) = 0,\\ \alpha(\xi_{n-2}^3) &= \alpha(\xi_{n-1}) = 3,\\ \alpha(\xi_n) &= 9,\\ \alpha(\xi_{n-1}^e x) &= \alpha(x) \quad \text{for } e \in \{0,1,2\} \text{ and } x \in P_n'.\end{aligned}$$

Proof First we check that α respects the comultiplication when restricted to P'_n. Since it is defined to be multiplicative on P'_n, it suffices to check that $\alpha(y) = \alpha(\Delta y)$ for y as each of the multiplicative generators. This is clear from the comultiplication formulas (6-2).

Now suppose $y = \xi_{n-2}x$, where $x \in P'_n$. We have

$$\Delta(\xi_{n-2}x) = (1 \otimes \xi_{n-2} + \xi_{n-2} \otimes 1)\Delta x = \sum (x' \otimes x''\xi_{n-2} + x'\xi_{n-2} \otimes x'')$$

and the α degrees of both sides agree since P'_n is a coalgebra. Similarly, if $y = \xi_{n-2}^2 x$ for $x \in P'_n$, we have

$$\begin{aligned} \alpha(\Delta y) &= \alpha((1 \otimes \xi_{n-2}^3 + 2\xi_{n-2} \otimes \xi_{n-2} + \xi_{n-2}^2 \otimes 1)(\Delta x)) \\ &= \alpha\Bigl(\sum x' \otimes \xi_{n-2}^2 x'' + 2\xi_{n-2}x' \otimes \xi_{n-2}x'' + \xi_{n-2}^2 x' \otimes x''\Bigr) = \alpha(\Delta x). \qquad \square \end{aligned}$$

6.3 The E_2 page of the $b_{10}^{-1} B_n$–based MPASS

The goal of this section is to prove the following:

Proposition 6.8 *If $b_{10}^{-4} h_{10} w_2^2 w_{n-1}^3$ is the target of a differential in the $b_{10}^{-1} B_n$–based MPASS calculating $b_{10}^{-1} \operatorname{Ext}_{P_n}^{*,*}(k,k)$, that differential must be*

$$d_4(w_n) = \pm b_{10}^{-4} h_{10} w_2^2 w_{n-1}^3.$$

The main task is to calculate enough of $E_2(k, B_n)$ to do a degree-counting argument (Proposition 6.16), where

$$B_n = P_n \,\square_D\, k = k[\xi_1^3, \xi_2, \xi_{n-2}, \xi_{n-1}, \xi_n]/(\xi_1^9, \xi_2^3, \xi_{n-2}^{27}, \xi_{n-1}^9, \xi_n^3).$$

As in the calculation of the E_2 page of the $b_{10}^{-1} B$–based MPASS (Section 5), the Künneth formula for the functor $b_{10}^{-1} \operatorname{Ext}_D^{*,*}(k, -)$ (Lemma 5.5) guarantees flatness of $(b_{10}^{-1} B_n)_{**}(b_{10}^{-1} B_n)$ over $(b_{10}^{-1} B_n)_{**}$. So we can use the formula

$$E_2 \cong \operatorname{Ext}^{*,*}_{(b_{10}^{-1} B_n)_{**} b_{10}^{-1} B_n}((b_{10}^{-1} B_n)_{**}, (b_{10}^{-1} B_n)_{**}), \tag{6-3}$$

where $(b_{10}^{-1} B_n)_{**} = b_{10}^{-1} \operatorname{Ext}_{P_n}^{*,*}(k, B_n) = R$ and

$$(b_{10}^{-1} B_n)_{**}(b_{10}^{-1} B_n) = b_{10}^{-1} \operatorname{Ext}_{P_n}^{*,*}(R, B_n^{\otimes 2}) \cong b_{10}^{-1} \operatorname{Ext}_D^{*,*}(k, B_n)$$

by the change-of-rings theorem. We will simultaneously determine the vector space structure and the comultiplication on $b_{10}^{-1} \operatorname{Ext}_D^{*,*}(k, B_n)$.

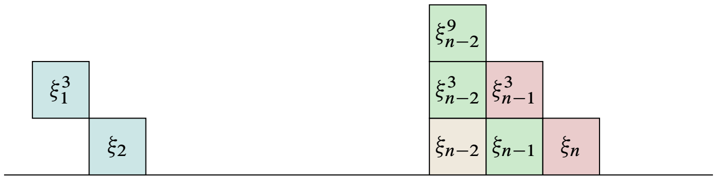


Figure 4: Illustration of the decomposition of B_n into tensor factors.

Remark 6.9 By (6-1) and the Künneth formula mentioned above, we have

$$E_1^{s,*}(k, B_n) \cong b_{10}^{-1} \operatorname{Ext}_D^{*,*}(k, \bar{B}_n)^{\otimes s}$$

and so the coproduct on $b_{10}^{-1} \operatorname{Ext}_D^{*,*}(k, B_n)$ coincides with d_1 on $E_1^{1,*}$.

We can write B_n as a tensor product

$$B_n = k[\xi_2, \xi_1^3]/(\xi_2^3, \xi_1^9) \otimes k[\xi_{n-2}]/\xi_{n-2}^3 \otimes k[\xi_{n-1}, \xi_{n-2}^3]/(\xi_{n-1}^3, \xi_{n-2}^{27}) \\ \otimes k[\xi_n, \xi_{n-1}^3]/(\xi_n^3, \xi_{n-1}^9)$$

illustrated in Figure 4.

Since we have a Künneth formula for $b_{10}^{-1} \operatorname{Ext}_D^{*,*}(k, -)$, it suffices to apply this functor to each of the four factors of B_n above.

Factor 1: $k[\xi_2, \xi_1^3]/(\xi_2^3, \xi_1^9)$ As a D–comodule, this decomposes as

$$\text{(6-4)} \quad k[\xi_2, \xi_1^3]/(\xi_2^3, \xi_1^9) \\ \cong \underbrace{k\{1\}}_{\cong k} \oplus \underbrace{k\{\xi_2, \xi_1^3\}}_{\cong M(1)} \oplus \underbrace{k\{\xi_2^2, \xi_1^3\xi_2, \xi_1^6\}}_{\cong D} \oplus \underbrace{k\{\xi_1^3\xi_2^2, \xi_1^6\xi_2\}}_{\cong M(1)} \oplus \underbrace{k\{\xi_1^6\xi_2^2\}}_{\cong k}.$$

(Recall $M(1)$ was defined to be the D–comodule $k[\xi_1]/\xi_1^2$, and every D–comodule is a sum of copies of k, $M(1)$ and D.) As a module over $R := E[h_{10}] \otimes k[b_{10}^{\pm 1}]$, this is generated by a class $e_2 = e(\xi_2)$ in $b_{10}^{-1} \operatorname{Ext}_D^{1,16}(k, k\{\xi_2, \xi_1^3\})$, a class $f_{20} = e(\xi_1^3\xi_2^2)$ in $b_{10}^{-1} \operatorname{Ext}_D^{1,48}(k, k\{\xi_1^3\xi_2^2, \xi_1^6\xi_2\})$ and a class c_2 in $b_{10}^{-1} \operatorname{Ext}_D^{0,56}(k, k\{\xi_1^6\xi_2^2\})$. As $b_{10}^{-1} \operatorname{Ext}_D^{*,*}(k, D) = 0$, we may ignore the free summands.

Using Lemma 5.7, we can give explicit representatives for the classes in

$$b_{10}^{-1} \operatorname{Ext}_D^{*,*}(k, k[\xi_2, \xi_1^3]/(\xi_2^3, \xi_1^9))$$

coming from the decomposition (6-4):

$$\begin{aligned} e_2 &:= e(\xi_2) = [\xi_1]\xi_2 - [\xi_1^2]\xi_1^3 \in \operatorname{Ext}_D^{1,16}(k, k[\xi_2, \xi_1^3]/(\xi_2^3, \xi_1^9)), \\ f_{20} &:= e(\xi_1^3\xi_2^2) = [\xi_1]\xi_1^3\xi_2^2 + [\xi_1^2]\xi_1^6\xi_2, \\ c_2 &= \xi_1^6\xi_2^2, \end{aligned}$$

satisfying relations $e_2^2 = 0 = f_{20}^2$ and $b_{10}c_2 = e_2 f_{20}$.

Lemma 6.10 *The classes e_2 and f_{20} are primitive in the coalgebra $b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k, B_n)$.*

Proof As described in Section 1.1, we can interpret the MPASS as a filtration spectral sequence on the cobar complex $C_{P_n}(k,k)$, where $[a_1 | \cdots | a_s]$ is in filtration n if at least n of the a_i are in $\overline{B}_n P_n$. The elements e_2 and f_{20} correspond to elements in $F^1/F^2 C^2_{P_n}(k,k)$ with the same formulas, and by Remark 6.9 it suffices to show that $d_1(e_2) = 0 = d_1(f_{20})$ in the filtration spectral sequence. One checks explicitly that $d_{\mathrm{cobar}}(e_2) = 0$, so it is a permanent cycle. This is not true of f_{20}, but we can write down explicit correcting terms in higher filtration,

$$f_{20} \equiv \widetilde{f}_{20} := [\xi_2 | \xi_2^2] + [\xi_2^2 | \xi_2] - [\xi_1\xi_2 | \xi_2\xi_1^3] + [\xi_1\xi_2^2 | \xi_1^3] + [\xi_1^2\xi_2 | \xi_1^6] + [\xi_1^2 | \xi_2\xi_1^6] + [\xi_1 | \xi_2^2\xi_1^3],$$

and then check that $d_{\mathrm{cobar}}(\widetilde{f}_{20}) = [\xi_1^3 | \xi_1^6 | \xi_1^3] + [\xi_1^3 | \xi_1^3 | \xi_1^6]$. This has filtration 3, and so $d_1(f_{20}) = 0$. □

So we've proved:

Proposition 6.11 *There is an isomorphism of Hopf algebras*

$$b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k, k[\xi_2, \xi_1^3]/(\xi_2^3, \xi_1^9)) \cong R \otimes E[e_2, f_{20}],$$

where e_2 and f_{20} are primitive.

We can summarize the degree information as follows:

element	s	t	u	$u'' = u - 6t$	α
1	0	0	0	0	0
h_{10}	0	1	4	–2	0
b_{10}	0	2	12	0	0
$e_2 = [\xi_1]\xi_2 - [\xi_1^2]\xi_1^3$	1	1	20	14	0
$f_{20} = [\xi_1]\xi_1^3\xi_2^2 + [\xi_1^2]\xi_1^6\xi_2$	1	1	48	42	0
$c_2 = \xi_1^6\xi_2^2$	1	0	56	56	0

Factor 2: $k[\xi_{n-2}]/\xi_{n-2}^3$ This decomposes as $k\{1\}\oplus k\{\xi_{n-2}\}\oplus k\{\xi_{n-2}^2\}$, so we have three R–module generators:

element	s	t	u	$u''=u-6t$	α
1	0	0	0	0	0
ξ_{n-2}	1	0	$2(3^{n-2}-1)$	$2(3^{n-2}-1)$	0
ξ_{n-2}^2	1	0	$2\cdot 2(3^{n-2}-1)$	$2\cdot 2(3^{n-2}-1)$	0

As a Hopf algebra we have

$$b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k,k[\xi_{n-2}]/\xi_{n-2}^3)\cong R\otimes D[\xi_{n-2}].$$

Factor 3: $k[\xi_{n-1},\xi_{n-2}^3]/(\xi_{n-1}^3,\xi_{n-2}^{27})$ Similarly to (6-4), for the third factor of B_n we have a D–comodule decomposition

$$\begin{aligned}&k[\xi_{n-1},\xi_{n-2}^3]/(\xi_{n-1}^3,\xi_{n-2}^{27})\\&\qquad\cong\underbrace{k\{1\}}_{\cong k}\oplus\underbrace{k\{\xi_{n-1},\xi_{n-2}^3\}}_{\cong M(1)}\oplus\underbrace{k\{\xi_{n-1}^2\xi_{n-2}^{21},\xi_{n-1}\xi_{n-2}^{24}\}}_{\cong M(1)}\oplus\underbrace{k\{\xi_{n-1}^2\xi_{n-2}^{24}\}}_{\cong k}\oplus F,\end{aligned}$$

where F is a free D–comodule, which gives the following R–module generators of $b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k,k[\xi_{n-1},\xi_{n-2}^3]/(\xi_{n-1}^3,\xi_{n-2}^{27}))$:

element	s	t	u	$u''=u-6t$	α
1	0	0	0	0	0
$e_{n-1}:=[\xi_1]\xi_{n-1}-[\xi_1^2]\xi_{n-2}^3$	1	1	$2(3^{n-1}+1)$	$2(3^{n-1}-2)$	3
$y_{n-1}:=[\xi_1]\xi_{n-1}^2\xi_{n-2}^{21}+[\xi_1^2]\xi_{n-1}\xi_{n-2}^{24}$	1	1	$2(3^{n+1}-21)$	$2(3^{n+1}-24)$	27
$z_{n-1}:=\xi_{n-1}^2\xi_{n-2}^{24}$	1	0	$2(3^{n+1}+3^{n-1}-26)$	$2(3^{n+1}+3^{n-1}-26)$	30

Lemma 6.12 *e_{n-1} is a permanent cycle in $E_r(k,B_n)$. In particular, $d_1(e_{n-1})=0$.*

Proof Use the filtration spectral sequence interpretation of the MPASS described in the proof of Lemma 6.10, where e_{n-1} has representative

$$[\xi_1\,|\,\xi_{n-1}]-[\xi_1^2\,|\,\xi_{n-2}^3]$$

in $C_{P_n}(k,k)$. It is clear that this is a cycle in $C_{P_n}(k,k)$, hence a permanent cycle in the spectral sequence. □

Factor 4: $k[\xi_n,\xi_{n-1}^3]/(\xi_n^3,\xi_{n-1}^9)$ There is a D–comodule decomposition

$$k[\xi_n,\xi_{n-1}^3]/(\xi_n^3,\xi_{n-1}^9) \cong \underbrace{k\{1\}}_{\cong k} \oplus \underbrace{k\{\xi_n,\xi_{n-1}^3\}}_{\cong M(1)} \oplus \underbrace{k\{\xi_n^2,\xi_{n-1}^3\xi_n,\xi_{n-1}^6\}}_{\cong D} \oplus \underbrace{k\{\xi_{n-1}^3\xi_n^2,\xi_{n-1}^6\xi_n\}}_{\cong M(1)} \oplus \underbrace{k\{\xi_{n-1}^6\xi_n^2\}}_{\cong k}.$$

The nonfree summands lead to R–module generators of

$$b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k,k[\xi_n,\xi_{n-1}^3]/(\xi_n^3,\xi_{n-1}^9)),$$

which have representatives (in order):

element	s	t	u	$u''=u-6t$	α
1	0	0	0	0	0
$e_n:=[\xi_1]\xi_n-[\xi_1^2]\xi_{n-1}^3$	1	1	$2(3^n+1)$	$2(3^n-2)$	9
$f_{n0}:=[\xi_1]\xi_{n-1}^3\xi_n^2-[\xi_1^2]\xi_{n-1}^6\xi_n$	1	1	$2(3^{n+1}-3)$	$2(3^{n+1}-6)$	27
$c_n:=\xi_{n-1}^6\xi_n^2$	1	0	$2(3^{n+1}+3^n-8)$	$2(3^{n+1}+3^n-8)$	36

Corollary 6.13 *There is an isomorphism of R–modules*

$$b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k,B_n) \cong R\{1,e_2,f_{20},c_2\}\otimes R\{1,\xi_{n-2},\xi_{n-2}^2\}\otimes R\{1,e_{n-1},y_{n-1},z_{n-1}\}\otimes R\{1,e_n,f_{n,0},c_n\}.$$

We have already computed part of the Hopf algebra structure on $b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k,B_n)=E_1^{1,*}(k,B_n)$ but do not need to finish this; we just need one more piece of information.

Lemma 6.14 *e_n is primitive in $b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k,B_n)$*

Proof Write $\psi(e_n)=\sum_i c[x_i \mid y_i]$, where $c\in R$ and $x_i,y_i\in b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k,B_n)$. As the cobar differential preserves the grading α (see Proposition 6.7) and ψ can be given in terms of the cobar differential (see eg Remark 6.9), ψ also preserves α. Since $\alpha(e_n)=9$, in order for $d_1(e_n)$ to have $\alpha=9$, we need $\alpha(x_i)+\alpha(y_i)=9$. Looking at α degrees in the above charts of R–module generators in $b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k,B_n)$, the only options are for $e_n \mid x_i$ or y_i, or for $e_{n-1}^2 \mid x_i$ or y_i. But $e_{n-1}^2=0$ by Lemma 5.7, and so the only option is for e_n to be primitive. □

Combining Lemmas 6.10, 6.12 and 6.14, we have:

Corollary 6.15 *In $b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k,B_n)$, the elements e_2, f_{20}, e_{n-1} and e_n are exterior generators in the Hopf algebra sense — they are primitive and square to zero.*

Now we have computed enough of $E_2(k, B_n)$ to show Proposition 6.8. If the element $b_{10}^{-4}h_{10}w_2^2w_{n-1}^3$ (which is in degree $\alpha=9$, $u'=2(3^n-8)$ and $u=2(3^n+1)$) is the target of a differential, it must be a d_r for $r\le 4$ (since the target is in filtration 5), and the source of that differential must have degree $\alpha=9$, $u'=2(3^n-5)$ and $u=2(3^n+1)$. Thus it suffices to prove Proposition 6.16.

Proposition 6.16 *The only element in $E_2(k, B_n)$ with $s\le 4$, $\alpha=9$, $u'=2(3^n-5)$ and $u=2(3^n+1)$ is $\pm w_n$.*

Proof There is a map $R\otimes E[e_2, f_{20}, e_{n-1}, e_n]\otimes D[\xi_{n-2}]\to b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k, B_n)$ that is an isomorphism on degree $u''<2(3^{n+1}-24)$ and induces a map on cobar complexes

$$C^s_{R\otimes E[e_2,f_{20},e_{n-1},e_n]\otimes D[\xi_{n-2}]}(R, R)\to C^s_{b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k,B_n)}(R, R).$$

We claim the map of cobar complexes is an isomorphism in degree

$$u''<-2+2(3^{n+1}-24)+14(s-1).$$

One can see this by noting that a minimal-degree element in $C^s_{b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k,B_n)}(R, R)$ not in the image is $h_{10}[y_{n-1}\mid e_2\mid\cdots\mid e_2]$, in degree

$$-2+2(3^{n+1}-24)+14(s-1).$$

(We use u'' degree here because it is additive with respect to multiplication within $b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k, B_n)=E_1^{1,*}$, whereas u' degree is additive with respect to multiplication of cohomology classes in $H^*E_1=E_2$.) Note that the desired degrees $u''=u'+6s=2(3^n-5)+6s$ fall into the region described here for every s.

Now we look at the map induced on Ext in this region. Since d_r differentials increase u'' degree by $6(r-1)$ (they preserve u and decrease t by $r-1$) and increase s by r, differentials originating in the region $u''<-2+2(3^{n+1}-24)+14(s-1)$ stay in the region, but there might be differentials originating outside the region hitting elements in the region. Instead of showing that the map on Ext is an isomorphism in a smaller region, note that this is already enough for our purposes: we want to check that $\operatorname{Ext}^{*,*}_{b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k,B_n)}(R, R)$ is zero in particular dimensions, and it suffices to check that in $\operatorname{Ext}^{*,*}_{R\otimes E[e_2,f_{20},e_{n-1},e_n]\otimes D[\xi_{n-2}]}(R, R)$.

We have

$$\operatorname{Ext}^{*,*}_{R\otimes E[e_2,f_{20},e_{n-1},e_n]\otimes D[\xi_{n-2}]}(R, R)\cong R[w_2, b_{20}, b_{n-2,0}, w_{n-1}, w_n]\otimes E[h_{n-2,0}],$$

where $w_i=[e_i]$, $b_{20}=[f_{20}]$ and $\operatorname{Ext}^{*,*}_{D[\xi_{n-2}]}(R, R)=R\otimes E[h_{n-2,0}]\otimes k[b_{n-2,0}]$.

Degree information is as follows:

element	s	t	u	u'	α
w_2	1	1	20	8	0
b_{20}	1	1	48	36	0
$h_{n-2,0}$	1	0	$2(3^{n-2}-1)$	$2(3^{n-2}-1)$	0
$b_{n-2,0}$	2	0	$2(3^{n-1}-3)$	$2(3^{n-1}-3)$	0
w_{n-1}	1	1	$2(3^{n-1}+1)$	$2(3^{n-1}-5)$	3
w_n	1	1	$2(3^n+1)$	$2(3^n-5)$	9
h_{10}	0	1	4	-2	0
b_{10}	0	2	12	0	0

Of course, w_n has the right degree. Any other monomial with the right degree must be in $R[w_2, b_{20}, b_{n-2,0}, w_{n-1}] \otimes E[h_{n-2,0}]$, and it is clear from looking at α degree above that it must have the form $w_{n-1}^3 x$ (where $x \in R[w_2, b_{20}, b_{n-2,0}] \otimes E[h_{n-2,0}]$). Since $u'(w_{n-1}^3) = 2(3^n-15)$, we need $u'(x) = 20$, which is not possible using w_2 in degree 8, b_{20} in degree 36, h_{10} in degree -2 (where $h_{10}^2 = 0$), and $h_{n-2,0}$ and $b_{n-2,0}$ in higher degree.

So the element must be $\pm b_{10}^N w_n$, and by checking u degree we see that the power N has to be zero. $\square$

6.4 Degree-counting in the ISS

Recall that $b_{10}^{-4} h_{10} w_2^2 w_{n-1}^3$ has $\alpha = 9$ and $u' = 2(3^n-8)$; if it were a permanent cycle, it would converge to an element of $b_{10}^{-1}\operatorname{Ext}_{P_n}^{a,b}(k,k)$ with stem $b-6a = 2(3^n-8)$ (see Definition 6.5) and $\alpha = 9$. The goal of this section is to prove:

Proposition 6.17 *The subvector space of $b_{10}^{-1}\operatorname{Ext}_{P_n}^{*,*}(k,k)$ consisting of elements in stem $2(3^n-8)$ and $\alpha = 9$ is zero.*

We will prove this using a (localized) Ivanovskii spectral sequence (ISS) computing $b_{10}^{-1}\operatorname{Ext}_{P_n}^{*,*}(k,k)$. In our case, the ISS is constructed by filtering the cobar complex for P_n by powers of the augmentation ideal. For example, $[\xi_n]$ is in filtration 1, and, in the Milnor diagonal

$$d_{\mathrm{cobar}}([\xi_n]) = [\xi_1 \mid \xi_{n-1}^3] + [\xi_2 \mid \xi_{n-2}^9],$$

$[\xi_1 \mid \xi_{n-1}^3]$ is in filtration 4 (since $[\xi_1]$ is in filtration 1 and $[\xi_{n-1}^3]$ is in filtration 3), and $[\xi_2 \mid \xi_{n-2}^9]$ is in filtration 10. In general, all of the multiplicative generators ξ_1,

ξ_2, ξ_{n-2}, ξ_{n-1} and ξ_n are primitive in the associated graded, ie they are in $\ker d_0$. To form the b_{10}–localized spectral sequence, take the colimit of multiplication by b_{10}. In Section A.2 we show that the (localized and unlocalized) ISS converges in our case.

So we have $E_0 \cong D[\xi_1, \xi_1^3, \xi_2, \xi_{n-2}, \xi_{n-2}^3, \xi_{n-2}^9, \xi_{n-1}, \xi_{n-1}^3, \xi_n]$ and

$$E_1^{\mathrm{ISS}} = E[h_{1i}, h_{20}, h_{n-2,j}, h_{n-1,i}, h_{n0}]_{i\in\{0,1\},\, j\in\{0,1,2\}} \\ \otimes k[b_{10}^{\pm 1}, b_{11}, b_{20}, b_{n-2,j}, b_{n-1,i}, b_{n,0}]_{i\in\{0,1\},\, j\in\{0,1,2\}}.$$

Here $h_{ij} = [\xi_i^{3^j}]$ has filtration 3^j and b_{ij} has filtration 3^{j+1}. To help with the degree-counting argument in Proposition 6.17, here is a table of the degrees of the multiplicative generators of the E_1 page:

element	s	t	u	$u' = u - 6t$	α
h_{10}	1	1	4	–2	0
b_{10}	3	2	12	0	0
h_{11}	3	1	12	6	0
b_{11}	9	2	36	24	0
h_{20}	1	1	16	10	0
b_{20}	3	2	48	36	0
$h_{n-2,0}$	1	1	$2(3^{n-2}-1)$	$2(3^{n-2}-4)$	0
$b_{n-2,0}$	3	2	$2(3^{n-1}-3)$	$2(3^{n-1}-9)$	0
$h_{n-2,1}$	3	1	$2(3^{n-1}-3)$	$2(3^{n-1}-6)$	3
$b_{n-2,1}$	9	2	$2(3^{n}-9)$	$2(3^{n}-15)$	9
$h_{n-2,2}$	9	1	$2(3^{n}-9)$	$2(3^{n}-12)$	9
$b_{n-2,2}$	27	2	$2(3^{n+1}-27)$	$2(3^{n+1}-33)$	27
$h_{n-1,0}$	1	1	$2(3^{n-1}-1)$	$2(3^{n-1}-4)$	3
$b_{n-1,0}$	3	2	$2(3^{n}-3)$	$2(3^{n}-9)$	9
$h_{n-1,1}$	3	1	$2(3^{n}-3)$	$2(3^{n}-6)$	9
$b_{n-1,1}$	9	2	$2(3^{n+1}-9)$	$2(3^{n+1}-15)$	27
$h_{n,0}$	1	1	$2(3^{n}-1)$	$2(3^{n}-4)$	9
$b_{n,0}$	3	2	$2(3^{n+1}-3)$	$2(3^{n+1}-9)$	27

Proof of Proposition 6.17 The argument has two parts:

(1) show the only generators (up to powers of b_{10}) in E_1^{ISS} in degree $u' = 2(3^n-8)$ and $\alpha = 9$ are $h_{10}h_{20}h_{n-2,2}$ and $h_{10}h_{11}h_{20}b_{n-2,1}$;

(2) show that those elements are targets of higher differentials in the b_{10}–local ISS.

From looking at α degrees we see that no monomial in E_1 in degree $u' = 2(3^n - 8)$ and $\alpha = 9$ can be divisible by $b_{n-2,2}$, $b_{n-1,1}$, or $b_{n,0}$, and moreover by looking at u' degree we see it is not possible for $b_{n-1,0}$, $h_{n-1,1}$ or $h_{n,0}$ to be a factor of such a monomial. The only monomial of the right degree divisible by $h_{n-2,2}$ is $b_{10}^N h_{10} h_{20} h_{n-2,2}$. Any remaining elements of the right degree are in

$$E[h_{10}, h_{11}, h_{20}, h_{n-2,0}, h_{n-2,1}, h_{n-1,0}] \otimes k[b_{10}^{\pm 1}, b_{11}, b_{20}, b_{n-2,0}, b_{n-2,1}].$$

Of these generators, only $h_{n-2,1}$, $h_{n-1,0}$ and $b_{n-2,1}$ have $\alpha > 0$. Since $h_{n-2,1}^2 = 0 = h_{n-1,0}^2$, a monomial with $\alpha = 9$ needs to be divisible by $b_{n-2,1}$. If $u'(b_{n-2,1}x) = 2(3^n - 8)$ then $u'(x) = 14$, and the only possibility is $x = b_{10}^N h_{10} h_{11} h_{20}$. (Here we are using the assumption $n \geq 5$ to determine that $u'(h_{n-2,0}) = 2(3^{n-2} - 4) \geq 46$, and the elements following it in the chart have greater degree.)

This concludes part (1) of the argument; for (2) it suffices to show

$$d_9(h_{10} h_{20} b_{n-1,0}) = h_{10} h_{20} h_{11} b_{n-2,1} - b_{10} h_{10} h_{20} h_{n-2,2}, \tag{6-5}$$

$$d_9(b_{10} h_{10} h_{n0}) = -b_{10} h_{10} h_{20} h_{n-2,2}. \tag{6-6}$$

First, we claim that $h_{10} h_{20}$ is a permanent cycle; it is represented by $[\xi_1 \mid \xi_2] - [\xi_1^2 \mid \xi_1^3] = w_2$, which we've seen is a permanent cycle in the cobar complex. The class $b_{n-1,0}$ has cobar representative $[\xi_{n-1} \mid \xi_{n-1}^2] + [\xi_{n-1}^2 \mid \xi_{n-1}]$ and

$$\begin{aligned} b_{n-1,0} \equiv [\xi_{n-1} \mid \xi_{n-1}^2] + [\xi_{n-1}^2 \mid \xi_{n-1}] - [\xi_1 \xi_{n-1} \mid \xi_{n-1} \xi_{n-2}^3] + [\xi_1 \xi_{n-1}^2 \mid \xi_{n-2}^3] \\ + [\xi_1^2 \xi_{n-1} \mid \xi_{n-2}^6] + [\xi_1^2 \mid \xi_{n-1} \xi_{n-2}^6] + [\xi_1 \mid \xi_{n-1}^2 \xi_{n-2}^3] \\ \in (F^3/F^4) C_{P_n}^2(k,k). \end{aligned}$$

Computing the cobar differential on this class (and remembering that $\xi_{n-3}^9 = 0$ in P_n), we see that $d_9(b_{n-1,0}) = h_{11} b_{n-2,1} - b_{10} h_{n-2,2}$. So

$$d_9(h_{10} h_{20} b_{n-1,0}) = h_{10} h_{20} d_9(b_{n-1,0}) = h_{10} h_{20} (h_{11} b_{n-1,1} - b_{10} h_{n-2,2}).$$

We have $h_{10} h_{n0} \equiv [\xi_1 \mid \xi_n] - [\xi_1^2 \mid \xi_{n-1}^3] = w_n \in F^2/F^3$ and there is a cobar differential

$$d_{\mathrm{cobar}}([\xi_1 \mid \xi_n] - [\xi_1^2 \mid \xi_{n-1}^3]) = -[\xi_1 \mid \xi_2 \mid \xi_{n-2}^9] + [\xi_1^2 \mid \xi_1^3 \mid \xi_{n-2}^9].$$

This implies (6-6). (We did not check that $h_{10} h_{20} h_{11} b_{n-2,1}$ and $h_{10} h_{20} b_{10} h_{n-2,2}$ survive to the E_9 page, because that is not necessary: we only have to check that these elements die somehow in the spectral sequence, and if they have already died before the E_9 page, then that is good enough for this argument.) □

7 Some results on higher differentials

In the case $r = 4$, the following proposition gives an explicit way to compute d_8 on any class, given our knowledge of d_4 from the previous section.

Proposition 7.1 *Suppose $\overline{x} \in E_2$ satisfies $d_{r'}(\overline{x}) = 0$ for $r' < r$ and $d_r(\overline{x}) = h_{10}\widetilde{y} \in E_r$. Also suppose $\overline{y}$ is an E_2 representative for $\widetilde{y}$ and $d_4(\overline{y}) = h_{10}\widetilde{z}$. Then $d_{r+4}(h_{10}\overline{x}) = b_{10}\widetilde{z}$.*

Note that the choice $\overline{y}$ does not matter, as two such choices differ (up to E_2 class) by a boundary.

One is tempted to use Massey product arguments, eg try to apply the Massey product differential and extension theorem [8, Theorem 4.5 and Corollary 4.6] to $\langle h_{10}, h_{10}, \widetilde{z}, h_{10}\rangle$, but the following explicit argument avoids Massey product technicalities.

Lemma 7.2 *Suppose $0 \neq \overline{x} \in E_2^{u'(x),s(x)}$ is not h_{10}–divisible, and define $\overline{y} \in E_r^{u'(x)-4,s(x)+r}$ such that $d_r(\overline{x}) = h_{10}\overline{y}$ and $d_{r'}(\overline{x}) = 0$ for $r' < r$. Furthermore, suppose $d_4(\overline{y}) = h_{10}\overline{z}$. Then there is a cobar representative $x \in F^{s(x)}$ of $b_{10}^N\overline{x}$ for some N, a cobar representative $y \in F^{s(x)+r}$ of $b_{10}^N\overline{y}$ and a cobar representative $z \in F^{s(x)+r+4}$ of $b_{10}^N\overline{z}$ such that*

$$d(x) = [\xi_1 \mid y] - [\xi_1^2 \mid z]. \tag{7-1}$$

Proof We prove this by induction on u'. The statement is trivially true for $u' < -2$, since there are no elements of E_2 in those degrees. So let $\overline{x} \in E_2$ with $u'(\overline{x}) \geq -2$, and assume the inductive hypothesis.

By Proposition 2.3, $d_r(\overline{x})$ has the form $h_{10}\overline{y}$. If $\overline{y}$ is not a permanent cycle, we abuse notation by letting $\overline{y}$ denote an E_2 representative. By Proposition 2.3, there is a nontrivial differential $d_R(\overline{y}) = h_{10}\overline{z}$ for some $R \geq 4$ such that $d_{r'}(\overline{y}) = 0$ for $r' < R$. Since $u'(\overline{y}) = u'(\overline{x}) - 4$, we may apply the inductive hypothesis to $\overline{y}$, obtaining a cobar representative y of $b_{10}^N\overline{y}$ for some N, a cobar representative $z \in F^{s(x)+r+R}$ of $b_{10}^N\overline{z}$ and a cobar element $w \in F^{s(x)+r+R+4}$ such that

$$d(y) = [\xi_1 \mid z] - [\xi_1^2 \mid w]. \tag{7-2}$$

If $\overline{y}$ is a permanent cycle, (7-2) holds with $z = 0 = w$.

Since $d_r(b_{10}^N\overline{x}) = b_{10}^N h_{10}\overline{y}$, there exists a cobar representative $x \in F^{s(x)}$ for $b_{10}^N\overline{x} \in E_2^{s=s(x)}$ such that $d(x) \equiv [\xi_1 \mid y] \pmod{F^{s(x)+r+1}}$. In particular, we may write

$$d(x) = [\xi_1 \mid y] - [\xi_1^2 \mid z] + x' \tag{7-3}$$

with $x' \in F^{s(x)+r+1}$. (Note that $[\xi_1^2 \mid z]$ is also in higher filtration than y, and this term is added because it simplifies the next calculation.)

Claim 7.3 *We may choose x and x' such that $x' \in F^{s(x)+r+5}$.*

Proof Applying d to (7-2), we have

$$0 = -[\xi_1 \mid d(z)] + [\xi_1 \mid \xi_1 \mid w] + [\xi_1^2 \mid d(w)].$$

Equating terms starting with ξ_1, we obtain $d(z) = [\xi_1 \mid w]$; equating terms starting with ξ_1^2, we obtain $d(w) = 0$. Applying d to (7-3), we have

$$0 = -[\xi_1 \mid d(y)] + [\xi_1 \mid \xi_1 \mid z] + [\xi_1^2 \mid d(z)] + d(x') = [\xi_1 \mid \xi_1^2 \mid w] + [\xi_1^2 \mid \xi_1 \mid w] + d(x'),$$

so $d(x') = -b_{10}w \in F^{s(x)+r+R+4} \subseteq F^{s(x)+8}$. So x' represents an element of $E_2^{s=s(x)+r+1}$. Since $u'(x') = u'(h_{10}y)$, Lemma 2.2 implies that if x' were nonzero in $E_2^{s(x')}$, then $s(x') \equiv s(h_{10}y) = s(x) + r \pmod 9$. In particular, x' is zero as an element of $E_2^{s(x)+r+1}$, so it must have a representative in higher filtration. Repeating this argument, we find x' is zero as an element of $E_2^{s(x)+r+i}$ for $1 \le i \le 5$. So we may write $x' + d(x_1) \in F^{s(x)+r+5}$, where $x_1 \in F^{s(x)+r}$. Thus, by adjusting the representative x by x_1, we may assume $x' \in F^{s(x)+r+5}$. $\square$

Then

$$\begin{aligned}
d(b_{10}x) &= [\xi_1 \mid \xi_1^2 \mid \xi_1 \mid y] + [\xi_1^2 \mid \xi_1 \mid \xi_1 \mid y] - [\xi_1 \mid \xi_1^2 \mid \xi_1^2 \mid z] \\
&\qquad - [\xi_1^2 \mid \xi_1 \mid \xi_1^2 \mid z] + b_{10}x', \\
d(b_{10}x - [\xi_1^2 \mid \xi_1^2 \mid y]) &= [\xi_1 \mid \xi_1^2 \mid \xi_1 \mid y] - [\xi_1 \mid \xi_1^2 \mid \xi_1^2 \mid z] - [\xi_1^2 \mid \xi_1 \mid \xi_1^2 \mid z] + b_{10}x' \\
&\qquad + [\xi_1 \mid \xi_1 \mid \xi_1^2 \mid y] - [\xi_1^2 \mid \xi_1^2 \mid \xi_1 \mid z] + [\xi_1^2 \mid \xi_1^2 \mid \xi_1^2 \mid w] \\
&=: [\xi_1 \mid \tilde{y}] - [\xi_1^2 \mid \tilde{z}],
\end{aligned}$$

where

$$\tilde{y} := b_{10}y - [\xi_1^2 \mid \xi_1^2 \mid z] + [\xi_1^2 \mid x'], \quad \tilde{z} := b_{10}z - [\xi_1^2 \mid \xi_1^2 \mid w] - [\xi_1 \mid x'].$$

By our assumptions on the filtrations of all the elements involved,

$$\tilde{y} \equiv b_{10} \pmod{F^{s(x)+r}} \quad \text{and} \quad \tilde{z} \equiv b_{10}z \pmod{F^{s(x)+r+5}},$$

so $\tilde{y}$ is a representative of $b_{10}^{N+1}\overline{y}$ and $\tilde{z}$ is a representative of $b_{10}^{N+1}\overline{z}$. $\square$

Proof of Proposition 7.1 Use Lemma 7.2 to write

$$d(x)=[\xi_1\mid y]-[\xi_1^2\mid z], \tag{7-4}$$

where x is a cobar representative for $b_{10}^N\overline{x}$, y is a cobar representative for $b_{10}^N\overline{y}$ and z is a cobar representative for $b_{10}^N\widetilde{z}$. Applying d to (7-4),

$$0=-[\xi_1\mid d(y)]+[\xi_1\mid \xi_1\mid z]-[\xi_1^2\mid d(z)].$$

Equating terms whose first component is ξ_1, we have $d(y)=[\xi_1\mid z]$; equating terms whose first component is ξ_1^2, we have $d(z)=0$. Then $[\xi_1|x]-[\xi_1^2|y]$ is a representative for $h_{10}\overline{x}$, and we have

$$d([\xi_1\mid x]-[\xi_1^2\mid y])=[\xi_1\mid \xi_1^2\mid z]+[\xi_1^2\mid \xi_1\mid z]=b_{10}z.$$

Thus, in the b_{10}–localized spectral sequence, $d_{r+4}(b_{10}^N h_{10}\overline{x})=b_{10}^N\widetilde{z}$ implies that $d_{r+4}(h_{10}\overline{x})=b_{10}\widetilde{z}$. □

Conjecture 7.4 The $K(\xi_1)$–based MPASS collapses at E_9.

Using computer calculations, we verified the conjecture for stems ≤ 600. However, it is not possible to rule out higher differentials based only on degree.

Proposition 7.5 *Assuming Conjecture 7.4, we have*

$$b_{10}^{-1}\operatorname{Ext}_P^{*,*}(k,k)\cong b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k,k[\widetilde{w}_2,\widetilde{w}_3,\dots]),$$

where $\widetilde{w}_n=b_{10}^{-1}w_n$ *and the* D*–coaction on the* E_2 *page is given by* $\psi(\widetilde{w}_n)=1\otimes\widetilde{w}_n+\xi_1\otimes h_{10}\widetilde{w}_2^2\widetilde{w}_{n-1}^3$ *for* $n\geq 3$.

Proof Let $\widetilde{W}=k[\widetilde{w}_2,\widetilde{w}_3,\dots]$. We have $d_4(\widetilde{w}_n)=h_{10}\widetilde{w}_2^2\widetilde{w}_{n-1}^3$. By Proposition 2.3 and Conjecture 7.4, the E_∞ page of the MPASS is obtained by taking the cohomology of E_2 by d_4 and d_8; more precisely, we have

$$E_\infty\cong\ker(d_4|_{W_+})/\operatorname{im}(d_8|_{W_+})\oplus\ker(d_8|_{W_-})/\operatorname{im}(d_4|_{W_-}).$$

If we let $\partial(x)=\frac{1}{h_{10}}d_4(x)$, then Proposition 7.1 says that $b_{10}\partial^2(x)=d_8(h_{10}x)$. Thus we may write down an isomorphism f of chain complexes,

$$\begin{tikzcd}
\cdots \arrow[r] & \widetilde{W} \arrow[r, "\partial"] \arrow[d, "f^{2n}"] & \widetilde{W} \arrow[r, "\partial^2"] \arrow[d, "f^{2n+1}"] & \widetilde{W} \arrow[r, "\partial"] \arrow[d, "f^{2n+2}"] & \cdots \\
\cdots \arrow[r] & \widetilde{W}\{b_{10}^n\} \arrow[r, "d_4"] & \widetilde{W}\{h_{10}b_{10}^n\} \arrow[r, "d_8"] & \widetilde{W}\{b_{10}^{n+1}\} &
\end{tikzcd}$$

By Lemma 4.2, the cohomology of the top complex is $b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k,\widetilde{W})$, and we have argued below that the cohomology of the bottom complex is E_∞. Thus we have an isomorphism of vector spaces $b_{10}^{-1}\operatorname{Ext}_P^{*,*}(k,k)\cong b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k,\widetilde{W})$.

It remains to show that this is an isomorphism of R–modules. We will just check that the induced map f_* on cohomology respects h_{10}–multiplication. If $\omega=[x]\in\widetilde{W}^{2n}$ is a cycle, then $h_{10}\omega$ is represented by $[x]\in\widetilde{W}^{2n+1}$. If $\nu=[y]\in\widetilde{W}^{2n+1}$ is a cycle, then $h_{10}\nu$ is represented by $[\partial y]\in\widetilde{W}^{2n}$. So $f_*^{2n+1}(h_{10}\omega)=[h_{10}b_{10}^n x]=h_{10}[b_{10}^n x]=h_{10}f_*^{2n}(\omega)$. For the other case, we need to show that $f_*^{2n+2}(h_{10}\nu)=[b_{10}^{n+1}(\partial y)]$ can be represented as $h_{10}\cdot[h_{10}b_{10}^n y]=h_{10}f_*^{2n+1}(\nu)$. This corresponds to a hidden multiplication in the MPASS. From the commutativity of the diagram we have $d_4([b_{10}^n y])=[h_{10}b_{10}^n\partial y]=h_{10}[b_{10}^n\partial y]$. The desired relation $h_{10}[h_{10}b_{10}^n y]=[b_{10}^{n+1}(\partial y)]$ follows from Lemma 7.6. $\square$

Lemma 7.6 *Suppose $d_r(\overline{x})=h_{10}\overline{y}$, where $\overline{x}\in W_+$ and $d_{r'}(\overline{x})=0$ for $r'<r$. Then there is a hidden multiplication $h_{10}\cdot(h_{10}\overline{x})=-b_{10}\overline{y}$.*

This is closely related to the Massey product shuffle $h_{10}(h_{10}\overline{x})=h_{10}\langle h_{10},h_{10},\overline{y}\rangle=\langle h_{10},h_{10},h_{10}\rangle\overline{y}$, though the following explicit argument avoids Massey product technicalities.

Proof Use Lemma 7.2 to find a representative x such that $d(x)=[\xi_1\mid y]-[\xi_1^2\mid z]$, where y is a representative for $\overline{y}$ and z is a representative for $\overline{z}$ such that $d_4(\overline{y})=h_{10}\overline{z}$. We use $[\xi_1\mid x]-[\xi_1^2\mid y]$ as a representative for $h_{10}\overline{x}$. Then $h_{10}\cdot(h_{10}\overline{x})$ is represented by $[\xi_1\mid\xi_1\mid x]-[\xi_1\mid\xi_1^2\mid y]$. Since $d([\xi_1^2\mid x])=-[\xi_1\mid\xi_1\mid x]-[\xi_1^2\mid\xi_1\mid y]+[\xi_1^2\mid\xi_1^2\mid z]$, we have

$$[\xi_1\mid\xi_1\mid x]-[\xi_1\mid\xi_1^2\mid y]+d(\xi_1^2\mid x)=-b_{10}y+[\xi_1^2\mid\xi_1^2\mid z]\equiv -b_{10}\overline{y}. \tag{7-5}$$ $\square$

8 Localized cohomology of a large quotient of P

In this section we will prove Theorem 1.5, a complete calculation of b_{10}–local cohomology of a small P–comodule. Using the change-of-rings theorem, this is equivalent to the following:

Theorem 8.1 *Let $D_{1,\infty}=k[\xi_1,\xi_2,\dots]/(\xi_1^3)$. Then*

$$b_{10}^{-1}\operatorname{Ext}_{D_{1,\infty}}^{*,*}(k,k)\cong E[h_{10},h_{20}]\otimes P[b_{10}^{\pm1},b_{20},w_3,w_4,\dots].$$

In particular, one can write

$$b_{10}^{-1}\operatorname{Ext}^{*,*}_{D_{1,\infty}}(k,k) \cong b_{10}^{-1}\operatorname{Ext}^{*,*}_{D}(k,k[h_{20},b_{20},w_3,w_4,\dots]/(h_{20}^2)),$$

where all the generators h_{20}, b_{20} *and* w_n *are* D*–primitive.*

Though $D_{1,\infty}$ seems reasonably close to P in size, the computation of its b_{10}–local cohomology is much simpler. In particular, attempting to apply the methods in this section (especially the explicit construction in Lemma 8.7) to computing $b_{10}^{-1}\operatorname{Ext}^{*,*}_{P}(k,k)$ quickly becomes intractable.

The strategy is to explicitly construct a map from the cobar complex $C_{D_{1,\infty}}(k,k)$ to another complex which is designed to have the right cohomology, and then show the map is a quasi-isomorphism. Note that the cobar complex is a dga under the concatenation product, so every element is a product of elements in degree 1. Thus if our target complex is a dga, it suffices to construct a map out of $C^1_{D_{1,\infty}}(k,k) = \overline{D}_{1,\infty}$, and then extend the map to all of $C^*_{D_{1,\infty}}(k,k)$ by multiplicativity. In order to ensure the resulting map is a map of complexes, there is a criterion that the map on degree 1 needs to satisfy:

Proposition 8.2 *Let* Γ *be a Hopf algebra over* k, Q^* *be a dga with augmentation* $k \to Q^*$ *and* $\theta\colon \overline{\Gamma} \to Q^1$ *be a* k*–linear map such that*

$$d_Q(\theta(x)) = \sum \theta(x')\theta(x'') \tag{8-1}$$

for all $x \in \overline{\Gamma}$*, where* $\sum x' \otimes x''$ *is the reduced diagonal* $\overline{\Delta}(x)$*. Then there is a map of dgas* $f\colon C^*_\Gamma(k,k) \to Q^*$ *sending* $[a_1|\cdots|a_n]$ *to* $\prod \theta(a_i)$*.*

Proof We just need to check that f commutes with the differential; that is, we have to check the following diagram commutes:

$$\begin{tikzcd}
C^n_\Gamma(k,k) \arrow[r, "f"] \arrow[d, "d_{\mathrm{cobar}}"'] & Q^n \arrow[d, "d_Q"] \\
C^{n+1}_\Gamma(k,k) \arrow[r, "f"] & Q^{n+1}
\end{tikzcd}$$

For $n=1$, this is precisely what the condition (8-1) guarantees. Commutativity for $n>1$ follows from the Leibniz rule. The map on $n=0$ is the augmentation. □

Remark 8.3 This is an example of the more general construction of *twisting cochains*; see [5, Section II.1]. A morphism θ satisfying (8-1) will be called a *twisting morphism*.

The target of our desired twisting morphism will be the complex $b_{10}^{-1}\widetilde{U}^* \otimes W'$, where

- $W' = k[w_3, w_4, \dots]$, with $u(w_n) = 2(3^n - 1)$, is in homological degree zero with zero differential, and
- $\widetilde{U}^* := UL^*(\xi_1) \otimes UL^*(\xi_2) \subset C^*_{D[\xi_1,\xi_2]}(k,k)$, where the sub-dga $UL^*(x) \subset C^*_{D[x]}(k,k)$ is defined below.

Definition 8.4 Given a height-3 truncated polynomial algebra $D[x]$, let $UL^*(x)$ be the sub-dga of $C^*_{D[x]}(k,k)$ multiplicatively generated by the elements $\alpha = [x]$, $\beta = [x^2]$ and $\gamma = [x \mid x^2] + [x^2 \mid x]$. This inherits from $C^*_{D[x]}(k,k)$ the differentials $d(\alpha) = 0$, $d(\beta) = -\alpha^2$ and $d(\gamma) = 0$, along with the relations $\alpha\beta + \beta\alpha = \gamma$, $\alpha^3 = 0$ and $\beta^2 = 0$.

Remark 8.5 This is (up to signs) the $p = 3$ case of a construction due to Moore: let UL^* be the dga which has multiplicative generators $a_1, \dots, a_{p-1}$ in degree 1 and $t_2, \dots, t_p$ in degree 2 with $d(a_i) = t_i$, subject to

$$a_1^2 = t_2, \qquad a_i^2 = 0 \quad \text{for } i \neq 1, \qquad a_1^p = 0,$$
$$a_i a_j = -a_j a_i \quad \text{for } i, j \neq 1, \qquad a_j a_1 = -a_1 a_j + t_{j+1}, \qquad a_i t_j = t_j a_i,$$
$$t_i t_j = t_j t_i.$$

This is a dga quasi-isomorphic to, and much smaller than, $C_{k[x]/x^p}(k,k)$. It also has the nice property that t_p (which, in the case $x = \xi_1$, represents b_{10}) is central.

Notation 8.6 Denote the generators of $UL^*(\xi_1)$ by $a_1 = [\xi_1]$, $a_2 = [\xi_1^2]$ and $b_{10} = [\xi_1 \mid \xi_1^2] + [\xi_1^2 \mid \xi_1]$, and the generators of $UL^*(\xi_2)$ by $q_1 = [\xi_2]$, $q_2 = [\xi_2^2]$ and $b_{20} = [\xi_2 \mid \xi_2^2] + [\xi_2^2 \mid \xi_2]$. (This definition of b_{10} and b_{20} does, of course, match up with the image of b_{10} and b_{20} along $\mathrm{Ext}_P^{*,*}(k,k) \to \mathrm{Ext}^{*,*}_{D[\xi_1,\xi_2](k,k)}$, and even $\mathrm{Ext}_P^{*,*}(k,k) \to \mathrm{Ext}^{*,*}_{D_{1,\infty}}(k,k)$.) Note that

$$H^*(\widetilde{U}) = H^*(C_{D[\xi_1,\xi_2]}(k,k)) = E[h_{10}, h_{20}] \otimes P[b_{10}, b_{20}].$$

So our target complex $b_{10}^{-1}\widetilde{U} \otimes W'$ has cohomology

$$H^*(b_{10}^{-1}\widetilde{U} \otimes W') = H^*(b_{10}^{-1}\widetilde{U}) \otimes W' = E[h_{10}, h_{20}] \otimes P[b_{10}^{\pm 1}, b_{20}] \otimes W'. \tag{8-2}$$

8.1 Defining $\theta\colon \overline{D}_{1,\infty} \to b_{10}^{-1}\widetilde{U} \otimes W'$

The definition of the map $\theta\colon \overline{D}_{1,\infty} \to b_{10}^{-1}\widetilde{U}^* \otimes W'$ is quite ad hoc, and will be done in several stages. The map will arise as a composition

$$D_{1,\infty} \to D' \to \widetilde{U}^* \otimes W' \to b_{10}^{-1}\widetilde{U}^* \otimes W',$$

where the first map is the natural surjection to

$$D' := k[\xi_1, \xi_2, \dots]/(\xi_1^3, \xi_2^9, \xi_3^9, \dots)$$

and the last map is the natural localization map; the main goal is to construct a map $D' \to \widetilde{U}^* \otimes W'$ satisfying the twisting morphism condition, and we begin by constructing a map out of a slightly smaller coalgebra.

Lemma 8.7 *Let*

$$C = k[\xi_1, \xi_2^3, \xi_3, \xi_4, \dots]/(\xi_1^3, \xi_2^9, \xi_3^9, \dots).$$

There is a twisting morphism $\theta\colon \overline{C} \to UL^1(\xi_1) \otimes W'$.

Proof For $n, m, k \geq 3$, define

$$\begin{aligned} \theta(\xi_1) &= a_1, & \theta(\xi_{n-1}^3 \xi_{m-1}^3) &= a_2 w_n w_m, \\ \theta(\xi_1^2) &= a_2, & \theta(\xi_n \xi_{m-1}^3) &= 0, \\ \theta(\xi_{n-1}^3) &= -a_1 w_n, & \theta(\xi_n \xi_m) &= 0, \\ \theta(\xi_n) &= a_2 w_n, & \theta(\xi_1^2 \xi_{n-1}^3) &= 0, \\ \theta(\xi_1 \xi_{n-1}^3) &= -a_2 w_n, & \theta(\xi_1 \xi_{n-1}^3 \xi_{m-1}^3) &= 0, \\ \theta(\xi_1 \xi_n) &= 0, & \theta(\xi_{n-1}^3 \xi_{m-1}^3 \xi_{k-1}^3) &= 0. \end{aligned}$$

It is a straightforward computation with the cobar differential to check that each of these does not violate the twisting morphism condition

$$d(\theta(x)) = \sum \theta(x') \cdot \theta(x''), \tag{8-3}$$

where $\overline{\Delta}(x) = \sum x' \otimes x''$. (Note that, in C, we have $\overline{\Delta}(\xi_{n-1}^3) = 0$ and $\overline{\Delta}(\xi_n) = \xi_1 | \xi_{n-1}^3$.) Now it suffices to prove the following:

Claim 8.8 *Defining* $\theta(X) = 0$ *for all monomials* X *except the ones listed above defines a twisting morphism.*

Define a (nonmultiplicative) grading ρ on C, where

$$\rho(1) = 0, \quad \rho(\xi_1) = 1, \quad \rho(\xi_1^2) = 2,$$
$$\rho(\xi_{n-1}^3) = 1, \quad \rho(\xi_{n-1}^6) = 2, \quad \rho(\xi_n) = 2, \quad \rho(\xi_n^2) = 4$$

for $n \geq 3$, and $\rho\bigl(\prod_i \xi_i^{a_i + 3b_i}\bigr) = \sum \rho(\xi_i^{a_i}) + \rho(\xi_i^{3b_i})$ (where $a_i, b_i \in \{0, 1, 2\}$). The reason for considering this grading is the following:

Claim 8.9 *Writing $\Delta(x) = \sum x' \otimes x''$, we have $\rho(x') + \rho(x'') \le \rho(x)$.*

Proof If $X = \prod \xi_i^{a_i+3b_i}$ for $a_i, b_i \in \{0,1,2\}$, consider the collection

$$\mathscr{T}_X = \{\xi_i^{a_i} : a_i \neq 0\} \cup \{\xi_i^{3b_i} : b_i \neq 0\}.$$

Use induction on $n := \#\mathscr{T}_X$. If $n = 1$, then it suffices to check explicitly the Milnor diagonal of each of the terms $\{\xi_1, \xi_1^2, \xi_{i-1}^3, \xi_{i-1}^6, \xi_i, \xi_i^2\}$. (In fact, we find $\rho(x) = \rho(x') + \rho(x'')$ for each of these terms.)

For general monomials a and b, we have

$$\rho(ab) \le \rho(a) + \rho(b). \tag{8-4}$$

By definition, if x and y are products of nonoverlapping subsets of $\mathscr{T}_X$, then

$$\rho(xy) = \rho(x) + \rho(y). \tag{8-5}$$

Write $X = xy$, where $x \in \mathscr{T}_X$ and y is a product of terms in $\mathscr{T}_X$ (different from x). Since $\Delta(xy) = \sum x'y' \mid x''y''$ it suffices to prove $\rho(x'y') + \rho(x''y'') \le \rho(xy)$. We have

$$\rho(x'y') + \rho(x''y'') \le \rho(x') + \rho(y') + \rho(x'') + \rho(y'') \le \rho(x) + \rho(y) = \rho(xy),$$

where the first inequality is by (8-4), the second inequality is by the inductive hypothesis, and the last equality is by (8-5). $\square$

So the monomials in C with degree 1 are ξ_1 and ξ_{n-1}^3 for $n \ge 3$, the monomials with ρ degree 2 are ξ_1^2, ξ_n, $\xi_{n-1}^3\xi_{m-1}^3$ and $\xi_1\xi_{n-1}^3$ for $n, m \ge 3$, and the monomials with degree 3 are $\xi_1^2\xi_{n-1}^3$, $\xi_1\xi_{n-1}^3\xi_{m-1}^3$, $\xi_{n-1}^3\xi_{m-1}^3\xi_{k-1}^3$, $\xi_1\xi_n$ and $\xi_{n-1}^3\xi_m$ for $n, m \ge 3$. Notice that θ has already been defined for these monomials above. So it remains to show that θ can be defined consistently for monomials with $\rho \ge 4$. In particular, we will show using induction on ρ degree that we can define $\theta(x) = 0$ if $\rho(x) \ge 3$ while preserving the twisting morphism condition (8-1).

Since we have already checked above that we can define $\theta(x) = 0$ on the monomials x with $\rho(x) = 3$, let $\rho(x) = n > 3$ and assume inductively that we have already defined $\theta(y) = 0$ if $3 \le \rho(y) \le n-1$. Any monomial y with $\rho(y) = 0$ is in k (and hence $\theta(y) = 0$), so we can assume that $\rho(x') < \rho(x)$ and $\rho(x'') < \rho(x)$. So, by the inductive hypothesis, we have $\sum \theta(x') \cdot \theta(x'') = 0$, and so we can set $\theta(x) = 0$ without violating (8-1). $\square$

Lemma 8.10 *One may extend* θ *constructed in* Lemma 8.7 *to a twisting morphism* $\overline{D'} \to \widetilde{U}^1 \otimes W'$ *by defining*

$$\theta(\xi_2)=q_1, \qquad \theta(\xi_2^2)=q_2, \qquad \theta(\xi_2 x)=0 \quad \text{for } x\in\overline{C}, \qquad \theta(\xi_2^2 x)=0 \quad \text{for } x\in\overline{C},$$

where $\overline{C}$ *is the cokernel of the unit map* $k \to C$.

Proof Note that ξ_2 is primitive in D', and C is a subcoalgebra of D', so we need to define θ on $\xi_2 C$ and $\xi_2^2 C$. It is straightforward to check that $\theta(\xi_2) = q_1$ and $\theta(\xi_2^2) = q_2$ is consistent with (8-1).

If $x = \xi_2 y$ for $y \in \overline{C}$, then every y', y'' in Δy is in C, and

$$\begin{aligned}\sum \theta(x')\cdot\theta(x'') &= \sum(\theta(\xi_2 y')\cdot\theta(y'')+\theta(y')\cdot\theta(\xi_2 y''))\\ &= \theta(\xi_2)\theta(y)+\theta(y)\theta(\xi_2)+\sum_{y',y''\notin k}(\theta(\xi_2 y')\cdot\theta(y'')+\theta(y')\cdot\theta(\xi_2 y''))\\ &= q_1\theta(y)+\theta(y)q_1+\sum_{y',y''\notin k}(\theta(\xi_2 y')\cdot\theta(y'')+\theta(y')\cdot\theta(\xi_2 y'')).\end{aligned}$$

Since $\theta(y) \in UL^1(\xi_1)\otimes W'$ and q_1 anticommutes with the generators a_1 and a_2 of $UL^1(\xi_1)$, we have $q_1\theta(y)+\theta(y)q_1 = 0$. Thus, defining $\theta(\xi_2 y) = 0$ does not violate (8-1).

Similarly, if $x = \xi_2^2 y$ for $y \in \overline{C}$, then

$$\begin{aligned}\sum \theta(x')\cdot\theta(x'') &= \sum(\theta(\xi_2^2 y')\cdot\theta(y'')+2\theta(\xi_2 y')\cdot\theta(\xi_2 y'')+\theta(y')\cdot\theta(\xi_2^2 y''))\\ &= \theta(\xi_2^2)\theta(y)+2\theta(\xi_2)\theta(\xi_2 y)+2\theta(\xi_2 y)\theta(\xi_2)+\theta(y)\theta(\xi_2^2)\\ &\qquad + \sum_{y',y''\notin k}(\theta(\xi_2^2 y')\cdot\theta(y'')+2\theta(\xi_2 y')\cdot\theta(\xi_2 y'')+\theta(y')\cdot\theta(\xi_2^2 y''))\\ &= \theta(\xi_2^2)\theta(y)+\theta(y)\theta(\xi_2^2)+\sum_{y',y''\notin k}(\theta(\xi_2^2 y')\theta(y'')+\theta(y')\theta(\xi_2^2 y'')),\end{aligned}$$

where in the third equality we use the fact that $0 = \theta(\xi_2 y) = \theta(\xi_2 y') = \theta(\xi_2 y'')$ (for $y', y'' \notin k$). Again, $\theta(\xi_2^2)\theta(y)+\theta(y)\theta(\xi_2^2) = q_2\theta(y)+\theta(y)q_2$, which is zero since $\theta(y)$ is in $UL^1(\xi_1)\otimes W'$ and q_2 anticommutes with the generators a_1 and a_2 of $UL^1(\xi_1)$. So it is consistent with (8-1) to define $\theta(\xi_2^2 y) = 0$. □

Now precompose with the surjection $q\colon D_{1,\infty} \to D'$ to obtain a twisting morphism

$$\theta\colon D_{1,\infty} \to D' \to \widetilde{U}^1 \otimes W'.$$

This remains a twisting morphism because it is a coalgebra map — in particular, q commutes with the coproduct — and so $d\big(\theta(q(x))\big) = \sum \theta(q(x)')\theta(q(x)'') = \sum \theta(q(x'))\theta(q(x''))$. So, by Proposition 8.2, we get an induced map

$$\theta'\colon C^*_{D_{1,\infty}}(k,k) \to \widetilde{U}^* \otimes W' \tag{8-6}$$

by extending θ multiplicatively using the concatenation product on the cobar complex.

8.2 Showing θ is a quasi-isomorphism via spectral sequence comparison

Our goal is to show that the map

$$\theta'\colon C^*_{D_{1,\infty}}(k,k) \to \widetilde{U}^* \otimes W'$$

induces an isomorphism in cohomology after inverting b_{10}.

To prove this, we define filtrations on $C^*_{D_{1,\infty}}(k,k)$ and on $\widetilde{U}^* \otimes W'$ in a way that makes θ' a filtration-preserving map; this induces a map of filtration spectral sequences. We compute the E_2 pages of both sides and show that θ' induces an isomorphism of E_2 pages, hence an isomorphism of E_∞ pages.

Let $B_{1,\infty} := k[\xi_2, \xi_3, \dots] = D_{1,\infty} \square_D k$. Define a decreasing filtration on $C^*_{D_{1,\infty}}(k,k)$, where $[a_1|\cdots|a_n]$ is in $F^s C^*_{D_{1,\infty}}(k,k)$ if at least s of the a_i are in $\ker(D_{1,\infty} \to D) = \bar{B}_{1,\infty} D_{1,\infty}$. Define a decreasing filtration on $\widetilde{U}^* \otimes W'$ by the multiplicative grading

- $|a_1| = |a_2| = |b_{10}| = 0$,
- $|q_1| = |q_2| = 1$,
- $|b_{20}| = 2$,
- $|w_n| = 1$.

Looking at the definition of θ in Lemmas 8.7 and 8.10, it is clear that θ is filtration-preserving, and hence so is θ'.

For the same reasons that the $b_{10}^{-1}B$–based MPASS coincides at E_1 with the filtration spectral sequence mentioned in Section 1.1, the $b_{10}^{-1}B_{1,\infty}$–based MPASS for computing $b_{10}^{-1}\operatorname{Ext}^{*,*}_{D_{1,\infty}}(k,k)$ coincides with the b_{10}–localized version of the filtration spectral sequence on $C^*_{D_{1,\infty}}(k,k)$ defined above. Our next goal is to calculate the E_2 page of (the b_{10}–localized version of) the filtration spectral sequence on $C^*_{D_{1,\infty}}(k,k)$, and using this correspondence we may instead calculate the MPASS E_2 term

$$E_2^{s,*} = b_{10}^{-1}\operatorname{Ext}^{s,*}_{b_{10}^{-1}\operatorname{Ext}^{*,*}_D(k,B_{1,\infty})}(b_{10}^{-1}\operatorname{Ext}^{*,*}_D(k,k), b_{10}^{-1}\operatorname{Ext}^{*,*}_D(k,k)). \tag{8-7}$$

So we need to compute $b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k, B_{1,\infty})$ and its coalgebra structure. The correspondence of spectral sequences further gives that

$$E_1^{1,*} = b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k, \bar{B}_{1,\infty}) \cong b_{10}^{-1} H^*(F^1/F^2 C^*_{D_{1,\infty}}(k,k)) \tag{8-8}$$

and the reduced diagonal on $b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k, B_{1,\infty})$ coincides with d_1 in the filtration spectral sequence.

Proposition 8.11 *As coalgebras, we have*

$$b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k, B_{1,\infty}) \cong b_{10}^{-1} E[e_3, e_4, \ldots] \otimes D[\xi_2],$$

ie e_n and ξ_2 are primitive and $\bar{\Delta}(\xi_2^2) = 2\xi_2 \otimes \xi_2$.

Proof The first task is to determine the D–comodule structure on $B_{1,\infty}$. Let ψ denote the D–coaction induced by the D–coaction on P, and $\partial\colon B_{1,\infty} \to B_{1,\infty}$ denote the operator defined by $\psi(x) = 1 \otimes x + \xi_1 \otimes \partial x - \xi_1^2 \otimes \partial^2 x$ (see Definition 4.1). For example, $\partial(\xi_n) = \xi_{n-1}^3$, $\partial(\xi_{n-1}^3) = 0$, and ∂ satisfies the Leibniz rule.

We have a coalgebra isomorphism $B_{1,\infty} \cong D[\xi_2] \otimes k[\xi_2^3, \xi_3, \xi_4, \ldots]$. Since 1, ξ_2 and ξ_2^2 are all primitive, $D[\xi_2]$ splits as D–comodule into three trivial D–comodules, generated by 1, ξ_2 and ξ_2^2, respectively. So it suffices to determine the D–comodule structure of $k[\xi_2^3, \xi_3, \xi_4, \ldots]$.

As part of the determination of the structure of $b_{10}^{-1}\operatorname{Ext}_D^{*,*}(k, B)$ in Section 4.1, we showed that there is a D–comodule decomposition

$$B \cong \bigoplus_{\substack{\xi_{n_1}\cdots\xi_{n_d} \\ n_i \geq 2 \text{ distinct}}} T(\langle \xi_{n_1} \cdots \xi_{n_d}; 1\rangle) \oplus F,$$

where F is a free D–comodule and $T(\langle \xi_{n_1} \cdots \xi_{n_d}; 1\rangle)$ is generated as a vector space by monomials of the form $\partial^{\varepsilon_1}(\xi_{n_1}) \cdots \partial^{\varepsilon_d}(\xi_{n_d})$ for $\varepsilon_i \in \{0,1\}$. I claim the surjection $f\colon B \to k[\xi_2^3, \xi_3, \xi_4, \ldots]$ takes F to another free summand: this map preserves the direct sum decomposition into summands of the form D, $M(1)$ and k, and the image of a free summand D must be either 0 or another free summand (just as there are no D*–module* maps $k = k[x]/(x) \to D$ or $M(1) = k[x]/(x^2) \to D$, there are no D–comodule maps $D \to k$ or $D \to M(1)$).

Furthermore, I claim that f acts as zero on summands $T(\langle \xi_{n_1} \cdots \xi_{n_d}; 1\rangle)$ where some $n_i = 2$, and is the identity otherwise. In the first case, every basis element $\partial^{\varepsilon_i}(\xi_2)\prod_{j\neq i}\partial^{\varepsilon_j}(\xi_{n_j})$ in $T(\langle \xi_{n_1} \cdots \xi_{n_d}; 1\rangle)$ has either the form $\xi_2 \prod_{j\neq i}\partial^{\varepsilon_j}(\xi_{n_j})$ or

$\xi_1^3 \prod_{j\neq i} \partial^{\varepsilon_j}(\xi_{n_j}) \in \xi_1^3 \cdot k[\xi_2^3, \xi_3, \xi_4, \dots]$, and these are sent to zero under f. If instead $n_i > 2$ for every i, then every term $\partial^{\varepsilon_1}(\xi_{n_1}) \cdots \partial^{\varepsilon_d}(\xi_{n_d})$ is in $k[\xi_2^3, \xi_3, \xi_4, \dots]$ and so f acts as the identity. So we have shown that there is a D–comodule isomorphism

$$B_{1,\infty} = \Bigg(\bigoplus_{\substack{\xi_{n_1}\cdots\xi_{n_d} \\ n_i \geq 3 \text{ distinct}}} T(\langle \xi_{n_1} \cdots \xi_{n_d}; 1\rangle) \oplus F' \Bigg) \otimes (k_1 \oplus k_{\xi_2} \oplus k_{\xi_2^2}),$$

where F' is a free D–comodule. So we have

$$\begin{aligned} b_{10}^{-1} \operatorname{Ext}_D^{*,*}(k, B_{1,\infty}) &\cong \bigoplus_{\substack{\xi_{n_1}\cdots\xi_{n_d} \\ n_i \geq 3 \text{ distinct}}} b_{10}^{-1} \operatorname{Ext}_D^{*,*}\big(k, T(\langle \xi_{n_1} \cdots \xi_{n_d}; 1\rangle) \otimes k\{1, \xi_2, \xi_2^2\}\big) \\ &\cong \bigoplus_{\substack{\xi_{n_1}\cdots\xi_{n_d} \\ n_i \geq 3 \text{ distinct}}} b_{10}^{-1} \operatorname{Ext}_D^{*,*}\big(k, T(\langle \xi_{n_1} \cdots \xi_{n_d}; 1\rangle)\big) \otimes k\{1, \xi_2, \xi_2^2\}. \end{aligned}$$

By Lemma 5.7, $b_{10}^{-1} \operatorname{Ext}_D^{d,*}\big(k, T(\langle \xi_{n_1} \cdots \xi_{n_d}; 1\rangle)\big)$ is generated by $e_{n_1} \cdots e_{n_d}$, where

$$e_n = [\xi_1]\xi_n - [\xi_1^2]\xi_{n-1}^3 \in b_{10}^{-1} \operatorname{Ext}_D^{1,2(3^n+1)}\big(k, T(\langle \xi_n; 1\rangle)\big)$$

is primitive. The map $B \to B_{1,\infty}$ gives rise to a map of MPASSs, and in particular a map $b_{10}^{-1} \operatorname{Ext}_D^{*,*}(k, B) \to b_{10}^{-1} \operatorname{Ext}_D^{*,*}(k, B_{1,\infty})$ of Hopf algebras over $b_{10}^{-1} \operatorname{Ext}_D^{*,*}(k, k)$ sending $e_n \mapsto e_n$ for $n \geq 3$, and $e_2 \mapsto h_{10} \cdot \xi_2$. In particular, we have

$$b_{10}^{-1} \operatorname{Ext}_D^{*,*}(k, B_{1,\infty}) \cong E[h_{10}, e_3, e_4, \dots] \otimes P[b_{10}^{\pm 1}] \otimes k\{1, \xi_2, \xi_2^2\} \tag{8-9}$$

and $e_n \in b_{10}^{-1} \operatorname{Ext}_D^{*,*}(k, B_{1,\infty})$ is primitive. To find the coproduct on the elements ξ_2 and ξ_2^2, use (8-8), in particular the fact that the (reduced) Hopf algebra diagonal corresponds to d_1 in the filtration spectral sequence. In particular, $\xi_2 \in b_{10}^{-1} \operatorname{Ext}_D^{0,16}(k, B_{1,\infty})$ corresponds to the element $[\xi_2] \in F^1/F^2 C^1_{D_{1,\infty}}(k, k)$, and we have $d_{\mathrm{cobar}}([\xi_2]) = [\xi_1 | \xi_1^3]$, which is zero in $C^*_{D_{1,\infty}}(k, k)$, so ξ_2 is primitive. Similarly, the cobar differential on $C^*_{D_{1,\infty}}(k, k)$ shows $\overline{\Delta}(\xi_2^2) = 2\xi_2 \otimes \xi_2$. Thus the tensor factor $k\{1, \xi_2, \xi_2^2\}$ is, as a coalgebra, a truncated polynomial algebra. This finishes the determination of the coalgebra structure of $b_{10}^{-1} \operatorname{Ext}_D^{*,*}(k, B_{1,\infty})$ in (8-9). □

The E_2 page (8-7) of the MPASS is the cohomology of the Hopf algebroid

$$\begin{aligned} &(b_{10}^{-1} \operatorname{Ext}_D^{*,*}(k, k), b_{10}^{-1} \operatorname{Ext}_D^{*,*}(k, B_{1,\infty})) \\ &\qquad = (E[h_{10}] \otimes P[b_{10}^{\pm 1}], E[h_{10}, e_3, e_4, \dots] \otimes P[b_{10}^{\pm 1}] \otimes D[\xi_2]), \end{aligned}$$

so we have:

Corollary 8.12 *The MPASS E_2 page is*

$$E_2^{**} \cong E[h_{10}, h_{20}] \otimes P[b_{10}^{\pm 1}, b_{20}, w_3, w_4, \dots].$$

Proposition 8.13 *The map θ' induces an isomorphism of E_2 pages after inverting b_{10}.*

Proof We first show that the E_2 pages of the filtration spectral sequences on $C_P^*(k,k)$ and $\widetilde{U}^* \otimes W'$ are abstractly isomorphic after inverting b_{10}. By the machinery of Section 1.1, it suffices to calculate the E_2 page for $\widetilde{U}^* \otimes W'$ and check that it coincides with the E_2 page of the MPASS from Corollary 8.12. Then we show that the map θ' induces this isomorphism.

In the associated graded, there is a differential $d_0(a_2) = -a_1^2$, but the corresponding differential on q_2 is a d_1. So the filtration spectral sequence ${}^U E_r$ computing $H^*(b_{10}^{-1}\widetilde{U}^* \otimes W')$ has E_0 page

$${}^U E_0 \cong b_{10}^{-1} UL^*(\xi_1) \otimes UL^*(\xi_2) \otimes W'$$

with differential $d_0(u_1 \otimes u_2 \otimes w) = d(u_1) \otimes u_2 \otimes w$. So

$${}^U E_1 \cong H^*(b_{10}^{-1} UL^*(\xi_1)) \otimes UL^*(\xi_2) \otimes W' \cong E[h_{10}] \otimes P[b_{10}^{\pm 1}] \otimes UL^*(\xi_2) \otimes W'$$

and the only remaining differential is generated by $d_1(q_2) = -q_1^2$, so

$${}^U E_2 \cong E[h_{10}] \otimes P[b_{10}^{\pm}] \otimes H^*(UL^*(\xi_2)) \otimes W' = E[h_{10}, h_{20}] \otimes P[b_{10}^{\pm 1}, b_{20}] \otimes W'.$$

Then $E_r \cong E_2$ for $r \geq 2$.

To show that θ' is an isomorphism, it suffices to show that $\theta'(h_{10}) = h_{10}$, $\theta'(b_{10}) = b_{10}$, $\theta'(h_{20}) = h_{20}$, $\theta'(b_{20}) = b_{20}$ and $\theta'(w_n) = b_{10} w_n$ for $n \geq 3$. We use the fact that θ' extends θ multiplicatively using the concatenation product in the cobar complex. So $\theta'([a_1 \mid \cdots \mid a_n]) = \prod \theta(a_i)$, and we have

$$\begin{aligned}
\theta'(h_{10}) &= \theta'([\xi_1]) = \theta(\xi_1) = a_1,\\
\theta'(b_{10}) &= \theta'([\xi_1 \mid \xi_1^2] + [\xi_1^2 \mid \xi_1]) = \theta(\xi_1)\theta(\xi_1^2) + \theta(\xi_1^2)\theta(\xi_1) = a_1 a_2 + a_2 a_1 = b_{10},\\
\theta'(h_{20}) &= \theta'([\xi_2]) = \theta(\xi_2) = q_1,\\
\theta'(b_{20}) &= \theta'([\xi_2 \mid \xi_2^2] + [\xi_2^2 \mid \xi_2]) = \theta(\xi_2)\theta(\xi_2^2) + \theta(\xi_2^2)\theta(\xi_2) = q_1 q_2 + q_2 q_1 = b_{20},\\
\theta'(w_n) &= \theta'([\xi_1 \mid \xi_n] - [\xi_1^2 \mid \xi_{n-1}^3]) = a_1 a_2 w_n + a_2 a_1 w_n = b_{10} w_n. \qquad \square
\end{aligned}$$

Proof of Theorem 8.1 In Section 8.1 we constructed a map $\theta'\colon C^*_{D_{1,\infty}}(k,k) \to \widetilde{U}^* \otimes W'$ which is filtration-preserving, where $C^*_{D_{1,\infty}}(k,k)$ has the filtration associated to the MPASS and $\widetilde{U}^* \otimes W'$ has the filtration constructed in Section 8.2. By Proposition 8.13, θ' induces an isomorphism of spectral sequences after inverting b_{10}, and so it induces an isomorphism in cohomology. Thus

$$b_{10}^{-1}\operatorname{Ext}^{*,*}_{D_{1,\infty}}(k,k) = b_{10}^{-1}H^*(C_{D_{1,\infty}}(k,k)) \cong b_{10}^{-1}H^*(b_{10}^{-1}\widetilde{U}^* \otimes W').$$

The result follows from (8-2). □

Appendix Convergence of localized spectral sequences

In this appendix, we study the convergence of two b_{10}–localized spectral sequences, the b_{10}–localized MPASS (the main subject of this paper) and the b_{10}–localized ISS (introduced in Section 6). In each case, the nonlocalized spectral sequences converges for straightforward reasons.

In general, there are two possible ways in which a localization of a convergent spectral sequence can fail to converge:

(1) There could be a b_{10}–tower x in E_∞ that does not appear in $b_{10}^{-1}E_\infty$ because it is broken into a series of b_{10}–torsion towers connected by hidden multiplications.

(2) There could be a b_{10}–tower x in $b_{10}^{-1}E_\infty$ that is not a permanent cycle in E_∞ because in the nonlocalized spectral sequence it supports a series of increasing-length differentials to b_{10}–torsion elements (so these differentials would be zero in $b_{10}^{-1}E_r$).

(The reverse of (2), where a sequence of torsion elements supports a differential that hits a b_{10}–tower, cannot happen: if $d_r(x) = y$ and $b_{10}^n x = 0$ in E_r, then $0 = d_r(b_{10}^n x) = b_{10}^n d_r(x) = b_{10}^n y$.)

A.1 Convergence of the $K(\xi_1)$–based MPASS

In this section we prove convergence of the B_Γ–based MPASS of Theorem 1.6 in the case that Γ is a quotient of P (in fact, the only property of Γ that is used is that $u(x) \geq u(\xi_1^3)$ for $u \in \Gamma$). The convergence argument will only rely on the form of the E_1 page.

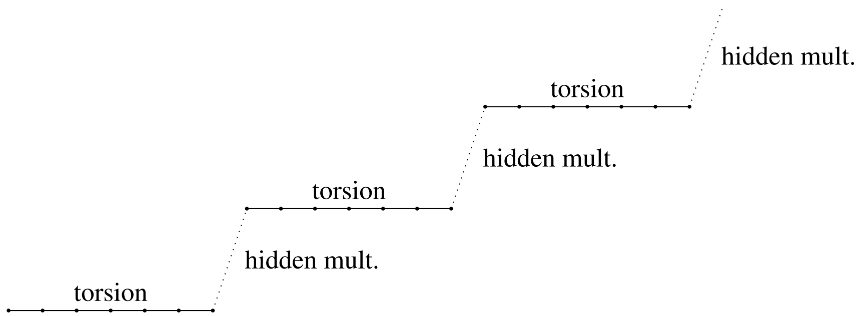


Figure 5: Illustration of (1): this represents a b_{10}–tower in "homotopy".

Proposition A.1 *For any nonnegatively graded Γ–comodule M, the b_{10}–localized $K(\xi_1)$–based MPASS*

$$E_1^{s,*} = b_{10}^{-1}\,\mathrm{Ext}_D^{*,*}(k, \overline{B}_\Gamma^{\otimes s} \otimes M) \Rightarrow b_{10}^{-1}\,\mathrm{Ext}_\Gamma^{*,*}(k, M) \tag{A-1}$$

converges.

The proof is a slight modification of [14, Propositions 4.4.1 and 4.2.6].

Recall our grading convention: $x \in E_1^{s,t,u}$ is an element in $\mathrm{Ext}_\Gamma^{t,u}(k, B_\Gamma \otimes \overline{B}_\Gamma^{\otimes s})$.

Lemma A.2 *Let M be a bounded-below graded D–comodule and suppose $u_M = \min\{u(x) : x \in M\}$. If $x \in \mathrm{Ext}_D^{*,*}(k, M)$ is a nonzero element of degree (s,t,u) and $x \neq 0$, then $u \geq u_M + 6t - 2$.*

Proof It suffices to check the cases $M = k$, $M = M(1) = k[\xi_1]/\xi_1^2$ and $M = D$. In the case $M = k$, we have $\mathrm{Ext}_D^{*,*}(k, k\{y\}) = E[h_{10}] \otimes k[b_{10}] \otimes k\{y\}$. In the case $M = M(1)$, write $M = k\{y, \partial y\}$; then $\mathrm{Ext}_D^{*,*}(k, M) = k[b_{10}] \otimes k\{\partial y, e(y)\}$, where $e(y) = [\xi_1]y - [\xi_1^2](\partial y)$. In the case $M = D$, $\mathrm{Ext}_D^{0,*}(k, D) \cong k$ is concentrated in homological degree zero. In each of these cases, we verify the desired statement, using the fact that $b_{10} \in E_1^{0,2,12}$ and $h_{10} \in E_1^{0,1,4}$. $\square$

Proposition A.3 *There is a vanishing plane in the E_1 page of (A-1): $E_1^{s,t,u} = 0$ if $u < 12s + 6t - 2$.*

Proof Recall $E_1^{s,t,*} = \mathrm{Ext}_\Gamma^{t,*}(k, \Gamma \,\square_D\, (\overline{B}_\Gamma^{\otimes s} \otimes M)) \cong \mathrm{Ext}_D^{*,*}(k, \overline{B}_\Gamma^{\otimes s} \otimes M)$. Since Γ is a quotient of P, if $x \in \overline{B}_\Gamma$ is nonzero then $u(x) \leq u(\xi_1^3) = 12$. Therefore a nonzero element $x \in \overline{B}_\Gamma^{\otimes s} \otimes M$ has $u \geq 12s$. By Lemma A.2, if $x \in E_1^{s,t,u}$ has degree (s,t,u), then $u \geq 12s + 6t - 2$. $\square$

Corollary A.4 *$d_r\colon E_r^{s,t,u} \to E_r^{s+r,t-r+1,u}$ is zero if $r > \frac{1}{6}(u-12s-6t-4)$.*

Proof Given $x \in E_r^{s,t,u}$, $d_r(x) \in E_r^{s',t',u'} = E_r^{s+r,t-r+1,u}$ will be zero because of the vanishing plane if $12s'+6t'-2-u' > 0$. But

$$12s'+6t'-2-u' = 12(s+r)+6(t-r+1)-2-u = (12s+6t+4-u)+6r,$$

which is >0 for r as indicated. □

Corollary A.5 *There is a vanishing line in* $\mathrm{Ext}_\Gamma^{*,*}(k,M)$*: if $x \in \mathrm{Ext}_\Gamma^{t',u}(k,M)$ and $u-6t'+2<0$, then $x=0$.*

Proof Permanent cycles in $E_1^{s,t,u}$ converge to elements in $\mathrm{Ext}_\Gamma^{s+t,u}(k,M)$. Any such x would then be represented by a permanent cycle in $E_1^{s,t,u}$ with $u-6(s+t)+2 < 0 \le 6s$ (since Adams filtrations are nonnegative), which falls in the vanishing region of Proposition A.3. □

Note that $b_{10} \in \mathrm{Ext}_\Gamma^{2,12}(k,M)$ acts parallel to this vanishing line.

Proof of Proposition A.1 Convergence of the nonlocalized MPASS follows from a general result by Palmieri [14, Proposition 1.4.3].

For convergence problem (1), suppose x has degree (s_x,t_x,u_x). If there were no multiplicative extensions, then $b_{10}^i x$ would have degree (s_x, t_x+2i, u_x+12i). But multiplicative extensions cause it to have the expected internal degree u and stem $s+t$, but higher s. That is, $b_{10}^i x$ has degree $(s_x+n_i, t_x+2i-n_i, u_x+12i)$ for some $n_i>0$, and because this scenario involves the existence of infinitely many multiplicative extensions, the sequence $(n_i)_i$ is increasing and unbounded above. This causes us to run afoul of the vanishing plane (Proposition A.3) for sufficiently large i:

$$\begin{aligned} 12s+6t-2-u &= 12(s_x+n_i)+6(t_x+2i-n_i)-2-(u_x+12i) \\ &= 12s_x+6t_x-2-u_x+6n_i, \end{aligned}$$

which is >0 for $i \gg 0$.

For convergence problem (2), the scenario is, more precisely, as follows: we have a b_{10}–periodic element $x \in \mathrm{Ext}_\Gamma^{*,*}(k,k)$, and a sequence of differentials $d_{r_i}(b_{10}^i x) = y_i \neq 0$, where every y_i is b_{10}–torsion. The sequence $(r_i)_i$ must be increasing and bounded above: if $b_{10}^{n_i} y_i = 0$ then $d_{r_i}(b_{10}^{n_i} x) = b_{10}^{n_i} y_i = 0$, and so if $b_{10}^{n_i} x$ is to support a differential $d_{r_{n_i}}$, we must have $r_{n_i} > r_i$. Note that the condition on r in Corollary A.4 is the same for all $b_{10}^i x$. So some of the r_i will be greater than this bound, contradicting the assumption that $d_{r_i}(b_{10}^i x) \neq 0$. □

A.2 Convergence of the b_{10}–local ISS

In this section, we consider the b_{10}–local ISS computing $b_{10}^{-1}\operatorname{Ext}^{*,*}_{P_n}(k,k)$. As discussed in Section 6.4, this is obtained by b_{10}–localizing a filtration spectral sequence on the cobar complex for P_n, where the filtration is defined by taking powers of the augmentation ideal. Let E_r^{ISS} denote the E_r page of the nonlocalized ISS and $b_{10}^{-1}E_r^{\mathrm{ISS}}$ denote the E_r page of the localized ISS.

Lemma A.6 *There is a slope $\frac{1}{4}$ vanishing line in E_1^{ISS} in (u,s) coordinates. That is, if $x \in E_1^{\mathrm{ISS}}$ has $s(x) > \frac{1}{4}u(x)$, then $x = 0$.*

Proof In Section 6.4 we computed the E_1 page:

$$E_1^{\mathrm{ISS}} = \bigotimes_{(i,j)\in I} E[h_{ij}] \otimes k[b_{ij}],$$

where

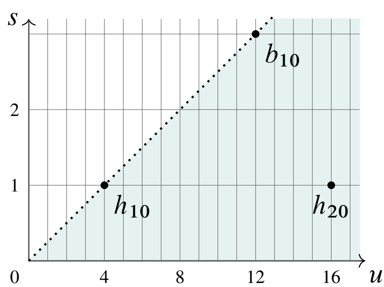
$$I = \{(1,0),(1,1),(2,0),(n-2,0),(n-2,1),(n-2,2),(n-1,0),(n-1,1),(n,0)\}.$$

These generators occur in the following degrees:

element	u	s	u/s
h_{ij}	$2(3^i-1)3^j$	3^j	$2(3^i-1)$
b_{ij}	$2(3^i-1)3^{j+1}$	3^{j+1}	$2(3^i-1)$

So we have $u/s \geq 2(3^1-1) = 4$, which proves the lemma. Note that b_{10}, in degrees $(u=12,\ s=3)$, acts parallel to the vanishing line. □

Here is a picture:



Differentials are vertical: d_r takes elements in degree (u,s) to degree $(u,s+r)$.

Proposition A.7 *The b_{10}–localized ISS converges to $b_{10}^{-1}\operatorname{Ext}^{*,*}_{P_n}(k,k)$.*

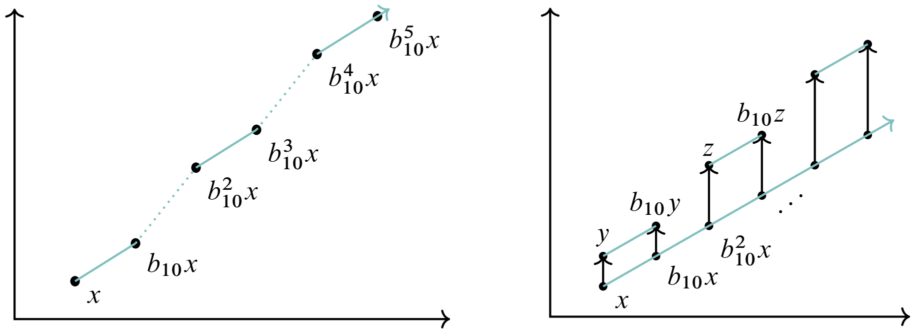


Figure 6: Convergence problems (1) and (2) for the ISS.

Proof The nonlocalized ISS converges because it is based on a decreasing filtration of the cobar complex that clearly satisfies both $\bigcap_s F^s C_{P_n}(k,k) = \{0\}$ and $\bigcup_s F^s C_{P_n}(k,k) = C_{P_n}(k,k)$.

The two convergence problems are illustrated in Figure 6.

In both of these cases, it is clear from the pictures that these cannot happen if there is a vanishing line of slope equal to the degree of b_{10}, as guaranteed by Lemma A.6. □

Remark A.8 The same proof shows that the ISS for $b_{10}^{-1}\operatorname{Ext}_P^{*,*}(k,k)$ converges; in particular, the vanishing line in Lemma A.6 goes through even with more h_{ij}'s and b_{ij}'s in the E_1 page.

References

[1] **J F Adams**, *A periodicity theorem in homological algebra*, Proc. Cambridge Philos. Soc. 62 (1966) 365–377 MR

[2] **T Barthel**, **D Heard**, **G Valenzuela**, *Local duality in algebra and topology*, Adv. Math. 335 (2018) 563–663 MR

[3] **E Belmont**, *A Cartan–Eilenberg spectral sequence for non-normal extensions*, J. Pure Appl. Algebra 224 (2020) art. id. 106216 MR

[4] **M Hovey**, **J H Palmieri**, **N P Strickland**, *Axiomatic stable homotopy theory*, Mem. Amer. Math. Soc. 610, Amer. Math. Soc., Providence, RI (1997) MR

[5] **D Husemoller**, **J C Moore**, **J Stasheff**, *Differential homological algebra and homogeneous spaces*, J. Pure Appl. Algebra 5 (1974) 113–185 MR

[6] **L N Ivanovskiĭ**, *Cohomologies of a Steenrod algebra*, Dokl. Akad. Nauk SSSR 157 (1964) 1284–1287 MR In Russian

[7] **H R Margolis**, *Spectra and the Steenrod algebra: modules over the Steenrod algebra and the stable homotopy category*, North-Holland Mathematical Library 29, North-Holland, Amsterdam (1983) MR

[8] **J P May**, *Matric Massey products*, J. Algebra 12 (1969) 533–568 MR

[9] **J P May**, **R J Milgram**, *The Bockstein and the Adams spectral sequences*, Proc. Amer. Math. Soc. 83 (1981) 128–130 MR

[10] **H R Miller**, *A localization theorem in homological algebra*, Math. Proc. Cambridge Philos. Soc. 84 (1978) 73–84 MR

[11] **H Miller**, **C Wilkerson**, *Vanishing lines for modules over the Steenrod algebra*, J. Pure Appl. Algebra 22 (1981) 293–307 MR

[12] **J Milnor**, *The Steenrod algebra and its dual*, Ann. of Math. 67 (1958) 150–171 MR

[13] **C Nassau**, *Yacop*, Sage code, version 1.0 (2018) `http://nullhomotopie.de/projects/yacop-page.html`

[14] **J H Palmieri**, *Stable homotopy over the Steenrod algebra*, Mem. Amer. Math. Soc. 716, Amer. Math. Soc., Providence, RI (2001) MR

[15] **J-C Renaud**, *The decomposition of products in the modular representation ring of a cyclic group of prime power order*, J. Algebra 58 (1979) 1–11 MR

Department of Mathematics, Northwestern University
Evanston, IL, United States

ebelmont@northwestern.edu

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