PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY, SERIES B
Volume 11, Pages 1–14 (January 5, 2024)
https://doi.org/10.1090/bproc/203

THE REDUCED RING OF THE $RO(C_2)$-GRADED C_2-EQUIVARIANT STABLE STEMS

EVA BELMONT, ZHOULI XU, AND SHANGJIE ZHANG

(Communicated by Julie Bergner)

ABSTRACT. We describe in terms of generators and relations the ring structure of the $RO(C_2)$-graded C_2-equivariant stable stems $\pi_\star^{C_2}$ modulo the ideal of all nilpotent elements. As a consequence, we also record the ring structure of the homotopy groups of the rational C_2-equivariant sphere $\pi_\star^{C_2}(\mathbb{S}_{\mathbb{Q}})$.

1. INTRODUCTION

One of the motivating problems in stable homotopy theory is to understand the stable homotopy groups of spheres, $\pi_n^s = \mathrm{colim}_{k\to\infty}[S^{k+n}, S^k]$. An early structural result by Nishida [Nis73] says that every nonzero element in π_n^s for $n > 0$ is nilpotent. It is also well known that $\pi_0^s \cong \mathbb{Z}$, and so we may repackage this result by saying that the *reduced ring* of π_*^s—namely π_*^s modulo the ideal of nilpotent elements—is isomorphic to $\mathbb{Z}$. In this paper we obtain an analogous result for the C_2-equivariant homotopy groups of spheres $\pi_\star^{C_2}$. Here $\star$ stands for the $RO(C_2)$-grading: $\pi_{s,w}^{C_2}$ is the group of homotopy classes of stable maps $S^{s-w} \wedge S^{w\sigma} \to S^0$, where $S^{w\sigma}$ denotes w copies of the sign representation and S^{s-w} has trivial action.

In Theorem 1.3, we give explicit generators and relations for $\pi_\star^{C_2}/\mathfrak{N}$, where $\mathfrak{N}$ is the ideal of nilpotent elements. The main input is the knowledge that the non-nilpotent elements are concentrated in specific computationally accessible degrees, combined with C_2-equivariant Adams spectral sequence computations. All of the main pieces of this calculation can be read off of previous work, but the result itself does not appear in the literature. Our contribution is to put the pieces together, handling a number of details along the way such as the determination of part of $\pi_{0,w}^{C_2}$, and turning 2-completed into integral information.

As Nishida's theorem can be thought of as the starting point of chromatic homotopy theory, our work answers an analogous question in C_2-equivariant chromatic homotopy theory. Other parts of equivariant chromatic homotopy theory have been studied, giving some expectation about the complexity of the answer. For example, for a finite group G, Balmer and Sanders [BS17] proved that G-equivariant thick prime ideals are exactly those pulled back from non-equivariant thick prime ideals via the geometric fixed points functors Φ^H for subgroups $H \leq G$. More to the point, the equivariant nilpotence theorem [BS17, BGH20] says that an element

Received by the editors January 25, 2023, and, in revised form, November 2, 2023, and November 12, 2023.

2020 *Mathematics Subject Classification.* Primary 55Q91, 55Q45.

The first author was supported by NSF Grant DMS-2204357. The second author was supported by NSF Grant DMS-2105462.

$x \in \pi_\star^G$ in the $RO(G)$-graded homotopy groups of the sphere is non-nilpotent if and only if $\Phi^H(x)$ is non-nilpotent for some subgroup H of G. At $G = C_2$, there are only two subgroups to consider, and we have an assembly problem whose pieces can be described simply, but has a fairly complicated answer (see Theorem 1.3).

1.1. **Main theorems.** We describe a $\mathbb{Z}$-additive basis for $\pi_\star^{C_2}/\mathfrak{N}$ in Theorem 1.1, which emphasizes the 2-divisibility of η^i and makes the module structure over the Burnside ring $\pi_{0,0}^{C_2}$ apparent; then we describe the ring structure of $\pi_\star^{C_2}/\mathfrak{N}$ in Theorem 1.3, which is our main purpose. We also include a description of the ring structure of the rationalization $\pi_\star^{C_2}(\mathbb{S}_\mathbb{Q})$ (Corollary 1.4).

Let $\rho \in \pi_{-1,-1}^{C_2}$ denote the Euler class $S^{0,0} \to S^{1,1}$ of the sign representation. The element $\eta \in \pi_{1,1}^{C_2}$ is defined so that $\Phi^{C_2}(\eta) = 2$ and its underlying non-equivariant map is $\Phi^e(\eta) = -\eta_{cl}$, where $\eta_{cl} \in \pi_1^s$ is the classical Hopf invariant one element. For more on the notation and choices of generators, see Section 2.2.

Theorem 1.1. *The reduced ring of the C_2-equivariant stable stems is concentrated in degrees (s, w) where $s = 0$ and w is even, or $s - w = 0$. Its underlying group structure is a free abelian group of rank 1 in each of those degrees, except for degree $(0,0)$ in which it has rank 2. The additive generators of $\pi_\star^{C_2}/\mathfrak{N}$ are*

$$\rho^i \text{ in degree } (-i,-i) \text{ for } i \geq 0$$
$$\omega_n \text{ for } n \in \mathbb{Z} \text{ in degree } (0,-2n)$$
$$\frac{\eta^i}{2^{n(i)}} \text{ in degree } (i,i) \text{ for } i \geq 1$$

where

$$n(i) = \begin{cases} 4t-1 & i = 8t \\ 4t & i = 8t+j \text{ for } j = 1,2,3,4 \\ 4t+1 & i = 8t+5 \\ 4t+2 & i = 8t+6 \\ 4t+3 & i = 8t+7. \end{cases}$$

In particular, for $i \geq 1$, η^i is uniquely $2^{n(i)}$-divisible in $\pi_\star^{C_2}/\mathfrak{N}$, and is not $2^{n(i)+1}$-divisible. Let $[C_2/e] \in A(C_2)$ denote the nontrivial generator. Then $[C_2/e] = \omega_0$ acts by multiplication by 2 on ω_n for all n, and acts as zero on $\frac{\eta^i}{2^{n(i)}}$ and ρ^i for $i \geq 1$.

Remark 1.2. In the C_2-Adams spectral sequence, the elements ω_n are represented by $\tau^{2n}h_0$ when $n \geq 0$ and $\frac{\gamma}{\tau^{-2n-1}}$ (which some authors notate as $\frac{\theta}{\tau^{-2n-2}}$) when $n < 0$. The Adams spectral sequence representatives for $\frac{\eta^i}{2^{n(i)}}$ can be read off of Table 1 in Section 4. Also, note η is the element in degree $(1,1)$, as $n(1) = 0$.

Theorem 1.3. *As a ring, $\pi_\star^{C_2}/\mathfrak{N}$ is generated by*

$$\rho \text{ in degree } (-1,-1)$$
$$\omega_n \text{ for } n \in \mathbb{Z} \text{ in degree } (0,-2n)$$
$$\frac{\eta^i}{2^{n(i)}} \text{ for } i \geq 1, i \equiv 1 \text{ or } 7 \pmod 8 \text{ in degree } (i,i)$$

subject to the following additional relations.

(a) $2 = \omega_0 + \rho\eta$

(b) $\omega_n \cdot \omega_m = 2 \cdot \omega_{n+m}$ *for* $n, m \in \mathbb{Z}$.

(c) *If* $i \equiv 1 \pmod 8$, *then*

$$\eta^3 \cdot \frac{\eta^i}{2^{n(i)}} = \rho^3 \cdot \frac{\eta^{i+6}}{2^{n(i+6)}}.$$

If $i \equiv 7 \pmod 8$, *then*

$$\eta \cdot \frac{\eta^i}{2^{n(i)}} = \rho \cdot \frac{\eta^{i+2}}{2^{n(i+2)}}.$$

(d) $0 = \rho \cdot \omega_n = \dfrac{\eta^i}{2^{n(i)}} \cdot \omega_n$ *for all* $n \in \mathbb{Z}$, $i \geq 1$.

The next statement is obtained by rationalizing Theorem 1.3. The additive structure of this ring is previously known; see for example [GQ22].

Corollary 1.4. *Let* $\mathbb{S}_\mathbb{Q}$ *denote the* C_2*-equivariant rational sphere. As a ring, we have*

$$\pi^{C_2}_\star(\mathbb{S}_\mathbb{Q}) \cong \pi^{C_2}_\star \otimes \mathbb{Q} \cong \mathbb{Q}[\rho, \eta, \omega_1, \omega_{-1}]/\sim,$$

where

$$|\rho| = (-1,-1), |\eta| = (1,1), |\omega_1| = (0,-2), |\omega_{-1}| = (0,2)$$

and $\sim$ *denotes the relations*

$$\omega_1 \cdot \omega_{-1} = 4 - 2\rho\eta$$
$$\rho \cdot \omega_{\pm 1} = \eta \cdot \omega_{\pm 1} = 0$$
$$\rho \cdot (2 - \rho\eta) = \eta \cdot (2 - \rho\eta) = 0.$$

For the rest of this paper, we will implicitly work over the reduced ring to avoid problems arising from nilpotent elements in the same degree. In particular, one should think of a non-nilpotent element as a coset.

1.2. **Notation.**

π^s_*: the classical stable homotopy groups of spheres.
$\pi^{C_2}_{s,w}$: the $RO(C_2)$-graded stable homotopy groups of spheres with $RO(C_2)$-degree $(s-w)+w\sigma$.
$\mathfrak{N}$: the nilradical of $\pi^{C_2}_{*,*}$.
$\pi^{C_2}_{s,w}/\mathfrak{N}$: the reduced ring of $RO(C_2)$-graded stable homotopy groups of spheres in degree (s,w).
$\pi^{\mathbb{R}}_{s,w}$: the $\mathbb{R}$-motivic homotopy groups in stem s and motivic weight w.
$\mathbb{M}^{C_2}_2$: the $RO(C_2)$-graded $H\underline{\mathbb{F}_2}$-homology of a point.
$\mathbb{M}^{\mathbb{C}}_2$: the $\mathbb{C}$-motivic $\mathbb{F}_2$-homology of a point.
$\mathbb{M}^{\mathbb{R}}_2$: the $\mathbb{R}$-motivic $\mathbb{F}_2$-homology of a point.
$\mathrm{Ext}^{s,f,w}_{\mathbb{C}}$: the E_2-page of the $\mathbb{C}$-motivic Adams spectral sequence in stem s, Adams filtration f, and motivic weight w.
$\mathrm{Ext}^{s,f,w}_{C_2}$: the E_2-page of the C_2-equivariant Adams spectral sequence in degree (s,f,w).
$\mathrm{Ext}^{s,f,w}_{\mathbb{R}}$: the E_2-page of the $\mathbb{R}$-motivic Adams spectral sequence in degree (s,f,w).
$\mathrm{Ext}^{s,f,w}_{NC}$: the summand of $\mathrm{Ext}^{s,f,w}_{C_2}$ from [GHIR20, §2,3].
E^+_r: the ρ-Bockstein spectral sequence which converges to $\mathrm{Ext}_{\mathbb{R}}$.
E^-_r: the ρ-Bockstein spectral sequence which converges to Ext_{NC}.
Φ^{C_2}: the C_2-geometric fixed point functor as in [BS17].

Φ^e: the forgetful functor taking a C_2-equivariant spectrum to its underlying non-equivariant spectrum.

1.3. **Organization.** In Section 2 we review general results about equivariant nilpotence and define several important elements. We also briefly discuss the $\mathbb{C}$-motivic, $\mathbb{R}$-motivic, and C_2-equivariant Adams spectral sequences, which will be used in later sections. In Section 3, we describe $\pi_\star^{C_2}/\mathfrak{N}$ in stem zero (i.e., $s = 0$). In Section 4, we describe $\pi_\star^{C_2}/\mathfrak{N}$ in coweight zero (i.e., $s - w = 0$). In Section 5 we assemble previous results to prove the main theorems. In Section 6 we discuss the p-completed ring structure and higher multiplicative structure in the reduced ring.

2. Notation and preliminaries

2.1. **Prior results on equivariant non-nilpotence.** Next we recall some theorems about C_2-equivariant non-nilpotent elements that form the starting point for our work. Iriye first proved an analogue of the Nishida nilpotence theorem in the equivariant context.

Theorem 2.1 ([Iri83]). *Let G be a finite group. Every torsion element of $\pi_\star^G$ is nilpotent.*

From now on, we specialize to $G = C_2$.

Theorem 2.2 ([Lin80], [GM95], [BS20], [GQ22]). *$\pi_{s,w}^{C_2}$ has infinite order if and only if*

$$s = w \quad \text{or} \quad s = 0 \ \text{ and } w \text{ is even}$$

Moreover, in these cases the rank of $\pi_{s,w}^{C_2}$ as a free abelian group is 1 unless $(s, w) = (0, 0)$, in which case the rank is 2.

Remark 2.3. Greenlees and Quigley [GQ22] studied the analogue of Theorem 2.2 for general finite groups.

Balmer and Sanders [BS17], and Barthel, Greenlees and Hausmann [BGH20] proved the following equivariant nilpotence theorem.

Theorem 2.4. *An element $\alpha \in \pi_\star^{C_2}$ is nilpotent if and only if both its geometric fixed points $\Phi^{C_2}(\alpha) \in \pi_*^s$ and its underlying non-equivariant map $\Phi^e(\alpha) \in \pi_*^s$ are nilpotent.*

By Theorem 2.1, Theorem 2.2, and Corollary 2.5, $\pi_\star^{C_2}/\mathfrak{N}$ is a free abelian group with ranks bounded above by those specified in Theorem 2.2. We will see that this upper bound is achieved in all degrees.

Corollary 2.5. *$\pi_\star^{C_2}/\mathfrak{N}$ is torsion-free.*

Proof. Suppose $x \in \pi_\star^{C_2}$ is non-nilpotent. By Theorem 2.4 and the fact (from Nishida's Nilpotence Theorem) that the non-nilpotent elements of π_*^s are concentrated in $\pi_0^s \cong \mathbb{Z}$, there is some $H \subseteq C_2$ such that $\Phi^H(x) \neq 0$ is in stem 0, hence non-torsion and non-nilpotent. We need to check that nx is non-nilpotent for every $n \neq 0$. But $\Phi^H((nx)^k) = n^k \Phi^H(x)^k$ is nonzero for every n, k, and, hence, so is $(nx)^k$. $\square$

2.2. **Some elements in $\pi_\star^{C_2}$.** Recall $RO(C_2)$ is generated by the trivial 1-dimensional representation and the sign representation σ. In accordance with the grading convention for $\mathbb{R}$-motivic homotopy theory, we use the double grading

$$\pi_{s,w}^{C_2}$$

to indicate the $RO(C_2)$-degree $(s-w)+w\sigma$. We call s the stem, w the weight, and $s-w$ the coweight.

In coweight 0, there is a map

$$\rho : S^{0,0} \to S^{1,1}$$

given by inclusion of fixed points, which is a non-nilpotent element in $\pi_{-1,-1}^{C_2}$. Equivalently, this is the Euler class of the sign representation σ of C_2, sometimes written as a_σ.

The nonequivariant Hopf map $\eta_{cl} : S^3 \to S^2$ can be modeled as the defining quotient map $\mathbb{C}^2 - \{0\} \to \mathbb{C}P^1$ for complex projective space. The quotient map is compatible with the action of complex conjugation so we get a C_2-equivariant homotopy class, and we define η in $\pi_{1,1}^{C_2}$ to be the *negative* of this map. This definition is the negative of that in [GHIR20] and for most other authors since it satisfies $\Phi^e(\eta) = -\eta_{cl}$, but we choose it here because it satisfies

$$\Phi^{C_2}(\eta) = 2$$

(see [GHIR20, Remark 10.8]). Since 2 is non-nilpotent, this relation implies that η is non-nilpotent.

For a finite group G, let $A(G)$ denote the Burnside ring. Segal [Seg71] proved that there is an isomorphism

$$A(G) \xrightarrow{\cong} \pi_0^G.$$

In the case $G = C_2$, we have $\pi_{0,0}^{C_2} = A(C_2)$ with free basis 1 and $[C_2/e]$, where 1 denotes the identity and $[C_2/e]$ denotes the generator of C_2, which means we have the relation $[C_2/e]^2 = 2[C_2/e]$. Note that elements of the Burnside ring are all non-nilpotent.

There is also a key relation in $\pi_{0,0}^{C_2}$ (see [GHIR20, Lemma 10.9]):

$$[C_2/e] = 2 - \rho \cdot \eta. \tag{2.6}$$

Moreover, $[C_2/e]$ is represented by h_0 in the C_2-equivariant Adams spectral sequence (see [GHIR20, Definition 11.6]).

2.3. **Review of relevant Adams spectral sequences.** We review the setup of the $\mathbb{C}$-motivic, $\mathbb{R}$-motivic, and C_2-equivariant spectral sequences; references for this material can be found in [DI10], [IWX23], [DI17], and [GHIR20]. The homology of a point in the $\mathbb{C}$-motivic, $\mathbb{R}$-motivic, and C_2-equivariant cases, respectively, is given by

$$\begin{aligned} \mathbb{M}_2^{\mathbb{C}} &= \mathbb{F}_2[\tau] \\ \mathbb{M}_2^{\mathbb{R}} &= \mathbb{F}_2[\rho,\tau] \\ \mathbb{M}_2^{C_2} &= \mathbb{F}_2[\rho,\tau] \oplus NC, \qquad NC = \bigoplus_{s\geq 0} \frac{\mathbb{F}_2[\tau]}{\tau^\infty}\left\{\frac{\gamma}{\rho^s}\right\} \end{aligned}$$

where $|\rho| = (-1,-1)$, $|\tau| = (0,-1)$, and $|\frac{\gamma}{\tau}| = (0,2)$. The algebra NC is called the *negative cone*, and it is generated by elements of the form $\frac{\gamma}{\rho^a\tau^b}$ for $a \geq 0, b \geq 1$, where $\rho \cdot \frac{\gamma}{\tau} = 0 = \tau \cdot \frac{\gamma}{\tau}$; note that γ is not itself an element in $\mathbb{M}_2^{C_2}$. There is a functor ("Betti realization") from the $\mathbb{R}$-motivic category to the C_2-equivariant genuine category that takes $\rho \in \pi^{\mathbb{R}}_{-1,-1}$ to the element of the same name $\rho \in \pi^{C_2}_{-1,-1}$, which is described in Section 2.2. Let $\mathcal{A}_{\mathbb{C}}$, $\mathcal{A}_{\mathbb{R}}$, and $\mathcal{A}_{C_2}$ denote the $\mathbb{C}$-motivic, $\mathbb{R}$-motivic, and C_2-equivariant Steenrod algebras, respectively. Then the $\mathbb{C}$-motivic, $\mathbb{R}$-motivic, and C_2-equivariant Adams spectral sequences have signatures

$$E_2^{s,f,w} = \mathrm{Ext}_{\mathbb{C}}^{s,f,w} := \mathrm{Ext}_{\mathcal{A}_{\mathbb{C}}}^{s,f,w}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}}) \Rightarrow (\pi^{\mathbb{C}}_{s,w})^{\wedge}_2$$
$$E_2^{s,f,w} = \mathrm{Ext}_{\mathbb{R}}^{s,f,w} := \mathrm{Ext}_{\mathcal{A}_{\mathbb{R}}}^{s,f,w}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}}) \Rightarrow (\pi^{\mathbb{R}}_{s,w})^{\wedge}_2$$
$$E_2^{s,f,w} = \mathrm{Ext}_{C_2}^{s,f,w} := \mathrm{Ext}_{\mathcal{A}_{C_2}}^{s,f,w}(\mathbb{M}_2^{C_2}, \mathbb{M}_2^{C_2}) \Rightarrow (\pi^{C_2}_{s,w})^{\wedge}_2$$

where f denotes homological degree, (s, w) in the equivariant case is described in Section 2.2, and (s, w) in both the $\mathbb{C}$-motivic and $\mathbb{R}$-motivic cases is the pair of total degree and motivic weight. Our grading convention follows that of [DI10, DI17, IWX23] which has been widely adopted in computational motivic homotopy theory, but differs from the classical convention where s denotes the filtration instead of the stem. Betti realization induces a map of spectral sequences from the second spectral sequence to the first, and Proposition 2.7 says that the map of E_2-pages $\mathrm{Ext}_{\mathbb{R}} \to \mathrm{Ext}_{C_2}$ is a split inclusion.

Proposition 2.7 ([GHIR20]).

(a) *The splitting* $\mathbb{M}_2^{C_2} \cong \mathbb{M}_2^{\mathbb{R}} \oplus NC$ *induces a splitting*

$$\mathrm{Ext}_{C_2} = \mathrm{Ext}_{\mathbb{R}} \oplus \mathrm{Ext}_{NC}$$

where $\mathrm{Ext}_{NC} = \mathrm{Ext}_{\mathcal{A}_{C_2}}(\mathbb{M}_2^{C_2}, NC)$.

(b) *There is a ρ-Bockstein spectral sequence converging to* Ext_{C_2} *such that a Bockstein differential d_r takes a class x of degree (s, f, w) to a class $d_r(x)$ of degree $(s - 1, f + 1, w)$. Under the splitting in part* (a)*, the spectral sequence decomposes as*

$$E_1^{+} = \mathrm{Ext}_{\mathbb{C}}[\rho] \Rightarrow \mathrm{Ext}_{\mathbb{R}}$$
$$E_1^{-} \Rightarrow \mathrm{Ext}_{NC}.$$

Guillou, Hill, Isaksen and Ravenel [GHIR20, §3] explicitly describe the $\mathbb{F}_2$-generators of E_1^-:

Proposition 2.8. *E_1^- is generated over $\mathbb{F}_2$ by elements of the following two types:*

- *Elements of the form $\frac{\gamma}{\rho^a \tau^b} x$ where $0 \leq a, 1 \leq b$, and x is an element of* $\mathrm{Ext}_{\mathbb{C}}$ *that is τ-free and not divisible by τ. If x has degree (s, f, w) in* $\mathrm{Ext}_{\mathbb{C}}$*, then $\frac{\gamma}{\rho^a \tau^b} x$ has degree $(s + a, f, w + a + b + 1)$.*
- *Elements of the form $\frac{Q}{\rho^a \tau^b} x$ where $0 \leq a, 0 \leq b \leq k$, and x is an element of* $\mathrm{Ext}_{\mathbb{C}}$ *that is τ-torsion and divisible by τ^k but not by τ^{k+1}. If x has degree (s, f, w) in* $\mathrm{Ext}_{\mathbb{C}}$*, then $\frac{Q}{\rho^a \tau^b} x$ has degree $(s + a + 1, f - 1, w + a + b + 1)$.*

Remark 2.9. There is an element h_0 in degree $(0, 1, 0)$ in $\mathrm{Ext}_{\mathbb{C}}$, and it survives the ρ-Bockstein spectral sequence that converges to $\mathrm{Ext}_{\mathbb{R}}$. Moreover, it also survives the $\mathbb{R}$-motivic Adams spectral sequence. In particular, it detects the homotopy element in the Burnside ring by convention. Combining facts in Section 2.2, the homotopy element $2 \in \pi^{C_2}_{0,0}$ is detected by $h_0 + \rho\eta$ in the E_∞-page.

3. Non-nilpotent elements in stem 0

By Theorem 2.2, there are two cases to consider. In this section, we determine $\pi_{s,w}^{C_2}$ modulo nilpotents when $s = 0$ and w is even. In Propositions 3.2, 3.3, and 3.5 we identify the additive generators in terms of the C_2-equivariant Adams spectral sequence. In Theorem 3.8 we give a multiplicative description.

3.1. **Additive generators in stem 0.** We remark here that the results in this subsection are largely known (see, for example, [BI22], [BI20], [DI17], [Hil11]), though not all of the exact statements have been written down before. In this subsection, we assume knowledge of the E_2-page of both the $\mathbb{R}$-motivic and C_2-equivariant Adams spectral sequences and describe the generators ω_n that appear in Theorem 1.1. Since these spectral sequences converge to 2-completed homotopy groups, results must be translated from the 2-completed to the integral context (Lemma 3.7).

We start by proving Lemma 3.1 used in the determination of $(\pi_{0,-2n}^{C_2})^\wedge_2$ for $n > 0$ using the Adams spectral sequence in Proposition 3.3.

Lemma 3.1. *Any class in negative stem in* Ext_{C_2} *is* ρ*-free.*

Proof. Since the $\mathbb{C}$-motivic Steenrod algebra is concentrated in nonnegative stems (see [DI10]), so is $\mathrm{Ext}_{\mathbb{C}}$. Using this fact and Proposition 2.8, we see that E_1^- is concentrated in nonnegative stems. By Proposition 2.7 it suffices to study $\mathrm{Ext}_{\mathbb{R}}$. By [BI22, Proposition 3.1], there is an isomorphism $\mathrm{Ext}_{\mathbb{R}}(S/\rho) \to \mathrm{Ext}_{\mathbb{C}}$ from the E_2-page of the $\mathbb{R}$-motivic Adams spectral sequence computing $\pi_{*,*}^{\mathbb{R}}(S/\rho)$, and $\mathrm{Ext}_{\mathbb{C}}$ is concentrated in nonnegative stems. Suppose $x \in \mathrm{Ext}_{\mathbb{R}}$ has stem $s < 0$ and has $\rho \cdot x = 0$. Then in the long exact sequence

$$\cdots \to \mathrm{Ext}_{\mathbb{R}}^{s,f-1,w-1}(S/\rho) \to \mathrm{Ext}_{\mathbb{R}}^{s,f,w} \xrightarrow{\rho} \mathrm{Ext}_{\mathbb{R}}^{s-1,f,w-1} \to \dots$$

the element x in the middle term pulls back to an element $\tilde{x} \in \mathrm{Ext}_{\mathbb{R}}(S/\rho) = \mathrm{Ext}_{\mathbb{C}}$ in stem $s < 0$, a contradiction. □

In Propositions 3.2, 3.3, and 3.5, we introduce the generators ω_n for $n \in \mathbb{Z}$. While we identify $\pi_{0,0}^{C_2}/\mathfrak{N}$ directly in Proposition 3.2, in the other cases we start by working in the 2-completed context because of our Adams spectral sequence methods.

Proposition 3.2. *We have* $\pi_{0,0}^{C_2}/\mathfrak{N} \cong \mathbb{Z}\{1, \omega_0\} \cong \mathbb{Z}\{1, \rho\eta\}$ *where* ω_0 *is detected by* h_0 *in the* C_2*-equivariant Adams spectral sequence. Moreover, we have* $\Phi^e(\omega_0) = 2$ *and* $2 = \omega_0 + \rho\eta$.

Proof. For $n = 0$, we saw in Section 2.2 that $\pi_{0,0}^{C_2}$ has $\mathbb{Z}$-module generators 1 and $[C_2/e] = 2 - \rho\eta$, which is detected by h_0 in the Adams spectral sequence. As $\Phi^e([C_2/e]) = 2$ these generators are both non-nilpotent, and so are all of their integer multiples. Let $\omega_0 := [C_2/e]$; then the above relation gives $2 = \omega_0 + \rho\eta$. □

Proposition 3.3. *For* $n > 0$ *there are nonzero elements* $\omega_n \in (\pi_{0,-2n}^{C_2})^\wedge_2$ *detected in the* C_2*-equivariant Adams spectral sequence by* $\tau^{2n} h_0$*. They are non-2-divisible and satisfy* $\Phi^e(\omega_n) = 2$.

Proof. Let $n > 0$. In the ρ-Bockstein spectral sequence computing $\mathrm{Ext}_{\mathbb{R}}$ there is a differential $d_1(\tau) = \rho h_0$ [DI17, Proposition 3.2], and so there are differentials $d_1(\tau^{2n+1}) = \rho\tau^{2n} h_0$. By [DI17, Lemma 3.4], $\tau^{2n} h_0$ is a nonzero permanent cycle in the ρ-Bockstein spectral sequence for every $n > 0$, and hence it represents a nontrivial class in $\mathrm{Ext}_{\mathbb{R}}^{0,1,-2n}$. By the splitting in Proposition 2.7, these are nontrivial classes in Ext_{C_2} as well.

We show that they are permanent cycles in the C_2-Adams spectral sequence. Recall the Adams differential d_r decreases stem by 1, increases filtration by r and preserves weight. Since every element in stem -1 is ρ-free by Lemma 3.1 and $\tau^{2n}h_0$ is ρ-torsion, $\tau^{2n}h_0$ cannot support any differential. Since $\tau^{2n}h_0$ is in filtration 1, it cannot be hit by a differential.

The functor Φ^e induces a map of Adams spectral sequences from the C_2-equivariant Adams spectral sequence to the classical Adams spectral sequence for the sphere, and $\Phi^e(\tau^{2n}h_0) = h_0$ where h_0 detects $2 \in \pi_0^s$. Choose a homotopy representative $\omega_n \in (\pi_{0,-2n}^{C_2})_2^\wedge$ such that $\Phi^e(\omega_n) = 2$. (Note that other choices have image $2u$ for $u \in \mathbb{Z}_2^\times$.) It remains to check that these classes are non-2-divisible in $(\pi_\star^{C_2})_2^\wedge$. Note that any homotopy class x such that $2 \cdot x \in \pi_\star^{C_2}$ is detected by $\tau^{2n}h_0$ would be detected in filtration 0. On the other hand, we claim that $\mathrm{Ext}_{C_2}^{0,0,<0} = 0$. Since $\mathrm{Ext}_{\mathbb{C}}^{*,0,*} = \mathbb{F}_2[\tau]$, Proposition 2.8 shows $\mathrm{Ext}_{NC}^{0,0,<0} = 0$. To see $\mathrm{Ext}_{\mathbb{R}}^{0,\leq 0,<0} = 0$, note $(E_1^+)^{0,0,<0} = \tau\mathbb{F}_2[\tau]$, and these classes all support Bockstein differentials [DI17, Proposition 3.2]. Thus ω_n is non-2-divisible for degree reasons. ☐

Remark 3.4. Balderrama, Culver and Quigley [BCQ21, Theorem 7.4.2] proved a stronger result which determines whether $\rho^r\tau^{2^n(4m+1)}h_i$ is a permanent cycle in $\mathrm{Ext}_{\mathbb{R}}$. This also implies Proposition 3.3.

Proposition 3.5. *For $n > 0$ there are nonzero elements $\omega_{-n} \in (\pi_{0,2n}^{C_2})_2^\wedge$ detected in the C_2-equivariant Adams spectral sequence by $\frac{\gamma}{\tau^{2n-1}}$. They are non-2-divisible and satisfy $\Phi^e(\omega_{-n}) = 2$.*

Proof. According to Proposition 2.8, there are generators $\frac{\gamma}{\tau^{2n-1}}$ for $n \geq 1$ in degree $(0, 2n)$ in E_1^-.

We claim that each $\frac{\gamma}{\tau^{2n-1}}$ is a permanent cycle in Ext_{NC}, and also a permanent cycle in Ext_{C_2}. Indeed, E_1^- and Ext_{NC} vanish in the (-1)-stem by Proposition 2.8 and $\mathrm{Ext}_{\mathbb{R}}$ vanishes in negative coweight: Morel's connectivity theorem [Mor05] implies that $\mathrm{Ext}_{\mathbb{C}}$ vanishes in negative coweight, and hence so does E_1^+. Therefore, the elements $\frac{\gamma}{\tau^{2n-1}}$ are permanent cycles in the ρ-Bockstein and Adams spectral sequences for degree reasons.

Moreover, $\frac{\gamma}{\tau^{2n-1}}$ is non-2-divisible because it is in Adams filtration zero. We have $\Phi^e(\frac{\gamma}{\tau^{2n-1}}) = h_0$ in the E_∞-page according to [Kon20, Proposition 3.5]; therefore, as in Proposition 3.3 we have $\Phi^e(\frac{\gamma}{\tau^{2n-1}}) = 2$, and we may choose homotopy classes ω_{-n} detected by these elements such that $\Phi^e(\omega_{-n}) = 2$. ☐

Corollary 3.6. *For $n \neq 0$, $(\pi_{0,-2n}^{C_2})_2^\wedge/\mathfrak{N}$ is generated by ω_n.*

Proof. Combine Theorem 2.2, Proposition 3.3, and Proposition 3.5. ☐

Now we need to show that the generators in Corollary 3.6 are in the image of the uncompleted homotopy groups.

Lemma 3.7. *For $n \neq 0$ let $T \subseteq \pi_{0,-2n}^{C_2}$ denote the torsion subgroup. Then the completion map*

$$\mathbb{Z} \cong \pi_{0,-2n}^{C_2}/T \to (\pi_{0,-2n}^{C_2})_2^\wedge/T \cong \mathbb{Z}_2$$

sends the generator to ω_n.

Proof. By Theorem 2.2 and Corollary 2.5, $\pi^{C_2}_{0,-2n}$ for $n \neq 0$ is a rank-1 abelian group.

Let $n > 0$. Consider the following diagrams for any prime p, where the vertical maps are completion.

$$\begin{array}{ccc} (\pi^{C_2}_{0,-2n})/T & \xrightarrow{\Phi^e} & \pi^s_0 \\ \downarrow & & \downarrow \\ (\pi^{C_2}_{0,-2n})^\wedge_p/T & \xrightarrow{\Phi^e} & (\pi^s_0)^\wedge_p \end{array} \qquad \begin{array}{ccc} \mathbb{Z} & \xrightarrow{\ell} & \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{Z}_p & \xrightarrow{\ell} & \mathbb{Z}_p \end{array}$$

For $p = 2$, we know from Proposition 3.3 that $\Phi^e(\omega_n) = 2$ and ω_n is not 2-divisible, and so ℓ can be expressed as 2 times a 2-adic unit. For p odd, we have $(\pi^{C_2}_{0,-2n})^\wedge_p \cong \mathbb{Z}_p\{\tau^{2n}\}$ (see [BS20, before Proposition 7.11]) and the underlying map of τ^{2n} is 1 up to p-adic units. This shows that ℓ is invertible in $\mathbb{Z}_p$. Putting together the information at all primes, we have $\ell = 2$. Since $\Phi^e(\omega_n) = 2$ its image in $\pi^{C_2}_{0,-2n}/T$ is a generator.

Next we consider the negative coweight case. The element $\omega_{-1} \in (\pi^{C_2}_{0,2})^\wedge_2$ is detected by an integral class $\frac{\gamma}{\tau} : S^{2,2} \to S^{2,2}/C_2 \simeq S^{2,0}$ which forgets to $2 \in \pi^s_0$ (see [Kon20, Definition 3.4]). Proposition 3.5 says that ω_{-1} is not divisible by 2 in $(\pi^{C_2}_{0,2})^\wedge_2$, and so it is not divisible by 2 in $\pi^{C_2}_{0,2}$. Thus ω_{-1} is a generator of $\pi^{C_2}_{0,2}/T$.

For $n > 1$, we will show that $(\omega_{-1})^n \in \pi^{C_2}_{0,2n}$ is divisible by (exactly) 2^n. Consider the following diagram.

$$\begin{array}{ccc} (\pi^{C_2}_{0,2n})/T & \xrightarrow{\Phi^e} & \pi^s_0 \\ \downarrow & & \downarrow \\ (\pi^{C_2}_{0,2n})^\wedge_2/T & \xrightarrow{\Phi^e} & (\pi^s_0)^\wedge_2 \end{array} \qquad \begin{array}{ccc} \mathbb{Z} & \xrightarrow{\ell} & \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{Z}_2 & \xrightarrow{\ell} & \mathbb{Z}_2 \end{array}$$

Since $\Phi^e((\omega_{-1})^n) = 2^n$ we know that 2^n is in the image of the top row, which implies $\ell \mid 2^n$. Moreover, Proposition 3.5 says that 2 is in the image of the bottom row, and 1 is not in the image. Thus $\ell = 2$, and $(\omega_{-1})^n$ is 2^n-divisible in $(\pi^{C_2}_{0,2n})/T$. $\square$

3.2. **Multiplicative relations in stem 0.**

Theorem 3.8. *As a ring, $\bigoplus_{n\in\mathbb{Z}} \pi^{C_2}_{0,2n}/\mathfrak{N}$ is generated by $1 \in \pi^{C_2}_{0,0}$ and $\omega_n \in \pi^{C_2}_{0,-2n}$ for $n \in \mathbb{Z}$ where $\Phi^e(\omega_n) = 2 \in \pi^s_0$, subject to the relations*

$$\omega_n \cdot \omega_m = 2 \cdot \omega_{n+m}$$

for all $n, m \in \mathbb{Z}$. Moreover, $\rho \cdot \omega_n = 0$ in $\pi^{C_2}_\star/\mathfrak{N}$ for all $n \in \mathbb{Z}$.

Proof. First we verify the additive statement. For $n = 0$, this is Proposition 3.2. For $n \neq 0$, Theorems 2.1 and 2.2 imply $\pi^{C_2}_{0,-2n} \cong \mathbb{Z} \oplus T_n$ for a torsion group T_n, and $\mathfrak{N} \supseteq T_n$. Since ω_n is non-nilpotent and torsion-free, we have $\pi^{C_2}_{0,-2n}/\mathfrak{N} \cong \pi^{C_2}_{0,-2n}/T_n \cong \mathbb{Z}$. Thus the additive description in these degrees follows from Lemma 3.7.

For the multiplicative relations, observe that for degree reasons, $\omega_n \cdot \omega_m \in \mathbb{Z}\{\omega_{n+m}\}$, and we have $\omega_n \cdot \omega_m = 2\omega_{n+m}$ because of the underlying maps $\Phi^e(\omega_n) = \Phi^e(\omega_m) = 2$ from Propositions 3.2, 3.3, and 3.5. For $n \neq 0$, we have $\rho \cdot \omega_n = 0$ in

$\pi_\star^{C_2}/\mathfrak{N}$ for degree reasons. For $n = 0$, the relation $\omega_0 \cdot \rho = 0$ can be observed in $\pi_{*,*}^{\mathbb{R}}$ (see [DI17, p. 2] using the expression for ω_0 in Proposition 3.2). □

Remark 3.9. Guillou, Hill, Isaksen and Ravenel computed the homotopy of ko_{C_2} and $k\mathbb{R}$ [GHIR20], where the Hurewicz images also detect most of these nonzero product relations.

Remark 3.10. William Balderrama also has an argument for the product relations in stem 0, using the transfer maps

$$tr_n : \pi_0^s \to \pi_{0,n}^{C_2}$$

and the double coset formula. His method has more potential for generalization to groups of larger order.

4. Non-nilpotent elements in coweight zero

Proposition 4.1. *For $i > 0$, $\pi_{i,i}^{C_2}/\mathfrak{N} \cong \mathbb{Z}$ is generated by an element x_i satisfying $2^{n(i)}x_i = \eta^i$ modulo $\mathfrak{N}$, where $n(i)$ is the function in Theorem 1.1. For $i < 0$, $\pi_{i,i}^{C_2}/\mathfrak{N} \cong \mathbb{Z}$ is generated by ρ^i.*

Proof. Landweber [Lan69] showed that the image of $\Phi^{C_2} : \pi_{i,i}^{C_2} \to \pi_0^s$ for $i > 0$ is $2^{b(i)}\mathbb{Z}$ for a certain function $b : \mathbb{Z} \to \mathbb{Z}$. It is straightforward to show that

$$b(i) = i - n(i).$$

For $i \geq 1$, let x_i denote an element of $\pi_{i,i}^{C_2}$ satisfying $\Phi^{C_2}(x_i) = 2^{b(i)}$. We have $\Phi^{C_2}(2^{i-b(i)}x_i - \eta^i) = 0$ which implies $2^{i-b(i)}x_i - \eta^i = 2^{n(i)}x_i - \eta^i$ is nilpotent. Moreover, if x is any other element such that $2^{n(i)}x - \eta^i$ is nilpotent, then $x = x_i$ in $\pi_\star^{C_2}/\mathfrak{N}$ since the reduced ring is uniquely divisible. Thus

$$\frac{\eta^i}{2^{n(i)}} =: x_i$$

is a well-defined element of $\pi_{i,i}^{C_2}/\mathfrak{N}$. To see it generates $\pi_{i,i}^{C_2}/\mathfrak{N}$, simply observe that it cannot be p-divisible for any prime p since $\Phi^{C_2}(x_i)/p$ is not in the image of Φ^{C_2} by definition of x_i.

For the statement about $\pi_{i,i}^{C_2}$ for $i < 0$, observe that ρ^i is in a non-nilpotent element in this group, and it must be a generator of $\mathbb{Z}$ because $\Phi^{C_2}(\rho^i) = 1$. In particular, ρ^i cannot be divisible by a non-unit in $\mathbb{Z}$. □

Lemma 4.2. *We have $2\eta = \rho\eta^2$ and $2\rho = \rho^2\eta$ in $\pi_\star^{C_2}$.*

Proof. As ρ, 2, and η are all in the image of $\pi_{*,*}^{\mathbb{R}} \to \pi_{*,*}^{C_2}$, the relations follow from the corresponding $\mathbb{R}$-motivic relations (see [DI17, p. 2]). □

Lemma 4.3. *For $0 \leq j \leq i$, we have that η^i is divisible by 2^j in $\pi_\star^{C_2}/\mathfrak{N}$ if and only if η^{i-j} is divisible by ρ^j in $\pi_\star^{C_2}/\mathfrak{N}$.*

Proof. First we show that $\pi_\star^{C_2}/\mathfrak{N}$ is uniquely η-divisible and ρ-divisible in stems $\neq 0$; i.e., for homogeneous elements $x, y \in \pi_{s,w}^{C_2}/\mathfrak{N}$ with $s \neq 0$, if $\eta x = \eta y$ then $x = y$, and similarly for ρ. If $\rho x = \rho y$, then $x - y$ is ρ-torsion, which implies $\Phi^{C_2}(x - y) = 0$. Since the stem is nonzero, $\Phi^e(x - y) = 0$, and so $x - y$ is nilpotent by Theorem 2.4. If $\eta x = \eta y$, then $\rho^2\eta x = \rho^2\eta y$, and using Lemma 4.2 we have $2\rho x = 2\rho y$. Since $\pi_\star^{C_2}/\mathfrak{N}$ is torsion-free, we have $\rho x = \rho y$, reducing to the previous case.

Suppose $\eta^i = 2^j x$ modulo $\mathfrak{N}$. Then $\eta^i \rho = 2^j \rho x = (\rho\eta)^j \rho x$ by Lemma 4.2 and $\eta^{i-j} = \rho^j x$ by η- and ρ-divisibility. This argument is reversible, establishing the converse. □

The next statement is about the $\mathbb{Z}[\rho]$-module structure of the coweight-zero part of the reduced ring, which is natural from the point of view of the Adams spectral sequence.

Proposition 4.4. *$\bigoplus_{n\in\mathbb{Z}} \pi_{n,n}^{C_2}/\mathfrak{N}$ is generated as a $\mathbb{Z}[\rho]$-module by*

$$\frac{\eta^i}{\rho^{m(i)}} \text{ for } i \geq 1$$

subject to the relations $2\eta = \rho\eta^2$ and $2\rho = \rho^2\eta$, where

$$m(i) = \begin{cases} i-1 & i \equiv 0 \text{ or } 1 \pmod 4 \\ i-2 & i \equiv 2 \pmod 4 \\ i-3 & i \equiv 3 \pmod 4. \end{cases}$$

Proof. The relations are in Lemma 4.2. It is a straightforward computation to use Lemma 4.3 to convert the 2-divisibility information modulo $\mathfrak{N}$ in Proposition 4.1 to η-divisibility information modulo $\mathfrak{N}$: for all i one must check that $m(i-n(i)) \geq n(i)$ but $m(i-(n(i)+1)) < n(i)+1$. □

Remark 4.5. Guillou and Isaksen [GI20] specify the Adams spectral sequence names for the elements $\eta^i/\rho^{m(i)}$. We reproduce this information in Table 1.

TABLE 1. Adams spectral sequence representatives for coweight 0 elements of $\pi_\star^{C_2}/\mathfrak{N}$

stem	name in $\pi_\star^{C_2}/\mathfrak{N}$	name in Ext_{C_2}
1	η	h_1
$8k-1$	$\frac{\eta^{8k-1}}{2^{4k-1}} = \frac{\eta^{4k}}{\rho^{4k-1}}$	$\frac{Q}{\rho^{4k-2}} h_1^{4k}$
$8k+1$	$\frac{\eta^{8k+1}}{2^{4k}} = \frac{\eta^{4k+1}}{\rho^{4k}}$	$\frac{Q}{\rho^{4k-1}} h_1^{4k+1}$

Remark 4.6. One can also prove the results in this section using the Mahowald invariants $M(2^i)$ for $i \geq 1$, which were calculated in [MR93], as input instead of Landweber's result. In the formulation given by Bruner and Greenlees [BG95], the Mahowald invariant describes the ρ-divisibility of C_2-equivariant elements after 2-completion; note that Proposition 4.4 is essentially a result about the ρ-divisibility of η^i for $i \geq 1$.

Remark 4.7. The elements in Proposition 4.4 comprise the Bredon-Landweber region of $\pi_\star^{C_2}$ described in [GI20]. In that paper, Guillou and Isaksen provide an alternate proof of the Mahowald invariants $M(2^i)$. They also illustrated this ρ-divisibility of η^i in [GI20, Figure 2].

5. Proofs of the main theorems

Proofs of Theorems 1.1 *and* 1.3. The additive generators in Theorem 1.1 are given in Propositions 3.2, 3.3, 3.5 and 4.1. By Lemma 4.2 we have

$$(2-\rho\eta)\cdot\eta = [C_2/e]\cdot\eta = 0, \quad (2-\rho\eta)\cdot\rho = [C_2/e]\cdot\rho = 0$$

which concludes the proof of Theorem 1.1. Theorem 1.3(a) and Theorem 1.3(b) are proved as Proposition 3.2 and Theorem 3.8. Theorem 1.3(c) follows from Lemma 4.3 and Proposition 4.4. More explicitly, Proposition 4.4 and the fact that

$$m(i+1) = \begin{cases} m(i)+1 & i\equiv 0 \pmod 4 \\ m(i) & i\equiv 1 \pmod 4 \\ m(i) & i\equiv 2 \pmod 4 \\ m(i)+3 & i\equiv 3 \pmod 4 \end{cases}$$

imply the relations that if $i\equiv 0 \pmod 4$, then

$$\eta\cdot\frac{\eta^i}{\rho^{m(i)}} = \rho\cdot\frac{\eta^{i+1}}{\rho^{m(i+1)}} \tag{5.1}$$

and if $i\equiv 1 \pmod 4$, then

$$\eta^3\cdot\frac{\eta^i}{\rho^{m(i)}} = \rho^3\cdot\frac{\eta^{i+3}}{\rho^{m(i+3)}}. \tag{5.2}$$

Since $\eta = \frac{\eta^1}{\rho^{m(1)}}$ and $\eta\cdot\frac{\eta^i}{\rho^{m(i)}} = \frac{\eta^{i+1}}{\rho^{m(i+1)}}$ if $i\equiv 1,2 \pmod 4$, we do not need to include $\frac{\eta^i}{\rho^{m(i)}}$ for $i\equiv 2,3 \pmod 4$ as ring generators. So the ring generators are $\frac{\eta^i}{\rho^{m(i)}}$ for $i\equiv 0,1 \pmod 4$, and by Lemma 4.3 they correspond to the elements $\frac{\eta^i}{2^{n(i)}}$ for $i\equiv 0,1 \pmod 7$. Similar calculations allow us to convert (5.1) and (5.2) into the relations in Theorem 1.3(c). The relations in Theorem 1.3(d) are true for degree reasons: any nontrivial element of $\pi_\star^{C_2}/\mathfrak{N}$ must have stem zero or coweight zero. □

Proof of Corollary 1.4. By Corollary 2.5, the rationalization of $\pi_\star^{C_2}/\mathfrak{N}$ in Theorem 1.3 is a subring of $\pi_\star^{C_2}(S_{\mathbb{Q}})$. Comparing the rank in each degree with Theorem 2.2, we see that the inclusion is in fact an equality. □

6. A few remarks

Remark 6.1 (Higher multiplicative structure)*.* The Toda bracket structure on $\pi_\star^{C_2}$ does not in general descend to a Toda bracket structure on $\pi_\star^{C_2}/\mathfrak{N}$. However, we may consider Toda brackets in $\pi_\star^{C_2}(\mathbb{S}_{\mathbb{Q}})$. For degree reasons, all three-fold Toda brackets in $\pi_\star^{C_2}(\mathbb{S}_{\mathbb{Q}})$ are 0.

There exists a nontrivial four-fold Toda bracket in $\pi_\star^{C_2}$,

$$\omega_1 \in \langle \omega_0^2, \rho, \omega_0, \rho\rangle.$$

To see this, use the ρ-Bockstein differential $d_1(\tau)=\rho h_0$ (where h_0 detects ω_0) to show that $\tau^2 h_0$ is in the Massey product $\langle h_0^2, \rho, h_0, \rho\rangle$ which has zero indeterminacy for degree reasons, and then use the Moss convergence Theorem [Mos70] to show $\omega_1 \in \langle \omega_0^2, \rho, \omega_0, \rho\rangle$. This Toda bracket in $\pi_\star^{C_2}$ does not contain zero, but it does contain zero after rationalization, as $\omega_1 = \frac{1}{4}\omega_0^2\cdot\omega_1$ in $\pi_\star^{C_2}(\mathbb{S}_{\mathbb{Q}})$.

For another example, the shuffle

$$\langle \omega_0^2, \rho, \omega_0, \rho\rangle \cdot \omega_{-1} = \omega_0^2 \cdot \langle \rho, \omega_0, \rho, \omega_{-1}\rangle$$

shows that $\langle \rho, \omega_0, \rho, \omega_{-1}\rangle$ contains 1 in $\pi^{C_2}_\star$. After rationalization this bracket equals $\pi^{C_2}_{0,0}(\mathbb{S}_\mathbb{Q})$, so in particular it also contains 0.

Remark 6.2. We have implicitly calculated the 2-completed ring structure of $(\pi^{C_2}_\star/\mathfrak{N})^\wedge_2$. It has the same generators and relations as a $\mathbb{Z}_2$-algebra as in Theorem 1.3. In particular, the information can be directly read off using the Mahowald invariant of powers of 2 and the C_2-Adams spectral sequence.

The p-completed ring structure $(\pi^{C_2}_\star/\mathfrak{N})^\wedge_p$ for odd primes p can also be deduced from Theorem 1.3.

References

[BCQ21] William Balderrama, Dominic Leon Culver, and James D. Quigley, *The motivic lambda algebra and motivic Hopf invariant one problem*, Preprint, `arXiv:2112.07479`, (2021).

[BG95] Robert Bruner and John Greenlees, *The Bredon-Löffler conjecture*, Experiment. Math. **4** (1995), no. 4, 289–297. MR1387694

[BGH20] Tobias Barthel, John P. C. Greenlees, and Markus Hausmann, *On the Balmer spectrum for compact Lie groups*, Compos. Math. **156** (2020), no. 1, 39–76, DOI 10.1112/s0010437x19007656. MR4036448

[BI20] Eva Belmont and Daniel C. Isaksen, *$\mathbb{R}$-motivic Adams charts*, `https://s.wayne.edu/isaksen/adams-charts`, 2020.

[BI22] Eva Belmont and Daniel C. Isaksen, *$\mathbb{R}$-motivic stable stems*, J. Topol. **15** (2022), no. 4, 1755–1793, DOI 10.1112/topo.12256. MR4461846

[BS17] Paul Balmer and Beren Sanders, *The spectrum of the equivariant stable homotopy category of a finite group*, Invent. Math. **208** (2017), no. 1, 283–326, DOI 10.1007/s00222-016-0691-3. MR3621837

[BS20] Mark Behrens and Jay Shah, *C_2-equivariant stable homotopy from real motivic stable homotopy*, Ann. K-Theory **5** (2020), no. 3, 411–464, DOI 10.2140/akt.2020.5.411. MR4132743

[DI10] Daniel Dugger and Daniel C. Isaksen, *The motivic Adams spectral sequence*, Geom. Topol. **14** (2010), no. 2, 967–1014, DOI 10.2140/gt.2010.14.967. MR2629898

[DI17] Daniel Dugger and Daniel C. Isaksen, *Low-dimensional Milnor-Witt stems over $\mathbb{R}$*, Ann. K-Theory **2** (2017), no. 2, 175–210, DOI 10.2140/akt.2017.2.175. MR3590344

[GHIR20] Bertrand J. Guillou, Michael A. Hill, Daniel C. Isaksen, and Douglas Conner Ravenel, *The cohomology of C_2-equivariant $\mathcal{A}(1)$ and the homotopy of* ko_{C_2}, Tunis. J. Math. **2** (2020), no. 3, 567–632, DOI 10.2140/tunis.2020.2.567. MR4041284

[GI20] Bertrand J. Guillou and Daniel C. Isaksen, *The Bredon-Landweber region in C_2-equivariant stable homotopy groups*, Doc. Math. **25** (2020), 1865–1880. MR4184454

[GM95] John P. C. Greenlees and J. Peter May, *Generalized Tate cohomology*, Mem. Amer. Math. Soc. **113** (1995), no. 543, viii+178, DOI 10.1090/memo/0543. MR1230773

[GQ22] John P. C. Greenlees and James D. Quigley, *Ranks of $RO(G)$-graded stable homotopy groups of spheres for finite groups G*, Proc. Amer. Math. Soc. Ser. B **10** (2023), 101–113, DOI 10.1090/bproc/140. MR4569751

[Hil11] Michael A. Hill, *Ext and the motivic Steenrod algebra over $\mathbb{R}$*, J. Pure Appl. Algebra **215** (2011), no. 5, 715–727, DOI 10.1016/j.jpaa.2010.06.017. MR2747214

[Iri83] Kouyemon Iriye, *The nilpotency of elements of the equivariant stable homotopy groups of spheres*, J. Math. Kyoto Univ. **22** (1982/83), no. 2, 257–259, DOI 10.1215/kjm/1250521814. MR666975

[IWX23] Daniel C. Isaksen, Guozhen Wang, and Zhouli Xu, *Stable homotopy groups of spheres: from dimension 0 to 90*, Publ. Math. Inst. Hautes Études Sci. **137** (2023), 107–243, DOI 10.1007/s10240-023-00139-1. MR4588596

[Kon20] Jia Kong, *The C_2-Effective Spectral Sequence for C_2-Equivariant Connective Real K-Theory*, ProQuest LLC, Ann Arbor, MI, 2021. Thesis (Ph.D.)–The University of Chicago. MR4297464

[Lan69] Peter S. Landweber, *On equivariant maps between spheres with involutions*, Ann. of Math. (2) **89** (1969), 125–137, DOI 10.2307/1970812. MR238313

[Lin80] Wen Hsiung Lin, *On conjectures of Mahowald, Segal and Sullivan*, Math. Proc. Cambridge Philos. Soc. **87** (1980), no. 3, 449–458, DOI 10.1017/S0305004100056887. MR556925

[Mor05] Fabien Morel, *The stable* $\mathbb{A}^1$*-connectivity theorems*, K-Theory **35** (2005), no. 1-2, 1–68, DOI 10.1007/s10977-005-1562-7. MR2240215

[Mos70] R. Michael F. Moss, *Secondary compositions and the Adams spectral sequence*, Math. Z. **115** (1970), 283–310, DOI 10.1007/BF01129978. MR266216

[MR93] Mark E. Mahowald and Douglas C. Ravenel, *The root invariant in homotopy theory*, Topology **32** (1993), no. 4, 865–898, DOI 10.1016/0040-9383(93)90055-Z. MR1241877

[Nis73] Goro Nishida, *The nilpotency of elements of the stable homotopy groups of spheres*, J. Math. Soc. Japan **25** (1973), 707–732, DOI 10.2969/jmsj/02540707. MR341485

[Seg71] Graeme B. Segal, *Equivariant stable homotopy theory*, Actes du Congrès International des Mathématiciens (Nice, 1970), Gauthier-Villars Éditeur, Paris, 1971, pp. 59–63. MR423340

Department of Mathematics, Applied Mathematics, and Statistics, Case Western Reserve University, Cleveland, Ohio 44106

Email address: `eva.belmont@case.edu`

Department of Mathematics, University of California San Diego, La Jolla, California 92093

Email address: `xuzhouli@ucsd.edu`

Department of Mathematics, University of California San Diego, La Jolla, California 92093

Email address: `shz046@ucsd.edu`